COMP 2804 — Assignment 1

Due: Thursday October 5, before 11:59pm, through cuLearn.

Assignment Policy:

- Your assignment must be submitted as a PDF file through cuLearn. Details about how to submit will be announced later.
- Late assignments will not be accepted.
- You are encouraged to collaborate on assignments, but at the level of discussion only. When writing your solutions, you should do so in your own words.
- Past experience has shown conclusively that those who do not put adequate effort into the assignments do not learn the material and have a probability near 1 of doing poorly on the exams.
- When writing your solutions, you must follow the guidelines below.
  - You must justify your answers.
  - The answers should be concise, clear and neat.
  - When presenting proofs, every step should be justified.

Question 1: On the first page of your assignment, write your name and student number.

Question 2: Let $f \geq 4$ and $m \geq 4$ be integers. The Carleton Computer Science program has f female students and m male students that are eligible to be a TA for COMP 2804. Determine the number of way to choose eight TAs out of these $f + m$ students, such that the number of female TAs is equal to the number of male TAs.

Question 3: A string of letters is called a *palindrome*, if reading the string from left to right gives the same result as reading the string from right to left. For example, *madam* and *racecar* are palindromes. Recall that there are five vowels in the English alphabet: a, e, i, o, and u.

In this question, we consider strings consisting of 28 characters, with each character being a lowercase letter. Determine the number of such strings that (i) start and end with the same letter, or (ii) are palindromes, or (iii) contain vowels only.

Question 4: Let $n \geq 1$ be an integer. A function $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is called *awesome*, if there is at least one integer i in $\{1, 2, \dots, n\}$ for which $f(i) = i$.

Determine the number of awesome functions.

Question 5: Let $n \geq 3$ be an integer and consider the set $S = \{1, 2, \ldots, n\}$. Let k be an integer with $2 \leq k \leq n-2$. In this question, we consider subsets A of S for which $|A| = k$ and $\{1,2\} \not\subseteq A$. Let N denote the number of such subsets.

- Use the Sum Rule to determine N.
- Use the Complement Rule to determine N.
- Use the above two results to prove that

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$

Question 6: Let $k \geq 1$ be an integer and consider a sequence $n_1, n_2, \ldots, n_k$ of positive integers. Use a combinatorial proof to show that

$$\binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_k}{2} \leq \binom{n_1 + n_2 + \cdots + n_k}{2}.$$

Hint: You will not get any marks if you use an induction proof. For each i with $1 \leq i \leq k$, consider the complete graph on n_i vertices. How many edges does this graph have?

Question 7: Let $n \geq 1$ be an integer, and let X and Y be two disjoint sets, each consisting of n elements. An ordered triple (A, B, C) of sets is called *cool*, if

$$A \subseteq X, B \subseteq Y, C \subseteq B, \text{ and } |A| + |B| = n.$$

- Let k be an integer with $0 \leq k \leq n$. Determine the number of cool triples (A, B, C) for which $|A| = k$.
- Let k be an integer with $0 \leq k \leq n$. Determine the number of cool triples (A, B, C) for which $|C| = k$.
- Use the above two results to prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 \cdot 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k}\binom{2n-k}{n}.$$

Question 8: In this question, we consider sequences consisting of five digits.

- Determine the number of 5-digit sequences $d_1d_2d_3d_4d_5$, whose digits are decreasing, i.e., $d_1 > d_2 > d_3 > d_4 > d_5$.
- Determine the number of 5-digit sequences $d_1d_2d_3d_4d_5$, whose digits are non-increasing, i.e., $d_1 \geq d_2 \geq d_3 \geq d_4 \geq d_5$.

  Hint: Consider the numbers $x_1 = d_1 - d_2, x_2 = d_2 - d_3, x_3 = d_3 - d_4, x_4 = d_4 - d_5, x_5 = d_5$. What do you know about $x_1 + x_2 + x_3 + x_4 + x_5$? You may use any result that was proven in class.

Question 9: Let $S_1, S_2, \ldots, S_{26}$ be a sequence consisting of 26 subsets of the set $\{1, 2, \ldots, 9\}$. Assume that each of these 26 subsets consists of at most three elements.

Use the Pigeonhole Principle to prove that there exist two distinct indices i and j, such that

$$\sum_{x \in S_i} x = \sum_{x \in S_j} x,$$

i.e., the sum of the elements in S_i is equal to the sum of the elements in S_j.
Hint: What are the possible values for $\sum_{x \in S_i} x$?