VIETNAM NATIONAL UNIVERSITY, HO CHI MINH CITY
UNIVERSITY OF TECHNOLOGY
FACULTY OF COMPUTER SCIENCE AND ENGINEERING

DIGITAL SIGNAL PROCESSING

Report #08

Lab 08 : Z - Transform

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1 Exercise 1 : Find the all possible signal $x(n)$ that have Z transforms as follows

a) $x(z) = \dfrac{1 - 1.5z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}}$

$1 - 1.5z^{-1} + 0.5z^{-2} = (1 - \alpha z^{-1})(1 - \beta z^{-1})$

$\rightarrow \begin{cases} \alpha + \beta = 1.5 \\ \alpha \cdot \beta = 0.5 \end{cases}$

$\rightarrow \begin{cases} \alpha = 0.5 \\ \beta = 1 \end{cases}$

$1 - 1.5z^{-1} = A(1 - z^{-1}) + B(1 - 0.5z^{-1})$

$\rightarrow \begin{cases} A + B = 1 \\ 0.5A + B = 1.5 \end{cases}$

$\rightarrow \begin{cases} A = -1 \\ B = 2 \end{cases}$

$\Rightarrow x(z) = \dfrac{-1}{1 - 0.5z^{-1}} + \dfrac{2}{1 - z^{-1}}$

$$Result: \begin{cases} x(n) = -0.5^n u(n) + 2u(n), & |z| > 1 \\ x(n) = -0.5^n u(n) + 2u(-n-1), & 0.5 < |z| < 1 \\ x(n) = -0.5^n u(-n-1) + 2u(-n-1), & |z| < 0.5 \end{cases}$$

b) $x(z) = \dfrac{1}{1 - z^{-1} + 0.25z^{-2}}$

$1 - z^{-1} + 0.25z^{-2} = (1 - \alpha z^{-1})(1 - \beta z^{-1})$

$\rightarrow \begin{cases} \alpha + \beta = 1 \\ \alpha \cdot \beta = 0.25 \end{cases}$

$\rightarrow \begin{cases} \alpha = 0.5 \\ \beta = 0.5 \end{cases}$

$\Rightarrow x(z) = \dfrac{1}{(1 - 0.5z^{-1})^2}$

$\Rightarrow x(z) = \dfrac{1}{1 - 0.5z^{-1}} + \dfrac{0.5z^{-1}}{(1 - 0.5z^{-1})^2}$

$$Result: \begin{cases} x(n) = 0.5^n u(n) + n \cdot 0.5^n u(n), & |z| > 0.5 \\ x(n) = -0.5^n u(-n-1) - n \cdot 0.5^n u(-n-1), & |z| < 0.5 \end{cases}$$

c) $x(z) = \dfrac{1}{3 - 10z^{-1} + 3z^{-2}}$

$3 - 10z^{-1} + 3z^{-2} = 3(1 - \alpha z^{-1})(1 - \beta z^{-1})$

$\Leftrightarrow x(z) = \dfrac{1}{3(1 - \frac{1}{3}z^{-1})(1 - 3z^{-1})}$

$1 = A(1 - \frac{1}{3}z^{-1}) + B(1 - 3z^{-1})$

$\rightarrow \begin{cases} \frac{1}{3}A + 3B = 0 \\ A + B = 1 \end{cases}$

$\rightarrow \begin{cases} A = \frac{9}{8} \\ B = -\frac{1}{8} \end{cases}$

$\Rightarrow x(z) = \dfrac{\frac{-1}{24}}{1 - \frac{1}{3}z^{-1}} + \dfrac{\frac{3}{8}}{1 - 3z^{-1}}$

$$Result: \begin{cases} x(n) = \frac{-1}{24} \cdot (\frac{1}{3})^n u(n) + \frac{3}{8} \cdot 3^n u(n), & |z| > 3 \\ x(n) = \frac{-1}{24} \cdot (\frac{1}{3})^n u(n) - \frac{3}{8} \cdot 3^n u(-n-1), & \frac{1}{3} < |z| < 3 \\ x(n) = \frac{-1}{24} \cdot (\frac{1}{3})^n u(-n-1) - \frac{3}{8} \cdot 3^n u(-n-1), & |z| < \frac{1}{3} \end{cases}$$

d) $x(z) = \dfrac{1}{2 - 3z^{-1} + 1z^{-2}}$

$\Leftrightarrow x(z) = \dfrac{1}{2 - 3z^{-1} + 1z^{-2}} = \dfrac{1}{z}(1 - 2z^{-1})(1 - z^{-1})$

$1 = A(1 - z^{-1}) + B(1 - 2z^{-1})$

$\rightarrow \begin{cases} A + 2B = 0 \\ A + B = 1 \end{cases}$

$\rightarrow \begin{cases} A = 2 \\ B = -1 \end{cases}$

$\Rightarrow x(z) = \dfrac{2}{1 - 2z^{-1}} + \dfrac{-1}{1 - z^{-1}}$

$$Result: \begin{cases} x(n) = 2^{n+1} u(n) - u(n), & |z| > 2 \\ x(n) = 2^{n+1} u(n) + u(-n-1), & 1 < |z| < 2 \\ x(n) = 2^{n+1} u(-n-1) - u(-n-1), & |z| < 1 \end{cases}$$

e) $x(z) = \dfrac{1 + 2z^{-1} + z^{-2}}{1 + 4z^{-1} + 4z^{-2}}$

$\Leftrightarrow x(z) = \dfrac{1 + 2z^{-1} + z^{-2}}{1 + 4z^{-1} + 4z^{-2}} = \dfrac{(1 + z^{-1})^2}{4(1 + \frac{1}{2}z^{-1})^2} = \left(\frac{1}{2} \cdot \dfrac{1 + z^{-1}}{1 + \frac{1}{2}z^{-1}}\right)^2$

$1 + z^{-1} = A + B(1 + \frac{1}{2}z^{-1})$

$\rightarrow \begin{cases} \frac{1}{2}B = 1 \\ A + B = 1 \end{cases}$

$\rightarrow \begin{cases} A = -1 \\ B = 2 \end{cases}$

$\Rightarrow x(z) = \left(1 - \dfrac{\frac{1}{2}}{1 + \frac{1}{2}z^{-1}}\right)^2$

$\Leftrightarrow x(z) = \frac{1}{4} \cdot \dfrac{1}{(1 + \frac{1}{2}z^{-1})^2} - \dfrac{1}{1 + \frac{1}{2}z^{-1}} + 1$

$\Leftrightarrow x(z) = \frac{1}{4} \cdot \left(\dfrac{1}{1 + \frac{1}{2}z^{-1}} - \dfrac{\frac{1}{2}z^{-1}}{(1 + \frac{1}{2}z^{-1})^2}\right) - \dfrac{1}{1 + \frac{1}{2}z^{-1}} + 1$

$\Leftrightarrow x(z) = \dfrac{\frac{1}{4}}{1 + \frac{1}{2}z^{-1}} - \dfrac{\frac{1}{8}z^{-1}}{(1 + \frac{1}{2}z^{-1})^2} - \dfrac{1}{1 + \frac{1}{2}z^{-1}} + 1$

$$Result: \begin{cases} x(n) = \frac{1}{4} \cdot (\frac{1}{2})^n u(n) - \frac{1}{4} \cdot n \cdot (\frac{1}{2})^n u(n) - (\frac{1}{2})^n u(n) + \delta(n), & |z| > \frac{1}{2} \\ x(n) = -\frac{1}{4} \cdot (\frac{1}{2})^n u(-n-1) + \frac{1}{4} \cdot n \cdot (\frac{1}{2})^n u(-n-1) + (\frac{1}{2})^n u(-n-1) + \delta(n), & |z| < \frac{1}{2} \end{cases}$$

f) $x(z) = \dfrac{2z^2 - 12z}{(z - 0.3)(z + 0.2)(z - 3)}$

$\Leftrightarrow x(z) = 2\left(\dfrac{z^2 - 6z}{z^3(1 - 0.3z^{-1})(1 + 0.2z^{-1})(1 - 3z^{-1})}\right) = 2\left(\dfrac{z^{-1} - 6z^{-2}}{(1 - 0.3z^{-1})(1 + 0.2z^{-1})(1 - 3z^{-1})}\right)$

$z^{-1} - 6z^{-2} = A(1 - 0.3z^{-1})(1 + 0.2z^{-1}) + B(1 + 0.2z^{-1})(1 - 3z^{-1}) + C(1 - 0.3z^{-1})(1 - 3z^{-1})$

$\Leftrightarrow z^{-1} - 6z^{-2} = A(1 - 0.1z^{-1} - 0.06z^{-2}) + B(1 - 2.8z^{-1} - 0.6z^{-2}) + C(1 - 3.3z^{-1} + 0.9z^{-2})$

$\rightarrow \begin{cases} A + B + C = 0 \\ 0.1A + 2.8B + 3.3C = -1 \\ 0.06A + 0.6B - 0.9C = 6 \end{cases}$

$$\rightarrow \begin{cases} A = -\frac{25}{72} \\ B = \frac{38}{9} \\ C = -\frac{31}{8} \end{cases}$$

$$\Leftrightarrow x(z) = \frac{\frac{76}{9}}{1 - 0.3z^{-1}} + \frac{-\frac{31}{4}}{1 + 0.2z^{-1}} + \frac{-\frac{25}{36}}{1 - 3z^{-1}}$$

$$Result: \begin{cases} x(n) = \frac{76}{9} \cdot 0.3^n u(n) - \frac{31}{4} \cdot (-0.2)^n u(n) - \frac{25}{36} \cdot 3^n u(n), & |z| > 3 \\ x(n) = \frac{76}{9} \cdot 0.3^n u(n) - \frac{31}{4} \cdot (-0.2)^n u(n) + \frac{25}{36} \cdot 3^n u(-n-1), & 0.3 < |z| < 3 \\ x(n) = \frac{76}{9} \cdot 0.3^n u(n) + \frac{31}{4} \cdot (-0.2)^n u(-n-1) + \frac{25}{36} \cdot 3^n u(-n-1), & 0.2 < |z| < 0.3 \\ x(n) = -\frac{76}{9} \cdot 0.3^n u(-n-1) + \frac{31}{4} \cdot (-0.2)^n u(-n-1) + \frac{25}{36} \cdot 3^n u(-n-1), & |z| < 0.2 \end{cases}$$

2 Exercise 2 : Compute the convolution $x(n) = x_1(n) * x_2(n)$ using Z and Inverse Z transform, where:

a) $x_1(n) = \{1^{\uparrow}, 1, 1, 1, 1\}, x_2(n) = \{1^{\uparrow}, 1, 1, 1\}$
$\Rightarrow x_1(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}, \qquad x_2(z) = 1 + z^{-1} + z^{-2} + z^{-3}$
$x(n) = x_1(n) * x_2(n) \longrightarrow x(z) = x_1(z) \cdot x_2(z)$
$\Leftrightarrow x(z) = (1 + z^{-1} + z^{-2} + z^{-3} + z^{-4})(1 + z^{-1} + z^{-2} + z^{-3})$
$\Leftrightarrow x(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + z^{-6}$
$Result: x(n) = x_1(n) * x_2(n) = Z^{-1}[x(z)] = \{1^{\uparrow}, 2, 3, 4, 3, 2, 1\}$

b) $x_1(n) = \{1^{\uparrow}, 2, 3, 4, 5\}, x_2(n) = \{1^{\uparrow}, 1, 1\}$
$\Rightarrow x_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}, \qquad x_2(z) = 1 + z^{-1} + z^{-2}$
$x(n) = x_1(n) * x_2(n) \longrightarrow x(z) = x_1(z) \cdot x_2(z)$
$\Leftrightarrow x(z) = (1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4})(1 + z^{-1} + z^{-2})$
$\Leftrightarrow x(z) = 1 + 3z^{-1} + 6z^{-2} + 9z^{-3} + 12z^{-4} + 9z^{-5} + 5z^{-6}$
$Result: x(n) = x_1(n) * x_2(n) = Z^{-1}[x(z)] = \{1^{\uparrow}, 3, 6, 9, 12, 9, 5\}$

c) $x_1(n) = \left(\frac{1}{5}\right)^n u(n), x_2(n) = 2^n u(n)$
$$\Rightarrow \begin{cases} x_1(z) = \frac{1}{1 - 0.2z^{-1}}, \quad |z| > 0.2 & x_2(z) = \frac{1}{1 - 2z^{-1}}, \quad |z| > 2 \\ 0.2A + 2B = 0 \end{cases}$$
$x(n) = x_1(n) * x_2(n) \longrightarrow x(z) = x_1(z) \cdot x_2(z)$
$$\Leftrightarrow x(z) = \frac{1}{(1 - 0.2z^{-1})(1 - 2z^{-1})}$$
$1 = A(1 - 0.2z^{-1}) + B(1 - 2z^{-1})$
$$\rightarrow \begin{cases} A + B = 1 \\ 0.2A + 2B = 0 \end{cases}$$
$$\rightarrow \begin{cases} A = \frac{10}{9} \\ B = -\frac{1}{9} \end{cases}$$
$$\Leftrightarrow x(z) = \frac{-\frac{1}{9}}{1 - 0.2z^{-1}} + \frac{\frac{10}{9}}{1 - 2z^{-1}}$$
$Result: x(n) = -\frac{1}{9} \cdot 0.2^n u(n) + \frac{10}{9} \cdot 2^n u(n), \qquad |z| > 2$

d) $x_1(n) = nu(n), x_2(n) = 2^n u(n-1)$

$$\Rightarrow \begin{cases} x_1(z) = \dfrac{z^{-1}}{(1-z^{-1})^2}, & |z| > 1 \\ x_2(z) = \dfrac{2z^{-1}}{1-2z^{-1}}, & |z| > 2 \end{cases}$$

$x(n) = x_1(n) * x_2(n) \longrightarrow x(z) = x_1(z) \cdot x_2(z)$

$$\Leftrightarrow x(z) = \frac{2z^{-2}}{(1-z^{-1})^2(1-2z^{-1})}$$

$2z^{-2} = A(1-z^{-1})^2 + B(1-2z^{-1})$

$$\Rightarrow \begin{cases} A + B = 0 \\ A = 2 \end{cases}$$

$$\Rightarrow \begin{cases} A = 2 \\ B = -2 \end{cases}$$

$$\Leftrightarrow x(z) = \frac{-2}{(1-z^{-1})^2} + \frac{2}{1-2z^{-1}}$$

$$\Leftrightarrow x(z) = \frac{-2}{1-z^{-1}} - \frac{2z^{-1}}{(1-z^{-1})^2} + \frac{2}{1-2z^{-1}}$$

$Result: x(n) = -2 \cdot u(n) - 2 \cdot nu(n) + 2^n u(n), \qquad |z| > 2$

3 Exercise 3 : Given an LTI system represented by the input - output description equation

a) Draw corresponding block diagram
$Y(z) = 0.7z^{-1}Y(z) + X(z)$

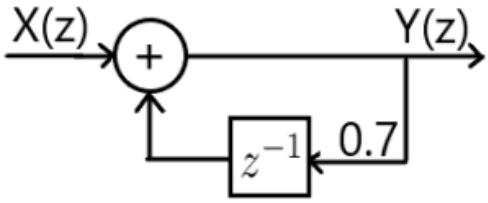

Figure 1: Block diagram

b) Determine $h(n)$ using Z and inverse Z transforms.
$y(n) = 0.7y(n-1) + x(n)$
$\Rightarrow Y(z)(1-0.7z^{-1}) = X(z)$

$$h(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.7z^{-1}}$$

$$\Rightarrow \begin{cases} h(n) = 0.7^n u(n), & |z| > 0.7 \\ h(n) = 0.7^n u(-n-1), & |z| < 0.7 \end{cases}$$

c) Determine $y(n)$ when $x(n) = u(n)$
$When\ x(n) = u(n): y(n) = 0.7y(n-1) + u(n)$

$$\Rightarrow Y(z)(1-0.7z^{-1}) = \frac{1}{1-z^{-1}}$$
$$\Rightarrow Y(z) = \frac{1}{(1-0.7z^{-1})(1-z^{-1})}, \qquad |z| > 1$$
$$1 = A(1-0.7z^{-1}) + B(1-z^{-1})$$
$$\rightarrow \begin{cases} A + B = 1 \\ 0.7A + B = 0 \end{cases}$$
$$\rightarrow \begin{cases} A = \frac{10}{3} \\ B = -\frac{7}{3} \end{cases}$$
$$\Leftrightarrow Y(z) = \frac{-\frac{7}{3}}{(1-0.7z^{-1})} + \frac{\frac{10}{3}}{(1-z^{-1})}$$
$$\Rightarrow y(n) = -\frac{7}{3} \cdot 0.7^n u(n) + \frac{10}{3} \cdot u(n), \qquad |z| > 1$$

4 Exercise 4 : Given an LTI system represented by the input - output description equation

a) Draw corresponding block diagram
$Y(z) = 0.5z^{-1}Y(z) + X(z) + z^{-1}X(z)$

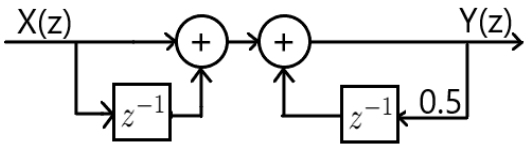

Figure 2: Block diagram

b) Determine $h(n)$ using Z and inverse Z transforms.
$y(n) = 0.5y(n-1) + x(n) + x(n-1)$
$\Rightarrow Y(z)(1-0.5z^{-1}) = X(z)(1+z^{-1})$

$$h(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-0.5z^{-1}} = \frac{1}{1-0.5z^{-1}} + 2\frac{0.5z^{-1}}{1-0.5z^{-1}}$$
$$\Rightarrow \begin{cases} h(n) = 0.5^n u(n) + 2 \cdot 0.5^n u(n-1), & |z| > 0.5 \\ h(n) = 0.5^n u(-n-1) + 2 \cdot 0.5^n u(-n), & |z| < 0.5 \end{cases}$$

c) Determine $y(n)$ when $x(n) = 2^n u(n)$
When $x(n) = 2^n u(n)$:
$Y(z)(1-0.5z^{-1}) = X(z)(1+z^{-1})$
$$\Leftrightarrow Y(z) = \frac{1+z^{-1}}{1-0.5z^{-1}} \cdot \frac{1}{1-2z^{-1}}, \qquad |z| > 2$$
$$1 + z^{-1} = A(1-0.5z^{-1}) + B(1-2z^{-1})$$
$$\rightarrow \begin{cases} A + B = 1 \\ 0.5A + 2B = 1 \end{cases}$$

$$\rightarrow \begin{cases} A = \frac{2}{3} \\ B = \frac{1}{3} \end{cases}$$
$$\Leftrightarrow Y(z) = \frac{\frac{1}{3}}{(1 - 0.5z^{-1})} + \frac{\frac{2}{3}}{(1 - 2z^{-1})}$$
$$\Rightarrow y(n) = \tfrac{1}{3} \cdot 0.5^n u(n) + \tfrac{2}{3} \cdot 2^n u(n), \qquad |z| > 2$$

5 Exercise 5 : Use SciLab to find $h(n)$ where

a) $H(z) = \dfrac{z^{-2}}{1 - 3z^{-1} + 2z^{-2}}, \qquad |z| > 2$

Replace $s = z^{-1} \Rightarrow H(s) = \dfrac{s^2}{1 - 3s^1 + 2s^2}$
after using Scilab like below,

```
--> s = poly(0, 's')
 s  =

  s

--> h = s^2/(1 - 3*s + 2*s^2)
 h  =


      s²
  ----------
  1 -3s +2s²

--> pfss(h)
 ans  =


      ans(1)


    1
  -----
  -1 +s

      ans(2)


   -0.25
  -------
  -0.5 +s

      ans(3)

 0.5
```

Figure 3: *Code used in Scilab to do partial-fraction expansion*

$$\Rightarrow H(z) = \frac{1}{-1 + s} + \frac{-0.25}{-0.5 + s} + 0.5$$
$$\textit{Replace } z = s^{-1} \Rightarrow H(z) = \frac{1}{-1 + z^{-1}} + \frac{-0.25}{-0.5 + z^{-1}} + 0.5 = \frac{-1}{1 - z^{-1}} + \frac{0.5}{1 - 2z^{-1}} + 0.5$$
$$\textit{Result} : \begin{cases} h(n) = -u(n) + 0.5 \cdot 2^n u(-n-1) + 0.5\delta(n), & 1 < |z| < 2 \\ h(n) = -u(-n-1) + 0.5 \cdot 2^n u(-n-1) + 0.5\delta(n), & |z| < 1 \end{cases}$$

b) $H(z) = \dfrac{z}{z^3 - 6z^2 + 11z - 6}, \qquad |z| < 2$

$$\Leftrightarrow H(z) = \frac{z^{-2}}{1 - 6z^{-1} + 11z^{-2} - 6z^{-3}}$$

Replace $s = z^{-1} \Rightarrow H(s) = \dfrac{s^3}{1 - 6s^1 + 11s^2 - 6s^3}$
after using Scilab like below:

```
--> h = s^3 / (1 - 6*s + 11*s^2 -6*s^3)
 h  =


          s³
   ----------------
   1 -6s +11s² -6s³

--> pfss(h)
 ans  =


       ans(1)


   -0.5
   -----
   -1 +s

       ans(2)


    0.25
   -------
   -0.5 +s

       ans(3)


    -0.0555556
   -------------
   -0.3333333 +s

       ans(4)

  -0.1666667
```

Figure 4: *Code used in Scilab to do partial-fraction expansion*

$$\Rightarrow H(s) = \frac{-0.5}{-1+s} + \frac{0.25}{-0.5+s} + \frac{-\frac{1}{18}}{-\frac{1}{3}+s} + \frac{1}{6} = \frac{0.5}{1-s} + \frac{-0.5}{1-2s} + \frac{\frac{1}{6}}{1-3s} + \frac{1}{6}$$

$$\textit{Replace } z = s^{-1} \Rightarrow H(z) = \frac{0.5}{1-z^{-1}} + \frac{-0.5}{1-2z^{-1}} + \frac{\frac{1}{6}}{1-3z^{-1}} + \frac{1}{6}$$

$$\textit{Result}: \begin{cases} h(n) = 0.5 \cdot u(n) - 0.5 \cdot 2^n u(-n-1) + \frac{1}{6} \cdot 3^n u(-n-1) + \frac{1}{6}\delta(n), & 1 < |z| < 2 \\ h(n) = 0.5 \cdot u(-n-1) - 0.5 \cdot 2^n u(-n-1) + \frac{1}{6} \cdot 3^n u(-n-1) + \frac{1}{6}\delta(n), & |z| < 1 \end{cases}$$