

A Tableau algorithm for \mathcal{ALCSCC}

Ryny Khy

January 12, 2021

- 1 Introduction
- 2 Difficulties with \mathcal{ALCSCC}
- 3 Dealing with successor constraints
- 4 Tableau for \mathcal{ALCSCC}
- 5 Conclusion

Motivation

- Reasoning in data base

Motivation

- Reasoning in data base
- Satisfiability check for reasoning

Motivation

- Reasoning in data base
- Satisfiability check for reasoning
- Tableau algorithm for satisfiability check

Tableau Algorithm

Main Idea (for \mathcal{ALCQ}):

$$x : A \sqcap B \sqcap \geq 2r.C$$

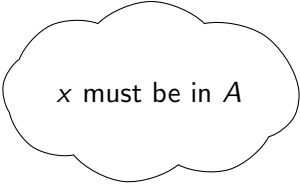
xx must be in A

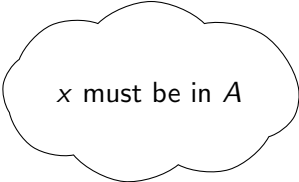
Tableau Algorithm

Main Idea (for \mathcal{ALCQ}):

$$x : A \sqcap B \sqcap \geq 2r.C$$



A



x must be in A

Tableau Algorithm

Main Idea (for \mathcal{ALCQ}):

$$x : A \sqcap B \sqcap \geq 2r.C$$



A

x must be in B

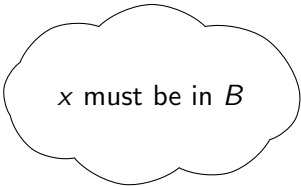
Tableau Algorithm

Main Idea (for \mathcal{ALCQ}):

$$x : A \sqcap B \sqcap \geq 2r.C$$



A, B



x must be in B

Tableau Algorithm

Main Idea (for \mathcal{ALCQ}):

$$x : A \sqcap B \sqcap \geq 2r.C$$



A, B

x has at least
two r -successor
in C

Tableau Algorithm

Main Idea (for \mathcal{ALCQ}):

$$x : A \sqcap B \sqcap \geq 2 r.C$$

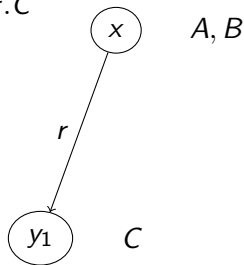
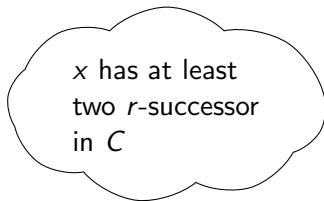
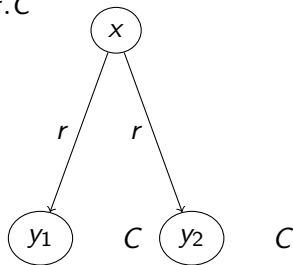


Tableau Algorithm

Main Idea (for \mathcal{ALCQ}):

$$x : A \sqcap B \sqcap \geq 2r.C$$

x has at least
two r -successor
in C



Goal

Tableau algorithm for \mathcal{ALCSCC} concepts

ALCSCC: successors

~~$\exists r.C$~~

~~$\forall r.C$~~

~~$\leq mr.C$~~

~~$\geq mr.C$~~

$$\boxed{\text{succ}(c)}$$

c: set constraint or cardinality constraint

ALCSCC: constraints

set constraint:

- $isChild \cap Female$
 $\subseteq Daughter$

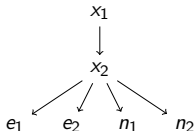
cardinality constraint

- $2\ dvd\ |hasLegs|$
- $|Edges| \leq |Nodes|$

ALCSCC: constraints

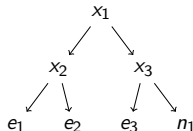
set constraint:

- $\text{succ}(2 \text{ dvd } |Edges|) \subseteq \text{succ}(|Edges| = |Nodes|)$



cardinality constraint

- $\text{succ}(2 \text{ dvd } |Edges|) \leq \text{succ}(|Edges| = |Nodes|)$



Problem with successors constraints

$$x : succ(|s| > 1) \sqcap succ(|r| \leq |s|) \sqcap succ(|r| > |s|)$$

Problem with successors constraints

$$x : succ(|s| > 1) \sqcap succ(|r| \leq |s|) \sqcap succ(|r| > |s|)$$

- endless loop of adding r - and s -successors

Problem with successors constraints

$$x : succ(|s| > 1) \sqcap succ(|r| \leq |s|) \sqcap succ(|r| > |s|)$$

- endless loop of adding r - and s -successors
- blocking?

Problem with blocking

$$x : succ(3 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 3 \cdot |r|) \sqcap succ(|r| > 1)$$

s-successors



r-successors



Problem with blocking

$$x : succ(3 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 3 \cdot |r|) \sqcap \underline{succ(|r| > 1)}$$

s-successors



r-successors



Problem with blocking

$$x : \underline{\text{succ}(3 \cdot |r| \leq 5 \cdot |s|)} \sqcap \text{succ}(5 \cdot |s| \leq 3 \cdot |r|) \sqcap \text{succ}(|r| > 1)$$

s-successors



r-successors



Problem with blocking

$$x : \underline{\text{succ}(3 \cdot |r| \leq 5 \cdot |s|)} \sqcap \text{succ}(5 \cdot |s| \leq 3 \cdot |r|) \sqcap \text{succ}(|r| > 1)$$

s-successors



r-successors



Problem with blocking

$$x : succ(3 \cdot |r| \leq 5 \cdot |s|) \sqcap \underline{succ(5 \cdot |s| \leq 3 \cdot |r|)} \sqcap succ(|r| > 1)$$

s-successors



r-successors



Problem with blocking

$$x : succ(3 \cdot |r| \leq 5 \cdot |s|) \sqcap \underline{succ(5 \cdot |s| \leq 3 \cdot |r|)} \sqcap succ(|r| > 1)$$

s-successors

1

r-successors

2

Problem with blocking

$$x : \underline{\text{succ}(3 \cdot |r| \leq 5 \cdot |s|)} \sqcap \text{succ}(5 \cdot |s| \leq 3 \cdot |r|) \sqcap \text{succ}(|r| > 1)$$

s-successors



r-successors



QFBAPA formula and solver

$$x : \text{succ}(3 \cdot |r| \leq 5 \cdot |s|) \sqcap \text{succ}(5 \cdot |s| \leq 3 \cdot |r|) \sqcap \text{succ}(|r| > 1)$$

QFBAPA formula and solver

$$x : \text{succ}(3 \cdot |r| \leq 5 \cdot |s|) \sqcap \text{succ}(5 \cdot |s| \leq 3 \cdot |r|) \sqcap \text{succ}(|r| > 1)$$



$$3 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 3 \cdot |X_r| \wedge |X_r| > 1$$

QFBAPA formula and solver

$$x : \text{succ}(3 \cdot |r| \leq 5 \cdot |s|) \sqcap \text{succ}(5 \cdot |s| \leq 3 \cdot |r|) \sqcap \text{succ}(|r| > 1)$$



$$3 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 3 \cdot |X_r| \wedge |X_r| > 1$$



$$X_r = \{r_1, r_2, \dots, r_5\} \text{ and } X_s = \{s_1, s_2, s_3\}$$

QFBAPA formula and solver

What about

$$3 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 3 \cdot |X_r| \wedge |X_r| > 1$$



$$|X_r| = 10 \text{ and } |X_s| = 6$$

QFBAPA formula and solver

What about

$$3 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 3 \cdot |X_r| \wedge |X_r| > 1$$



$$|X_r| = 100 \text{ and } |X_s| = 60$$

QFBAPA formula and solver

→ Upper bound

→ ILP

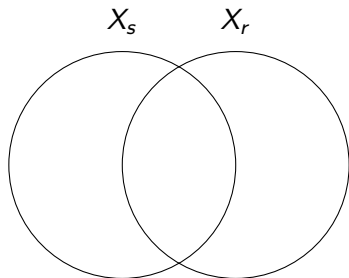
Venn regions

Problem: Are the variables disjoint or not?

Venn regions

Problem: Are the variables disjoint or not?

Solution: Venn region



Venn regions

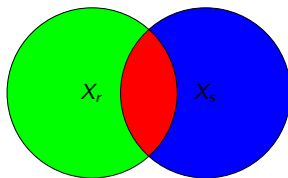
Two variables X_s and $X_r \rightarrow$ four Venn regions:

$$v_1 = X_s \cap X_r$$

$$v_2 = X_s^{\neg} \cap X_r$$

$$v_3 = X_s \cap X_r^{\neg}$$

$$v_4 = X_s^{\neg} \cap X_r^{\neg}$$



Venn regions

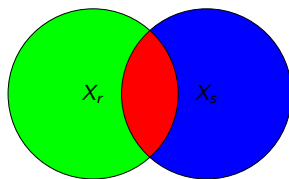
Two variables X_s and $X_r \rightarrow$ four Venn regions:

$$v_1 = X_s \cap X_r$$

$$v_2 = X_s^{\neg} \cap X_r$$

$$v_3 = X_s \cap X_r^{\neg}$$

$$v_4 = X_s^{\neg} \cap X_r^{\neg}$$



$$X_s = v_1 \cup v_3 \text{ and } X_r = v_1 \cup v_2$$

Venn regions

Franz Baader ¹:

For every QFBAPA formula ϕ there is a number N , which is polynomial in the size of ϕ and can be computed in polynomial time such, that for every solution σ of ϕ there exists a solution σ' of ϕ such that:

- $|\{v \mid v \text{ is a Venn region and } \sigma'(v) \neq \emptyset\}| \leq N$
- $\{v \mid v \text{ is a Venn region and } \sigma'(v) \neq \emptyset\} \subseteq \{v \mid v \text{ is a Venn region and } \sigma(v) \neq \emptyset\}$

¹A New Description Logic with Set Constraints and Cardinality Constraints on Role Successors. Springer International Publishing: 43-59, 2017

Venn regions

In short:

2^n Venn regions,

n : amount of set variables



N Venn regions,

N : polynomial in the size of formula



Transforming formula into ILP

$$3 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 3 \cdot |X_r| \wedge |X_r| > 1$$



$$-5 \cdot |X_s| + 3 \cdot |X_r| \leq 0$$

$$5 \cdot |X_s| - 3 \cdot |X_r| \leq 0$$

$$|X_r| > 1$$

Transforming formula into ILP



$$-5 \cdot |X_s| + 3 \cdot |X_r| \leq 0$$

$$5 \cdot |X_s| - 3 \cdot |X_r| \leq 0$$

$$|X_r| > 1$$



$$-5 \cdot |X_s| + 3 \cdot |X_r| + l_1 = 0$$

$$5 \cdot |X_s| - 3 \cdot |X_r| + l_2 = 0$$

$$|X_r| - l_3 = 2$$

Transforming formula into ILP



$$-5 \cdot |X_s| + 3 \cdot |X_r| + l_1 = 0$$

$$5 \cdot |X_s| - 3 \cdot |X_r| + l_2 = 0$$

$$|X_r| - l_3 = 2$$



$$-5 \cdot |v_1 \cup v_3| + 3 \cdot |v_1 \cup v_2| + l_1 = 0$$

$$5 \cdot |v_1 \cup v_3| - 3 \cdot |v_1 \cup v_2| + l_2 = 0$$

$$|v_1 \cup v_2| - l_3 = 2$$

ILP

$$\begin{array}{l}
 -5 \cdot |X_s| + 2 \cdot |X_r| + l_1 \\
 5 \cdot |X_s| - 2 \cdot |X_r| + l_2 \\
 |X_r| - l_3
 \end{array}
 \begin{pmatrix}
 & v_1 & v_2 & v_3 & v_4 & l_1 & l_2 & l_3 \\
 -5 & +3 & 3 & -5 & 0 & 1 & 0 & 0 \\
 5 & -3 & -3 & 5 & 0 & 0 & 1 & 0 \\
 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1
 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Upper bound

Christos H. Papadimitriou²:

For each ILP $Ax = b$, A $m \times n$ matrix, $x \in \mathbb{N}^n$ and $b \in \mathbb{R}^m$ there exists an upper bound M and a solution $x' \in \{0, \dots, M\}^n$ such that

$$Ax' = b$$

²On the Complexity of Integer Programming. J. ACM, 28(4):765-768, Oct. 1981.

Tableau for \mathcal{ALCSCC}

Now we consider the rules for the tableau algorithm.

Decomposing rules

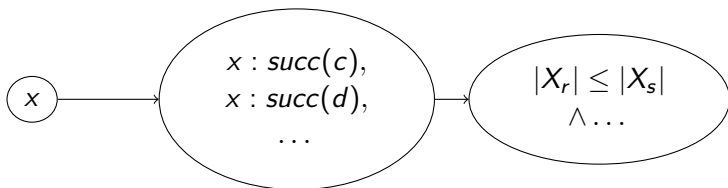
- \sqcap -rule:
If $x : A \sqcap B$ is in ABox then $x : A$ and $x : B$ must be in ABox
- \sqcup -rule:
If $x : A \sqcup B$ is in ABox then $x : A$ or $x : B$ must be in ABox

Decomposing rules

- \sqcap -rule:
If $x : A \sqcap B$ is in ABox then $x : A$ and $x : B$ must be in ABox
- \sqcup -rule:
If $x : A \sqcup B$ is in ABox then $x : A$ or $x : B$ must be in ABox

Always apply when possible (higher priority)

successor-rule

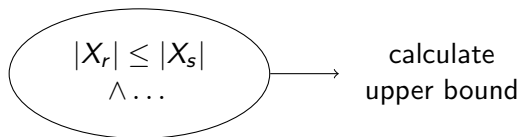


rule not
applied yet

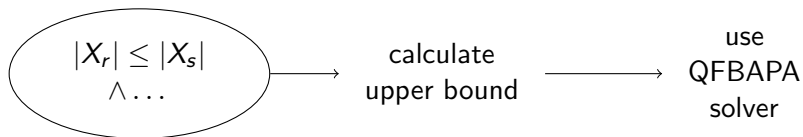
set of
successor constraints

QFBAPA formula

successor-rule



successor-rule



successor-rule

use QFBAPA solver

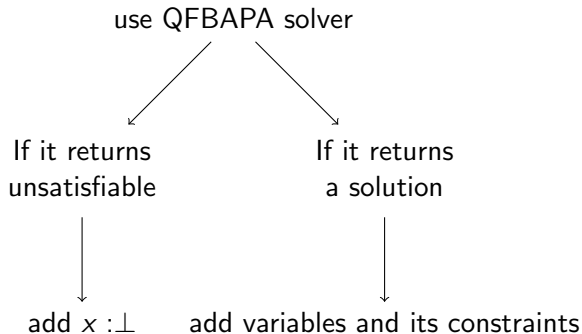


If it returns
unsatisfiable



add $x : \perp$

successor-rule



Example: Concept

pancake : $\text{succ}(|\text{contains} \cap \text{Milk}| \geq |\text{contains} \cap \text{Flavour}|) \sqcap$

$\text{succ}(|\text{contains} \cap \text{Milk}| \leq 2 \cdot |\text{contains} \cap \text{Flavour}|) \sqcap$

$\text{succ}(|\text{contains} \cap \text{Egg}| \geq 1)$

Example: Decomposing

We use the \sqcap -rule twice and add

Example: Decomposing

We use the \sqcap -rule twice and add

$$pancake : succ(|contains \cap Milk| \geq |contains \cap Flavour|),$$

$$pancake : succ(|contains \cap Milk| \leq 2 \cdot |contains \cap Flavour|),$$

$$pancake : succ(|contains \cap Egg| \geq 1)$$

Example: *successor*-rule

$$\begin{aligned} & succ(|contains \cap Milk| \geq |contains \cap Flavour|) \\ & succ(|contains \cap Milk| \leq 2 \cdot |contains \cap Flavour|) \\ & succ(|contains \cap Egg| \geq 1) \end{aligned}$$

Example: *successor*-rule

$$\begin{aligned} & succ(|contains \cap Milk| \geq |contains \cap Flavour|) \\ & succ(|contains \cap Milk| \leq 2 \cdot |contains \cap Flavour|) \\ & succ(|contains \cap Egg| \geq 1) \end{aligned}$$



$$\begin{aligned} & |X_{contains} \cap X_{Milk}| \geq |X_{contains} \cap X_{Flavour}| \wedge \\ & |X_{contains} \cap X_{Milk}| \leq 2 \cdot |X_{contains} \cap X_{Flavour}| \wedge \\ & |X_{contains} \cap X_{Egg}| \geq 1 \end{aligned}$$

Example: ILP

- 4 set variables $\rightarrow 2^4 = 16$ Venn regions

Example: ILP

- 4 set variables $\rightarrow 2^4 = 16$ Venn regions
- BUT: every successor is a *contains*-successor

Example: ILP

- 4 set variables $\rightarrow 2^4 = 16$ Venn regions
- BUT: every successor is a *contains*-successor
- Assume *Milk*, *Flavour* and *Egg* are disjoint

Example: ILP

- 4 set variables $\rightarrow 2^4 = 16$ Venn regions
- BUT: every successor is a *contains*-successor
- Assume *Milk*, *Flavour* and *Egg* are disjoint

$$\begin{aligned} |X_{\text{contains}} \cap X_{\text{Milk}}| - |X_{\text{contains}} \cap X_{\text{Flavour}}| - l_1 &= 0 \\ |X_{\text{contains}} \cap X_{\text{Milk}}| - 2 \cdot |X_{\text{contains}} \cap X_{\text{Flavour}}| + l_2 &= 0 \\ |X_{\text{contains}} \cap X_{\text{Egg}}| - l_3 &= 1 \end{aligned}$$

Example: ILP

$$\begin{array}{l}
 \text{Milk} > \text{Flavour} \\
 \text{not too much milk} \\
 \text{Eggs}
 \end{array}
 \begin{pmatrix}
 X_M & X_F & X_E & X_c & l_1 & l_2 & l_3 \\
 1 & -1 & 0 & 1 & -1 & 0 & 0 \\
 1 & -2 & 0 & 1 & 0 & 1 & 0 \\
 0 & 0 & 1 & 1 & 0 & 0 & -1
 \end{pmatrix}
 \cdot
 \begin{pmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
 x_6 \\
 x_7
 \end{pmatrix}
 =
 \begin{pmatrix}
 0 \\
 0 \\
 1
 \end{pmatrix}$$

Example: Solution of the solver

Assume the solver returns

$$X_{Milk} = \{cup_m\},$$

$$X_{Flavour} = \{cup_f\}$$

$$X_{egg} = \{egg1, egg2\} \text{ and}$$

$$X_{contains} = X_{Milk} \cup X_{Flavour} \cup X_{Egg}$$

Example: Solution of the solver

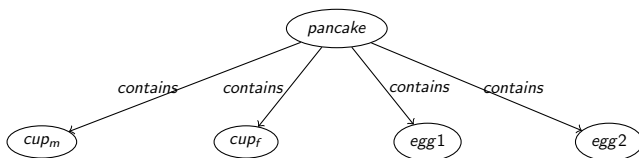
Assume the solver returns

$$X_{Milk} = \{cup_m\},$$

$$X_{Flavour} = \{cup_f\}$$

$$X_{egg} = \{egg1, egg2\} \text{ and}$$

$$X_{contains} = X_{Milk} \cup X_{Flavour} \cup X_{Egg}$$



- Tableau for $ALCSCC$
- 2ExpSpace because of upper bound

- Tableau for *ALCSCC*
- 2ExpSpace because of upper bound

For the future:

- use/find smaller upper bound
- extend tableau for whole knowledge base
- tableau without QFBAPA solver