# A Tableau algorithm for $\mathcal{ALCSCC}$

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# Tableau Algorithm

Main Idea:

$$x : A \sqcap B \sqcap \exists r.C$$

$x$ must be in $A$

$x$ must be in $B$

$x$ must have an $r$-successor in $C$

# $\mathcal{ALCSCC}$: successors

$\cancel{\exists r.C}$ $\cancel{\forall r.C}$ $\cancel{\leq m\, r.C}$ $\cancel{\geq m\, r.C}$

$$\boxed{succ(c)}$$

$c$: **set constraint** or **cardinality constraint**

# $\mathcal{ALCSCC}$: constraints

**set constraint:**

- $r \subseteq s$
- $C \cap r \subseteq D$
- $succ(C \cap r) \subseteq succ(D)$

**cardinality constraint**

- $2\,dvd\,|r|$
- $|C \cap r| \leq |D|$
- $|succ(C \cap r)| \leq |succ(D)|$

# Problem with successors constraints

$$x : \; succ(|s| > 1) \sqcap succ(|r| = |s|) \sqcap succ(|r| > |s|)$$

| $s$-successors | $r$-successors |
| --- | --- |
| 0 | 0 |

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$$x : \underline{succ(|s| > 1)} \sqcap succ(|r| = |s|) \sqcap succ(|r| > |s|)$$

| $s$-successors | $r$-successors |
| --- | --- |
| 1 | 0 |

# Problem with successors constraints

$$x:\ succ(|s| > 1) \sqcap \underline{succ(|r| = |s|)} \sqcap succ(|r| > |s|)$$

| $s$-successors | $r$-successors |
|---|---|
| 1 | 1 |

# Problem with successors constraints

$$x : \ succ(|s| > 1) \sqcap succ(|r| = |s|) \sqcap \underline{succ(|r| > |s|)}$$

| $s$-successors | $r$-successors |
|---|---|
| 1 | 2 |

# Problem with blocking

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

| $s$-successors | $r$-successors |
| --- | --- |
| 0 | 0 |

# Problem with blocking

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap \underline{succ(|r| > 1)}$$

| $s$-successors | $r$-successors |
| --- | --- |
| 0 | 1 |

# Problem with blocking

$$x : \underline{succ(2 \cdot |r| \leq 5 \cdot |s|)} \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

| $s$-successors | $r$-successors |
| --- | --- |
| 1 | 1 |

# QFBAPA formula and solver

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

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$$\downarrow$$

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

# QFBAPA formula and solver

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

$$\downarrow$$

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 5 \text{ and } |X_s| = 2$$

# QFBAPA formula and solver

What about

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 10 \text{ and } |X_s| = 4$$

# QFBAPA formula and solver

What about

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 100 \text{ and } |X_s| = 40$$

# QFBAPA formula and solver

We have infinite possible solutions

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Do we need all of them?

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We have infinite possible solutions

Do we need all of them?

No: Consider formula as ILP and calculate an upper bound

# ILP and upper bound

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$\begin{aligned} -5 \cdot |X_s| + 2 \cdot |X_r| &\leq 0 \\ 5 \cdot |X_s| - 2 \cdot |X_r| &\leq 0 \\ -|X_r| &< -1 \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} -5 \cdot |X_s| + 2 \cdot |X_r| + l_1 &= 0 \\ 5 \cdot |X_s| - 2 \cdot |X_r| + l_2 &= 0 \\ -|X_r| + l_3 &= -1 \end{aligned}$$

# ILP and upper bound

**Problem**: Are the variables disjoint or not?

# ILP and upper bound

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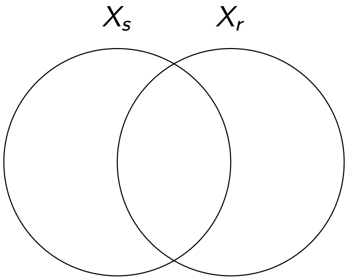
**Solution**: Venn region



# Venn regions

Two variables $X_s$ and $X_r$ $\rightarrow$ four Venn regions:

$$
\begin{aligned}
v_1 &= X_s \cap X_r \\
v_2 &= X_s^{\neg} \cap X_r \\
v_3 &= X_s \cap X_r^{\neg} \\
v_4 &= X_s^{\neg} \cap X_r^{\neg}
\end{aligned}
$$

# Venn regions

Two variables $X_s$ and $X_r$ $\rightarrow$ four Venn regions:

$$\begin{aligned} v_1 &= X_s \cap X_r \\ v_2 &= X_s^{\neg} \cap X_r \\ v_3 &= X_s \cap X_r^{\neg} \\ v_4 &= X_s^{\neg} \cap X_r^{\neg} \end{aligned}$$

$X_s = v_1 \cup v_3$ and $X_r = v_1 \cup v_2$

# ILP

$$\begin{array}{c c}
 & \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & l_1 & l_2 & l_3 \end{array} \\
\begin{array}{c} -5\cdot|X_s|+2\cdot|X_r|+l_1 \\ 5\cdot|X_s|-2\cdot|X_r|+l_2 \\ -|X_r|+l_3 \end{array} &
\begin{pmatrix} -5+2 & 2 & -5 & 0 & 1 & 0 & 0 \\ 5-3 & -2 & 5 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{array} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$$

# ILP and upper bound

Christos H. Papadimitriou[1]:

There exists an upper bound $M$ such that for each ILP $Ax = b$, $A$ a $m \times n$ matrix, $x \in \mathbb{N}^n$ and $b \in \mathbb{R}^m$ there exists a solution $x' \in \{0, \ldots, M\}^n$.

[1] On the Complexity of Integer Programming. J. ACM,28(4):765-768, Oct. 1981.

# Tableau algorithm

- $\sqcap$-rule:
- $\sqcup$-rule: