Mixing Times of the Generalized Rook's Walk

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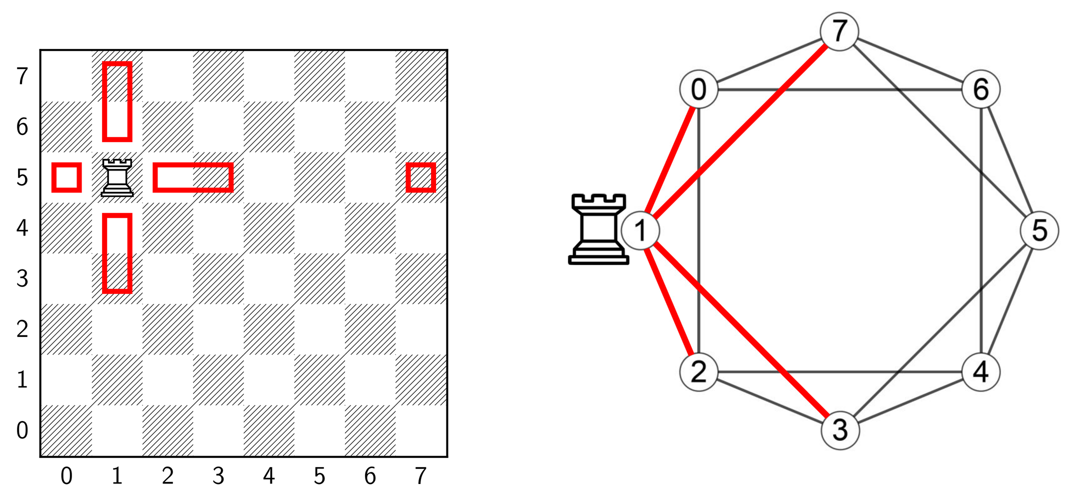
Introduction

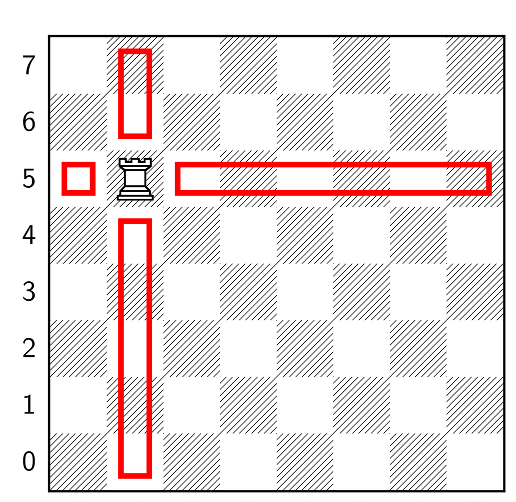
Our project investigates the mixing time of two classes of Markov Chains using the path coupling method: two restrictions on the rook's walk. These are equivalent to random walks on Cartesian powers of circulant graphs.



As the number of moves increases, the probability distribution of the rook's position approaches the uniform distribution. In our work, we bound the mixing time or the minimal number of moves for the distribution of the rook at time t to be sufficiently close to the limiting distribution.

Background

Mixing Time of a Rook's Walk [K, 2012] uses the spectral method to bound the mixing time of the traditional rook.



Result: $t_{mix}(\epsilon) \leq \frac{d(n-1)}{n} \log\left(\frac{n^d}{\epsilon}\right)$

Mixing Times for the Rook's Walk Via Path Coupling [MORS, 2014] uses path coupling to bound the mixing time of the traditional rook.

Result: $t_{mix}(\epsilon) \leq \left\lceil \frac{d(n-1)}{n-2} \log\left(\frac{2d}{\epsilon}\right) \right\rceil$

Notice that path coupling gives a tighter upper bound on the mixing time than the spectral method.

Path Coupling Method

A **coupling** of two distributions μ and ν is a pair of random variables (X, Y) on a common probability space with marginals μ and ν. The following inequality relates the distance between distributions and their coupling.

Coupling Inequality [LPW, 2009]:

$$\|P^t(x,\cdot) - P^t(y,\cdot)\|_{TV} \leq \mathbb{P}(X_t \neq Y_t)$$

The rate at which the coupling is expected to contract, or come together, gives us information about the mixing time.

Mixing Time Theorem [LPW, 2009]:

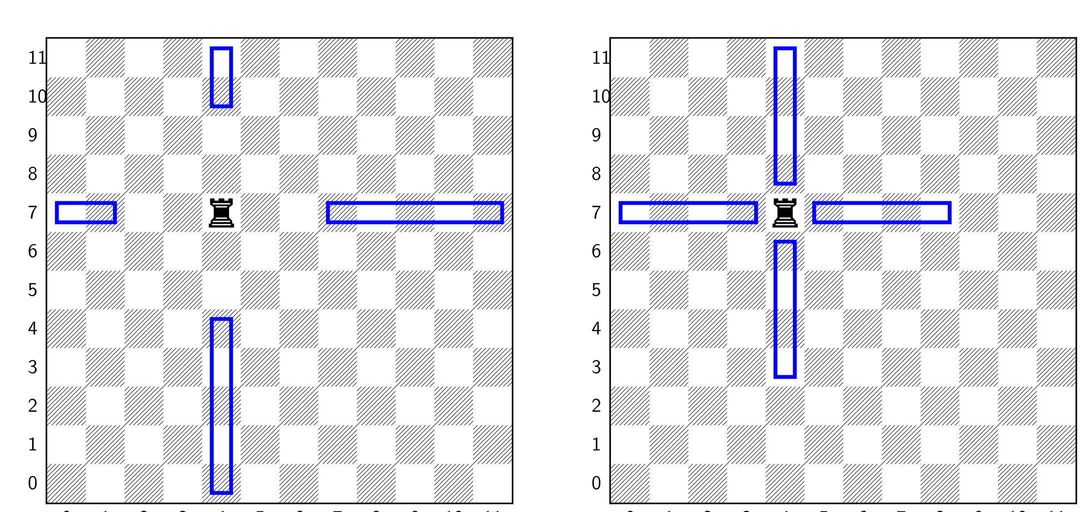
$$t_{mix}(\epsilon) \leq \frac{1}{1-m} \log\left(\frac{diam(\Omega)}{\epsilon}\right)$$

With *mean coupling distance* $m = \frac{E[\rho(X_t, Y_t)]}{\rho(x_{t-1}, y_{t-1})}$
This bound only holds if we have **contraction** $(m < 1)$.
We use **path coupling**, which allows us to only consider neighboring pairs, rather than arbitrary pairs in the mean coupling distance m.

Restrictions

In our work, we consider two restrictions on the rook's movement: near restriction and far restriction. The rook's set of legal moves K are restricted by the restriction term r as defined below.

Near restriction: $K = \{r+1, r+2, r+3, ..., \lfloor \frac{n}{2} \rfloor\}$
Far restriction: $K = \{1, 2, 3, ..., \lfloor \frac{n}{2} \rfloor - r\}$



Example: $n = 12$, $r = 2$

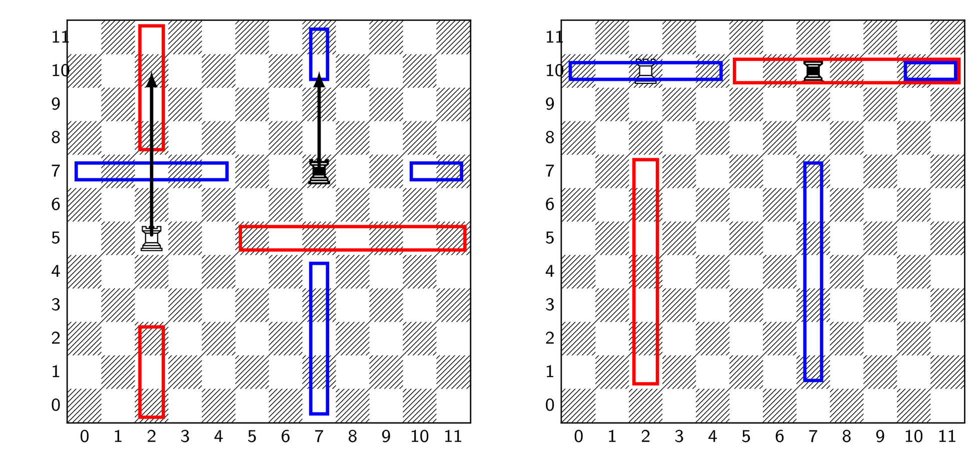
Near restriction: $K = \{r+1 = 3, 4, 5, 6 = \lfloor \frac{n}{2} \rfloor\}$
Far restriction: $K = \{1, 2, 3, 4 = \lfloor \frac{n}{2} \rfloor - r\}$

Notice that unlike near restriction, in far restriction the accessible squares are interrupted by the rook's current position.

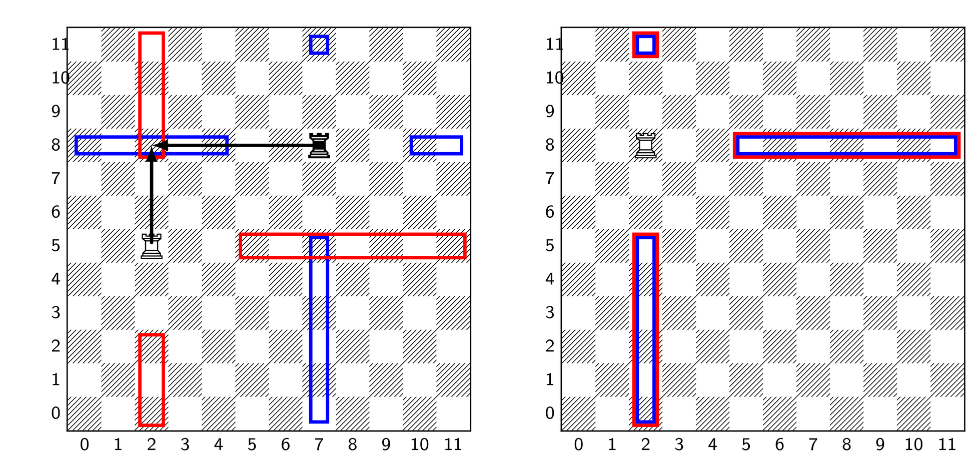
Coupling for Near Restriction

We defined the coupling as a one-to-one correspondence between two rooks' legal moves for any pair. There are 3 types of moves the rooks can make:

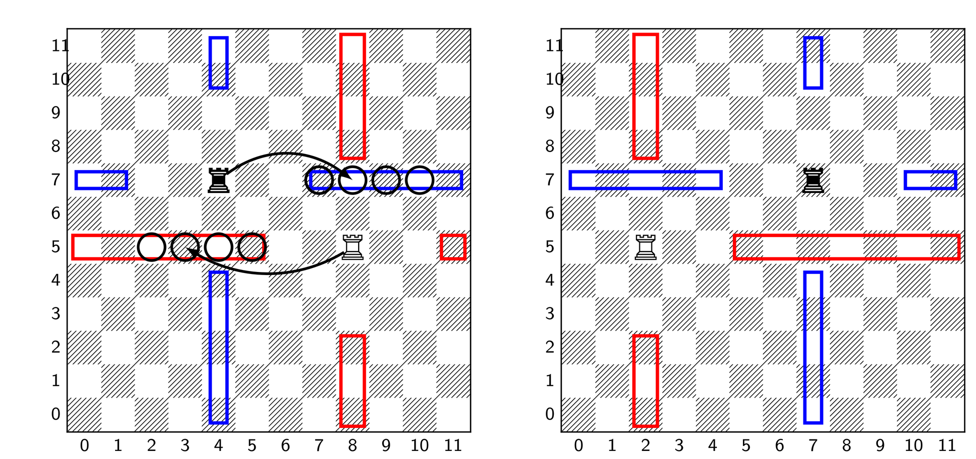
- Collapse 1 Dimension



- Collapse 2 Dimensions



- Maintain Distance

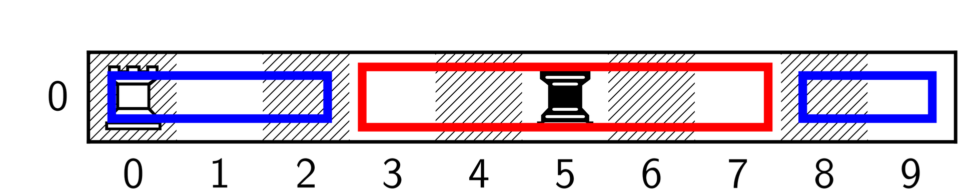


With this coupling, for near restriction we have the mean coupling distance:

$$m = \left(\frac{d-1}{d}\right) \cdot 1 + \left(\frac{2r+1}{d(n-2r-1)}\right) \cdot 1 = 1 - \frac{n-4r-2}{d(n-2r-1)}$$

Contraction for Near Restriction

Our coupling only gives meaningful results about the mixing time if it contracts. For near restriction, we proved that there are no cases where $m > 1$, but there are cases with no matches where $m = 1$.



To guarantee at least one match, we must consider only cases of near restriction where $n \geq 4r + 4$ for even n, and $n \geq 4r + 3$ for odd n.

Results

1. Near Restriction: $r < \frac{n}{4}$

$$t_{mix}(\epsilon) \leq \left\lceil \frac{d(n-2r-1)}{n-4r-2} \log\left(\frac{2d}{\epsilon}\right) \right\rceil$$

2. Far Restriction: $r < \frac{n}{6}$

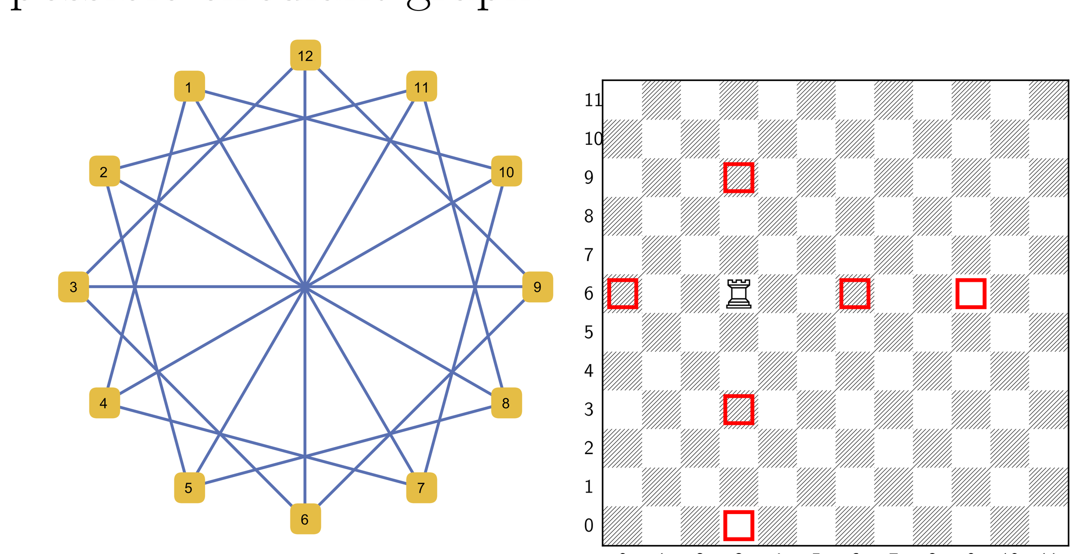
$$t_{mix}(\epsilon) \leq \left\lceil \frac{d(n-2r)}{n-4r} \log\left(\frac{2d}{\epsilon}\right) \right\rceil, \text{ for even } n.$$

$$t_{mix}(\epsilon) \leq \left\lceil \frac{d(n-2r-1)}{n-4r-2} \log\left(\frac{2d}{\epsilon}\right) \right\rceil, \text{ for odd } n.$$

Remark: As $n \to \infty$, all mixing time bounds converge to the bound of $\lceil d \cdot log(\frac{2d}{\epsilon}) \rceil$.

Future Work

We would like to continue investigating restrictions on the rook's walk to generalize our results to any possible circulant graph.



Example: $n = 12, K = \{3, 6\}$

References

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