Caltech

Theory of Stochastic Gravitational Wave Backgrounds

mathematical formalism and potential sources

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Outline

- Common formalism for stochastic backgrounds
  - plane wave expansion
  - Intensity and Stokes' parameters
  - GW energy density
  - Energy spectra
- Overview of GW(B) sources
  - GW emitters across the spectrum
  - key features: merger rate, energy emission, anisotropy
- Focus: backgrounds from compact binary coalescences
  - properties of the CBC background in LIGO/Virgo
  - observation timeline

Stochastic GWB formalism

a free-for-all ensemble of plane waves.

A gravitational-wave “background”

incoherent superposition/ stochastic generation → **unresolved** → **stochastic variables**

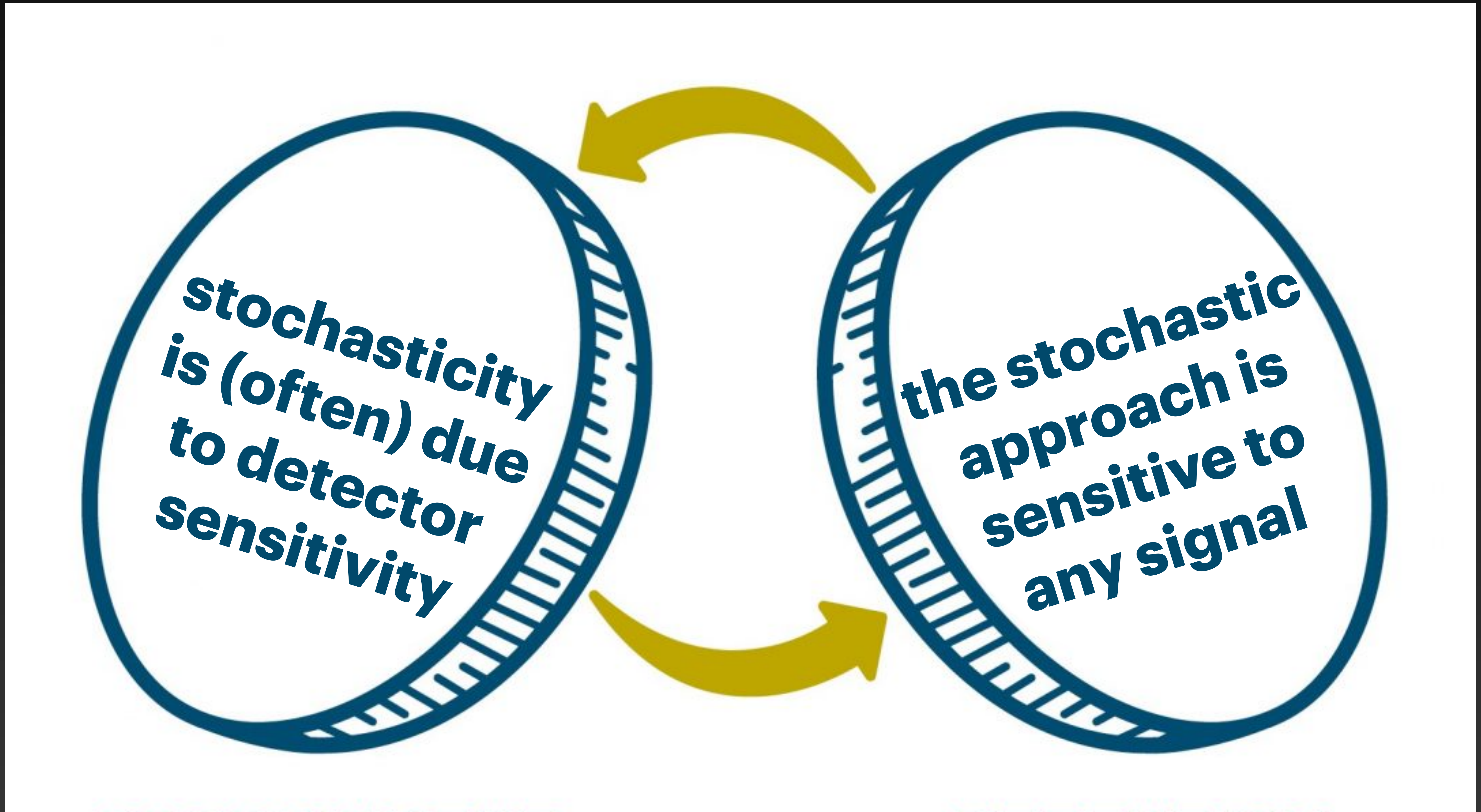
many, many, many gravitational waves...

which the detector/ our methods can’t fully parse...

need a one-size-fits-all solution!

A gravitational-wave “background”

incoherent superposition/ stochastic generation ⟶ **unresolved** ⟶ **stochastic variables**



A gravitational-wave "background"

incoherent superposition/ stochastic generation ⟶ **unresolved** ⟶ **stochastic variables**

<u>target for match-filtered searches</u>

1 gravitational wave: $h_{ij}(t) = \sum_A h_A(t)\, \epsilon_{ij}^A$

superposition of many waves: $h_{ij}(t,x) = \int_f df \int_{S^2} d\hat{n}\, \tilde{h}_A(f,\hat{n})\, \epsilon_{ij}^A(\hat{n})\, e^{i2\pi f(t-\hat{n}\cdot x)}$

total GW intensity: $I(f) \sim \frac{1}{2} \sum_{\text{waves}} \sum_A |\tilde{h}_A(f)|^2$

<u>target for stochastic searches</u>

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total GW intensity: $I(f) \sim \frac{1}{2} \sum_{\text{waves}} \sum_A |\tilde{h}_A(f)|^2$

target for stochastic searches

Plane wave expansion

integral over all signal frequencies

plane wave coefficients

time evolution

$$h_{ij}(t,x) = \int_f df \int_{S^2} d\hat{n}\, \tilde{h}_A(f,\hat{n})\, \epsilon^A{}_{ij}(\hat{n})\, e^{i\,2\pi f(t-\hat{n}\cdot x)}$$

strain tensor

"measured" at point $\vec{x}$

integral over all sky directions

polarisation tensor

plain wave mode

If $\tilde{h}_A(f,\hat{n})$ are mean-zero Gaussian fields, the signal is fully described by its **second moments**:

$$\langle h_+(f,\hat{n})\, h_+^\star(f',\hat{n}')\rangle + \langle h_\times(f,\hat{n})\, h_\times^\star(f',\hat{n}')\rangle = \delta^{(2)}(\hat{n},\hat{n}')\,\delta(f-f')\, I(f,\hat{n})$$

ensemble averages over all realisations of the field

target for stochastic searches: variance of the field

AIR et al.

Stokes' parameters

The full set of second moments may be written as

$$\begin{pmatrix} \langle h_+(f,\hat{n})\, h_+^\star(f',\hat{n}')\rangle & \langle h_+(f,\hat{n})\, h_\times^\star(f',\hat{n}')\rangle \\ \langle h_\times(f,\hat{n})\, h_+^\star(f',\hat{n}')\rangle & \langle h_\times(f,\hat{n})\, h_\times^\star(f',\hat{n}')\rangle \end{pmatrix} = \delta^{(2)}(\hat{n},\hat{n}')\,\delta(f-f')\times$$

$$\times\begin{pmatrix} I(f,\hat{n})+Q(f,\hat{n}) & U(f,\hat{n})-iV(f,\hat{n}) \\ U(f,\hat{n})+iV(f,\hat{n}) & I(f,\hat{n})-Q(f,\hat{n}) \end{pmatrix}$$

STOKES' PARAMETERS: I Q U V

I: total (unpolarised) intensity

Q, U: linear polarisation intensity

V: circular polarisation intensity

Stokes' parameters

The full set of second moments may be written as

$$\begin{pmatrix} \langle h_+(f,\hat n)\, h_+^\star(f',\hat n')\rangle & \langle h_+(f,\hat n)\, h_\times^\star(f',\hat n')\rangle \\ \langle h_\times(f,\hat n)\, h_+^\star(f',\hat n')\rangle & \langle h_\times(f,\hat n)\, h_\times^\star(f',\hat n')\rangle \end{pmatrix} = \delta^{(2)}(\hat n,\hat n')\,\delta(f-f')\times$$

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Any SGWB is (almost always) expected to be unpolarised

STOKES' PARAMETERS: I Q U V

Intensity Stokes' parameter

I → total (unpolarised) intensity

Q, U → linear polarisation intensity

V → circular polarisation intensity

GW energy density: a splinter in the Universe

Universe energy content

$$H(z) = H_0 \left(\Omega_{\rm R}\,(1+z)^4 + \Omega_{\rm M}\,(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda\right)^{1/2}$$

$$\Omega_\gamma + \Omega_\nu + \Omega_{\rm GW} + \cdots$$

GWs act as light degrees of freedom of the early Universe $\longrightarrow$ can be constrained by CMB measurements '15 :

$$\boxed{h_0^2\,\Omega_{\rm GW} < 1.2 \times 10^{-6}}$$

$\longrightarrow$ may safely be considered a perturbation (for cosmology).

GW energy density: the spectrum

Isaacson '68

GW energy density: $\rho_{\text{GW}} = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle = \frac{4\pi^2}{G}\int_0^\infty df \int_{S^2} d\hat{n}\, f^2 I(f)$

fractional GW energy density per log-f bin

$$\boxed{\Omega_{\text{GW}}(f) := \frac{1}{\rho_c}\frac{d\rho_{GW}}{d\ln f} = \frac{32\pi^3}{3H_0^2} f^3\, I(f)}$$

Allen & Ottewill '97

COSMOLOGY AND ASTROPHYSICS: GW history of the universe.

number of events in unit comoving volume - redshifted to *observed* number

from Phinney '01 : $\rho_{\text{GW}} = \int_0^\infty \frac{df}{f}\int_0^\infty dz\, \frac{N(z)}{1+z}\, f_s \frac{dE_{\text{GW}}}{df_s}$

energy radiated per event per source-frame frequency

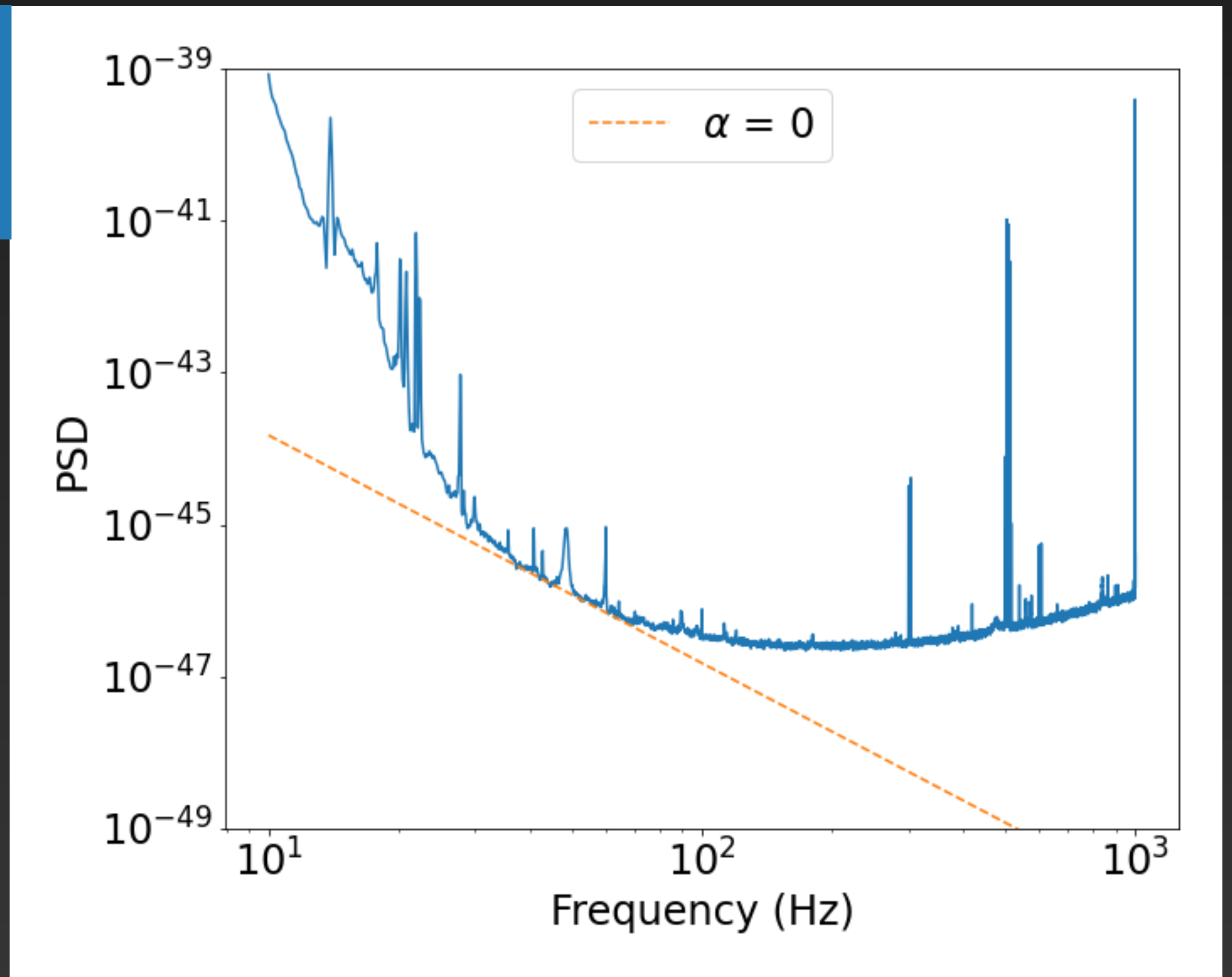
Stochastic background spectral shapes

Spectral shape usually assumed a power law:

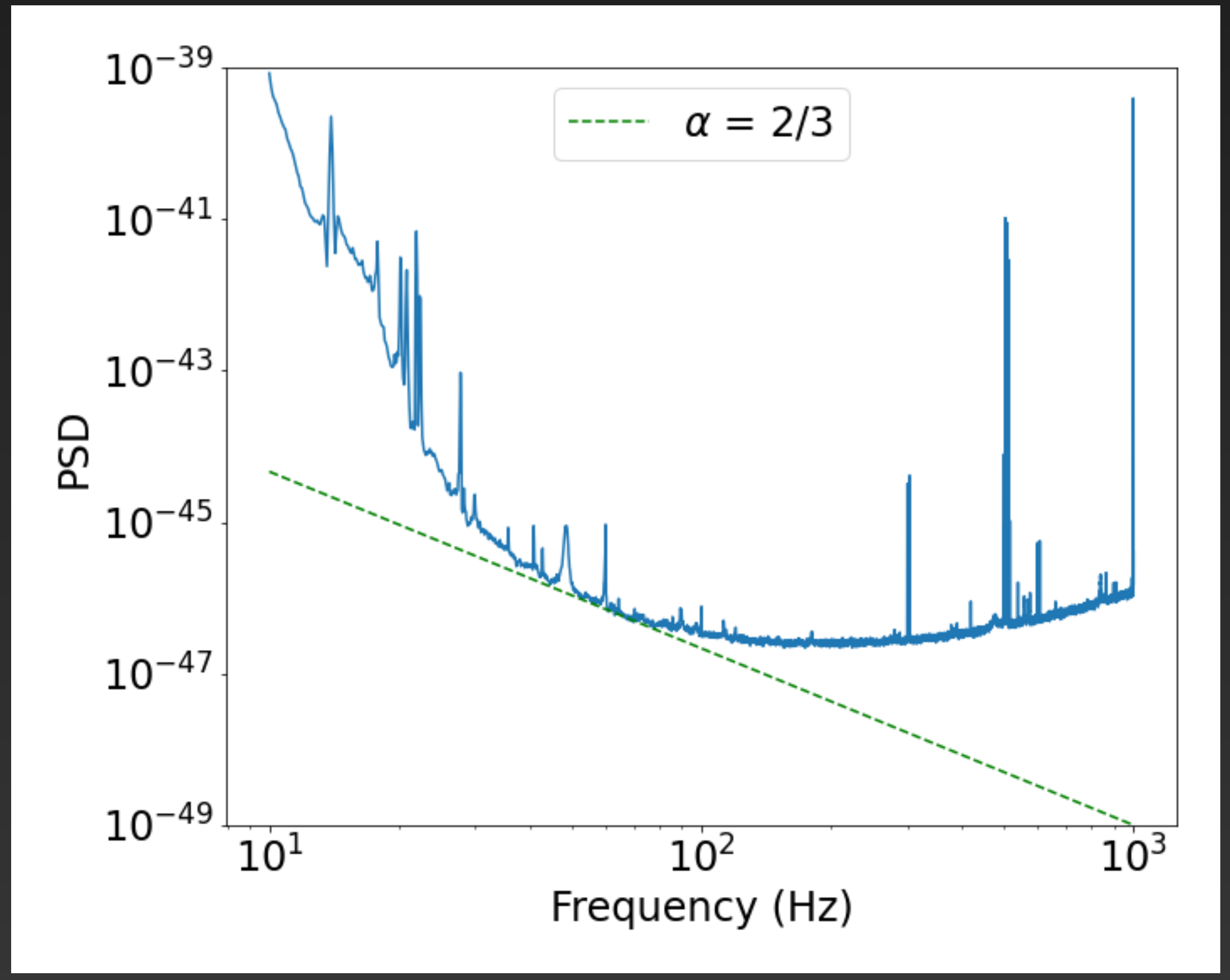
$$\boxed{\Omega_{\mathrm{GW}} = \Omega_{\mathrm{GW}}(f_{\mathrm{ref}})\left(\frac{f}{f_{\mathrm{ref}}}\right)^{\alpha}}$$

LIGO NOISE PSD

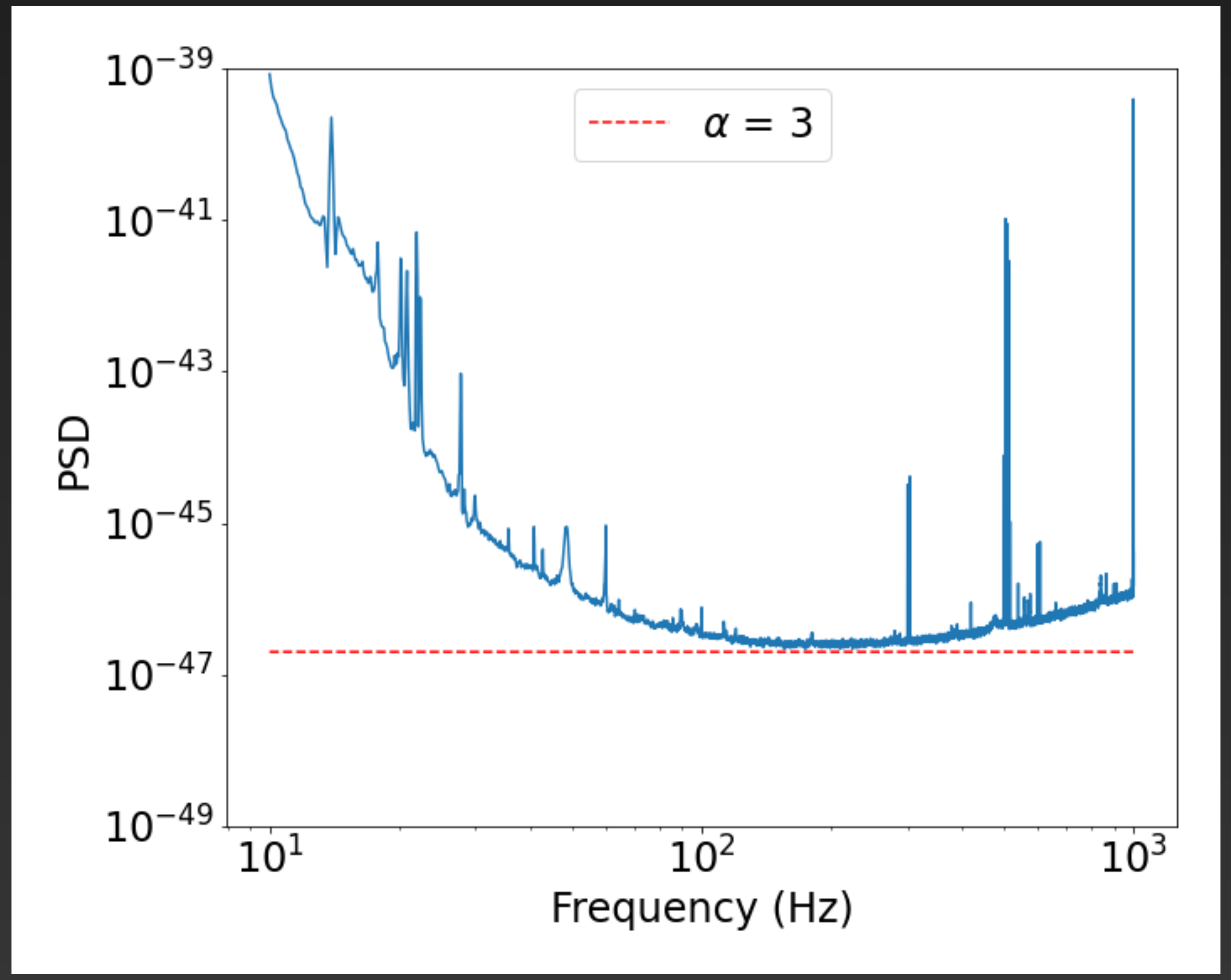
"cosmological"



"inspiral/astro"



"best fit"



Another spectrum: characteristic strain

is the amplitude density over logarithmic frequency interval:

$$h_c(f) = \sqrt{16\pi f I(f)}$$

$$\mathrm{PSD}(f) = 16\pi I(f) \longrightarrow h_c(f) = \sqrt{f}\,\mathrm{ASD}(f)$$

Total GW energy: $\mathcal{E}_{\rm tot} = \int df\, \mathrm{ASD}^2(f) = \int d\log f\, h_c^2$

see e.g. Phinney '01 and Mingarelli et al. '19.

Different quantities, same physics.

- **Results/cosmology:** $$\Omega_{\mathrm{GW}}(f) = \frac{32\pi^3}{3H_0^2} f^3 I(f)$$
  Contribution to the Universe's energy density
- **Ground-based interferometers:** $$I(f) \sim \sum_A \langle h_A(f)\, h_A^\star(f) \rangle$$
  Intensity
- **Pulsar Timing Arrays:** $$h_c(f) = \sqrt{16\pi f I(f)}$$
  Characteristic strain

Different quantities, same physics.

- **Results/cosmology:** $\Omega_{\rm GW}(f) = \Omega_{\rm GW}(f_{\rm ref})\left(\dfrac{f}{f_{\rm ref}}\right)^{\alpha}$

  Power-law index α, *reference frequency* $f_{\rm ref}$

- **Ground-based interferometers:** $I(f) = I(f_{\rm ref})\left(\dfrac{f}{f_{\rm ref}}\right)^{\alpha-3}$

- **Pulsar Timing Arrays:** $h_c(f) = h_c(f_{\rm ref})\left(\dfrac{f}{f_{\rm ref}}\right)^{\alpha/2-1}$

... for more details, see [Mingarelli et al. '19].

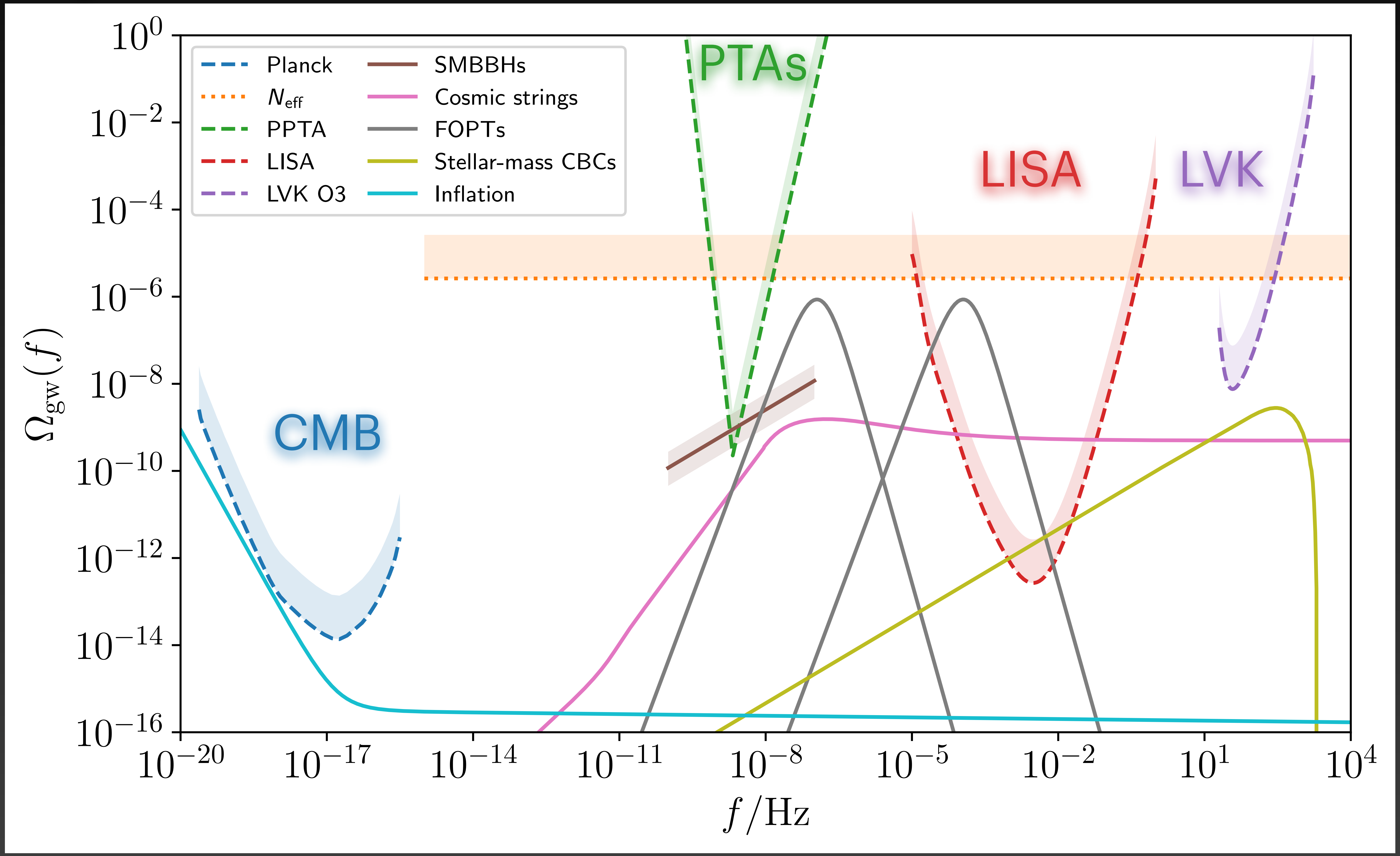
GW(B) sources across the spectrum

Those you can’t determine, are stochastic.

Gravitational-Wave Background Sources

incoherent superposition/ stochastic generation → **unresolved** → **stochastic variables**

Primordial



Astrophysical

From AIR et al.

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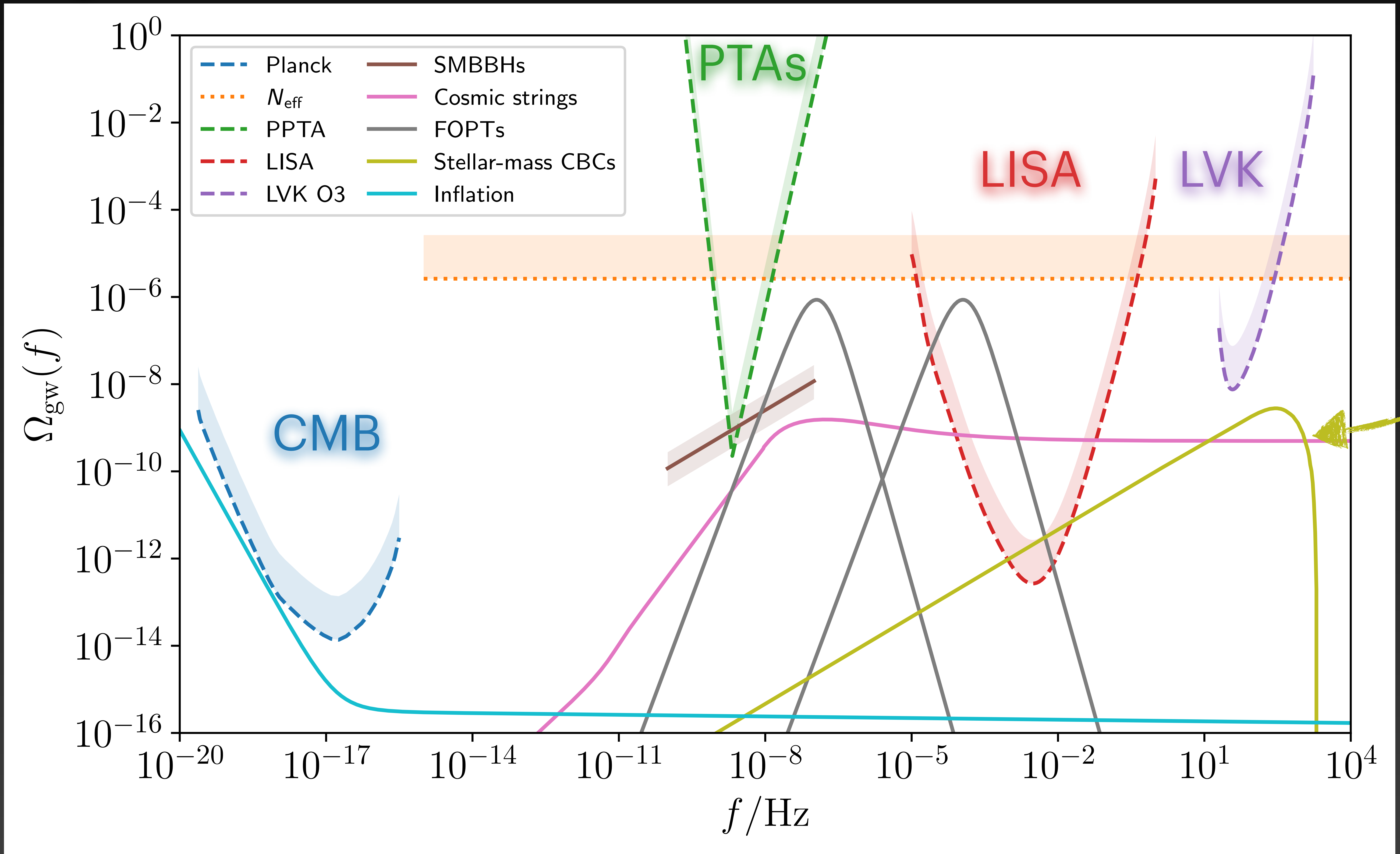

Astrophysical

stellar mass compact binary coalescences

From AIR et al.

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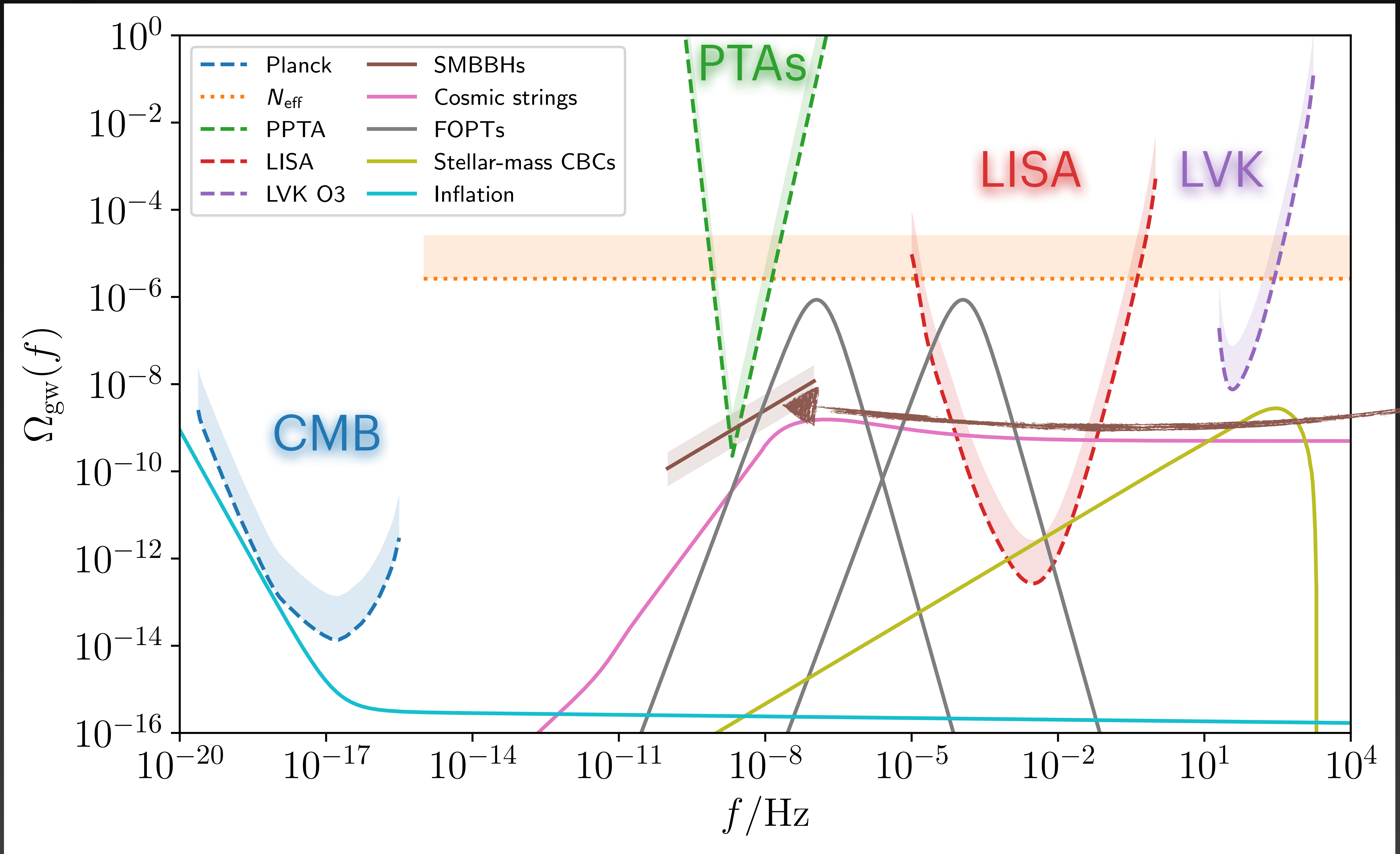

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stellar mass compact binary coalescences

supermassive black hole binary inspirals

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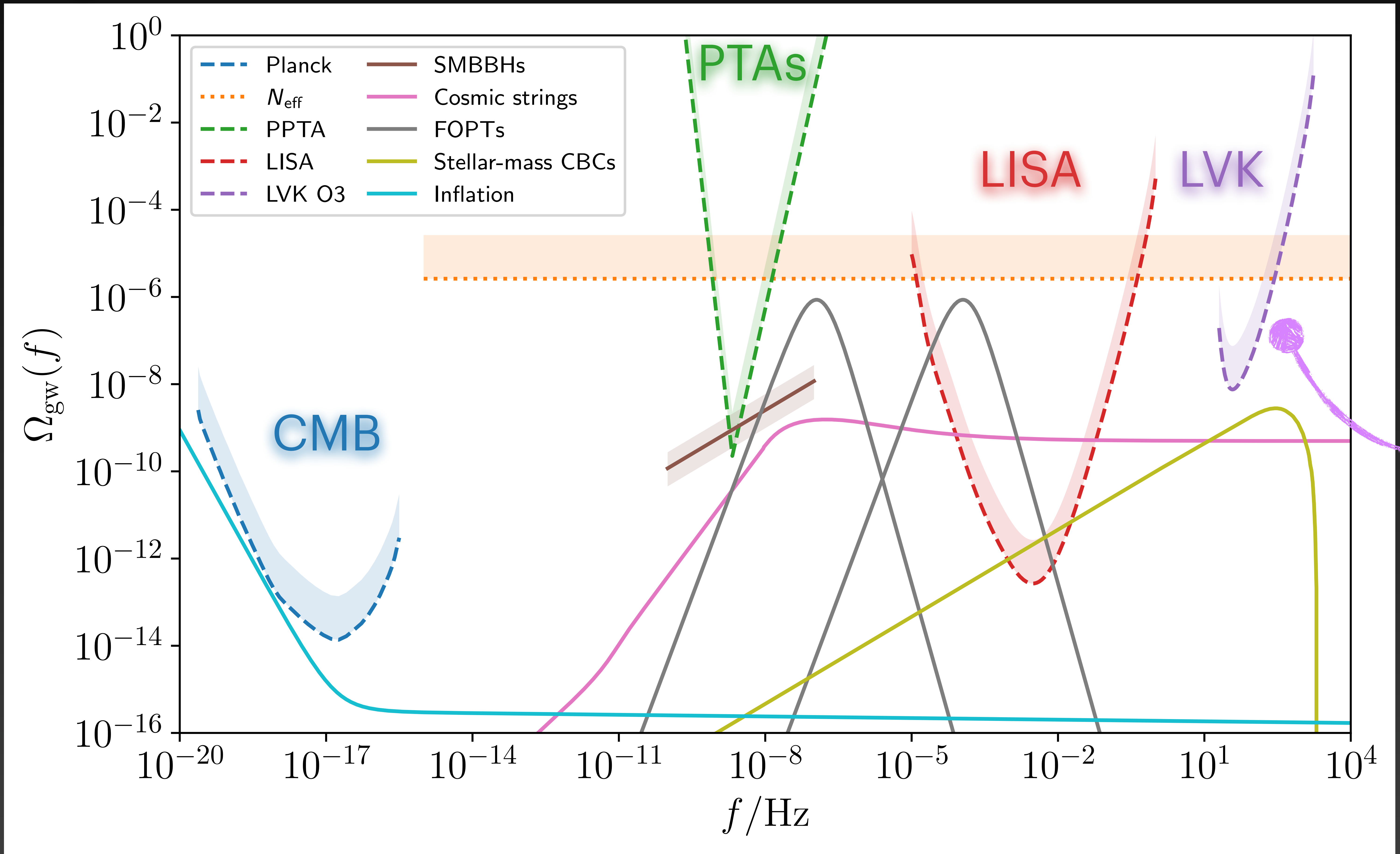

Astrophysical

stellar mass black hole and neutron star CBCs

supermassive black hole binary inspirals

core-collapse supernovas

From [AIR et al.]

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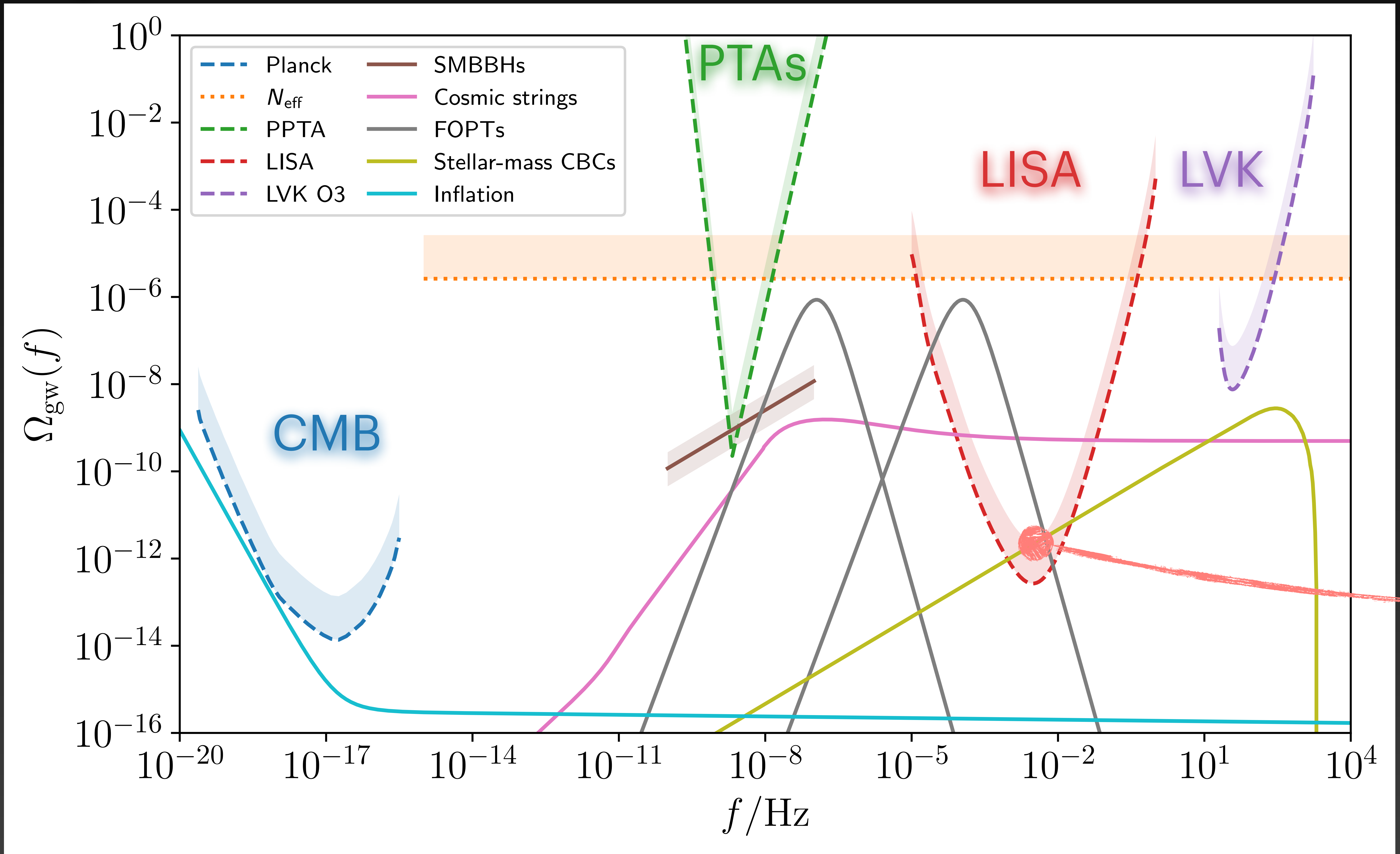

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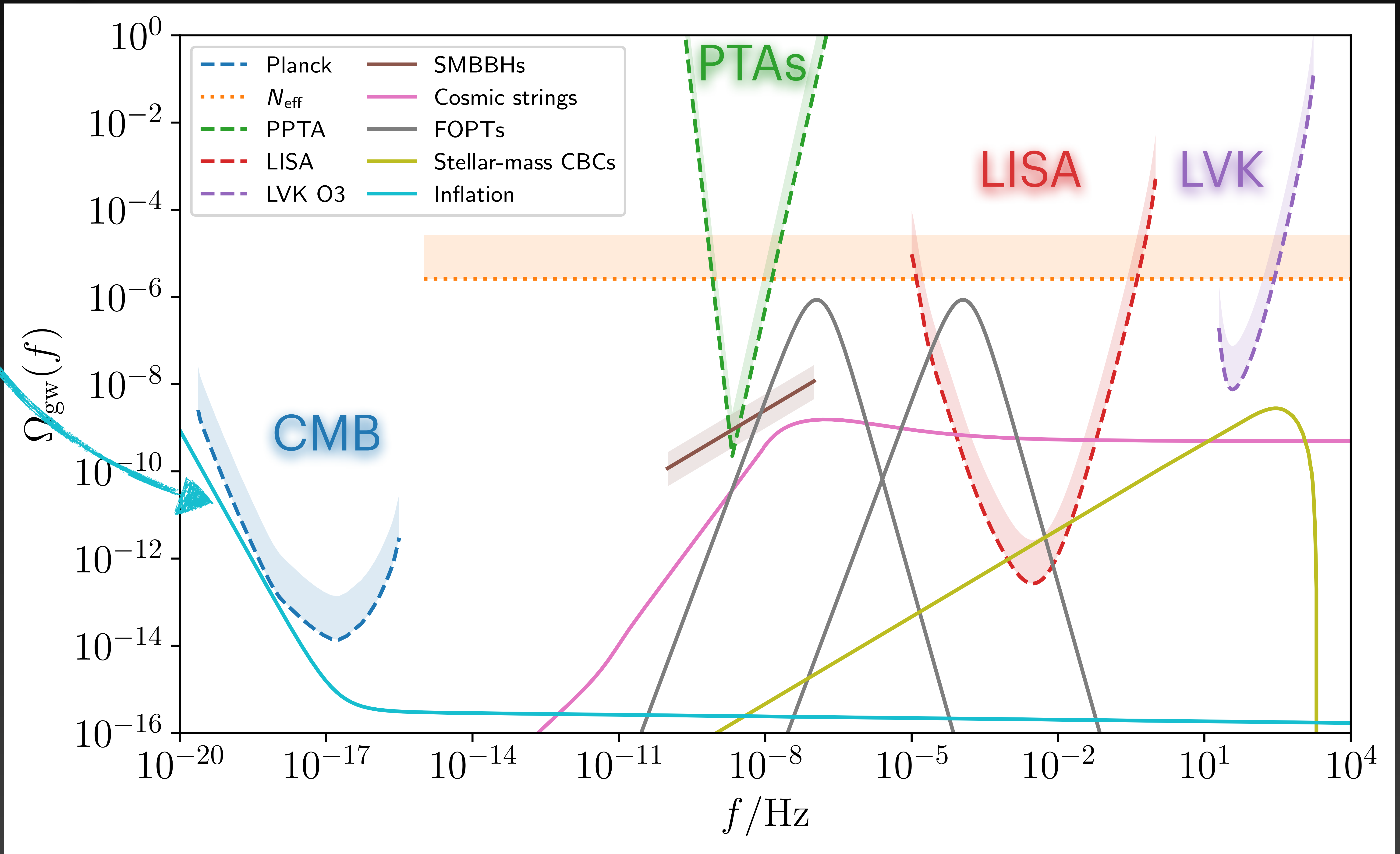
Primordial

GWs from inflation



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supermassive black hole binary inspirals

core-collapse supernovas

binary white dwarfs

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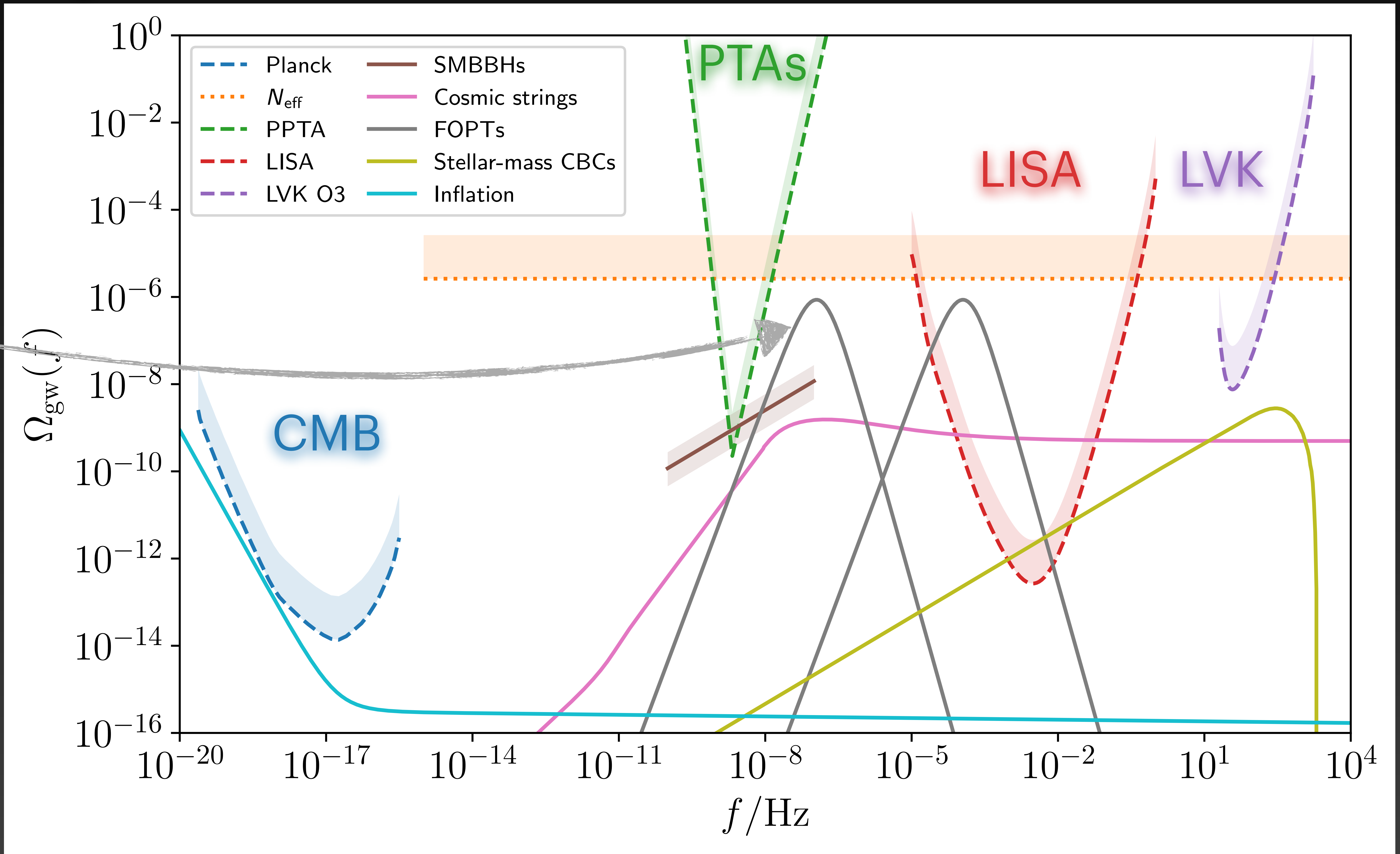
Primordial

GWs from inflation

first order phase transitions



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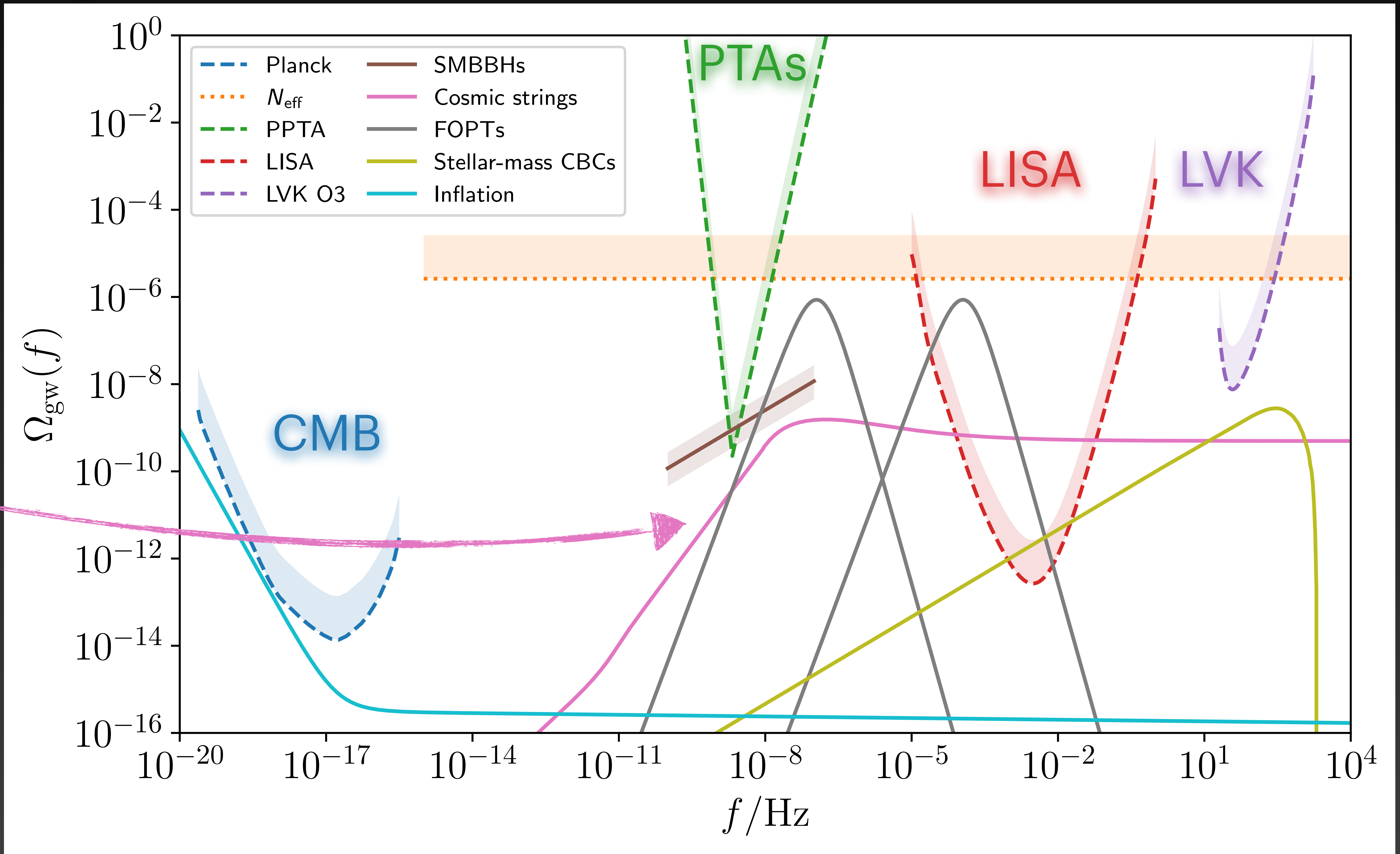
cosmic strings



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GWs from inflation

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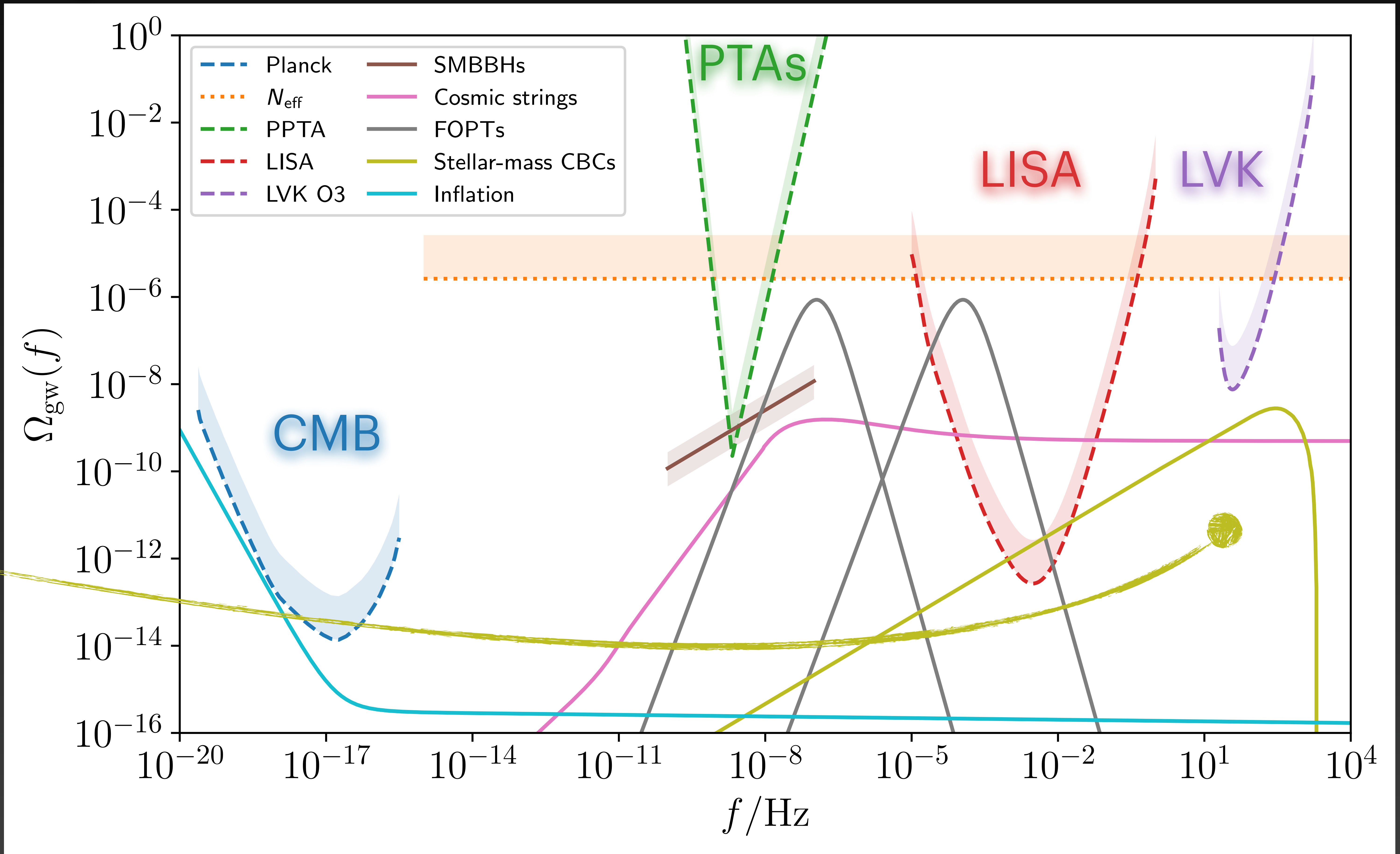
primordial black holes



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From AIR et al.

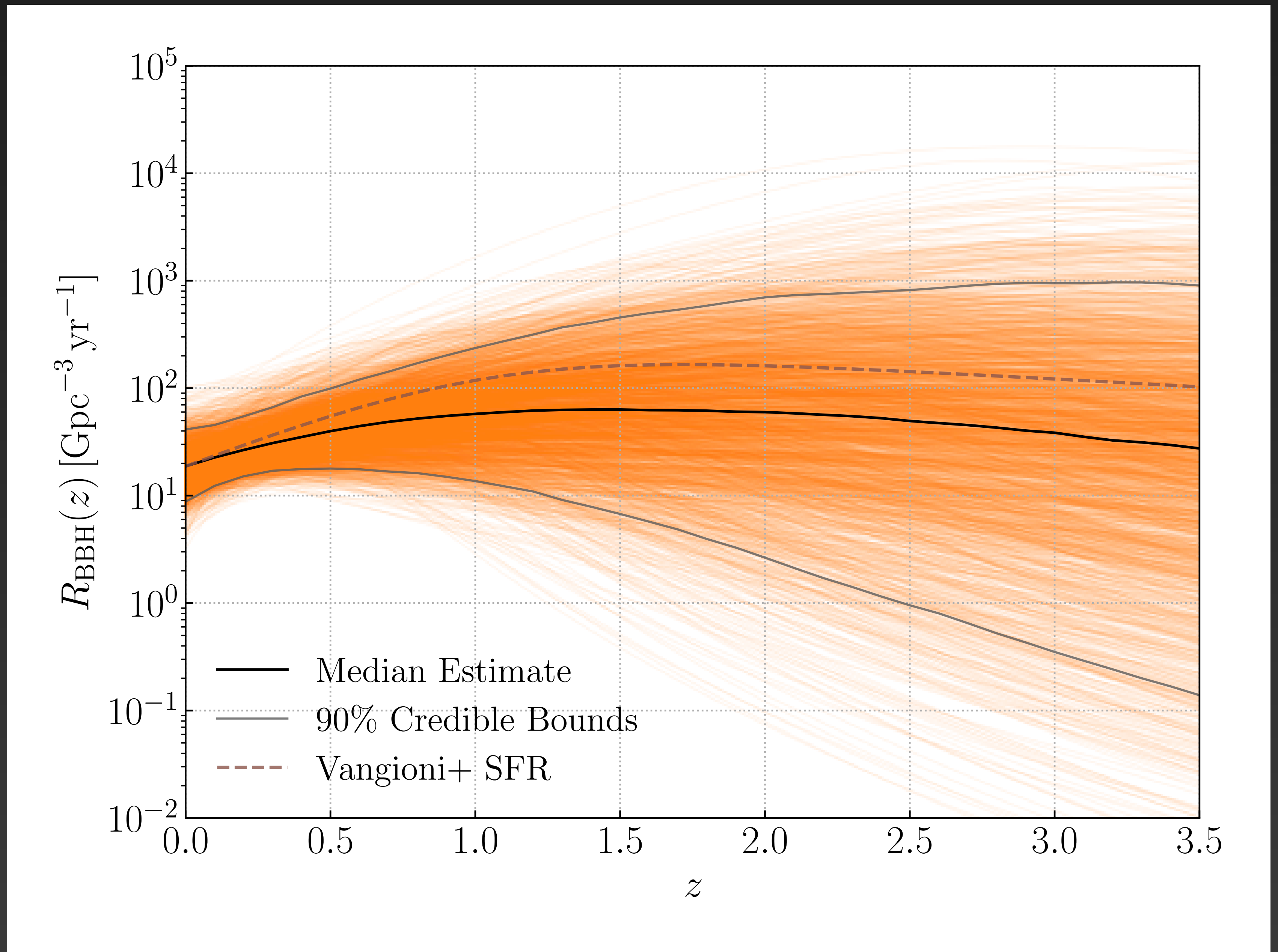
Stochastic sources: event rate

from Phinney '01 : $N(z) = R(z)\frac{dt_r}{dz} \equiv \frac{R(z)}{(1+z)H(z)}$

$$\Omega_{\rm GW}(f) = \frac{f}{\rho_c}\int_0^{z_{\rm max}} dz \frac{R(z)}{(1+z)H(z)}\left[\frac{dE_{\rm GW}}{df_s}\right]_{f_s=f(1+z)}$$

$$R(z) = \mathscr{C}(\alpha,\beta,z_p)\frac{R_0\,(1+z)^\alpha}{1+\left(\frac{1+z}{1+z_p}\right)^{\alpha+\beta}}$$

$\longleftrightarrow$ star formation rate



eg, from LVK for BBHs

Callister et al. '16, Callister et al. '20

Stochastic sources: compact binary spectrum
(super massive black hole binaries)

exercise: what is the spectral shape and amplitude of $\Omega_{\rm BSMBH}$?

from Sesana '08:

$$\frac{dE_{\rm GW}}{d\ln f_{\rm s}} = \frac{\pi^{2/3}}{3} G^{2/3} \mathcal{M}^{5/3} f_{\rm s}^{2/3}$$

single binary inspiral

$$\left[\frac{dE_{\rm GW}}{df_{\rm s}}\right]_{f_{\rm s}} \longrightarrow \left\langle \frac{dE_{\rm GW}}{df_{\rm s}} \right\rangle_{f_{\rm s}}$$

average over population parameters

$$N = \int_{\mathcal{M}_{\rm min}}^{\mathcal{M}_{\rm max}} \frac{dN}{d\mathcal{M}} d\mathcal{M}$$

number density parametrised by mass

$$\longrightarrow \boxed{\Omega_{\rm GW}^{\rm CB} \propto f^{2/3} \int_0^{z_{\rm max}} dz \, \frac{1}{(1+z)^{1/3}} \int_{\mathcal{M}_{\rm min}}^{\mathcal{M}_{\rm max}} d\mathcal{M} \frac{dN}{d\mathcal{M}} \mathcal{M}^{5/3}}$$

Stochastic sources: anisotropic spectra

$$\Omega_{\mathrm{GW}}(f, \hat{n}) := \frac{1}{\rho_c} \frac{d\rho_{\mathrm{GW}}}{d \ln f \, d\hat{n}} \longrightarrow \Omega_{\mathrm{GW}}(f, \hat{n}) \approx E(f)\, \Omega_{\mathrm{GW}}(\hat{n})$$

spectral shape

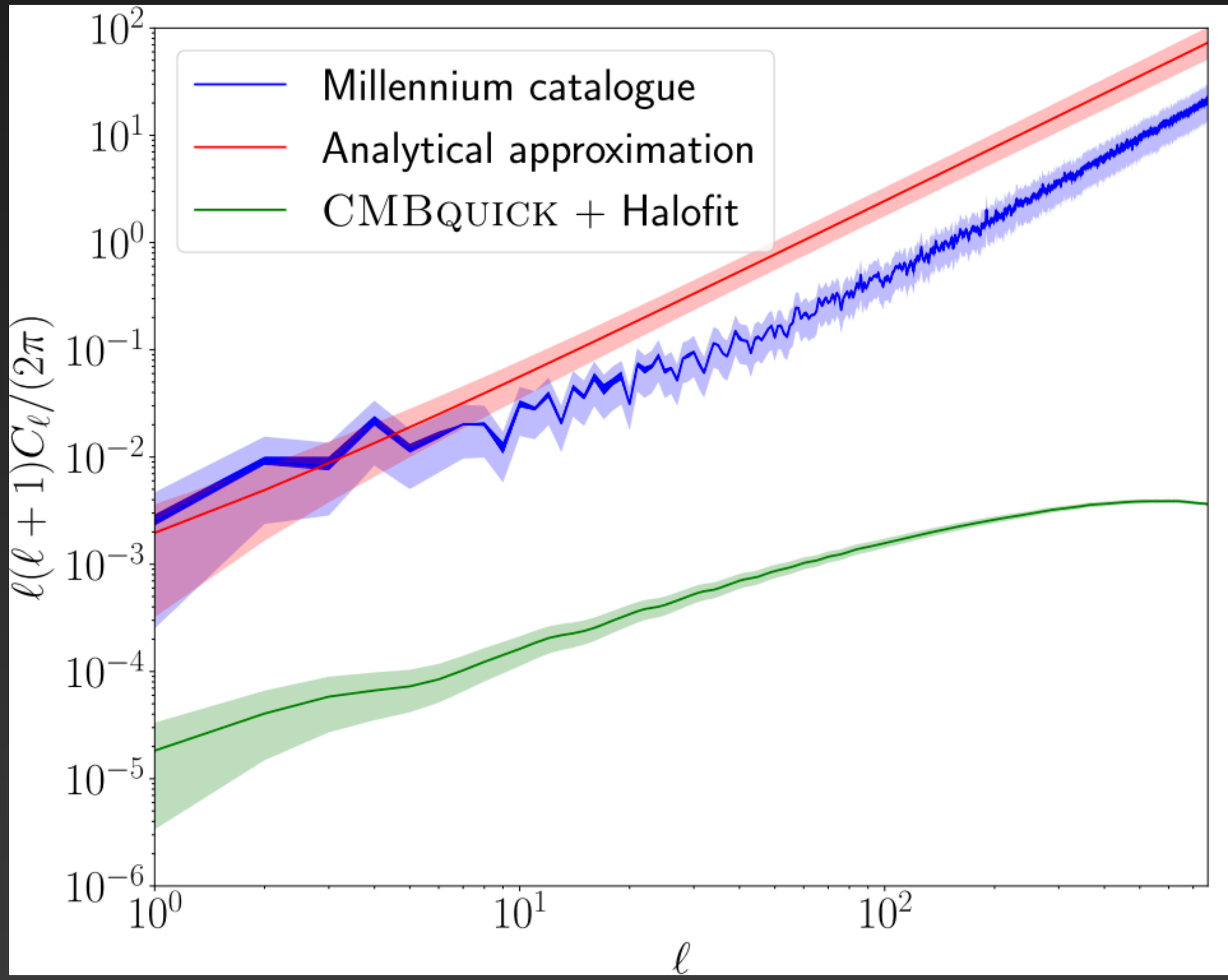
power on the sky

$$\Omega_{\mathrm{GW}}(\hat{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \Omega^{\mathrm{GW}}_{\ell m} Y_{\ell m}(\hat{n})$$

spherical harmonic expansion

assuming Gaussianity

$$\boxed{C^{\Omega}_{\ell} = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{+\ell} \langle \Omega^{\mathrm{GW}}_{\ell m} \Omega^{\mathrm{GW}}_{\ell m} \rangle}$$



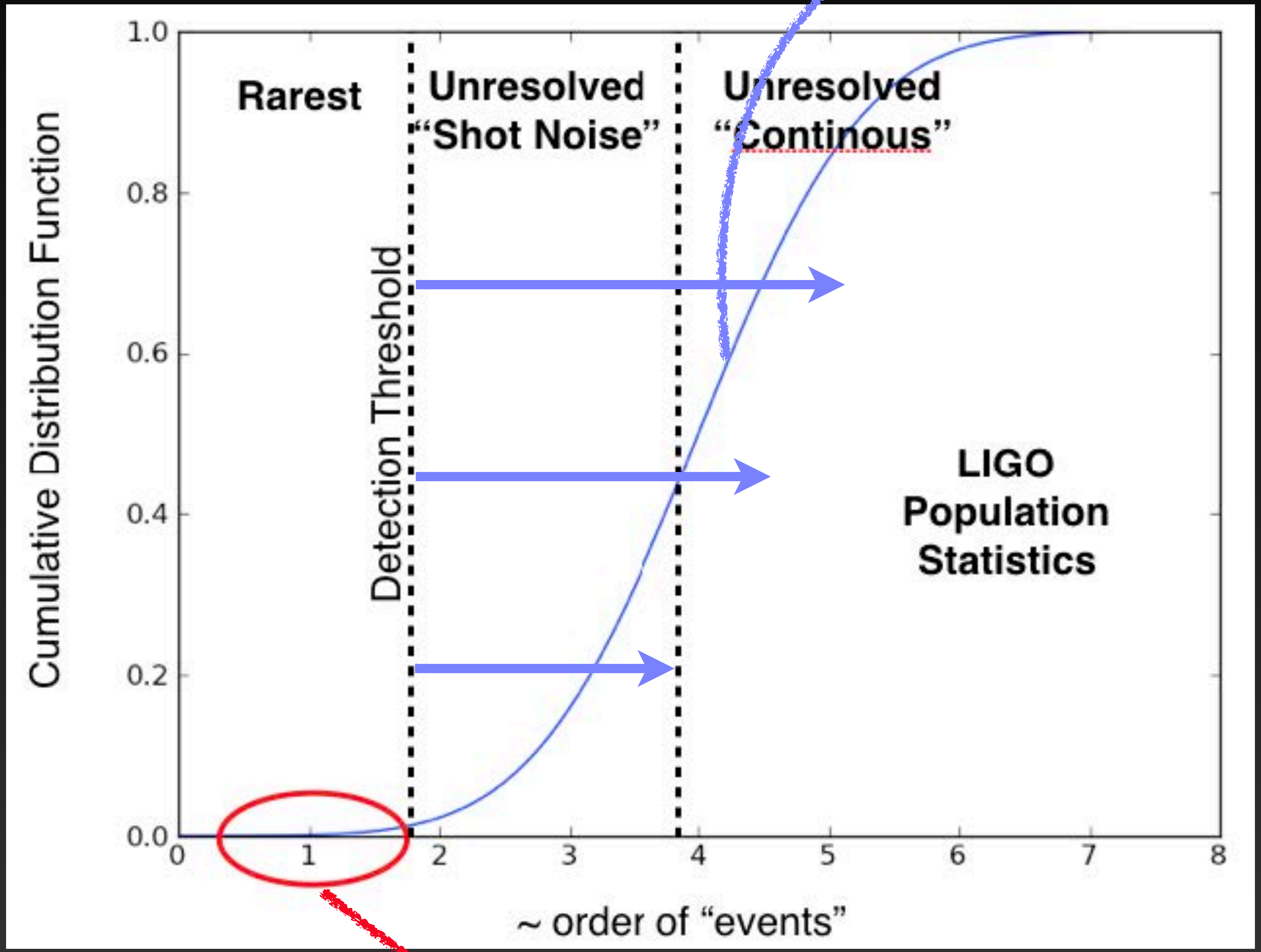
Expected CBC GWB angular power spectrum from Cusin et al. & Jenkins et al.

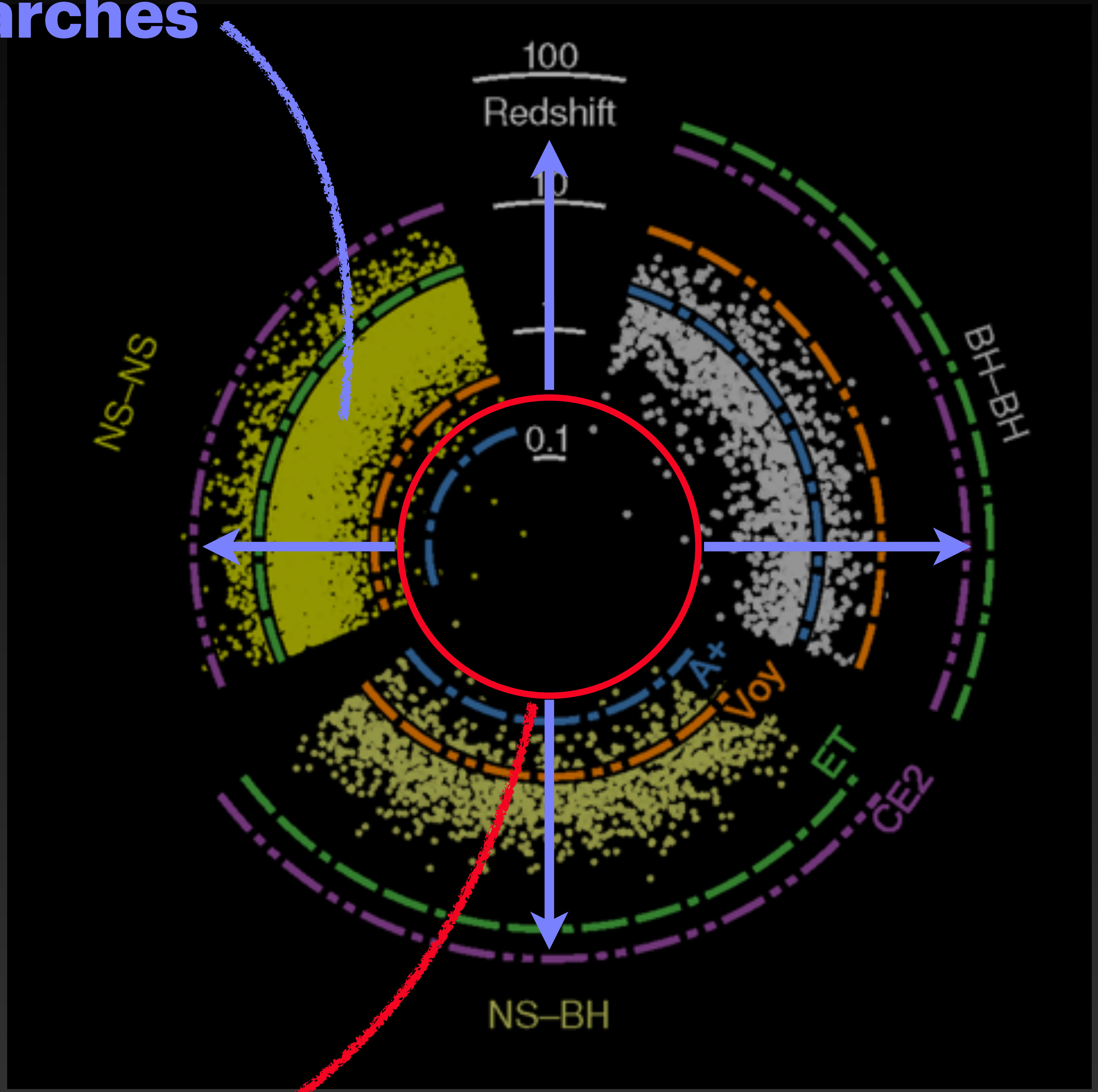
The SGWB from CBCs in LIGO

it's not *really* a background, and not *really* stochastic.

The CBC stochastic GW background in L/V

target for stochastic searches





Contaldi + AIR, ICL

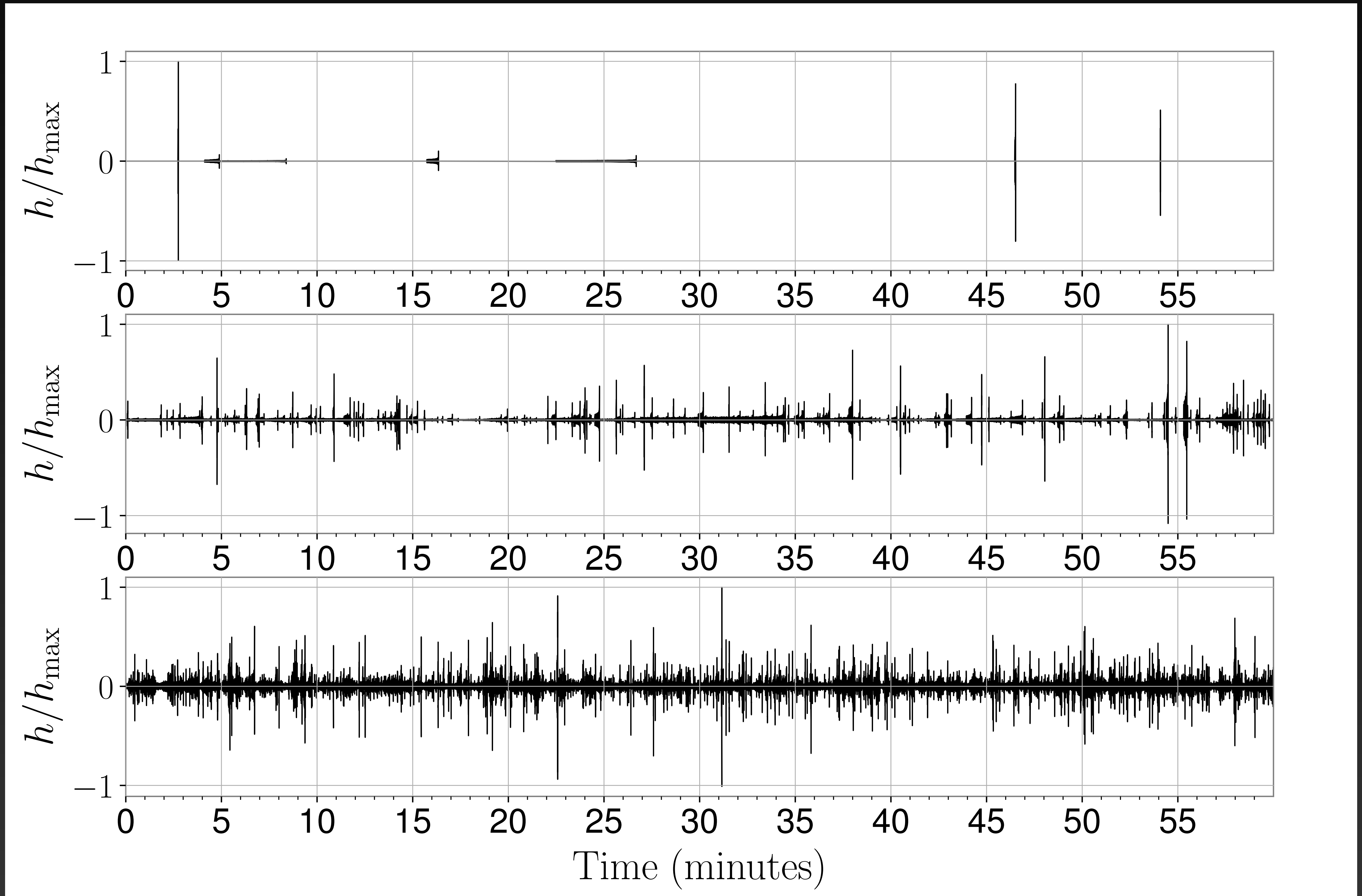
target for match-filtered searches

Hall + Vitale, MIT

Observing CBCs in the time domain

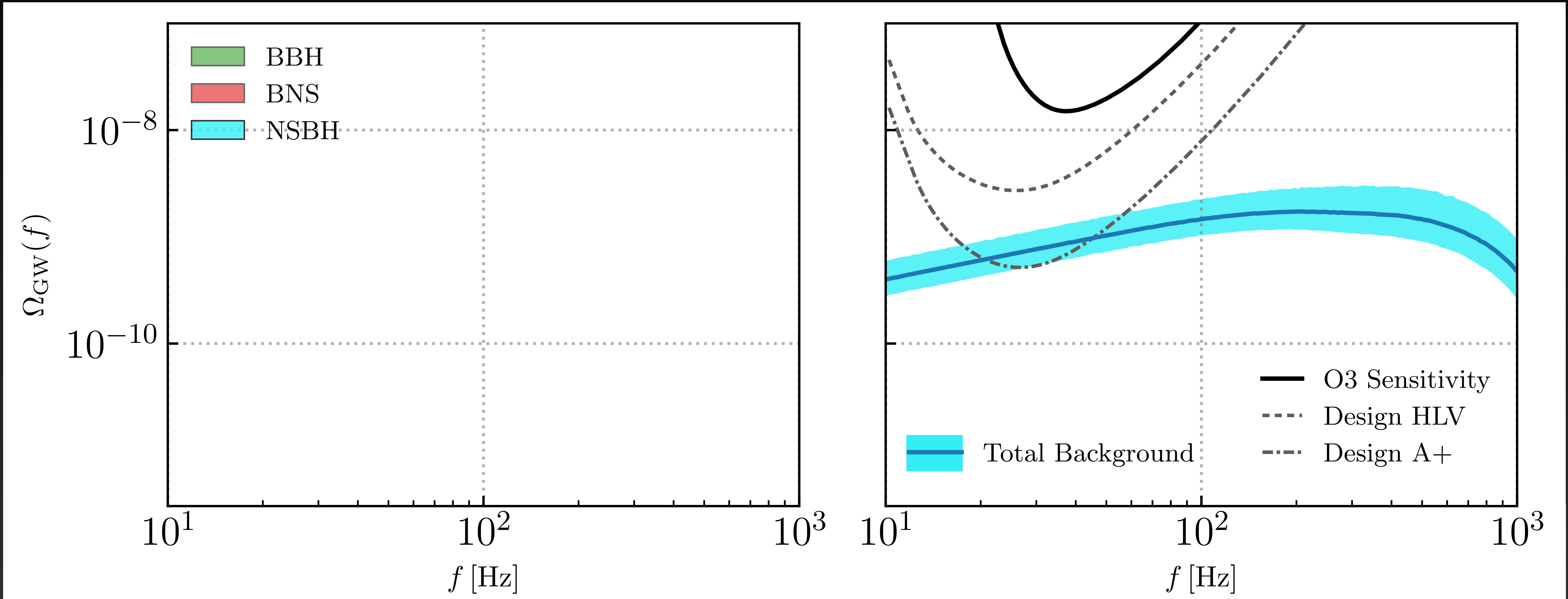
Gaussianity/non-Gaussianity of continuous/intermittent GWBs

current CBC signal due to detector sensitivity



We can use different models to search for Gaussian/non backgrounds!

Observing the background: the spectrum



CBC SGWB from [LVK GWTC-3 populations paper]

How long until detection?→*depending on sensitivity/methods*, before 3G!


... Thanks!