Markov chain example

Problem

When a tourist arrives in a country capital W they want to visit also two main cities: G in the north and K in the south.
From survey data, when in W, there is the probability p of the tourist to go to K, 0 to G and 1-p to remain in W.
When in K, p to go to G, 1-p to return to W, and 0 to continue to stay in K. Finally, when in G, 1-p to go to K, 0 to go to W and p to continue in G.
Hint: think about mathematical formalism for transition problems.

Assuming the start of the trip in W, what are the probabilities of being in W, K or G after n days? Please provide a closed form formula

Define transition matrix

In[347]:= $\texttt{trans[p\_] :=} \begin{pmatrix} 1-p & p & 0 \\ 1-p & 0 & p \\ 0 & 1-p & p \end{pmatrix};$

Starting value

In[348]:=
```
x0 = (1 0 0);
```

Probabilities after n steps

In[349]:=
```
transn[p_, n_] := MatrixPower[trans[p], n];
```

In[350]:=
```
xn[p_, n_] := Dot[x0, transn[p, n]];
```

Probability to stay in W

In[351]:=
```
TraditionalForm@FullSimplify[xn[p, n][[1]][[1]],
  p ≥ 0 && p ≤ 1 && Element[p, Reals] && n ≥ 0 && Element[n, Integers]]
```

Out[351]//TraditionalForm=

$$\frac{p\left((-(p-1)\,p)^{n/2}+(-1)^{n+1}\,(-(p-1)\,p)^{\frac{n+1}{2}}+(-(p-1)\,p)^{\frac{n+1}{2}}+\left(-\sqrt{(1-p)\,p}\right)^{n}+2\,p-4\right)+2}{2\,(p-1)\,p+2}$$

Probability to stay in K

In[352]:=
```
TraditionalForm@FullSimplify[xn[p, n]〚1〛〚2〛,
  p ≥ 0 && p ≤ 1 && Element[p, Reals] && n ≥ 0 && Element[n, Integers]]
```

Out[352]//TraditionalForm=

$$\left(p\left((p-1)^2\,(-(p-1)\,p)^{n/2}+(-1)^n\,p\,(-(p-1)\,p)^{\frac{n+1}{2}}-p\,(-(p-1)\,p)^{\frac{n+1}{2}}+(p-1)^2\left(\left(-\sqrt{(1-p)\,p}\right)^n-2\right)\right)\right)\Big/ (2\,(p-1)\,((p-1)\,p+1))$$

Probability to stay in G

In[353]:=
```
TraditionalForm@FullSimplify[xn[p, n]〚1〛〚3〛,
  p ≥ 0 && p ≤ 1 && Element[p, Reals] && n ≥ 0 && Element[n, Integers]]
```

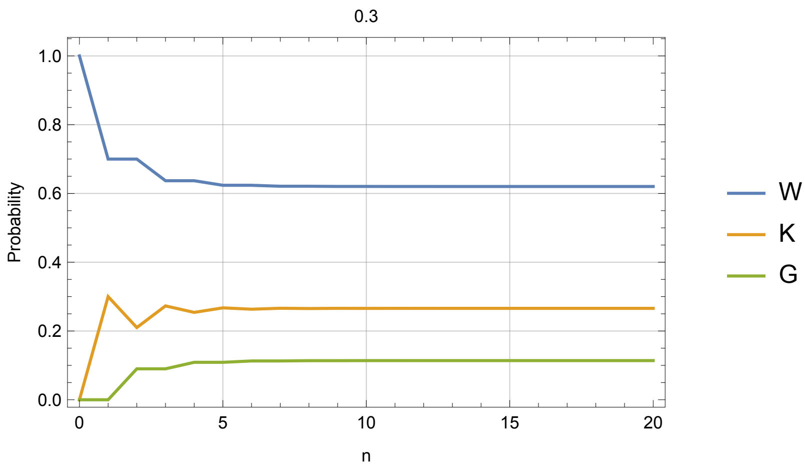
Out[353]//TraditionalForm=

$$\left(p\left(\left((-1)^{n+1}+1\right)(-(p-1)\,p)^{\frac{n+1}{2}}-(p-1)\,p\left((-(p-1)\,p)^{n/2}+\left(-\sqrt{(1-p)\,p}\right)^n-2\right)\right)\right)\Big/(2\,(p-1)\,((p-1)\,p+1))$$

Example: plot of probability vs n for a given p

In[354]:=
```
range = 20;
pr = 0.3;
pts = Outer[{#1, #2} &, {pr}, Range[0, range]]〚1〛;
probs = Flatten[Apply[xn, pts, {1}], 1];
ListLinePlot[{probs〚All, 1〛, probs〚All, 2〛, probs〚All, 3〛},
 DataRange → {0, range}, Frame → True, GridLines → Automatic,
 FrameLabel → {{"Probability", ""}, {"n", pr}},
 PlotLegends → {"W", "K", "G"}]
```

Out[358]=



Is there a distribution of tourists between W, K and G that remains invariant in time?

The stationary states, i.e. distribution of states invariant in time, are the eigenstates of the transition matrix

In[35]:=
```
inv[p_] := Eigenvectors[trans[p]];
vals[p_] := Eigenvalues[trans[p]];
```

Check first eigenstate

In[37]:=
```
TraditionalForm@FullSimplify[vals[p][[1]], p ≥ 0 && p ≤ 1 && Element[p, Reals]]
```

Out[37]//TraditionalForm=

$$1$$

In[38]:=
```
TraditionalForm@FullSimplify[inv[p][[1]], p ≥ 0 && p ≤ 1 && Element[p, Reals]]
```

Out[38]//TraditionalForm=

$$\{1, 1, 1\}$$

In[39]:=
```
TraditionalForm@FullSimplify[Dot[transn[p, n], inv[p][[1]]] / Power[vals[p][[1]], n],
  p ≥ 0 && p ≤ 1 && Element[p, Reals] && n ≥ 0 && Element[n, Integers]]
```

Out[39]//TraditionalForm=

$$\{1, 1, 1\}$$

Check second eigenstate

In[40]:=
```
TraditionalForm@FullSimplify[vals[p][[2]], p ≥ 0 && p ≤ 1 && Element[p, Reals]]
```

Out[40]//TraditionalForm=

$$-\sqrt{(1-p)\,p}$$

In[41]:=
```
TraditionalForm@FullSimplify[inv[p][[2]], p ≥ 0 && p ≤ 1 && Element[p, Reals]]
```

Out[41]//TraditionalForm=

$$\left\{\frac{p\sqrt{-(p-1)\,p}}{(p-1)^2}, \frac{p+\sqrt{-(p-1)\,p}}{p-1}, 1\right\}$$

In[42]:=
```
TraditionalForm@FullSimplify[Dot[transn[p, n], inv[p][[2]]] / Power[vals[p][[2]], n],
  p ≥ 0 && p ≤ 1 && Element[p, Reals] && n ≥ 0 && Element[n, Integers]]
```

Out[42]//TraditionalForm=

$$\left\{\frac{p\sqrt{-(p-1)\,p}}{(p-1)^2}, \frac{p+\sqrt{-(p-1)\,p}}{p-1}, 1\right\}$$

Check third eigenstate

In[43]:=
```
TraditionalForm@FullSimplify[vals[p][[3]], p ≥ 0 && p ≤ 1 && Element[p, Reals]]
```

Out[43]//TraditionalForm=

$$\sqrt{-(p-1)\,p}$$

In[44]:=
```
TraditionalForm@FullSimplify[inv[p][[3]], p ≥ 0 && p ≤ 1 && Element[p, Reals]]
```
Out[44]//TraditionalForm=

$$\left\{\frac{(-(p-1)\,p)^{3/2}}{(p-1)^3}, \frac{p-\sqrt{(1-p)\,p}}{p-1}, 1\right\}$$

In[45]:=
```
TraditionalForm@FullSimplify[Dot[transn[p, n], inv[p][[3]]] / Power[vals[p][[3]], n],
  p ≥ 0 && p ≤ 1 && Element[p, Reals] && n ≥ 0 && Element[n, Integers]]
```
Out[45]//TraditionalForm=

$$\left\{\frac{(-(p-1)\,p)^{3/2}}{(p-1)^3}, \frac{p-\sqrt{(1-p)\,p}}{p-1}, 1\right\}$$

Eigenstates are ok. Now normalise them such that sum of probabilities is 1 to get actual distributions

First distribution

In[46]:=
```
TraditionalForm@
 FullSimplify[inv[p][[1]] / (inv[p][[1]][[1]] + inv[p][[1]][[2]] + inv[p][[1]][[3]]),
  p ≥ 0 && p ≤ 1 && Element[p, Reals]]
```
Out[46]//TraditionalForm=

$$\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$$

Second distribution

In[47]:=
```
TraditionalForm@
 FullSimplify[inv[p][[2]] / (inv[p][[2]][[1]] + inv[p][[2]][[2]] + inv[p][[2]][[3]]),
  p ≥ 0 && p ≤ 1 && Element[p, Reals]]
```
Out[47]//TraditionalForm=

$$\left\{\frac{p\left(p+\sqrt{-(p-1)\,p}\right)}{(1-2\,p)^2}, \frac{2\,(p-1)\,p-\sqrt{(1-p)\,p}}{(1-2\,p)^2}, \frac{(p-1)^2}{(2\,p-1)\left(p+\sqrt{-(p-1)\,p}-1\right)}\right\}$$

Third distribution

In[48]:=
```
TraditionalForm@
 FullSimplify[inv[p][[3]] / (inv[p][[3]][[1]] + inv[p][[3]][[2]] + inv[p][[3]][[3]]),
  p ≥ 0 && p ≤ 1 && Element[p, Reals]]
```
Out[48]//TraditionalForm=

$$\left\{\frac{p\left(p-\sqrt{(1-p)\,p}\right)}{(1-2\,p)^2}, \frac{2\,(p-1)\,p+\sqrt{-(p-1)\,p}}{(1-2\,p)^2}, -\frac{(p-1)^2}{(2\,p-1)\left(-p+\sqrt{-(p-1)\,p}+1\right)}\right\}$$