A Tableau algorithm for $\mathcal{ALCSCC}$

Ryny Khy

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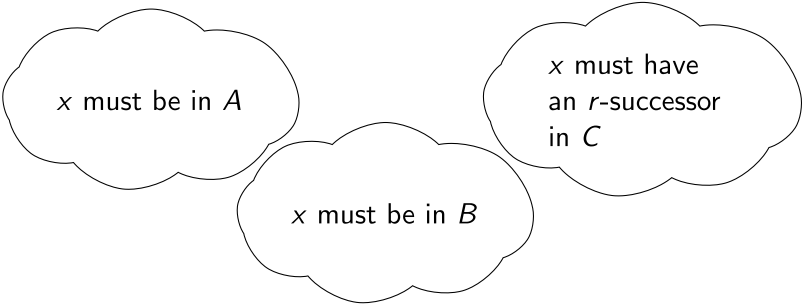
Tableau Algorithm

Main Idea:

$$x : A \sqcap B \sqcap \exists r.C$$



$\mathcal{ALCSCC}$: successors

$\cancel{\exists r.C}$ $\cancel{\forall r.C}$ $\cancel{\leq m\, r.C}$ $\cancel{\geq m\, r.C}$

$$\boxed{succ(c)}$$

c: **set constraint** or a **cardinality constraint**

$\mathcal{ALCSCC}$: constraints

set constraint:

- $r \subseteq s$
- $C \cap r \subseteq D$
- $succ(C \cap r) \subseteq succ(D)$

cardinality constraint

- $2\ dvd\ |r|$
- $|C \cap r| \leq |D|$
- $|succ(C \cap r)| \leq |succ(D)|$

Problem with successors constraints

$$x: \; succ(|s| > 1) \sqcap succ(|r| = |s|) \sqcap succ(|r| > |s|)$$

s-successors

0

r-successors

0

Problem with successors constraints

$$x : \underline{succ(|s| > 1)} \sqcap succ(|r| = |s|) \sqcap succ(|r| > |s|)$$

s-successors	r-successors
1	0

Problem with successors constraints

$$x : \ succ(|s| > 1) \sqcap \underline{succ(|r| = |s|)} \sqcap succ(|r| > |s|)$$

s-successors	r-successors
1	1

Problem with successors constraints

$$x : \; succ(|s| > 1) \sqcap succ(|r| = |s|) \sqcap \underline{succ(|r| > |s|)}$$

s-successors	r-successors
1	2

Problem with blocking

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

s-successors	r-successors
0	0

Problem with blocking

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap \underline{succ(|r| > 1)}$$

s-successors	r-successors
0	1

Problem with blocking

$$x : \underline{succ(2 \cdot |r| \leq 5 \cdot |s|)} \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

s-successors

1

r-successors

1

QFBAPA formula and solver

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

QFBAPA formula and solver

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

$$\downarrow$$

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \land 5 \cdot |X_s| \leq 2 \cdot |X_r| \land |X_r| > 1$$

QFBAPA formula and solver

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

$$\downarrow$$

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 5 \text{ and } |X_s| = 2$$

QFBAPA formula and solver

What about

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 10 \text{ and } |X_s| = 4$$

QFBAPA formula and solver

What about

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 100 \text{ and } |X_s| = 40$$

QFBAPA formula and solver

We have infinite possible solutions

QFBAPA formula and solver

We have infinite possible solutions

Do we need all of them?

QFBAPA formula and solver

We have infinite possible solutions

Do we need all of them?

No: Consider formula as ILP and calculate an upper bound

ILP and upper bound

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$\begin{aligned} -5 \cdot |X_s| + 2 \cdot |X_r| &\leq 0 \\ 5 \cdot |X_s| - 2 \cdot |X_r| &\leq 0 \\ -|X_r| &< -1 \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} -5 \cdot |X_s| + 2 \cdot |X_r| + l_1 &= 0 \\ 5 \cdot |X_s| - 2 \cdot |X_r| + l_2 &= 0 \\ -|X_r| + l_3 &= -1 \end{aligned}$$

ILP and upper bound

Problem: Are the variables disjoint or not?

ILP and upper bound

Problem: Are the variables disjoint or not?

Solution: Venn region

Venn regions

Two variables X_s and X_r $\rightarrow$ four Venn regions:

$$
\begin{aligned}
v_1 &= X_s \cap X_r \\
v_2 &= X_s^{\neg} \cap X_r \\
v_3 &= X_s \cap X_r^{\neg} \\
v_4 &= X_s^{\neg} \cap X_r^{\neg}
\end{aligned}
$$

Venn regions

Two variables X_s and X_r $\rightarrow$ four Venn regions:

$$
\begin{aligned}
v_1 &= X_s \cap X_r \\
v_2 &= X_s^{\neg} \cap X_r \\
v_3 &= X_s \cap X_r^{\neg} \\
v_4 &= X_s^{\neg} \cap X_r^{\neg}
\end{aligned}
$$

$X_s = v_1 \cup v_3$ and $X_r = v_1 \cup v_2$

ILP

$$\begin{array}{rc} & \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & l_1 & l_2 & l_3 \end{array} \\ \begin{array}{r} -5\cdot|X_s|+2\cdot|X_r|+l_1 \\ 5\cdot|X_s|-2\cdot|X_r|+l_2 \\ -|X_r|+l_3 \end{array} & \begin{pmatrix} -5+2 & 2 & -5 & 0 & 1 & 0 & 0 \\ 5-3 & -2 & 5 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$$