1 Preliminaries

Let $\mathbf{C}$ be a set of concept names and $\mathbf{R}$ be a set of role names such that $\mathbf{C}$ and $\mathbf{R}$ are disjoint.
$\mathcal{ALCQ}$ concepts are:

- all concept names
- if C, D are concepts, r a role and $n \in \mathbb{N}$ then:
  - $\neg C$
  - $C \sqcup D$
  - $C \sqcap D$
  - $\geq n\, r.C$
  - $\leq n\, r.C$

$\mathcal{ALC}$ is sublogic of $ALCQ$

$$\forall r.C = \leq 0\, r.\neg C \qquad \exists r.C = \geq 1\, r.C$$

$QFBAPA$: Let T be a set of symbols

- set terms over T are:
  - empty set $\emptyset$ and universal set $\mathcal{U}$
  - every set symbol in T
  - if s, t are set terms then also $s \cap t$, $s \cup t$ and $s^{\neg}$
- set constraints over T are
  - $s \subseteq t$ and $s \not\subseteq t$
  - $s = t$ and $s \neq t$

  where s, t are set terms
- cardinality terms over T are:
  - every number $n \in \mathbb{N}$
  - $|s|$ if s is a set term
  - if k, l are cardinality terms then also $k + l$ and $n \cdot k$, $n \in \mathbb{N}$
- cardinality constraints over T are:
  - $k = l$ and $k \neq l$
  - $k < l$ and $k \not< l$
  - $k \leq l$ and $k \not\leq l$
  - $n\ dvd\ k$ and $n\ \neg dvd\ k$

where k, l are cardinality terms and $n \in \mathbb{N}$

$\mathcal{ALCSCC}$ concepts are:

- all concept names
- if C, D are concepts then:
  - $\neg C$
  - $C \sqcup D$
  - $C \sqcap D$
- $succ(c)$ if c is a set or cardinality constraint over $\mathcal{ALCSCC}$ concepts and role names

$\mathcal{ALCQ}$ is sublogic of $ALCSCC$:

$$\leq n\, r.\neg C = succ(|r \cap C| \leq n) \qquad \geq n\, r.\neg C = succ(|r \cap C| \geq n)$$

The set S is a set of constraints of the form:

$$x : C \qquad x.r.y \qquad (x, y) : s$$

where C is a concept, r a role name, s a set term and x, y variables. The constraint $(x, y) : s$ denotes that y is a successor of x, while y must hold s

2 Tableau

Definition 1 (Merge)**.** *Merging* y_1 and y_2 results in one variable y such that for each $i \in \{1, 2\}$

- $x.r.y_i$, we add $x.r.y$
- $y_i.r.z$, we add $y.r.z$
- $(x, y_i) : s$, we add $(x, y) : s$
- $y_i : C$, we add $y : C$

Delete all constraints where y_1 and/or y_2 occur.

Note that by merging two successors other constraints might become violated:

$$\begin{aligned} S = \{&x : succ(|r \cap A| = 1) \sqcap succ(|r \cap B| = 1) \sqcap succ(|r| > 1), \\ &y_1 : A, y_2 : B, x.r.y_1, x.r.y_2\} \end{aligned} \tag{1}$$

If we merge y_1 and y_2 then the constraint $x : succ(|r| > 1)$ which was satisfied becomes violated. But not only constraints regarding x might become violated after merging two successors of x:

$$S = \{x : succ(|r| \leq 1), x.r.y_1, x.r.y_2,\\ y_1 : succ(|s| \leq 1), y_2 : succ(|s| \leq 1), y_1.s.z_1, y_2.s.z_2\} \tag{2}$$

We see that the first constraint is violated and therefore merging y_1 and y_2 would solve the problem but on the other hand the constraints regarding y_1 and y_2 become violated:

$$S = \{x : succ(|r| \leq 1), x.r.y, y : succ(|s| \leq 1), y.s.z_1, y.s.z_2\} \tag{3}$$

To solve this problem we merge z_1 and z_2.

Definition 2 (Tableau)**.** Let S be a set of constraints. Conjunction binds stronger than disjunction: $s \cup t \cap u = s \cup (t \cap u)$.

1. $\sqcap$-rule: In S is $x : C_1 \sqcap C_2$ but not both $x : C_1$ and $x : C_2$
   $\rightarrow S := S \cup \{x : C_1, x : C_2\}$
2. $\sqcup$-rule: In S is $x : C_1 \sqcup C_2$ but neither $x : C_1$ or $x : C_2$
   $\rightarrow S := S \cup \{x : C_1\}$ or $S := S \cup \{x : C_2\}$
3. *choose*-rule: In S are $x : succ(k \leq l)$, $y : C$ or $x.r.y$ and
   C or r occur in k but $(x, y) : k \notin S$
   $\rightarrow S := S \cup \{(x, y) : k\}$ or $S := S \cup \{(x, y) : \neg k\}$
4. *cardinality*-rule: In S are either
   $x : succ(k = l)$ and $k > l$,
   $x : succ(k \leq l)$ and $k > l$ or
   $x : succ(k < l)$ and $k \geq l$ or
   $x : succ(n\, dvd\, l)$ and $mod(l, n) \neq 0$ then
   a) if there is a set term s in l
      $\rightarrow$ introduce new variable y and $S := S \cup \{(x, y) : x\}$
   b) if $l \in \mathbb{N}$ does not contain a set term then merge two successor $y_1 \neq y_2$ of x for which $(x, y_1) : k \in S$ and $(x, y_2) : k \in S$ if no other constraints regarding x become violated
5. *set*-rule: In S are $x : succ(c_1 \subseteq c_2)$ and $(x, y) : c_1$ but not $(x, y) : c_2$
   $\rightarrow S := S \cup \{(x, y) : c_2\}$
6. *set.term*-rule (Repeat until inapplicable): In S is $(x, y) : s$ and
   a) $s = s_1 \cap s_2$ but $\{(x, y) : s_1,\ (x, y) : s_2\} \not\subseteq S$ then
      $\rightarrow S := S \cup \{(x, y) : s_1,\ (x, y) : s_2\}$
   b) $s = s_1 \cup s_2$ and neither $\{(x, y) : s_1\} \subseteq S$ nor $S\{(x, y) : s_2\} \subset S$ then
      $\rightarrow$ either $S := S \cup \{(x, y) : s_1\}$ or $S := S \cup \{(x, y) : s_2\}$

c) $s = r$ and $x.r.y \notin S$ then
$\rightarrow S := S \cup \{x.r.y\}$

d) $s = C$ and $y : C \notin S$, where C is a $\mathcal{ALCSCC}$ concepts then
$\rightarrow S := S \cup \{y : C\}$

Note that:

- $4b$ is never applicable for $n\, dvd\, l$
- $n_1\, dvd\, n_2 \cdot l$ and $n_1 \neg dvd\, n_2$ then $n_1\, dvd\, l$ eventually

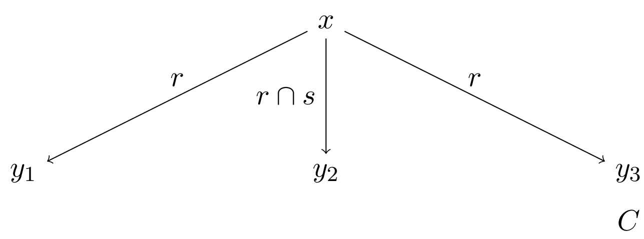
Example for $4b$:

$$S = \{x : succ(|r| = 1) \sqcap succ(|r \cap s| = 1) \sqcap succ = (|r \cap C| = 1|\}$$

After rule 1 (two times):

$$\begin{aligned} S = \{&x : succ(|r| = 1) \sqcap succ(|r \cap s| = 1) \sqcap succ = (|r \cap C| = 1| \\ &x : succ(|r| = 1), x : succ(|r \cap s| = 1), x : succ = (|r \cap C| = 1|\} \end{aligned}$$

If we try to satisfy at least two of the new constraints by the Tableau-algorithm above we end up with at least one constraint being violated. Let say we use the rules on the three new constraints sequentially. Then we have



After using rule 4b two times we have the variable x and its only $r \cap s$-successor y which is of the concept C. We could use this rule because we do not violate any other constraints. If we look again on the first example

$$S = \{x : succ(|r \cap A| = 1) \sqcap succ(|r \cap B| = 1) \sqcap succ(|r| > 1), y_1 : A, y_2 : B, x.r.y_1, x.r.y_2\}$$

we see that we can not use rule 4b here, because otherwise $succ(|r| > 1)$ becomes violated.