**23.** **(a)** Use `PhaseLines` to investigate the bifurcation diagram for the differential equation

$$\frac{dy}{dt} = ay - y^3,$$

where $a$ is a parameter. Describe the different types of phase lines that occur.

**(b)** What are the bifurcation values for the one-parameter family in part (a)?

**(c)** Use `PhaseLines` to investigate the bifurcation diagram for the differential equation

$$\frac{dy}{dt} = r + ay - y^3,$$

where $r$ is a positive parameter. How does the bifurcation diagram change from the $r = 0$ case (see part (a))?

**(d)** Suppose $r$ is negative in the equation in part (c). How does the bifurcation diagram change?

# 1.8 LINEAR EQUATIONS

In Section 1.2 we developed an analytic method for finding explicit solutions to separable differential equations. Although many interesting problems lead to separable equations, most differential equations are not separable. The qualitative and numerical techniques we developed in Sections 1.3–1.6 apply to a much wider range of problems. It would be nice if we could also extend our analytic methods by developing ways to find explicit solutions of equations that are not separable.

Unfortunately, there is no general technique for computing explicit solutions that works for every differential equation. Although we know from the Existence Theorem that every reasonable differential equation has solutions, we have no guarantee that these solutions are made up of familiar functions such as polynomials, sines, cosines, and exponentials. In fact, they usually are not. Over the centuries, mathematicians have dealt with this dilemma by developing numerous specialized techniques for various types of differential equations. Today these techniques are available to us as one-line commands in sophisticated computer packages such as Maple and *Mathematica*. Nevertheless, you should be familiar with a few of the standard analytic techniques that apply to the most commonly encountered types of equations. In this section and Section 1.9, we develop two of the standard techniques for solving the most important type of differential equation—the *linear* differential equation.

## Linear Differential Equations

A first-order differential equation is **linear** if it can be written in the form

$$\frac{dy}{dt} = a(t)y + b(t),$$

where $a(t)$ and $b(t)$ are arbitrary functions of $t$. Examples of linear equations include

$$\frac{dy}{dt} = t^2 y + \cos t,$$

where $a(t) = t^2$ and $b(t) = \cos t$, and

$$\frac{dy}{dt} = \frac{e^{4\sin t}}{t^3 + 7t} y + 23t^3 - 7t^2 + 3,$$

where $a(t) = e^{4\sin t}/(t^3 + 7t)$ and $b(t) = 23t^3 - 7t^2 + 3$.

Sometimes it is necessary to do a little algebra in order to see that an equation is linear. For example, the differential equation

$$\frac{dy}{dt} - 3y = ty + 2$$

can be rewritten as

$$\frac{dy}{dt} = (t + 3)y + 2.$$

In this form we see that the equation is linear with $a(t) = t + 3$ and $b(t) = 2$.

Some differential equations fit into several categories. For example, the equation

$$\frac{dy}{dt} = 2y + 8$$

is linear with $a(t) = 2$ and $b(t) = 8$. (Both $a(t)$ and $b(t)$ are constant functions of $t$.) It is also autonomous and consequently separable.

The term *linear* refers to the fact that the dependent variable $y$ appears in the equation only to the first power. The differential equation

$$\frac{dy}{dt} = y^2$$

is not linear because $y^2$ cannot be rewritten in the form $a(t)y + b(t)$, no matter how $a(t)$ and $b(t)$ are chosen.

Of course, there is nothing magical about the names of the variables. The equation

$$\frac{dP}{dt} = e^{2t} P - \sin t$$

is linear with $a(t) = e^{2t}$ and $b(t) = -\sin t$. Also,

$$\frac{dw}{dt} = (\sin t)w$$

is both linear ($a(t) = \sin t$ and $b(t) = 0$) and separable. However,

$$\frac{dz}{dt} = t \sin z$$

is not linear but is separable.

### Additional terminology for linear equations

Linear differential equations come in two flavors. If $b(t) = 0$ for all $t$, then the equation is said to be **homogeneous** or *unforced*. Otherwise it is **nonhomogeneous** or *forced*. For example,

$$\frac{dy}{dt} = (\sin 2t)y$$

is homogeneous, and

$$\frac{dy}{dt} = y + \sin 2t$$

is nonhomogeneous.

A first-order linear differential equation is a *constant-coefficient* equation if $a(t)$ is a constant. In other words, the linear equation is a constant-coefficient equation if it has the form

$$\frac{dy}{dt} = \lambda y + b(t),$$

where $\lambda$ is a constant.

## Linearity Principles

Linear differential equations are important for many reasons. They are used to model a wide range of phenomena such as the decay of radioactive elements, the cooling of a cup of coffee, and the mixing of chemicals in a solution. In fact, when we start the modeling process, we almost always try a linear model first. Not only do we want to keep the model as simple as possible, but we also want to exploit the fact that the solutions to a linear equation are all related in a simple way. Given one or two nontrivial solutions, we get the rest by using the appropriate linearity principle.

### The homogeneous case

There are two linearity principles, one for homogeneous equations and a different one for nonhomogeneous equations. We begin with the homogeneous case.

LINEARITY PRINCIPLE If $y_h(t)$ is a solution of the homogeneous linear equation

$$\frac{dy}{dt} = a(t)y,$$

then any constant multiple of $y_h(t)$ is also a solution. That is, $ky_h(t)$ is a solution for any constant $k$. ■

We verify this theorem simply by checking that $ky_h(t)$ satisfies the differential equation. In other words, if $y_h(t)$ is a solution, then

$$\frac{dy_h}{dt} = a(t)y_h$$

for all $t$. If $k$ is a constant, then

$$\begin{aligned}\frac{d(ky_h)}{dt} &= k\frac{dy_h}{dt}\\ &= ka(t)y_h\\ &= a(t)(ky_h).\end{aligned}$$

We conclude that $ky_h(t)$ is also a solution to $dy/dt = a(t)y$.

This theorem is not very surprising. A homogeneous linear equation

$$\frac{dy}{dt} = a(t)y$$

is separable. Separating variables yields

$$\int \frac{1}{y}\,dy = \int a(t)\,dt,$$

and if we integrate the left-hand side, we get $\ln|y| + c = \int a(t)\,dt$, where $c$ is a constant of integration. Exponentiating both sides, removing the absolute value sign, and rewriting the constant produces

$$y(t) = ke^{\int a(t)\,dt},$$

where $k$ is an arbitrary constant. In this form, we can see that the nonzero solutions are constant multiples of each other. (Note that the equilibrium solution $y(t) = 0$ for all $t$ is a solution to *every* homogeneous equation.)

For example, consider the homogeneous equation

$$\frac{dy}{dt} = (\cos t)y.$$

All solutions are constant multiples of

$$y(t) = e^{\int \cos t\,dt} = e^{\sin t}.$$

In other words, the general solution of this equation is $y(t) = ke^{\sin t}$, where $k$ is an arbitrary constant (see Figure 1.93).

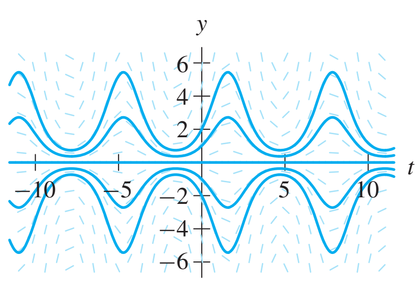



Figure 1.93
The slope field and graphs of various solutions to

$$\frac{dy}{dt} = (\cos t)y.$$

Note that the solutions are constant multiples of one another.

Remember that the Linearity Principle applies only to homogeneous linear equations. For example, it is easy to check that $y_1(t) = 1/(1-t)$ is a solution of the nonlinear equation

$$\frac{dy}{dt} = y^2$$

and that $y_2(t) = 2y_1(t) = 2/(1-t)$ is *not* a solution (see Exercise 17).

## The nonhomogeneous case

Although the Linearity Principle does not hold for a nonhomogeneous linear equation (see Exercises 18 and 34), there is a nice relationship between its solutions and the solutions to its *associated homogeneous equation*.

EXTENDED LINEARITY PRINCIPLE Consider the nonhomogeneous equation

$$\frac{dy}{dt} = a(t)y + b(t)$$

and its associated homogeneous equation

$$\frac{dy}{dt} = a(t)y.$$

1. If $y_h(t)$ is *any* solution of the homogeneous equation and $y_p(t)$ is *any* solution of the nonhomogeneous equation ("$p$" stands for particular), then $y_h(t) + y_p(t)$ is also a solution of the nonhomogeneous equation.
2. Suppose $y_p(t)$ and $y_q(t)$ are two solutions of the nonhomogeneous equation. Then $y_p(t) - y_q(t)$ is a solution of the associated homogeneous equation.

Therefore, if $y_h(t)$ is nonzero, $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation. ■

If $ky_h(t)$ is the general solution of the homogeneous equation, then the first half of the Extended Linearity Principle says that

$$ky_h(t) + y_p(t)$$

is a solution of the nonhomogeneous equation for *any* value of the constant $k$. The second half of the Extended Linearity Principle says that *any* solution $y_q(t)$ of the nonhomogeneous equation can be written as

$$ky_h(t) + y_p(t)$$

for some value of $k$. Therefore, $ky_h(t) + y_p(t)$ is the general solution of the nonhomogeneous equation. We often summarize this observation by saying that

> "The general solution of the nonhomogeneous equation is the sum of the general solution of the homogeneous equation and one solution of the nonhomogeneous equation."

For example, consider the nonhomogeneous equation

$$\frac{dy}{dt} = (\cos t)y + \tfrac{1}{5}(1 - t\cos t).$$

We have already seen that the general solution to its associated homogeneous equation $dy/dt = (\cos t)y$ is $y(t) = ke^{\sin t}$, where $k$ is an arbitrary constant. It is also easy to verify that $y_p(t) = t/5$ is a solution to the nonhomogeneous equation (see Exercise 32). Once we have the particular solution $y_p(t) = t/5$, the Extended Linearity Principle tells us that the general solution of the nonhomogeneous equation is

$$y(t) = \frac{t}{5} + ke^{\sin t},$$

where $k$ is an arbitrary constant (see Figure 1.94).

We can verify the Extended Linearity Principle by substituting the functions into the differential equation just as we did when we verified the Linearity Principle earlier in this section (see Exercise 33).

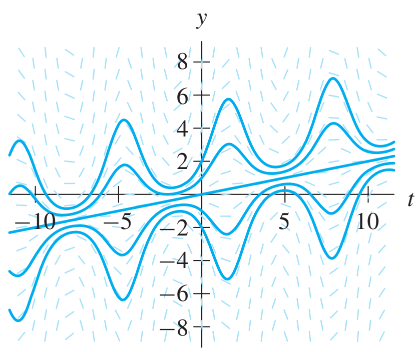



Figure 1.94
The slope field and graphs of various solutions to

$$\frac{dy}{dt} = (\cos t)y + \tfrac{1}{5}(1 - t\cos t).$$

We obtain these graphs by taking the graphs in Figure 1.93 and adding them to the graph of $y = t/5$.

## Solving Linear Equations

We now have a three-step procedure for solving linear equations. First, we find the general solution of the homogeneous equation, separating variables if necessary. Then we find one “particular” solution of the nonhomogeneous equation. Finally, we obtain the general solution of the nonhomogeneous equation by adding the general solution of the homogeneous equation to the particular solution of the nonhomogeneous equation.

In theory, we could solve any linear differential equation using this procedure. In practice, however, this technique is used only for special linear equations such as constant-coefficient equations. The limitation is caused by the fact that the second step requires that we produce a particular solution of the nonhomogeneous equation. If $a(t)$ is not a constant, this step can be quite difficult. If $a(t)$ is a constant, then we can sometimes succeed using a time-honored mathematical technique. We guess.

## The lucky guess

For example, consider the nonhomogeneous linear equation

$$\frac{dy}{dt} = -2y + e^t.$$

The associated homogeneous equation is $dy/dt = -2y$, and its general solution is $y(t) = ke^{-2t}$. (You could solve this homogeneous equation by separating variables, but its general solution should be second nature by now. See page 6.)

The hardest part of guessing a solution to the nonhomogeneous equation is deciding what to guess, and this task is made easier if we rewrite the equation so that all terms that involve $y$ are on the left-hand side. In other words, we rewrite the equation in question as

$$\frac{dy}{dt} + 2y = e^t.$$

Now we need to guess a function $y_p(t)$ such that, if we insert $y_p(t)$ into the left-hand side of the equation, out pops $e^t$ on the right-hand side. We probably should not guess sines or cosines for $y_p(t)$ because the left-hand side would still involve trigonometric functions after the computation. Similarly, polynomials would not work. What we need to guess is an exponential function. Guessing $y_p(t) = e^t$ seems to be a natural choice because its derivative is also $e^t$. Unfortunately, when we compute

$$\frac{dy_p}{dt} + 2y_p,$$

we get $e^t + 2e^t$, which does not equal $e^t$. Close, but no cigar.

This guess $y_p(t) = e^t$ almost worked. We were only off by the constant factor of 3. Perhaps we should guess a constant multiple of $e^t$, and in fact, perhaps we should let the differential equation tell us what the constant should be. In other words, we should replace the guess $y_p(t) = e^t$ with the guess $y_p(t) = \alpha e^t$, where $\alpha$ is a constant to be named later. This method is called the Method of the Undetermined Coefficient: We must determine the coefficient $\alpha$ so that $y_p(t) = \alpha e^t$ is a solution of the nonhomogeneous equation.

Starting with this more flexible guess of $y_p(t) = \alpha e^t$, we check to see if it works. We substitute $y_p(t)$ into $dy/dt + 2y$ and obtain

$$\begin{aligned}\frac{dy_p}{dt} + 2y_p &= \alpha e^t + 2\alpha e^t \\ &= 3\alpha e^t.\end{aligned}$$

In order for $y_p(t)$ to be a solution, $3\alpha e^t$ must equal $e^t$. That is, $3\alpha = 1$, which implies $\alpha = 1/3$. Therefore, the guess of $y_p(t) = e^t/3$ is a solution, and the general solution of $dy/dt = -2y + e^t$ is

$$y(t) = ke^{-2t} + \tfrac{1}{3}e^t,$$

where $k$ is an arbitrary constant.

## Another lucky guess

In the previous example, we guessed $y_p(t) = \alpha e^t$ because the equation was

$$\frac{dy}{dt} + 2y = b(t),$$

where $b(t)$ was an exponential involving $e^t$. Now let's consider a nonhomogeneous equation where $b(t)$ is a trigonometric function. For example,

$$\frac{dy}{dt} + 2y = \cos 3t.$$

Then the general solution of the homogeneous equation is still $y(t) = ke^{-2t}$. However, guessing an exponential will not work for this equation. This time we try

$$y_p(t) = \alpha \cos 3t + \beta \sin 3t.$$

Note that the simpler guesses of $y_p(t) = \alpha \cos 3t$ and $y_p(t) = \alpha \sin 3t$ are destined to fail because we end up with both sines and cosines when we compute $dy/dt + 2y$ (see Exercise 13).

To determine $\alpha$ and $\beta$, we substitute $y_p(t)$ into $dy/dt + 2y$ and obtain

$$\begin{aligned}\frac{dy_p}{dt} + 2y_p &= \frac{d(\alpha \cos 3t + \beta \sin 3t)}{dt} + 2(\alpha \cos 3t + \beta \sin 3t)\\ &= -3\alpha \sin 3t + 3\beta \cos 3t + 2\alpha \cos 3t + 2\beta \sin 3t\\ &= (-3\alpha + 2\beta) \sin 3t + (2\alpha + 3\beta) \cos 3t.\end{aligned}$$

In order for $y_p(t)$ to be a solution, we must find $\alpha$ and $\beta$ so that

$$(-3\alpha + 2\beta) \sin 3t + (2\alpha + 3\beta) \cos 3t = \cos 3t$$

for all $t$. To accomplish this, we solve the simultaneous algebraic equations

$$\begin{cases} -3\alpha + 2\beta = 0 \\ 2\alpha + 3\beta = 1 \end{cases}$$

for $\alpha$ and $\beta$. We obtain $\alpha = 2/13$ and $\beta = 3/13$. So

$$y_p(t) = \tfrac{2}{13} \cos 3t + \tfrac{3}{13} \sin 3t$$

is a solution of the nonhomogeneous equation.

Therefore, the general solution of $dy/dt + 2y = \cos 3t$ is

$$y(t) = ke^{-2t} + \tfrac{2}{13} \cos 3t + \tfrac{3}{13} \sin 3t,$$

where $k$ is an arbitrary constant. Solutions for several different initial conditions are shown in Figure 1.95.

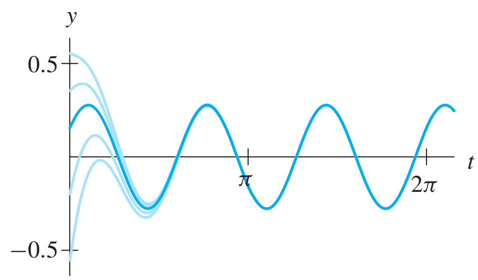



**Figure 1.95**
Graphs of several solutions of

$$\frac{dy}{dt} + 2y = \cos 3t.$$

Note that all of these graphs tend to merge relatively quickly.

### How lucky do you need to be?

After a little practice, you will find that there really isn't much luck involved. If $b(t)$ is made up of nice functions (sines, cosines, exponentials, ... ), you guess a particular solution made up of the same types of functions. If you make an inappropriate guess (for example, forgetting the $\beta \sin 3t$ term in the second example), then it will be impossible to find choices of the constants that make the guess a solution. If that happens, simply refine the original guess based on what you learned from the previous computation.

Also, you should be careful to avoid a common mistake. Throughout this process, it is important to remember that the undetermined constants are treated as constants during the differentiation step. Do not force a guess of the wrong form to work by turning $\alpha$ (or any other undetermined constant) into a nonconstant function $\alpha(t)$ during the last step in the computation.

## Qualitative Analysis

The previous example gives a great deal of insight into the qualitative behavior of solutions of many nonhomogeneous, linear differential equations. Note that the general solution of the associated homogeneous equation, $ke^{-2t}$, tends to zero quickly. Consequently, every solution is eventually close to the particular solution

$$y_p(t) = \tfrac{2}{13}\cos 3t + \tfrac{3}{13}\sin 3t.$$

We see this clearly in Figure 1.95, where solutions with different initial conditions tend toward the same periodic function. (This periodic solution is called a **steady-state solution** because every solution tends toward it in the long term. Note that this steady-state solution oscillates in a periodic fashion unlike an equilibrium solution that remains constant for all time.)

We could have predicted some of this behavior without computation. If we look at the slope field for this equation (see Figure 1.96), we see that for $y > 1/2$, the slopes are negative, and for $y < -1/2$, the slopes are positive. Graphs of solutions with initial conditions that are outside the interval $-1/2 \le y \le 1/2$ eventually enter the strip of the $ty$-plane determined by the inequalities $-1/2 \le y \le 1/2$. The detailed behavior of solutions near the $t$-axis is harder to see from the slope field. However, it is clear that solutions oscillate in some manner.

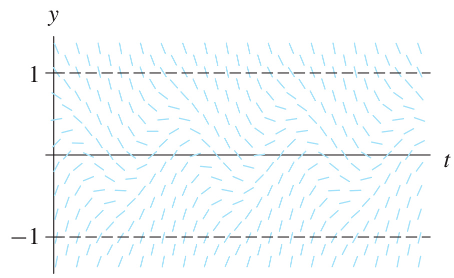



Figure 1.96
Slope field of

$$\frac{dy}{dt} = -2y + \cos 3t.$$

Note that, if $y \geq 1$, then $dy/dt \leq -1$. Similarly, if $y \leq -1$, the $dy/dt \geq 1$. Hence, any solution that enters the strip $-1 \leq y \leq 1$ remains in that strip as $t \to \infty$.

Looking again at the general solution

$$y(t) = ke^{-2t} + \tfrac{2}{13}\cos 3t + \tfrac{3}{13}\sin 3t,$$

we see that the long-term behavior of the solution is an oscillation with period $2\pi/3$ (see Figure 1.95). Note that this period is the same as the period of $\cos 3t$. However, the amplitude and the phase (that is, the locations of the maxima and minima) for the solution are not exactly the same as the amplitude and phase of $\cos 3t$. (We study the amplitude and phase of solutions to linear equations in detail in Chapter 4.)

These same ideas hold for any nonhomogeneous equation of the form

$$\frac{dy}{dt} = \lambda y + b(t)$$

as long as $\lambda$ is negative. As before, the homogeneous equation associated with this equation is

$$\frac{dy}{dt} = \lambda y,$$

whose general solution is $ke^{\lambda t}$. If $\lambda < 0$, these functions tend to zero exponentially fast. If one solution of the nonhomogeneous equation is $y_p(t)$, then the general solution of the nonhomogeneous equation is

$$y(t) = ke^{\lambda t} + y_p(t),$$

and we see that all solutions are close to $y_p(t)$ for large $t$. In other words, the solution of the homogeneous part of the equation tends to zero, and all solutions merge toward $y_p(t)$ over the long term.

The fact that all solutions converge over time definitely relies on the fact that $\lambda$ is negative. If $\lambda \geq 0$, very different behavior is possible (see Exercises 25–28).

## Second Guessing

Sometimes, our first guess may not work no matter how reasonable it is. If this happens, we simply guess again.

Consider the equation

$$\frac{dy}{dt} = -2y + 3e^{-2t}.$$

To compute the general solution, we first note that the general solution of the homogeneous equation is $y(t) = ke^{-2t}$. To find a particular solution of the nonhomogeneous equation we rewrite the equation as

$$\frac{dy}{dt} + 2y = 3e^{-2t}$$

and guess $y_p(t) = \alpha e^{-2t}$, with $\alpha$ as the undetermined coefficient. Substituting this guess into $dy/dt + 2y$, we get

$$\begin{aligned} \frac{dy_p}{dt} + 2y_p &= \frac{d(\alpha e^{-2t})}{dt} + 2\alpha e^{-2t} \\ &= -2\alpha e^{-2t} + 2\alpha e^{-2t} \\ &= 0. \end{aligned}$$

This is upsetting. No matter how we pick the coefficient $\alpha$, we always get zero when we substitute $y_p(t)$ into $dy/dt + 2y$. None of the solutions of the nonhomogeneous equation are of the form $y_p(t) = \alpha e^{-2t}$. We failed because our guess, $\alpha e^{-2t}$, is a solution of the associated homogeneous equation. When we substitute $y_p(t) = \alpha e^{-2t}$ into $dy/dt + 2y$, we are guaranteed to get zero.

Our guess must contain a factor of $e^{-2t}$ to have any hope of being a solution. Unfortunately, there is a wide variety of possible choices. We need a second guess for $y_p(t)$ that contains an $e^{-2t}$ term, is not a solution of the homogeneous equation, and is as simple as possible. Guesses of the form $\alpha e^{-2t} \sin t$ or $\alpha e^{bt}$ are clearly destined to fail. We need a guess whose derivative has one term that is just like itself and another term that involves $e^{-2t}$. The Product Rule suggests a product of $t$ and our first guess, so we try

$$y_p(t) = \alpha t e^{-2t},$$

where $\alpha$ is our undetermined coefficient. The derivative of $y_p(t)$ is $\alpha(1 - 2t)e^{-2t}$, and substituting this derivative into $dy/dt + 2y$, we obtain

$$\begin{aligned} \frac{dy_p}{dt} + 2y_p &= \alpha(1 - 2t)e^{-2t} + 2\alpha t e^{-2t} \\ &= \alpha e^{-2t}. \end{aligned}$$

Since we want $dy/dt + 2y$ to be $3e^{-2t}$, the guess $y_p(t) = \alpha t e^{-2t}$ is a solution if $\alpha = 3$. (This calculation illustrates why multiplying our first guess by $t$ is a good idea.) The general solution to this nonhomogeneous equation is

$$y(t) = ke^{-2t} + 3te^{-2t},$$

where $k$ is an arbitrary constant.

### Rule of thumb for second guessing

The last example indicates what is so unsatisfying about guessing techniques. How did we know to make the second guess a product of $t$ and our first guess? The answer is that we have either seen a similar problem before or we can figure out at least the form of the guess by another technique. Methods for arriving at the second guess with less guesswork but more computation are given in Exercise 23 of Section 1.9, in Chapter 6, and in Exercises 17 and 18 of Appendix B.

## EXERCISES FOR SECTION 1.8

In Exercises 1–6, find the general solution of the equation specified.

**1.** $\dfrac{dy}{dt} = -4y + 9e^{-t}$

**2.** $\dfrac{dy}{dt} = -4y + 3e^{-t}$

**3.** $\dfrac{dy}{dt} = -3y + 4\cos 2t$

**4.** $\dfrac{dy}{dt} = 2y + \sin 2t$

**5.** $\dfrac{dy}{dt} = 3y - 4e^{3t}$

**6.** $\dfrac{dy}{dt} = \dfrac{y}{2} + 4e^{t/2}$

In Exercises 7–12, solve the given initial-value problem.

**7.** $\dfrac{dy}{dt} + 2y = e^{t/3}, \quad y(0) = 1$

**8.** $\dfrac{dy}{dt} - 2y = 3e^{-2t}, \quad y(0) = 10$

**9.** $\dfrac{dy}{dt} + y = \cos 2t, \quad y(0) = 5$

**10.** $\dfrac{dy}{dt} + 3y = \cos 2t, \quad y(0) = -1$

**11.** $\dfrac{dy}{dt} - 2y = 7e^{2t}, \quad y(0) = 3$

**12.** $\dfrac{dy}{dt} - 2y = 7e^{2t}, \quad y(0) = 3$

**13.** Consider the nonhomogeneous linear equation

$$\frac{dy}{dt} + 2y = \cos 3t.$$

To find a particular solution, it is pretty clear that our guess must contain a cosine function, but it is not so clear that the guess must also contain a sine function.

**(a)** Guess $y_p(t) = \alpha \cos 3t$ and substitute this guess into the equation. Is there a value of $\alpha$ such that $y_p(t)$ is a solution?

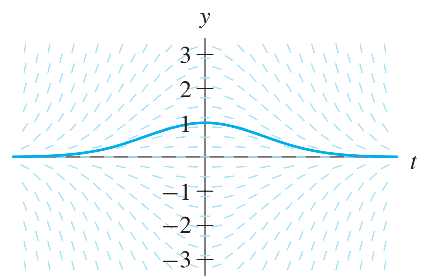
**(b)** Write a brief paragraph explaining why the proper guess for a particular solution is $y_p(t) = \alpha \cos 3t + \beta \sin 3t$.

**14.** Consider the nonhomogeneous linear equation

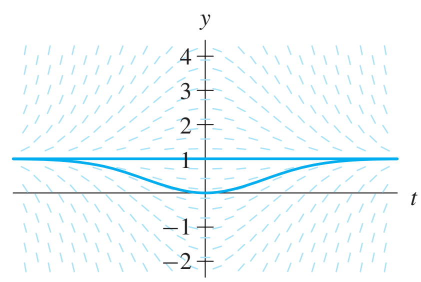
$$\frac{dy}{dt} = \lambda y + \cos 2t.$$

To find its general solution, we add the general solution of the associated homogeneous equation and a particular solution $y_p(t)$ of the nonhomogeneous equation. Briefly explain why it does not matter which solution of the nonhomogeneous equation we use for $y_p(t)$.

**15.** The graph to the right is the graph of a solution of a homogeneous linear equation $dy/dt = a(t)y$. Give rough sketches of the graphs of the solutions to this equation that satisfy the initial conditions $y(0) = 0$, $y(0) = 2$, $y(0) = 3$, $y(0) = -1$, and $y(0) = -2.5$.




**16.** The two graphs to the right are graphs of solutions of a nonhomogeneous linear equation $dy/dt = a(t)y + b(t)$. Give rough sketches of the graphs of the solutions to this equation that satisfy the initial conditions $y(0) = 2$, $y(0) = 3.5$, $y(0) = -1$, and $y(0) = -2$.




**17.** Consider the nonlinear differential equation $dy/dt = y^2$.

**(a)** Show that $y_1(t) = 1/(1-t)$ is a solution.

**(b)** Show that $y_2(t) = 2/(1-t)$ is not a solution.

**(c)** Why don't these two facts contradict the Linearity Principle?

**18.** Consider the nonhomogeneous linear equation $dy/dt = -y + 2$.

**(a)** Compute an equilibrium solution for this equation.

**(b)** Verify that $y(t) = 2 - e^{-t}$ is a solution for this equation.

**(c)** Using your results in parts (a) and (b) and the Uniqueness Theorem, explain why the Linearity Principle does not hold for this equation.

**19.** Consider a nonhomogeneous linear equation of the form

$$\frac{dy}{dt} + a(t)y = b_1(t) + b_2(t),$$

that is, $b(t)$ is written as a sum of two functions. Suppose that $y_h(t)$ is a solution of the associated homogeneous equation $dy/dt + a(t)y = 0$, that $y_1(t)$ is a solution of the equation $dy/dt + a(t)y = b_1(t)$, and that $y_2(t)$ is a solution of the equation $dy/dt + a(t)y = b_2(t)$. Show that $y_h(t) + y_1(t) + y_2(t)$ is a solution of the original nonhomogeneous equation.

**20.** Consider the nonhomogeneous linear equation

$$\frac{dy}{dt} + 2y = 3t^2 + 2t - 1.$$

In order to find the general solution, we must guess a particular solution $y_p(t)$. Since the right-hand side is a quadratic polynomial, it is reasonable to guess a quadratic for $y_p(t)$, so let

$$y_p(t) = at^2 + bt + c,$$

where $a$, $b$, and $c$ are constants. Determine values for these constants so that $y_p(t)$ is a solution.

In Exercises 21–24, find the general solution and the solution that satisfies the initial condition $y(0) = 0$.

**21.** $\dfrac{dy}{dt} + 2y = t^2 + 2t + 1 + e^{4t}$

**22.** $\dfrac{dy}{dt} + y = t^3 + \sin 3t$

**23.** $\dfrac{dy}{dt} - 3y = 2t - e^{4t}$

**24.** $\dfrac{dy}{dt} + y = \cos 2t + 3\sin 2t + e^{-t}$

In Exercises 25–28, give a brief qualitative description of the behavior of solutions. Note that we only give partial information about the functions in the differential equation, so your description must allow for various possibilities. Be sure to deal with initial conditions of different sizes and to discuss the long-term behavior of solutions.

**25.** $\dfrac{dy}{dt} + 2y = b(t)$, where $-1 < b(t) < 2$ for all $t$.

**26.** $\dfrac{dy}{dt} - 2y = b(t)$, where $-1 < b(t) < 2$ for all $t$.

**27.** $\dfrac{dy}{dt} + y = b(t)$, where $b(t) \to 3$ as $t \to \infty$.

**28.** $\dfrac{dy}{dt} + ay = \cos 3t + b$, where $a$ and $b$ are positive constants.

**29.** A person initially places \$1,000 in a savings account that pays interest at the rate of 1.1% per year compounded continuously. Suppose the person arranges for \$20 per week to be deposited automatically into the savings account.

- **(a)** Write a differential equation for $P(t)$, the amount on deposit after $t$ years (assume that "weekly deposits" is close enough to "continuous deposits" so that we may model the balance with a differential equation.)
- **(b)** Find the amount on deposit after 5 years.

**30.** A student has saved \$70,000 for her college tuition. When she starts college, she invests the money in a savings account that pays 1.5% interest per year, compounded continuously. Suppose her college tuition is \$30,000 per year and she arranges with the college that the money will be deducted from her savings account in small payments. In other words, we assume that she is paying continuously. How long will she be able to stay in school before she runs out of money?

**31.** A college professor contributes \$5,000 per year into her retirement fund by making many small deposits throughout the year. The fund grows at a rate of 7% per year compounded continuously. After 30 years, she retires and begins withdrawing from her fund at a rate of \$3000 per month. If she does not make any deposits after retirement, how long will the money last? [*Hint*: Solve this in two steps, before retirement and after retirement.]

**32.** Verify that the function $y(t) = t/5$ satisfies the nonhomogeneous linear equation

$$\frac{dy}{dt} = (\cos t)y + \tfrac{1}{5}(1 - t\cos t).$$

**33.** In this exercise, we verify the Extended Linearity Principle for the nonhomogeneous equation

$$\frac{dy}{dt} = a(t)y + b(t).$$

**(a)** Let $y_h(t)$ be a solution of the associated homogeneous equation and let $y_p(t)$ be any solution of the nonhomogeneous equation. Show that $y_h(t) + y_p(t)$ satisfies the nonhomogeneous equation by calculating $d(y_h + y_p)/dt$.

**(b)** Assume that $y_p(t)$ and $y_q(t)$ are solutions to the nonhomogeneous equation. Show that $y_p(t) - y_q(t)$ is a solution to the associated homogeneous equation by computing $d(y_p - y_q)/dt$.

**34.** Suppose that every constant multiple of a solution is also a solution for a first-order differential equation $dy/dt = f(t, y)$, where $f(t, y)$ is continuous on the entire $ty$-plane. What can be said about the differential equation?

## 1.9 INTEGRATING FACTORS FOR LINEAR EQUATIONS

In Section 1.8 we described a guessing technique for solving certain first-order nonhomogeneous linear differential equations. In this section we develop a different analytic method for solving these equations. It is more general than the technique of the previous section, so it can be applied successfully to more equations. It also avoids "guessing." Unfortunately, this method involves the calculation of an integral, which may be a problem as we will see. It is also not as amenable to qualitative analysis. At the end of this section, we discuss the pros and cons of both methods.

## Integrating Factors

Given a nonhomogeneous linear differential equation

$$\frac{dy}{dt} = a(t)y + b(t),$$

how can we go about finding the general solution? There is a clever trick that turns an equation of this form into a differential equation that can be solved by integration. As with many techniques in mathematics, the cleverness of this trick might leave you with that "how could I ever think of something like this?" feeling. The thing to remember is that differential equations have been around for more than 300 years. Given three centuries, it is not so surprising that mathematicians were able to discover and refine a slick way to treat these equations.

### The idea behind the method

We begin by rewriting the nonhomogeneous equation as

$$\frac{dy}{dt} + g(t)y = b(t),$$

where $g(t) = -a(t)$. We use this form and change the notation for two reasons. The form of the left-hand side of the equation suggests this method, and replacing $-a(t)$ by $g(t)$ avoids a number of annoying minus signs in the calculations.

After staring at this equation for a while (a couple of decades or so), we notice that, with sufficiently poor eyesight, the left-hand side looks somewhat like what we get when we differentiate using the Product Rule. That is, the Product Rule says that the derivative of the product of $y(t)$ and a function $\mu(t)$ is

$$\frac{d(\mu(t)\,y(t))}{dt} = \mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t).$$

Note that one term on the right-hand side has $dy/dt$ in it and the other term has $y$ in it just like the left-hand side of our nonhomogeneous linear equation.

Here's the clever part. Multiply both sides of the original differential equation by an (as yet unspecified) function $\mu(t)$. We obtain the new differential equation

$$\mu(t)\frac{dy}{dt} + \mu(t)g(t)y = \mu(t)b(t)$$

whose left-hand side looks even more like the derivative of a product of two functions. For the moment, let's *assume* that we have a function $\mu(t)$ so that the left-hand side actually *is* the derivative of the product $\mu(t)\,y(t)$. That is, suppose we have found a function $\mu(t)$ that satisfies

$$\frac{d(\mu(t)\,y(t))}{dt} = \mu(t)\frac{dy}{dt} + \mu(t)g(t)y.$$

Then the new differential equation is just

$$\frac{d(\mu(t)\,y(t))}{dt} = \mu(t)b(t).$$

How does this help? We can integrate both sides of this equation with respect to $t$ to obtain

$$\mu(t)\,y(t) = \int \mu(t)\,b(t)\,dt,$$

and consequently,

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)\,b(t)\,dt.$$

That is, assuming we have such a $\mu(t)$ and can evaluate $\int \mu(t)\,b(t)\,dt$, we can compute our solution $y(t)$.

## Finding the integrating factor

This derivation of $y(t)$ is based on one pretty big assumption. How can we find a function $\mu(t)$ such that

$$\frac{d(\mu(t)\,y(t))}{dt} = \mu(t)\frac{dy}{dt} + \mu(t)g(t)y(t)$$

in the first place?

Applying the Product Rule to the left-hand side, we see that the desired $\mu(t)$ must satisfy

$$\mu(t)\frac{dy}{dt} + \frac{d\mu}{dt}y(t) = \mu(t)\frac{dy}{dt} + \mu(t)g(t)y(t).$$

Canceling the $\mu(t)(dy/dt)$ term on both sides leaves

$$\frac{d\mu}{dt}y(t) = \mu(t)\,g(t)\,y(t).$$

So, if we find a function $\mu(t)$ that satisfies the equation

$$\frac{d\mu}{dt} = \mu(t)\,g(t),$$

we get our desired $\mu(t)$. However, this last equation is just $d\mu/dt = g(t)\mu$, which is a homogeneous linear differential equation, and we already know that

$$\mu(t) = e^{\int g(t)\,dt}.$$

(See page 113 for the derivation of this solution.)

Given this formula for $\mu(t)$, we now see that this strategy is going to work. The function $\mu(t)$ is called an **integrating factor** for the original nonhomogeneous equation because we can solve the equation by integration if we multiply it by the factor $\mu(t)$. In other words, whenever we want to determine an explicit solution to

$$\frac{dy}{dt} + g(t)y = b(t),$$

we first compute the integrating factor $\mu(t)$. Then we solve the equation by multiplying both sides by $\mu(t)$ and integrating. Note that, when we calculate $\mu(t)$, there is an arbitrary constant of integration in the exponent. Since we only need one integrating factor $\mu(t)$ to solve the equation, we choose the constant to be whatever is most convenient. That choice is usually zero.

To see this method at work, let's look at some examples. The method looks very general. However, because there are two integrals to calculate, we may get stuck before we obtain an explicit solution.

## Complete success

Consider the nonhomogeneous linear equation

$$\frac{dy}{dt} + \frac{2}{t}y = t - 1.$$

First we compute the integrating factor

$$\mu(t) = e^{\int g(t)\,dt} = e^{\int (2/t)\,dt} = e^{2\ln t} = e^{\ln(t^2)} = t^2.$$

Remember that the idea behind this method is to multiply both sides of the differential equation by $\mu(t)$ so that the left-hand side of the new equation is the result of the Product Rule. In this case, multiplying by $\mu(t) = t^2$ yields

$$t^2\frac{dy}{dt} + 2ty = t^2(t - 1).$$

Note that the left-hand side is the derivative of the product of $t^2$ and $y(t)$. In other words, this equation is the same as

$$\frac{d}{dt}(t^2 y) = t^3 - t^2.$$

Integrating both sides with respect to $t$ yields

$$t^2 y = \frac{t^4}{4} - \frac{t^3}{3} + k,$$

where $k$ is an arbitrary constant. The general solution is

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{k}{t^2}.$$

Of course, we can check that these functions satisfy the differential equation by substituting them back into the equation.

It is important to note the role of constants of integration in this example. When we calculated $\mu(t) = t^2$, we ignored the constant because we only need one integrating factor. However, after we multiplied both sides of the original equation by $\mu(t)$ and integrated, it was important to include the constant of integration on the right-hand side. If we had omitted that constant, we would have computed just one solution to the nonhomogeneous equation rather than the general solution.

This example is also a good illustration of the Extended Linearity Principle. Note that $k/t^2$ is the general solution of the associated homogeneous equation

$$\frac{dy}{dt} = -\frac{2}{t}\,y,$$

and

$$y(t) = \frac{t^2}{4} - \frac{t}{3}$$

is one solution of the nonhomogeneous equation.

## Problems with the integration

The previous example was chosen carefully. Another linear equation which does not look any more difficult is

$$\frac{dy}{dt} = t^2 y + t - 1.$$

We rewrite the differential equation as

$$\frac{dy}{dt} - t^2 y = t - 1$$

and compute the integrating factor

$$\mu(t) = e^{\int -t^2\,dt} = e^{-t^3/3}.$$

Next we multiply both sides by $\mu(t)$ and obtain

$$e^{-t^3/3}\frac{dy}{dt} - t^2 e^{-t^3/3}\,y = e^{-t^3/3}(t-1).$$

Note that the left-hand side is the derivative of the product of $e^{-t^3/3}$ and $y(t)$, so we have

$$\frac{d}{dt}\left(e^{-t^3/3}\,y\right) = e^{-t^3/3}(t-1).$$

Integrating both sides yields

$$e^{-t^3/3}\,y = \int e^{-t^3/3}(t-1)\,dt$$

but then we are stuck. It turns out that the integral on the right-hand side of this equation is not expressible in terms of the familiar functions (sin, cos, ln, and so on), so we cannot obtain explicit formulas for the solutions.

This example indicates what can go wrong with techniques that involve the calculation of explicit integrals. Even reasonable-looking functions can quickly lead to complicated integrating factors and integrals. On the other hand, we can express the solution in terms of integrals with respect to $t$, and although many integrals are impossible to calculate explicitly, many others are possible. Indeed, as we have mentioned before, there are a number of computer programs that are quite good at calculating the indefinite integrals involved in this technique.

## Mixing Problems Revisited

In Section 1.2 we considered a model of the concentration of a substance in solution. Typically in these problems we have a container in which there is a certain amount of fluid (such as water or air) to which a contaminant is added at some rate. The fluid is kept well mixed at all times. If the total volume of fluid is kept fixed, then the resulting differential equation for the amount of contaminant is autonomous and can be solved either by separating variables or by the Extended Linearity Principle along with a guessing technique. If the total volume of fluid changes with time, then the differential equation is nonautonomous and must be solved using an integrating factor.

### A polluted pond

Consider a pond that has an initial volume of 10,000 cubic meters. Suppose that at time $t = 0$, the water in the pond is clean and that the pond has two streams flowing into it, stream A and stream B, and one stream flowing out, stream C (see Figure 1.97). Suppose 500 cubic meters per day of water flow into the pond from stream A, 750 cubic meters per day flow into the pond from stream B, and 1250 cubic meters flow out of the pond via stream C.

At time $t = 0$, the water flowing into the pond from stream A becomes contaminated with road salt at a concentration of 5 kilograms per 1000 cubic meters. Suppose the water in the pond is well mixed so the concentration of salt at any given time is constant. To make matters worse, suppose also that at time $t = 0$ someone begins dumping trash into the pond at a rate of 50 cubic meters per day. The trash settles to the bottom of the pond, reducing the volume by 50 cubic meters per day. To adjust for the incoming trash, the rate that water flows out via stream C increases to 1300 cubic meters per day and the banks of the pond do not overflow.

The description looks very much like the mixing problems we have already considered (where "pond" replaces "vat" and "stream" replaces "pipe"). The new element here is that the total volume is not constant. Because of the dumping of trash, the volume decreases by 50 cubic meters per day.

If we let $S(t)$ be the amount of salt (in kilograms) in the pond at time $t$, then $dS/dt$ is the difference between the rate that salt enters the pond and the rate that salt

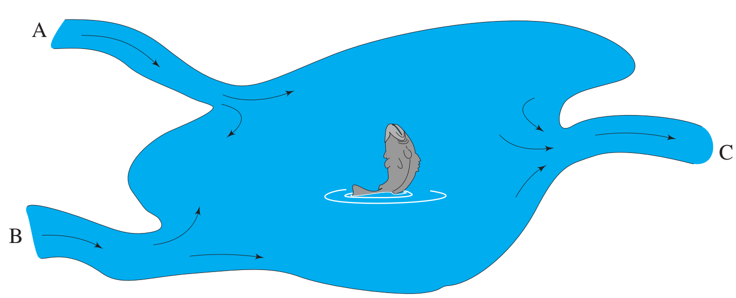



Figure 1.97
Schematic of the pond with three streams.

leaves the pond. Salt enters the pond from stream A only. The rate at which it enters is the product of its concentration in the water and the rate at which the water flows in through stream A. Since the concentration is 5 kilograms per 1000 cubic meters and the rate that water flows into the pond from stream A is 500 cubic meters per day, the rate at which salt enters the pond is $(500)(5/1000) = 5/2$ kilograms per day. The rate at which the salt leaves the pond via stream C is the product of its concentration *in the pond* and the rate at which water flows out of the pond. The rate at which water flows out is 1300 cubic meters per day. To determine the concentration, we note that it is the quotient of the amount $S$ of salt in the pond by the volume $V$. Because the volume is initially 10,000 cubic meters and it decreases by 50 cubic meters per day, we know that $V(t) = 10{,}000 - 50t$. Hence, the concentration is $S/(10{,}000 - 50t)$, and the rate at which salt flows out of the pond is

$$1300\left(\frac{S}{10{,}000 - 50t}\right),$$

which simplifies to $26S/(200 - t)$. Therefore, the differential equation that models the amount of salt in the pond is

$$\frac{dS}{dt} = \frac{5}{2} - \frac{26S}{200 - t}.$$

This model is valid only as long as there is water in the pond—that is, as long as the volume $V(t) = 10{,}000 - 50t$ is positive. So the differential equation is valid for $0 \le t < 200$. Because the water is clean at time $t = 0$, the initial condition is $S(0) = 0$.

Since this equation is nonautonomous, we solve this initial-value problem using an integrating factor. Rewriting the differential equation as

$$\frac{dS}{dt} + \left(\frac{26}{200 - t}\right) S = \frac{5}{2}$$

indicates that the integrating factor is

$$\mu(t) = e^{\int \frac{26}{200-t}\, dt} = e^{-26 \ln(200-t)} = e^{\ln\left((200-t)^{-26}\right)} = (200 - t)^{-26}.$$

Multiplying both sides by $\mu(t)$ gives

$$(200 - t)^{-26}\frac{dS}{dt} + 26(200 - t)^{-27} S = \frac{5}{2}(200 - t)^{-26}.$$

By the Product Rule, this equation is the same as the differential equation

$$\frac{d}{dt}\left((200 - t)^{-26} S\right) = \frac{5}{2}(200 - t)^{-26}.$$

Integrating both sides yields

$$\begin{aligned}(200 - t)^{-26} S &= \frac{5}{2}\int (200 - t)^{-26}\, dt \\ &= \frac{5}{2}\frac{(200 - t)^{-25}}{25} + c,\end{aligned}$$

where $c$ is an arbitrary constant. Solving for $S$, we obtain the general solution

$$S = \frac{200 - t}{10} + c(200 - t)^{26}.$$

Using the initial condition $S(0) = 0$, we find that $c = -20/200^{26}$ and the particular solution for the initial-value problem is

$$S = \frac{200 - t}{10} - 20\left(\frac{200 - t}{200}\right)^{26}.$$

This is an unusual-looking expression because of the large number $200^{26}$. However, the graph reveals that its behavior is not at all unusual (see Figure 1.98). The amount of salt in the pond rises fairly quickly, reaching a maximum close to $S = 20$ at $t \approx 25$. After that time, the amount of salt decreases almost linearly, reaching zero at $t = 200$.

The behavior of this solution is quite reasonable if we recall that the pond starts out containing no salt and that eventually it is completely filled with trash. (It contains no salt or water at time $t = 200$.) As we mentioned above, the concentration of salt in the pond water is given by $C(t) = S(t)/V(t) = S(t)/(10{,}000 - 50t)$. Graphing $C(t)$, we see that it increases asymptotically toward 0.002 kilograms per cubic meter even as the water level decreases (see Figure 1.99).

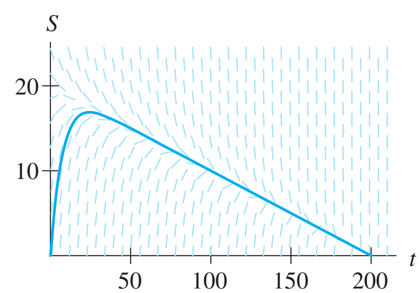



Figure 1.98
Graph of the solution of $dS/dt = 5/2 - 26S/(200 - t)$, with $S(0) = 0$.

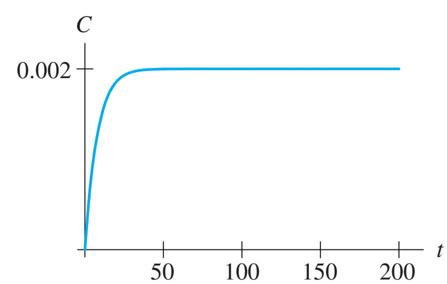



Figure 1.99
Graph of concentration of salt versus time for the solution graphed in Figure 1.98.

## Comparing the Methods of Solution for Linear Equations

There is an old saying that goes

> "If the only tool you have is a hammer, then every problem looks like a nail."

If you know only one method for solving linear differential equations, then you certainly save time thinking about which method to use when confronted with such an equation. However, we have two and each method has its advantages and disadvantages.

Which method should you use for a given linear differential equation? Trying to guess a solution to the nonhomogeneous equation that we just solved would be a nightmare. Hence, the method of integrating factors is the only reasonable choice for that equation.

On the other hand, consider a linear equation such as

$$\frac{dv}{dt} + 0.4v = 3\cos 2t,$$

which is typical for the voltage over a capacitor in an RC circuit with a periodic voltage source (see Section 1.4). The integrating factor for this equation is $\mu(t) = e^{0.4t}$. Therefore, the integral you must compute is

$$\int e^{0.4t}(3\cos 2t)\,dt.$$

This integral can certainly be done by hand using integration by parts but it would take some effort.

If you use a guessing technique, you would guess a particular solution of the form

$$v_p(t) = \alpha\cos 2t + \beta\sin 2t$$

and solve for $\alpha$ and $\beta$. The computation requires some algebra but not much calculus (see Exercise 21).

So which method is better for this equation? Both end up with the same general solution but the guessing method is arguably faster. One advantage of the guessing method is that it exploits the Extended Linearity Principle and we see the qualitative behavior of the solutions more directly. We know the general solution of the homogeneous equation is $ke^{-0.4t}$, which tends to zero in an exponential fashion, and over the long term, all solutions converge to the periodic solution $v_p(t)$ (see Figure 1.100).

In theory, the method of integrating factors works more generally but the integrals involved might be difficult or impossible to do. The guessing technique described in the previous section avoids the integration but is only practical for certain linear equations such as constant-coefficient equations with relatively simple functions $b(t)$.

Most importantly, you need to understand what it means to be a linear equation and the implications of the Linearity and Extended Linearity Principles. It is also important that you remember the clever idea behind the development of integrating factors. Each of the methods teaches us something about linear differential equations.

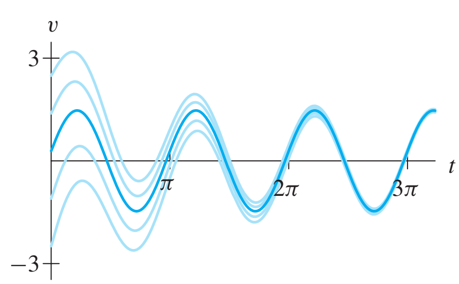



**Figure 1.100**
Graphs of various solutions of

$$\frac{dv}{dt} + 0.4v = 3\cos 2t.$$

Note that all solutions converge to the solution

$$v_p(t) = \tfrac{15}{52}\cos 2t + \tfrac{75}{52}\sin 2t$$

over the long term (see Exercise 21).