粒子物理基础期末复习

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1 2022

1. (a) Write down the three names of the fundamental interactions and draw the corresponding fundamental processes in the standard model.

   (b) Write down all the fundamental particles in the standard model and indicate their electric charges, spins, and masses as well as the possible interactions they may be involved.

   (c) Write down at least five: (a) pseudoscalar mesons; (b) vector mesons; and (c) baryons in terms of the fundamental particles, respectively. (For example: proton $\equiv \mathrm{p} = \mathrm{uud}$.)

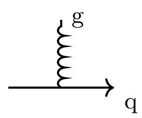
   (d) Write down the gauge groups of strong and electroweak interactions of the standard model.

   (e) Write down at least five names of conservation laws.

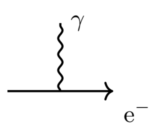
   Solution:

   (a) Names of the three fundamental interactions:

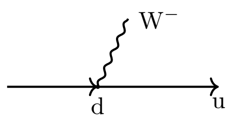
   - Strong interaction: it binds quarks inside hadrons. It is carried by the gluon g. Group: $\mathrm{SU}(3)_\mathrm{C}$.



   - Electromagnetic interaction: it acts between electric charges. It is carried by the photon γ. Group: $\mathrm{U}(1)_\mathrm{EM}$.



   - Weak interaction: it causes some particle decays and changes flavor. It is carried by $\mathrm{W}^\pm$ and Z^0. Group: $\mathrm{SU}(2)_\mathrm{L}$.



   (b) Fundamental particles and short notes:

   - Quarks (six types): $\mathrm{u, d, s, c, b, t}$.

- Charges: $\mathrm{u,c,t}$ have charge $+\frac{2}{3}\mathrm{e}$; $\mathrm{d,s,b}$ have charge $-\frac{1}{3}\mathrm{e}$.
- Spin: all quarks are spin-$\frac{1}{2}$ (fermions).
- Interactions: quarks feel strong, electromagnetic, and weak forces.
- Example composition: $\mathrm{p} \equiv \mathrm{p} = \mathrm{uud}, \quad \mathrm{n} \equiv \mathrm{n} = \mathrm{udd}$.

- Leptons (six types): e, μ, τ and their neutrinos $\nu_{\mathrm{e}}, \nu_\mu, \nu_\tau$.
  - Charges: e, μ, τ have -1 unit of charge ($-1\mathrm{e}$); neutrinos are neutral.
  - Spin: leptons are spin-$\frac{1}{2}$.
  - Interactions: charged leptons feel electromagnetic and weak; neutrinos feel weak only.
- Gauge bosons (force carriers):
  - Gluon: g, spin-1, carries color, mediates strong force.
  - Photon: γ, spin-1, mediates electromagnetic force.
  - $\mathrm{W}^{\pm}$, Z^0, spin-1, mediate weak force.
- Scalar boson:
  - Higgs boson: H, spin-0, responsible for mass generation.
- Note on masses:
  - Quarks and charged leptons have masses varying widely (e.g., $m_{\mathrm{e}} \approx 0.511\mathrm{MeV}/c^2$, $m_{\mathrm{t}} \approx 173\mathrm{GeV}/c^2$).
  - Neutrinos have extremely small, non-zero masses (massless in the original Standard Model).
  - Photon and Gluon are massless ($m = 0$). W/Z and Higgs are massive.

(c) Examples of hadron lists (at least five each, as requested):

- (a) Pseudoscalar mesons (examples): $\pi^{\pm}, \pi^0, \mathrm{K}^{\pm}, \mathrm{K}^0, \eta$.
- (b) Vector mesons (examples): $\rho^{\pm}, \rho^0, \omega, \phi, \mathrm{K}^*$.
- (c) Baryons (examples, show quark content):

$$\mathrm{p} = \mathrm{uud}, \quad \mathrm{n} = \mathrm{udd}, \quad \Lambda = \mathrm{uds}, \quad \Sigma^+ = \mathrm{uus}, \quad \Xi^0 = \mathrm{uss}. \tag{1}$$

(d) Gauge groups of the Standard Model:

$$\mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}. \tag{2}$$

(e) At least five conservation laws:

- Energy conservation.
- Momentum conservation.
- Electric charge conservation.
- Baryon number conservation (approximate in SM; no observed violation).
- Lepton number conservation (separately for each family approximately; note: neutrino oscillation mixes flavors).

- (Extra common laws) Angular momentum, color charge (local), CPT symmetry.

2. Examine the following processes, and state for each one whether it is possible or impossible. In the former case, state which interaction is responsible — strong, electromagnetic or weak; in the latter case cite a conservation law that prevents its from occurring.

   (a) $\mu^- \to \mathrm{e}^- + \gamma$

   (b) $\mu^- \to \gamma + \gamma$

   (c) $\mu^- \to \pi^- + \nu_\mu$

   (d) $\tau^- \to \pi^- + \nu_\tau$

   (e) $\pi^0 \to \mathrm{e}^+ + \mathrm{e}^-$

   (f) $\pi^0 \to \mathrm{e}^+ + \mu^-$

   (g) $\pi^- \to \mu^- + \bar{\nu}_\mu$

   (h) $\mathrm{K}^+ \to \pi^+ + \nu_\tau$

   (i) $\mathrm{n} \to \bar{\mathrm{p}} + \mathrm{e}^+ + \nu_\mathrm{e}$

   (j) $\mathrm{e}^- \to \mathrm{e}^- + \gamma$

   (k) $\mathrm{e}^+ + \mathrm{e}^- \to \gamma$

   (l) $\mathrm{e}^+ + \mathrm{e}^- \to \gamma + \gamma$

   Solution:

   (a) Impossible. Lepton family numbers are not conserved ($L_\mu : 1 \neq 0$, $L_\mathrm{e} : 0 \neq 1$).

   (b) Impossible. Electric charge is not conserved ($Q : -1 \neq 0$).

   (c) Impossible. Energy is not conserved ($m_\mu < m_\pi$).

   (d) Possible. Weak interaction.

   (e) Possible. Electromagnetic interaction.

   (f) Impossible. Lepton family numbers are not conserved ($L_\mathrm{e} : 0 \neq -1$, $L_\mu : 0 \neq 1$).

   (g) Possible. Weak interaction.

   (h) Impossible. Lepton number is not conserved ($L : 0 \neq 1$).

   (i) Impossible. Baryon number is not conserved ($B : 1 \neq -1$).

   (j) Impossible. Energy and momentum cannot be conserved.

   (k) Impossible. Momentum cannot be conserved.

   (l) Possible. Electromagnetic interaction.

3. A photon γ hits an electron at rest, producing an electron-positron pair.

   (a) What is the reaction?

   (b) What is the minimum energy of the incident photon?

   Solution:

   (a) The reaction is:

   $$\gamma + \mathrm{e}^- \to \mathrm{e}^- + \mathrm{e}^+ + \mathrm{e}^-. \tag{3}$$

   (b) We use the conservation of four-momentum. Let P_γ be the four-momentum of the photon. Let P_e be the four-momentum of the initial electron at rest. Let P_f be the total four-momentum of the final particles.

   The conservation law is:

   $$P_\gamma + P_\mathrm{e} = P_\mathrm{f}. \tag{4}$$

Square both sides (invariant mass squared):

$$(P_\gamma + P_\mathrm{e})^2 = P_\mathrm{f}^2. \tag{5}$$

Expand the left side:

$$P_\gamma^2 + P_\mathrm{e}^2 + 2P_\gamma \cdot P_\mathrm{e} = P_\mathrm{f}^2. \tag{6}$$

We know the following facts:

- The photon is massless, so $P_\gamma^2 = 0$.
- The initial electron is at rest, so $P_\mathrm{e}^2 = m_\mathrm{e}^2$.
- The dot product is $P_\gamma \cdot P_\mathrm{e} = E_\gamma m_\mathrm{e}$.

For the minimum energy (threshold), all three final particles (two electrons and one positron) move together with the same velocity. The invariant mass of the final state is the sum of their masses:

$$\sqrt{P_\mathrm{f}^2} = m_\mathrm{e} + m_\mathrm{e} + m_\mathrm{e} = 3m_\mathrm{e}. \tag{7}$$

So:

$$P_\mathrm{f}^2 = (3m_\mathrm{e})^2 = 9m_\mathrm{e}^2. \tag{8}$$

Substitute these values into the expanded equation:

$$0 + m_\mathrm{e}^2 + 2E_\gamma m_\mathrm{e} = 9m_\mathrm{e}^2. \tag{9}$$

Subtract m_e^2 from both sides:

$$2E_\gamma m_\mathrm{e} = 8m_\mathrm{e}^2. \tag{10}$$

Divide by $2m_\mathrm{e}$:

$$E_\gamma = 4m_\mathrm{e}. \tag{11}$$

The minimum energy is $4m_\mathrm{e}$.

4. Consider the charged pion-neutron scattering $\pi^\pm \mathrm{n} \to \pi^\pm \mathrm{n}$, which contains three processes:

   (a) $\pi^+ + \mathrm{n} \to \pi^+ + \mathrm{n}$

   (b) $\pi^- + \mathrm{n} \to \pi^- + \mathrm{n}$

   (c) $\pi^+ + \mathrm{n} \to \pi^0 + \mathrm{p}$

   (a) Use isospin invariance to calculate the cross sections of (a), (b) and (c) in terms of the isospin amplitudes of $\mathcal{M}_1$ and $\mathcal{M}_3$.

   (b) Find the ratio of the total cross sections: $\dfrac{\sigma_\mathrm{tot}(\pi^+ + \mathrm{n})}{\sigma_\mathrm{tot}(\pi^- + \mathrm{n})}$.

   Solution:

(a) We use the isospin decomposition. The nucleon (p, n) has isospin $I = \frac{1}{2}$. The pion (π) has isospin $I = 1$. We look up the Clebsch-Gordan coefficients to decompose the states $|I_\pi, I_{3\pi}\rangle\,|I_N, I_{3N}\rangle$ into total isospin states $|I, I_3\rangle$.

Process (a): $\pi^+ + \mathrm{n} \to \pi^+ + \mathrm{n}$. The initial state is $|\pi^+\mathrm{n}\rangle = |1, +1\rangle\,|1/2, -1/2\rangle$. From the table:

$$|\pi^+\mathrm{n}\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2}, +\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}, +\frac{1}{2}\right\rangle. \tag{12}$$

The final state is the same. The amplitude is the sum of the contributions from $I = \frac{3}{2}$ ($\mathcal{M}_3$) and $I = \frac{1}{2}$ ($\mathcal{M}_1$):

$$A_a = \sqrt{\frac{1}{3}}\sqrt{\frac{1}{3}}\mathcal{M}_3 + \sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}\mathcal{M}_1 = \frac{1}{3}\mathcal{M}_3 + \frac{2}{3}\mathcal{M}_1. \tag{13}$$

The cross section is proportional to the square of the amplitude:

$$\sigma_a \propto |A_a|^2 = \left|\frac{1}{3}\mathcal{M}_3 + \frac{2}{3}\mathcal{M}_1\right|^2. \tag{14}$$

Process (b): $\pi^- + \mathrm{n} \to \pi^- + \mathrm{n}$. The initial state is $|\pi^-\mathrm{n}\rangle = |1, -1\rangle\,|1/2, -1/2\rangle$. This corresponds to a pure state with total isospin $I = \frac{3}{2}$ and $I_3 = -\frac{3}{2}$:

$$|\pi^-\mathrm{n}\rangle = \left|\frac{3}{2}, -\frac{3}{2}\right\rangle. \tag{15}$$

The amplitude depends only on $\mathcal{M}_3$:

$$A_b = \mathcal{M}_3. \tag{16}$$

The cross section is:

$$\sigma_b \propto |A_b|^2 = |\mathcal{M}_3|^2. \tag{17}$$

Process (c): $\pi^+ + \mathrm{n} \to \pi^0 + \mathrm{p}$. The initial state is the same as in (a):

$$|\mathrm{initial}\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2}, +\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}, +\frac{1}{2}\right\rangle. \tag{18}$$

The final state is $|\pi^0\mathrm{p}\rangle = |1, 0\rangle\,|1/2, +1/2\rangle$. From the table:

$$|\pi^0\mathrm{p}\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}, +\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2}, +\frac{1}{2}\right\rangle. \tag{19}$$

The amplitude is:

$$A_c = \sqrt{\frac{1}{3}}\sqrt{\frac{2}{3}}\mathcal{M}_3 - \sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}}\mathcal{M}_1 = \frac{\sqrt{2}}{3}(\mathcal{M}_3 - \mathcal{M}_1). \tag{20}$$

The cross section is:

$$\sigma_c \propto |A_c|^2 = \frac{2}{9}|\mathcal{M}_3 - \mathcal{M}_1|^2. \tag{21}$$

(b) We calculate the total cross sections. For $\pi^+ + \mathrm{n}$, the total cross section is the sum of the elastic (a) and charge exchange (c) processes:

$$\sigma_{\mathrm{tot}}(\pi^+ + \mathrm{n}) = \sigma_a + \sigma_c. \tag{22}$$

Substitute the expressions (ignoring the common proportionality factor):

$$\begin{aligned}\sigma_{\text{tot}}(\pi^+ + \text{n}) &= \left|\frac{1}{3}\mathcal{M}_3 + \frac{2}{3}\mathcal{M}_1\right|^2 + \frac{2}{9}|\mathcal{M}_3 - \mathcal{M}_1|^2 \\ &= \frac{1}{9}(\mathcal{M}_3 + 2\mathcal{M}_1)^2 + \frac{2}{9}(\mathcal{M}_3 - \mathcal{M}_1)^2 \\ &= \frac{1}{9}(\mathcal{M}_3^2 + 4\mathcal{M}_1^2 + 4\mathcal{M}_3\mathcal{M}_1) + \frac{2}{9}(\mathcal{M}_3^2 + \mathcal{M}_1^2 - 2\mathcal{M}_3\mathcal{M}_1) \\ &= \frac{1}{9}(3\mathcal{M}_3^2 + 6\mathcal{M}_1^2) \\ &= \frac{1}{3}|\mathcal{M}_3|^2 + \frac{2}{3}|\mathcal{M}_1|^2.\end{aligned} \tag{23}$$

For $\pi^- + \text{n}$, only the elastic process (b) is possible (charge conservation prevents charge exchange to π^0):

$$\sigma_{\text{tot}}(\pi^- + \text{n}) = \sigma_b = |\mathcal{M}_3|^2. \tag{24}$$

The ratio is:

$$\frac{\sigma_{\text{tot}}(\pi^+ + \text{n})}{\sigma_{\text{tot}}(\pi^- + \text{n})} = \frac{\frac{1}{3}|\mathcal{M}_3|^2 + \frac{2}{3}|\mathcal{M}_1|^2}{|\mathcal{M}_3|^2} = \frac{1}{3} + \frac{2}{3}\left|\frac{\mathcal{M}_1}{\mathcal{M}_3}\right|^2. \tag{25}$$

Note: If the scattering is dominated by the $\Delta(1232)$ resonance ($I = \frac{3}{2}$), then $\mathcal{M}_3 \gg \mathcal{M}_1$, and the ratio becomes:

$$\text{Ratio} \approx \frac{1}{3}. \tag{26}$$

5. Write down the color, spin, and flavor wave functions for

(a) π^+,

(b) n (neutron).

Solution:

(a) Pion (π^+):

- **Color**: The meson is a color singlet.

$$\psi_{\text{color}} = \frac{1}{\sqrt{3}}(\text{r}\bar{\text{r}} + \text{g}\bar{\text{g}} + \text{b}\bar{\text{b}}). \tag{27}$$

- **Spin**: The pion has spin 0. The spins of the two quarks are antiparallel (singlet).

$$\chi_{\text{spin}} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{28}$$

- **Flavor**: The π^+ is made of u and $\bar{\text{d}}$.

$$\phi_{\text{flavor}} = |\text{u}\rangle\,\bar{\text{d}}. \tag{29}$$

(b) Neutron (n):

- **Color**: The baryon is a color singlet. It is totally antisymmetric.

$$\psi_{\text{color}} = \frac{1}{\sqrt{6}}(\text{rgb} - \text{rbg} + \text{gbr} - \text{grb} + \text{brg} - \text{bgr}). \tag{30}$$

- **Spin-Flavor**: The total wave function must be antisymmetric. Since the color part is antisymmetric and the spatial part is symmetric ($L = 0$), the spin-flavor part must be **symmetric**.

  We construct the symmetric state from the mixed-symmetric spin (χ) and flavor (ϕ) states. The flavor states for n (udd) are:

$$\phi_\rho = \frac{1}{\sqrt{2}}(\mathrm{du} - \mathrm{ud})\mathrm{d}, \quad \phi_\lambda = \frac{1}{\sqrt{6}}(2\mathrm{ddu} - \mathrm{udd} - \mathrm{dud}). \tag{31}$$

  (Note: ϕ_ρ is antisymmetric in 1-2, ϕ_λ is symmetric in 1-2).

  The spin states for spin-$\frac{1}{2}$ ($|\uparrow\rangle$) are:

$$\chi_\rho = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)|\uparrow\rangle, \quad \chi_\lambda = \frac{1}{\sqrt{6}}(2|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle). \tag{32}$$

  The combined symmetric spin-flavor wave function is:

$$\Psi_{\mathrm{SF}} = \frac{1}{\sqrt{2}}(\phi_\rho\chi_\rho + \phi_\lambda\chi_\lambda). \tag{33}$$

6. Optional Question:

   (a) Massless neutrinos couple to the Z^0 particle but not to the photon. Calculate the differential cross section for $\nu\nu \to \nu\nu$ scattering in the center-of-mass frame due to Z^0-exchange, assuming that the Z^0 is massless and has a photon-like propagator. (There are two diagrams to consider.) Express your answer in terms of scattering angle and center-of-mass energy.

   (b) If the Z^0 were massive, how would you expect the above calculation to be modified? Take $M_Z = 91\mathrm{GeV}$, $g = 0.57\mathrm{e}$, and $E_{\mathrm{cm}} = 100\mathrm{GeV}$, and compute the total cross-section in cm^2.

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1. (a) Write down the three names of the fundamental interactions and draw the corresponding fundamental processes in standard model.

   (b) Write down all the fundamental particles in the standard model and indicate their electric charges, spins, and masses as well as the possible interactions they may be involved.

   (c) Write down at least five: (a) pseudoscalar mesons; (b) vector mesons; and (c) baryons in terms of the fundamental particles, respectively. (For example: proton $\equiv$ p = uud.)

   (d) Write down the gauge groups of strong and electroweak interactions of the standard model.

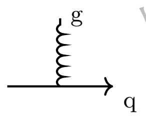
   (e) Write down at least five names of conservation laws.

   (f) Some neutral particles are their own antiparticles. For example, the photon: $\bar{\gamma} = \gamma$. Why is the neutron not its own antiparticle? What is about the case for a neutrino?

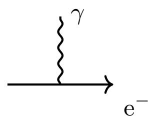
   Solution:

   (a) Names of the three fundamental interactions:

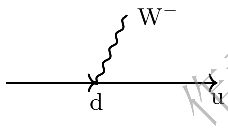
- Strong interaction: it binds quarks inside hadrons. It is carried by the gluon g. Group: $\mathrm{SU}(3)_{\mathrm{C}}$.



- Electromagnetic interaction: it acts between electric charges. It is carried by the photon γ. Group: $\mathrm{U}(1)_{\mathrm{EM}}$.



- Weak interaction: it causes some particle decays and changes flavor. It is carried by $\mathrm{W}^\pm$ and Z^0. Group: $\mathrm{SU}(2)_{\mathrm{L}}$.



(b) Fundamental particles and short notes:

- Quarks (six types): $\mathrm{u}, \mathrm{d}, \mathrm{s}, \mathrm{c}, \mathrm{b}, \mathrm{t}$.
  - Charges: $\mathrm{u}, \mathrm{c}, \mathrm{t}$ have charge $+\frac{2}{3}\mathrm{e}$; $\mathrm{d}, \mathrm{s}, \mathrm{b}$ have charge $-\frac{1}{3}\mathrm{e}$.
  - Spin: all quarks are spin-$\frac{1}{2}$ (fermions).
  - Interactions: quarks feel strong, electromagnetic, and weak forces.
  - Example composition: $\mathrm{p} \equiv \mathrm{p} = \mathrm{uud}, \quad \mathrm{n} \equiv \mathrm{n} = \mathrm{udd}$.
- Leptons (six types): e, μ, τ and their neutrinos $\nu_{\mathrm{e}}, \nu_\mu, \nu_\tau$.
  - Charges: e, μ, τ have -1 unit of charge ($-1\mathrm{e}$); neutrinos are neutral.
  - Spin: leptons are spin-$\frac{1}{2}$.
  - Interactions: charged leptons feel electromagnetic and weak; neutrinos feel weak only.
- Gauge bosons (force carriers):
  - Gluon: g, spin-1, carries color, mediates strong force.
  - Photon: γ, spin-1, mediates electromagnetic force.
  - $\mathrm{W}^\pm$, Z^0, spin-1, mediate weak force.
- Scalar boson:
  - Higgs boson: H, spin-0, responsible for mass generation.
- Note on masses:
  - Quarks and charged leptons have masses varying widely (e.g., $m_{\mathrm{e}} \approx 0.511\mathrm{MeV}/c^2$, $m_{\mathrm{t}} \approx 173\mathrm{GeV}/c^2$).
  - Neutrinos have extremely small, non-zero masses (massless in the original Standard Model).

– Photon and Gluon are massless ($m = 0$). W/Z and Higgs are massive.

(c) Examples of hadron lists (at least five each, as requested):

- (a) Pseudoscalar mesons (examples): $\pi^{\pm}, \pi^0, \mathrm{K}^{\pm}, \mathrm{K}^0, \eta$.
- (b) Vector mesons (examples): $\rho^{\pm}, \rho^0, \omega, \phi, \mathrm{K}^*$.
- (c) Baryons (examples, show quark content):

$$\mathrm{p} = \mathrm{uud}, \quad \mathrm{n} = \mathrm{udd}, \quad \Lambda = \mathrm{uds}, \quad \Sigma^+ = \mathrm{uus}, \quad \Xi^0 = \mathrm{uss}. \tag{34}$$

(d) Gauge groups of the Standard Model:

$$\mathrm{SU(3)_C} \times \mathrm{SU(2)_L} \times \mathrm{U(1)_Y}. \tag{35}$$

(e) At least five conservation laws:

- Energy conservation.
- Momentum conservation.
- Electric charge conservation.
- Baryon number conservation (approximate in SM; no observed violation).
- Lepton number conservation (separately for each family approximately; note: neutrino oscillation mixes flavors).
- (Extra common laws) Angular momentum, color charge (local), CPT symmetry.

(f)
- Neutron: No. The neutron (n) has baryon number $B = 1$ (udd), while the antineutron ($\bar{\mathrm{n}}$) has $B = -1$ ($\bar{\mathrm{u}}\bar{\mathrm{d}}\bar{\mathrm{d}}$). They are distinct.
- Neutrino: It depends. In the Standard Model (Dirac fermions), neutrinos are distinct from antineutrinos ($L = 1$ vs $L = -1$). If they are Majorana fermions, they are their own antiparticles.

2. Examine the following processes, and state for each one whether it is possible or impossible. In the former case, state which interaction is responsible — strong, electromagnetic or weak; in the latter case cite a conservation law that prevents it from occurring.

(a) $\pi^0 \to \gamma + \gamma$

(b) $\pi^0 \to \gamma + \gamma + \gamma$

(c) $\mu^- \to \pi^- + \nu_\mu$

(d) $\tau^- \to \pi^- + \nu_\tau$

(e) $\mathrm{p} + \mathrm{p} \to \mathrm{e}^+ + \mathrm{e}^-$

(f) $\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{p} + \mathrm{p} + \bar{\mathrm{p}}$

(g) $\mathrm{p} \to \mathrm{n} + \mathrm{e}^+ + \nu_\mathrm{e}$

(h) $\mathrm{K}^+ \to \pi^+ + \pi^+ + \pi^-$

(i) $\mathrm{n} \to \bar{\mathrm{p}} + \mathrm{e}^+ + \nu_\mathrm{e}$

(j) $\mathrm{e}^- \to \mathrm{e}^- + \gamma$

(k) $\mathrm{e}^+ + \mathrm{e}^- \to \gamma$

(l) $\mathrm{e}^+ + \mathrm{e}^- \to \gamma + \gamma$

Solution:

(a) Possible. Electromagnetic interaction.

(b) Impossible. Charge conjugation parity (C-parity) is not conserved. π^0 has $C = +1$. The system of 3γ has $C = (-1)^3 = -1$.

(c) Impossible. Energy is not conserved. The mass of μ^- (105.7 MeV) is smaller than the mass of π^- (139.6MeV).

(d) Possible. Weak interaction.

(e) Impossible. Baryon number is not conserved. The initial baryon number is $B = 1+1 = 2$. The final baryon number is $B = 0$.

(f) Possible. Strong interaction.

(g) Impossible. Energy is not conserved. The mass of p (938.3MeV) is smaller than the mass of n (939.6MeV).

(h) Possible. Weak interaction.

(i) Impossible. Baryon number is not conserved. The initial baryon number is $B = 1$. The final baryon number is $B = -1$.

(j) Impossible. Energy and momentum cannot be conserved simultaneously.

(k) Impossible. Energy and momentum cannot be conserved simultaneously.

(l) Possible. Electromagnetic interaction.

3. An experiment is performed to search for evidence of the reaction $\mathrm{p} + \mathrm{p} \to H + \mathrm{K}^+ + \mathrm{K}^-$.

   (a) What are the values of electric charge, lepton number and baryon number of the particle H? How many quarks must H contain?

   (b) A theoretical calculation for the mass of this state H yields a predicted value of $m_H = 2150\text{MeV}$, What is the minimum value of incident-beam proton momentum necessary to produce this state? (Assume that the target protons are at rest.)

Solution:

(a) We use the conservation laws for the reaction $\mathrm{p} + \mathrm{p} \to H + \mathrm{K}^+ + \mathrm{K}^-$.

- **Electric Charge** (Q):
  Initial charge: $Q_\mathrm{i} = 1 + 1 = 2$.
  Final charge: $Q_\mathrm{f} = Q_H + 1 + (-1) = Q_H$.
  Conservation $Q_\mathrm{i} = Q_\mathrm{f}$ implies:
  $$Q_H = 2. \tag{36}$$

- **Baryon Number** (B):
  Initial baryon number: $B_\mathrm{i} = 1 + 1 = 2$.
  Final baryon number: $B_\mathrm{f} = B_H + 0 + 0 = B_H$.
  Conservation $B_\mathrm{i} = B_\mathrm{f}$ implies:
  $$B_H = 2. \tag{37}$$

- **Lepton Number** (L):
  Initial lepton number: $L_\mathrm{i} = 0 + 0 = 0$;
  Final lepton number: $L_\mathrm{f} = L_H + 0 + 0 = L_H$.
  Conservation $L_\mathrm{i} = L_\mathrm{f}$ implies:
  $$L_H = 0. \tag{38}$$

- **Number of Quarks**:

  A baryon ($B = 1$) contains 3 quarks. The particle H has baryon number $B = 2$. It is a dibaryon.

  The number of quarks is:

$$N_{\text{quarks}} = 2 \times 3 = 6. \tag{39}$$

(b) We calculate the threshold momentum.

Let P_{beam} be the four-momentum of the incident proton.

Let P_{target} be the four-momentum of the target proton (at rest).

Let P_{final} be the total four-momentum of the final particles $(H, \mathrm{K}^+, \mathrm{K}^-)$.

The conservation of four-momentum is:

$$P_{\text{beam}} + P_{\text{target}} = P_{\text{final}}. \tag{40}$$

We square both sides to find the invariant mass squared s:

$$(P_{\text{beam}} + P_{\text{target}})^2 = P_{\text{final}}^2. \tag{41}$$

We expand the left side:

$$P_{\text{beam}}^2 + P_{\text{target}}^2 + 2P_{\text{beam}} \cdot P_{\text{target}} = m_{\mathrm{p}}^2 + m_{\mathrm{p}}^2 + 2E_{\text{beam}} m_{\mathrm{p}}. \tag{42}$$

For the minimum energy (threshold), all final particles move together with the same velocity. The invariant mass of the final state is the sum of their masses:

$$\sqrt{P_{\text{final}}^2} = M_{\text{final}} = m_H + m_{\mathrm{K}^+} + m_{\mathrm{K}^-}. \tag{43}$$

We substitute the values:

$$2m_{\mathrm{p}}^2 + 2E_{\text{beam}} m_{\mathrm{p}} = (m_H + 2m_{\mathrm{K}})^2. \tag{44}$$

We solve for the beam energy E_{beam}:

$$E_{\text{beam}} = \frac{(m_H + 2m_{\mathrm{K}})^2 - 2m_{\mathrm{p}}^2}{2m_{\mathrm{p}}}. \tag{45}$$

We use the values: $m_{\mathrm{p}} \approx 938.3\text{MeV}$, $m_{\mathrm{K}} \approx 493.7\text{MeV}$, $m_H = 2150\text{MeV}$.

$$M_{\text{final}} = 2150 + 2(493.7) = 3137.4\text{MeV}. \tag{46}$$

$$E_{\text{beam}} = \frac{(3137.4)^2 - 2(938.3)^2}{2(938.3)} = \frac{9843279 - 1760814}{1876.6} \approx 4307\text{MeV}. \tag{47}$$

The beam momentum is:

$$|\vec{p}_{\text{beam}}| = \sqrt{E_{\text{beam}}^2 - m_{\mathrm{p}}^2} = \sqrt{4307^2 - 938.3^2} \approx 4204\text{MeV}. \tag{48}$$

The minimum momentum is approximately $4.2\text{GeV}/c$.

4. (a) Write down the possible values of (I_1, I_3), where $I(I_3)$ stands for the isospin (its third component), for the following particles or systems:

i. π^+, π^-, and π^0;

ii. ρ^+, ρ^-, and ρ^0;

iii. $\pi^+\pi^-, \pi^+\pi^0, \pi^-\pi^0$, and $\pi^0\pi^0$.

(b) Using the isospin invariance to examine the following processes and state for each one whether it is possible or impossible:

i. $\rho^+ \to \pi^+\pi^0$;

ii. $\rho^- \to \pi^-\pi^0$;

iii. $\rho^0 \to \pi^0\pi^0$;

iv. $\rho^0 \to \pi^+\pi^-$.

Solution:

(a) We list the values of (I, I_3).

i. **Pions** (π): The pion is an isospin triplet ($I = 1$).

- π^+: $(1, +1)$.
- π^0: $(1, 0)$.
- π^-: $(1, -1)$.

ii. **Rho mesons** (ρ): The rho meson is an isospin triplet ($I = 1$).

- ρ^+: $(1, +1)$.
- ρ^0: $(1, 0)$.
- ρ^-: $(1, -1)$.

iii. **Two-pion systems**: We add the isospins of two pions ($I = 1$ and $I = 1$). The total isospin I can be $0, 1, 2$. The third component is $I_3 = I_{3a} + I_{3b}$.

- $\pi^+\pi^-$: $I_3 = +1 + (-1) = 0$. Possible $I \in \{0, 1, 2\}$.
  Values: $(0, 0), (1, 0), (2, 0)$.
- $\pi^+\pi^0$: $I_3 = +1 + 0 = +1$. Possible $I \in \{1, 2\}$ (since $I \geq |I_3|$).
  Values: $(1, +1), (2, +1)$.
- $\pi^-\pi^0$: $I_3 = -1 + 0 = -1$. Possible $I \in \{1, 2\}$.
  Values: $(1, -1), (2, -1)$.
- $\pi^0\pi^0$: $I_3 = 0 + 0 = 0$. For identical bosons, the Clebsch-Gordan coefficient for $I = 1$ vanishes ($\langle 1, 0; 1, 0 | 1, 0\rangle = 0$). Only symmetric isospin states are allowed if the spatial part is symmetric (or antisymmetric if spatial is antisymmetric). Generally for $I_3 = 0+0$, the possible total isospins are $I \in \{0, 2\}$.
  Values: $(0, 0), (2, 0)$.

(b) We analyze the decays using isospin invariance and Bose statistics.

The decay is $\rho \to \pi + \pi$.

Conservation of Isospin: The ρ meson has $I = 1$. The strong interaction conserves isospin. So the final $\pi\pi$ system must have total isospin $I = 1$.

Angular Momentum and Parity: The ρ has spin $S=1$. Pions have spin $S=0$. Conservation of angular momentum ($1 \to 0+0+L$) requires the orbital angular momentum to be $L=1$.

Bose Statistics: Pions are bosons. The total wave function must be symmetric under the exchange of the two pions.

$$\Psi_{\text{total}} = \psi_{\text{space}} \times \psi_{\text{isospin}}. \tag{49}$$

- The spatial part with $L=1$ is antisymmetric: $\psi_{\text{space}}(-x) = (-1)^1\psi_{\text{space}}(x) = -\psi_{\text{space}}(x)$.
- To make Ψ_{total} symmetric, the isospin part ψ_{isospin} must be **antisymmetric**.
- For two isospin-1 particles, the $I=1$ state is antisymmetric, while $I=0$ and $I=2$ are symmetric.

Conclusion: The decay is possible only if the $\pi\pi$ pair is in the $I=1$ state.

i. $\rho^+ \to \pi^+\pi^0$:

$I_3 = 1+0 = 1$. The system $\pi^+\pi^0$ can have $I=1$ or $I=2$. The $I=1$ state is allowed.

Possible. (Strong interaction).

ii. $\rho^- \to \pi^-\pi^0$:

$I_3 = -1+0 = -1$. The system $\pi^-\pi^0$ can have $I=1$ or $I=2$. The $I=1$ state is allowed.

Possible. (Strong interaction).

iii. $\rho^0 \to \pi^0\pi^0$:

$I_3 = 0+0 = 0$. The system consists of two identical bosons.

The spatial wave function is antisymmetric ($L=1$).

The isospin wave function for $\pi^0\pi^0$ is symmetric (since they are identical states).

The total wave function would be antisymmetric, which violates Bose statistics.

Alternatively, the Clebsch-Gordan coefficient for coupling two $I=1$ states to $I=1$ with z-components 0 is zero. The $\pi^0\pi^0$ system has no $I=1$ component.

Impossible. (Forbidden by Bose statistics and Isospin conservation).

iv. $\rho^0 \to \pi^+\pi^-$:

$I_3 = 1+(-1) = 0$. The system $\pi^+\pi^-$ is a superposition of $I=0,1,2$.

The $I=1$ component exists and is antisymmetric in isospin. Combined with the antisymmetric spatial part ($L=1$), the total state is symmetric.

Possible. (Strong interaction).

5. (a) Write down the color, spin, and flavor wave functions for n (neutron).

   (b) Calculate the neutron magnetic moment in terms of those of quarks.

Solution:

(a) Neutron Wave Function (n):

- **Color**: The neutron is a color singlet. The wave function is totally antisymmetric under the exchange of color indices.

$$\psi_{\text{color}} = \frac{1}{\sqrt{6}}(\text{rgb} - \text{rbg} + \text{gbr} - \text{grb} + \text{brg} - \text{bgr}). \tag{50}$$

- **Spin-Flavor**:

  The total wave function of fermions (quarks) must be totally antisymmetric.

$$\Psi_{\text{total}} = \psi_{\text{space}} \times \psi_{\text{color}} \times \Psi_{\text{spin-flavor}}. \tag{51}$$

  The ground state has orbital angular momentum $L = 0$, so ψ_{space} is symmetric. ψ_{color} is antisymmetric. Therefore, $\Psi_{\text{spin-flavor}}$ must be **symmetric** under the exchange of any two quarks.

  We construct the symmetric state using the mixed-symmetric basis for three particles.

  The flavor states for n (udd) are:

$$\phi_\rho = \frac{1}{\sqrt{2}}(\text{du} - \text{ud})\text{d}, \quad \phi_\lambda = \frac{1}{\sqrt{6}}(2\text{ddu} - \text{udd} - \text{dud}). \tag{52}$$

  (Here ϕ_ρ is antisymmetric in 1-2, ϕ_λ is symmetric in 1-2).

  The spin states for spin-$\frac{1}{2}$ are:

$$\chi_\rho = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\,|\uparrow\rangle\,, \quad \chi_\lambda = \frac{1}{\sqrt{6}}(2\,|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle). \tag{53}$$

  The symmetric spin-flavor wave function is:

$$\Psi_{\text{SF}} = \frac{1}{\sqrt{2}}(\phi_\rho\chi_\rho + \phi_\lambda\chi_\lambda). \tag{54}$$

(b) Neutron Magnetic Moment:

We calculate the magnetic moment $\mu_{\text{n}} = \langle \text{n}|\hat{\mu}_z|\text{n}\rangle$. The operator is $\hat{\mu}_z = \sum_{i=1}^{3} \mu_i \sigma_{zi}$.

Instead of integrating the full wave function, we use a simpler physical argument based on symmetry.

The neutron consists of two d quarks and one u quark (udd). The two d quarks are identical fermions. In the ground state ($L = 0$), their spatial wave function is symmetric. Their color wave function is antisymmetric (part of the total color singlet). Therefore, their combined spin-flavor wave function must be symmetric. Since they have the same flavor (d), the flavor part is symmetric. Thus, the spin part of the two d quarks must be **symmetric**. Two spin-$\frac{1}{2}$ particles in a symmetric spin state form a triplet ($S = 1$). So, the two d quarks couple to spin $S_{\text{dd}} = 1$.

The total spin of the neutron is $S = \frac{1}{2}$. We couple the d-pair ($S = 1$) with the u quark ($S = \frac{1}{2}$) to get total spin $\frac{1}{2}$.

Using Clebsch-Gordan coefficients for $1 \otimes \frac{1}{2} \to \frac{1}{2}$:

$$|\text{n}\uparrow\rangle = \sqrt{\frac{2}{3}}\,|1,1\rangle_{\text{dd}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle_{\text{u}} - \sqrt{\frac{1}{3}}\,|1,0\rangle_{\text{dd}}\left|\frac{1}{2},+\frac{1}{2}\right\rangle_{\text{u}}. \tag{55}$$

Let's write out the spin states explicitly:

- $|1,1\rangle_{\mathrm{dd}} = |\uparrow\uparrow\rangle_{\mathrm{dd}}$.
- $|1,0\rangle_{\mathrm{dd}} = \dfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_{\mathrm{dd}} + |\downarrow\uparrow\rangle_{\mathrm{dd}})$.

The magnetic moment operator is $\hat{\mu}_z = \mu_{\mathrm{u}}\sigma_{z\mathrm{u}} + \mu_{\mathrm{d}}(\sigma_{z\mathrm{d}1} + \sigma_{z\mathrm{d}2})$.

We evaluate the expectation value for each term in the superposition.

Term 1: $|1\rangle = |\uparrow\uparrow\rangle_{\mathrm{dd}} \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle_{\mathrm{u}}$. Probability $P_1 = \left(\sqrt{\dfrac{2}{3}}\right)^2 = \dfrac{2}{3}$.

$$\mu_1 = \mu_{\mathrm{d}}(1+1) + \mu_{\mathrm{u}}(-1) = 2\mu_{\mathrm{d}} - \mu_{\mathrm{u}}. \tag{56}$$

Term 2: $|2\rangle = \dfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)_{\mathrm{dd}} \left|\dfrac{1}{2}, +\dfrac{1}{2}\right\rangle_{\mathrm{u}}$. Probability $P_2 = \left(-\sqrt{\dfrac{1}{3}}\right)^2 = \dfrac{1}{3}$.

The dd pair has total $S_z = 0$, so $\sigma_{z\mathrm{d}1} + \sigma_{z\mathrm{d}2} = 0$. The u quark has spin up.

$$\mu_2 = \mu_{\mathrm{d}}(0) + \mu_{\mathrm{u}}(1) = \mu_{\mathrm{u}}. \tag{57}$$

The total magnetic moment is the weighted sum:

$$\mu_{\mathrm{n}} = P_1\mu_1 + P_2\mu_2 = \frac{2}{3}(2\mu_{\mathrm{d}} - \mu_{\mathrm{u}}) + \frac{1}{3}(\mu_{\mathrm{u}}). \tag{58}$$

Simplify the expression:

$$\mu_{\mathrm{n}} = \frac{4}{3}\mu_{\mathrm{d}} - \frac{2}{3}\mu_{\mathrm{u}} + \frac{1}{3}\mu_{\mathrm{u}} = \frac{4}{3}\mu_{\mathrm{d}} - \frac{1}{3}\mu_{\mathrm{u}}. \tag{59}$$