Mathematics for Data Science

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Motivation

- **Feature Selection and Dimensionality Reduction:** Echelon form helps identify linearly dependent features, and the rank indicates intrinsic dimensionality, aiding in dimensionality reduction techniques like PCA.
- **Solving Linear Systems:** Echelon form simplifies solving linear equations, crucial for algorithms like linear regression and least squares solutions.
- **Matrix Decompositions:** Concepts of rank and echelon form are essential for matrix decompositions (e.g., SVD, eigenvalue decomposition), which are used in regularization and feature extraction.

Motivation (contd.)

- **Optimization and Numerical Stability:** Understanding matrix rank and structure improves numerical stability and optimization efficiency in algorithms like gradient descent.
- **Computational Efficiency:** Echelon form and rank help in efficiently handling sparse matrices and optimizing matrix operations, leading to faster and more efficient machine learning algorithms.

Elementary Row Operations

There are three elementary row operations on an $m \times n$ matrix A over the field F:

1. multiplication of any row of A by a non-zero scalar c, $(R_j \to cR_j; c \neq 0)$
2. replacement of the r^{th} row of A by row plus c times row s, c is any scalar and $r \neq s$, $(R_r \to R_r + cR_s)$
3. interchange of any two rows of A $(R_i \leftrightarrow R_j, i \neq j)$

Definition

If A and B are $m \times n$ matrices over the field F, we say that B is **row equivalent** to A if B can be obtained from any A by a finite sequence of elementary row operations.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 4 & 4 \end{bmatrix}$. Using elementary row operations, transform A into an identity matrix.

Solution: $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 4 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 2 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1}$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{R_3 \to \frac{-1}{2} R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix}$

Example

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 + 2R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - \frac{5}{2} R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Echelon Matrix

A matrix A is called an **echelon matrix**, or is said to be in echelon form, if the following two conditions hold:

1. All zero rows, if any, are at the bottom of the matrix.
2. Each leading nonzero entry in a row is to the right of the leading nonzero entry in the preceding row. That is, if rows $1, 2, ..., r$ are the non-zero rows of the matrix, and if the leading non-zero entry i occurs in column k_i, $i = 1, 2, ..., r$ then $k_1 < k_2 < ... < k_r$.

Example

1. $\begin{bmatrix} 0 & 1 & 3 & 7 & 5 \\ 0 & 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & 8 \\ 0 & 0 & 4 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 5 & 8 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}$

Example (1) are in Echelon forms but (2) are not in Echelon forms.

Problems

Transform the following matrix into their echelon form using elementary row operations.

- $\begin{bmatrix} 2 & 4 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \end{bmatrix}$

Solution

$$\begin{bmatrix} 2 & 4 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 2 & 4 & -3 & -2 \\ 0 & 1 & -1 & -7 \\ 1 & 3 & -2 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{2} R_1}$$

$$\begin{bmatrix} 2 & 4 & -3 & -2 \\ 0 & 1 & -1 & -7 \\ 0 & 1 & -\frac{1}{2} & 3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 2 & 4 & -3 & -2 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & \frac{1}{2} & 10 \end{bmatrix}$$

Rank of a Matrix

Let $A = [a_{ij}]_{m \times n}$. The rank of a matrix A (denoted by $r(A)$ or $rank(A)$) is defined as the number of non-zero rows in the echelon form of A. It is also defined as the maximum no. of linearly independent rows or columns of the matrix A.

Examples

- $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}, r(A) = 2$
- $B = \begin{bmatrix} 2 & 2 & 3 \\ -2 & -2 & -3 \\ 6 & 6 & 9 \end{bmatrix}, r(B) = 1$

Problems

Find the rank of the following matrices

- $\begin{bmatrix} 1 & 3 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$
- $\begin{bmatrix} 2 & -1 & 4 & 5 \\ 4 & 2 & 2 & 1 \end{bmatrix}$

Solution

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}_{3\times 2} \xrightarrow{R_2 \to R_2 + R_1, R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 3 \\ 0 & 5 \\ 0 & -5 \end{bmatrix}_{3\times 2} \xrightarrow{R_3 \to R_3 + R_2}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}_{3\times 2}, \ r(A) = 2$$

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ r(B) = 1 \text{ For}$$

$$A_{m\times n}, r(A) \leq \min\{m, n\}$$

Solution

$$A = \begin{bmatrix} 2 & -1 & 4 & 5 \\ 4 & 2 & 2 & 1 \end{bmatrix}_{2\times 4}, r(A) \leq 2$$

$$\sim \begin{bmatrix} 2 & -1 & 4 & 5 \\ 0 & 4 & -6 & -9 \end{bmatrix} \text{(by } R_2 \to R_2 - 2R_1\text{)}$$

$$r(A) = 2$$

Problem

- Find the conditions/values of α and β for which the matrix

$$\begin{bmatrix} \alpha & 1 & 2 \\ 0 & 2 & \beta \\ 1 & 3 & 6 \end{bmatrix}$$

has (i) rank $= 1$
(ii) rank $= 2$
(iii) rank $= 3$

Solution

$$A = \begin{bmatrix} \alpha & 1 & 2 \\ 0 & 2 & \beta \\ 1 & 3 & 6 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 3 & 6 \\ 0 & 2 & \beta \\ \alpha & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \alpha R_1}$$

$$\begin{bmatrix} 1 & 3 & 6 \\ 0 & 2 & \beta \\ 0 & 1-3\alpha & 2-6\alpha \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1-3\alpha}{2} R_2}$$

$$\begin{bmatrix} 1 & 3 & 6 \\ 0 & 2 & \beta \\ 0 & 0 & \frac{(1-3\alpha)(4-\beta)}{2} \end{bmatrix}$$

Solution

For *Rank*(*A*)**=1:** no choice of α or β can ensure this. Rank-1 is not possible.
For *Rank*(*A*)**=2:** $\alpha = \frac{1}{3}$ or $\beta = 4$
For *Rank*(*A*)**=3:** $\alpha \neq \frac{1}{3}$ and $\beta \neq 4$

Properties of Rank of a Matrix

- Rank of only a zero matrix is zero.
- Elementary row and column operations on a matrix are rank-preserving.
- $r(A) = r(A^T)$
- $r(cA) = r(A), c$ is any non-zero scalar.
- If $A = [a_{ij}]_{m \times n}$, then $r \leq \min\{m, n\}$
- Let $A = [a_{ij}]_{n \times n}$, $r(A) = n$ iff $|A| \neq 0$

Thank You

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