1.4 NUMERICAL TECHNIQUE: EULER'S METHOD

The geometric concept of a slope field as discussed in the previous section is closely related to a fundamental numerical method for approximating solutions to a differential equation. Given an initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

we can get a rough idea of the graph of its solution by first sketching the slope field in the ty-plane and then, starting at the initial value (t_0, y_0), sketching the solution by drawing a graph that is tangent to the slope field at each point along the graph. In this section we describe a numerical procedure that automates this idea. Using a computer or a calculator, we obtain numbers and graphs that approximate solutions to initial-value problems.

Numerical methods provide quantitative information about solutions even if we cannot find their formulas. There is also the advantage that most of the work can be done by machine. The disadvantage is that we obtain only approximations, not precise solutions. If we remain aware of this fact and are prudent, numerical methods become powerful tools for the study of differential equations. It is not uncommon to turn to numerical methods even when it is possible to find formulas for solutions. (Most of the graphs of solutions of differential equations in this text were drawn using numerical approximations even when formulas were available.)

The numerical technique that we discuss in this section is called *Euler's method*. A more detailed discussion of the accuracy of Euler's method as well as other numerical methods is given in Chapter 7.

Stepping along the Slope Field

To describe Euler's method, we begin with the initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

Since we are given $f(t, y)$, we can plot its slope field in the ty-plane. The idea of the method is to start at the point (t_0, y_0) in the slope field and take tiny steps dictated by the tangents in the slope field.

We begin by choosing a (small) **step size** Δt. The slope of the approximate solution is updated every Δt units of t. In other words, for each step, we move Δt units along the t-axis. The size of Δt determines the accuracy of the approximate solution as well as the number of computations that are necessary to obtain the approximation.

Starting at (t_0, y_0), our first step is to the point (t_1, y_1) where $t_1 = t_0 + \Delta t$ and (t_1, y_1) is the point on the line through (t_0, y_0) with slope given by the slope field at (t_0, y_0) (see Figure 1.31). At (t_1, y_1) we repeat the procedure. Taking a step whose size along the t-axis is Δt and whose direction is determined by the slope field at (t_1, y_1), we reach the new point (t_2, y_2). The new time is given by $t_2 = t_1 + \Delta t$ and (t_2, y_2) is on the line segment that starts at (t_1, y_1) and has slope $f(t_1, y_1)$. Continuing, we use the slope field at the point (t_k, y_k) to determine the next point (t_{k+1}, y_{k+1}). The sequence of values $y_0, y_1, y_2, \ldots$ serves as an approximation to the solution at the times $t_0, t_1, t_2, \ldots$. Geometrically, we think of the method as producing a sequence of tiny line segments connecting (t_k, y_k) to (t_{k+1}, y_{k+1}) (see Figure 1.32). Basically, we are stitching together little pieces of the slope field to form a graph that approximates our solution curve.

This method uses tangent line segments, given by the slope field, to approximate the graph of the solution. Consequently, at each stage we make a slight error (see Figure 1.32). Hopefully, if the step size is sufficiently small, these errors do not get out of hand as we continue to step, and the resulting graph is close to the desired solution.

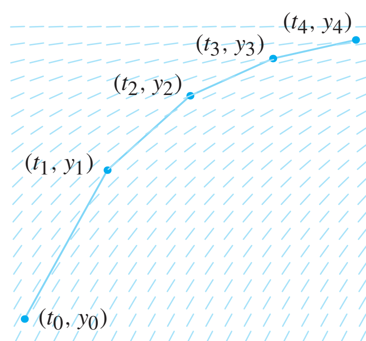

Figure 1.31
Stepping along the slope field.

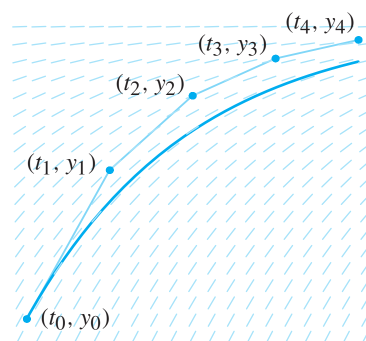

Figure 1.32
The graph of a solution and its approximation obtained using Euler's method.

Euler's Method

To put Euler's method into practice, we need a formula for determining (t_{k+1}, y_{k+1}) from (t_k, y_k). Finding t_{k+1} is easy. We specify the step size Δt at the outset, so

$$t_{k+1} = t_k + \Delta t.$$

To obtain y_{k+1} from (t_k, y_k), we use the differential equation. We know that the slope of the solution to the equation $dy/dt = f(t, y)$ at the point (t_k, y_k) is $f(t_k, y_k)$, and Euler's method uses this slope to determine y_{k+1}. In fact, the method determines the point (t_{k+1}, y_{k+1}) by assuming that it lies on the line through (t_k, y_k) with slope $f(t_k, y_k)$ (see Figure 1.33).

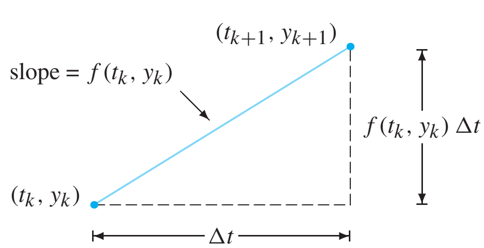

Figure 1.33
Euler's method uses the slope at the point (t_k, y_k) to approximate the solution for $t_k \le t \le t_{k+1}$.

Now we can use our basic knowledge of slopes to determine y_{k+1}. The formula for the slope of a line gives

$$\frac{y_{k+1} - y_k}{t_{k+1} - t_k} = f(t_k, y_k).$$

Since $t_{k+1} = t_k + \Delta t$, the denominator $t_{k+1} - t_k$ is just Δt, and therefore we have

$$\frac{y_{k+1} - y_k}{\Delta t} = f(t_k, y_k)$$

$$y_{k+1} - y_k = f(t_k, y_k)\,\Delta t$$

$$y_{k+1} = y_k + f(t_k, y_k)\,\Delta t.$$

This is the formula for Euler's method (see Figures 1.33 and 1.34).

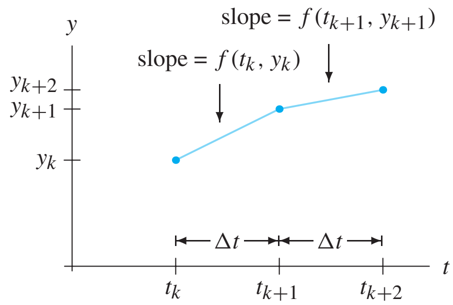

Figure 1.34
Two successive steps of Euler's method. Note that the slope used in the $k+1$st step is $f(t_k, y_k)$, and this slope determines y_{k+1} by the formula

$$y_{k+1} = y_k + f(t_k, y_k)\Delta t.$$

The slope used at the $k+2$nd step is $f(t_{k+1}, y_{k+1})$, and y_{k+2} is determined similarly.

Euler's method for $\frac{dy}{dt} = f(t, y)$

Given the initial condition $y(t_0) = y_0$ and the step size Δt, compute the point (t_{k+1}, y_{k+1}) from the preceding point (t_k, y_k) as follows:

1. Use the differential equation to compute the slope $f(t_k, y_k)$.

2. Calculate the next point (t_{k+1}, y_{k+1}) using the formulas

$$t_{k+1} = t_k + \Delta t$$

and

$$y_{k+1} = y_k + f(t_k, y_k)\Delta t.$$

Approximating an Autonomous Equation

To illustrate Euler's method, we first use it to approximate the solution to a differential equation whose solution we already know. In this way, we are able to compare the approximation we obtain to the known solution. Consequently, we are able to gain some insight into the effectiveness of the method in addition to seeing how it is implemented.

Consider the initial-value problem

$$\frac{dy}{dt} = 2y - 1, \quad y(0) = 1.$$

This equation is separable, and by separating and integrating we obtain the solution

$$y(t) = \frac{e^{2t} + 1}{2}.$$

In this example, $f(t, y) = 2y - 1$, so Euler's method is given by

$$y_{k+1} = y_k + (2y_k - 1)\Delta t.$$

To illustrate the method, we start with a relatively large step size of $\Delta t = 0.1$ and approximate the solution over the interval $0 \le t \le 1$. In order to approximate the solution over an interval whose length is 1 with a step size of 0.1, we must compute ten iterations of the method. The initial condition $y(0) = 1$ provides the initial value $y_0 = 1$. Given $\Delta t = 0.1$, we have $t_1 = t_0 + 0.1 = 0 + 0.1 = 0.1$. We compute the y-coordinate for the first step by

$$y_1 = y_0 + (2y_0 - 1)\Delta t = 1 + (1)\,0.1 = 1.1.$$

Thus the first point (t_1, y_1) on the graph of the approximate solution is (0.1, 1.1).

To compute the y-coordinate y_2 for the second step, we now use y_1 rather than y_0. That is,

$$y_2 = y_1 + (2y_1 - 1)\Delta t = 1.1 + (1.2)\,0.1 = 1.22,$$

and the second point for our approximate solution is $(t_2, y_2) = (0.2, 1.22)$.

Continuing this procedure, we obtain the results given in Table 1.3. After ten steps, we obtain the approximation of $y(1)$ by $y_{10} = 3.596$. (Different machines use different algorithms for rounding numbers, so you may get slightly different results on

Table 1.3
Euler's method (to three decimal places) for $dy/dt = 2y - 1$, $y(0) = 1$ with $\Delta t = 0.1$

k	t_k	y_k	$f(t_k, y_k)$
0	0	1	1
1	0.1	1.100	1.20
2	0.2	1.220	1.44
3	0.3	1.364	1.73
4	0.4	1.537	2.07
5	0.5	1.744	2.49
6	0.6	1.993	2.98
7	0.7	2.292	3.58
8	0.8	2.650	4.30
9	0.9	3.080	5.16
10	1.0	3.596	

your computer or calculator. Keep this fact in mind whenever you compare the numerical results presented in this book with the results of your calculation.) Since we know that

$$y(1) = \frac{e^2+1}{2} \approx 4.195,$$

the approximation y_{10} is off by slightly less than 0.6. This is not a very good approximation, but we'll soon see how to avoid this (usually). The reason for the error can be seen by looking at the graph of the solution and its approximation. The slope field for this differential equation always lies below the graph (see Figure 1.35), so we expect our approximation to come up short.

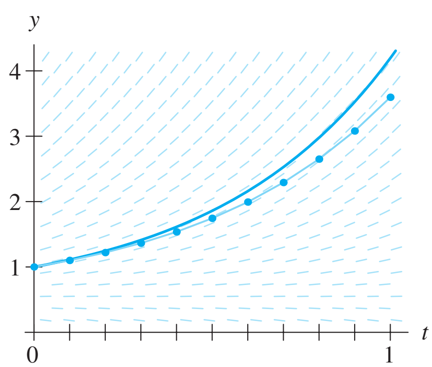

Figure 1.35
The graph of the solution to

$$\frac{dy}{dt} = 2y - 1$$

with $y(0) = 1$ and the approximation produced by Euler's method with $\Delta t = 0.1$.

Using a smaller step size usually reduces the error, but more computations must be done to approximate the solution over the same interval. For example, if we halve the step size in this example ($\Delta t = 0.05$), then we must calculate twice as many steps, since $t_1 = 0.05$, $t_2 = 0.1, \ldots, t_{20} = 1.0$. Again we start with $(t_0, y_0) = (0, 1)$ as specified by the initial condition. However, with $\Delta t = 0.05$, we obtain

$$y_1 = y_0 + (2y_0 - 1)\Delta t = 1 + (1)\,0.05 = 1.05.$$

This step yields the point $(t_1, y_1) = (0.05, 1.05)$ on the graph of our approximate solution. For the next step, we compute

$$y_2 = y_1 + (2y_1 - 1)\Delta t = 1.05 + (1.1)\,0.05 = 1.105.$$

Now we have the point $(t_2, y_2) = (1.1, 1.105)$. This type of calculation gets tedious fairly quickly, but luckily calculations such as these are perfect for a computer or a calculator. For $\Delta t = 0.05$, the results of Euler's method are given in Table 1.4.

If we carefully compare the final results of our two computations, we see that, with $\Delta t = 0.1$, we approximate $y(1) \approx 4.195$ with $y_{10} = 3.596$. With $\Delta t = 0.05$, we approximate $y(1)$ with $y_{20} = 3.864$. The error in the first approximation is slightly less than 0.6, whereas the error in the second approximation is 0.331. Roughly speaking we halve the error by halving the step size. This type of improvement is typical of Euler's method. (We will be much more precise about how the error in Euler's method is related to the step size in Chapter 7.)

Table 1.4
Euler's method (to three decimal places) for $dy/dt = 2y - 1$, $y(0) = 1$ with $\Delta t = 0.05$

k	t_k	y_k	$f(t_k, y_k)$
0	0	1	1
1	0.05	1.050	1.100
2	0.10	1.105	1.210
3	0.15	1.166	1.331
⋮	⋮	⋮	⋮
19	0.95	3.558	6.116
20	1.00	3.864	

With the even smaller step size of $\Delta t = 0.01$, we must do much more work since we need 100 steps to go from $t = 0$ to $t = 1$. However, in the end, we obtain a much better approximation to the solution (see Table 1.5).

This example illustrates the typical trade-off that occurs with numerical methods. There are always decisions to be made such as the choice of the step size Δt. Lowering Δt often results in a better approximation—at the expense of more computation.

Table 1.5
Euler's method (to four decimal places) for $dy/dt = 2y - 1$, $y(0) = 1$ with $\Delta t = 0.01$

k	t_k	y_k	$f(t_k, y_k)$
0	0	1	1
1	0.01	1.0100	1.0200
2	0.02	1.0202	1.0404
3	0.03	1.0306	1.0612
⋮	⋮	⋮	⋮
98	0.98	3.9817	6.9633
99	0.99	4.0513	7.1026
100	1.00	4.1223	

A Nonautonomous Example

Note that it is the value $f(t_k, y_k)$ of the right-hand side of the differential equation at (t_k, y_k) that determines the next point (t_{k+1}, y_{k+1}). The last example was an autonomous differential equation, so the right-hand side $f(t_k, y_k)$ depended only on y_k. However, if the differential equation is nonautonomous, the value of t_k also plays a role in the computations.

To illustrate Euler's method applied to a nonautonomous equation, we consider the initial-value problem

$$\frac{dy}{dt} = -2ty^2, \quad y(0) = 1.$$

This differential equation is also separable, and we can separate variables to obtain the solution

$$y(t) = \frac{1}{1+t^2}.$$

We use Euler's method to approximate this solution over the interval $0 \le t \le 2$. The value of the solution at $t = 2$ is $y(2) = 1/5$. Again, it is interesting to see how close we come to this value with various choices of Δt. The formula for Euler's method is

$$y_{k+1} = y_k + f(t_k, y_k)\,\Delta t = y_k - (2t_k y_k^2)\Delta t$$

with $t_0 = 0$ and $y_0 = 1$. We begin by approximating the solution from $t = 0$ to $t = 2$ using just four steps. This involves so few computations that we can perform the arithmetic "by hand." To cover an interval of length 2 in four steps, we must use $\Delta t = 2/4 = 1/2$. The entire calculation is displayed in Table 1.6.

Table 1.6
Euler's method for $dy/dt = -2ty^2$, $y(0) = 1$ with $\Delta t = 1/2$

k	t_k	y_k	$f(t_k, y_k)$
0	0	1	0
1	1/2	1	−1
2	1	1/2	
	−1/2		
3	3/2	1/4	−3/16
4	2	5/32	

Note that we end up approximating the exact value $y(2) = 1/5 = 0.2$ by $y_4 = 5/32 = 0.15625$. Figure 1.36 shows the graph of the solution as compared to the results of Euler's method over this interval.

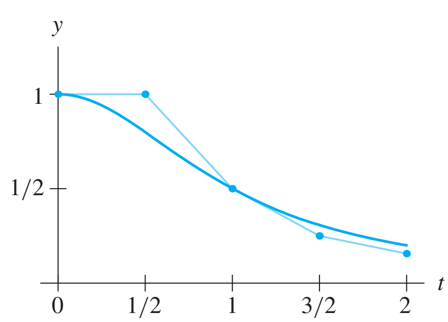

Figure 1.36
The graph of the solution to the initial-value problem

$$\frac{dy}{dt} = -2ty^2, \quad y(0) = 1,$$

and the approximation produced by Euler's method with $\Delta t = 1/2$.

As before, choosing smaller values of Δt yields better approximations. For example, if $\Delta t = 0.1$, the Euler approximation that gives the exact value $y(2) = 0.2$ is

Table 1.7
Euler's method (to four decimal places) for $dy/dt = -2ty^2$, $y(0) = 1$ with $\Delta t = 0.1$

k	t_k	y_k
0	0	1
1	0.1	1.0000
2	0.2	0.9800
3	0.3	0.9416
⋮	⋮	⋮
19	1.9	0.2101
20	2.0	0.1933

Table 1.8
Euler's method (to six decimal places) for $dy/dt = -2ty^2$, $y(0) = 1$ with $\Delta t = 0.001$

k	t_k	y_k
0	0	1
1	0.001	1.000000
2	0.002	0.999998
3	0.003	0.999994
⋮	⋮	⋮
1999	1.999	0.200097
2000	2	0.199937

$y_{20} = 0.1933$. If $\Delta t = 0.001$, we need to compute 2000 steps, but the approximation improves to $y_{2000} = 0.199937$ (see Tables 1.7 and 1.8).

Note that the convergence of the approximation to the actual value is slow. We computed 2000 steps and obtained an answer that is only accurate to three decimal places. In Chapter 7, we present more complicated algorithms for numerical approximation of solutions. Although the algorithms are more complicated from a conceptual point of view, they obtain better accuracy with less computation.

An RC Circuit with Periodic Input

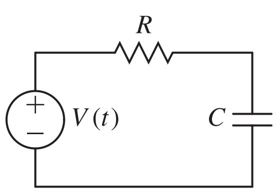

Figure 1.37
Circuit diagram with resistor, capacitor, and voltage source.

Recall from Section 1.3 that the voltage v_c across the capacitor in the simple circuit shown in Figure 1.37 is given by the differential equation

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{RC}$$

where R is the resistance, C is the capacitance, and $V(t)$ is the source or input voltage. We have seen how we can use slope fields to give a qualitative sketch of solutions. Using Euler's method we can also obtain numerical approximations of the solutions.

Suppose we consider a circuit where $R = 0.5$ and $C = 1$ (see the footnote on page 44 in Section 1.3 for a comment regarding our choice of units in these examples). Then the differential equation is

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{0.5} = 2(V(t) - v_c).$$

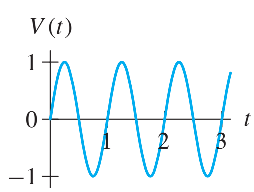

Figure 1.38
Graph of $V(t) = \sin(2\pi t)$, the input voltage.

To understand how the voltage v_c varies if the voltage source $V(t)$ is periodic in time, we consider the case where $V(t) = \sin(2\pi t)$. Consequently, the voltage oscillates between -1 and 1 once each unit of time (see Figure 1.38). The differential equation is now

$$\frac{dv_c}{dt} = -2v_c + 2\sin(2\pi t).$$

From the slope field for this equation (see Figure 1.39), we might predict that the solutions oscillate. Using Euler's method applied to this equation for several different initial conditions, we see that the solutions do indeed oscillate. In addition, we see that they also approach each other and collect around a single solution (see Figure 1.40). This uniformity of long-term behavior is not so easily predicted from the slope field alone.

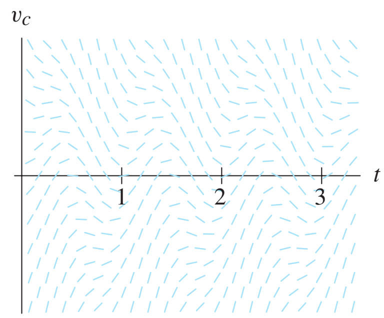

Figure 1.39
Slope field for
$dv_C/dt = -2v_C + 2\sin(2\pi t)$.

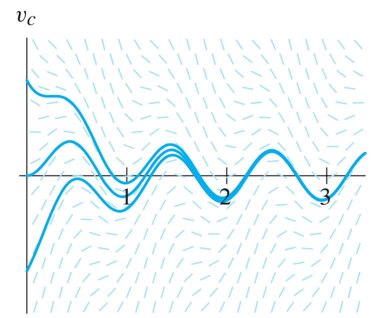

Figure 1.40
Graphs of approximate solutions to
$dv_C/dt = -2v_C + 2\sin(2\pi t)$ obtained using Euler's method.

Errors in Numerical Methods

By its very nature, any numerical approximation scheme is inaccurate. For instance, in each step of Euler's method, we almost always make an error of some sort. These errors can accumulate and sometimes lead to disastrously wrong approximations. As an example, consider the differential equation

$$\frac{dy}{dt} = e^t \sin y.$$

There are equilibrium solutions for this equation if $\sin y = 0$. In other words, any constant function of the form $y(t) = n\pi$ for any integer n is a solution.

Using the initial value $y(0) = 5$ and a step size $\Delta t = 0.1$, Euler's method yields the approximation graphed in Figure 1.41. It seems that something must be wrong. At first, the solution tends toward the equilibrium solution $y(t) = \pi$, but then just before $t = 5$ something strange happens. The graph of the approximation jumps dramatically. If we lower Δt to 0.05, we still find erratic behavior, although t is slightly greater than 5 before this happens (see Figure 1.42).

The difficulty arises in Euler's method for this equation because of the term e^t on the right-hand side. It becomes very large as t increases, and consequently slopes in the slope field are quite large for large t. Even a very small step in the t-direction throws us far from the actual solution.

This problem is typical of the use of numerics in the study of differential equations. Numerical methods, when they work, work beautifully. But they sometimes fail. We must always be aware of this possibility and be ready with an alternate approach.

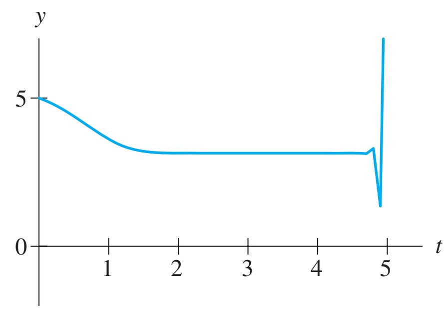

Figure 1.41
Euler's method applied to

$$\frac{dy}{dt} = e^t \sin y$$

with $\Delta t = 0.1$

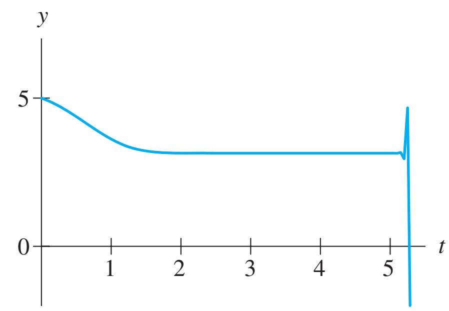

Figure 1.42
Euler's method applied to

$$\frac{dy}{dt} = e^t \sin y$$

with $\Delta t = 0.05$.

In the next section we present theoretical results that help identify when numerical approximations have gone awry.

The Big Three

We have now introduced examples of all three of the fundamental methods for attacking differential equations—the analytic, the numeric, and the qualitative approaches. Which method is the best depends both on the differential equation in question and on what we want to know about the solutions. Often all three methods "work," but a great deal of labor can be saved if we think first about which method gives the most direct route to the information we need.

EXERCISES FOR SECTION 1.4

In Exercises 1–4, use `EulersMethod` to perform Euler's method with the given step size Δt on the given initial-value problem over the time interval specified. Your answer should include a table of the approximate values of the dependent variable. It should also include a sketch of the graph of the approximate solution.

1. $\dfrac{dy}{dt} = 2y + 1, \quad y(0) = 3, \quad 0 \le t \le 2, \quad \Delta t = 0.5$

2. $\dfrac{dy}{dt} = t - y^2, \quad y(0) = 1, \quad 0 \le t \le 1, \quad \Delta t = 0.25$

3. $\dfrac{dy}{dt} = y^2 - 4t, \quad y(0) = 0.5, \quad 0 \le t \le 2, \quad \Delta t = 0.25$

4. $\dfrac{dy}{dt} = \sin y, \quad y(0) = 1, \quad 0 \le t \le 3, \quad \Delta t = 0.5$