1 Preliminaries

Let **C** be a set of concept names and **R** a set of role names such that they are disjoint.

Definition 1 ($QFBAPA$)**.** Let T be a set of symbols

- set terms over T are:
  - empty set $\emptyset$ and universal set $\mathcal{U}$
  - every set symbol in T
  - if s, t are set terms then also $s \cap t$, $s \cup t$ and $s^{\neg}$
- set constraints over T are
  - $s \subseteq t$ and $s \not\subseteq t$
  - $s = t$ and $s \neq t$

  where s, t are set terms
- cardinality terms over T are:
  - every number $n \in \mathbb{N}$
  - $|s|$ if s is a set term
  - if k, l are cardinality terms then also $k + l$ and $n \cdot k$, $n \in \mathbb{N}$
- cardinality constraints over T are:
  - $k = l$ and $k \neq l$
  - $k < l$ and $k \geq l$
  - $k \leq l$ and $k > l$
  - n *dvd* k and n $\neg$*dvd* k

  where k, l are cardinality terms and $n \in \mathbb{N}$

For readability we use $\lesseqgtr$ to address the comparison symbols $=, \leq, \geq, <, >$. The negation $\not\lesseqgtr$ address the symbols $\neq, >, <, \geq, \leq$ respectively.

Definition 2 ($\mathcal{ALCSCC}$)**.** Concepts are:

- all concept names
- $succ(c)$ if c is a set or cardinality constraint over $\mathcal{ALCSCC}$ concepts and role names
- if C, D are concepts then:
  - $\neg C$
  - $C \sqcup D$
  - $C \sqcap D$

Definition 3 (Negation Normal Form)**.** A concept is in *negation normal form* (NNF) if the negation sign $\neg$ appears only in front of a concept name or above a role name. Let C be a arbitrary concept. Its NNF is obtained by applying the following rules

- $\neg\top \to \bot$
- $\neg\bot \to \top$
- $\neg\neg C \to C$
- $\neg(C \sqcap D) \to \neg C \sqcup \neg D$
- $\neg(C \sqcup D) \to \neg C \sqcap \neg D$
- $(C)^\neg \to \neg C$
- $\neg succ(c) \to succ(\neg c)$
- $\neg(k \lesseqgtr l) \to k \ggless l$
- $\neg(n\ dvd\ k) \to n\ \neg dvd\ k$
- $\neg(n\ \neg dvd\ k) \to n\ dvd\ k$
- $\neg(s_1 \subseteq s_2) \to s_1 \not\subseteq s_2 \to s_1 \subseteq s_2^\neg$
- $(s \cap t)^\neg \to s^\neg \cup t^\neg$
- $(s \cup t)^\neg \to s^\neg \cap t^\neg$
- $(s^\neg)^\neg \to s$

With $NNF(C)$ we denote the concept which is obtained by applying the rules above on C until none is applicable any more.

Definition 4 (Positive and Negative Sign)**.** A concept name C or a role name r has a *positive sign* if there is no negation sign in front or above it. It has a *negative sign* otherwise.

The set S is a finite set of assertions of the form $x : C$ and $(x, y) : s$, where C is a concept, s a set term and x, y variables. The set $Var(S)$ is the set of variables occurring in S.

Definition 5 (Interpretation)**.** An *interpretation* $\mathcal{I} = (\cdot^\mathcal{I}, \cdot^\mathcal{I})$ over $\mathcal{ALCSCC}$ consists of a non-empty set $\Delta^\mathcal{I}$ and a mapping $\cdot^\mathcal{I}$ which maps:

- $\emptyset$ to $\emptyset^\mathcal{I}$
- $\mathcal{U}$ to $\mathcal{U}^\mathcal{I} \subseteq \Delta^\mathcal{I}$
- every concept names $A \in \mathbf{C}$ to $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$
- every role name $r \in \mathbf{R}$ to $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, such that every element in $\Delta^\mathcal{I}$ has a finite number of successors.

The set $r^\mathcal{I}(x)$ contains all elements y such that $(x, y) \in r^\mathcal{I}$ e.g. it contains all r-successors of x.
For compound concepts the mapping $\cdot^\mathcal{I}$ is extended inductively as follows

- $(C \sqcap D)^\mathcal{I} := C^\mathcal{I} \sqcap D^\mathcal{I}$, $(C \sqcup D)^\mathcal{I} := C^\mathcal{I} \sqcup D^\mathcal{I}$
- $(\neg C)^\mathcal{I} := \Delta^\mathcal{I} \backslash C^\mathcal{I}$
- $(s \cap t)^\mathcal{I} := s^\mathcal{I} \cap t^\mathcal{I}$, $(s \cup t)^\mathcal{I} := s^\mathcal{I} \cup t^\mathcal{I}$

- $(s^{\neg})^{\mathcal{I}} := \mathcal{U}^{\mathcal{I}} \backslash s^{\mathcal{I}}$
- $|s|^{\mathcal{I}} := |s^{\mathcal{I}}|$
- $(k+l)^{\mathcal{I}} := (k^{\mathcal{I}} + l^{\mathcal{I}})$, $(n \cdot k)^{\mathcal{I}} := n \cdot k^{\mathcal{I}}$
- $succ(c)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | \text{the mapping } \cdot^{\mathcal{I}_x} \text{ satisfies } c\}$

The mapping $\cdot^{\mathcal{I}_x}$ maps $\emptyset$ to $\emptyset^{\mathcal{I}}$, $\mathcal{U}$ to $\mathcal{U}^{\mathcal{I}_x} := \{\bigcup_{r \in \mathbf{R}} r^{\mathcal{I}}(x)\}$, every concept C occurring in c to $C^{\mathcal{I}_x} := C^{\mathcal{I}} \cap \mathcal{U}^{\mathcal{I}_x}$ and every role name r occurring in c to $r^{\mathcal{I}_x} := r^{\mathcal{I}}$.
The mappings satisfies for the set terms s, t and the cardinality terms k, l

- $s = t$ iff $s^{\mathcal{I}} = t^{\mathcal{I}}$
- $s \subseteq t$ iff $s^{\mathcal{I}} \subseteq t^{\mathcal{I}}$
- $k \lesseqgtr l$ iff $k^{\mathcal{I}} \lesseqgtr l^{\mathcal{I}}$
- $n \, dvd \, l$ iff $\exists m \in \mathbb{N} : n \cdot m = l^{\mathcal{I}}$

An *assignment* $\pi_{\mathcal{I}} : Var(S) \to \Delta^{\mathcal{I}}$ satisfies

- $x : C$ iff $\pi_{\mathcal{I}}(x) \in C^{\mathcal{I}}$
- $(x, y) : s$ iff $(\pi_{\mathcal{I}}(x), \pi_{\mathcal{I}}(y)) \in s^{\mathcal{I}}$

A constraint c is satisfiable if there exists an interpretation $\mathcal{I}$ and an assignment $\pi_{\mathcal{I}}$ such that $\pi_{\mathcal{I}}$ satisfies c. $\pi_{\mathcal{I}}$ satisfies a constraint set S if $\pi_{\mathcal{I}}$ satisfies every constraint in S. S is satisfiable if there exists an interpretation $\mathcal{I}$ and an assignment $\pi_{\mathcal{I}}$ such that $\pi_{\mathcal{I}}$ satisfies S.

Definition 6 (Number of successors)**.** Let S be a set of constraints, $\mathcal{I}$ be an interpretation, $\pi_{\mathcal{I}} : Var(S) \to \Delta^{\mathcal{I}}$ be an assignment, x be a variable and k be a cardinality term. The number of successors of x in k in the interpretation $\mathcal{I}$ is denoted by

$$n_{\mathcal{I}}(x_{S_k}) := \begin{cases} n_{\mathcal{I}}(x_{S_i}) + n_{\mathcal{I}}(x_{S_j}) & \text{if } k = i + j \\ n_{\mathcal{I}}(x_{S_i}) \cdot n_{\mathcal{I}}(x_{S_j}) & \text{if } k = i \cdot j \\ n_{\mathcal{I}}(x_{S_l}) \cdot n & \text{if } k = l \cdot n \\ |s^{\mathcal{I}} \cap \bigcup_{r \in \mathbf{R}} r^{\mathcal{I}}(x)| & \text{if } k = |s| \end{cases}$$

where i, j, l are cardinality terms, $n \in \mathbb{N}$ and s is a set term.
A constraint regarding a variable x in an interpretation $\mathcal{I}$ is *violated* if

- $x : succ(k \lesseqgtr n)$ and $n_{\mathcal{I}}(x_{S_k}) \not\lesseqgtr n$
- $x : succ(k \lesseqgtr l)$ and $n_{\mathcal{I}}(x_{S_k}) \not\lesseqgtr n_{\mathcal{I}}(x_{S_l})$
- $x : succ(n \, dvd \, k)$ and $mod(n_{\mathcal{I}}(x_{S_k}), n) \neq 0$

where $n \in \mathbb{N}$.

2 Tableau

For the algorithm we assume that the constraints in the constraint set are in NNF.

Definition 7 (Merge)**.** *Merging* y_1 and y_2 results in one variable y: replace all occurrence of y_1 and y_2 with y.

Note that by merging two successors other constraints might become violated:

$$S = \{x : succ(|r \cap A| = 1) \sqcap succ(|r \cap B| = 1) \sqcap succ(|r| > 1), \\ y_1 : A, y_2 : B, x.r.y_1, x.r.y_2\} \tag{1}$$

If we merge y_1 and y_2 then the constraint $x : succ(|r| > 1)$ which was satisfied becomes violated.
But not only constraints regarding x might become violated after merging two successors of x:

$$S = \{x : succ(|r| \leq 1), x.r.y_1, x.r.y_2, \\ y_1 : succ(|s| \leq 1), y_2 : succ(|s| \leq 1), y_1.s.z_1, y_2.s.z_2\} \tag{2}$$

We see that the first constraint is violated and therefore merging y_1 and y_2 to y would solve the problem but on the other hand the constraints regarding y become violated:

$$S = \{x : succ(|r| \leq 1), x.r.y, y : succ(|s| \leq 1), y.s.z_1, y.s.z_2\}$$

To solve this problem we have to merge z_1 and z_2.
For the next definition we define first properties of the following notations:

- Conjunction binds stronger than disjunction: $s \cup t \cap u = s \cup (t \cap u)$
- if k, l are cardinality terms then $k = l$ replaces $k \leq l$ and $k \geq l$
- if s, t are set terms then $s = t$ replaces $s \subseteq t$ and $s \supseteq t$

Definition 8 (Induced Interpretation $\mathcal{I}_I$)**.** An interpretation $\mathcal{I}_I$ can be induced from a constraint set S by the following steps:

- for each variable $x \in Var(S)$ we introduce $x^{\mathcal{I}_I}$ and add it to $\Delta^{\mathcal{I}_I}$
- for each $x : C$ such that C is a concept name we add $x^{\mathcal{I}_I}$ to $C^{\mathcal{I}_I}$
- for each $(x, y) : C$ such that C is a concept name we add $y^{\mathcal{I}_I}$ to $C^{\mathcal{I}_I}$
- for each $(x, y) : r$ such that r is a role name we add $(x^{\mathcal{I}_I}, y^{\mathcal{I}_I})$ to $r^{\mathcal{I}_I}$

With $\mathcal{I}_I$ we can count how many successors a variable has during the Tableau-algorithm. Regarding the cardinality constraints $k \leq l$ and $k < l$ we have to be *safe* before adding or merging variables:

- It is safe to introduce a new variable y if there is no set term s which occurs in k and l with the same sign
- It is safe to merge two variable if by merging them we do not violate any constraint regarding the predecessor in the induced interpretation of the resulting constraint set

Definition 9 (Tableau)**.** Let S be a set of constraints in NNF.

1. $\sqcap$-rule: S contains $x : C_1 \sqcap C_2$ but not both $x : C_1$ and $x : C_2$
   $\rightarrow S := S \cup \{x : C_1, x : C_2\}$
2. $\sqcup$-rule: S contains $x : C_1 \sqcup C_2$ but neither $x : C_1$ nor $x : C_2$
   $\rightarrow S := S \cup \{x : C_1\}$ or $S := S \cup \{x : C_2\}$
3. *choose*-rule: S contains $x : succ(k \leq l)$ and $(x, y) : k'$ with k' in k but $(x, y) : s \notin S$ for some $|s|$ occurring in k
   $\rightarrow$ for all $|s|$ in k, in which k' occurs, either $S := S \cup \{(x, y) : s\}$ or $S := S \cup \{(x, y) : s^{\neg}\}$ and then jump to rule **??**
4. *choose-a-role*-rule: S contains $(x, y) : s$ but for no role name r with a positive sign we have $(x, y) : r \notin S$
   $\rightarrow$ choose one rule name $r \in \mathbf{R}$ which does not occur with a negative sign in s and add $(x, y) : r$ to S
5. *cardinality*-rule: S contains $x : succ(c)$, with $c \in \{k \leq l,\ k < l,\ n\, dvd\, l\}$, such that c is violated in $\mathcal{I}_I$ regarding x
   a) if there is a set term $|s|$ in l such that it is safe to add variables regarding c
      $\rightarrow$ introduce new variable y and $S := S \cup \{(x, y) : s\}$, then jump to rule **??**
   b) if $l \in \mathbb{N}$ does not contain a set term and we have two successor $y_1 \neq y_2$ of x such that for a $|s|$ in k we have $(x, y_1) : s \in S$ and $(x, y_2) : s \in S$:
      $\rightarrow$ if it is safe to merge the two variables then merge them
6. *set*-rule: S contains $x : succ(s_1 \subseteq s_2)$ and $(x, y) : s_1$ but not $(x, y) : s_2$
   $\rightarrow S := S \cup \{(x, y) : s_2\}$ and then jump to rule **??**
7. *set.term*-rule (Repeat until inapplicable): In S is $(x, y) : s$ and
   a) $s = s_1 \cap s_2$ but $\{(x, y) : s_1,\ (x, y) : s_2\} \not\subseteq S$, then
      $\rightarrow S := S \cup \{(x, y) : s_1,\ (x, y) : s_2\}$
   b) $s = s_1 \cup s_2$ and neither $\{(x, y) : s_1\} \subseteq S$ nor $S\{(x, y) : s_2\} \subset S$, then
      $\rightarrow$ either $S := S \cup \{(x, y) : s_1\}$ or $S := S \cup \{(x, y) : s_2\}$
   c) $s = C$ and $y : C \notin S$, where C is an $\mathcal{ALCSCC}$ concepts, then
      $\rightarrow S := S \cup \{y : C\}$

Note that:

- s in **??** can also be of the form $t^{\neg}$.
- **??** is never applicable for $n \, dvd \, l$
- if $n_1 \, dvd \, n_2 \cdot l$ and $mod(n_2, n_1) \neq 0$ then $n_1 \, dvd \, l$ eventually

Definition 10 (Clash)**.** A constraint set S contains a *clash* if

- $\{x : \bot\} \subseteq S$ or
- $\{x : A, x : \neg A\} \subseteq S$ or
- $\{x : succ(c)\} \subseteq S$ and c is violated regarding x

and no more rules are applicable.

Definition 11 (Derived Set)**.** A *derived set* is a constraint set S' without clashes where only rule **??** is not applicable.

The first rule decompose the conjunction and the second rule adds non-deterministically the right constraint. The third rule is important because we need to know of every successor what kind of role successors they are and in which concepts they are. We use $n_{\mathcal{I}}(x_{S_k})$ to count the successors of x in k which is important for detecting and avoiding violations of constraints. Now there might be a successor y which satisfies only some part of k in the given S such that $n_{\mathcal{I}}(x_{S_k})$ does not count y. However there might be an interpretation $\mathcal{I}'$ and the assignment $\pi_{\mathcal{I}'}$ such that $n_{\mathcal{I}'}(x_{S_k})$ counts y and hence $n_{\mathcal{I}}(x_{S_k}) \neq n_{\mathcal{I}'}(x_{S_k})$. However the Tableau-algorithm should be able to construct every interpretation of S. Therefore this rule adds non-deterministically either $(x, y) : s$ or $(x, y) : s^{\neg}$ which are the only two possibilities.
The *choose-a-role*-rule is necessary because for a constraint $x : succ(c)$ we might have no role name with a positive sign in c. Which means we know x must have some successors but we can not decide which role-successor it is. As example we have

Example 1.

$$\mathbf{R} = \{r, s\}$$
$$S = \{x : succ(|r^{\neg}| \geq 1)\}$$

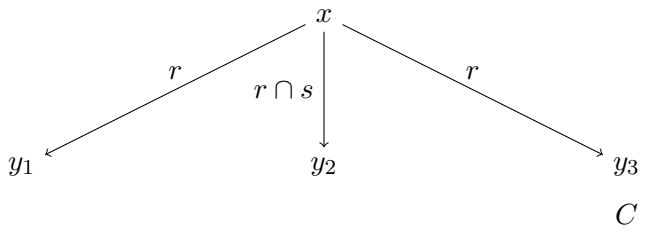
It states that x have at least one successor which is not a r-successor. Since $\mathbf{R}$ only contains r and s we know that the successors must be s-successors. With that rule we would pick either r or s. We can not pick r because $r^{\neg}$ occurs in the constraint. Therefore we have to pick s.

We consider now two examples to explain the rule. **??**

Example 2. Consider the following example

$$S = \{x : succ(|r| = 1) \sqcap succ(|r \cap s| = 1) \sqcap succ = (|r \cap C| = 1|)$$
$$x : succ(|r| = 1), x : succ(|r \cap s| = 1), x : succ = (|r \cap C| = 1|)\}$$

If we try to satisfy at least two of the new constraints by the Tableau-algorithm above we end up with at least one constraint being violated. We use the rules on the three new constraints sequentially. Then we have



After using rule **??** two times we have the variable x and its only $r \cap s$-successor y which is of the concept C. We could use this rule because we do not violate any other constraints.

This condition helps to prevent an infinite chain of rule application:

Example 3.

$$S = \{x : succ(|r| < 2) \sqcap succ(|r| \geq 2)\}$$

First we apply the rules **??** and **??** two times to add two r-successors for x hence $x : succ(|r| < 2)$ is not satisfied any more. If we ignore the condition in rule **??** and apply it then we merge the two successors leading to $x : succ(|r| < 2)$ being satisfied but $x : succ(|r| \geq 2)$ being violated. Then we apply the rules **??** and **??** again leading to $x : succ(|r| < 2)$ being violated again and so on. By the condition in **??** we can not use the rule which means the algorithm terminates with a clash stating that the constraint set is unsatisfiable.

We also know that the application of rule **??** eventually terminates because the number of concept names and role names are finite in S (since S is finite).

3 Correctness

For the correctness proof of the Tableau-algorithm we have to show that

- for every input the Tableau-algorithm terminates
- If no more rules are applicable on a clash-free constraint set S then S is satisfiable
- For every interpretation $\mathcal{I}$ of a satisfiable constraint set S we can find a chain of rule application such that the induced interpretation $\mathcal{I}_I$ is equal to $\mathcal{I}$

First we prove that the tableau algorithm terminates.

Proposition 1. Let C be a concept in negation normal form. Then there is no infinite chain of applications of any tableau rules issuing from $\{x : C\}$.

To prove this we map any derived set S to an element $\Psi(S)$ from a set Q. We then show that the elements in Q can be ordered by a well-founded relation $\prec$. A well-founded relation says that there is no infinite decreasing chain. If we can show that by obtaining a derived set S' from another set S we have $\Psi(S') \prec \Psi(S)$ then the algorithm terminates. The elements in Q are finite multisets of septuples and the elements of the septuples are either integers or mutlisets of integers. For two septuples $q = (q_1, \ldots, q_7)$ and $q' = (q'_1, \ldots, q'_7)$ it holds $q \prec q'$ if for the first i, $1 \leq i \leq 7$, for which q_i and q'_i differs it holds that $q_i \prec q'_i$ (also called lexicographical ordering). For two mutlisets of integers q_i and q'_i it holds $q'_i \prec q_i$ if q'_i can be obtained from q_i by replacing an integer c in q'_i by a finite number of integers which are all smaller than c. The relation $\prec$ for those multisets is also well-founded because we work with integers. That means from a multiset $\{0, \ldots, 0\}$ we can not obtain a smaller multiset because we would have to replace at least one 0 with integers which are smaller.
For a concept C its size $size(C)$ is inductively defined as

- 0, if C is $\bot$
- 1, if C is a primitive concept of $\mathbf{C}$
- $size(\neg C) = size(C)$
- $size(succ(c)) = 1 + \sum_{C \in \mathbf{C} \text{ occurs in c}} size(C)$
- $size(C \sqcap D) = size(C \sqcup D) = size(C) + size(D)$

The number $n_{sc}(x)$ denotes the number of constraints of the form $x : succ(s_1 \subseteq s_2)$ for a variable x. Let y be a successor of x. The number $n_{sc}(x, y)$ denotes the number of set constraints of the form $x : succ(c_1 \subseteq c_2)$ where $(x, y) : s_1 \in S$ and $(x, y) : s_2 \in S$ hold.
The septuples in Q are defined as follows

Definition 12. Let S be a constraint set. The multiset $\Psi(S)$ consist of septuples $\psi_S(x)$ for each variable x. The component of the septuples are structured as follows

- the first component is a non-negative integer $max\{size(C) \mid x : C \in S\}$
- the second component is a multiset of integers containing for each $x : C \sqcap D$, on which the $\sqcap$-rule is applicable, the non-negative integer $size(C \sqcap D)$ (respectively for $C \sqcup D$)
- the third component is a multiset which denotes for every $x : succ(k \leq l)$ the integer $n_{\mathcal{I}_I}(x_{S_k}) \trianglelefteq n_{\mathcal{I}_I}(x_{S_l})$
- the fourth component is a multiset of integers in which for each successor y of x we have $n_{sc}(x) - n_{sc}(x, y)$
- the fifth component denotes the number of all successors of x in S
- the sixth component is a multiset of integers containing for each $x : succ(k \leq n) \in S$ the number of all successors y of x such that we have $(x, y) : k'$, k' occurs in k but for at least one $|s|$ in k we have neither $(x, y) : s \in S$ nor $(x, y) : \neg s \in S$

- the seventh component is the number of successors y of x such that for every role name r with a positive sign we have $(x, y) : r$

Lemma 1. The following properties hold

1. For any concept C we have $size(C) \geq size(NNF(\neg C))$
2. Any variable y in a derived set S has at most one predecessor x in S
3. If $(x, y) : r \in S$ for a $r \in \mathbf{R}$ (and y is a introduced variable) then

$$max\{size(C) \mid x : C \in S\} > max\{size(D) \mid y : D \in S\}$$

Proof.

1. By induction over the number of applications to compute the negation normal form we have $size(C) = size(NNF(\neg C))$. Because $\neg succ(k \geq 0)$ can be replace by $\bot$ which is *smaller* than $\neg succ(k \geq 0)$, we have $size(C) \geq size(NNF(\neg C))$. This can be done because $\neg succ(k \leq 0) = succ(k < 0)$ which is impossible to satisfy and therefore $\neg succ(k \leq 0) = \bot$.
2. If y is a newly introduced variable, then it can only be introduced by exactly one variable x which is y's only predecessor. If two variables are merged together by rule **??** then both variables must have the same predecessor x by the condition of that rule.
3. By the second fact we know that x is the only predecessor of y. When y is introduced by applying **??** on a constraint $x : succ(k \lesseqgtr l)$ then we have $y : C$ for every concept C occurring in l (for $\neg C$ we have $y : \neg C$). We know that $size(succ(k \lesseqgtr l))$ is greater then $size(C) =: max\{size(D) \mid y : D \in S\}$ therefore Lemma 1.3 holds. A new constraint $y : D$ can occur either because rule **??** or **??** are applicable on $y : C$ with $C = D \sqcap D'$ or $C = D \sqcup D'$, which neither raise $max\{size(D) \mid y : D \in S\}$, or because rule **??** is applicable but that also does not raise $max\{size(D) \mid y : D \in S\}$: If rule **??** is applicable on $x : succ(k \leq l)$ then for every added constraint $y : D$ the concept D must occur in k and therefore $size(succ(k \leq)) > size(D)$. If y gets merged together with another variable z, then y and z must have the same predecessor which means that all concept sizes regarding z are also smaller then $max\{size(C) \mid x : C \in S\}$.

□

From the next Lemma we can conclude that the Tableau-algorithm terminates.

Lemma 2. If S' is a derived set obtained from the derived set S, then $\Psi(S') \prec \Psi(S)$

The following proof is sectioned by the definition of obtaining a derived set.

Proof.

1. S' is obtained by the application of rule **??** on $x : C \sqcap D$:
   The first component remains the same because $size(C) < size(C \sqcup D)$ and $size(D) < size(C \sqcap D)$. The second component decreases because rule **??** can not be applied on $x : C \sqcap D$ any more meaning that the corresponding entry in the multiset is removed. If C (or D) happens to be a disjunction ($C' \sqcup D'$) or a conjunction ($C' \sqcap D'$) then the second component also becomes smaller because $size(C')$ and $size(D')$ are always smaller than the disjunction or conjunction of them and therefore also smaller than $size(C \sqcap D)$. Hence the entry for $size(C \sqcap D)$ can be replace by the smaller $size(C' \sqcup D')$ or $size(C' \sqcap D')$.
   Consider now a tuple $\psi_S(y)$ such that $x \neq y$. $\psi_S(y)$ can only be affected if x is a successor of y. The first and second component of $\psi_S(y)$ remain unaffected because both are independent from x. The third component can decrease but never increase: By adding constraint for x the number $n_{sc}(y, x)$ might increases and hence the component decreases. The fourth, fifth and sixth component also remain unchanged because the number of y's successors does not change. The sixth also do not change because we do not add a constraint of the form $(y, x) : s$. Hence $\psi_S(y)$ does not change.
   This means that we can obtain $\Psi(S')$ from $\Psi(S)$ by replacing $\psi_S(x)$ with the smaller tuple $\psi_{S'}(x)$.

2. S' is obtained by the application of rule **??** on $x : C \sqcup D$:
   similar to above

3. S' is obtained by the application of rule **??** on $(x, y) : s$:
   The first and second component remains unchanged. Also the third and fifth component because we do not add new successors. The fourth component can decrease but never increase: By adding a constraint $(x, y) : r, r \in \mathbf{R}$ we can only increase $n_{sc}(x, y)$ and therefore can only decrease the multiset. With a similar reasoning the sixth component can decrease but never increase. The seventh component always decreases because y is no an r-successor of x.
   Let z be a variable such that $z \neq y$. The element $\psi_S(z)$ can only change if z is a predecessor of y. But by Lemma 1.2 that means that $z = x$.
   We can obtained $\Psi(S')$ from $\Psi(S)$ by replacing $\psi_S(x)$ with the smaller $\psi_{S'}(x)$.

4. S' is obtained by the application of rule **??** on $x : succ(k < l)$ or $x : succ(k \leq l)$:
   The first and second component of $\psi_S(x)$ remain unchanged. The third component also remains unchanged: Because we can apply rule **??** we have $l \in \mathbb{N}$ and therefore $n_{\mathcal{I}_I}(x_{S_k}) > l$ which means the integer in this multiset is 0. By merging two successor we have $n_{\mathcal{I}_I}(x_{S_k}) \geq n_{\mathcal{I}_I}(x_{S_l})$ which means the asymmetrical difference $n_{\mathcal{I}_I}(x_{S_k}) \trianglelefteq n_{\mathcal{I}_I}(x_{S_l})$ is still 0. The fourth component can decreases: By merging two successor y_1, y_2 the two corresponding entries in the multiset are removed and a new one is added. The new variable y has all constraints of the two successors which means that for some constraints $x : succ(s_1 \subseteq s_2)$, such that $(x, y_1) : s_1 \in S$ and $(x, y_2) : s_2 \in S$ but $(x, y_1) : s_1 \notin S$ and $(x, y_2) : s_2 \notin S$, we have after the rule application $(x, y) : s_1 \in S$ and $(x, y) : s_2 \in S$ which means that $n_{sc}(x, y) >$

$n_{sc}(x, y_1)$ and $n_{sc}(x, y) > n_{sc}(x, y_2)$. Therefore $n_{sc}(x) - n_{sc}(x, y)$ is smaller than $n_{sc}(x) - n_{sc}(x, y_1)$ or $n_{sc}(x) - n_{sc}(x, y_2)$. The fourth component can not increases because that would mean that by merging two successors we had lost constraints regarding y_1 and y_2. The fifth component decreases because we have one successor less and therefore $\psi_{S'}(x)$ is smaller than $\psi_S(x)$. The new tuple $\psi_{S'}(y)$ is also smaller than $\psi_S(x)$ because y has the same constraints of the two merged successors whose first component are always smaller than the first component of $\psi_S(x)$ because of Lemma 1.3.
No other tuples $\psi_S(z)$ are affected because otherwise z must be a predecessor of y and by Lemma 1.2 $z = x$.
Therefore $\Psi(S')$ can be obtained from $\Psi(S)$ by deleting the tuples of the two merged successors and by replacing $\psi(x)$ with the smaller tuples $\psi_{S'}(x)$ and $\psi_{S'}(y)$.

5. S' is obtained by the application of rule **??** on $x : succ(k \leq l)$ for a successor y and of rule **??**
   After rule **??** we have either $(x, y) : s$ or $(x, y) : s^{\neg}$ for all $|s|$ in k. Whether it is $(x, y) : s$ or $(x, y) : s^{\neg}$ the first two component do not change because we do not add any new constraint regarding x. The third and fifth component also does not change because we do not add any new successors for x. The fourth component might decreases but never increases: By adding constraints we can only increase the number $n_{sc}(x, y)$ which means that $n_{sc}(x) - n_{sc}(x, y)$ decreases. The sixth component of $\psi_S(x)$ decreases because y does not hold the condition of the fifth component any more. Hence $\psi_{S'} \prec \psi_S(x)$.
   For any variable z such that $z \neq y$. The tuple $\psi_S(z)$ is unaffected. It can only be affected by the rules if z is a predecessor of y. But by Lemma 1.2 that would mean that $z = x$.
   Because y is a successor of x we know by Lemma 1.3 that the first component of $\psi_{S'}(y)$ is smaller than the first component of $\psi_{S'}(x)$ and therefore $\psi_{S'}(y) \prec \psi_{S'}(x)$. Since the first component of $\psi_{S'}(x)$ does not change we also have $\psi_{S'}(y) \prec \psi_S(x)$. We can obtained $\Psi(S')$ from $\Psi(S)$ by deleting $\psi_S(y)$ and replacing $\psi_S(x)$ by the two smaller septuples $\psi_{S'}(x)$ and $\psi_{S'}(y)$.

6. S' is obtained by the application of rule **??** on $x : succ(k < l)$, $x : succ(k \leq l)$ or $x : succ(n \, dvd \, l)$ and rule **??**:
   For a set term s which occurs as $|s|$ in l we introduce a new variable y and add $(x, y) : s$. The first two component of $\psi_S(x)$ remains unchanged. Because we can apply this rule we have $n_{\mathcal{I}_I}(x_{S_k}) > n_{\mathcal{I}_I}(x_{S_l})$ and we have no set term s which occurs in k and in l with the same sign. That means that by adding a new successor to l it can never be a successor to k, too. Therefore only $n_{\mathcal{I}_I}(x_{S_l})$ increases which means $n_{\mathcal{I}_I}(x_{S_k}) \trianglelefteq n_{\mathcal{I}_I}(x_{S_l})$ decreases and hence also the third component.
   In S' exists now a new tuple $\psi_{S'}(y)$. But since it was introduced by the constraint $x : succ(c)$, $c \in \{k < l, k \leq l, n \, dvd \, l\}$, the first component of it is always smaller then the first component of $\psi_S(x)$.
   For any variable z such that $z \neq y$. The tuple $\psi_S(z)$ is unaffected. It can only be

affected by the rules if z is a predecessor of y. But by Lemma 1.2 that would mean that $z = x$.
Altogether $\Psi(S')$ can be obtained from $\Psi(S)$ by replacing $\psi_S(x)$ with the two smaller tuples $\psi_{S'}(x)$ and $\psi_{S'}(y)$.

7. S' is obtained by the application of rule **??** on $x : succ(s_1 \subseteq s_2)$ and rule **??**:
After rule **??** S contains $(x, y) : s_2$. Then rule **??** is applied until inapplicable. After rule **??** we can have multiple $(x, y) : r$, $r \in \mathbf{R}$, and/or $y : C$. The first and second component do not change. The third component also does not change because we do not add more successors. The fourth component always decreases because the number $n_{sc}(x, y)$ increases. For any $y : C$ $\psi_S(x)$ remains unchanged but we know that the first component of $\psi'_S(y)$ is smaller then the first component of $\psi_S(x)$ by Lemma 1.3.
For any variable z such that $z \neq y$ the tuple $\psi_S(z)$ is unaffected. It can only be affected by the rules if z is a predecessor of y. But by Lemma 1.2 that would mean that $z = x$.
Therefore $\Psi(S')$ can be obtained from $\Psi(S)$ by deleting $\psi_S(y)$ and by replacing $\psi_S(x)$ with the two smaller septuples $\psi_{S'}(x)$ and $\psi_{S'}(y)$.

□

Lemma 3. If the Tableau-algorithm terminates without a clash then S is satisfiable

Proof. Let $\mathcal{I}_I$ be the induced interpretation of the constraint set S' created by the Tableau-algorithm from S. We construct an assignment $\pi_{\mathcal{I}_I}$ as follows: For every variable x let $\pi_{\mathcal{I}_I}(x) = x^{\mathcal{I}}$. By the definition of induced interpretation $\pi_{\mathcal{I}_I}$ satisfies S'. □

Lemma 4. For every interpretation $\mathcal{I}$ of a satisfiable constraint set S we can find a chain of rule application such that the induced interpretation $\mathcal{I}_I$ is equal to $\mathcal{I}$

Proof. blub? □