COMP1805A (Summer 2017) – "Discrete Structures I"

Tutorial 7

1. Show that $3n^2 - 2n + 17$ is $O(n^2)$ and $O(n^3)$ and $\Omega(n)$ and $\Omega(n^2)$.

…is $O(n^2)$ **for $n \geq 1$**

$$3n^2 - 2n + 17 \leq 3n^2 + 17 \leq 3n^2 + 17n^2 = 20n^2$$

…is $O(n^3)$ **for $n \geq 1$**

$$3n^2 - 2n + 17 \leq 3n^2 + 17 \leq 3n^2 + 17n^2 = 20n^2 \leq 20n^3$$

…is $\Omega(n)$ **for $n \geq 1$**

$$3n^2 - 2n + 17 \geq 3n^2 - 2n \geq 3n^2 - 2n^2 = n^2 \geq n$$

…is $\Omega(n^2)$ **for $n \geq 1$**

$$3n^2 - 2n + 17 \geq 3n^2 - 2n \geq 3n^2 - 2n^2 = n^2$$

2. Assume that the domain of all functions is the positive integers, and recall the definitions for O, Ω, and Θ in order to prove or disprove each of the following.
   a. $f(n) = n^2 - 3n + 4$ $\quad g(n) = n^2$
   b. $f(n) = 3n^2 - 6$ $\quad g(n) = n^2 / 300$
   c. $f(n) = 4n^2 \log n - 4n - 4$ $\quad g(n) = n^2$
   d. $f(n) = 10 \log n$ $\quad g(n) = n^2$

2. We provide the definition of O, Ω and Θ below. The domain of all functions is the positive integers.
$f(n) \in O(g(n))$ provided that $f(n) \leq cg(n), \forall n \geq d$, for constants $c, d > 0$.
$f(n) \in O(g(n))$ provided that $\lim_{n\to\infty} f(n)/g(n) \leq c$ for constant $c > 0$.
$f(n) \in \Omega(g(n))$ provided that $f(n) \geq cg(n), \forall n \geq d$, for constants $c, d > 0$.
$f(n) \in \Omega(g(n))$ provided that $\lim_{n\to\infty} f(n)/g(n) \geq c$ for constant $c > 0$.
$f(n)$ is $\Theta(g(n))$ provided that $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

Determine if $f(n)$ is $O(g(n))$ or $f(n)$ is $\Omega(g(n))$ or $f(n)$ is $\Theta(g(n))$ for the following:

- $f(n) = n^2 - 3n + 4$, $g(n) = n^2$

  Solution: We will show that $f(n) = O(g(n))$.

$$\begin{aligned} f(n) &= n^2 - 3n + 4 \\ &\le n^2 + 4 \\ &\le n^2 + 4n^2 && \text{for } n \ge 1 \\ &= 5n^2 \\ &= 5 \cdot g(n) \end{aligned}$$

  Therefore, $f(n) = O(g(n))$. Now we will show that $f(n) = \Omega(g(n))$.

$$\begin{aligned} f(n) &= n^2 - 3n + 4 \\ &\ge n^2 - 3n \\ &= \frac{1}{2}n^2 + \left(\frac{1}{2}n^2 - 3n\right) \\ &\ge \frac{1}{2}n^2 && \text{for } n \ge 6 \text{, since } \frac{1}{2}n^2 \ge 3n \text{ for all } n \ge 6 \\ &= \frac{1}{2} \cdot g(n) \end{aligned}$$

  Therefore, $f(n) = \Omega(g(n))$. Since $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, $f(n) = \Theta(g(n))$.

- $f(n) = 3n^2 - 6$, $g(n) = n^2/300$

  Solution: We will show that $f(n) = O(g(n))$.

$$\begin{aligned} f(n) &= 3n^2 - 6 \\ &\le 3n^2 \\ &= \frac{900}{300} \cdot n^2 \\ &= 900 \cdot g(n) \end{aligned}$$

  Therefore, $f(n) = O(g(n))$. Now we will show that $f(n) = \Omega(g(n))$.

$$\begin{aligned} f(n) &= 3n^2 - 6 \\ &= n^2 + (2n^2 - 6) \\ &\ge n^2 && \text{for } n \ge 2 \text{, since } 2n^2 \ge 6 \text{ for all } n \ge 2 \\ &\ge \frac{1}{300} \cdot n^2 \\ &= g(n) \end{aligned}$$

  Therefore, $f(n) = \Omega(g(n))$. Since $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$, $f(n) = \Theta(g(n))$.

- $f(n) = 4n^2 \log n - 4n - 4$, $g(n) = n^2$

  Solution: We will show that $f(n)$ is not $O(g(n))$. Assume that $f(n) = O(g(n))$. Then there exist constants $c > 0$ and $k \geq 1$ such that

$$\begin{aligned} f(n) &\leq c \cdot g(n) && \text{for } n \geq k \\ 4n^2 \log n - 4n - 4 &\leq c \cdot n^2 \\ 4 \log n - \frac{4}{n} - \frac{4}{n^2} &\leq c \end{aligned}$$

  Note that

$$\log n = 4 \log n - 2 \log n - \log n \leq 4 \log n - \frac{4}{n} - \frac{4}{n^2}$$

  for $n \geq 4$ since $2 \log n \geq 4/n$ and $\log n \geq 4/n^2$ for all $n \geq 4$. From our assumption we have

$$\log n \leq c \qquad\qquad \text{for } n \geq \max 4, k$$

  But for all $n > 2^c$ we have $\log n > \log 2^c = c$ which contradicts our assumption. Therefore, $f(n)$ is not $O(g(n))$. Now we will show that $f(n) = \Omega(g(n))$.

$$\begin{aligned} f(n) &= 4n^2 \log n - 4n - 4 \\ &\geq 4n^2 - 4n - 4 \\ &= n^2 + (2n^2 - 4n) + (n^2 - 4) \\ &\geq n^2 && \text{for } n \geq 2, \text{ since } 2n^2 \geq 4n \text{ and } n^2 \geq 4 \text{ for all } n \geq 2 \\ &= g(n) \end{aligned}$$

  Therefore, $f(n) = \Omega(g(n))$.

- $f(n) = 10 \log n$, $g(n) = n^2$

  Solution: We will show that $f(n) = O(g(n))$.

$$\begin{aligned} f(n) &= 10 \log n \\ &\leq 10n^2 \\ &= 10 \cdot g(n) \end{aligned}$$

Therefore, $f(n) = O(g(n))$. Now we will show that $f(n)$ is not $\Omega(g(n))$. Assume that $f(n) = \Omega(g(n))$. Then there exist constants $c > 0$ and $k \geq 1$ such that

$$\begin{aligned} f(n) &\geq c \cdot g(n) & \text{for } n \geq k \\ 10 \log n &\geq c \cdot n^2 \\ \frac{10 \log n}{n^2} &\geq c \end{aligned}$$

Note that

$$\frac{10}{n} = \frac{10n}{n^2} \geq \frac{10 \log n}{n^2}.$$

Then from our assumption we have

$$\frac{10}{n} \geq c$$

but for all $n > 10/c$ we have

$$\frac{10}{n} < \frac{10}{10/c} = c$$

which contradicts our assumption. Therefore, $f(n)$ is not $\Omega(g(n))$.