Gravitational Waves from Individual SMBBHs (Continuous Waves)

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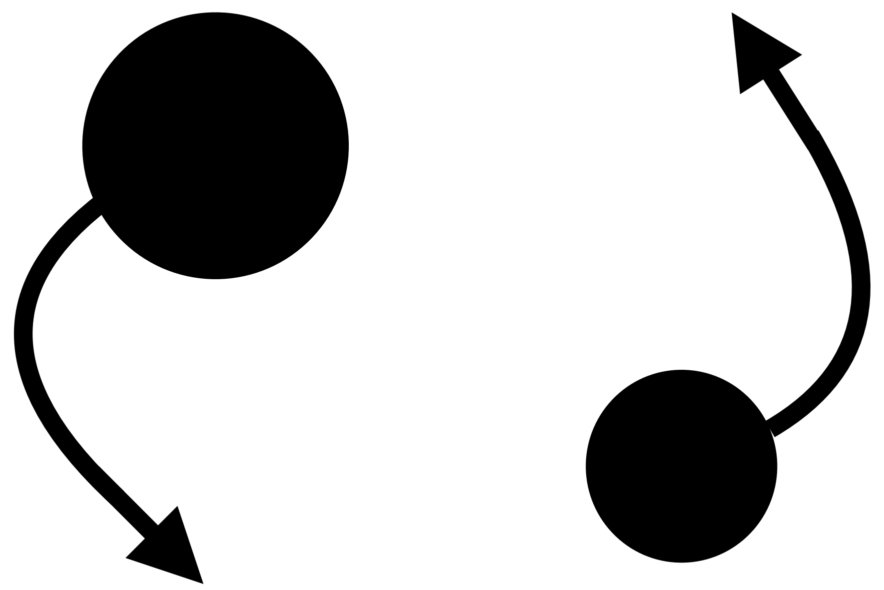
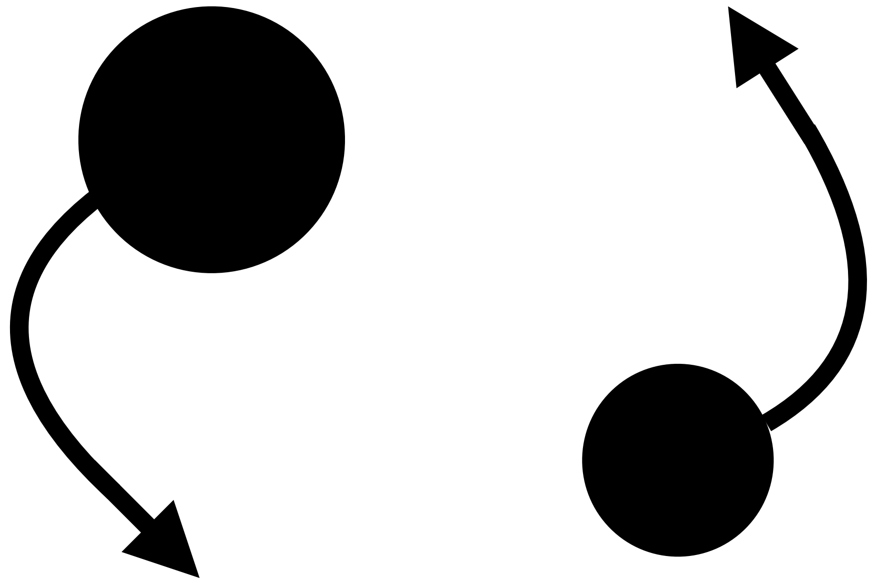


For pulsar timing arrays, **continuous waves** come from individual circular supermassive binary black holes.

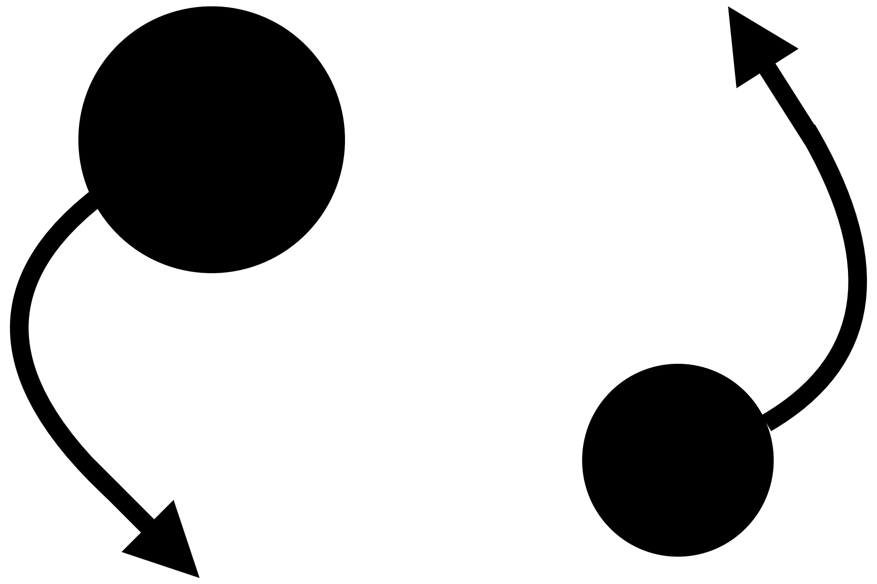
Image credits: J. Cuadra, D. Madison, S. Burke-Spolaor



Consider a binary: $$M = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

How is the orbital frequency related to the masses and orbital separation?



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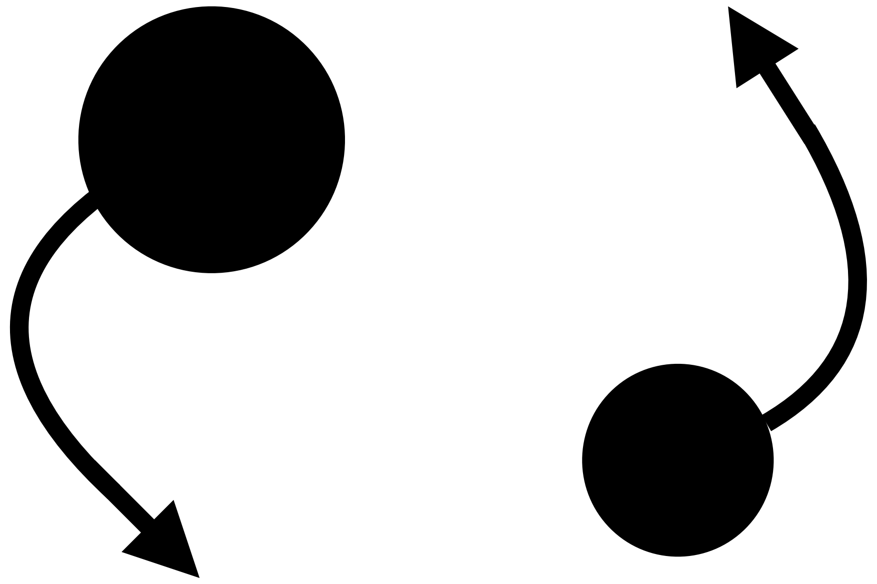
The orbital separation is related to the orbital frequency:

$$\mu\, a\, \omega^2 = \frac{M\mu}{a^2} \Rightarrow \omega = \sqrt{\frac{M}{a^3}}$$



Pulsar timing arrays see gravitational waves with frequencies between a few and a few hundred nanohertz. What are some typical orbital separations for these systems?

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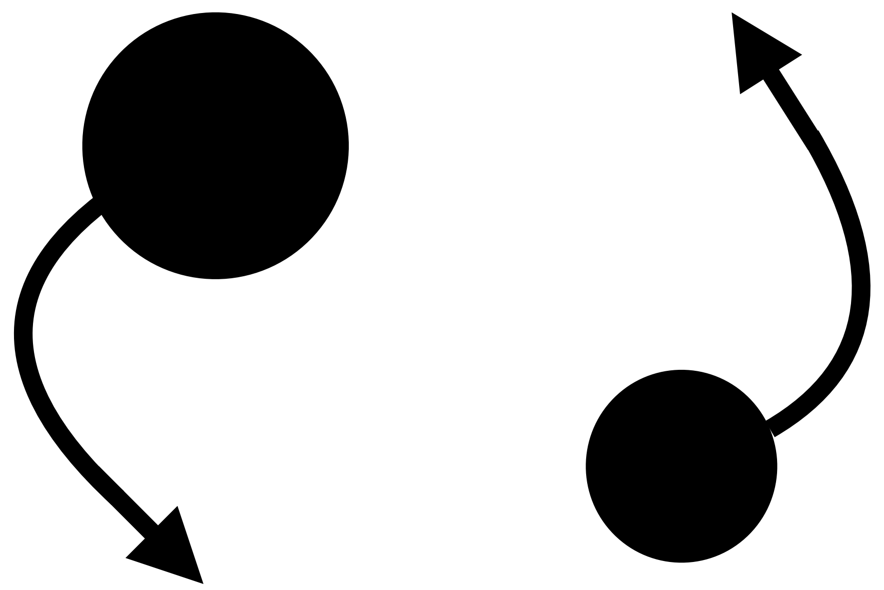
Geometric units:

$$1\ M_\odot = 5\ \mu\mathrm{s}$$

$$1\ \mathrm{pc} = 3\ \mathrm{yr}$$

$$1\ \mathrm{yr} = 3 \times 10^7\ \mathrm{s}$$

$$1/(1\ \mathrm{yr}) = 3 \times 10^{-8}\ \mathrm{Hz}$$



Pulsar timing arrays see gravitational waves with frequencies between a few and a few hundred nanohertz. What are some typical orbital separations for these systems?

$$\omega = \sqrt{\frac{M}{a^3}}$$

$$M = 10^9\ M_\odot,\ \omega = 10^{-8}\ \mathrm{Hz} \Rightarrow a = 0.04\ \mathrm{pc}$$

$$M = 10^{10}\ M_\odot,\ \omega = 10^{-8}\ \mathrm{Hz} \Rightarrow a = 0.1\ \mathrm{pc}$$

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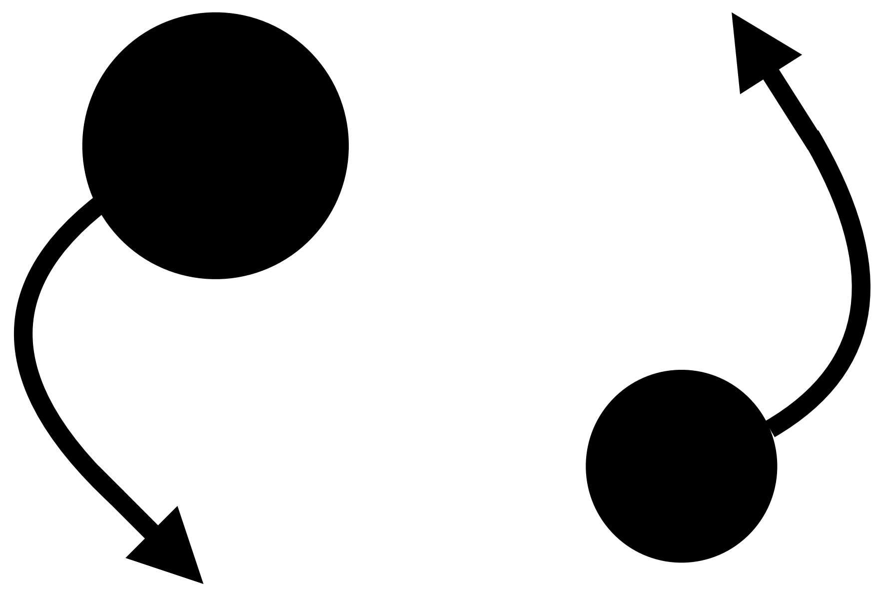

Consider a binary: $M = m_1 + m_2$

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The orbital separation is related to the orbital frequency:

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Gravitational waves carry away energy:

$$\frac{dE}{dt} = -\frac{32}{5}\frac{\mu^2 M^3}{a^5}$$

How does the orbital separation change as the binary loses energy?
How does the orbital frequency change?

How does the orbital separation change as the binary loses energy?

$$
\begin{aligned}
E &= \frac{1}{2}\mu a^2\omega^2 - \frac{M\mu}{a} \\
&= \frac{1}{2}\frac{M\mu}{a} - \frac{M\mu}{a} = -\frac{1}{2}\frac{M\mu}{a}
\end{aligned}
$$

$$
\begin{aligned}
\frac{da}{dt} &= \frac{dE}{dt}\left(\frac{dE}{da}\right)^{-1} \\
&= \left(-\frac{32}{5}\frac{\mu^2 M^3}{a^5}\right)\left(\frac{M\mu}{2a^2}\right)^{-1} \\
&= -\frac{64}{5}\frac{\mu M^2}{a^3}
\end{aligned}
$$

How does the orbital frequency change?

$$\begin{aligned}
\frac{d\omega}{dt} &= \frac{da}{dt}\left(\frac{d\omega}{da}\right) \\
&= \left(-\frac{64}{5}\frac{\mu M^2}{a^3}\right)\left(-\frac{3}{2}M^{1/2}a^{-5/2}\right) \\
&= \frac{96}{5}\mu M^{5/2}a^{-11/2} \\
&= \frac{96}{5}\mu M^{2/3}\omega^{11/3} \\
&= \frac{96}{5}\mathcal{M}^{5/3}\omega^{11/3}
\end{aligned}$$

$$\begin{aligned}
\mathcal{M} &\equiv \mu^{3/5}M^{2/5} \\
&= M\left[\frac{q}{(1+q)^2}\right]^{3/5}
\end{aligned}$$

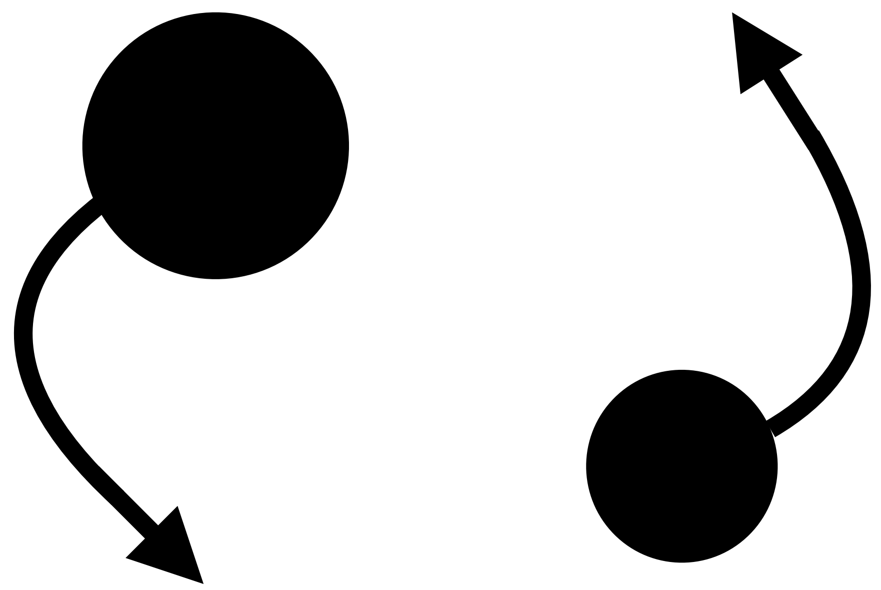
chirp mass

How does the orbital frequency change?

$$\frac{d\omega}{dt} = \frac{96}{5}\mathcal{M}^{5/3}\omega^{11/3}$$

$$\Rightarrow \int \omega^{-11/3}\, d\omega = \frac{96}{5}\mathcal{M}^{5/3}\int dt$$

$$\Rightarrow \omega(t) = \omega_0\left[1 - \frac{256}{5}\mathcal{M}^{5/3}\omega_0^{8/3}(t - t_0)\right]^{-3/8}$$

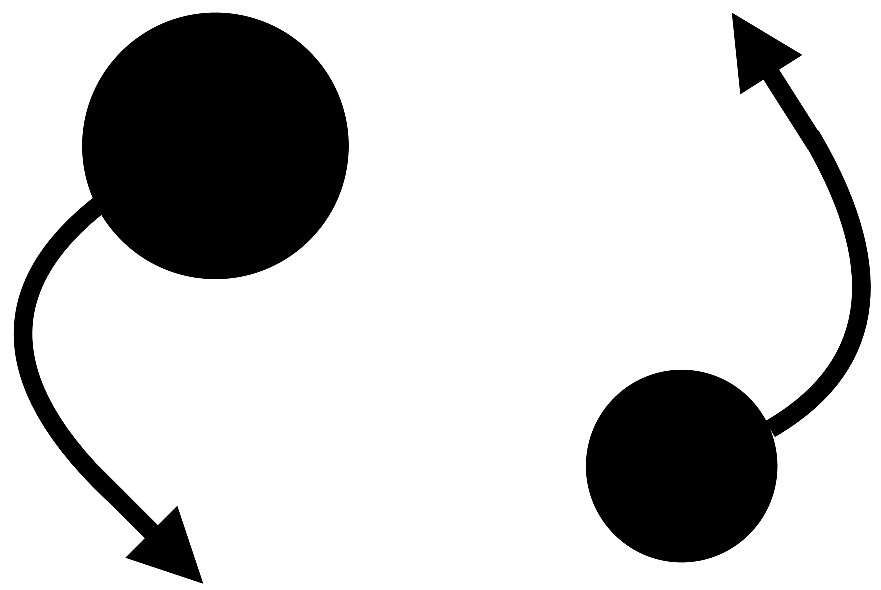


The GWs emitted depend on the second time-derivative of the quadrupole moment:

$$h^{ij} = \frac{2}{r}\ddot{I}^{ij}(t-r)$$

$$x^i(t) = (a\cos\Phi(t), a\sin\Phi(t), 0) \qquad \frac{d\Phi}{dt} = \omega$$

$$I^{ij} = \begin{pmatrix} \frac{\mu a^2}{2}\left[1+\cos 2\Phi(t)\right] & \frac{\mu a^2}{2}\sin 2\Phi(t) & 0 \\ \frac{\mu a^2}{2}\sin 2\Phi(t) & \frac{\mu a^2}{2}\left[1-\cos 2\Phi(t)\right] & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

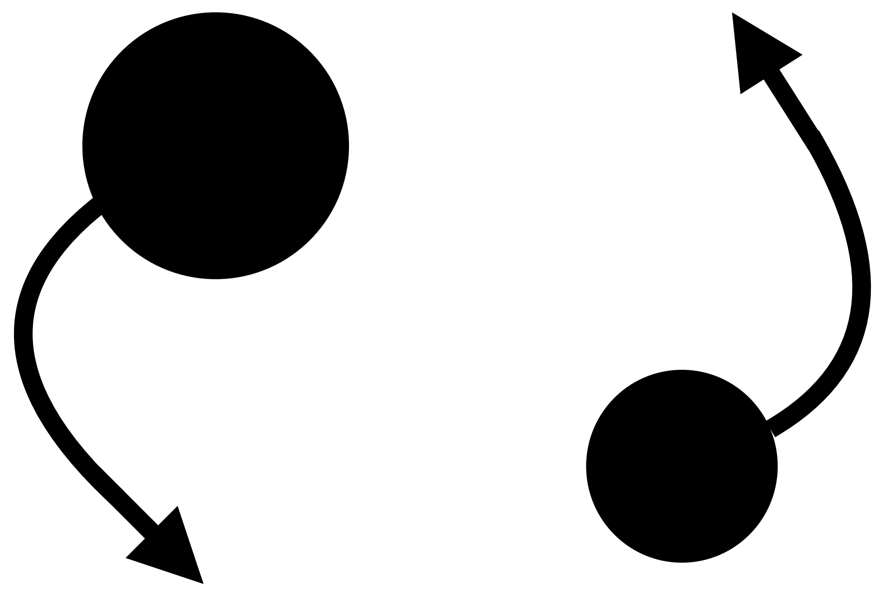


The GWs emitted depend on the second time-derivative of the quadrupole moment:

$$h^{ij} = \frac{2}{r}\ddot{I}^{ij}(t-r)$$

$$\ddot{I}^{ij} = \begin{pmatrix} -2\mu a^2\omega^2\cos 2\Phi(t) & -2\mu a^2\omega^2\sin 2\Phi(t) & 0 \\ -2\mu a^2\omega^2\sin 2\Phi(t) & 2\mu a^2\omega^2\cos 2\Phi(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$h^{ij} = -\frac{4\mu a^2\omega^2}{r}\begin{pmatrix} \cos 2\Phi(t-r) & \sin 2\Phi(t-r) & 0 \\ \sin 2\Phi(t-r) & -\cos 2\Phi(t-r) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



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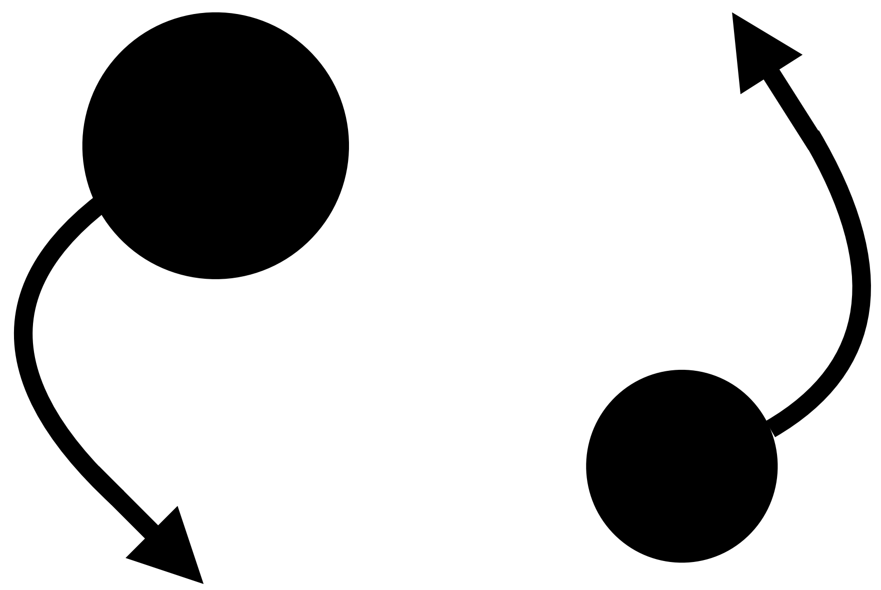
$$\begin{aligned} h_+(t) &= -\frac{4\mu a^2\omega^2}{r}\cos 2\Phi(t) = -\frac{4\mathcal{M}^{5/3}\omega^{2/3}}{r}\cos 2\Phi(t) \\ h_\times(t) &= -\frac{4\mu a^2\omega^2}{r}\sin 2\Phi(t) = -\frac{4\mathcal{M}^{5/3}\omega^{2/3}}{r}\sin 2\Phi(t) \end{aligned}$$



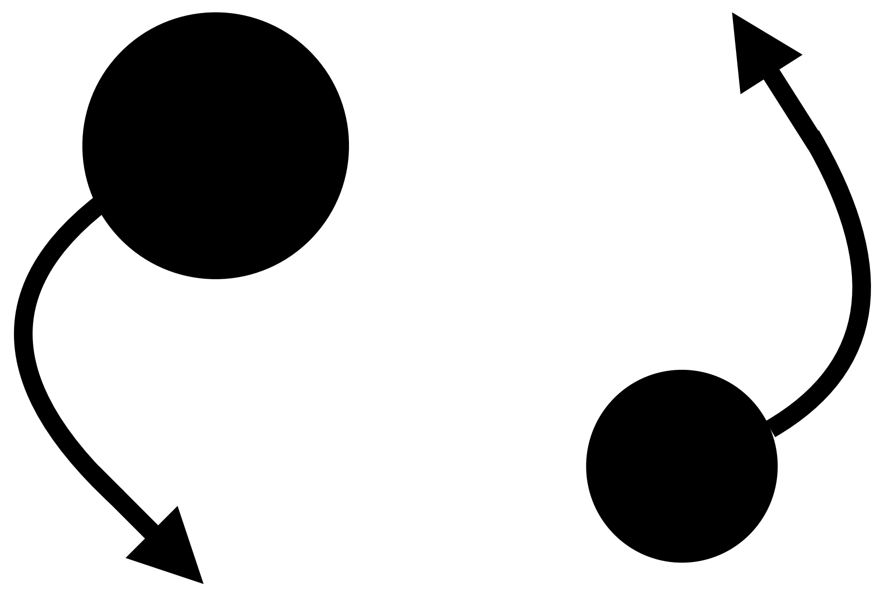
$$
\begin{aligned}
h_+(t) &= -\frac{4\mathcal{M}^{5/3}\omega(t)^{2/3}}{r}\cos 2\Phi(t) \\
h_\times(t) &= -\frac{4\mathcal{M}^{5/3}\omega(t)^{2/3}}{r}\sin 2\Phi(t)
\end{aligned}
$$

Note that the GW frequency is twice the orbital frequency:

$$\omega_{\rm gw} = 2\omega\,, \;\; f_{\rm gw} = \frac{\omega_{\rm gw}}{2\pi} = \frac{\omega}{\pi}$$

We find the orbital phase by integrating the orbital frequency:

$$
\begin{aligned}
\Phi(t) &= \Phi_0 + \int_0^t \omega(t)\,dt \\
&= \Phi_0 + \frac{1}{32}\mathcal{M}^{-5/3}\left[\omega_0^{-5/3} - \omega(t)^{-5/3}\right]
\end{aligned}
$$



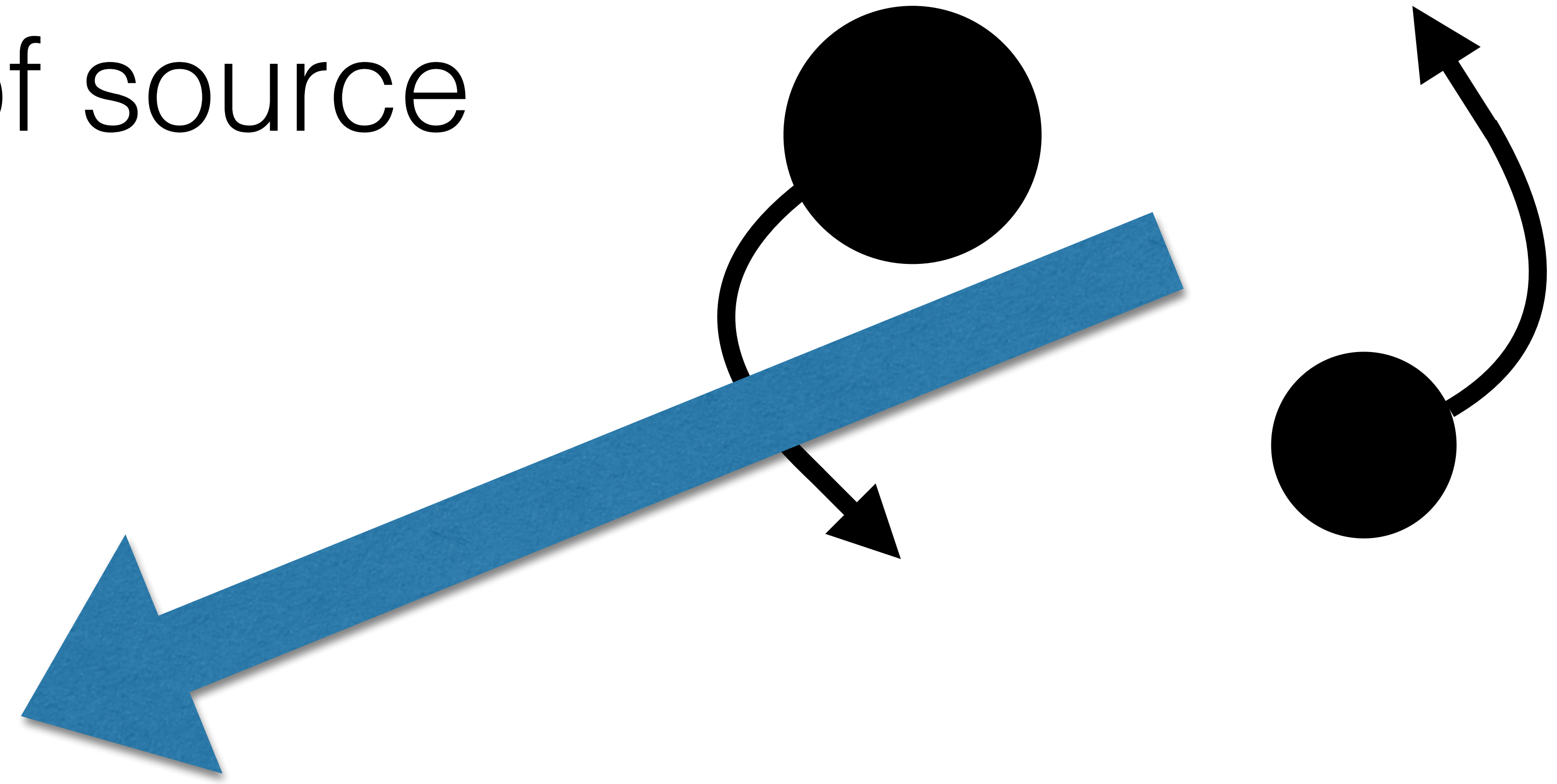
We derived all of this assuming a binary in the xy-plane. We need to introduce an inclination angle, to allow for orbits that are inclined with respect to the xy-plane.

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{pmatrix}$$

$$\begin{aligned} h_+(t) &= -\frac{2\mathcal{M}^{5/3}\omega(t)^{2/3}}{r}\cos 2\Phi(t)\left(1+\cos^2 i\right) \\ h_\times(t) &= -\frac{4\mathcal{M}^{5/3}\omega(t)^{2/3}}{r}\sin 2\Phi(t)\cos i \end{aligned}$$

Sky position of source

(θ, φ)



$$\hat{\Omega} = -\hat{n}$$

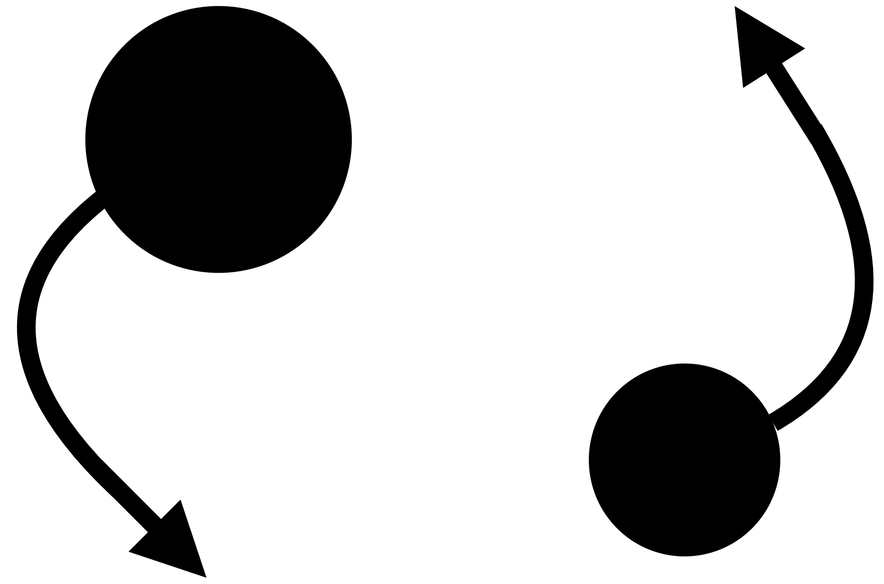
$$\hat{n} = \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}$$

Reference: Taylor et al., 2016

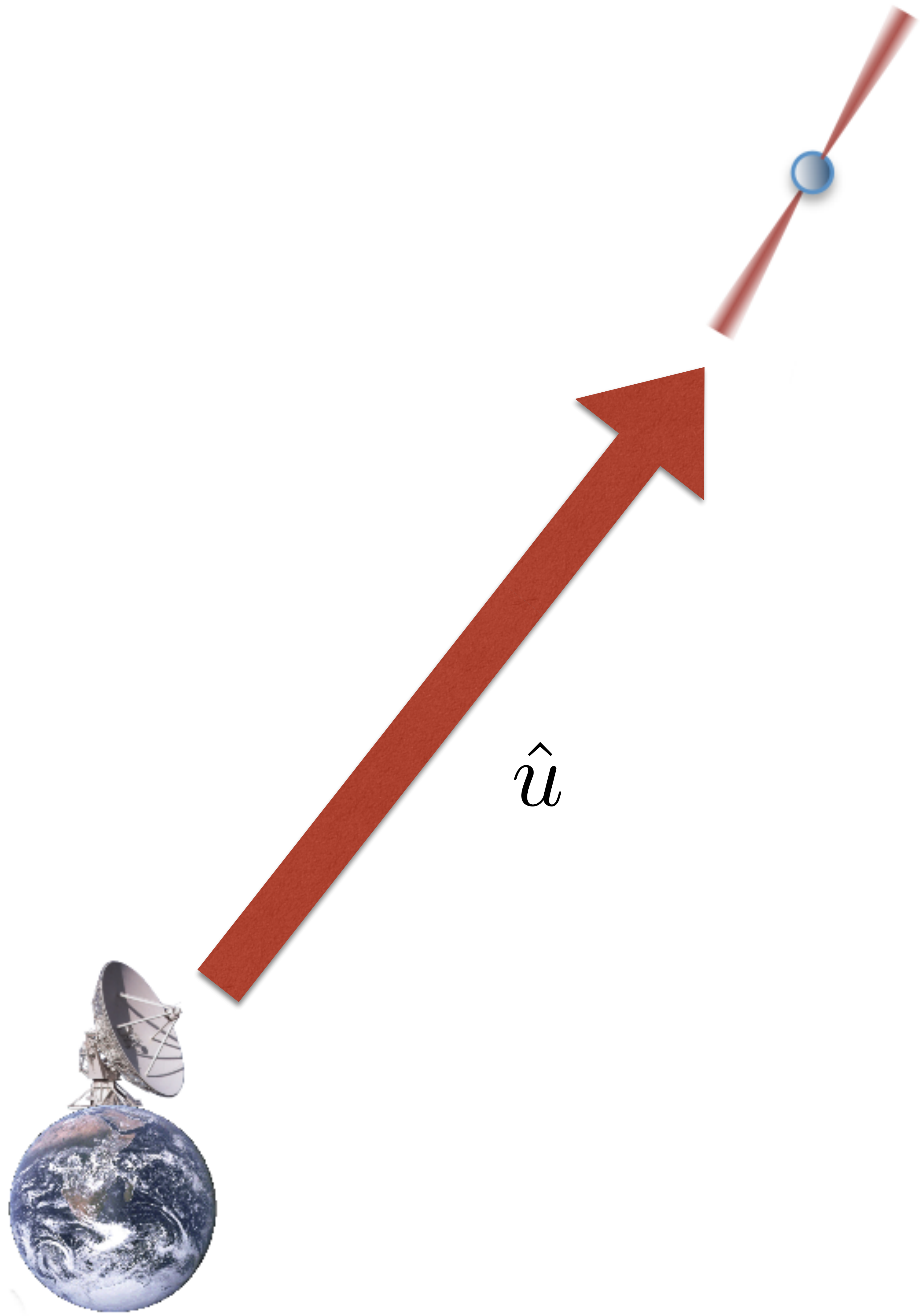
includes dependence on
sky position and
polarization angle

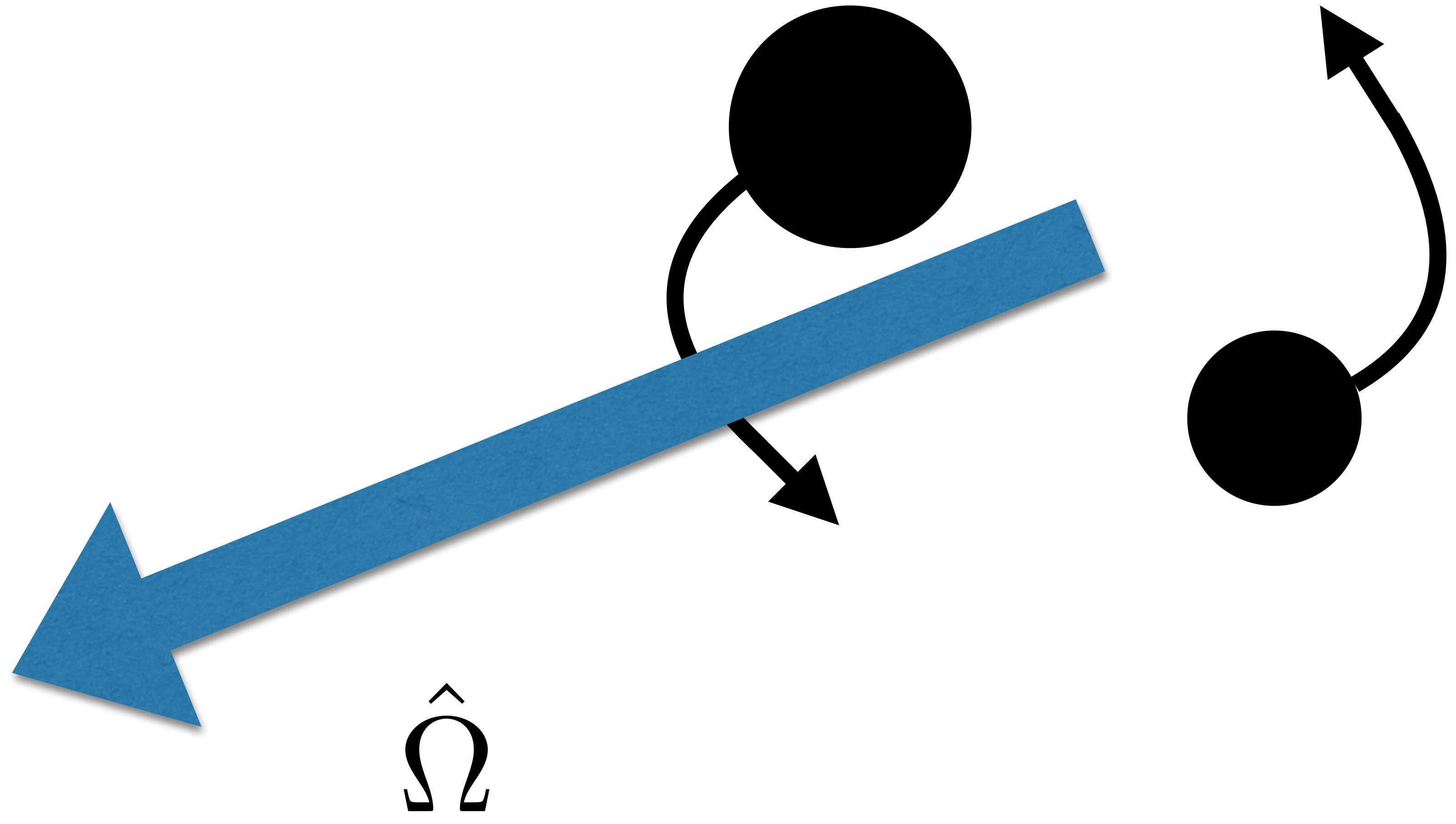
Define GW polarization tensors:

$$
\begin{aligned}
e^{+}_{ab}(\hat{\Omega}) &= \hat{p}_a\hat{p}_b - \hat{q}_a\hat{q}_b \\
e^{\times}_{ab}(\hat{\Omega}) &= \hat{p}_a\hat{q}_b + \hat{q}_a\hat{p}_b
\end{aligned}
$$



$$
\begin{aligned}
\hat{p} &= (\cos\psi\cos\theta\cos\phi - \sin\psi\sin\phi)\ \hat{x} \\
&\quad +(\cos\psi\cos\theta\sin\phi + \sin\psi\cos\phi)\ \hat{y} \\
&\quad -\cos\psi\sin\theta\ \hat{z} \\
\hat{q} &= (\sin\psi\cos\theta\cos\phi + \cos\psi\sin\phi)\ \hat{x} \\
&\quad +(\sin\psi\cos\theta\sin\phi - \cos\psi\cos\phi)\ \hat{y} \\
&\quad -\sin\psi\sin\theta\ \hat{z}
\end{aligned}
$$





redshift of radio signal from pulsar:

$$z = \frac{\delta\nu}{\nu} = \frac{\nu_p - \nu_e}{\nu_e}$$

Therefore, the redshift of the radio pulse induced by GWs is

$$\begin{aligned} z(t,\hat{\Omega}) &\approx \log(\nu_p/\nu_e) \\ &= \sum_{A=+,\times} F^A(\hat{\Omega})\,\Delta h_A(t) \end{aligned}$$

The antenna pattern functions are

$$F^A(\hat{\Omega}) \equiv \frac{1}{2}\frac{\hat{u}^a\hat{u}^b}{1+\hat{\Omega}\cdot\hat{u}}\,e^A_{ab}(\hat{\Omega})$$

The GW induced residuals are

$$s(t,\hat{\Omega}) \quad = \quad \int_0^t z(t,\hat{\Omega})dt = \sum_{A=+,\times} F^A(\hat{\Omega}) \int_0^t \Delta h_A(t)\; dt$$

$$h_+(t) \quad = \quad -\frac{2\mathcal{M}^{5/3}\omega(t)^{2/3}}{r}\cos 2\Phi(t)\left(1+\cos^2 i\right)$$

$$h_\times(t) \quad = \quad -\frac{4\mathcal{M}^{5/3}\omega(t)^{2/3}}{r}\sin 2\Phi(t)\cos i$$

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We can do this integral approximately because the orbital frequency evolves slowly:

$$\begin{aligned} \sin 2\Phi(t) \quad &\to \quad -\frac{1}{2\omega(t)} \cos 2\Phi(t) \\ \cos 2\Phi(t) \quad &\to \quad \frac{1}{2\omega(t)} \sin 2\Phi(t) \end{aligned}$$

Putting it all together,

$$s(t,\hat{\Omega}) \quad = \quad \sum_{A=+,\times} F^A(\hat{\Omega})[s_A(t) - s_A(t_p)]$$

$$\begin{aligned} s_+(t) &= \frac{\mathcal{M}^{5/3}}{d_L\,\omega(t)^{1/3}}\left[-\sin 2\Phi(t)\left(1+\cos^2 i\right)\right] \\ s_\times(t) &= \frac{\mathcal{M}^{5/3}}{d_L\,\omega(t)^{1/3}} 2\cos 2\Phi(t)\cos i \end{aligned}$$

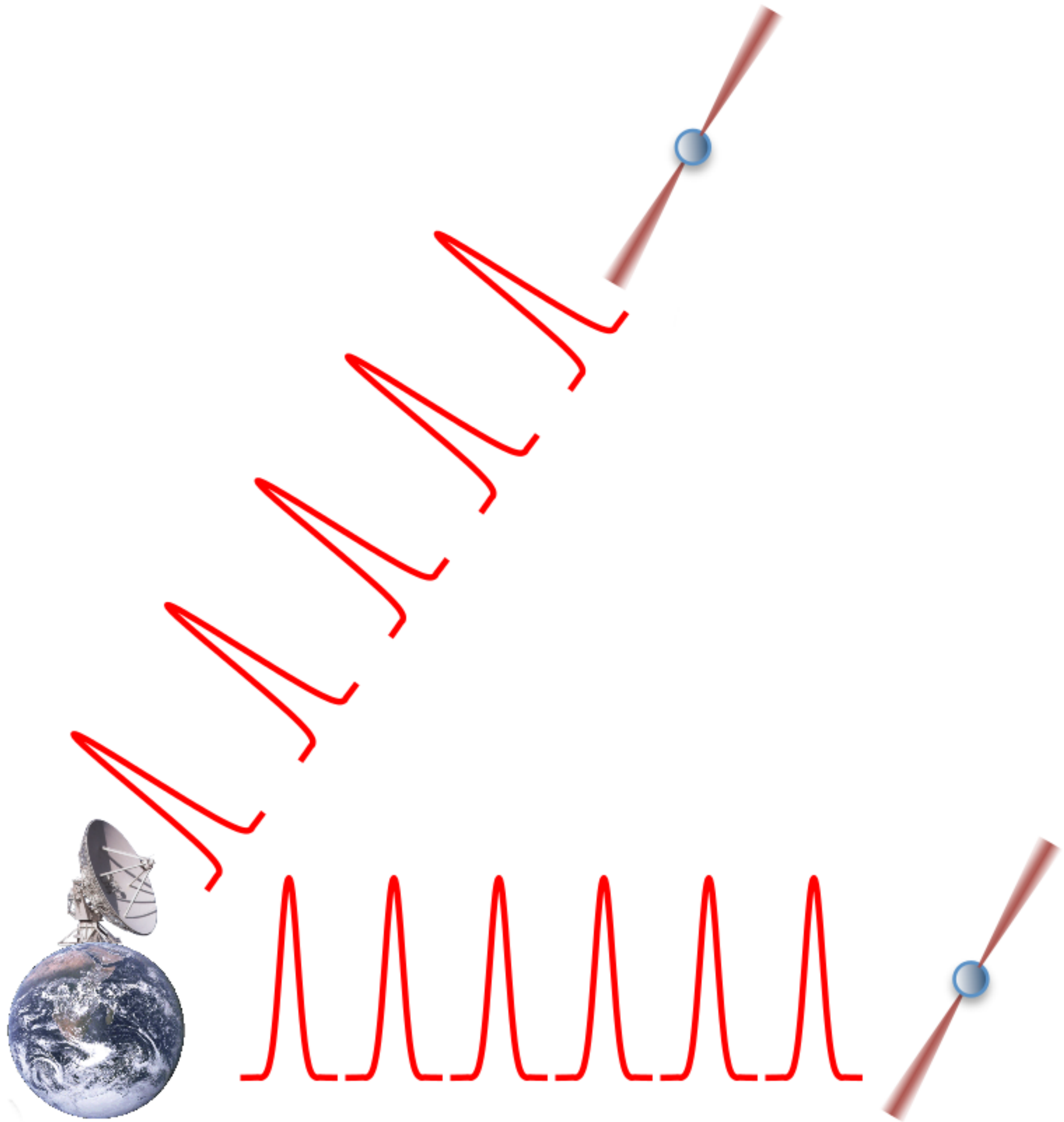
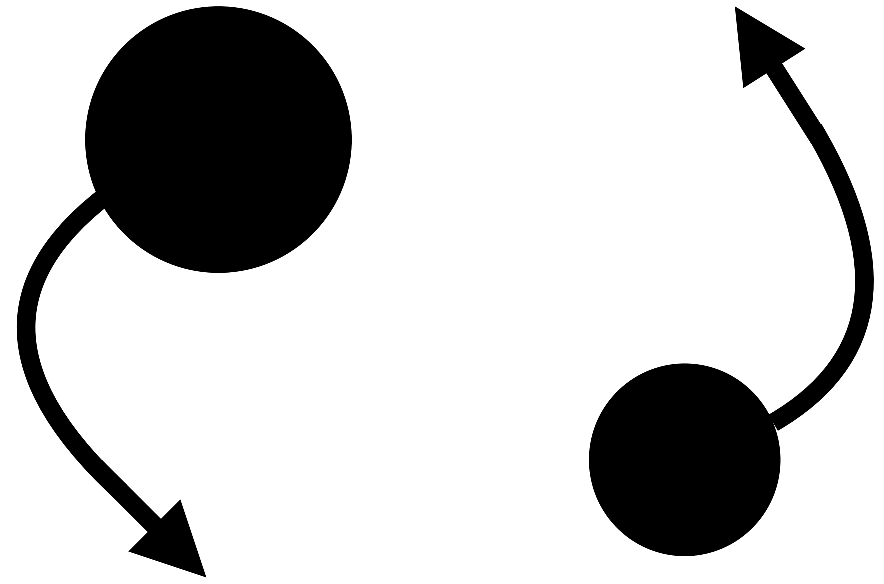


Figure credit: NANOGrav (modified)



antenna pattern

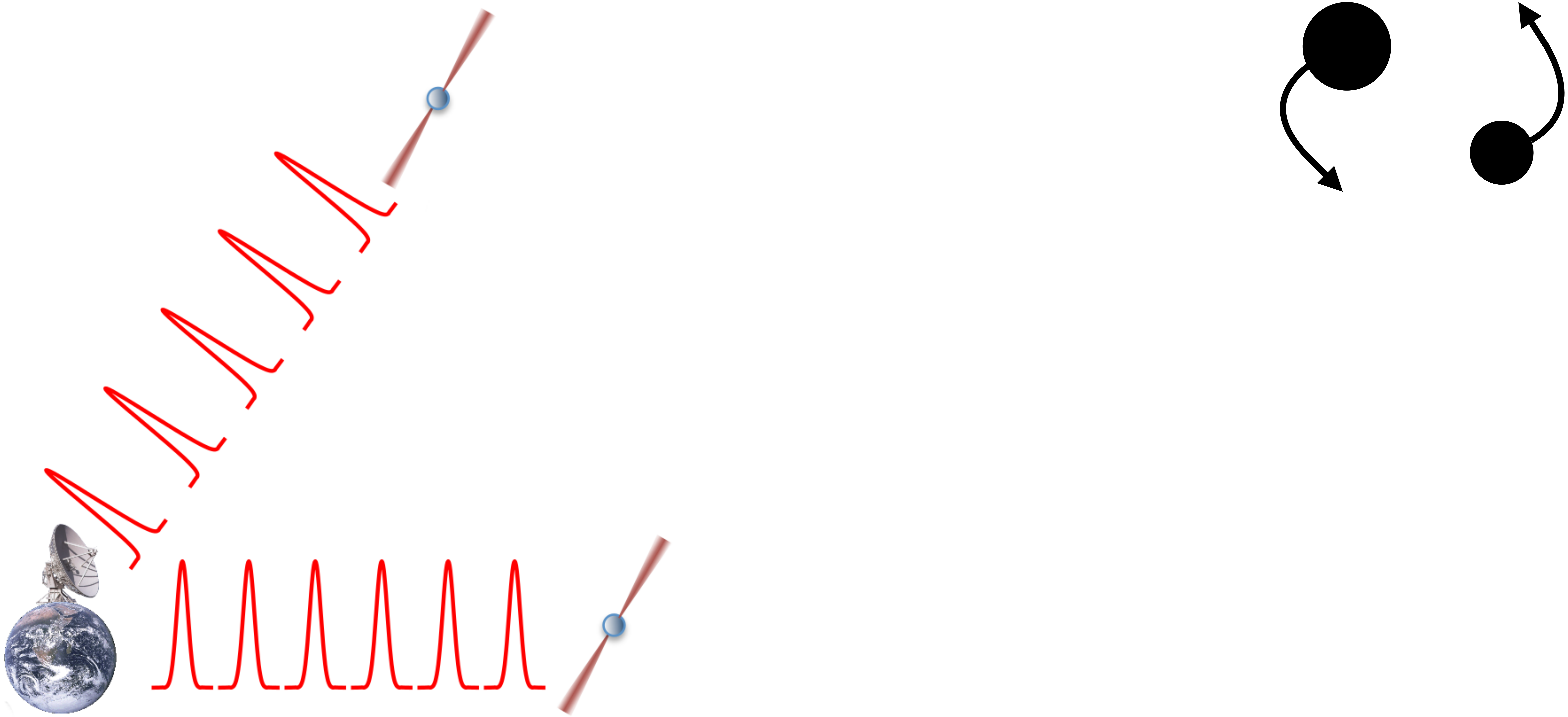
$$s_{+,\times}(t) = F^{+,\times}(\hat{\Omega})\left[s_{+,\times}(t) - s_{+,\times}(t_p)\right]$$

Earth term

pulsar term

The CW signal has eight parameters that go into the Earth term:

$$\vec{\lambda} = \{\theta, \varphi, \psi, i, \Phi_0, \mathcal{M}, \omega_0, d_L\}$$



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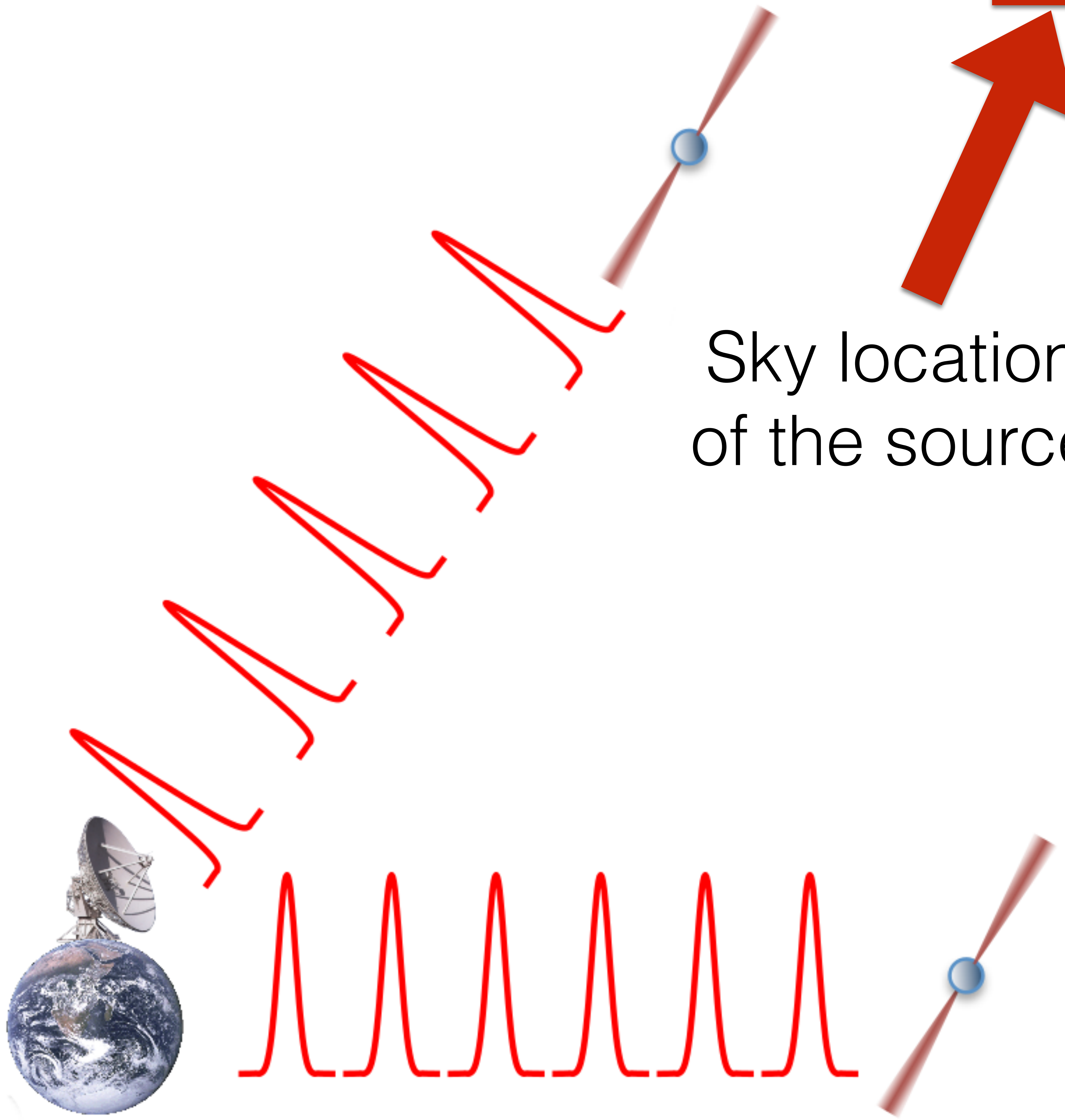
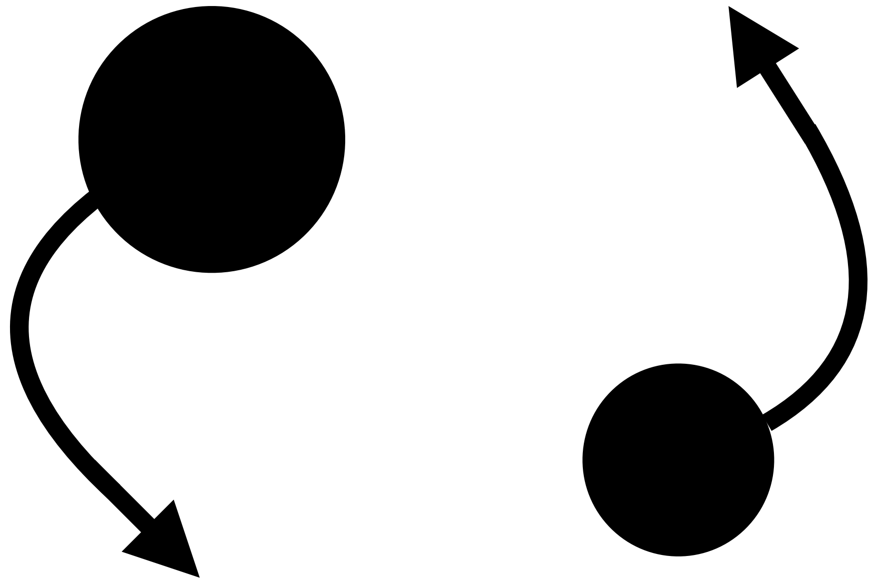


Sky location of the source

Orientation of the binary



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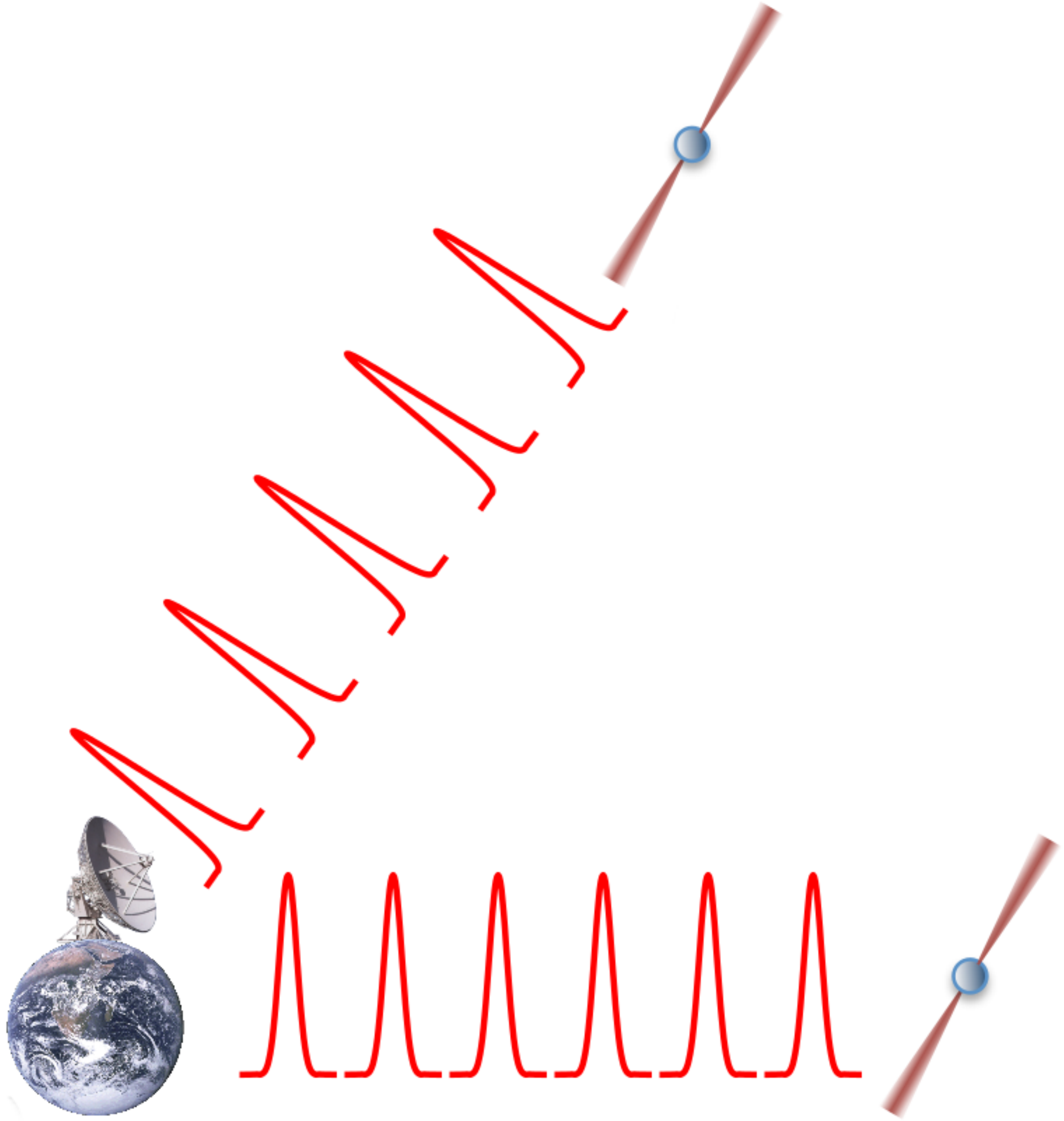
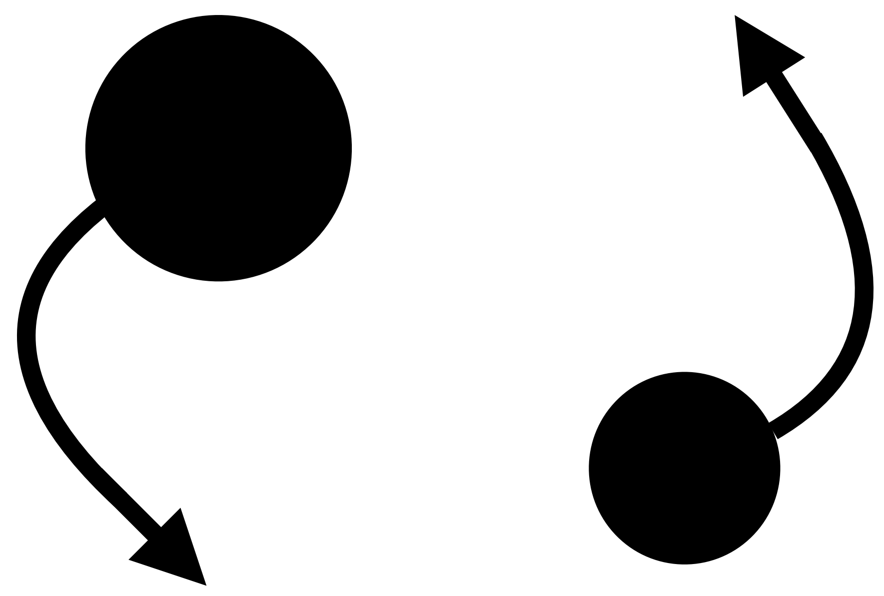
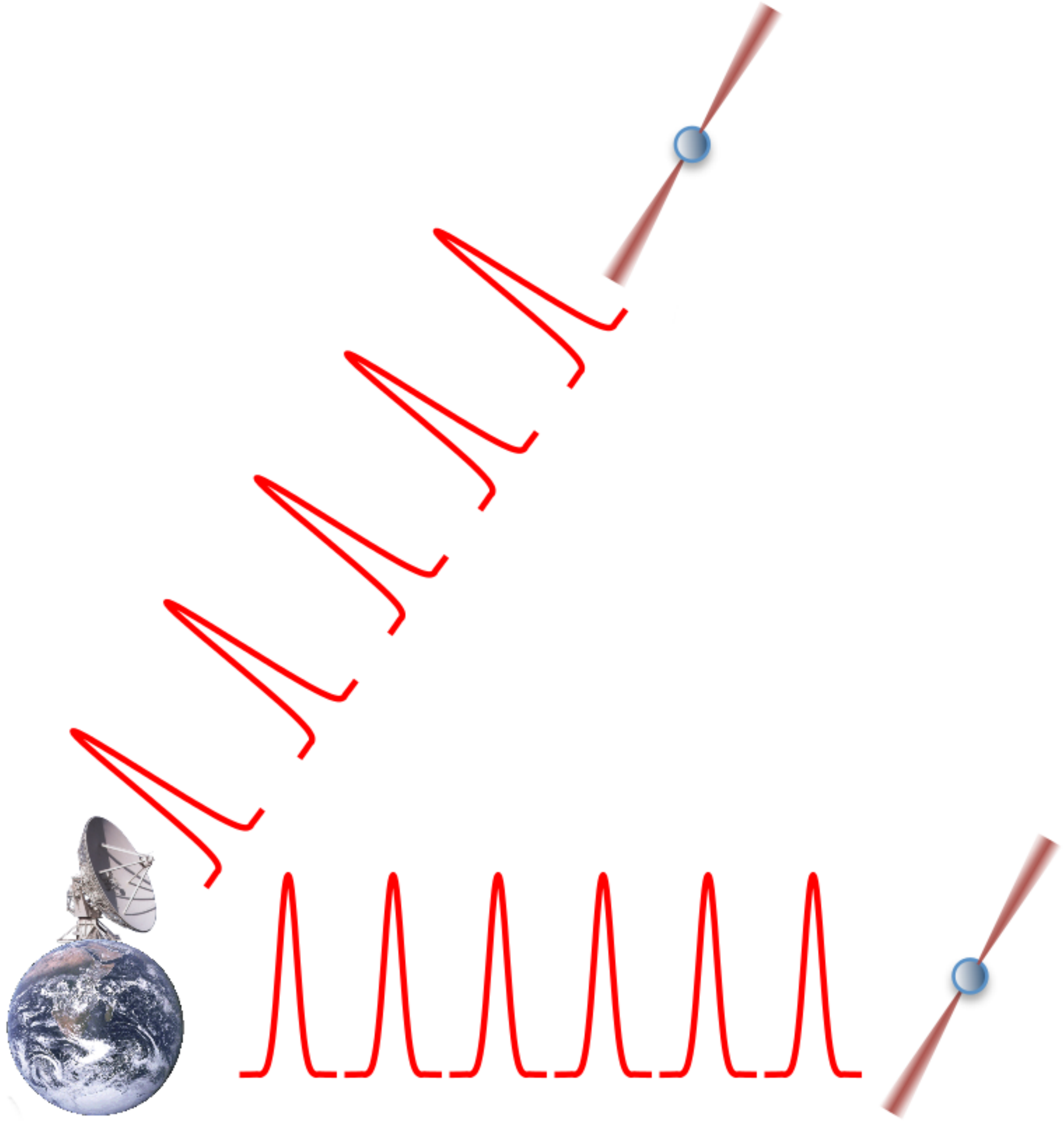


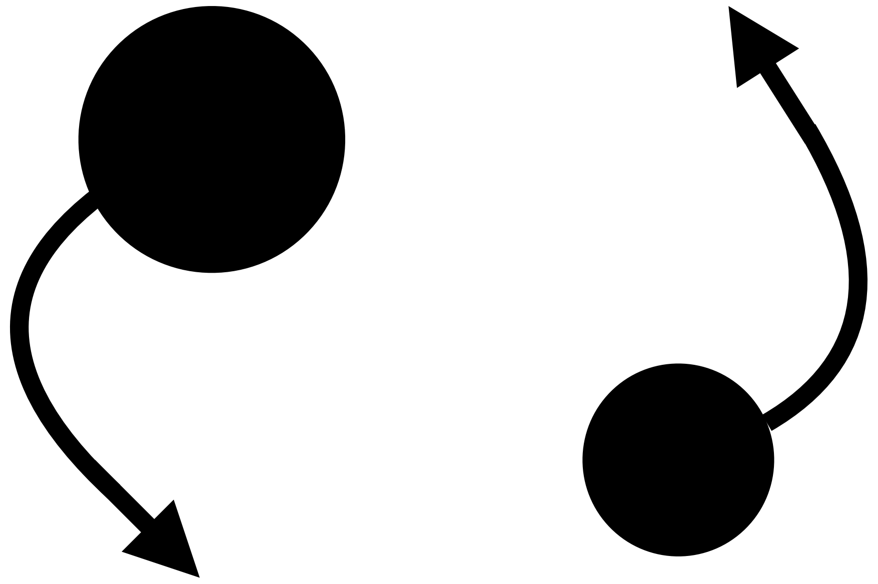


Orbital frequency and phase of the binary

The CW signal has eight parameters that go into the Earth term:

$$\vec{\lambda} = \{\theta, \varphi, \psi, i, \Phi_0, \boxed{\mathcal{M}, \omega_0, d_L}\}$$





Amplitude of the signal

We often parametrize the signal in terms of the strain amplitude instead of the luminosity distance to the source:

$$h_0 = \frac{2\mathcal{M}^{5/3}\omega_0^{2/3}}{d_L} = \frac{2\mathcal{M}^{5/3}(\pi f_{\rm gw})^{2/3}}{d_L}$$

When we include the pulsar terms, we must take into account two things:

- The light travel time between the Earth and the pulsar
- The phase difference between the Earth term and pulsar term

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- The light travel time between the Earth and the pulsar

$$t_p = t - L\left(1 + \hat{\Omega} \cdot \hat{u}\right)$$

For our pulsars, $t - t_p \sim 10^3 \text{ yrs}$

Therefore, there is a difference in the frequency of the Earth term and pulsar term.

How much does the orbital frequency change for our sources?

Consider a few different values for the GW frequency (Earth term) and the chirp mass.

$$\frac{d\omega}{dt} = \frac{96}{5}\mathcal{M}^{5/3}\omega^{11/3}$$

Geometric units:

$$1\ M_\odot = 5\ \mu\mathrm{s}$$

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How much does the orbital frequency change for our sources?

Consider a few different values for the GW frequency (Earth term) and the chirp mass.

$$\mathcal{M} = 10^9 \ M_\odot, \omega = 10^{-8} \ \mathrm{Hz} \Rightarrow \frac{d\omega}{dt} = 1.3 \times 10^{-22}$$

$$\mathcal{M} = 10^{10} \ M_\odot, \omega = 10^{-8} \ \mathrm{Hz} \Rightarrow \frac{d\omega}{dt} = 6 \times 10^{-21}$$

$$\mathcal{M} = 10^9 \ M_\odot, \omega = 10^{-7} \ \mathrm{Hz} \Rightarrow \frac{d\omega}{dt} = 6 \times 10^{-19}$$

$$\mathcal{M} = 10^{10} \ M_\odot, \omega = 10^{-7} \ \mathrm{Hz} \Rightarrow \frac{d\omega}{dt} = 3 \times 10^{-17}$$

We can measure the chirp mass of the binary from the frequency difference between the Earth term and the pulsar term:

$$\begin{aligned}\Delta\omega &= \omega_0 - \omega(t_{p,0}) \\ &= \omega_0 \left\{ 1 - \left[1 - \frac{256}{5} \mathcal{M}^{5/3} \omega_0^{8/3} (t_{p,0} - t_0) \right]^{-3/8} \right\}\end{aligned}$$

$$t_{p,0} - t_0 = -L\left(1 + \hat{\Omega}\cdot\hat{u}\right)$$

When we include the pulsar terms, we must take into account two things:

- The phase difference between the Earth term and pulsar term

$$\Phi(t) = \Phi_0 + \Phi_p + \frac{1}{32}\mathcal{M}^{-5/3}\left[\omega(t_{p,0})^{-5/3} - \omega(t_p)^{-5/3}\right]$$

You can compute the phase difference using the pulsar distance:

$$t_{p,0} = t_0 - L\left(1 + \hat{\Omega}\cdot\hat{u}\right)$$

$$\Phi_p = \frac{1}{32}\mathcal{M}^{-5/3}\left[\omega_0^{-5/3} - \omega(t_{p,0})^{-5/3}\right]$$

But the uncertainty in a pulsar distance is much larger than a GW wavelength, so in practice we treat the pulsar phase as a free parameter.

What are the GW wavelengths for PTAs in pc?

How are pulsar distances measured? What are the typical uncertainties on pulsar distances?

What are the GW wavelengths for PTAs in pc?

$$f_{\rm gw} = 3 \times 10^{-9}~\text{Hz} \Rightarrow \lambda = 10^{17}~\text{m} = 3~\text{pc}$$

$$f_{\rm gw} = 3 \times 10^{-7}~\text{Hz} \Rightarrow \lambda = 10^{15}~\text{m} = 0.03~\text{pc}$$

How are pulsar distances measured? What are the typical uncertainties on pulsar distances?

Timing parallax, VLBI parallax, DM distance, astrometry (if there is a WD companion)

Typical uncertainties are ~10-100 pc

When distances are measured by measuring the parallax, the error on the parallax is not equal to the error on the distance:

$$\pi = \frac{1}{d} \Rightarrow \Delta\pi = -\frac{1}{d^2}\Delta d$$

$$\Rightarrow |\Delta d| = d^2 \, |\Delta\pi|$$

Putting this all together, the CW search requires:

- 8 intrinsic parameters
- 1 pulsar distance parameter per pulsar
- 1 pulsar phase parameter per pulsar
- 2 red noise parameters per pulsar

For the NANOGrav 12.5-year data set, this requires

$$8 + 4 \times \mathrm{N}_{\mathrm{psr}} = 8 + 4 \times 45 = 188 \text{ parameters}$$

We can do frequentist searches for CWs using the F_e and F_p statistics.

We can derive it in a way that is similar to how we derive the optimal statistic:

$$\mathcal{L}(\mathbf{s}|\delta t) = \frac{1}{\sqrt{\det 2\pi C}} \exp\left[-\frac{1}{2}(\delta t - \tilde{s})^T C^{-1} (\delta t - \tilde{s})\right]$$

post-fit signal model

$$\log \Lambda = \log \mathcal{L}(s|\delta t) - \log \mathcal{L}(0|\delta t)$$

We want to find the signal parameters that maximize the log-likelihood ratio.

For the F_e statistic, we only consider the Earth terms, and we write the signal template as

Basis functions are trig functions, include antenna pattern

$$s_\alpha = s^e_\alpha(t, \hat{\Omega}) = \sum_{i=1}^{4} \left[a_i \, A^i_\alpha(t, \theta, \varphi, \omega_0) \right]$$

one of these per pulsar

coefficients are functions of extrinsic parameters (inclination, GW polarization angle, strain amplitude, etc.)

For the F_p statistic, we include pulsar terms, but we assume that the GW frequency is the same at all of the pulsars and the Earth (i.e., the GW frequency does not evolve).

Basis functions

$$s_\alpha(t,\hat{\Omega}) = b_{1,\alpha}\frac{\sin(2\omega_0 t)}{\omega_0^{1/3}} + b_{2,\alpha}\frac{\cos(2\omega_0 t)}{\omega_0^{1/3}}$$

one of these per pulsar

Coefficients contain all the information about the source and its location and orientation

For the F_p statistic, we include the Earth terms and pulsar terms, and we write the signal template as

$$s_\alpha(t, \hat{\Omega}) = b_{1,\alpha} \frac{\sin(2\omega_0 t)}{\omega_0^{1/3}} + b_{2,\alpha} \frac{\cos(2\omega_0 t)}{\omega_0^{1/3}}$$

Find the coefficients that maximize the log-likelihood ratio for a source with a given frequency:

$$P_\alpha^i = (\delta t_\alpha | \tilde{B}_\alpha^i)$$

$$Q_\alpha^{ij} = (\tilde{B}_\alpha^i | \tilde{B}_\alpha^j)$$

$$\log \Lambda = \sum_{\alpha=1}^{M} \left[b_{i\alpha} P_\alpha^i - \frac{1}{2} Q_\alpha^{ij} b_{i\alpha} b_{j\alpha} \right]$$

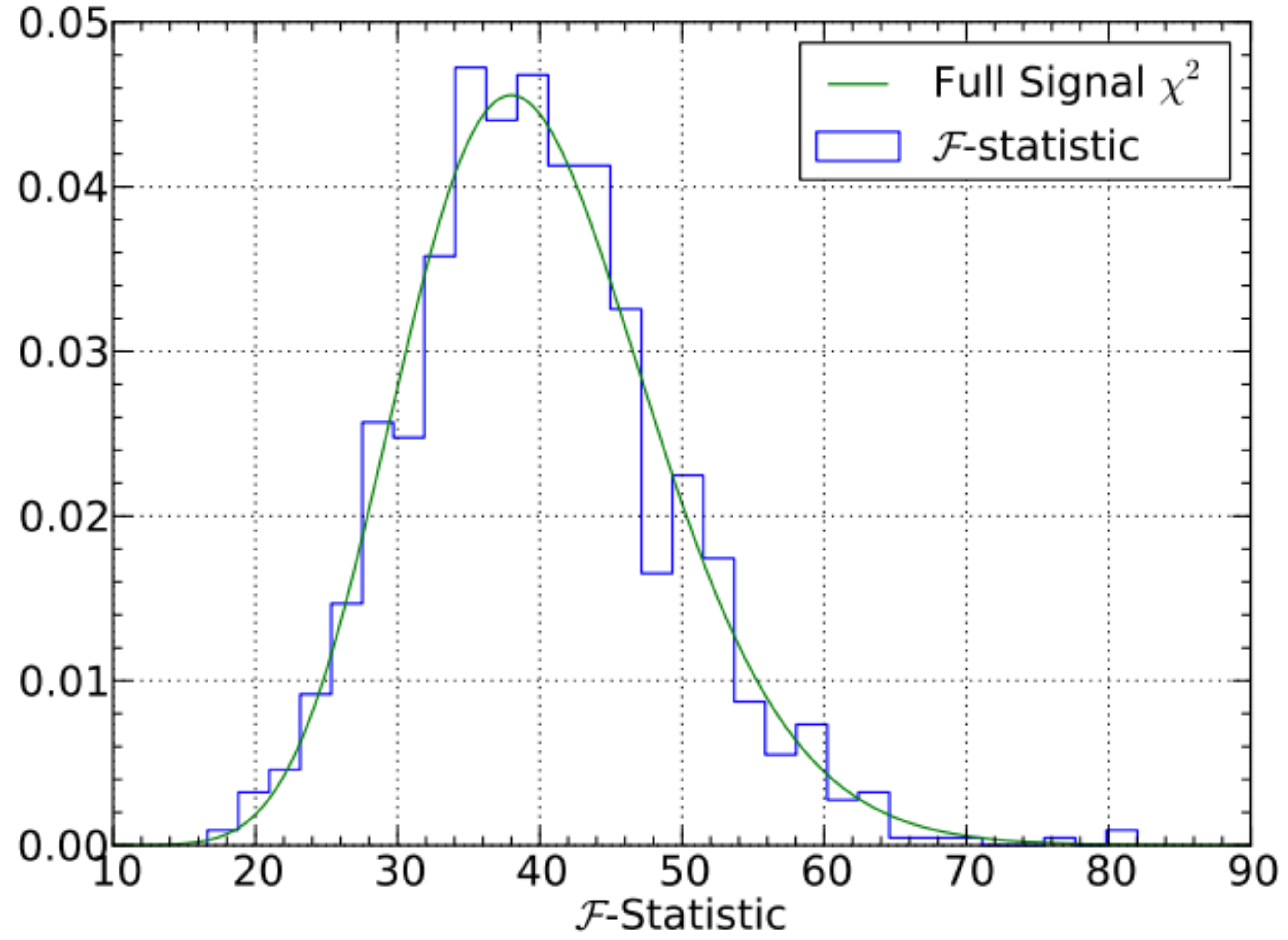
$$\hat{b}_{i\beta} = Q_{ik}^\beta P_\beta^k$$

Putting these coefficients back into the log-likelihood ratio gives the *F*$_p$-statistic:

$$2\mathcal{F}_p = \sum_{\alpha=1}^{M} P^i_\alpha Q^\alpha_{ij} P^j_\alpha$$

It follows a chi-squared distribution with 2M degrees of freedom:

$$\langle 2\mathcal{F}_p \rangle = 2M + \rho^2 = 2M + (\tilde{s}|\tilde{s})$$



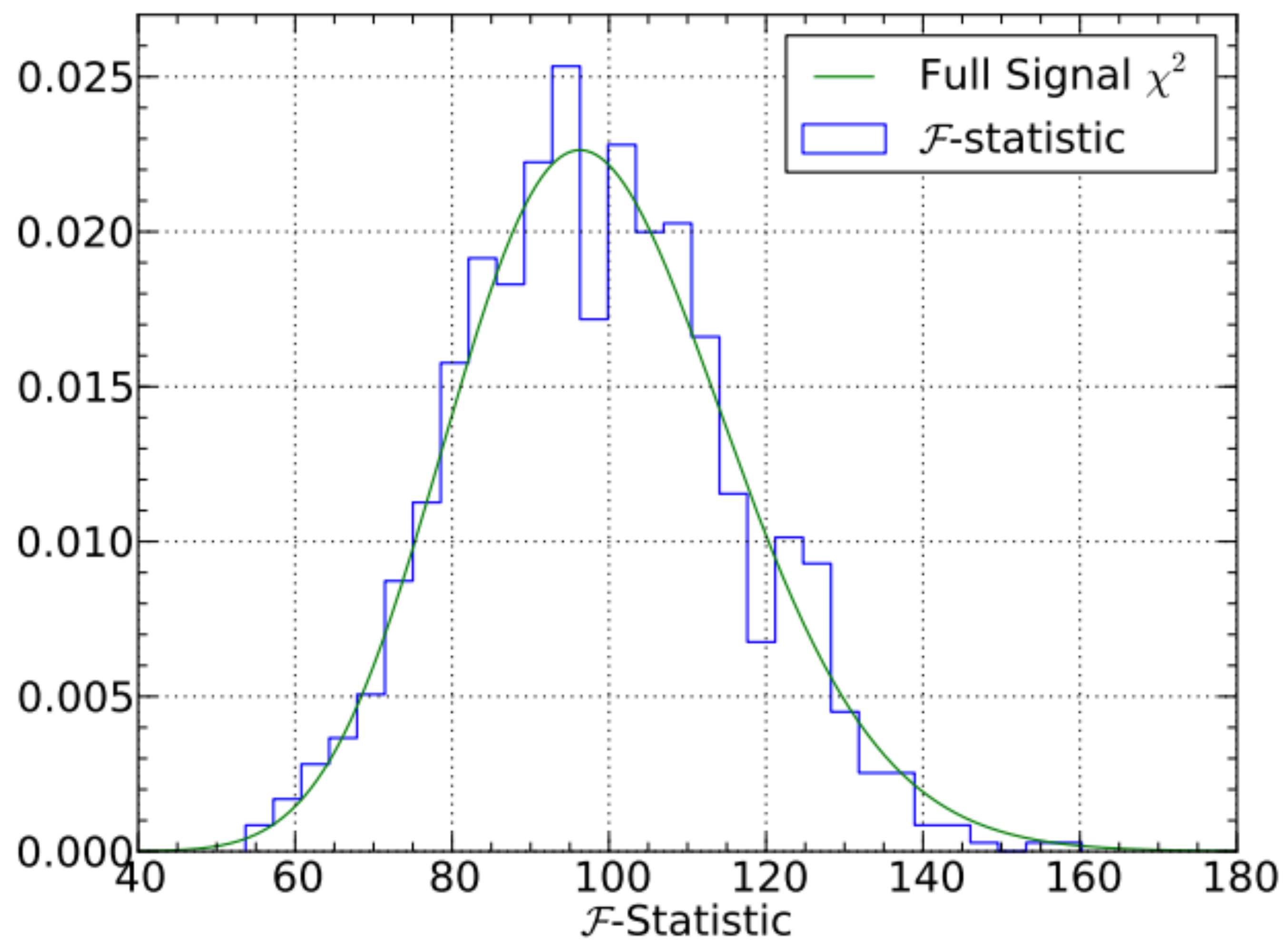


Figure from Ellis, Siemens, & Creighton (2012).

Summary

The CW search is conceptually simple, but difficult in practice.

The signal model involves many parameters in order to describe the position, orientation, and evolution of the source.

Including the pulsar term requires adding $2N_{psr}$ parameters to the model.

The large number of parameters and complex likelihood surface make using MCMC challenging (but not impossible!)

Frequentist methods can be used, which are faster to compute, but carry their own limitations, and require using simplified signal models.