42. Suppose you are having a dinner party for a large group of people, and you decide to make 2 gallons of chili. The recipe calls for 2 teaspoons of hot sauce per gallon, but you misread the instructions and put in 2 tablespoons of hot sauce per gallon. (Since each tablespoon is 3 teaspoons, you have put in 6 teaspoons per gallon, which is a total of 12 teaspoons of hot sauce in the chili.) You don't want to throw the chili out because there isn't much else to eat (and some people like hot chili), so you serve the chili anyway. However, as each person takes some chili, you fill up the pot with beans and tomatoes without hot sauce until the concentration of hot sauce agrees with the recipe. Suppose the guests take 1 cup of chili per minute from the pot (there are 16 cups in a gallon), how long will it take to get the chili back to the recipe's concentration of hot sauce? How many cups of chili will have been taken from the pot?

43. In Exercise 12 of Section 1.1, we saw that the velocity v of a freefalling skydiver is well modeled by the differential equation

$$m\frac{dv}{dt} = mg - kv^2,$$

where m is the mass of the skydiver, g is the gravitational constant, and k is the drag coefficient determined by the position of the driver during the dive.

(a) Find the general solution of this differential equation.

(b) Confirm your answer to Exercise 12 of Section 1.1 by calculating the limit of $v(t)$ as $t \to \infty$.

1.3 QUALITATIVE TECHNIQUE: SLOPE FIELDS

Finding an analytic expression (in other words, finding a formula) for a solution to a differential equation is often a useful way to describe a solution of a differential equation. However, there are other ways to describe solutions, and these alternative representations are frequently easier to understand and use. In this section we focus on geometric techniques for representing solutions, and we develop a method for visualizing the graphs of the solutions to the differential equation

$$\frac{dy}{dt} = f(t, y).$$

The Geometry of $dy/dt = f(t, y)$

If the function $y(t)$ is a solution of the equation $dy/dt = f(t, y)$ and if its graph passes through the point (t_1, y_1) where $y_1 = y(t_1)$, then the differential equation says that the derivative dy/dt at $t = t_1$ is given by the number $f(t_1, y_1)$. Geometrically, this equality of dy/dt at $t = t_1$ with $f(t_1, y_1)$ means that the slope of the tangent line to the graph of $y(t)$ at the point (t_1, y_1) is $f(t_1, y_1)$ (see Figure 1.10). Note that there is nothing special about the point (t_1, y_1) other than the fact that it is a point on the graph

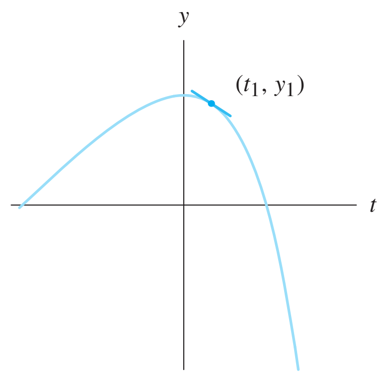

Figure 1.10
Slope of the tangent at the point (t_1, y_1) is given by the value of $f(t_1, y_1)$.

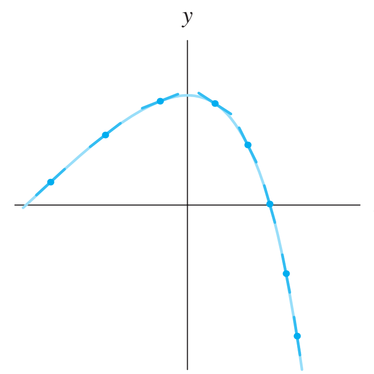

Figure 1.11
If $y = y(t)$ is a solution, then the slope of any tangent must equal $f(t, y)$.

of the solution $y(t)$. The equality of dy/dt and $f(t, y)$ must hold for all t for which $y(t)$ satisfies the differential equation. In other words, the values of the right-hand side of the differential equation yield the slopes of the tangents at all points on the graph of $y(t)$ (see Figure 1.11).

Slope Fields

This simple geometric observation leads to our main device for the visualization of the solutions to a first-order differential equation

$$\frac{dy}{dt} = f(t, y).$$

If we are given the function $f(t, y)$, we obtain a rough idea of the graphs of the solutions to the differential equation by sketching its corresponding **slope field**. We make this sketch by selecting points in the ty-plane and computing the numbers $f(t, y)$ at these points. At each point (t, y) selected, we use $f(t, y)$ to draw a minitangent line whose slope is $f(t, y)$ (see Figure 1.12). These minitangent lines are also called slope marks. Once we have a lot of slope marks, we can visualize the graphs of the solutions. For example, consider the differential equation

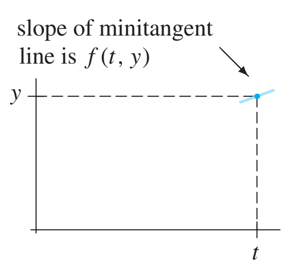

Figure 1.12
The slope of the minitangent at the point (t, y) is determined by the right-hand side $f(t, y)$ of the differential equation.

$$\frac{dy}{dt} = y - t.$$

In other words, the right-hand side of the differential equation is given by the function $f(t, y) = y - t$. To get some practice with the idea of a slope field, we sketch its slope field by hand at a small number of points. Then we discuss a computer-generated version of this slope field.

Generating slope fields by hand is tedious, so we consider only the nine points in the ty-plane. For example, at the point $(t, y) = (1, -1)$, we have $f(t, y) = f(1, -1) = -1 - 1 = -2$. Therefore we sketch a "small" line segment with slope -2 centered at

Nancy Kopell (1942–) received her doctorate in mathematics at the University of California, Berkeley, where she wrote her thesis under the direction of Stephen Smale. She is one of the leading figures in the world in the use of differential equations to model natural phenomena. Kopell has employed techniques similar to those that we study in this book to tackle such diverse problems as spontaneous pattern formation in chemical systems and the networks of neurons that govern rhythmic motion in animals and other oscillations in the central nervous system.

For her work, she has received numerous awards, including a MacArthur Fellowship "genius grant" in 1990. In 1996, she was elected to the National Academy of Sciences. She is currently professor of mathematics and founding director of the Center for BioDynamics (and the authors' colleague) at Boston University.

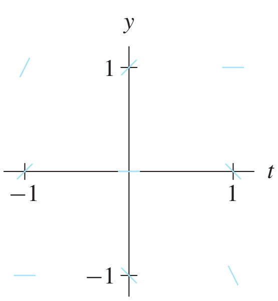

Figure 1.13
A "sparse" slope field generated from Table 1.2.

the point $(1, -1)$ (see Figure 1.13). To sketch the slope field for all nine points, we use the function $f(t, y)$ to compute the appropriate slopes. The results are summarized in Table 1.2. Once we have these values, we use them to give a sparse sketch of the slope field for this equation (see Figure 1.13).

Sketching slope fields is best done using a computer. Figure 1.14 is a sketch of the slope field for this equation over the region $-3 \le t \le 3$ and $-3 \le y \le 3$ in the ty-plane. We calculated values of the function $f(t, y)$ over 25×25 points (625 points) in that region.

A glance at this slope field suggests that the graph of one solution is a line passing through the points $(-1, 0)$ and $(0, 1)$. Solutions corresponding to initial conditions that are below this line seem to increase until they reach an absolute maximum. Solutions corresponding to initial conditions that are above the line seem to increase more and more rapidly.

In fact, in Section 1.8 we will learn an analytic technique for finding solutions of this equation. We will see that the general solution consists of the family of functions

$$y(t) = t + 1 + ce^t,$$

where c is an arbitrary constant. (At this point it is important to emphasize that, even though we have not studied the technique that gives us these solutions, we can still

Table 1.2
Selected slopes corresponding to the differential equation $dy/dt = y - t$

(t, y)	$f(t, y)$	(t, y)	$f(t, y)$	(t, y)	$f(t, y)$
$(-1, 1)$	2	$(0, 1)$	1	$(1, 1)$	0
$(-1, 0)$	1	$(0, 0)$	0	$(1, 0)$	-1
$(-1, -1)$	0	$(0, -1)$	-1	$(1, -1)$	-2

check to see whether these functions are indeed solutions. If $y(t) = t + 1 + ce^t$, then $dy/dt = 1 + ce^t$. Also $f(t, y) = y - t = (t + 1 + ce^t) - t = 1 + ce^t$. Hence all of these functions are solutions.)

In Figure 1.15 we sketch the graphs of these functions with $c = -2, -1, 0, 1, 2, 3$. Note that each of these graphs is tangent to the slope field. Also note that, if $c = 0$, the graph is a line whose slope is 1. It goes through the points $(-1, 0)$ and $(0, 1)$.

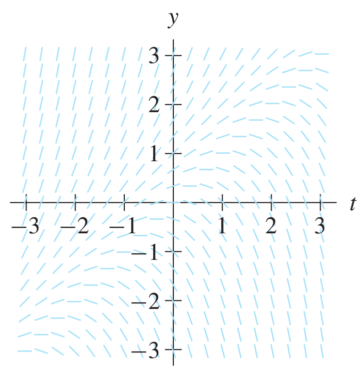

Figure 1.14
A computer-generated version of the slope field for $dy/dt = y - t$.

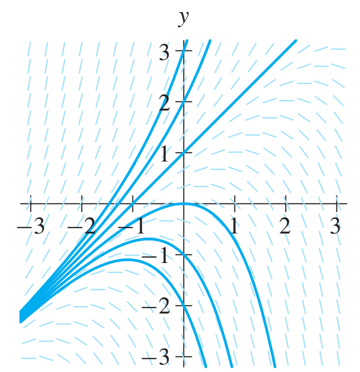

Figure 1.15
The graphs of six solutions to $dy/dt = y - t$ superimposed on its slope field.

Important Special Cases

From an analytic point of view, differential equations of the forms

$$\frac{dy}{dt} = f(t) \quad \text{and} \quad \frac{dy}{dt} = f(y)$$

are somewhat easier to consider than more complicated equations because they are separable. The geometry of their slope fields is equally special.

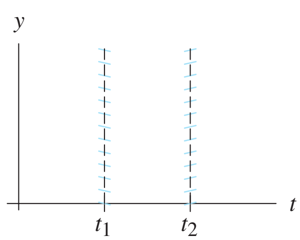

Figure 1.16
If the right-hand side of the differential equation is a function of t alone, that is,

$$\frac{dy}{dt} = f(t),$$

then the slope marks in the slope field are determined solely by their t-coordinate

Slope fields for $dy/dt = f(t)$

If the right-hand side of the differential equation in question is solely a function of t, or in other words, if $dy/dt = f(t)$, the slope at any point is the same as the slope of any other point with the same t-coordinate (see Figure 1.16).

Geometrically, this implies that all of the slope marks on each vertical line are parallel. Whenever a slope field has this geometric property for all vertical lines throughout the domain in question, we know that the corresponding differential equation is really an equation of the form

$$\frac{dy}{dt} = f(t).$$

(Note that finding solutions to this type of differential equation is the same thing as finding an antiderivative of $f(t)$ in calculus.)

For example, consider the slope field shown in Figure 1.17. We generated this slope field from the equation

$$\frac{dy}{dt} = 2t,$$

and from calculus we know that

$$y(t) = \int 2t\, dt = t^2 + c,$$

where c is the constant of integration. Hence the general solution of the differential equation consists of functions of the form

$$y(t) = t^2 + c.$$

In Figure 1.18 we have superimposed graphs of such solutions on this field. Note that all of these graphs simply differ by a vertical translation. If one graph is tangent to the slope field, we can get infinitely many graphs—all tangent to the slope field—by translating the original graph either up or down.

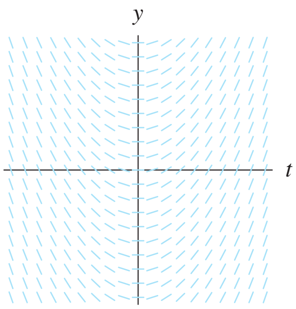

Figure 1.17
The slope field for

$$\frac{dy}{dt} = 2t.$$

Note the parallel slopes along vertical lines.

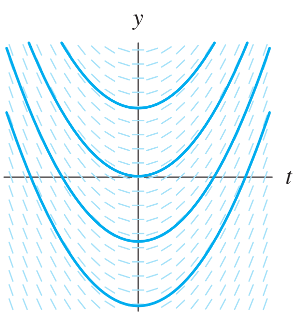

Figure 1.18
Graphs of solutions to

$$\frac{dy}{dt} = 2t$$

superimposed on its slope field.

Slope fields for autonomous equations

In the case of an autonomous differential equation

$$\frac{dy}{dt} = f(y),$$

the right-hand side of the equation does not depend on the independent variable t. The slope field in this case is also somewhat special. Here, the slopes that correspond to two different points with the same y-coordinate are equal. That is, $f(t_1, y) = f(t_2, y) = f(y)$ since the right-hand side of the differential equation depends only on y. In other words, the slope field of an autonomous equation is parallel along each horizontal line (see Figure 1.19).

For example, the slope field for the autonomous equation

$$\frac{dy}{dt} = 4y(1 - y)$$

is given in Figure 1.20. Note that, along each horizontal line, the slope marks are parallel. In fact, if $0 < y < 1$, then dy/dt is positive, and the tangents suggest that a solution

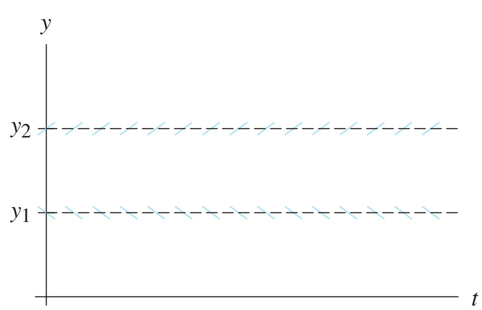

Figure 1.19
If the right-hand side of the differential equation is a function of y alone, that is, if

$$\frac{dy}{dt} = f(y),$$

then the slope marks in the slope field are determined solely by their y-coordinate.

with $0 < y < 1$ is increasing. On the other hand, if $y < 0$ or if $y > 1$, then dy/dt is negative and any solution with either $y < 0$ or $y > 1$ is decreasing.

We have equilibrium solutions at $y = 0$ and at $y = 1$ since the right-hand side of the differential equation vanishes along these lines. The slope field is horizontal all along these two horizontal lines, and therefore we know that these lines are the graphs of solutions. Solutions whose graphs are between these two lines are increasing. Solutions that are above the line $y = 1$ or that are below the line $y = 0$ are decreasing (see Figure 1.21).

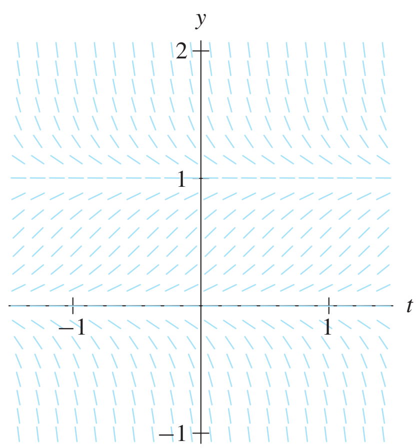

Figure 1.20
The slope field for $dy/dt = 4y(1 - y)$.

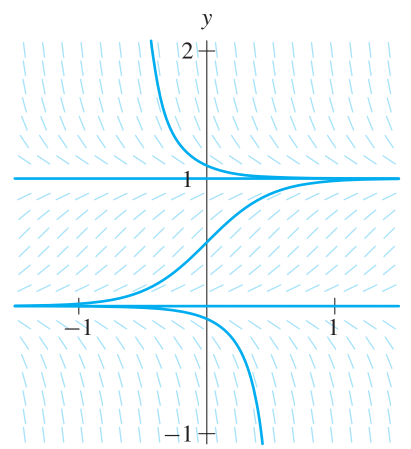

Figure 1.21
The graphs of five solutions superimposed on the slope field for $dy/dt = 4y(1 - y)$.

The fact that autonomous equations produce slope fields that are parallel along horizontal lines indicates that we can get infinitely many solutions from one solution simply by translating the graph of the given solution left or right (see Figure 1.22). We will make extensive use of this simple geometric observation about the solutions to autonomous equations in Section 1.6.

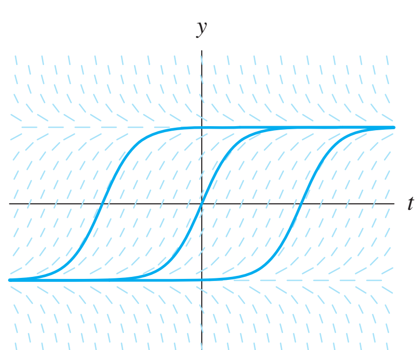

Figure 1.22
The graphs of three solutions to an autonomous equation, that is, an equation of the form

$$\frac{dy}{dt} = f(y).$$

Note that each graph is a horizontal translate of the others.

Analytic versus Qualitative Analysis

For the autonomous equation

$$\frac{dy}{dt} = 4y(1-y),$$

we could have used the analytic techniques of the previous section to find explicit formulas for the solutions. In fact, we can perform all of the required integrations to determine the general solution (see Exercise 17 on page 34). However, these integrations are complicated, and the formulas that result are by no means easy to interpret. This points out the power of geometric and qualitative methods for solving differential equations. With very little work, we gain a lot of insight into the behavior of solutions. Although we cannot use qualitative methods to answer specific questions, such as what the exact value of the solution is at any given time, we can use these methods to understand the long-term behavior of a solution.

These ideas are especially important if the differential equation in question cannot be handled by analytic techniques. As an example, consider the differential equation

$$\frac{dy}{dt} = e^{y^2/10}\sin^2 y.$$

This equation is autonomous and hence separable. To solve this equation analytically, we must evaluate the integrals

$$\int \frac{dy}{e^{y^2/10}\sin^2 y} = \int dt.$$

However, the integral on the left-hand side cannot be evaluated so easily. Thus we resort to qualitative methods. The right-hand side of this differential equation is positive except if $y = n\pi$ for any integer n. These special lines correspond to equilibrium solutions of the equation. Between these equilibria, solutions must always increase. From the slope field, we expect that their graphs either lie on one of the horizontal lines $y = n\pi$ or increase from one of these lines to the next higher as $t \to \infty$ (see Figure 1.23). Hence we can predict the long-term behavior of the solutions even though we cannot explicitly solve the equation.

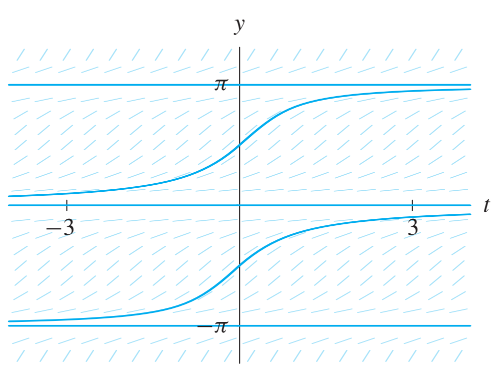

Figure 1.23
The slope field and graphs of solutions for the differential equation

$$\frac{dy}{dt} = e^{y^2/10}\sin^2 y.$$

The lines $y = n\pi$ are the graphs of the equilibrium solutions, and between these lines, all solutions are increasing.

Although the computer pictures of solutions of this differential equation are convincing, some subtle questions remain. For example, how do we *really* know that these pictures are correct? In particular, for $dy/dt = e^{y^2/10} \sin^2 y$, how do we know that the graphs of solutions do not cross the horizontal lines that are the graphs of the equilibrium solutions (see Figure 1.23)? Such a solution could not cross these lines at a nonzero angle since we know that the tangent line to the solution must be horizontal. But what prevents certain solutions from crossing these lines tangentially and then continuing to increase?

For the differential equation

$$\frac{dy}{dt} = 4y(1-y)$$

we can eliminate these questions because we can evaluate all of the integrals and check the accuracy of the pictures using analytic techniques. But using analytic techniques to check our qualitative analysis does not work if we cannot find explicit solutions. Besides, having to resort to analytic techniques to check the qualitative results defeats the purpose of using these methods in the first place. In Section 1.5 we discuss powerful theorems that answer many of these questions without undue effort.

The Mixing Problem Revisited

Recall that in the previous section (page 32) we found precise analytic solutions for the differential equation

$$\frac{dS}{dt} = \frac{2000 - 3S}{100},$$

where S describes the amount of sugar in a vat at time t. We found that the general solution of this equation was

$$S(t) = ce^{-0.03t} + \frac{2000}{3},$$

where c is an arbitrary constant.

Using the slope field of this equation, we can easily derive a qualitative description of these solutions. In Figure 1.24, we display the slope field and graphs of selected solutions. Note that, as expected, the slope field is horizontal if $S = 2000/3$, the equilibrium solution. Slopes are positive if $S < 2000/3$ and negative if $S > 2000/3$. So

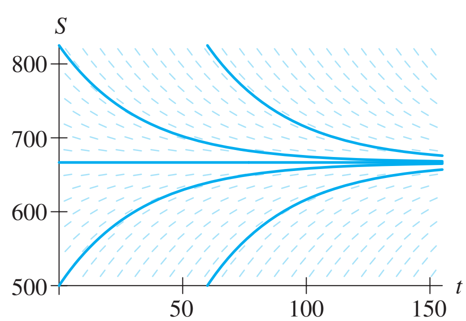

Figure 1.24
The slope field and graphs of a few solutions of

$$\frac{dS}{dt} = \frac{2000 - 3S}{100}.$$

The horizontal line is the graph of the equilibrium solution $S(t) = 2000/3$ for all t. Solutions below the equilibrium value are increasing, and solutions above that value are decreasing.

we expect solutions to tend toward the equilibrium solution as t increases. This qualitative analysis indicates that, no matter what the initial amount of sugar, the amount of sugar in the vat tends to $2000/3$ as $t \to \infty$. Of course, we obtain the same information by taking the limit of the general solution as $t \to \infty$, but it is nice to see the same result in a geometric setting. Furthermore, in other examples, taking such a limit may not be as easy as in this case, but qualitative methods may still be used to determine the long-term behavior of the solutions.

An RC Circuit

The simple electric circuit pictured in Figure 1.25 contains a capacitor, a resistor, and a voltage source. The behavior of the resistor is specified by a positive parameter R (the "resistance"), and the behavior of the capacitor is specified by a positive parameter C (the "capacitance").* The input voltage across the voltage source at time t is denoted by $V(t)$. This voltage source could be a constant source such as a battery, or it could be a source that varies with time such as alternating current. In any case, we consider $V(t)$ to be a function that is specified by the circuit designer. In other words, it is part of the design of the circuit.

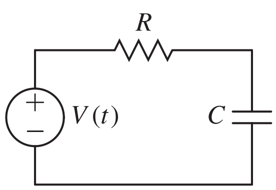

Figure 1.25
Circuit diagram with resistor, capacitor, and voltage source.

The quantities that specify the behavior of the circuit at a particular time t are the current $i(t)$ and the voltage across the capacitor $v_c(t)$. In this example we are interested in the voltage $v_c(t)$ across the capacitor. From the theory of electric circuits, we know that $v_c(t)$ satisfies the differential equation

$$RC\frac{dv_c}{dt} + v_c = V(t).$$

If we rewrite this in our standard form $dv_c/dt = f(t, v_c)$, we have

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{RC}.$$

We use slope fields to visualize solutions for four different types of voltage sources $V(t)$. (If you don't know anything about electric circuits, don't worry; Paul, Bob, and Glen don't either. In examples like this, all we need to do is accept the differential equation and "go with it.")

Zero input

If $V(t) = 0$ for all t, the equation becomes

$$\frac{dv_c}{dt} = \frac{-v_c}{RC}.$$

A slope field for a particular choice of R and C is given in Figure 1.26. We see clearly that all solutions "decay" toward $v_c = 0$ as t increases. If there is no voltage source,

*The usual units are ohms for resistance and farads for capacitance. In this section and in Section 1.4, we chose values of of R and C so that the numbers in the examples work out nicely. A 1 farad capacitor would be extremely large.

the voltage across the capacitor $v_c(t)$ decays to zero. This prediction for the voltage agrees with what we obtain analytically since the general solution of this equation is $v_c(t) = v_0 e^{-t/RC}$, where v_0 is the initial voltage across the capacitor. (Note that this equation is essentially the same as the exponential growth model that we studied in Section 1.1, and consequently we can solve it analytically by either guessing the correct form of a solution or by separating variables—see Exercise 20.)

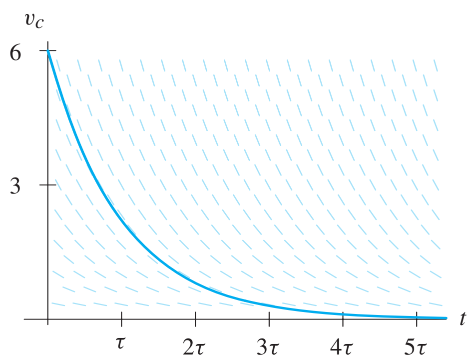

Figure 1.26
Slope field for

$$\frac{dv_c}{dt} = -\frac{v_c}{RC}$$

with $R = 0.5$ and $C = 1$, and the graph of the solution with initial value $v_c(0) = 6$. The time constant τ for this equation exponentially decaying solution is $\tau = 0.5$. (For more information about time constants, see Exercise 9 in Section 1.1.)

Constant nonzero voltage source

Suppose $V(t)$ is a nonzero constant K for all t. The equation for voltage across the capacitor becomes

$$\frac{dv_c}{dt} = \frac{K - v_c}{RC}.$$

This equation is autonomous with one equilibrium solution at $v_c = K$. The slope field for this equation shows that all solutions tend toward this equilibrium as t increases (see Figure 1.27). Given any initial voltage $v_c(0)$ across the capacitor, the voltage $v_c(t)$ tends to the value $v = K$ as time increases.

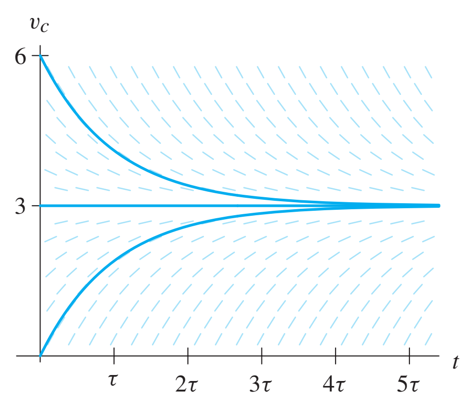

Figure 1.27
Slope field for

$$\frac{dv_c}{dt} = \frac{K - v_c}{RC}$$

for $R = 0.5$, $C = 1$, and $K = 3$, and the graphs of three solutions with different initial conditions. The time constant for this equation is the same as the time constant for the equation in Figure 1.26.

We could find a formula for the general solution by separating variables and integrating, but we leave this as an exercise (see Exercise 21).

On-off voltage source

Suppose $V(t) = K > 0$ for $0 \le t < 3$, but at $t = 3$, this voltage is "turned off," perhaps by someone flicking a light switch. Then $V(t) = 0$ for $t > 3$. Our differential equation is

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{RC} = \begin{cases} \dfrac{K - v_c}{RC} & \text{for } 0 \le t < 3; \\[2ex] \dfrac{-v_c}{RC} & \text{for } t > 3. \end{cases}$$

The right-hand side is given by two different formulas depending on the value of t. We can see this discontinuity in the slope field for this equation (see Figure 1.28). It resembles Figures 1.26 and 1.27 pasted together along the vertical line $t = 3$. Since the differential equation is not defined at $t = 3$, we must add an additional assumption to our model. We assume that the voltage $v_c(t)$ is a continuous function at $t = 3$.

The particular solution with the initial condition $v_c(0) = K$ is constant for $t < 3$, but for $t > 3$ it decays exponentially. Solutions with $v_c(0) \ne K$ move toward K for $t < 3$, but then decay toward zero for $t > 3$. We could find formulas for the solutions by first calculating $v_c(t)$ for $t \le 3$, and then using the value $v_c(3)$ to solve the equation for $t > 3$ (see Section 1.2). We again leave this derivation as an exercise (see Exercise 22).

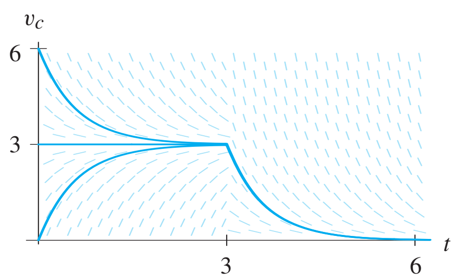

Figure 1.28
Slope field for

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{RC}$$

for $V(t)$, which "turns off" at $t = 3$ for $R = 0.5$, $C = 1$, and $K = 3$, along with graphs of three solutions with different initial conditions.

A flashing light

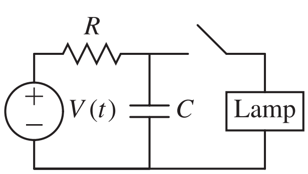

Figure 1.29
Circuit diagram for a flashing light.

The circuit in Figure 1.25 can be modified to produce a flashing light such as those that are used on cell phone towers and flashing road hazard signs (see Figure 1.29). The switch periodically opens and closes. It is open for an interval of time T_O. (The letter o stands for open. The constant T_O is *not* an initial value.) After the switch is open for the time interval T_O, the switch closes and remains closed for a different (and shorter) interval T_C, where the letter c stands for closed. When the switch is open, the capacitor is charging according to the equation

$$\frac{dv_c}{dt} = \frac{V(t) - v_c}{RC},$$

where $V(t)$ is the voltage source. If $V(t)$ is a constant K and v_0 is the initial value $v_c(0)$, then $v_c(t)$ satisfies the initial-value problem

$$\frac{dv_c}{dt} = \frac{K - v_c}{RC}, \quad v_c(0) = v_0.$$

The voltage v_c satisfies this equation for $0 \le t \le T_O$.

At time T_O, the switch closes and the light turns on. While the lamp is lit, it acts as a resistor in parallel with the other resistor. Let R_L be the resistance that is due solely to the lamp, then it can be shown that the differential equation that governs v_c over the time interval $T_O < t < T_O + T_C$ is

$$\frac{dv_c}{dt} = \frac{K}{RC} - \left(\frac{R + R_L}{RR_L C}\right) v_c.$$

Note that increased resistance due to the lamp causes v_c to decrease faster than it increased when the switch was open.

The light switch remains closed over the interval $T_O < t < T_O + T_C$, and we pick T_C so that $v_c(T_O + T_C) = v_0$. In other words, we pick T_C so that the voltage v_c is periodic with period $T_O + T_C$ (see Figure 1.30). For this example, the slope field is discontinuous along infinitely many vertical lines, that is, the lines $t = T_O$, $t = T_O + T_C$, $t = 2T_O + T_C$, $t = 2(T_O + T_C)$, ... (see Figure 1.30).

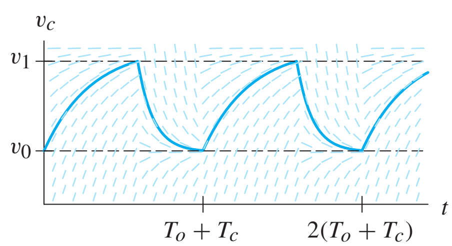

Figure 1.30
The graph of the solution and its corresponding slope field for the flashing light example in the case where $R = 0.5$, $R_L = 0.25$, $C = 1$, and $K = 2.5$. In this case, we have made $T_O = 1$ and $T_C = 0.7$, so the solution is periodic with period 1.7.

Combining Qualitative with Quantitative Results

When only knowledge of the qualitative behavior of the solution is required, sketches of solutions obtained from slope fields can sometimes suffice. In other applications it is necessary to know the exact value (or almost exact value) of the solution with a given initial condition. In these situations analytic and/or numerical methods can't be avoided. But even then, it is nice to have graphs of the solutions.

EXERCISES FOR SECTION 1.3

In Exercises 1–6, sketch the slope fields for the differential equation as follows:

(a) Pick a few points (t, y) with both $-2 \le t \le 2$ and $-2 \le y \le 2$ and plot the associated slope marks without the use of technology.

(b) Use `HPGSolver` to check these individual slope marks.

(c) Make a more detailed drawing of the slope field and then use `HPGSolver` to confirm your answer.

For more details about `HPGSolver` and other programs that are part of the `DETools` package, see the description of `DETools` inside the front cover of this book.

1. $\dfrac{dy}{dt} = t^2 + t$ **2.** $\dfrac{dy}{dt} = t^2 + 1$ **3.** $\dfrac{dy}{dt} = 1 - 2y$

4. $\dfrac{dy}{dt} = 4y^2$ **5.** $\dfrac{dy}{dt} = 2y(1 - y)$ **6.** $\dfrac{dy}{dt} = y + t + 1$