Mixing Times of the Generalized Rook's Walk

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Outline

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  - Mixing Time
  - Coupling
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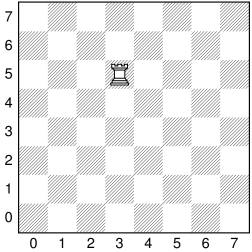
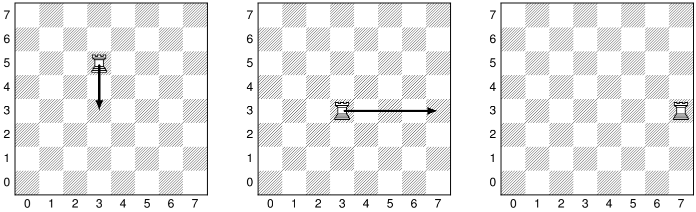
  - Couplings
  - Mixing Time Bounds / Long Term Behavior
  - Circulant Graphs

Motivation



Question: How long does it take for the rook to ”forget” its starting position?

Markov Chains



Markov chain: a sequence of random variables/vectors $X_1, X_2, \ldots$ such that

$$\mathbb{P}[X_{t+1} = x_{t+1} | X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0] = \mathbb{P}[X_{t+1} = x_{t+1} | X_t = x_t]$$

Where $x_0, \ldots, x_t$ are states at time t, and Ω is the state space.

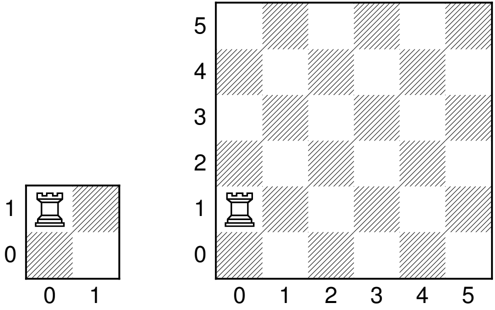
Distribution at time t: $P^t(x, \cdot) = \mathbb{P}(X_t = \cdot | X_0 = x)$

Convergence Theorem

For irreducible and aperiodic Markov chains

$$P^t(x,\cdot) \Longrightarrow \pi \qquad \text{as} \qquad t \to \infty$$

where π is the unique stationary distribution of the chain, i.e. $\pi P = \pi$.



Mixing Time

- Total variation distance

$$\|\mu - \nu\|_{\mathrm{TV}} := \max_{A \subset \Omega} |\mu(A) - \nu(A)| = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)|$$

- Distance to stationarity

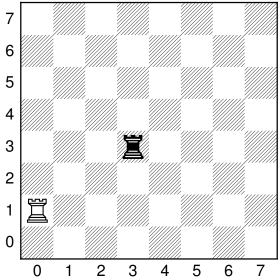
$$d(t) := \max_{x \in \Omega} \|P^t(x, \cdot) - \pi\|_{\mathrm{TV}}$$

- Mixing time: a measure of the convergence rate of the chain to its stationary distribution.

$$t_{\mathrm{mix}}(\varepsilon) := \min\{t : d(t) \le \varepsilon\}$$

Coupling

- A **coupling** of two distributions μ and ν is a pair of random variables (X, Y) on a common source of randomness with marginals μ and ν.
- We define a metric ρ to be the minimum moves required for a rook to reach another to measure the distance between the coupled rooks.



Coupling Markov Chains

Mixing Time Theorem:

$$t_{mix}(\varepsilon) \leq \frac{1}{1 - \frac{E[\rho(X_t, Y_t)]}{\rho(x_{t-1}, y_{t-1})}} \log\left(\frac{\max_{x,y} E[\rho(X_0, Y_0)]}{\varepsilon}\right)$$

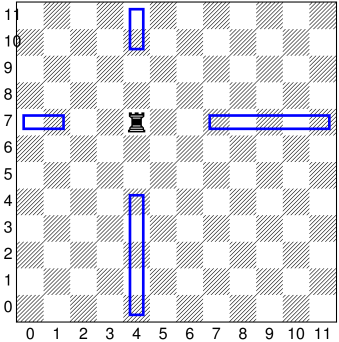
Mean Coupling distance: $\frac{E[\rho(X_t, Y_t)]}{\rho(x_{t-1}, y_{t-1})}$

$\max_{x,y} E[\rho(X_0, Y_0)] = diam(\Omega)$

Path Coupling uses the triangle inequality to allow us to only consider neighboring pairs in the mean coupling distance.

Definition: Near Restriction

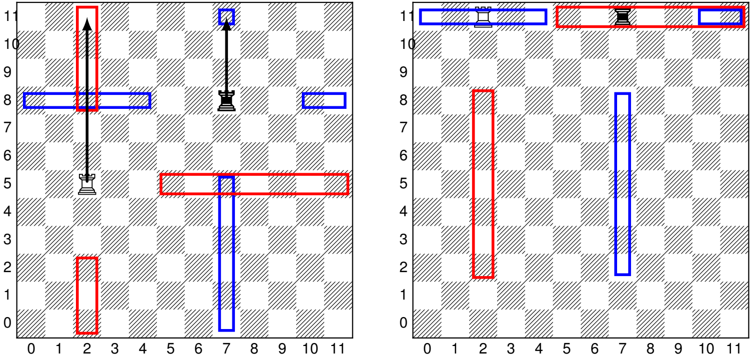
Legal moves: $K = \{r+1, r+2, ..., \lfloor \frac{n}{2} \rfloor\}$, where r is the restriction.



Example: $n = 12$, $r = 2$, so $K = \{3, 4, 5, 6\}$

Coupling: Near Restriction

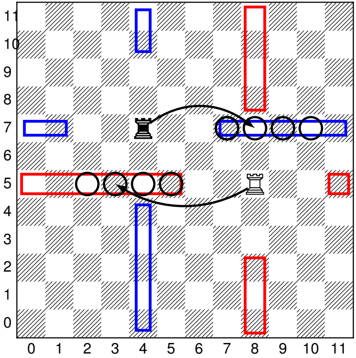
Coupling Rule: If the white rook moves to a square which is accessible to black, they collapse a dimension.

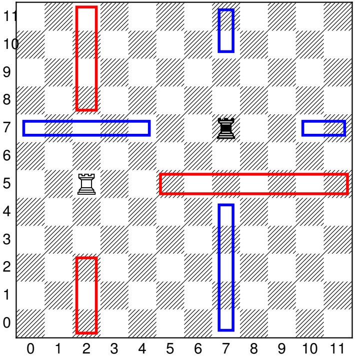


$$n = 12,\ r = 2, \text{ so } K = \{3, 4, 5, 6\},$$

Coupling: Near Restriction

Coupling Rule: If the white rook moves to a square which is inaccessible to black, black moves to maintain their distance $\rho(X_t, Y_t)$.

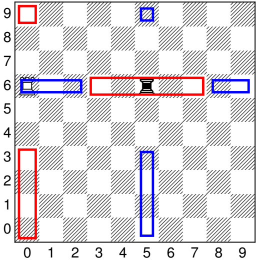




$n = 12$, $r = 2$, so $K = \{3, 4, 5, 6\}$,

Near Restriction: Contraction

$$E[\rho(X_t, Y_t)|\rho(x_{t-1}, y_{t-1}) = 1] = 1$$

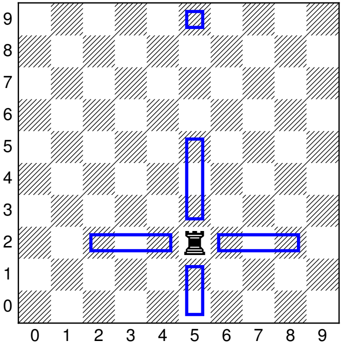


Contraction Result [OSWZ'16]

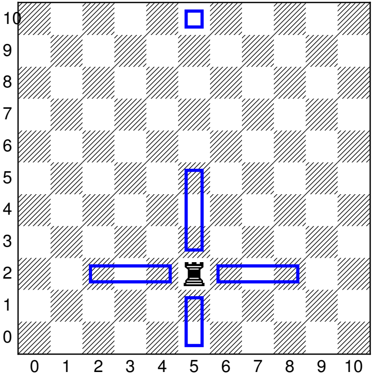
If $n \geq 4r + 4$ for even n and
$n \geq 4r + 3$ for odd n,
then our defined coupling for near restriction contract.

Definition: Far Restricted Rook's Walk

Legal Moves: $K = \{1, 2, ..., \lfloor \frac{n}{2} \rfloor - r\}$



$n = 10$, $r = 2$, so $K = \{1, 2, 3\}$



$n = 11$, $r = 2$, so $K = \{1, 2, 3\}$

Summary: Generalized Rook's Walk

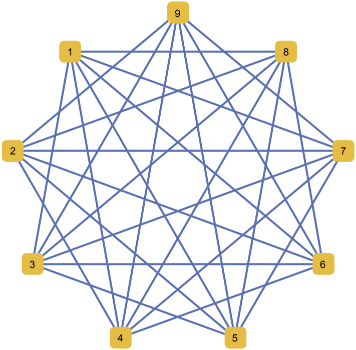
1. **No Restriction**: $r = 0$
   $t_{mix}(\varepsilon) \leq \left\lceil \frac{d(n-1)}{n-2} log(\frac{2d}{\varepsilon}) \right\rceil$
2. **Near Restriction**: $r < \frac{n}{4}$
   $t_{mix}(\varepsilon) \leq \left\lceil \frac{d(n-2r-1)}{n-4r-2} \log\left(\frac{2d}{\varepsilon}\right) \right\rceil$
3. **Far Restriction**: $r < \frac{n}{6}$
   $t_{mix}(\varepsilon) \leq \left\lceil \frac{d(n-2r)}{n-4r} \log\left(\frac{2d}{\varepsilon}\right) \right\rceil$, for even n.
   $t_{mix}(\varepsilon) \leq \left\lceil \frac{d(n-2r-1)}{n-4r-2} \log\left(\frac{2d}{\varepsilon}\right) \right\rceil$, for odd n.

Remark: As $n \to \infty$, all mixing time bounds converge to the bound of $\left\lceil d \cdot log(\frac{2d}{\varepsilon}) \right\rceil$.

Circulant Graphs

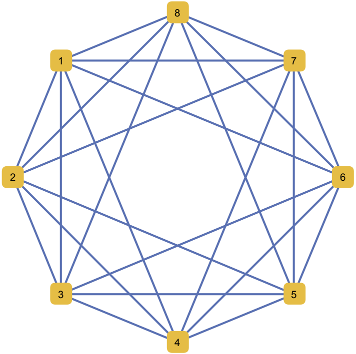
Each rook's walk is isomorphic to a random walk on a Cartesian power of a circulant graph.

Near Restriction



$n = 9, K = \{2, 3, 4\}$

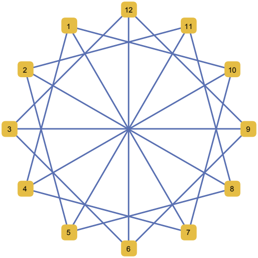
Far Restriction

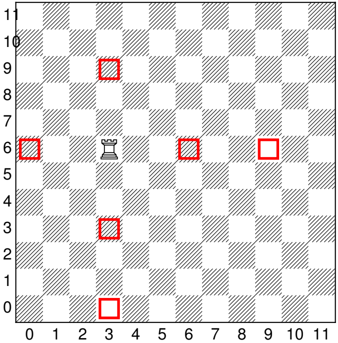


$n = 8, K = \{1, 2, 3\}$

Conclusion and Future Directions

- Generalize the rook's walk to any possible circulant graph.





$$n = 12, K = \{3, 6\}$$

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