Appendix A

Calculation of the system's transfer functions

To simplify the model, we assume that it can be represented as an inverted pendulum attached to a cart. Calculations about inverted pendulum attached to a cart is common and can be found on several websites on the internet, this will help ensure that these calculations are correct. The following calculations were inspired by MathWorks [12].

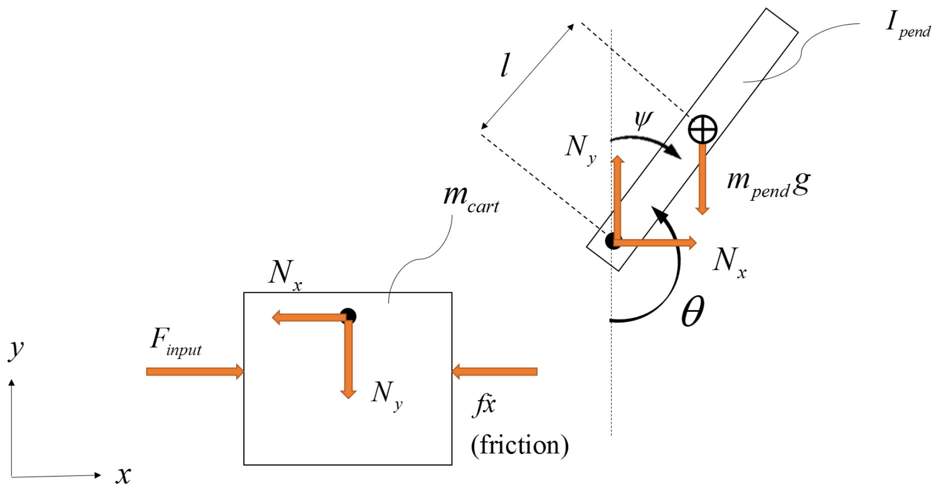

Figure A.1. Exposure of the simplified system with cart and pendulum

A.1 Equations for the cart

The only equation needed from the cart is the one who sums the forces in the x direction

$$F_{input} = m_{cart}\ddot{x} + f\dot{x} + N_x \tag{A.1}$$

where F_{input} is an applied force, m_{cart} is the mass of the cart, f is a constant of friction and N_x is the contact force in the axis between cart and pendulum in x-direction. See Figure A.1.

A.2 Equations for the pendulum

The sum of forces in the x direction on the pendulum is

$$N_x = m_{pend}\ddot{x} + m_{pend}l\ddot{\theta}\cos\theta - m_{pend}l\dot{\theta}^2\sin\theta \tag{A.2}$$

where m_{pend} is the mass and l is the distance to the center of mass. Variable θ is the angle between a vertical line and the pendulum. See Figure A.1 for details. The sum of all forces perpendicular to the pendulum is

$$N_y\sin\theta + N_x\cos\theta - m_{pend}g\sin\theta = m_{pend}l\ddot{\theta} + m_{pend}\ddot{x}\cos\theta \tag{A.3}$$

where N_y is the force in y direction and g is the gravity constant. Summarization the torque acting on the center of the pendulum gives the following equation

$$-N_yl\sin\theta - N_xl\cos\theta = I_{pend}\ddot{\theta} \tag{A.4}$$

where I_{pend} is the moment of inertia of the pendulum.

A.3 Combining the equations

Insert equation (A.2) in (A.1) gives the following equation

$$F_{input} = (m_{cart} + m_{pend})\ddot{x} + f\dot{x} + m_{pend}l\ddot{\theta}\cos\theta - m_{pend}l\dot{\theta}^2\sin\theta \tag{A.5}$$

Combine equations (A.2) and (A.3)

$$(I_{pend} + m_{pend}l^2)\ddot{\theta} + m_{pend}gl\sin\theta = -m_{pend}l\ddot{x}\cos\theta \tag{A.6}$$

A.4 Linearising the equations

Equation (A.5) and (A.6) is necessary to get transfer functions for the position x and the angle deviation ψ. To compute the transfer functions the equations need to be linearised. A proper equilibrium point would be when the pendulum is in upright position. The angle will represent the deviation of the pendulum from the equilibrium. The following approximations for small deviations will be used in the nonlinear equations (A.5) and (A.6).

$$\cos\theta = \cos(\pi + \psi) \approx -1 \tag{A.7}$$

$$\sin\theta = \sin(\pi + \psi) \approx -\psi \tag{A.8}$$

$$\dot{\theta}^2 = \dot{\psi}^2 \approx 0 \tag{A.9}$$

Linearization with (A.7), (A.8) and (A.9) in (A.5) and (A.6) leads to the following approximated linear equations where F_{input} has been substituted for the more general control effort u_{input}.

$$(I_{pend} + m_{pend}l^2)\ddot{\psi} - m_{pend}gl\psi = m_{pend}l\ddot{x} \tag{A.10}$$

$$u_{input} = (m_{cart} + m_{pend})\ddot{x} + f\dot{x} - m_{pend}l\ddot{\psi} \tag{A.11}$$

A.5 Laplace transform

To obtain the transfer functions, equations (A.10) and (A.11) is transformed to the Laplace domain, the transformation is here denoted by upper case letters.

$$(I_{pend} + m_{pend}l^2)\Psi(s)s^2 - m_{pend}gl\Psi(s) = m_{pend}lX(s)s^2 \tag{A.12}$$

$$U_{input}(s) = (m_{cart} + m_{pend})X(s)s^2 + fX(s)s - m_{pend}l\Psi(s)s^2 \tag{A.13}$$

A transfer function is a relationship between a single input and a single output, therefore it is needed to solve $X(s)$ from equation (A.12)

$$X(s) = \left[\frac{I_{pend} + m_{pend}l^2}{m_{pend}l} - \frac{g}{s^2}\right]\Psi(s) \tag{A.14}$$

Substitute (A.14) into (A.12) gives

$$U_{input}(s) = (m_{cart}+m_{pend})\left[\frac{I_{pend} + m_{pend}l^2}{m_{pend}l} - \frac{g}{s^2}\right]\Psi(s)s^2+f\left[\frac{I_{pend} + m_{pend}l^2}{m_{pend}l} - \frac{g}{s^2}\right]\Psi(s)s-m_{pend}l\Psi(s)s \tag{A.15}$$

If equation (A.15) is rearranged we get the transfer function $G_{\psi}(s)$ as the relation between $\Psi(s)$ and $U_{input}(s)$ as seen in (A.16).

$$\Psi(s) = \underbrace{\frac{\frac{m_{pend}l}{q}s}{s^3 + \frac{f(I_{pend}+m_{pend}l^2)}{q}s^2 - \frac{(m_{cart}+m_{pend})m_{pend}gl}{q}s - \frac{fm_{pend}gl}{q}}}_{G_\Psi(s)} U_{input}(s) \quad \text{(A.16)}$$

where

$$q = \left[(m_{cart} + m_{pend})(I_{pend} + m_{pend}l^2) - (m_{pend}l)^2\right] \quad \text{(A.17)}$$

The transfer function $G_x(s)$ that describes the cart position $X(s)$ looks as

$$X(s) = \underbrace{\frac{\frac{(I_{pend}+m_{pend}l^2)s^2 - gm_{pend}l}{q}}{s^4 + \frac{f(I_{pend}+m_{pend}l^2)}{q}s^3 - \frac{(m_{cart}+m_{pend})m_{pend}gl}{q}s^2 - \frac{fm_{pend}gl}{q}s}}_{G_x(s)} U_{input}(s) \quad \text{(A.18)}$$

A.6 State Space Modelling

It is possible to present the system in state space form. The matrix form is

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} = A \begin{bmatrix} x \\ \dot{x} \\ \psi \\ \dot{\psi} \end{bmatrix} + Bu_{input} \quad \text{(A.19)}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I_{pend}+m_{pend}l^2)f}{I_{pend}(m_{cart}+m_{pend})+m_{cart}m_{pend}l^2} & \frac{{m_{pend}}^2gl^2}{I_{pend}(m_{cart}+m_{pend})+m_{cart}m_{pend}l^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-m_{pend}lf}{I_{pend}(m_{cart}+m_{pend})+m_{cart}m_{pend}l^2} & \frac{m_{pend}gl(m_{cart}+m_{pend})}{I_{pend}(m_{cart}+m_{pend})+m_{cart}m_{pend}l^2} & 0 \end{bmatrix} \quad \text{(A.20)}$$

$$B = \begin{bmatrix} 0 \\ \frac{I_{pend}+m_{pend}l^2}{I_{pend}(m_{cart}+m_{pend})+m_{cart}m_{pend}l^2} \\ 0 \\ \frac{m_{pend}l}{I_{pend}(m_{cart}+m_{pend})+m_{cart}m_{pend}l^2} \end{bmatrix} \quad \text{(A.21)}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{(A.22)}$$