Eastern Economy Edition

DISCRETE MATHEMATICAL STRUCTURES

Sixth Edition

PHI

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25. Complete the following statement. A generic Venn diagram for three sets has ______ regions. Describe them in words.

26. **(a)** If $A = \{3, 7\}$, find $P(A)$.

(b) What is $|A|$? **(c)** What is $|P(A)|$?

27. If $P(B) = \{\{\ \}, \{m\}, \{n\}, \{m, n\}\}$, then find B.

28. **(a)** If $A = \{3, 7, 2\}$, find $P(A)$.

(b) What is $|A|$? **(c)** What is $|P(A)|$?

29. If $P(B) = \{\{a\}, \{\ \}, \{c\}, \{b, c\}, \{a, b\}, \ldots\}$ and $|P(B)| = 8$, then $B =$ ______.

In Exercises 30 through 32, draw a Venn diagram that represents these relationships.

30. $A \subseteq B$, $A \subseteq C$, $B \not\subseteq C$, and $C \not\subseteq B$

31. $x \in A$, $x \in B$, $x \notin C$, $y \in B$, $y \in C$, and $y \notin A$

32. $A \subseteq B$, $x \notin A$, $x \in B$, $A \not\subseteq C$, $y \in B$, $y \in C$

33. Describe all the subset relationships that hold for the sets given in Example 3.

34. Show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

35. The statement about sets in Exercise 34 can be restated as "Any subset of ______ is also a subset of any set that contains ______."

36. Suppose we know that set A has n subsets, $S_1, S_2, \ldots, S_n$. If set B consists of the elements of A and one more element so $|B| = |A| + 1$, show that B must have $2n$ subsets.

37. Compare the results of Exercises 12, 13, 26, and 28 and complete the following: Any set with two elements has ______ subsets. Any set with three elements has ______ subsets.

(d) $C \cap \{\ \}$ (e) $C \cup D$ (f) $C \cap D$

In Exercises 5 through 8, let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 4, 6, 8\}$, $B = \{2, 4, 5, 9\}$, $C = \{x \mid x$ *is a positive integer and* $x^2 \leq 16\}$, *and* $D = \{7, 8\}$.

5. Compute

(a) $A \cup B$ (b) $A \cup C$ (c) $A \cup D$

(d) $B \cup C$ (e) $A \cap C$ (f) $A \cap D$

(g) $B \cap C$ (h) $C \cap D$

6. Compute

(a) $A - B$ (b) $B - A$ (c) $C - D$

(d) $\overline{C}$ (e) $\overline{A}$ (f) $A \oplus B$

(g) $C \oplus D$ (h) $B \oplus C$

7. Compute

(a) $A \cup B \cup C$ (b) $A \cap B \cap C$

(c) $A \cap (B \cup C)$ (d) $(A \cup B) \cap D$

(e) $\overline{A \cup B}$ (f) $\overline{A \cap B}$

8. Compute

(a) $B \cup C \cup D$ (b) $B \cap C \cap D$

(c) $A \cup A$ (d) $A \cap \overline{A}$

(e) $A \cup \overline{A}$ (f) $A \cap (\overline{C} \cup D)$

In Exercises 9 and 10, let $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, c, f, g\}$, $B = \{a, e\}$, *and* $C = \{b, h\}$.

9. Compute

(a) $\overline{A}$ (b) $\overline{B}$ (c) $\overline{A \cup B}$

(d) $\overline{A \cap B}$ (e) $\overline{U}$ (f) $A - B$

(c) $w \in$ ...

13. Identify th...

(a) $x \in A$...

(c) $z \in A$...

14. Describe ... unions an... descriptio...

15. Let A, ... $|C| = 6$, ... and $|B \cap$ ...

In Exercises ...

16. (a) $A =$ ...

(b) $A =$ ...

17. (a) $A =$ ...

(b) $A =$ ...

18. (a) $A =$ ... $B =$ ...

(b) $A =$ ... $B =$ ...

36. Choose $w \in D \cap (E \cup F)$.
(a) $w \in D$ (b) $w \in E$ (c) $w \in F$
(d) $w \notin D$ (e) $w \in F \cup E$
(f) $w \in (D \cap E) \cup (D \cap F)$

37. Choose $t \in \overline{D \cap E}$.
(a) $t \in D$ (b) $t \in E$ (c) $t \notin D$
(d) $t \notin E$ (e) $t \in D \cup E$

38. Choose $x \in \overline{A} \cup (B \cap C)$.
(a) $x \in A$ (b) $x \in B$ (c) $x \in C$
(d) $x \in A \cup B$ (e) $x \in (\overline{A} \cup B) \cap (\overline{A} \cup C)$

39. Complete the following proof that $A \cap B \subseteq A$. Suppose $x \in A \cap B$. Then x belongs to ______. Thus $A \cap B \subseteq A$.

40. (a) Draw a Venn diagram to represent the situation $C \subseteq A$ and $C \subseteq B$.
(b) To prove $C \subseteq A \cup B$, we should choose an element from which set?
(c) Prove that if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cup B$.

41. (a) Draw a Venn diagram to represent the situation $A \subseteq C$ and $B \subseteq C$.
(b) To prove $A \cup B \subseteq C$, we should choose an element from which set?
(c) Prove that if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

42. Prove that $A - (A - B) \subseteq B$.

43. Suppose that $A \oplus B = A \oplus C$. Does this guarantee that $B = C$? Justify your conclusion.

44. Prove that $A - B = A \cap \overline{B}$.

45. If $A \cup B = A \cup C$, must $B = C$? Explain.

46. If $A \cap B = A \cap C$, must $B = C$? Explain.

47. Prove that if $A \subseteq B$ and $C \subseteq D$, then $A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$.

48. When is $A - B = B - A$? Explain.

49. Explain the last term in the sum in Theorem 3. Why is $|A \cap B \cap C|$ added and $|B \cap C|$ subtracted?

50. Write the four-set version of Theorem 3; that is, $|A \cup B \cup C \cup D| = \cdots$.

51. Describe in words the n-set version of Theorem 3.

1.3 Sequences

Some of the most important sets arise in connection with sequences. A **sequence** is simply a list of objects arranged in a definite order; a first element, second element, third element, and so on. The list may stop after n steps, $n \in \mathbb{N}$, or it may go on forever. In the first case we say that the sequence is **finite**, and in the second case we say that it is **infinite**. The elements may all be different, or some may be repeated.

Example 1 The sequence 1, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1 is a finite sequence with repeated items. The digit zero, for example, occurs as the second, third, fifth, seventh, and eighth elements of the sequence. ◆

Example 2 The list 3, 8, 13, 18, 23, ... is an infinite sequence. The three dots in the expression mean "and so on," that is, continue the pattern established by the first few elements. ◆

Example 3 Another infinite sequence is 1, 4, 9, 16, 25, ..., the list of the squares of all positive integers. ◆

It may happen that how a sequence is to continue is not clear from the first few terms. Also, it may be useful to have a compact notation to describe a sequence. Two kinds of formulas are commonly used to describe sequences. In Example 2, a natural description of the sequence is that successive terms are produced by adding 5 to the previous term. If we use a subscript to indicate a term's position in the sequence, we can describe the sequence in Example 2 as $a_1 = 3$, $a_n = a_{n-1} + 5$, $2 \le n$. A formula, like this one, that refers to previous terms to define the next term is called **recursive**. Every recursive formula must include a starting place.

On the other hand, in Example 3 it is easy to describe a term using only its position number. In the nth position is the square of n; $b_n = n^2$, $1 \le n$. This type of formula is called **explicit**, because it tells us exactly what value any particular term has.

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26. (a) Give an example of a set T, $|T| = 6$, and two partitions of T.

(b) For the set T in part (a), give a nonempty collection of subsets for which T has no exact cover.

For Exercises 27 through 29, use $A = \{a, b, c, \ldots, z\}$.

27. Give a partition $\mathcal{P}$ of A such that $|\mathcal{P}| = 4$ and one element of $\mathcal{P}$ contains only the letters needed to spell your first name.

28. Give a partition $\mathcal{P}$ of A such that $|\mathcal{P}| = 3$ and each element of $\mathcal{P}$ contains at least five elements.

29. Is it possible to have a partition $\mathcal{P}$ of A such that $\mathcal{P} = \{A_1, A_2, \ldots, A_{10}\}$ and $\forall i\ |A_i| \geq 3$?

30. If $B = \{0, 3, 6, 9, \ldots\}$, give a partition of B containing

(a) two infinite subsets.

(b) three infinite subsets.

31. List all partitions of $A = \{1, 2, 3\}$.

32. List all partitions of $B = \{a, b, c, d\}$.

33. The number of partitions of a set with n elements into k subsets satisfies the recurrence relation

$$S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

with initial conditions $S(n, 1) = S(n, n) = 1$. Find the number of partitions of a set with three elements into two

4.2 Relations and Digraphs

The notion of a relation betwee[n] itively clear (a formal definition human males and B is the set of ther) can be defined between A a

relation R) is the number of
the number of $b \in A$ such that $(a, b) \in R$.

What this means, in terms of the digraph of R, is that the in-degree of a vertex is the number of edges terminating at the vertex. The out-degree of a vertex is the number of edges leaving the vertex. Note that the out-degree of a is $|R(a)|$.

Example 21 Consider the digraph of Figure 4.4. Vertex 1 has in-degree 3 and out-degree 2. Also consider the digraph shown in Figure 4.5. Vertex 3 has in-degree 4 and out-degree 2, while vertex 4 has in-degree 0 and out-degree 1. ◆

Example 22 Let $A = \{a, b, c, d\}$, and let R be the relation on A that has the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Construct the digraph of R, and list in-degrees and out-degrees of all vertices.

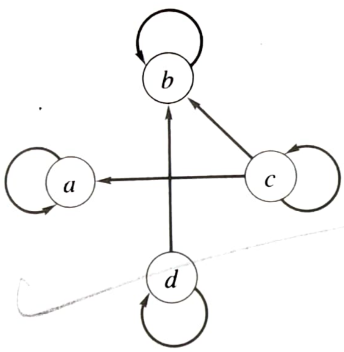


Figure 4.6

Solution

The digraph of R is shown in Figure 4.6. The following table gives the in-degrees and out-degrees of all vertices. Note that the sum of all in-degrees must equal the sum of all out-degrees.

	a	***b***	***c***	***d***
In-degree	2	3	1	1
Out-degree	1	1	3	2

◆

Example 23 Let $A = \{1, 4, 5\}$, and let R be given by the digraph shown in Figure 4.7. Find $\mathbf{M}_R$ and R.

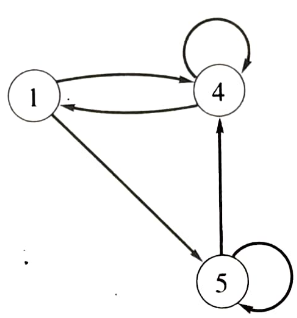


Figure 4.7

Solution

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad R = \{(1, 4), (1, 5), (4, 1), (4, 4), (5, 4), (5, 5)\}$$ ◆

If R is a relation on a set A, and B is a subset of A, the **restriction of R to B** is $R \cap (B \times B)$.

Example 24 Let $A = \{a, b, c, d, e, f\}$ and $R = \{(a, a), (a, c), (b, c), (a, e), (b, e), (c, e)\}$. Let $B = \{a, b, c\}$. Then

$$B \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

and the restriction of R to B is $\{(a, a), (a, c), (b, c)\}$. ◆

2. For the relation ... lowing ordered pairs belong to R?

(a) $(2, 0)$ (b) $(0, 2)$ (c) $(0, 3)$

(d) $(0, 0)$ (e) $\left(1, \frac{3\sqrt{3}}{2}\right)$ (f) $(1, 1)$

3. Let $A = \mathbb{Z}^+$, the positive integers, and R be the relation defined by $a\ R\ b$ if and only if $2a \leq b + 1$. Which of the following ordered pairs belong to R?

(a) $(2, 2)$ (b) $(3, 2)$ (c) $(6, 15)$

(d) $(1, 1)$ (e) $(15, 6)$ (f) (n, n)

In Exercises 4 through 12, find the domain, range, matrix, and, when $A = B$, the digraph of the relation R.

4. $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$,
$R = \{(a, 1), (a, 2), (b, 1), (c, 2), (d, 1)\}$

5. $A =$ {daisy, rose, violet, daffodil, peony},
$B =$ {red, white, purple, yellow, blue, pink, orange}
$R =$ {(daisy, red), (violet, pink), (rose, purple), (daffodil, white)}

6. $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 6, 8, 9\}$; $a\ R\ b$ if and only if $b = a^2$.

7. $A = \{1, 2, 3, 4, 8\} = B$; $a\ R\ b$ if and only if $a = b$.

8. $A = \{1, 2, 3, 4, 8\}$, $B = \{1, 4, 6, 9\}$; $a\ R\ b$ if and only if $a \mid b$.

9. $A = \{1, 2, 3, 4, 6\} = B$; $a\ R\ b$ if and only if a is a multiple of b.

10. $A = \{1, 2, 3, 4, 5\} = B$; $a\ R\ b$ if and only if $a \leq b$.

11. $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8\}$; $a\ R\ b$ if and only if $b < a$.

12. $A = \{1, 2, 3, 4, 8\} = B$; $a\ R\ b$ if and only if $a + b \leq 9$.

14. Let ... $a\ R$ ... Ran(

15. Let ... $a\ R$ ... Ran

16. Let ... for e

(a)

(c)

17. Let ... the

(a)

18. Let ... the

(a)

19. Let ... the

(a)

(c)

20. Let ... A_1 a

Example 4 Let $A = \{1, 2, 3, 4, 5, 6\}$. Let R be the relation whose digraph is shown in Figure 4.12. Figure 4.13 shows the digraph of the relation R^2 on A. A line connects two vertices in Figure 4.13 if and only if they are R^2-related, that is, if and only if there is a path of length two connecting those vertices in Figure 4.12. Thus

$$\begin{array}{llllll}
1\ R^2\ 2 & \text{since} & 1\ R\ 2 & \text{and} & 2\ R\ 2 \\
1\ R^2\ 4 & \text{since} & 1\ R\ 2 & \text{and} & 2\ R\ 4 \\
1\ R^2\ 5 & \text{since} & 1\ R\ 2 & \text{and} & 2\ R\ 5 \\
2\ R^2\ 2 & \text{since} & 2\ R\ 2 & \text{and} & 2\ R\ 2 \\
2\ R^2\ 4 & \text{since} & 2\ R\ 2 & \text{and} & 2\ R\ 4 \\
2\ R^2\ 5 & \text{since} & 2\ R\ 2 & \text{and} & 2\ R\ 5 \\
2\ R^2\ 6 & \text{since} & 2\ R\ 5 & \text{and} & 5\ R\ 6 \\
3\ R^2\ 5 & \text{since} & 3\ R\ 4 & \text{and} & 4\ R\ 5 \\
4\ R^2\ 6 & \text{since} & 4\ R\ 5 & \text{and} & 5\ R\ 6.
\end{array}$$

In a similar way, we can construct the digraph of R^n for any n. ◆

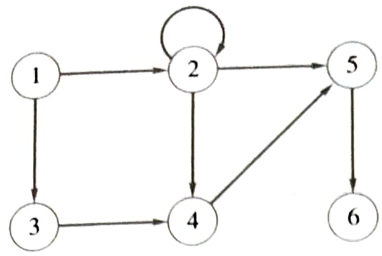


Figure 4.12

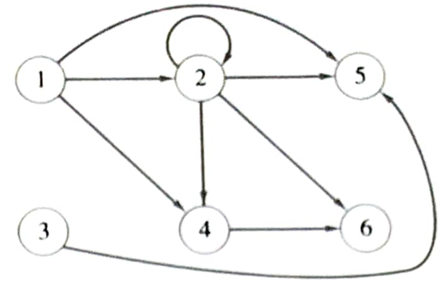


Figure 4.13

Example 5 Let $A = \{a, b, c, d, e\}$ and

$$R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}.$$

Compute **(a)** R^2; **(b)** R^∞.

Solution

(a) The digraph of R is shown in Figure 4.14.

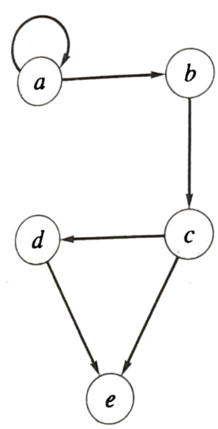


Figure 4.14

$$\begin{array}{llllll}
a\ R^2\ a & \text{since} & a\ R\ a & \text{and} & a\ R\ a. \\
a\ R^2\ b & \text{since} & a\ R\ a & \text{and} & a\ R\ b. \\
a\ R^2\ c & \text{since} & a\ R\ b & \text{and} & b\ R\ c. \\
b\ R^2\ e & \text{since} & b\ R\ c & \text{and} & c\ R\ e. \\
b\ R^2\ d & \text{since} & b\ R\ c & \text{and} & c\ R\ d. \\
c\ R^2\ e & \text{since} & c\ R\ d & \text{and} & d\ R\ e.
\end{array}$$

Hence

$$R^2 = \{(a, a), (a, b), (a, c), (b, e), (b, d), (c, e)\}.$$

(b) To compute R^∞, we need all ordered pairs of vertices for which there is a path of any length from the first vertex to the second. From Figure 4.14 we see that

$$R^\infty = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, c),\\ (b, d), (b, e), (c, d), (c, e), (d, e)\}.$$

For example, $(a, d) \in R^\infty$, since there is a path of length 3 from a to d: a, b, c, d. Similarly, $(a, e) \in R^\infty$, since there is a path of length 3 from a to e: a, b, c, e as well as a path of length 4 from a to e: a, b, c, d, e. ◆

If $|R|$ is large, it can be tedious and perhaps difficult to compute R^∞, or even R^2, from the set representation of R. However, $\mathbf{M}_R$ can be used to accomplish these tasks more efficiently.

Let R be a relation on a finite set $A = \{a_1, a_2, \ldots, a_n\}$, and let $\mathbf{M}_R$ be the $n \times n$ matrix representing R. We will show how the matrix $\mathbf{M}_{R^2}$, of R^2, can be computed from $\mathbf{M}_R$.

THEOREM 1 If R is a relation on $A = \{a_1, a_2, \ldots, a_n\}$, then $\mathbf{M}_{R^2} = \mathbf{M}_R \odot \mathbf{M}_R$ (see Section 1.5).

Proof

Let $\mathbf{M}_R = \left[\, m_{ij} \,\right]$ and $\mathbf{M}_{R^2} = \left[\, n_{ij} \,\right]$. By definition, the i, jth element of $\mathbf{M}_R \odot \mathbf{M}_R$ is equal to 1 if and only if row i of $\mathbf{M}_R$ and column j of $\mathbf{M}_R$ have a 1 in the same relative position, say position k. This means that $m_{ik} = 1$ and $m_{kj} = 1$ for some

k, $1 \le k \le n$. By definition of the matrix $\mathbf{M}_R$, the preceding conditions mean that $a_i \; R \; a_k$ and $a_k \; R \; a_j$. Thus $a_i \; R^2 \; a_j$, and so $n_{ij} = 1$. We have therefore shown that position i, j of $\mathbf{M}_R \odot \mathbf{M}_R$ is equal to 1 if and only if $n_{ij} = 1$. This means that $\mathbf{M}_R \odot \mathbf{M}_R = \mathbf{M}_{R^2}$. ∎

For brevity, we will usually denote $\mathbf{M}_R \odot \mathbf{M}_R$ simply as $(\mathbf{M}_R)^2_\odot$ (the symbol $\odot$ reminds us that this is not the usual matrix product).

4.3 Exercises

For Exercises 1 through 8, let R be the relation whose digraph is given in Figure 4.16.

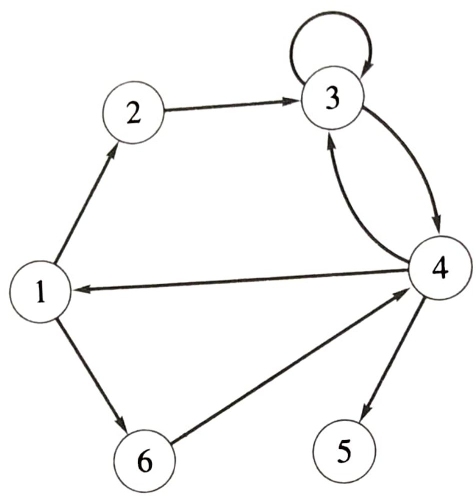


Figure 4.16

1. List all paths of length 1.

2. (a) List all paths of length 2 starting from vertex 2.

(b) List all paths of length 2.

3. (a) List all paths of length 3 starting from vertex 3.

(b) List all paths of length 3.

4. Find a cycle starting at vertex 2.

5. Find a cycle starting at vertex 6.

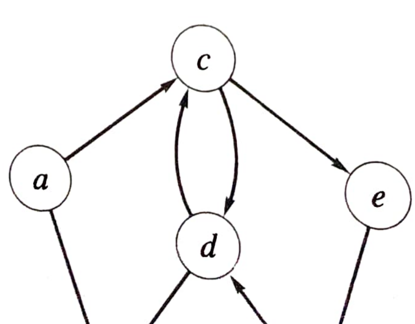
6. Draw the digraph of R^2.

7. Find $\mathbf{M}_{R^2}$.

8. (a) Find R^∞.

(b) Find $\mathbf{M}_{R^\infty}$.

For Exercises 9 through 16, let R be the relation whose digraph is given in Figure 4.17.



14. Find a cycle starting

15. Draw the digraph of

16. Find $\mathbf{M}_{R^2}$. Is this re
cise 15?

17. (a) Find $\mathbf{M}_{R^\infty}$.

(b) Find R^∞.

18. Let R and S be rel

19. Let R be a relatio
that $\mathbf{M}_{R^*} = \mathbf{M}_{R}$
matrix.

*In Exercises 20 throug
is given in Figure 4.18*

Figur

20. If π_1: 1, 2, 4,
$\pi_2 \circ \pi_1$.

21. If π_1: 1, 7, 5
$\pi_2 \circ \pi_1$.

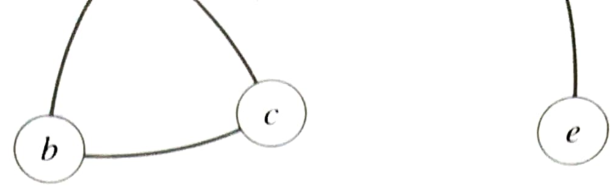


Figure 4.25

29. Let R be a symmetric relation given by its matrix $\mathbf{M}_R$. Describe a procedure for using $\mathbf{M}_R$ to determine if the graph of R is connected.

30. Let R be a relation on A and $B \subseteq A$. Which relational properties of R would be inherited by the restriction of R to B?

31. Prove or disprove that if a relation on a set A is transitive and irreflexive, then it is asymmetric.

32. Prove or disprove that if a relation R on A is transitive, then R^2 is also transitive.

33. Let R be a nonempty relation on a set A. Suppose that R is symmetric and transitive. Show that R is not irreflexive.

34. Prove that if a relation R on a set A is symmetric, then the relation R^2 is also symmetric.

35. Prove by induction that if a relation R on a set A is symmetric, then R^n is symmetric for $n \geq 1$.

36. Define a relation on $\mathbb{Z}^+$ that is reflexive, symmetric, and transitive and has not been defined previously.

37. Define a relation on the set $\{a, b, c, d\}$ that is

(a) reflexive and symmetric, but not transitive.

(b) reflexive and transitive, but not symmetric.

38. Define a relation on the set $\{a, b, c, d\}$ that is

(a) irreflexive and transitive, but not symmetric.

(b) antisymmetric and reflexive, but not transitive.

39. Define a relation on the set $\{a, b, c, d\}$ that is

(a) transitive, reflexive, and symmetric.

(b) asymmetric and transitive.

40. Give a direct proof of Theorem 1 of this section.

Example 3 Let $A = \{a, b, c, d, e\}$ and let R and S be two relations on A whose corresponding digraphs are shown in Figure 4.38. Then the reader can verify the following facts:

$$\overline{R} = \{(a,a), (b,b), (a,c), (b,a), (c,b), (c,d), (c,e), (c,a), (d,b), (d,a), (d,e), (e,b), (e,a), (e,d), (e,c)\}$$

$$R^{-1} = \{(b,a), (e,b), (c,c), (c,d), (d,d), (d,b), (c,b), (d,a), (e,e), (e,a)\}$$

$$R \cap S = \{(a,b), (b,e), (c,c)\}.$$

♦

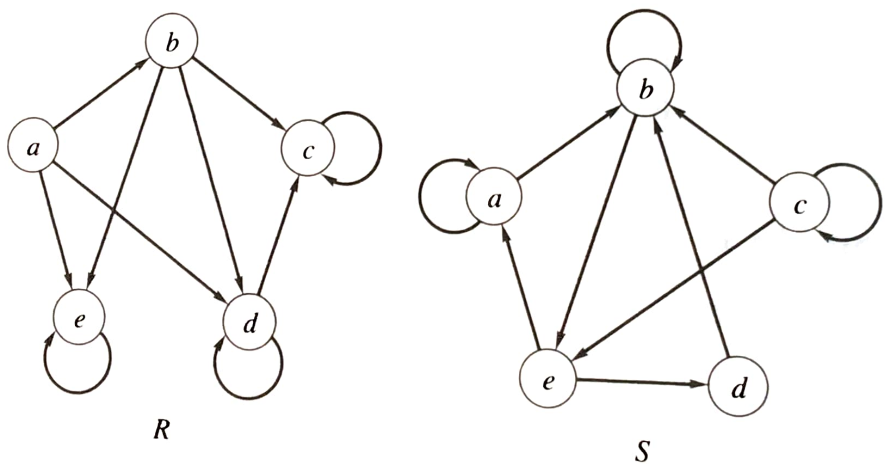


Figure 4.38

4.7 Exercises

In Exercises 1 and 2, let R and S be the given relations from A to B. Compute **(a)** $\overline{R}$; **(b)** $R \cap S$; **(c)** $R \cup S$; **(d)** S^{-1}.

1. $A = B = \{1, 2, 3\}$
$R = \{(1, 1), (1, 2), (2, 3), (3, 1)\}$
$S = \{(2, 1), (3, 1), (3, 2), (3, 3)\}$

2. $A = \{a, b, c\}$; $B = \{1, 2, 3\}$
$R = \{(a, 1), (b, 1), (c, 2), (c, 3)\}$
$S = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$

3. Let A = a set of people. Let $a\ R\ b$ if and only if a and b are brothers; let $a\ S\ b$ if and only if a and b are sisters. Describe $R \cup S$.

4. Let A = a set of people. Let $a\ R\ b$ if and only if a is older than b; let $a\ S\ b$ if and only if a is a brother of b. Describe $R \cap S$.

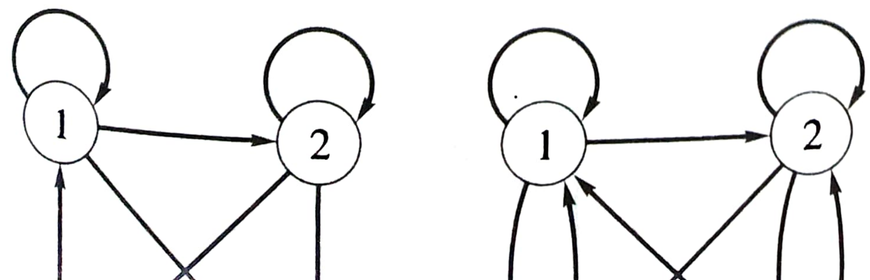
5. Let A = a set of people. Let $a\ R\ b$ if and only if a is the father of b; let $a\ S\ b$ if and only if a is the mother of b. Describe $R \cup S$.

6. Let $A = \{2, 3, 6, 12\}$ and let R and S be the following relations on A: $x\ R\ y$ if and only if $2 \mid (x - y)$; $x\ S\ y$ if and only if $3 \mid (x - y)$. Compute

(a) $\overline{R}$ **(b)** $R \cap S$ **(c)** $R \cup S$ **(d)** S^{-1}.

In Exercises 7 and 8, let R and S be two relations whose corresponding digraphs are shown in Figures 4.40 and 4.41. Compute **(a)** $\overline{R}$; **(b)** $R \cap S$; **(c)** $R \cup S$; **(d)** S^{-1}.

7.



VIKAS

DISCRETE MATHEMATICS

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De Morgan's Law Let A and B be two sets. Then

i. $(A \cap B)^c = A^c \cup B^c$

ii. $(A \cap B)^c = A^c \cap B^c$

Proof Let $x \in (A \cap B)^c$

$$\Rightarrow x \notin A \cap B$$

$$\Rightarrow x \notin A \text{ or } x \notin B$$

$$\Rightarrow x \in A^c \text{ or } x \notin B^c$$

$$\Rightarrow x \in A^c \cup B^c$$

$$\therefore (A \cap B)^c \subseteq A^c \cup B^c \qquad (1)$$

Now let $y \in A^c \cup B^c$

$$\Rightarrow y \in A^c \text{ or } y \in B^c$$

$$\Rightarrow y \notin A \text{ or } y \notin B$$

$$\Rightarrow y \notin A \cap B$$

$$\Rightarrow y \in (A \cap B)^c$$

$$\Rightarrow A^c \cup B^c \subseteq (A \cap B)^c \qquad (2)$$

From (1) and (2), $(A \cap B)^c = A^c \cup B^c$.

Similarly we can prove that the other identity $(A \cup B)^c = A^c \cap B^c$.

EXERCISE II

1. Suppose the universal set $\mu = \{1, 2, ..., 10\}$

   Express the following sets with bit strings where ith bit in the string is 1 if i is in the set and 0 otherwise.

   a. $A = \{4, 5, 6\}$ b. $B = \{1, 2, 9, 10)$ c. $C = \{2, 3, 7, 8, 9\}$

   Use bit strings to find

   i. $A \cup B$ ii. $A \cup C$ iii. $B \cup C$ iv. $A \cap B$

   v. $A \cap C$ vi. $B \cap C$ vii. $A \cup B \cup C$ viii. $A \cap B \cap C'$

2. Let $\mu = \{1, 2, ..., 10\}$ be the universal set.

   The subsets are $A = \{1, 7, 8\}$, $B = \{2, 4, 6\}$, $C = \{1, 3, 6, 8, 9\}$

   Find the minterms and maxterms generated by A, B, C.

3. Let $\{A_1, A_2, ..., A_n\}$ be a partition of a set X and let B be a non-empty subset of X.

   Prove that $\{A_i \cap B \,/\, A_i \cap B = \varnothing\}$ is a partition of $X \cap B$.

RELATION BETWEEN SETS

Cartesian Product

Let A and B be sets. The cartesian product of A and B is defined as

$$A \times B = \{a, b) \setminus a \in A; b \in B\}$$

i.e., the set of all ordered pairs (a_i, b_j) for every $a_i \in A$; $b_j \in B$.

Example $A = \{1, 2\}$, $B = \{a, b, c\}$

$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}$$

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\} \text{ (Clearly } A \times B \neq B \times A)$$

Note: We can represent the cartesian product as a rectangular array having n rows and m columns labelled in order as $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ respectively.

Binary Relation

A binary relation R from a set A to a set B is a subset R of the cartesian product $A \times B$

Example

1. Let $A = B = N$, the set of natural numbers.

   i. Define the relation R as '='.

   Now $R = \{(1, 1), (2, 2), (3, 3), ...\}$

   $\subseteq N \times N$

   $\therefore$ R is a binary relation.

   ii. Define R as '<'

   then $R = \{(1, 2), (2, 3), (3, 4), ... (1, 3), (2, 4), (3, 5), ...\}$

   $\subseteq N \times N$

   $\therefore$ R is a binary relation.

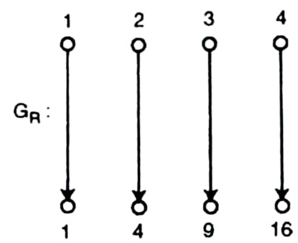
ii. The relation array can be viewed graphically as elements of sets represented by models, and an ordered pair is represented by an edge between the vertices that correspond to the pair elements, with an arrow pointing to the second element of the pair.

Example 1.5 Let $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 9, 16\}$ and the relation $R = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$. Draw the relation graph.

Solution First we shall write the relation matrix M_R.

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now we shall draw the relation graph G_R.

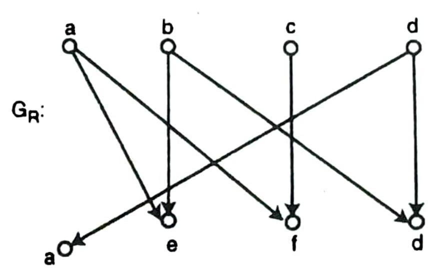


Example 1.6 Let $A = \{a, b, c, d\}$, $B = \{a, e, f, d\}$ and let $R = \{(a, e), (a, f), (b, e), (c, f), (b, d), (d, d), (d, a)\}$. Draw the relation graph.

Solution First we shall write the relation matrix M_R.

$$M_R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The relation graph G_R is given as

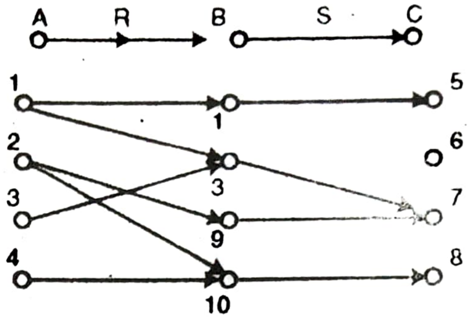


Composition of Two Relations

Let R be a binary relation from the set A to the set B and S be a binary relation from the set B to the set C, then the ordered pair (R, S) is said to be composable. If (R, S) is a composable pair of binary relations, the composite $R \circ S$ and R and S, is a binary relation from the set A to the set C, such that, for $a \in A$ and $c \in C$ $a\,(R \circ S)_C$ if for some $b \in B$, both aRb and bSc.

Example 1.7 $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 9, 10\}$ $C = \{5, 6, 7, 8\}$, $R = \{(1, 1), (1, 3), (2, 9), (2, 10), (3, 3), (4, 10)\}$ $S = \{(1, 5), (3, 7), (9, 7), (10, 8)\}$. Find $R \circ S$ and its relation graph.

Solution

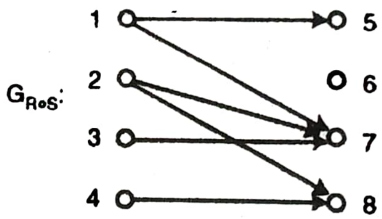


$R \circ S = \{(1,5), (1,7), (2,7), (2,8), (3,7), (4,8)\}$

The corresponding matrix is

$$M_{R \circ S} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the corresponding relation graph $G_{R \circ S}$ is,



Properties of a Binary Relation

Let R be a relation on a set A (i.e., $R \subseteq A \times A$). R is called

i. reflexive if $aRa, \forall\ a \in A$

ii. symmetric if aRb then $bRa, \forall\ a,b \in A$.

iii. transitive if aRb and bRc then $aRc, \forall\ a,b,c \in A$.

iv. irreflexive if $a \not R a$, and $a \in$ A.

v. antisymmetric if aRb then $b \not R a$, and $a,b \in A$.

Equivalence relation A relation R on a set A is called an equivalence relation if R is reflexive, symmet and transitive.

Example Let N be the set of natural numbers. Define R on N as

$$R = \{(x,y) / x + y \text{ is even}, x, y \in N\}$$

Proof Let $x \in N$. Now $x + x = 2x$.

Clearly $2x$ is even. Therefore R is reflexive. Let $x, y \in N$ and $x + y$ is even . Clearly $y + x$ is also even and he R is symmetric.

Now if $x + y$ is even and $y + z$ is even then we have to prove that $x + z$ is even.

Since x + y and y + z are even, both $(x + y)$ and $(y + z)$ are divisible by 2

SOLVED PROBLEMS

1. The relation R on a set is represented by

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric or antisymmetric?

Solution In the matrix M_R, the diagonal elements are 1. Therefore R is reflexive. Since the matrix M_R is symmetric, the relation R is also symmetric.

2. The relation R and R_1 on a set is represented by

a. $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ b. $M_{R1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Are the relations R and R_1 reflexive, symmetric, antisymmetric, and or transitive ?

Solution

a. Since the matrix M_R is symmetric and its diagonal entries are 1. The relation R is symmetric and reflexive. Since R is not antisymmetric R is transitive.

b. The relation R_1 is not reflexive.

R_1 is symmetric [$\because$ M_{R1} is symmetric] and R_1 is transitive.

3. Draw the relation graph for the following relations.

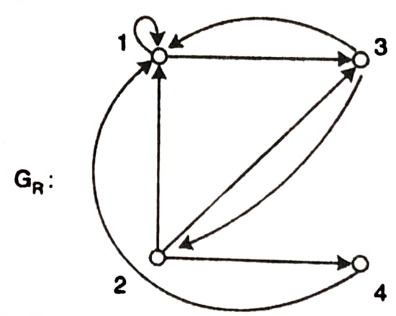
a. $R = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$ on the set $X = \{1, 2, 3, 4\}$.

b. $R_1 = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$ on the set $Y = \{1, 2, 3\}$

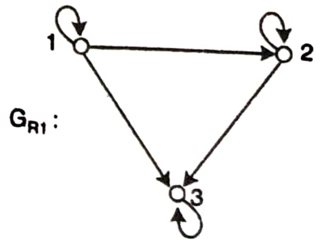
Solution

a. The relation graph G_R of R is drawn as:

The vertices of G_R are 1, 2, 3, 4.



b. The relation graph G_{R1} of R_1 is drawn as.



4. Let R be the relation represented by

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the relation matrices representing (a) R^{-1} (b) R^c (c) R^2

Solution

a. To get the inverse relation matrix (M^{R-1}) of a relation matrix (M_R) just write the transpose of M_R.

$$\therefore M_{R-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

b. To find the complement relation matrix, replace 0 by 1 and 1 by 0 in the given relation matrix.

$$\therefore M_{R^c} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

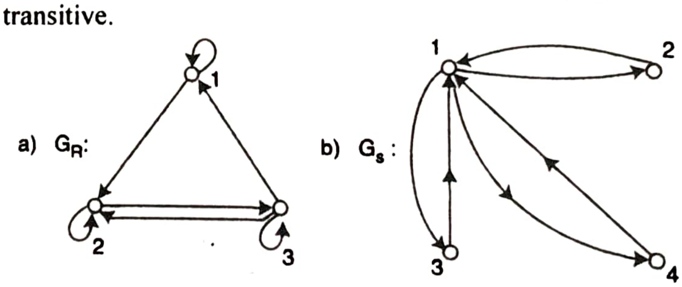
c. To find the relation matrix of R^2

Now $R^2 = R \circ R$.

If the relation matrix M_R is known, then $M_{R^2} = M_R \cdot M_R$ (the matrix multiplication)

$$\therefore M_{R^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

5. Find whether the relations for the directed graphs shown in the following figures are reflexive, symmetric, antisymmetric and \or transitive.



Solution

a. In G_R, there are loops at every vertex of the relation graph and hence it is reflexive.

It is neither symmetric nor antisymmetric since there is an edge between 1 and 2 but not from 2 to 1, but there are edges connecting 2 and 3 in both directions.

Moreover the relation is not transitive, since there is an edge from 1 to 2 and 2 to 3, but no edge from 1 to 3.

b. Since loops are not present in G_S, this relation is not reflexive. Further it is symmetric and not antisymmetric.

Moreover the relation is not transitive.

11. Let $X = \{1, 2, 3, ..., 7\}$ and $R = \{(x, y) / x - y$ is divisible by $3\}$
    Prove that R is an equivalance relation and draw the relation graph.
12. Let $x = \{1, 2, 3, 4, 5\}$ and let $P = \{(1, 2), (3), (4, 5)\}$. Prove that the partition P defines an equivalance relation on X.
13. A relation on a set that is reflexive and symmetric is called a compatible relation.
    a. Let A be a set of people and R be a binary relation on A such that $(a, b) \in R$ if a is a friend of b. Prove that R is a compatible relation
    b. Let A be a set of Latin words and R be a binary relation on A such that two words in A are related if they have one or more letters in common. Prove that R is a compatible relation.

Closures of a Relation

Let R be any relation on a set A. (R may be or may not be symmetric, reflexive and transitive). Let S be any other relation on A such that S contains R, and S is a subset of every relation containing R. Then S is called closure of R.

Reflexive closure Let R be a relation on a set X. The reflexive closure S of R is obtained by adding to R all pairs of the form (a,a) which are not in R, $a \in A$. Now S is reflexive, contains R and is in any reflexive relation containing R.

Example 1.9 $R = \{(a, a), (a, b), (b, a), (c, b)\}$ on the set $A = \{a, b, c\}$. Find the reflexive closure of R.

Solution By inspection, we can see that $(b, b), (c, c)$ are not in R.

$\therefore S = \{(a, a), (a, b), (b, a), (c, b), (b, b), (c, c)\}$

Then S is the reflexive closure of R.

Note: To obtain a reflexive closure of a relation R, just add the diagonal relation elements to R. i.e., Let $D = \{(a,c) / a \in A\}$, the diagonal relation S is the reflexive closure of R, then $S = R \cup D$

Symmetric closure Let R be a relation on set A. The symmetric closure S of R is obtained by adding to R all pairs of the form (b, a), if $(a, b) \in R$ and $(b, a) \notin R$

In other words, symmetric closure of R can be obtained as $S = R \cup R^{-1}$

Example 1.10 Let $R = \{(a, a), (a, b), (b, b), (c, c), (c, b)\}$ on $\{a, b, c\}$. Find the symmetric closure of R.

Solution Clearly R is not symmetric.

Now $S = R \cup R^{-1} = \{(a, a), (a, b), (b, b), (c, a), (c, b), (b, a), (a, c), (b, c)\}$

Clearly S is symmetric and it contains R and is in any symmetric relation containing R.

Paths in Directed Graphs

Now we shall study about paths in directed graphs and relationship between a relation on a set, before transitive closure.

Path A path from vertex a to vertex b in a directed graph G is a sequence of one or more edges (v_0, v_1), (v_1, v_2), ...,(v_{n-1}, v_n) in G, with $v_0 = a$; $v_n = b$ i.e., a sequence of edges whose terminal vertex is same as the intial vertex of the next edge in this path. This path is denoted by $v_0, v_1, ..., v_n$ of length n.

Let R be a relation on a set $A = \{1, 2, .., n\}$ and G_R be the corresponding relation graph whose vertices are $a = 1, v_1 = 2, ..., b = n$. There is a path in G_R from a to b if there is a sequence of vertices a, $v_1, v_2, ..., v_{n-1}, b$ with $(a, v_1) \in R, (v_1, v_2) \in R, (v_2, v_3) \in R, ..., (v_{n-1}, b) \in R$