Math 130 Project: Hilbert Space Frame Theory

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1 Introduction

Imagine that I know the location of some free pizza, and I want to send that information to you.

If I write the location in some coordinate system as $\mathbf{x} = (x_1, x_2)$. One possible way would be just sending you x_1 and x_2, or equivalently $\mathbf{x} \bullet (1,0)$ and $\mathbf{x} \bullet (0,1)$.

Now, if the information is sent over some noisy channels, this can be very bad: if one of the coordinates is lost, then you can not receive my complete message of the location. However, I could send you each coordinate twice to protect against the lost. From the cost and effect perspective, this is not optimal since that would require four transmissions.

In search for an optimal solution, I would like to do something in between. Now, I send you $\mathbf{x} \bullet (1,0)$, $\mathbf{x} \bullet (0,1)$, and $\mathbf{x} \bullet (1,1)$ instead, then any two of these will allow you to recover $\mathbf{x}$: the location. That is because the set $\{(1,0),(0,1),(1,1)\}$ forms a **Frame** over $\mathbb{R}^2$

Remark 1. *Frame is an "over-complete" basis for some Hilbert space $\mathcal{H}$.*

2 Hilbert Space Definition

A **Hilbert space** $\mathcal{H}$ is an **inner product space** which is **complete** relative to the induced norm (i.e. all Cauchy sequences in $\mathcal{H}$ converge).

2.1 Inner Product Space

An **inner product space** is a vector space V over $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$ together with a function $\langle -, - \rangle : V \times V \to \mathbb{F}$ (the **inner product**) satisfying:

1. Linear in the first argument: $\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$

2. Conjugate symmetry: $\langle v, b_1 w_1 + b_2 w_2 \rangle = \overline{b_1} \langle v, w_1 \rangle + \overline{b_2} \langle v_2, w_2 \rangle$

3. Non-negativity: $\langle v, v\rangle \geq 0$; and $\langle v, v\rangle = 0 \iff v = 0$.

Example 2.1.1. $\mathbb{R}^d$ or $\mathbb{C}^d$ with the standard dot product $\langle v, w\rangle = v \bullet w = \sum v_i \overline{w_i}$.

Example 2.1.2. $L^2[0,1]$, the functions $f : [0,1] \to \mathbb{F}$ such that $\int_0^1 |f(x)|^2 dx$ converges, with $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx$.

2.2 Metric Space

An **metric space** is an order pair(M, d) where M is a set and d is a metric on the set M such that for any $x, y, z \in M$ satisfying:

1. Symmetry: $d(x, y) = d(y, x)$
2. Non-negativity: $d(x, y) \geq 0$; and $d(x, x) = 0 \iff x = 0$.
3. Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$

Remark 2. *The inner product $\langle -, -\rangle$ generates a **norm** on V by $||v|| = \langle v, v\rangle^{1/2}$, which makes the vector space V into a **metric space**.*

Recall from the definition of Hilbert Space, both of the examples of inner product spaces are also Hilbert spaces.

3 Frame

3.1 Frame Definition

In a Hilbert space $\mathcal{H}$, a **frame** is a subset $F = \{\varphi_i\}_{i\in I}$ satisfies the following conditions:

1. The elements of F span $\mathcal{H}$.
2. There exist uniform positive constants A, B such that, for all $x \in \mathcal{H}$:

$$A||x||^2 \leq \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \leq B||x||^2$$

Alternately, the definition of a frame can be considered as the following function:

$$t_F(x) = \frac{1}{||x_i||^2} \sum_{i\in I} |\langle x, \varphi_i\rangle|^2$$

so

$$A \leq t_F(x) \leq B$$

This leaves out constant A and B as upper and lower bounds of the inequality from definition. People are interested in the best possible constants A, B for a given frame, which are as follows:

$$A_F = \inf_{x \neq 0 \in \mathcal{H}} t_F(x)$$
$$B_F = \sup_{x \neq 0 \in \mathcal{H}} t_F(x)$$

Remark 3. *If both I and $\dim(\mathcal{H})$ are finite, then condition 1 implies condition 2.*
Thus, in $\mathbb{R}^d$, ***any finite spanning set is a frame****! This is why we like to think of a frame as an "over-complete" basis.*

Example 3.1.1. If $\{\varphi_i\}$ happen to form an **orthonormal basis**, then $\sum_{i \in I} |\langle x, \varphi_i \rangle|^2 = ||x||^2$, so this is a frame with $A = B = 1$.

Note that since $t_F(ax) = t_F(x)$ for all nonzero scalars a, it suffices to consider $||x|| = 1$ in the equations above. This also means that if $\dim(\mathcal{H})$ is finite, the inf and sup above are actually attained.

3.2 Tight(good) Frame

A frame is **tight** if $\frac{B_F}{A_F} = 1$.
The intuition here is to make them as close as possible, so compare with the ratio, and want it $\frac{B_F}{A_F} \approx 1$

Application: Prove that the expensive Mercedes-Benz has tight frame.

Proof. The **Mercedes-Benz frame** consists of the following three frame vectors in $\mathbb{R}^2$:

$$\varphi_1 = (0,1) \quad \varphi_2 = \left(\sqrt{3}/2, -1/2\right) \quad \varphi_3 = \left(-\sqrt{3}/2, -1/2\right)$$

Take any nonzero $x = (x_1, x_2)$, we have:

$$\begin{aligned} t_F(x) &= \frac{1}{||x_i||^2} \sum_{i \in I} |\langle x, \varphi_i \rangle|^2 \\ &= \frac{(x_2)^2 + (\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2)^2 + (-\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2)^2}{x_1^2 + x_2^2} \\ &= \frac{3}{2} \end{aligned}$$

So $A_F = B_F = \frac{3}{2}$, meaning the Mercedes-Benz frame is tight , since $\frac{B_F}{A_F} = 1$.

3.3 Loose(bad) Frame

A frame is **loose** if frame vectors are too close with each others.

Application: Fix an angle $0 < \alpha \ll \frac{\pi}{4}$, and let $F = \{\varphi_i\}_{i \in I}$ consist of the following three frame vectors:

$$\varphi_1 = (1, 0) \quad \varphi_2 = (\cos\alpha, \sin\alpha) \quad \varphi_3 = (\cos\alpha, -\sin\alpha)$$

Then if taking $x = (1, 0)$,

$$t_F(1, 0) = 1^2 + (\cos\alpha)^2 + (\cos\alpha)^2 = 1 + 2\cos^2\alpha$$

now, if taking $x = (0, 1)$,

$$t_F(0, 1) = 0^2 + (\sin\alpha)^2 + (\sin\alpha)^2 = \sin^2\alpha$$

So $A_F \leq \sin^2\alpha$, $B_F \geq 1 + 2\cos^2\alpha$. This means

$$\frac{B_F}{A_F} \geq \frac{1 + 2\cos^2\alpha}{\sin^2\alpha} \to \infty \quad (\alpha \to 0)$$

The main problem here is that as $\alpha \to 0$, the vector x could be approximately orthogonal to all three frame vectors at the same time, so A_F could get very small. This is not an issue in the Mercedes-Benz frame, because the symmetry of that frame guarantees that that no vector could get too close or too far from all of the frame vectors at once.

In short, the ratio $\frac{B_F}{A_F}$measures **how far from symmetry** the frame is.

4 Application: Signal Process

Just like sending pizza location to you at the beginning.

When transmitting a signal, the process is very similar. The elements we wish to describe are functions, and the basis is typically the celebrated **Fourier basis**, which consists (up to normalizations) of

$$f(t) = \{\sin(nt), \cos(nt) | n \in \mathbb{Z}_+\}$$

In this context, the study of frames has powerful applications to signal processing, wavelets, and data compression.

5 Conclusion

This course project has broadened my horizons of viewing things in Hilbert space. It is not just about "geometry", points, lines... Through the study of Frame Theory in Hilbert space, I feel it is more about discovering a relationship which has connected with all the subjects we have learned. However, sometimes a picture is worth thousand lines of proofs, and it can give a good intuition of some abstract concept. Just like understanding the optimal tightness of a frame in Hilbert space, we visualize it through the Mercedes-Benz frame. The key through this paper and further research is to discover an optimal way to get a tight frame in higher dimension, and this might lead to the famous sphere packing problem.

Finally, thank you Professor Conway for a wonderful semester and reading my paper!

References

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