Your submission **must be created using Microsoft Word, Google Docs, or LaTeX**.

Your submission **must be saved as a single "pdf"** document and **have the name "a2.lastname.firstname.pdf"**

Do not compress your submission into a "zip" file.

Late assignments will not be accepted and will receive a mark of 0.

Submissions **written by hand**, **compressed into an archive**, or submitted in the **wrong format** (i.e., are not "pdf" documents) **will receive a mark of 0.**

Always show all of your work.

The due date for this assignment is May 23th, 2017, by 11:30pm.

1. Determine whether or not the following arguments are valid. If they are valid, then state the rules of inference used to prove validity. If they are invalid, outline precisely why they are invalid. [6 marks]

   a. If it is snowing then I bring my hat to work. I bring my hat. Therefore it must be snowing.

Solution: Invalid.
Let s = "It is snowing" and h = "I bring my hat". The argument is then:

$s \rightarrow h$
h
$\therefore s$

Consider the case where I bring my hat with me every day, but it doesn't snow every day. So, we don't actually know if it is snowing or not because I always have my hat and we cannot conclude that it is snowing.

   b. Everyone who has a PC plays computer games. Everyone taking COMP1501 next semester plays computer games. Therefore every student taking COMP1501 next semester has a PC.

Let P(x) be "x has a pc", G(x) be "x plays games ", C(x) be "x is taking 1501". The argument is then:

$\forall x\ P(x) \rightarrow G(x)$
$\forall x\ C(x) \rightarrow G(x)$

∴ ∀x C(x) → P(x)

Universal instantiation would allow the first and second lines to become P(me) → G(me) and C(me) → G(me), respectively. Consider the case where I do not have a PC but I do play computer games and I am taking 1501. For this case P(me) → G(me) would be true (since the antecedent is false) and C(me) → G(me) would also be true because both the antecedent and consequent are true. The conclusion C(me) → P(me), however, would be false because the antecedent is true and the consequent is false. For this case all premises are true but the conclusion is false, so the argument is invalid.

For your own edification, here is a case that you might have tried but that wouldn't have worked. Consider the case where I do not have a PC and I do not play games but I am taking 1501. Although P(me) → G(me) would still be true because of the false antecedent, C(me) → G(me) would be false because the antecedent is true but the consequent is false. Since argument validity only applies when the premises are true, I cannot use this case to show invalidity because not all the premises are true in this case.

2. Prove that $\sqrt{9}+\sqrt{7}$ is an irrational number, using a proof by contradiction **without** prime factorization. [30 marks]

Proof by contradiction.

Assume $\sqrt{9}+\sqrt{7}$ is rational.

Let r_i be used to denote unspecified rational numbers.

Assume $\sqrt{9}+\sqrt{7}=r_1$

$\pm 3+\sqrt{7}=r_1$ by math

$\sqrt{7}=r_1 \pm 3$ by math

Lemma 1

Prove 3 is a rational number

(please note this lemma is unnecessary because 3 is an integer and, thus, a rational)

3 = 3/1

3/1 is in lowest form

(because there is no common integer by which numerator and denominator are both divisible)

3 = 3/1 ∧ 3/1 is in lowest form

∴ 3 is rational

$\sqrt{7}=r_1 \pm 3$ by math

$\sqrt{7}=r_1+r_2$ by Lemma 1

$\sqrt{7}=r_3$ by the closure of rationals under addition

$\sqrt{7}=\frac{a}{b} \wedge \frac{a}{b}$ is in lowest form by the definition of rationals

$\sqrt{7}=\frac{a}{b}$ by simplification

$7 = a^2 / b^2$
$7b^2 = a^2$
a^2 is divisible by 7
$a^2 = 7k$

Lemma 2

Prove a^2 is divisible by 7 $\rightarrow$ a is divisible by 7
Prove $\neg a^2$ is divisible by 7 $\vee$ a is divisible by 7 by implication equivalence

Proof by contradiction

Assume $\neg(\neg a^2$ is divisible by 7 $\vee$ a is divisible by 7)

$\neg\neg a^2$ is divisible by 7 $\wedge$ $\neg$a is divisible by 7	by demorgan's law
a^2 is divisible by 7 $\wedge$ $\neg$a is divisible by 7	by double negation
$a^2 = 7k \wedge \neg$a is divisible by 7	by definition

Remember that a number x being divisible by 7 means x mod 7 == 0
So a number x being NOT divisible by 7 means x mod 7 == 1, 2, 3, 4, 5, or 6
Also remember that x mod y = z means x = yk + z

Proof by Cases

Case 1 of 6:

$a^2 = 7k \wedge a = 7k + 1$	
$a = 7k + 1$	by simplification
$a^2 = (7k + 1) (7k + 1)$	by math
$a^2 = 49k^2 + 14k + 1$	by math
$a^2 = 7(7k^2 + 2k) + 1$	by math
$a^2 = 7j + 1$	by math
$a^2 = 7k$	by simplification (of assumption)
$a^2 = 7k \wedge a^2 = 7j + 1$	by conjunction
False	by negation

Case 2 of 6:

$a^2 = 7k \wedge a = 7k + 2$	
$a = 7k + 2$	by simplification
$a^2 = (7k + 2) (7k + 2)$	by math
$a^2 = 49k^2 + 28k + 4$	by math
$a^2 = 7(7k^2 + 4k) + 4$	by math
$a^2 = 7j + 4$	by math
$a^2 = 7k$	by simplification (of assumption)
$a^2 = 7k \wedge a^2 = 7j + 4$	by conjunction
False	by negation

Case 3 of 6:

$a^2 = 7k \wedge a = 7k + 3$	
$a = 7k + 3$	by simplification
$a^2 = (7k + 3) (7k + 3)$	by math
$a^2 = 49k^2 + 42k + 9$	by math

$a^2 = 7(7k^2 + 6k) + 9$	by math
$a^2 = 7j + 9$	by math
$a^2 = 7k$	by simplification (of assumption)
$a^2 = 7k \wedge a^2 = 7j + 9$	by conjunction
False	by negation

**Alternatively, you could also split the remainder into a number that is divisible by seven and one that is not… i.e. $a^2 = 49k^2 + 42k + 9 = 49k^2 + 47k + 7 + 2 = a^2 = 7(7k^2 + 6k + 1) + 2$ and get $a^2 = 7j + 2$*

Repeat for remaining 3 remainders (cases 4, 5, and 6 not shown here but are required)

back to original proof

$a^2 = 7k$	repeating from before Lemma 2
$a = 7j$	by Lemma 2
$7b^2 = a^2$	repeating from before Lemma 2
$7b^2 = (7j)^2$	by math (i.e., substitute)
$7b^2 = 49j^2$	by math (i.e., substitute)
$b^2 = 7j^2$	by math (i.e., substitute)
b^2 is divisible by 7	by definition
b is divisible by 7	by Lemma 2
b is divisible by 7 $\wedge$ a is divisible by 7	by conjunction
$\frac{a}{b}$ is NOT in lowest form	by definition
$\frac{a}{b}$ is in lowest form	from before
$\frac{a}{b}$ is in lowest form $\wedge$ $\frac{a}{b}$ is NOT in lowest form	by conjunction
False	

$\therefore \sqrt{9} + \sqrt{7}$ is irrational

3. Prove that $\sqrt{25} + \sqrt{5}$ is an irrational number, using a proof by contradiction and prime factorization.

[10 marks]

1. Assume: $\sqrt{25} + \sqrt{5}$ is a rational number
2. $\sqrt{25} + \sqrt{5} = \frac{a}{b} \wedge$ (a and b are in lowest form) by definition of a rational number
3. $\sqrt{25} + \sqrt{5} = \frac{a}{b}$ by simplification
4. $\pm 5 + \sqrt{5} = \frac{a}{b}$ by math
5. $\sqrt{5} = \frac{a}{b} \pm 5$ by math
6. 5 is rational because $5 = \frac{p}{q}$ where p and q are integers in simplest form and $q \neq 0$. i.e. $\frac{2}{1}$ or so 2 is rational. (this step is unnecessary)
7. $\sqrt{5} = \frac{a}{b}$ by rational numbers are closed under addition and subtraction

8. $5 = \frac{a^2}{b^2}$ (by math)
9. $5b^2 = a^2$ (by math)

Lemma 1: 5 is a factor of a^2 (or a^2 is a multiple of 5). We need to assert the same about a.
Let's look at the **prime factorization** of a. Any number can be written as a product of primes. (Exs. 6=2*3; 10=2*5, 20=2*2*5, etc.

i. $a = f_1 \cdot f_2 \cdot \ldots \cdot f_n$ product of primes
ii. $a^2 = (f_1 \cdot f_2 \cdot \ldots \cdot f_n)^2 = (f_1 \cdot f_2 \cdot \ldots \cdot f_n)(f_1 \cdot f_2 \cdot \ldots \cdot f_n) = f_1 \cdot f_1 \cdot f_2 \cdot f_2 \cdot \ldots \cdot f_n \cdot f_n$ (by math)
iii. $a^2 = f_1 \cdot f_1 \cdot f_2 \cdot f_2 \cdot \ldots \cdot 5 \cdot \ldots \cdot f_n \cdot f_n$ Since we know that 5 is a factor in a^2
iv. a=5k, where k is some integer since 5 is a factor of a^2, then it must be one in a as well.

End of Lemma 1. Note that the proof of the lemma is not necessary. It is shown here for the purpose of explanation.

10. $5b^2 = (5k)^2$ By substitution from lemma 1
11. $5b^2 = 25k^2$ By math
12. $b^2 = 5k^2$ By math
13. b =5g for some integer g By lemma 1
14. a and b are both multiples of 5 by conjunction of 4 (lemma 1) and 13
15. a and b are not in lowest form by 14
16. a and b is in lowest form by simplification (of 2)
17. (a and b is in lowest form) ∧ (a and b is not in lowest form) by conjunction
18. false by negation

Therefore, $\sqrt{25} + \sqrt{5}$ is NOT a rational number. i.e. the sum is irrational.

4. Prove, by indirect proof, that if n is an integer and n^3+5 is odd, then n is even. Show all your work (no calculators allowed). [5 marks]

   Solution:

   Contrapositive: If n is odd, then n^3+5 is even.

   Assume $n=2k+1$ for some integer k.

   $n^3+5 = (2k+1)^3 +5 = (2k+1)(2k+1)(2k+1)+5$
   $= (4k^2+2k+2k+1)(2k+1)+5$
   $= (4k^2+4k+1)(2k+1)+ 5$
   $= (8k^3+4k^2+8k^2+4k+2k+1)+ 5$
   $= (8k^3+12k^2+6k+1)+ 5$
   $=8k^3+12k^2+6k+6$
   $=2(4k^3+6k^2+3k+3)$

Since n^3+5 is two times an integer, it is even and the contrapositive is true. Thus, the original statement is true as well.

5. Find the error in the following proof that every positive integer equals the next largest positive integer: "Proof: Let P(n) be the proposition '$n=n+1$'. Assume that P(n) is true, so that $n=n+1$. Add 1 to both sides of this equation to obtain $n+1=n+2$. Since this is the statement P(n+1), it follows that P(n) is true for all positive integers n." [1 mark]

   Solution: No basis case was done.

6. For integer x, such that $-2 \leq x \leq 2$, prove that $y<0$, where $y=x^4 -36x^2 +9x-5$. [6 marks]

   Case 1: -2, $y = x^4-36x^2 +9x-5 = (-2)^4 -36(-2)^2 +9(-2)-5 = -151$

   Case 2: -1, $y= x^4-36x^2 +9x-5 = (-1)^4 -36(-1)^2 +9(-1)-5= -49$

   Case 3: 0, $y= x^4-36x^2 +9x-5 = 0^4 -36(0)^2 +9(0)-5= -5$

   Case 4: +1, $y= x^4-36x^2 +9x-5 = 1^4 -36(1)^2 +9(1)-5= -31$

   Case 5: +2, $y= x^4-36x^2 +9x-5 = 2^4 -36(2)^2 +9(2)-5= -115$

Negative (less than zero) in all cases.

7. Prove by induction that $1+3+5+\ldots+(2n-1) = n^2$, for all positive integers n. [5 marks]

   Base step: $n=1$: $2(1)-1=1=1^2$
   True for basis step.

   Inductive Hypothesis: Assume true for n=k: $1+3+5+\ldots+(2k-1) = k^2$, for some positive integer $k \geq 1$.

   Inductive Step: We need to show that the statement holds for $n=k+1$:
   $1+3+5+\ldots+(2k-1)+ (2(k+1)-1)= (k+1)^2$
   LHS
   $1+3+5+\ldots+(2k-1)+ (2(k+1)-1)$
   $= \mathbf{1+3+5+\ldots+(2k-1)}+ (2k+2-1)$
   $= \boldsymbol{k^2} +2k+1$ (recursion)
   $= (k+1)^2$

   So the statement holds for $n=k+1$

8. What is the power set of $\{1, 6, 8\}$? [1 mark]

Solution: $\mathcal{P}\{1, 6, 8\} = \{ \emptyset, \{1\}, \{6\}, \{8\}, \{1, 6\}, \{1, 8\}, \{6, 8\}, \{1, 6, 8\} \}$

9. List explicitly the members of the following sets. [2 marks]

   a. $\{i \mid 0 \leq i < 21 \textit{ and either } i \textit{ begins with either a vowel or the letter } 't' \}$
   Solution **{one, two, three, eight, ten, eleven, twelve, thirteen, eighteen, twenty}**

   b. $\{j \mid j \textit{ is the first name of a } COMP1805 \textit{ teaching assistant that starts with a vowel}\}$
   Solution **{Enzo}**

10. Let $S = \{4, 7, \{1,7\}, \{monkey, banana\}, \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}\}$ and T$= \{\{1, 7, 20\}, \{5\}, 4, \{fruit\}, monkey\}$ and answer the following questions. [3 marks]

    a. Which set has the larger cardinality? **Solution: they both have 5**

    b. What is the intersection, $S \cap T$? **Solution: {4}**

    c. What is the cardinality of the union $S \cup T$? **Solution: 9**

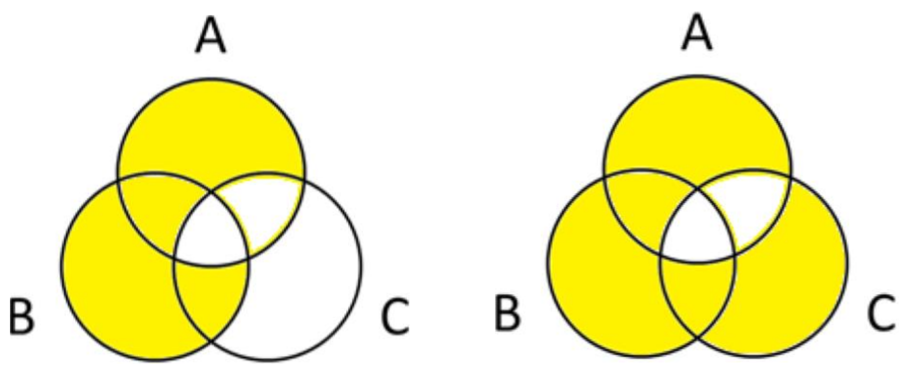
11. Determine whether or not the following is valid. Justify your answer by using membership tables. [5 marks]

$$(A - C) \cup (B - A) = (A \cup B \cup C) - (C \cap A)$$

A	B	C	$A - C$	$B - A$	$(A - C) \cup (B - A)$	$A \cup B \cup C$	$(C \cap A)$	$(A \cup B \cup C) - (C \cap A)$
1	1	1	0	0	0	1	1	0
1	1	0	1	0	1	1	0	1
1	0	1	0	0	0	1	1	0
1	0	0	1	0	1	1	0	1
0	1	1	0	1	1	1	0	1
0	1	0	0	1	1	1	0	1
0	0	1	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0

12. Draw the Venn Diagrams for the following set: [2 marks]

$$(A - C) \cup (B - A) = (A \cup B \cup C) - (C \cap A)$$



13. What is the intersection, A, of the set of all the digits that appear in your student number, B, and the set of all even numbers, C? Draw the Venn Diagram for sets A, B, and C. [3 marks]

This changes with your student number, but if your student number was 100123456...