23. The following systems are models of the populations of pairs of species that either *compete* for resources (an increase in one species decreases the growth rate of the other) or *cooperate* (an increase in one species increases the growth rate of the other). For each system, identify the variables (independent and dependent) and the parameters (carrying capacity, measures of interaction between species, etc.) Do the species compete or cooperate? (Assume all parameters are positive.)

(a)
$$\frac{dx}{dt} = \alpha x - \alpha\frac{x^2}{N} + \beta xy$$
$$\frac{dy}{dt} = \gamma y + \delta xy$$

(b)
$$\frac{dx}{dt} = -\gamma x - \delta xy$$
$$\frac{dy}{dt} = \alpha y - \beta xy$$

1.2 ANALYTIC TECHNIQUE: SEPARATION OF VARIABLES

What Is a Differential Equation and What Is a Solution?

A first-order differential equation is an equation for an unknown function in terms of its derivative. As we saw in Section 1.1, there are three types of "variables" in differential equations—the independent variable (almost always t for time in our examples), one or more dependent variables (which are functions of the independent variable), and the parameters. This terminology is standard but a bit confusing. The dependent variable is actually a function, so technically it should be called the dependent function.

The standard form for a first-order differential equation is

$$\frac{dy}{dt} = f(t, y).$$

Here the right-hand side typically depends on both the dependent and independent variables, although we often encounter cases where either t or y is missing.

A **solution** of the differential equation is a function of the independent variable that, when substituted into the equation as the dependent variable, satisfies the equation for all values of the independent variable. That is, a function $y(t)$ is a solution if it satisfies $dy/dt = y'(t) = f(t, y(t))$. This terminology doesn't tell us how to find solutions, but it does tell us how to check whether a candidate function is or is not a solution. For example, consider the simple differential equation

$$\frac{dy}{dt} = y.$$

We can easily check that the function $y_1(t) = 3e^t$ is a solution, whereas $y_2(t) = \sin t$ is not a solution. The function $y_1(t)$ is a solution because

$$\frac{dy_1}{dt} = \frac{d(3e^t)}{dt} = 3e^t = y_1 \quad \text{for all } t.$$

On the other hand, $y_2(t)$ is not a solution since

$$\frac{dy_2}{dt} = \frac{d(\sin t)}{dt} = \cos t,$$

and certainly the function $\cos t$ is not the same function as $y_2(t) = \sin t$.

Checking that a given function is a solution to a given equation

If we look at a more complicated equation such as

$$\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t},$$

then we have considerably more trouble finding a solution. On the other hand, if somebody hands us a function $y(t)$, then we know how to check whether or not it is a solution.

For example, suppose we meet three differential equations textbook authors—say Paul, Bob, and Glen—at our local espresso bar, and we ask them to find solutions of this differential equation. After a few minutes of furious calculation, Paul says that

$$y_1(t) = 1 + t$$

is a solution. Glen then says that

$$y_2(t) = 1 + 2t$$

is a solution. After several more minutes, Bob says that

$$y_3(t) = 1$$

is a solution. Which of these functions is a solution? Let's see who is right by substituting each function into the differential equation.

First we test Paul's function. We compute the left-hand side by differentiating $y_1(t)$. We have

$$\frac{dy_1}{dt} = \frac{d(1+t)}{dt} = 1.$$

Substituting $y_1(t)$ into the right-hand side, we find

$$\frac{(y_1(t))^2 - 1}{t^2 + 2t} = \frac{(1+t)^2 - 1}{t^2 + 2t} = \frac{t^2 + 2t}{t^2 + 2t} = 1.$$

The left-hand side and the right-hand side of the differential equation are identical, so Paul is correct.

To check Glen's function, we again compute the derivative

$$\frac{dy_2}{dt} = \frac{d(1+2t)}{dt} = 2.$$

With $y_2(t)$, the right-hand side of the differential equation is

$$\frac{(y_2(t))^2 - 1}{t^2 + 2t} = \frac{(1+2t)^2 - 1}{t^2 + 2t} = \frac{4t^2 + 4t}{t^2 + 2t} = \frac{4(t+1)}{t+2}.$$

The left-hand side of the differential equation does not equal the right-hand side for all t since the right-hand side is not the constant function 2. Glen's function is *not* a solution.

Finally, we check Bob's function the same way. The left-hand side is

$$\frac{dy_3}{dt} = \frac{d(1)}{dt} = 0$$

because $y_3(t) = 1$ is a constant. The right-hand side is

$$\frac{y_3(t)^2 - 1}{t^2 + 2t} = \frac{1-1}{t^2 + 2t} = 0.$$

Both the left-hand side and the right-hand side of the differential equation approaches zero for all t. Hence, Bob's function *is* a solution of the differential equation.

The lessons we learn from this example are that a differential equation may have solutions that look very different from each other algebraically and that (of course) not every function is a solution. Given a function, we can test to see whether it is a solution by just substituting it into the differential equation and checking to see whether the left-hand side is identical to the right-hand side. This is a very nice aspect of differential equations: *We can always check our answers.* So we should never be wrong.

Initial-Value Problems and the General Solution

When we encounter differential equations in practice, they often come with **initial conditions**. We seek a solution of the given equation that assumes a given value at a particular time. A differential equation along with an initial condition is called an **initial-value problem**. Thus the usual form of an initial-value problem is

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0.$$

Here we are looking for a function $y(t)$ that is a solution of the differential equation *and* assumes the value y_0 at time t_0. Often, the particular time in question is $t = 0$ (hence the name *initial condition*), but any other time could be specified.

For example,

$$\frac{dy}{dt} = 12t^3 - 2\sin t, \quad y(0) = 3,$$

is an initial-value problem. To solve this problem, note that the right-hand side of the differential equation depends only on t, not on y. We are looking for a function whose derivative is $12t^3 - 2\sin t$. This is a typical antidifferentiation problem from calculus, so all we need to do is to integrate this expression. We find

$$\int (12t^3 - 2\sin t)\, dt = 3t^4 + 2\cos t + c,$$

where c is a constant of integration. Thus the solution must be of the form

$$y(t) = 3t^4 + 2\cos t + c.$$

We now use the initial condition $y(0) = 3$ to determine c by

$$3 = y(0) = 3 \cdot 0^4 + 2\cos 0 + c = 0 + 2 \cdot 1 + c = 2 + c.$$

Thus $c = 1$, and the solution to this initial-value problem is $y(t) = 3t^4 + 2\cos t + 1$.

The expression

$$y(t) = 3t^4 + 2\cos t + c$$

is called the **general solution** of the differential equation because we can use it to solve any initial-value problem whatsoever. For example, if the initial condition is $y(0) = \pi$, then we choose $c = \pi - 2$ to solve the initial-value problem $dy/dt = 12t^3 - 2\sin t$, $y(0) = \pi$.

Separable Equations

Now that we know how to check that a given function is a solution to a differential equation, the question is: How can we get our hands on a solution in the first place? Unfortunately, it is rarely the case that we can find explicit solutions of a differential equation. Many differential equations have solutions that cannot be expressed in terms of known functions such as polynomials, exponentials, or trigonometric functions. However, there are a few special types of differential equations for which we can derive explicit solutions, and in this section we discuss one of these types of differential equations.

The typical first-order differential equation is given in the form

$$\frac{dy}{dt} = f(t, y).$$

The right-hand side of this equation generally involves both the independent variable t and the dependent variable y (although there are many important examples where either the t or the y is missing). A differential equation is called **separable** if the function $f(t, y)$ can be written as the product of two functions: one that depends on t alone and another that depends only on y. That is, a differential equation is separable if it can be written in the form

$$\frac{dy}{dt} = g(t)h(y).$$

For example, the differential equation

$$\frac{dy}{dt} = yt$$

is clearly separable, and the equation

$$\frac{dy}{dt} = y + t$$

is not. We might have to do a little work to see that an equation is separable. For instance,

$$\frac{dy}{dt} = \frac{t+1}{ty+t}$$

is separable since we can rewrite the equation as

$$\frac{dy}{dt} = \frac{(t+1)}{t(y+1)} = \left(\frac{t+1}{t}\right)\left(\frac{1}{y+1}\right).$$

Two important types of separable equations occur if either t or y is missing from the right-hand side of the equation. The differential equation

$$\frac{dy}{dt} = g(t)$$

is separable since we may regard the right-hand side as $g(t) \cdot 1$, where we consider 1 as a (very simple) function of y. Similarly,

$$\frac{dy}{dt} = h(y)$$

is also separable. This last type of differential equation is said to be **autonomous**. Many of the most important first-order differential equations that arise in applications (including all of our models in the previous section) are autonomous. For example, the right-hand side of the logistic equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

depends on the dependent variable P alone, so this equation is autonomous.

How to solve separable differential equations

To find explicit solutions of separable differential equations, we use a technique familiar from calculus. To illustrate the method, consider the differential equation

$$\frac{dy}{dt} = \frac{t}{y^2}.$$

There is a temptation to solve this equation by simply integrating both sides of the equation with respect to t. This yields

$$\int \frac{dy}{dt}\,dt = \int \frac{t}{y^2}\,dt,$$

and, consequently,

$$y(t) = \int \frac{t}{y^2}\,dt.$$

Now we are stuck. We can't evaluate the integral on the right-hand side because we don't know the function $y(t)$. In fact, that is precisely the function we wish to find. We have simply replaced the differential equation with an *integral equation*.

We need to do something to this equation *before* we try to integrate. Returning to the original differential equation

$$\frac{dy}{dt} = \frac{t}{y^2},$$

we first do some "informal" algebra and rewrite this equation in the form

$$y^2\,dy = t\,dt.$$

That is, we multiply both sides by $y^2\,dt$. Of course, it makes no sense to split up dy/dt by multiplying by dt. However, this should remind you of the technique of integration known as u-substitution in calculus. We will soon see that substitution is exactly what we are doing here.

We now integrate both sides: the left with respect to y and the right with respect to t. We have

$$\int y^2\,dy = \int t\,dt,$$

which yields

$$\frac{y^3}{3} = \frac{t^2}{2} + c.$$

Technically there is a constant of integration on both sides of this equation, but we can lump them together as a single constant c on the right. We may rewrite this expression as

$$y(t) = \left(\frac{3t^2}{2} + 3c\right)^{1/3};$$

and since c is an arbitrary constant, we may write this even more compactly as

$$y(t) = \left(\frac{3t^2}{2} + k\right)^{1/3},$$

where k is an arbitrary constant. As usual, we can check that this expression really is a solution of the differential equation, so despite the questionable separation we just performed, we do obtain infinitely many solutions.

Note that this process yields many solutions of the differential equation. Each choice of the constant k gives a different solution.

What is really going on in our informal algebra

If you read the previous example closely, you probably became nervous at one point. Treating dt as a variable is a tip-off that something a little more complicated is actually going on. Here is the real story.

We began with a separable equation

$$\frac{dy}{dt} = g(t)h(y),$$

and then rewrote it as

$$\frac{1}{h(y)}\frac{dy}{dt} = g(t).$$

This equation actually has a function of t on each side of the equals sign because y is a function of t. So we really should write it as

$$\frac{1}{h(y(t))}\frac{dy}{dt} = g(t).$$

In this form, we can integrate both sides with respect to t to get

$$\int \frac{1}{h(y(t))}\frac{dy}{dt}\,dt = \int g(t)\,dt.$$

Now for the important step: We make a "u-substitution" just as in calculus by replacing the function $y(t)$ by the new variable, say y. (In this case, the substitution is actually a y-substitution.) Of course, we must also replace the expression $(dy/dt)\,dt$ by dy. The method of substitution from calculus tells us that

$$\int \frac{1}{h(y(t))}\frac{dy}{dt}\,dt = \int \frac{1}{h(y)}\,dy,$$

and therefore we can combine the last two equations to obtain

$$\int \frac{1}{h(y)}\,dy = \int g(t)\,dt.$$

Hence, we can integrate the left-hand side with respect to y and the right-hand side with respect to t.

Separating variables and multiplying both sides of the differential equation by dt is simply a notational convention that helps us remember the method. It is justified by the argument above.

Missing Solutions

If it is possible to separate variables in a differential equation, it appears that solving the equation reduces to a matter of computing several integrals. This is true, but there are some hidden pitfalls, as the following example shows. Consider the differential equation

$$\frac{dy}{dt} = y^2.$$

This is an autonomous and hence separable equation, and its solution looks straightforward. If we separate and integrate as usual, we obtain

$$\int \frac{dy}{y^2} = \int dt$$

$$-\frac{1}{y} = t + c$$

$$y(t) = -\frac{1}{t+c}.$$

We are tempted to say that this expression

$$y(t) = -\frac{1}{t+c}$$

is the general solution. However, we cannot solve all initial-value problems with solutions of this form. In fact, we have $y(0) = -1/c$, so we cannot use this expression to solve the initial-value problem $y(0) = 0$.

What's wrong? Note that the right-hand side of the differential equation vanishes if $y = 0$. So the constant function $y(t) = 0$ is a solution to this differential equation. In other words, in addition to those solutions that we derived using the method of separation of variables, this differential equation possesses the equilibrium solution $y(t) = 0$ for all t, and it is this equilibrium solution that satisfies the initial-value problem $y(0) = 0$. Even though it is "missing" from the family of solutions that we obtain by separating variables, it is a solution that we need if we want to solve every initial-value problem for this differential equation. Thus the general solution consists of functions of the form $y(t) = -1/(t+c)$ together with the equilibrium solution $y(t) = 0$.

Getting Stuck

As another example, consider the differential equation

$$\frac{dy}{dt} = \frac{y}{1+y^2}.$$

As before, this equation is autonomous. So we first separate variables to obtain

$$\left(\frac{1+y^2}{y}\right) dy = dt.$$

Then we integrate

$$\int \left(\frac{1}{y} + y\right) dy = \int dt,$$

which yields

$$\ln|y| + \frac{y^2}{2} = t + c.$$

But now we are stuck; there is no way to solve the equation

$$\ln|y| + \frac{y^2}{2} = t + c$$

for y alone. Thus we cannot generate an *explicit* formula for y. We do, however, have an **implicit form** for the solution which, for many purposes, is perfectly acceptable.

Even though we don't obtain explicit solutions by separating variables for this equation, we can find one explicit solution. The right-hand side is zero if $y = 0$. Thus the constant function $y(t) = 0$ for all t is an equilibrium solution. Note that this equilibrium solution does not appear in the implicit solution we derived from the method of separation of variables.

There is another problem that arises with this method. It is often impossible to perform the necessary integrations. For example, the differential equation

$$\frac{dy}{dt} = \sec(y^2)$$

is autonomous. Separating variables and integrating we get

$$\int \frac{1}{\sec(y^2)}\, dy = \int dt.$$

In other words,

$$\int \cos(y^2)\, dy = \int dt.$$

The integral on the left-hand side is difficult, to say the least. (In fact, there is a special function that was defined just to give us a name for this integral.) The lesson is that, even for autonomous equations

$$\frac{dy}{dt} = f(y),$$

carrying out the required algebra or integration is frequently impossible. We will not be able to rely solely on analytic tools and explicit solutions when studying differential equations, even if we can separate variables.

A Savings Model

Suppose we deposit \$5000 in a savings account with interest accruing at the rate of 2% compounded continuously. If we let $A(t)$ denote the amount of money in the account at time t, then the differential equation for A is

$$\frac{dA}{dt} = 0.02A.$$

As we saw in the previous section, the general solution to this equation is the exponential function

$$A(t) = ce^{0.02t},$$

where $c = A(0)$. Thus $A(t) = 5000e^{0.02t}$ is our particular solution.

Assuming interest rates never change, after 10 years we will have

$$A(10) = 5000e^{0.2} \approx 6107$$

dollars in this account. That is a nice little nest egg, so we decide we should have some fun in life. We decide to withdraw \$500 (mad money) from the account each year in a continuous way beginning in year 10. How long will this money last? Will we ever go broke?

The differential equation for $A(t)$ must change, but only beginning in year 10. For $0 \leq t \leq 10$, our previous model works fine. However, for $t > 10$, the differential equation becomes

$$\frac{dA}{dt} = 0.02A - 500.$$

Thus we really have a differential equation of the form

$$\frac{dA}{dt} = \begin{cases} 0.02A & \text{for } t < 10; \\ 0.02A - 500 & \text{for } t > 10, \end{cases}$$

whose right-hand side consists of two pieces.

To solve this two-part equation, we solve the first part and determine $A(10)$. We just did that and obtained $A(10) \approx 6107$. Then we solve the second equation using $A(10) \approx 6107$ as the initial value. This equation is also separable, and we have

$$\int \frac{dA}{0.02A - 500} = \int dt.$$

We calculate this integral using substitution and the natural logarithm function. Let $u = 0.02A - 500$. Then $du = 0.02\,dA$, or $50\,du = dA$ since $0.02 = 1/50$. We obtain

$$\int \frac{50\,du}{u} = t + c_1$$

$$50 \ln |u| = t + c_1$$

$$50 \ln |0.02A - 500| = t + c_1,$$

for some constant c_1.

At $t = 10$, we know that $A \approx 6107$. Thus at $t = 10$,

$$\frac{dA}{dt} = 0.02A - 500 \approx -377.9 < 0.$$

In other words, we are withdrawing at a rate that exceeds the rate at which we are earning interest. Since dA/dt at $t = 10$ is negative, A will decrease and $0.02A - 500$ remains negative for all $t > 10$. If $0.02A - 500 < 0$, then

$$|0.02A - 500| = -(0.02A - 500) = 500 - 0.02A.$$

Consequently, we have

$$50 \ln(500 - 0.02A) = t + c_1.$$

Since dividing by 50 is the same as multiplying 0.02, we get

$$\ln(500 - 0.02A) = 0.02(t + c_1).$$

We exponentiate and obtain

$$
\begin{aligned}
500 - 0.02A &= e^{0.02(t+c_1)} \\
&= c_2 e^{0.02t}
\end{aligned}
$$

where $c_2 = e^{0.02c_1}$. Solving for A, we have

$$
\begin{aligned}
A &= \frac{500 - c_2 e^{0.02t}}{0.02} \\
&= 50\left(500 - c_2 e^{0.02t}\right) = 25000 - c_3 e^{0.02t},
\end{aligned}
$$

where $c_3 = 50c_2$. (Although we have been careful to spell out the relationships among the constants c_1, c_2, and c_3, we need only remember that c_3 is a constant that is determined from the initial condition.)

Now we use the initial condition to determine c_3. We know that

$$
6107 \approx A(10) = 25000 - c_3 e^{0.02(10)} \approx 25000 - c_3(1.2214).
$$

Solving for c_3, we obtain $c_3 \approx 15468$. Our solution for $t \geq 10$ is

$$
A(t) \approx 25000 - 15468e^{0.02t}.
$$

We see that

$$
\begin{aligned}
A(11) &\approx 5726 \\
A(12) &\approx 5336
\end{aligned}
$$

and so forth. Our account is being depleted, but not by that much. In fact, we can find out just how long the good times will last by asking when our money will run out. In other words, we solve the equation $A(t) = 0$ for t. We have

$$
0 = 25000 - 15468e^{0.02t},
$$

which yields

$$
t = 50 \ln\left(\frac{25000}{15468}\right) \approx 24.01.
$$

After letting the \$5000 accumulate interest for ten years, we can withdraw \$500 per year for more than twenty years.

A Mixing Problem

The name *mixing problem* refers to a large collection of different problems where two or more substances are mixed together at various rates. Examples range from the mixing of pollutants in a lake to the mixing of chemicals in a vat to the diffusion of cigar smoke in the air in a room to the blending of spices in a serving of curry.

Mixing in a vat

Consider a large vat containing sugar water that is to be made into soft drinks (see Figure 1.9). Suppose:

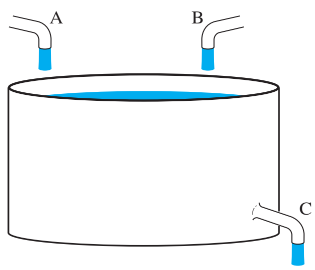

Figure 1.9
Mixing vat.

- The vat contains 100 gallons of liquid. Moreover, the amount flowing in is the same as the amount flowing out, so there are always 100 gallons in the vat.
- The vat is kept well mixed, so the sugar concentration is uniform throughout the vat.
- Sugar water containing 5 tablespoons of sugar per gallon enters the vat through pipe A at a rate of 2 gallons per minute.
- Sugar water containing 10 tablespoons of sugar per gallon enters the vat through pipe B at a rate of 1 gallon per minute.
- Sugar water leaves the vat through pipe C at a rate of 3 gallons per minute.

To make the model, we let t be time measured in minutes (the independent variable). For the dependent variable, we have two choices. We could choose either the total amount of sugar, $S(t)$, in the vat at time t measured in tablespoons, or $C(t)$, the concentration of sugar in the vat at time t measured in tablespoons per gallon. We develop the model for S, leaving the model for C as an exercise for the reader.

Using the total sugar $S(t)$ in the vat as the dependent variable, the rate of change of S is the difference between the amount of sugar being added and the amount of sugar being removed. The sugar entering the vat comes from pipes A and B and can be easily computed by multiplying the number of gallons per minute of sugar mixture entering the vat by the amount of sugar per gallon. The amount of sugar leaving the vat through pipe C at any given moment depends on the concentration of sugar in the vat at that moment. The concentration is given by $S/100$, so the sugar leaving the vat is the product of the number of gallons leaving per minute (3 gallons per minute) and the concentration ($S/100$). The model is

$$\frac{dS}{dt} = \underbrace{2 \cdot 5}_{\substack{\text{sugar in}\\ \text{from pipe A}}} + \underbrace{1 \cdot 10}_{\substack{\text{sugar in}\\ \text{from pipe B}}} - \underbrace{3 \cdot \frac{S}{100}}_{\substack{\text{sugar out}\\ \text{from pipe C}}}.$$

That is,

$$\frac{dS}{dt} = 20 - \frac{3S}{100} = \frac{2000 - 3S}{100}.$$

To solve this equation analytically, we separate and integrate. We find

$$\frac{dS}{2000 - 3S} = \frac{dt}{100}$$

$$\frac{\ln|2000 - 3S|}{-3} = \frac{t}{100} + c_1$$

$$\ln|2000 - 3S| = -\frac{3t}{100} - 3c_1$$

$$\ln|2000 - 3S| = -0.03t + c_2,$$

where $c_2 = -3c_1$. Exponentiating we obtain

$$|2000 - 3S| = e^{(-0.03t + c_2)} = c_3 e^{-0.03t},$$

where $c_3 = e^{c_2}$. Note that this means that c_3 is a positive constant. Now we must be careful. Removing the absolute value signs yields

$$2000 - 3S = \pm c_3 e^{-0.03t},$$

where we choose the plus sign if $S(t) < 2000/3$ and the minus sign if $S(t) > 2000/3$. Therefore we may write this equation more simply as

$$2000 - 3S = c_4 e^{-0.03t},$$

where c_4 is an arbitrary constant (positive, negative, or zero). Solving for S yields the general solution

$$S(t) = ce^{-0.03t} + \frac{2000}{3},$$

where $c = -c_4/3$ is an arbitrary constant. We can determine the precise value of c if we know the exact amount of sugar that is initially in the vat. Note that, if $c = 0$, the solution is simply $S(t) = 2000/3$, an equilibrium solution.

EXERCISES FOR SECTION 1.2

1. Bob, Glen, and Paul are once again sitting around enjoying their nice, cold glasses of iced cappucino when one of their students asks them to come up with solutions to the differential equation

$$\frac{dy}{dt} = \frac{y+1}{t+1}.$$

After much discussion, Bob says $y(t) = t$, Glen says $y(t) = 2t + 1$, and Paul says $y(t) = t^2 - 2$.

(a) Who is right?

(b) What solution should they have seen right away?

2. Make up a differential equation of the form

$$\frac{dy}{dt} = 2y - t + g(y)$$

that has the function $y(t) = e^{2t}$ as a solution.

3. Make up a differential equation of the form $dy/dt = f(t, y)$ that has $y(t) = e^{t^3}$ as a solution. (Try to come up with one whose right-hand side $f(t, y)$ depends explicitly on both t and y.)