A Tableau algorithm for $\mathcal{ALCSCC}$

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Motivation

- Reasoning in data base

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- Satisfiability check for reasoning

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- Satisfiability check for reasoning
- Tableau algorithm for satisfiability check

Tableau Algorithm

Main Idea (for $\mathcal{ALCQ}$):

$$x : A \sqcap B \sqcap \exists r.C$$

x

x must be in A

Tableau Algorithm

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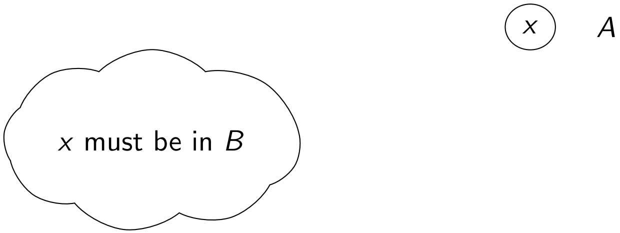

Tableau Algorithm

Main Idea (for $\mathcal{ALCQ}$):

$$x : A \sqcap B \sqcap \exists r.C$$

x $\quad A, B$

x must be in B

Tableau Algorithm

Main Idea (for $\mathcal{ALCQ}$):

$$x : A \sqcap B \sqcap \exists r.C$$

x $\quad A, B$

x must have
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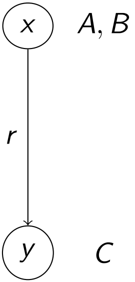

Goal

Tableau algorithm for $\mathcal{ALCSCC}$ concepts

$\mathcal{ALCSCC}$: successors

$\cancel{\exists r.C}$ $\cancel{\forall r.C}$ $\cancel{\leq m\, r.C}$ $\cancel{\geq m\, r.C}$

$$\boxed{succ(c)}$$

c: **set constraint** or **cardinality constraint**

$\mathcal{ALCSCC}$: constraints

set constraint:

- $hasChild \cap Female \subseteq Mother$
- $succ(2\ dvd\ |Edges|) \subseteq succ(|Edges| = |Nodes|)$

cardinality constraint

- $2\ dvd\ |hasLegs|$
- $|Edges| \leq |Nodes|$
- $|succ(C \cap r)| \leq |succ(D)|$

Problem with successors constraints

$$x : \, succ(|s| > 1) \sqcap succ(|r| \leq |s|) \sqcap succ(|r| > |s|)$$

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- endless loop of adding r- and s-successors

Problem with successors constraints

$$x : \ succ(|s| > 1) \sqcap succ(|r| \leq |s|) \sqcap succ(|r| > |s|)$$

- endless loop of adding *r*- and *s*-successors
- blocking?

Problem with blocking

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

s-successors	r-successors
0	0

Problem with blocking

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap \underline{succ(|r| > 1)}$$

s-successors	r-successors
0	1

Problem with blocking

$$x : \underline{succ(2 \cdot |r| \leq 5 \cdot |s|)} \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

s-successors

0

r-successors

1

Problem with blocking

$$x : \underline{succ(2 \cdot |r| \leq 5 \cdot |s|)} \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

s-successors

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r-successors

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QFBAPA formula and solver

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

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$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

QFBAPA formula and solver

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$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$X_r = \{r_1, r_2, \ldots, r_5\} \text{ and } X_s = \{s_1, s_2\}$$

QFBAPA formula and solver

What about

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 10 \text{ and } |X_s| = 4$$

QFBAPA formula and solver

What about

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 100 \text{ and } |X_s| = 40$$

QFBAPA formula and solver

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$\rightarrow$ upper bound

QFBAPA formula and solver

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$$|X_r| = 100 \text{ and } |X_s| = 40$$

$\rightarrow$ upper bound
$\rightarrow$ ILP

Venn regions

Problem: Are the variables disjoint or not?

Venn regions

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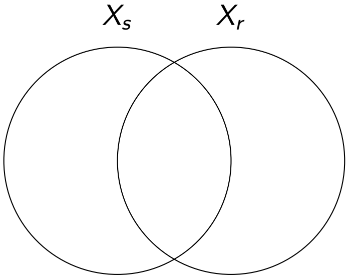
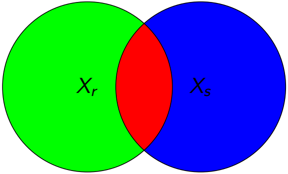
Solution: Venn region



Venn regions

Two variables X_s and X_r $\rightarrow$ four Venn regions:

$v_1 = X_s \cap X_r$
$v_2 = X_s^\neg \cap X_r$
$v_3 = X_s \cap X_r^\neg$
$v_4 = X_s^\neg \cap X_r^\neg$



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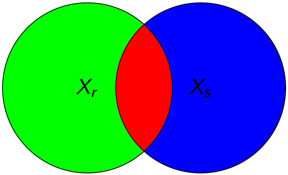

$X_s = v_1 \cup v_3$ and $X_r = v_1 \cup v_2$

Venn regions

Franz Baader [1]:

For every QFBAPA formula ϕ there is a number N, which is polynomial in the size of ϕ and can be computed in polynomial time such, that for every solution σ of ϕ there exists a solution σ' of ϕ such that:

- $|\{v|v \text{ is a Venn region and } \sigma'(v) \neq \emptyset\}| \leq N$
- $\{v|v \text{ is a Venn region and } \sigma'(v) \neq \emptyset\} \subseteq \{v|v \text{ is a Venn region and } \sigma(v) \neq \emptyset\}$

[1] A New Description Logic with Set Constraints and Cardinality Constraints on Role Successors. Springer International Publishing: 43-59, 2017

Venn regions

In short:

2^n Venn regions,	N Venn regions,
n : amount of set variables	N : polynomial in the size of formula
	

Transforming formula into ILP

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$\begin{aligned} -5 \cdot |X_s| + 2 \cdot |X_r| &\leq 0 \\ 5 \cdot |X_s| - 2 \cdot |X_r| &\leq 0 \\ |X_r| &> 1 \end{aligned}$$

Transforming formula into ILP

$$\downarrow$$

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-5 \cdot |X_s| + 2 \cdot |X_r| &\leq 0 \\
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|X_r| &> 1
\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}
-5 \cdot |X_s| + 2 \cdot |X_r| + l_1 &= 0 \\
5 \cdot |X_s| - 2 \cdot |X_r| + l_2 &= 0 \\
|X_r| - l_3 &= 2
\end{aligned}$$

Transforming formula into ILP

$$\downarrow$$

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$$\downarrow$$

$$\begin{aligned} -5 \cdot |v_1 \cup v_3| + 2 \cdot |v_1 \cup v_2| + l_1 &= 0 \\ 5 \cdot |v_1 \cup v_3| - 2 \cdot |v_1 \cup v_2| + l_2 &= 0 \\ |v_1 \cup v_2| - l_3 &= 2 \end{aligned}$$

ILP

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & l_1 & l_2 & l_3 \end{array} \\
\begin{array}{c} -5\cdot|X_s|+2\cdot|X_r|+l_1 \\ 5\cdot|X_s|-2\cdot|X_r|+l_2 \\ |X_r|-l_3 \end{array} &
\begin{pmatrix} -5+2 & 2 & -5 & 0 & 1 & 0 & 0 \\ 5-2 & -2 & 5 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}
\end{array}
\cdot
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}
$$

Upper bound

Christos H. Papadimitriou[2]:

For each ILP $Ax = b$, A $m \times n$ matrix, $x \in \mathbb{N}^n$ and $b \in \mathbb{R}^m$ there exists an upper bound M and a solution $x' \in \{0, \ldots, M\}^n$ such that

$$Ax' = b$$

[2] On the Complexity of Integer Programming. J. ACM,28(4):765-768, Oct. 1981.

Decomposing rules

- $\sqcap$-rule:
  If $x : A \sqcap B$ is in ABox then $x : A$ and $x : B$ must be in ABox
- $\sqcup$-rule:
  If $x : A \sqcup B$ is in ABox then $x : A$ or $x : B$ must be in ABox

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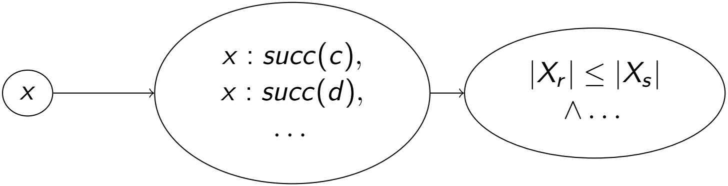
Always apply when possible (higher priority)

successor-rule



rule not applied yet

set of successor constraints

QFBAPA formula

successor-rule

$$|X_r| \leq |X_s| \wedge \dots \longrightarrow \text{calculate upper bound}$$

successor-rule

$|X_r| \leq |X_s| \wedge \ldots$ $\longrightarrow$ calculate upper bound $\longrightarrow$ use QFBAPA solver

successor-rule

use QFBAPA solver

If it returns unsatisfiable

add $x : \perp$

successor-rule

use QFBAPA solver

If it returns unsatisfiable → add $x :\perp$

If it returns a solution → add variables and its constraints

Example: Concept

$$\mathit{fruitcake} : \mathit{succ}(|\mathit{need} \cap \mathit{Cake}| \geq 1) \sqcap$$

$$\mathit{succ}(|\mathit{need} \cap \mathit{succ}(|\mathit{is} \cap (\mathit{Strawberry} \cup \mathit{Tangerine} \cup \mathit{Apple})| \geq 1)| \geq 1)$$

Example: Decomposing

Apply $\sqcap$-rule first:

$$\textit{fruitcake} : \textit{succ}(|\textit{need} \cap \textit{Cake}| \geq 1) \sqcap$$

$$\textit{succ}(|\textit{need} \cap \textit{succ}(|\textit{is} \cap (\textit{Strawberry} \cup \textit{Tangerine} \cup \textit{Apple})| \geq 1)| \geq 1),$$

Example: Decomposing

Apply $\sqcap$-rule first:

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$$\mathit{succ}(|\mathit{need} \cap \mathit{succ}(|\mathit{is} \cap (\mathit{Strawberry} \cup \mathit{Tangerine} \cup \mathit{Apple})| \geq 1)| \geq 1),$$

$$\mathit{fruitcake} : \mathit{succ}(|\mathit{need} \cap \mathit{Cake}| \geq 1), \mathit{fruitcake} : \mathit{succ}(|\mathit{need} \cap$$

$$\mathit{succ}(|\mathit{is} \cap (\mathit{Strawberry} \cup \mathit{Tangerine} \cup \mathit{Apple})| \geq 1)| \geq 1)$$

Consider successor constraints

Let $c = |is \cap (Strawberry \cup Tangerine \cup Apple)| \geq 1$

$$fruitcake : succ(|need \cap Cake| \geq 1),$$
$$fruitcake : succ(|need \cap succ(c)| \geq 1)$$

Consider successor constraints

Let $c = |is \cap (Strawberry \cup Tangerine \cup Apple)| \geq 1$

$$\begin{gathered} fruitcake : succ(|need \cap Cake| \geq 1), \\ fruitcake : succ(|need \cap succ(c)| \geq 1) \end{gathered}$$

$$\downarrow$$

$$\begin{gathered} |X_{need} \cap X_{Cake}| \geq 1 \wedge \\ |X_{need} \cap X_{succ(c)}| \geq 1 \end{gathered}$$

Example: successor constraints

Assume solver returns:
$X_{need} \cap X_{Cake} = \{cake\}$ and $X_{need} \cap X_{succ(c)} = \{fruit\}$

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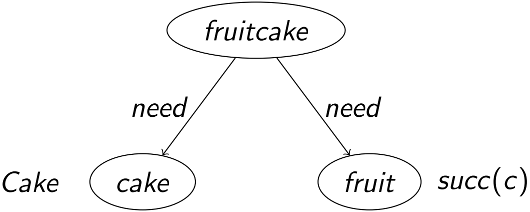

Example: individual name *fruit*

Same procedure for $\mathit{fruit} : \mathit{succ}(c)$:

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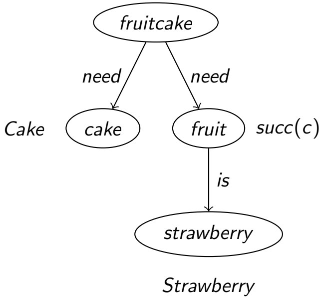

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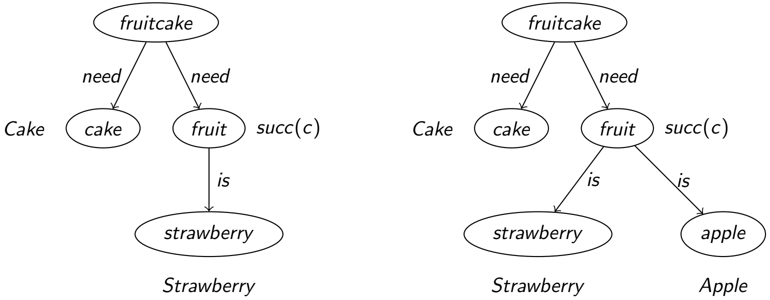

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- 2ExpSpace because of upper bound

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For the future:

- use/find smaller upper bound
- extend tableau for whole knowledge base
- tableau without QFBAPA solver