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2.

Let $f \geq 4$ and $m \geq 4$ be integers.
Let f represent female students, eligible to TA for COMP 2804.
Let m represent male students, eligible to TA for COMP 2804.
Let n represent the number of ways to choose eight TA's out of $f + m$ students, such that the number of female TAs is equal to the number of male TAs.

$n = f \cdot m$

We choose 4 female TAs from female students $\binom{f}{4}$ and we choose 4 male TAs from male students $\binom{m}{4}$. We then use the product rule ($n = (n_1) \cdot (n_2) \cdot (n_3) \ldots (n_m)$) to determine the number of ways to choose eight TAs out of $f + m$ students.

$$n = \binom{f}{4}\binom{m}{4}$$

$\therefore$ there are $n = \binom{f}{4}\binom{m}{4}$ ways to choose eight TAs out of f + m students, such that the number of female TAs is equal to the number of male TAs.

3.

Let $|A|$ represent the number of strings that start and end with the same letter.
Let $|B|$ represent the number of string that are palindromes.
Let $|C|$ represent the number of strings that only contain vowels.

i) $|A| = 26^{27}$
There are 26 letters to choose from, in a string of 27 characters. There are 27 characters to choose from because the start letter choice also counts as your end letter choice.

ii) $|B| = 26^{14}$
There are 26 letters to choose from, in a string of 14 characters. There are 14 characters to choose from because it is a palindrome (symmetrical string). Every character choice counts as two choices.

iii) $|C| = 5^{28}$
There are 5 letters to choose from, in a string of 28 characters. There are 5 letters to choose from because the string only contains vowels, and the string consists of 28 characters by default.

$|A \cap B| = 26^{14}$
There are 26 letters to choose from, in a string of 14 characters. There are 14 characters to choose from because the string is a palindrome, which encompasses the same trait as set $|A|$ (start letter = end letter).

$|A \cap C| = 5^{27}$
There are 5 letters to choose from, in a string of 27 characters. There are 5 letters to choose from because the string only contains vowels, and the string has the first and last letter as the same.

$|B \cap C| = 5^{14}$
There are 5 letters to choose from, in a string of 14 characters. There are 5 letters to choose from because the string only contains vowels, and the string is a palindrome, where each choice is reflected twice.

$|A \cap B \cap C| = 5^{14}$
There are 5 letters to choose from, in a string of 14 characters. There are 5 letters to choose from because vowels can only be used, the string is a palindrome (length/2) and the first and last letter are the same (like a palindrome).

By using the Inclusion/Exclusion principle we can deduce the number of strings that satisfy all three conditions (i, ii, iii).

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$|A \cup B \cup C| = 26^{27} + 26^{14} + 5^{28} - 26^{14} - 5^{27} - 5^{14} + 5^{14}$
$|A \cup B \cup C| = 26^{27} + 5^{28} - 5^{27}$

$\therefore$ the number of strings that satisfy the set i or ii or iii is: $26^{27} + 5^{28} - 5^{27}$

4.
Let $n \geq 1$ be an integer.
Let n_{total} represent the total number of possible functions mapped, with no restrictions.
Let n_{NA} represent the total number of possible functions mapped, where $f(i) \neq i$ (is <u>not awesome</u>).
Let n_A represent the total number of possible functions mapped, where there is at least one integer i in $\{1,2,3 \dots, n\}$ for which $f(i) = i$ (is <u>awesome</u>).

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ n \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \\ . \\ . \\ . \\ n \end{pmatrix}$$

$n_{total} = n^n$
The total number of functions mapped is n^n because you have n elements mapping to n elements.

$n_{NA} = (n-1)^n$
The total number of <u>not awesome</u> functions mapped is $(n-1)^n$ because for every n elements, n cannot map to itself, therefore you are always removing one element.

Using the complement rule $|A| = |U| - |U \backslash A|$ we can determine the awesome functions by subtracting the not awesome functions from all functions mapped.
$n_A = n_{total} - n_{NA}$
$n_A = n^n - ((n-1)^n)$

$\therefore$ using we have proved the number of awesome functions is equivalent to $n^n - ((n-1)^n)$.

5.

Let $n \geq 4$ be an integer. Set $S = \{1,2, \dots, n\}$. Let k be an integer with $2 \leq k \leq n-2$. A of S for which $\lvert A\rvert = k$ and $\{1,2\} \subseteq A$.	$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$

$\binom{n-2}{k}$
This set represents when 1 and 2 is not in the subset $|A|$.

$2\binom{n-2}{k-1}$
This set represents when 1 or 2 is in the subset $|A|$.

$\binom{n-2}{k-2}$
This set represents when 1 and 2 is in the subset $|A|$.

Determining N from the Sum Rule:
$N = \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-1}$

Determining N from the Complement Rule:
$N = \binom{n}{k} - \binom{n-2}{k-2}$

Using the above two results we can prove original equation:

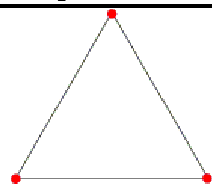
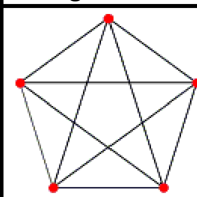
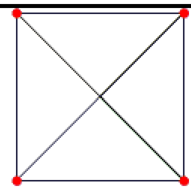
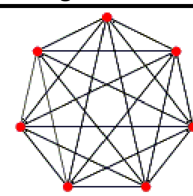
LHS	RHS
$N = N$	$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$
$\binom{n}{k} - \binom{n-2}{k-2} = \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-1}$	…
$\binom{n}{k} - \binom{n-2}{k-2} = \binom{n-2}{k} + 2\binom{n-2}{k-1}$	…
$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$	$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}$

∴ through combining our two results from the Sum Rule and Complement Rule, we have shown that those results prove the original equation.

6.
$k \geq 1$. Sequence $n_1, n_2, \dots, n_k$.
$\binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_k}{2} \leq \binom{n_1 + n_2 + \cdots + n_k}{2}$

$n_e = \left(\frac{n_v\,(n_v - 1)}{2}\right)$
To determine the number of edges (n_e) on a graph this formula can be used, where n_v represents the number of vertices on a graph.

By comparing nodes and vertices with the following chart we prove the above equation:

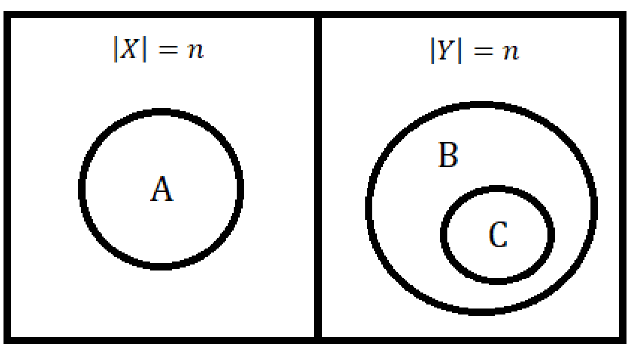
$\binom{n_1}{2}+\binom{n_2}{2}+\cdots+\binom{n_k}{2}$		$\leq$	$\binom{n_1+n_2+\cdots+n_k}{2}$
LHS		RHS	
# Of Vertices	# Of Edges	# Of Vertices	# Of Edges
2	1 Edges	2	1 Edges
3	 3 Edges	3	 5 Edges
4	 6 Edges	4	 7 Edges

If you add the number of edges from the first and second row (1+3=4), they will always be less than or equal to the number of edges of the second row (5).

7.

Let $n \geq 1$ be an integer. X and Y be two disjoint sets, each consisting of n elements.
$A \subseteq X, B \subseteq Y, C \subseteq B, |A| + |B| = n$

Let k be an integer with $0 \leq k \leq n$. Determine the number of cool triples (A, B, C) for which $|A| = k$.



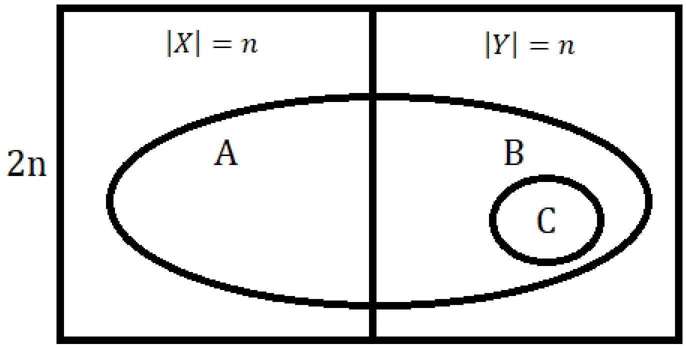
$|A| = \binom{n}{k}$
You are choosing k elements from n, which is in set $|X|$.

$|B| = \binom{n}{n-k}$
If set $|A| = k$, and $|A| + |B| = n$, then $|B| = n - k$. You are choosing $n - k$ elements from n, which is in set $|Y|$.

$|C| = 2^{n-k}$
The number of subsets is determined using 2^n, where $n = |B|$, which we defined as $n - k$. Therefore, the number of subsets for $|C|$ is 2^{n-k}.

$\therefore$ when $|A| = k$ the number of cool triples is $\binom{n}{k}\binom{n}{n-k} \cdot 2^{n-k}$ OR $\binom{n}{k}^2 \cdot 2^{n-k}$.

Let k be an integer with $0 \leq k \leq n$. Determine the number of cool triples (A, B, C) for which $|C| = k$.



$|C| = \binom{n}{k}$
In this case $|C| = k$, so you are choosing k elements from n.

To determine our element choices for $|A \cap B|$ we must use the complement rule:
$|U| - |U \backslash A|$
$|X| - |Y \backslash C|$
$n + (n - k)$
$2n - k$
We removed the $|C|$ set from $|Y|$ to determine $|A \cap B|$.

$|A \cap B| = \binom{2n-k}{n}$
You are choosing n elements from sets $|A \cap B|$, and are excluding k, which is $|C|$.

$\therefore$ when $|C| = k$ the number of cool triples is $\binom{n}{k}\binom{2n-k}{n}$
$\therefore$ The two above results have shown that $\binom{n}{k}^2 \cdot 2^{n-k} = \binom{n}{k}\binom{2n-k}{n}$.

8.
$d_1 d_2 d_3 d_4 d_5$, whose digits are decreasing, i.e., $d_1 > d_2 > d_3 > d_4 > d_5$

A digit is comprised of numbers 0 to 9, where $n = 10$.
A 5-digit decreasing sequence means you must choose 5 distinct numbers, where $k = 5$.

$\therefore$ if $\binom{n}{k}$, $\binom{10}{5}$ is the number of 5-digit sequences whose digits are decreasing.

$d_1 d_2 d_3 d_4 d_5$, whose digits are non-increasing, i.e., $d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5$

If we consider the numbers: $x_1 = d_1 - d_2, x_2 = d_2 - d_3, x_3 = d_3 - d_4, x_4 = d_4 - d_5, x_5 = d_5$, and we provide $d_1 d_2 d_3 d_4 d_5$ with values such as: $d_1 = 9, d_2 = 6, d_3 = 4,\ d_4 = 4, d_5 = 2$.

The values we receive will be:
$x_1 = d_1 - d_2 = 9 - 6 = 3$
$x_2 = d_2 - d_3 = 6 - 4 = 2$
$x_3 = d_3 - d_4 = 4 - 4 = 0$
$x_4 = d_4 - d_5 = 4 - 2 = 2$
$x_5 = d_5 = 2$

The sum of all the values (3+2+0+2+2) will always equal 9 and that will be our maximum possible digit, n. We also know the number of digits we are adding is 5 digits, which will be our k value.

We can use the inequality formula $\binom{n+k}{k}$ to help determine the number of 5-digit sequences that are non-increasing.

∴ if $\binom{n+k}{k}$, $\binom{14}{5}$ is the number of 5-digit sequences whose digits are non-increasing.

9.
Let $S_1, S_2, \ldots, S_{26}$ be sequence of 26 subsets of the set {1, 2…,9}. The 26 subsets consist of at most three elements.

If we begin to map the subsets we can show that there exists a sum from 0 = {empty set} up to the sum of 24 = {7,8,9}. The elements in the subset are distinct digits from 1 to 9, which creates the upper bound for the sum of 24, and lower bound for the empty set of 0. The total number of sums or *pigeonholes* is 25 (including the empty set).

It is not possible for there to be a 26th sum because the lower bound was 0 and the upper bound was 24. This brings 25 sums or *pigeonholes* and that cannot fit the 26th subset or *pigeon*. There are more subsets than there are sums proving there is an extra subset or *pigeon*.