A Tableau algorithm for $\mathcal{ALCSCC}$

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Tableau Algorithm

Main Idea:

$$x : A \sqcap B \sqcap \exists r.C$$

x must be in A

x must have
an r-successor
in C

x must be in B

Tableau Algorithm

Main Idea:

$$x : A \sqcap \neg A \sqcap \exists r.C$$

x must be in A

x must not be in B

x must have an r-successor in C

$\mathcal{ALCSCC}$: successors

$$\cancel{\exists r.C} \qquad \cancel{\forall r.C} \qquad \cancel{\leq m\, r.C} \qquad \cancel{\geq m\, r.C}$$

$$\boxed{succ(c)}$$

c: **set constraint** or **cardinality constraint**

$\mathcal{ALCSCC}$: constraints

set constraint:

- $r \subseteq s$
- $C \cap r \subseteq D$
- $succ(C \cap r) \subseteq succ(D)$

cardinality constraint

- $2\, dvd\, |r|$
- $|C \cap r| \leq |D|$
- $|succ(C \cap r)| \leq |succ(D)|$

Problem with successors constraints

$$x : succ(|s| > 1) \sqcap succ(|r| = |s|) \sqcap succ(|r| > |s|)$$

s-successors   r-successors

(0)   (0)

Problem with successors constraints

$$x : \underline{succ(|s| > 1)} \sqcap succ(|r| = |s|) \sqcap succ(|r| > |s|)$$

s-successors

r-successors

$\left(1\right)$

$\left(0\right)$

Problem with successors constraints

$$x : \; succ(|s| > 1) \sqcap \underline{succ(|r| = |s|)} \sqcap succ(|r| > |s|)$$

s-successors   r-successors

Problem with successors constraints

$$x : succ(|s| > 1) \sqcap succ(|r| = |s|) \sqcap \underline{succ(|r| > |s|)}$$

s-successors   r-successors

$\bigcirc 1$ $\bigcirc 2$

Problem with blocking

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

s-successors

r-successors

$\left(\,0\,\right)$ $\qquad$ $\left(\,0\,\right)$

Problem with blocking

$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap \underline{succ(|r| > 1)}$

s-successors

r-successors

$\begin{pmatrix} 0 \end{pmatrix}$

$\begin{pmatrix} 1 \end{pmatrix}$

Problem with blocking

$$x : \underline{succ(2 \cdot |r| \leq 5 \cdot |s|)} \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

s-successors

r-successors

$\left(1\right)$

$\left(1\right)$

QFBAPA formula and solver

$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$

QFBAPA formula and solver

$$x : succ(2 \cdot |r| \le 5 \cdot |s|) \sqcap succ(5 \cdot |s| \le 2 \cdot |r|) \sqcap succ(|r| > 1)$$

$$\downarrow$$

$$2 \cdot |X_r| \le 5 \cdot |X_s| \wedge 5 \cdot |X_s| \le 2 \cdot |X_r| \wedge |X_r| > 1$$

QFBAPA formula and solver

$$x : succ(2 \cdot |r| \leq 5 \cdot |s|) \sqcap succ(5 \cdot |s| \leq 2 \cdot |r|) \sqcap succ(|r| > 1)$$

$$\downarrow$$

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 5 \text{ and } |X_s| = 2$$

QFBAPA formula and solver

What about

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 10 \text{ and } |X_s| = 4$$

QFBAPA formula and solver

What about

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$|X_r| = 100 \text{ and } |X_s| = 40$$

QFBAPA formula and solver

We have infinite possible solutions

QFBAPA formula and solver

We have infinite possible solutions

Do we need all of them?

QFBAPA formula and solver

We have infinite possible solutions

Do we need all of them?

No: Consider formula as ILP and calculate an upper bound

ILP and upper bound

$$2 \cdot |X_r| \leq 5 \cdot |X_s| \wedge 5 \cdot |X_s| \leq 2 \cdot |X_r| \wedge |X_r| > 1$$

$$\downarrow$$

$$-5 \cdot |X_s| + 2 \cdot |X_r| \leq 0$$
$$5 \cdot |X_s| - 2 \cdot |X_r| \leq 0$$
$$-|X_r| < -1$$

$$\downarrow$$

$$-5 \cdot |X_s| + 2 \cdot |X_r| + l_1 = 0$$
$$5 \cdot |X_s| - 2 \cdot |X_r| + l_2 = 0$$
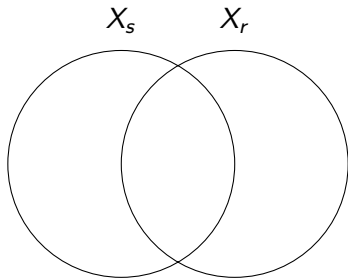$$-|X_r| + l_3 = -1$$

ILP and upper bound

Problem: Are the variables disjoint or not?

ILP and upper bound

Problem: Are the variables disjoint or not?

Solution: Venn region

Venn regions

Two variables X_s and $X_r \rightarrow$ four Venn regions:

$$v_1 = X_s \cap X_r$$
$$v_2 = X_s^\neg \cap X_r$$
$$v_3 = X_s \cap X_r^\neg$$
$$v_4 = X_s^\neg \cap X_r^\neg$$

Venn regions

Two variables X_s and $X_r \to$ four Venn regions:

$$v_1 = X_s \cap X_r$$
$$v_2 = X_s^\neg \cap X_r$$
$$v_3 = X_s \cap X_r^\neg$$
$$v_4 = X_s^\neg \cap X_r^\neg$$

$X_s = v_1 \cup v_3$ and $X_r = v_1 \cup v_2$

ILP

$$
\begin{array}{c}
\begin{array}{ccccccc}
v_1 & v_2 & v_3 & v_4 & l_1 & l_2 & l_3
\end{array} \\
\begin{array}{c}
-5 \cdot |X_s| + 2 \cdot |X_r| + l_1 \\
5 \cdot |X_s| - 2 \cdot |X_r| + l_2 \\
-|X_r| + l_3
\end{array}
\left(
\begin{array}{ccccccc}
-5+2 & 2 & -5 & 0 & 1 & 0 & 0 \\
5-3 & -2 & 5 & 0 & 0 & 1 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 1
\end{array}
\right) \cdot
\left(
\begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{array}
\right)
\end{array}
$$

$$
= \left(\begin{array}{ccc} 0 & 0 & -1 \end{array} \right)
$$

ILP and upper bound

Christos H. Papadimitriou[1]:

There exists an upper bound M such that for each ILP $Ax = b$, A
a $m \times n$ matrix, $x \in \mathbb{N}^n$ and $b \in \mathbb{R}^m$ there exists a solution
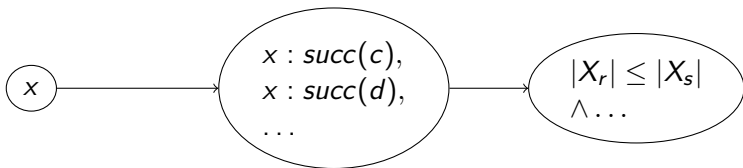$$x' \in \{0, \ldots, M\}^n.$$

[1] On the Complexity of Integer Programming. J. ACM,28(4):765-768, Oct. 1981.

Tableau algorithm

- $\sqcap$-rule:
  If $x : A \sqcap B$ is in ABox then $x : A$ and $x : B$ must be in ABox
- $\sqcup$-rule:
  If $x : A \sqcup B$ is in ABox then $x : A$ or $x : B$ must be in ABox
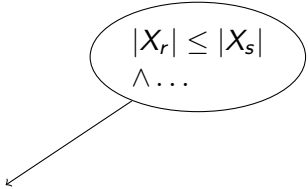
Tableau algorithm

- *successor*-rule:

rule not applied yet   set of successor assertions   QFBAPA formula

Tableau algorithm

- *successor*-rule:

$$\boxed{\begin{array}{c} |X_r| \leq |X_s| \\ \wedge \ldots \end{array}}$$

calculate upper bound

Tableau algorithm

- *successor*-rule:

$$|X_r| \leq |X_s|$$
$$\wedge \dots$$

calculate upper bound

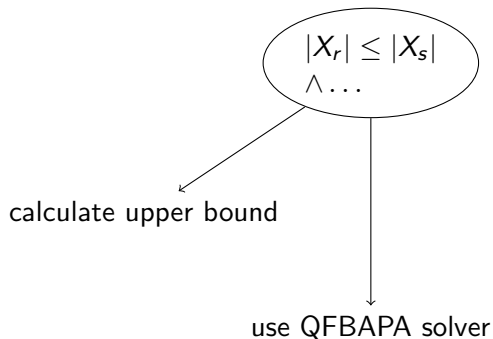use QFBAPA solver

Tableau algorithm

- *successor*-rule:

$$|X_r| \leq |X_s|$$
$$\wedge \dots$$

calculate upper bound
add variables accordingly

use QFBAPA solver