Specification for Tutorial 4

1. Determine which of the following arguments are valid and which ones are invalid. Explain why.

   a. If n is an integer and $n^2 > 36$ then $n > 6$.

   Solution:

   This is False. Notice that if we set $n = -7$, then n is an integer (since -7 is an integer) and $n^2 = 49$ which is greater than 36. However, n is NOT greater than 6 since -7 is NOT greater than 6.

   b. Anyone who owns a computer knows how to use a mouse. Christelle owns a computer. Therefore, Christelle knows how to use a mouse.

   Solution:

   Let the universe of discourse be all people. Let $C(x)$ be the proposition that x owns a computer. Let $M(x)$ be that x knows how to use a mouse.
   Logically, the premise are:

   - $\forall x(C(x) \rightarrow M(x))$
   - $C(Christelle)$

   The conclusion is $M(Christelle)$. To show this argument is valid, we need to show that if every statement in the premise is True, the conclusion MUST be True.

   This argument is valid. We now show this. Since $\forall x(C(x) \rightarrow M(x))$ is True, this means that $C(Christelle) \rightarrow M(Christelle)$ must be True since if something is True for ALL people, it is true for Christelle (who is a person). This is **universal instantiation**. Now, since $C(Christelle)$ is True (it is part of the Premise) AND $C(Christelle) \rightarrow M(Christelle)$ is True, Modus Ponens allows us to conclude that $M(Christelle)$ must be True.

   c. All computers need electricity. My textbook is not a computer. Therefore, my textbook does not need electricity.

   Solution:

Let the universe of discourse be all objects. Let $C(x)$ be the proposition that x is a computer. Let $E(x)$ be that x needs electricity. Let t be my textbook. Logically, the premise are:

- $\forall x(C(x) \rightarrow E(x))$
- $\neg C(t)$

The conclusion is $\neg E(t)$. To determine whether or not this argument is valid, we need to determine if all the premise allow us to derive the conclusion.

The argument is not valid. We show this by showing that the premise can be True and the conclusion False. Set $\neg E(t)$ to False, and set $C(x)$ to False for all x. With these truth values, let us verify the truth values of the premises.

$\forall x(C(x) \rightarrow E(x))$ is True because we set $C(x)$ to False for all x. An implication is True if the antecedent of the implication is False.

The second premise $\neg C(t)$ is True since $C(t)$ is set to False (we set $C(x)$ to False for all x, which includes t).

Finally, the conclusion is False since $\neg E(t)$ is set to False. So all of the premise statements are True and the conclusion is False which shows that the argument is invalid.

d. A commodore-64 is a fast computer. Moussa's computer is not a commodore-64. Therefore, Moussa's computer is not fast.

Solution:

Let the universe of discourse be all computers. Let $C(x)$ be the proposition that x is a commodore-64. Let $F(x)$ be that x is Fast. Logically, the premise are:

- $\forall x(C(x) \rightarrow F(x))$
- $\neg C(Moussa)$

The conclusion is $\neg F(Moussa)$. To determine whether or not this argument is valid, we need to determine if all the premise allow us to derive the conclusion.

This argument is not valid. To show this, we show that the premise can be True and conclusion False. Set $\neg F(Moussa)$ to False, and $C(x)$ to False for all x. Now $\forall x(C(x) \rightarrow F(x))$ is True because $C(x)$ is False for all x. The statment $\neg C(Moussa)$ is True since $C(Moussa)$ is False becaues C(x) is False for all people and Moussa is a person. Finally, $\neg F(Moussa)$ is False. So, we found a situation where the Premise is True and the conclusion is False. Thus, the argument is not valid.

2. An integer n is even if and only if there exists an integer k such that n = 2k and an integer n is odd if and only if there exists an integer k such that n = 2k + 1. Using these operational definitions for evenness and oddness, prove each of the following:

a. The sum of two odd integers is even.

Solution:

Let x and y be odd integers. From the definition of odd, we know there exists integers u and v such that $x = 2u + 1$ and $y = 2v + 1$. Adding x and y together yields

$$x + y = 2u + 1 + 2v + 1 = 2(u + v + 1) = 2z,$$

where $z = u + v + 1$ is an integer. Therefore, by the definition of even, we see that $x + y$ is even. Since x and y are arbitrary odd integers, this must hold for all odd integer. ❑

b. The product of two odd integers is odd.

Solution:

Let x and y be odd integers. From the definition of odd, we know there exists integers u and v such that $x = 2u+1$ and $y = 2v+1$. Multiplying these numbers together yields

$$x \times y = (2u + 1)(2v + 1) = 4uv + 2u + 2v + 1 = 2(2uv + u + v) + 1 = 2z + 1,$$

where $z = 2uv + u + v$ is an integer. Therefore, by the definition of odd, we see that $x \times y$ is odd. Since x and y are arbitrary odd integers, this must hold for all odd integer. ❑

3. Prove that if x is an integer and $x^3 + 35$ is odd then x is even...

a. ...using a direct proof.

Solution:

For a direct proof, we need to show that given that the statement x is an integer AND $x^3 + 35$ is odd is True, we must use this to derive that x is even must be True. We proceed as follows. If $x^3 + 35$ is odd, then x^3 must be even since by the previous question the sum of two odd integers is even. Now, if x^3 is even, then this means that $x^2 \times x$ is even. The product of two odd numbers CANNOT be even. An argument similar to the above shows this. Therefore, either x^2 is even or x is even. If x^2 is even, then x must be even since again, the product of two odd numbers is odd. Therefore, x must be even.

b. ...using a proof by contradiction.

To prove this by contradiction, we assume that the statement "if x is an integer and x^3+35 is odd then x is even" is False. This means that we assume the following is True: x is an integer and x^3+35 is odd and x is not even. Since we assume that x is odd, then $x = 2k+1$ for some integer k. $x^3+35 = (2k+1)^3+35 = 8k^3+12k^2+6k+36 = 2(4k^3+6k^2+3k+18)$. Note that this is even which contradicts our assumption that x^3+35 is odd. Therefore, the statement "if x is an integer and x^3+35 is odd then x is even" must be True.

4. Prove by induction that, for all n, (a+ aq+… _+aq^{n-1} = a $\frac{q^n-1}{q-1}$).

Solution:

$$(\forall n)\ (a + aq + \cdots + aq^{n-1} = a\frac{q^n-1}{q-1})$$

1. n=1 , $\quad a = a\,\frac{q^1-1}{q-1} = a$

2. n=k , $\quad a + aq + \cdots + aq^{k-1} = a\frac{q^k-1}{q-1}$

3. n=k+1 , $\quad \underbrace{a + aq + \cdots + aq^{k-1}}_{a\frac{q^k-1}{q-1}} + aq^k = a\frac{q^k-1+q^{k+1}-q^k}{q-1} = a\frac{q^{k+1}-1}{q-1}$