Overlap reduction functions (ORFs) (or, what is the Hellings and Downs curve?)

Joe Romano, Texas Tech University

13 July 2022

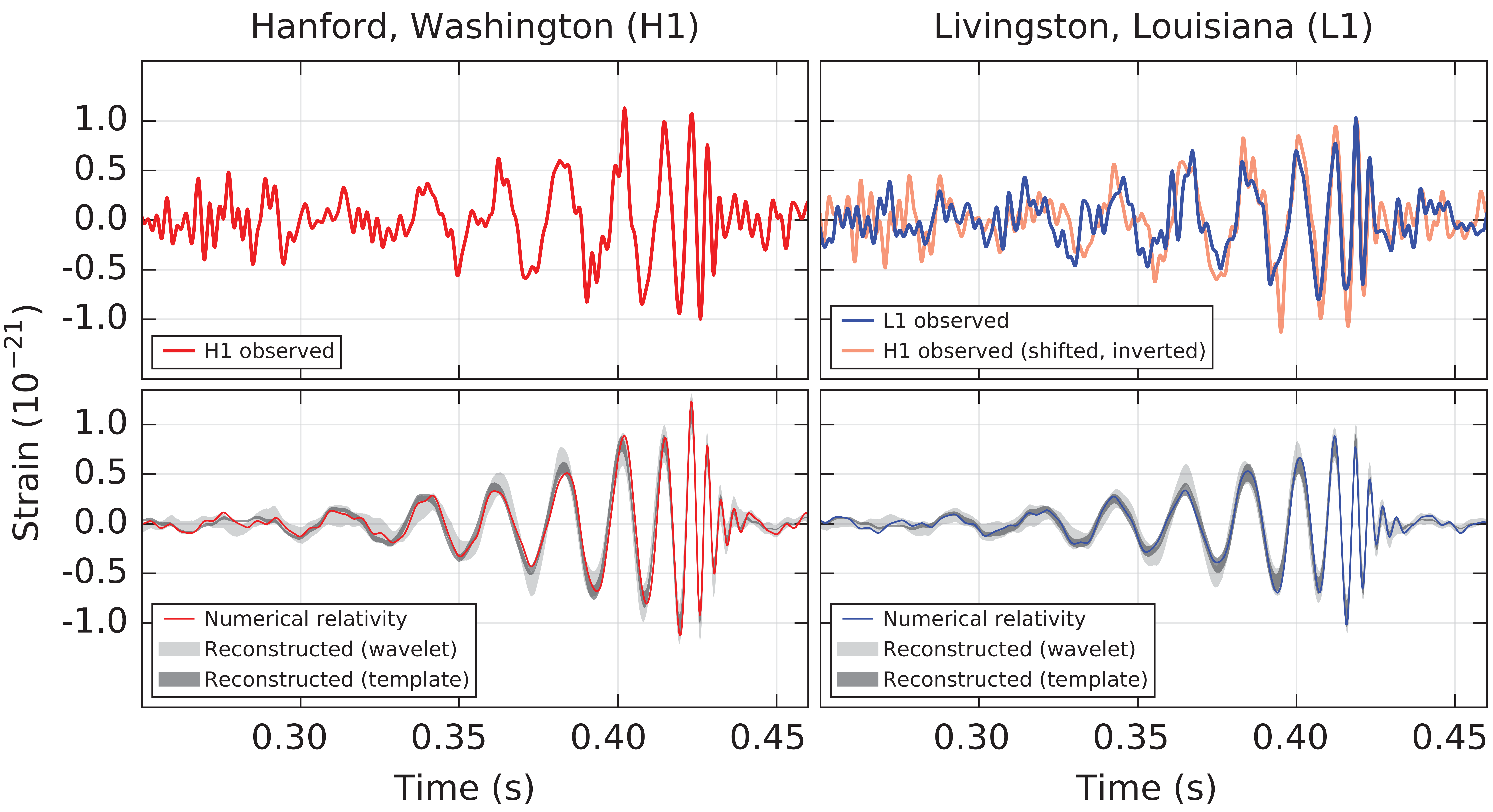
(VIPER Summer School, Vanderbilt University)

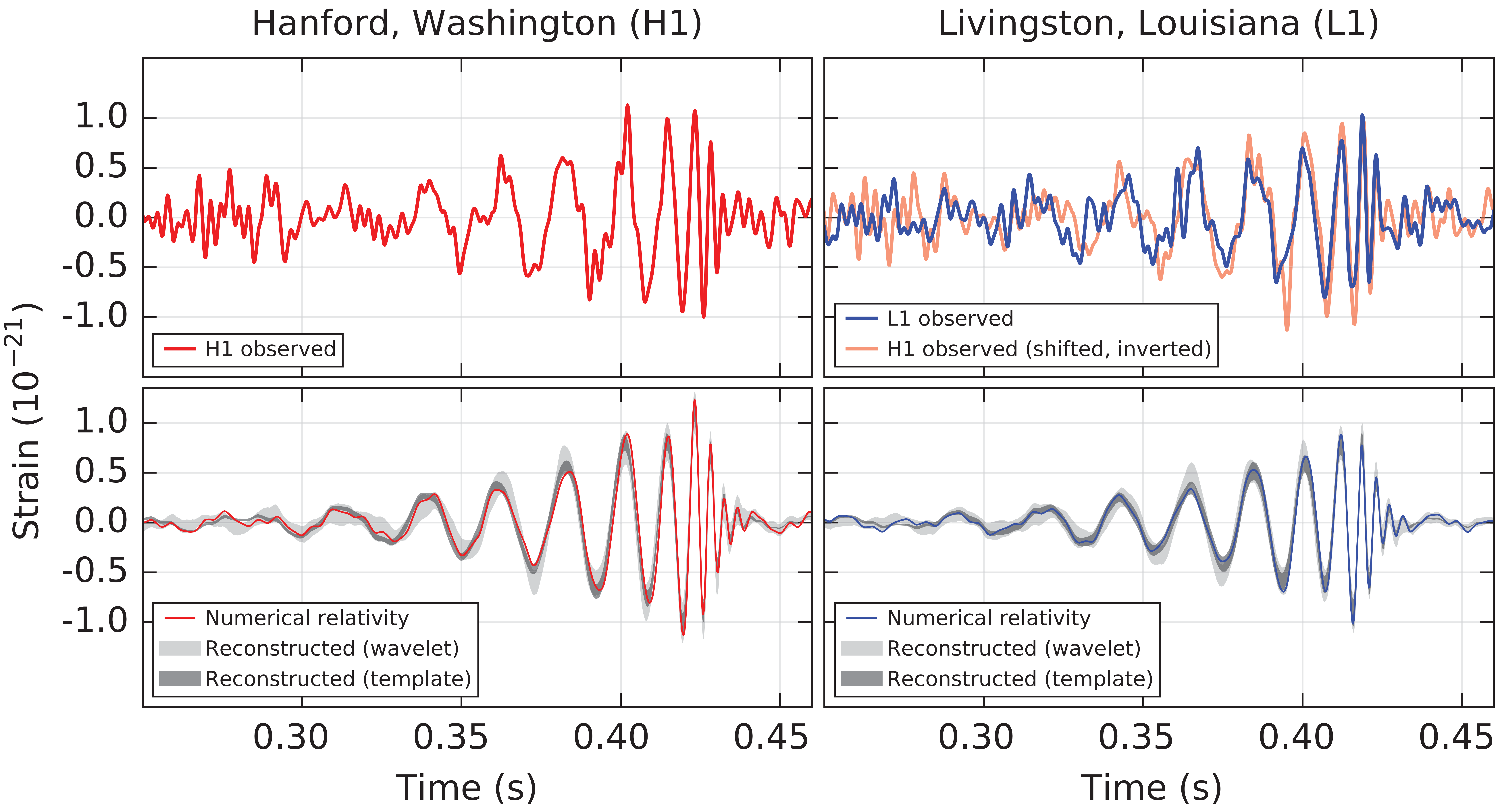
References

- Hellings and Downs, ApJ 265, L39 (1983)
- Jenet and Romano, AJP 83, 635 (2015)
- Cornish and Sesana, C&QG 30, 224005 (2013)
- Allen, arXiv:2205.05637 (2022)
- Allen and Romano, to appear (2022)

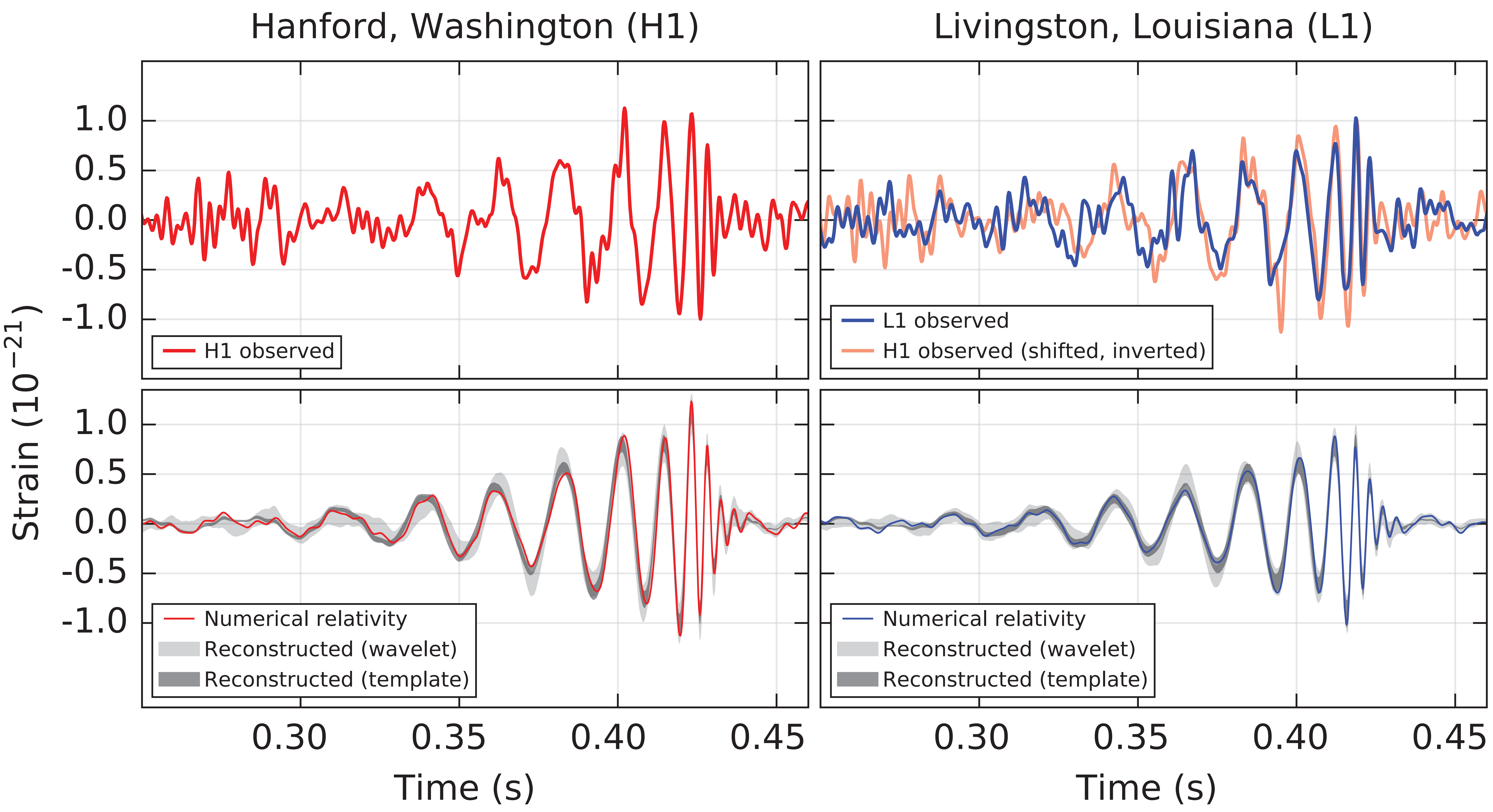
- Advertisement for Nima Laal's tutorial later this afternoon!!

GW150914: why was it such a convincing detection?


Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
1.0
0.5
0.0
-0.5
-1.0
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
0.30
0.35
0.40
0.45
Time (s)



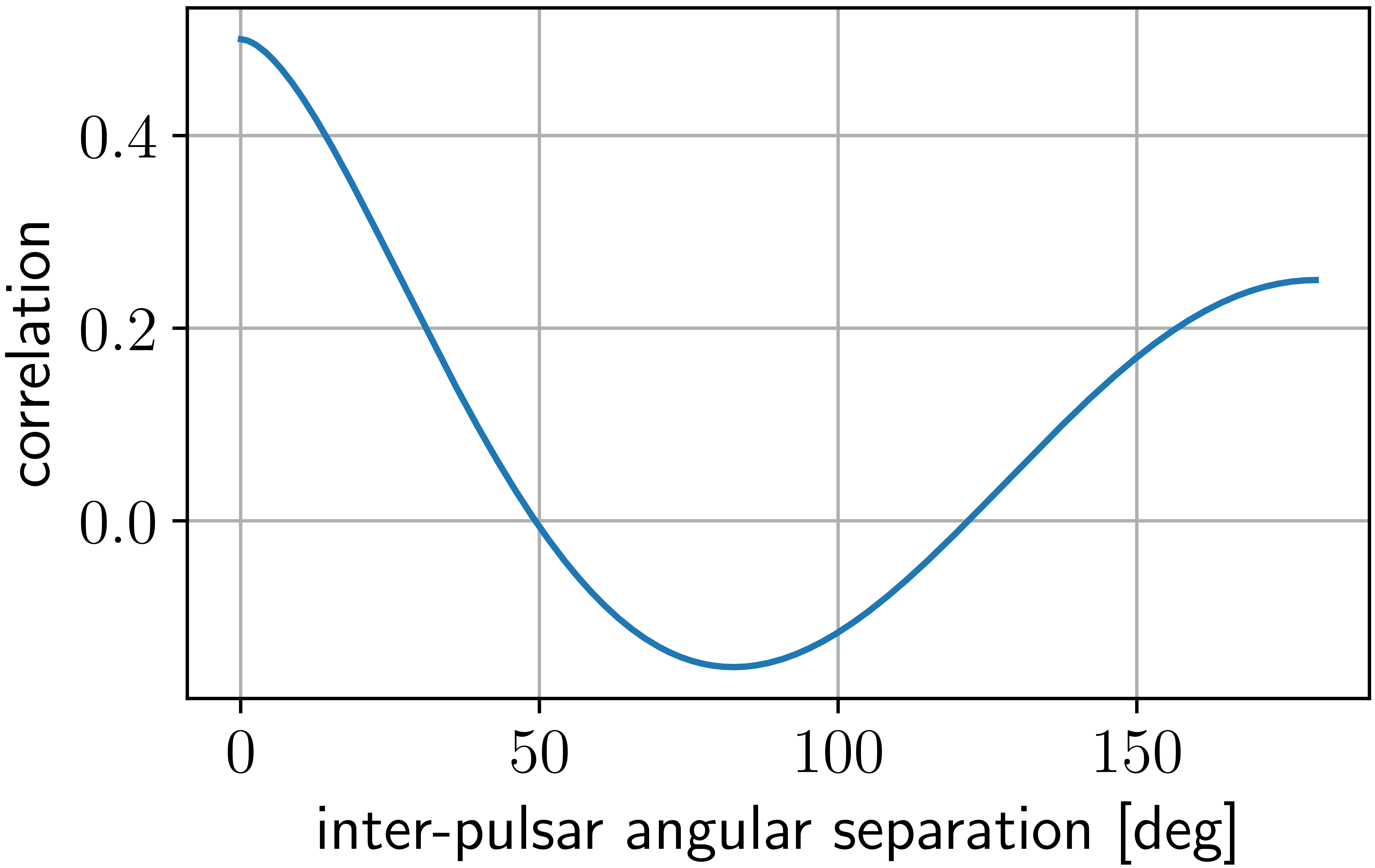
- observed data agrees with predicted GW signal from BBH merger



- **observed data agrees with predicted GW signal from BBH merger**
- **observed data are consistent across detectors**

What plays the role of a chirp waveform for PTA searches for a GW background?

Hellings and Downs correlation (1983)



General definition of an ORF (correlation coefficient)

- The ORF between two detectors a and b is defined as the transfer function relating the **expected cross-correlated power** in the two detectors to the strain power in the GWB:

$$C_{ab}(f) = \Gamma_{ab}(f) S_h(f) \quad \text{where } a, b \text{ label detectors}$$

- The ORF encodes the reduction in sensitivity due to the physical **separation** and **misalignment** of the two detectors
- Can be written as the **sky and polarization-averaged product of detector response functions**:

$$\Gamma_{ab}(f) = \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_A R_a^A(f, \hat{k}) R_b^{A\star}(f, \hat{k}) \qquad \text{where} \quad \tilde{r}_a(f) = \int d^2\Omega_{\hat{k}} \sum_{A=+,\times} h_A(f, \hat{k}) R_a^A(f, \hat{k})$$

Simple example: Omni-directional detectors

$$R_a^A(f,\hat{k}) = G\,\mathrm{e}^{-i2\pi f\hat{k}\cdot\vec{x}_a/c}$$

$$\Gamma_{ab}(f) = \frac{1}{8\pi}\int \mathrm{d}^2\Omega_{\hat{k}} \sum_A R_a^A(f,\hat{k})R_b^{A\star}(f,k)$$

$$= \frac{G^2}{4\pi}\int \mathrm{d}^2\Omega_{\hat{k}}\,\mathrm{e}^{-i\frac{2\pi f D_{ab}}{c}\hat{k}\cdot\hat{\Delta}x_{ab}}$$

$$= \frac{G^2}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{1}\mathrm{d}(\cos\theta)\,\mathrm{e}^{i\frac{2\pi f D_{ab}}{c}\cos\theta}$$

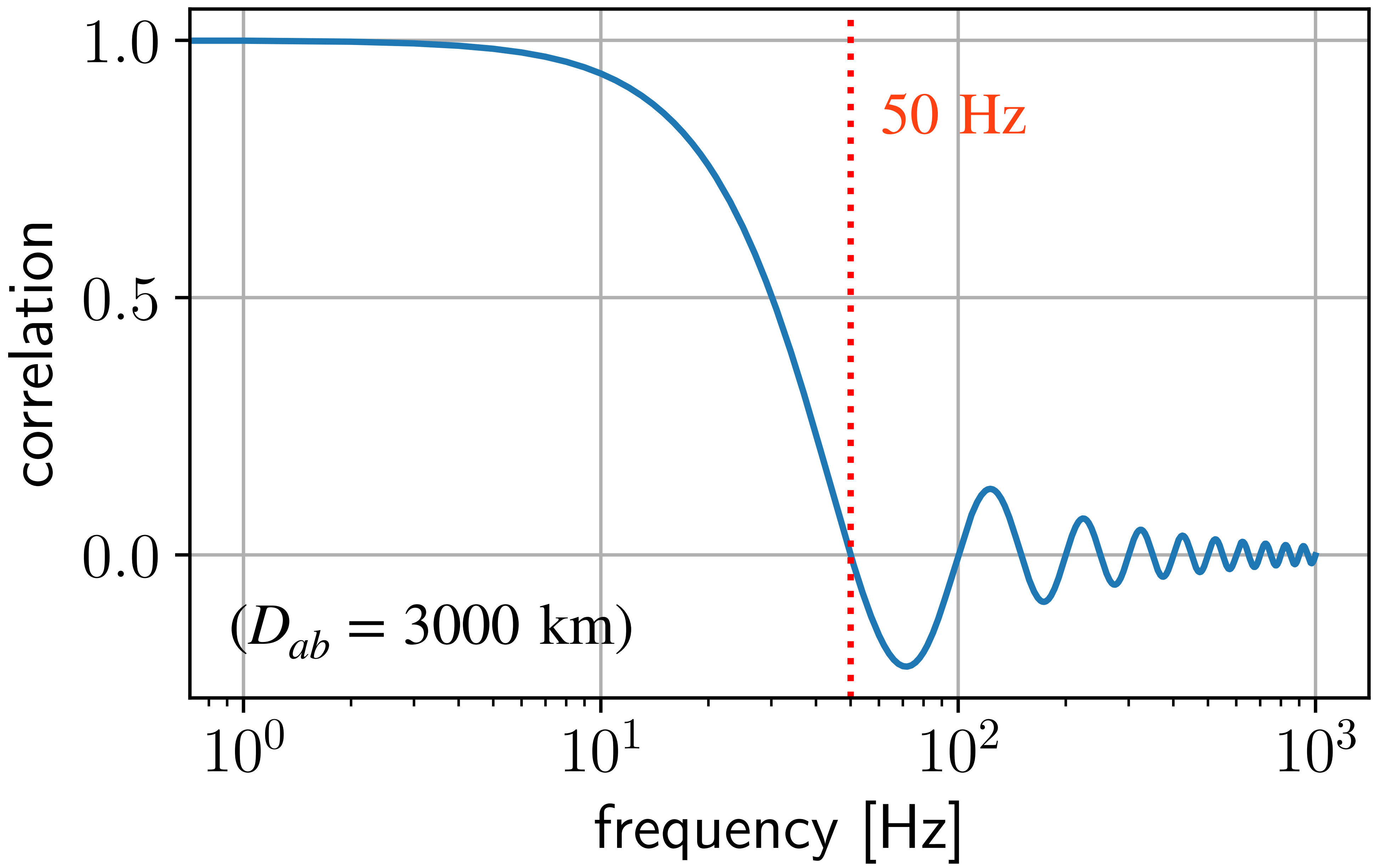
$$\dots$$

$$= G^2\,\mathrm{sinc}\left(\frac{2\pi f D_{ab}}{c}\right)$$

where

$$\vec{x}_a - \vec{x}_b \equiv D_{ab}\,\hat{\Delta}x_{ab}$$

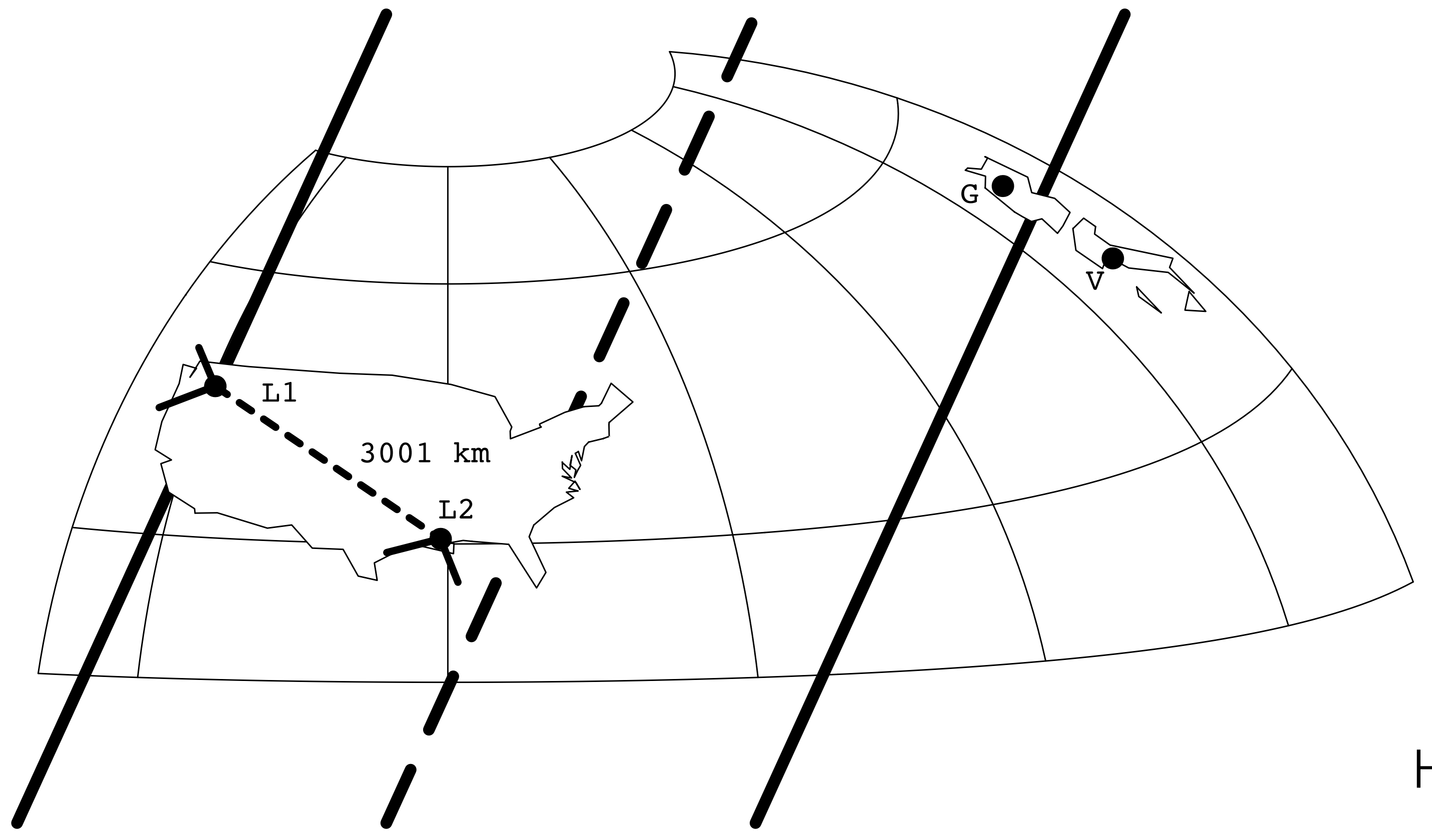
$$\mathrm{sinc}\,x \equiv \frac{\sin x}{x}$$



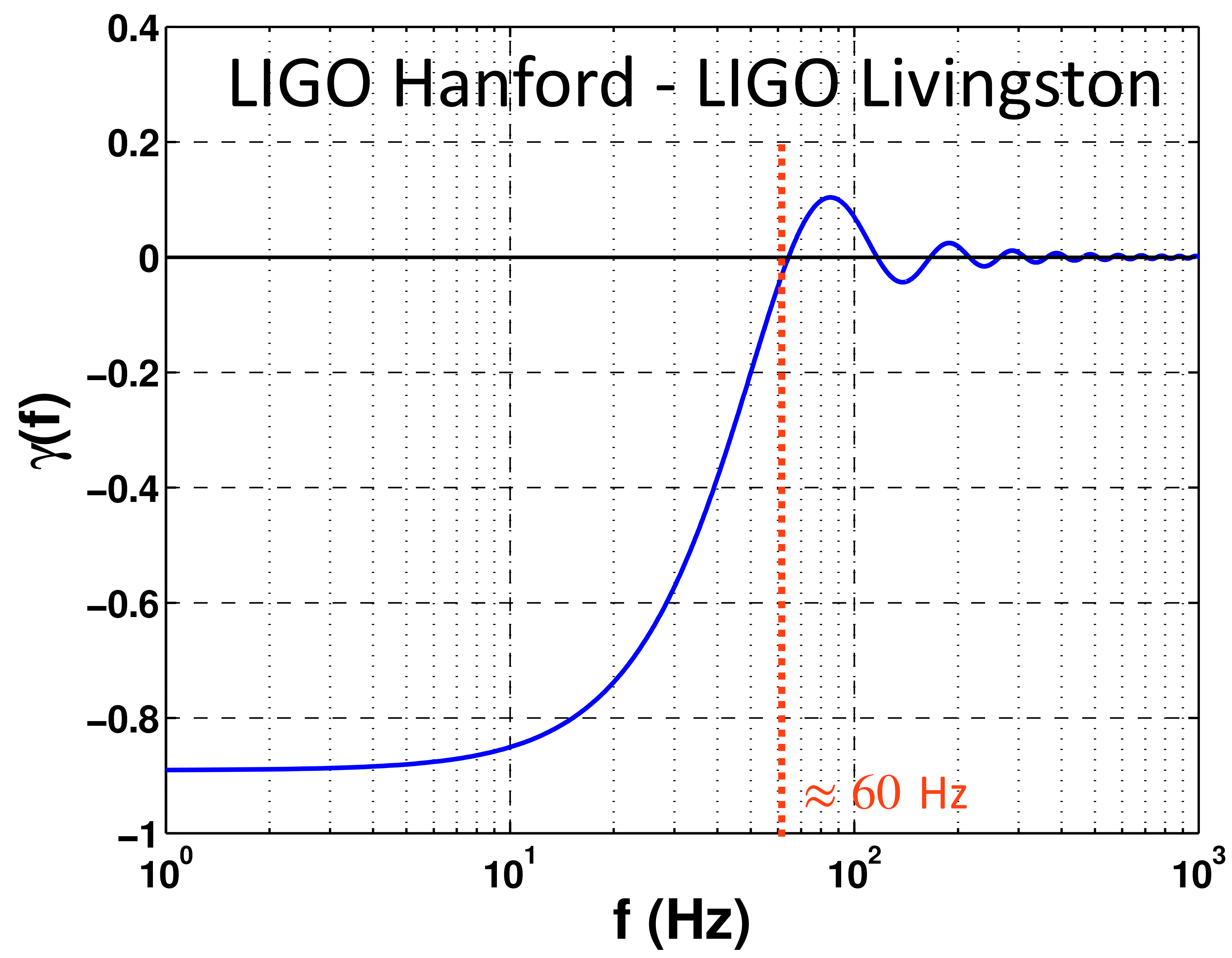
Example: Ground-based interferometers

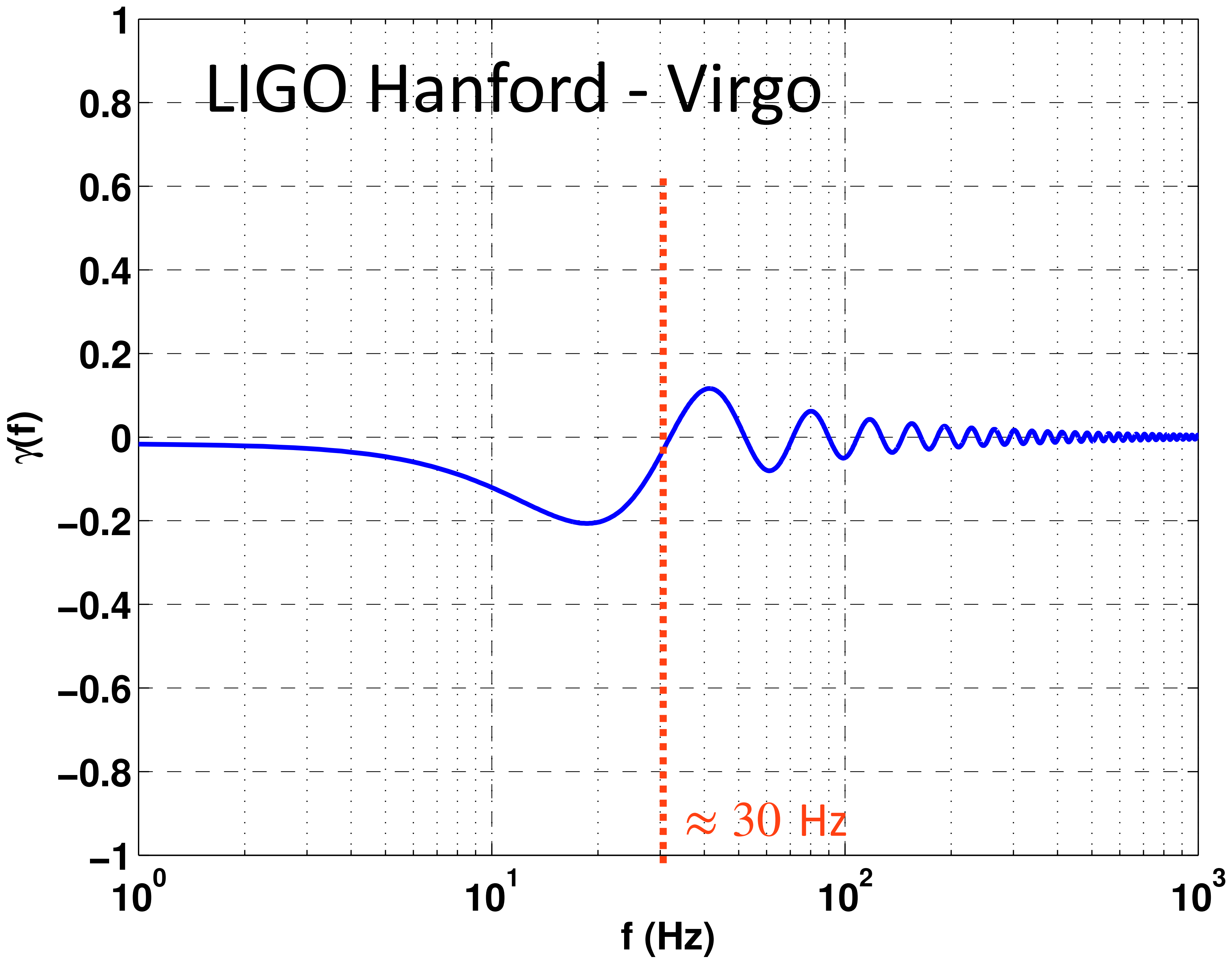
$$R_a^A(f,\hat{k}) \simeq \frac{1}{2}(u_a^i u_a^j - v_a^i v_a^j) e_{ij}^A(\hat{k})\, \mathrm{e}^{-i2\pi f \hat{k}\cdot \vec{x}_a/c}$$

(short-antenna limit)



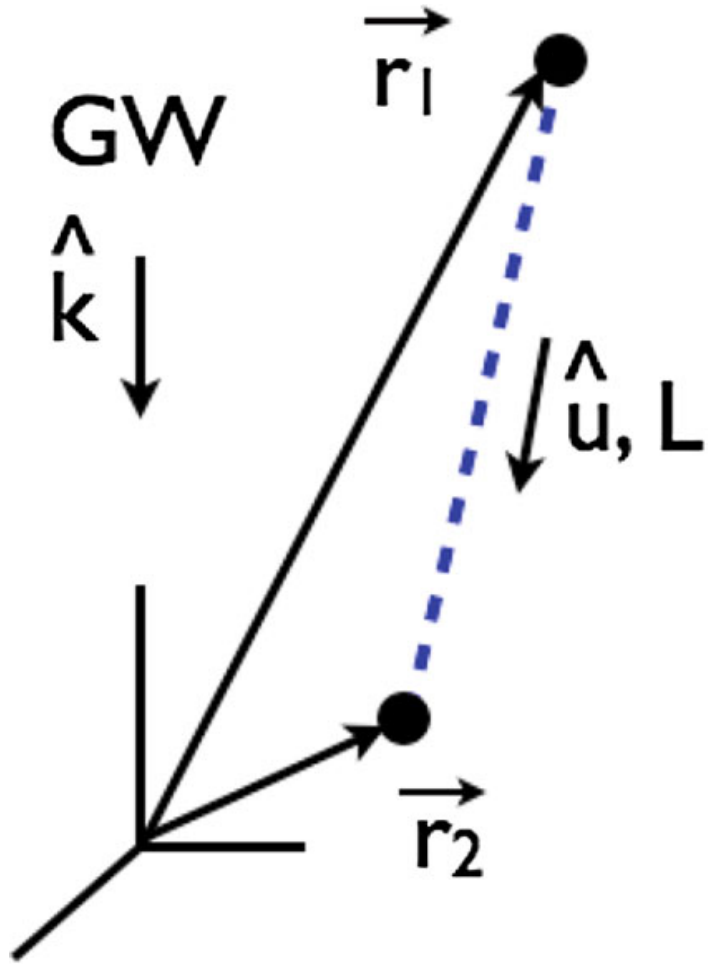
(B. Allen, Les Houches 1995)





Example: Pulsar timing arrays

- Response is non-trivial:



$$r(t) \equiv \Delta T(t) = \frac{1}{2c} u^i u^j \int_0^L \mathrm{d}s\, h_{ij}(t(s), \vec{x}(s))$$

$$t(s) = (t - L/c) + s/c\,, \qquad \vec{x}(s) = \vec{r}_1 + s\hat{u}$$

$$R^A(f, \hat{k}) = \frac{1}{i2\pi f} \frac{1}{2} \frac{u^i u^j}{1 - \hat{k}\cdot\hat{u}} e^A_{ij}(\hat{k}) \left[1 - \mathrm{e}^{-\frac{i2\pi f L}{c}(1 - \hat{k}\cdot\hat{u})}\right] \mathrm{e}^{-i2\pi f \hat{k}\cdot\vec{r}_2/c}$$

- Simplifications for redshift response $\Delta\nu/\nu = \mathrm{d}\Delta T/\mathrm{d}t$:

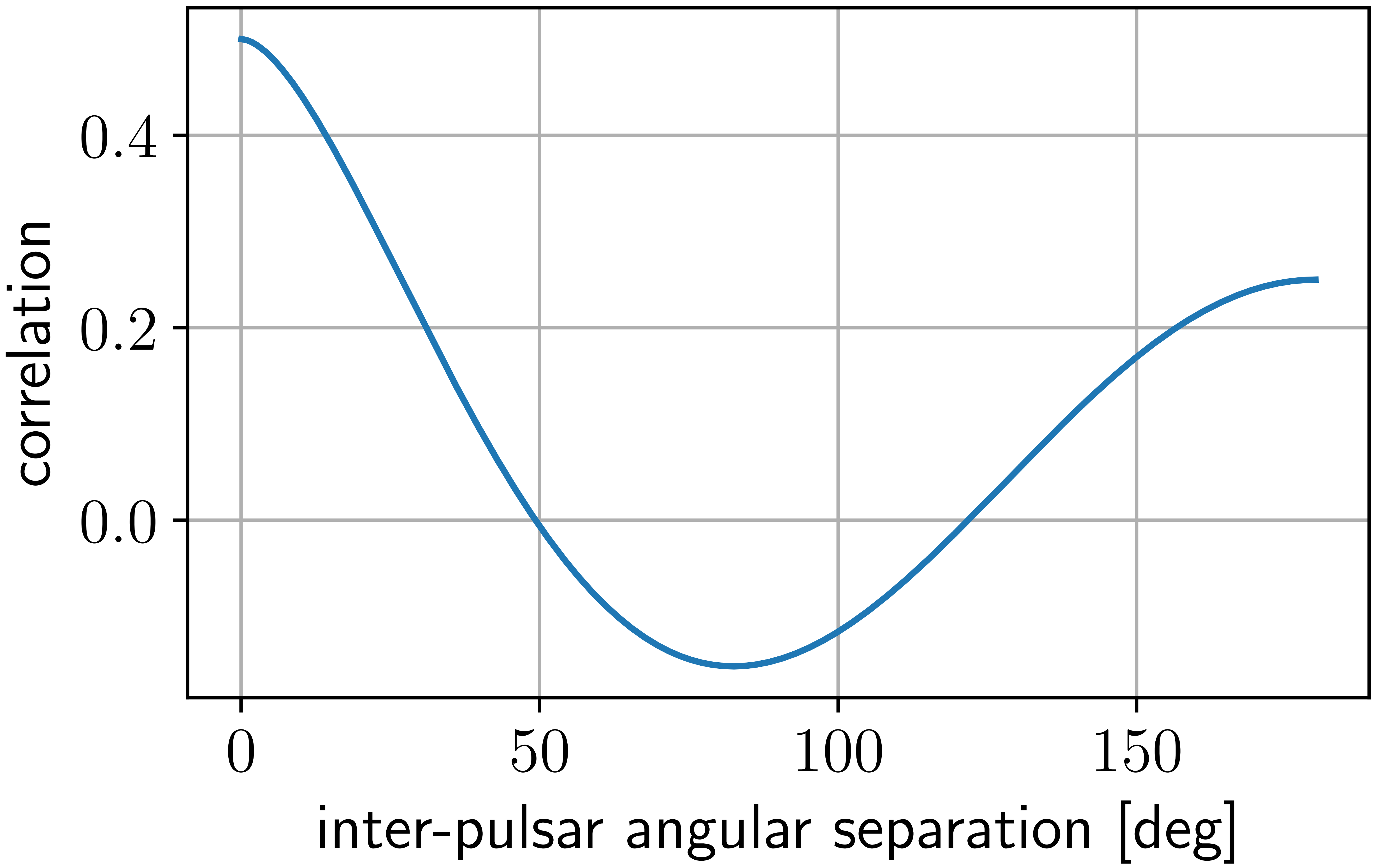
$$R^A_a(f, \hat{k}) = F^A_a(k)\left[1 - \mathrm{e}^{-\frac{i2\pi f L_a}{c}(1+\hat{k}\cdot\hat{p}_a)}\right] \qquad F^A_a(\hat{k}) \equiv \frac{1}{2} \frac{p^i_a p^j_a}{1 + \hat{k}\cdot\hat{p}_a} e^A_{ij}(\hat{k})$$

$$\Gamma_{ab}(f) = \frac{1}{8\pi} \int \mathrm{d}^2\Omega_{\hat{k}} \sum_A F^A_a(\hat{k}) F^A_b(\hat{k}) \left[1 - \cancel{\mathrm{e}^{-\frac{i2\pi f L_a}{c}(1+\hat{k}\cdot\hat{p}_a)}}\right] \left[1 - \cancel{\mathrm{e}^{+\frac{i2\pi f L_b}{c}(1+\hat{k}\cdot\hat{p}_b)}}\right] \simeq \frac{1}{8\pi} \int \mathrm{d}^2\Omega_{\hat{k}} \sum_A F^A_a(\hat{k}) F^A_b(\hat{k}) [1 + \delta_{ab}]$$

Hellings and Downs correlation

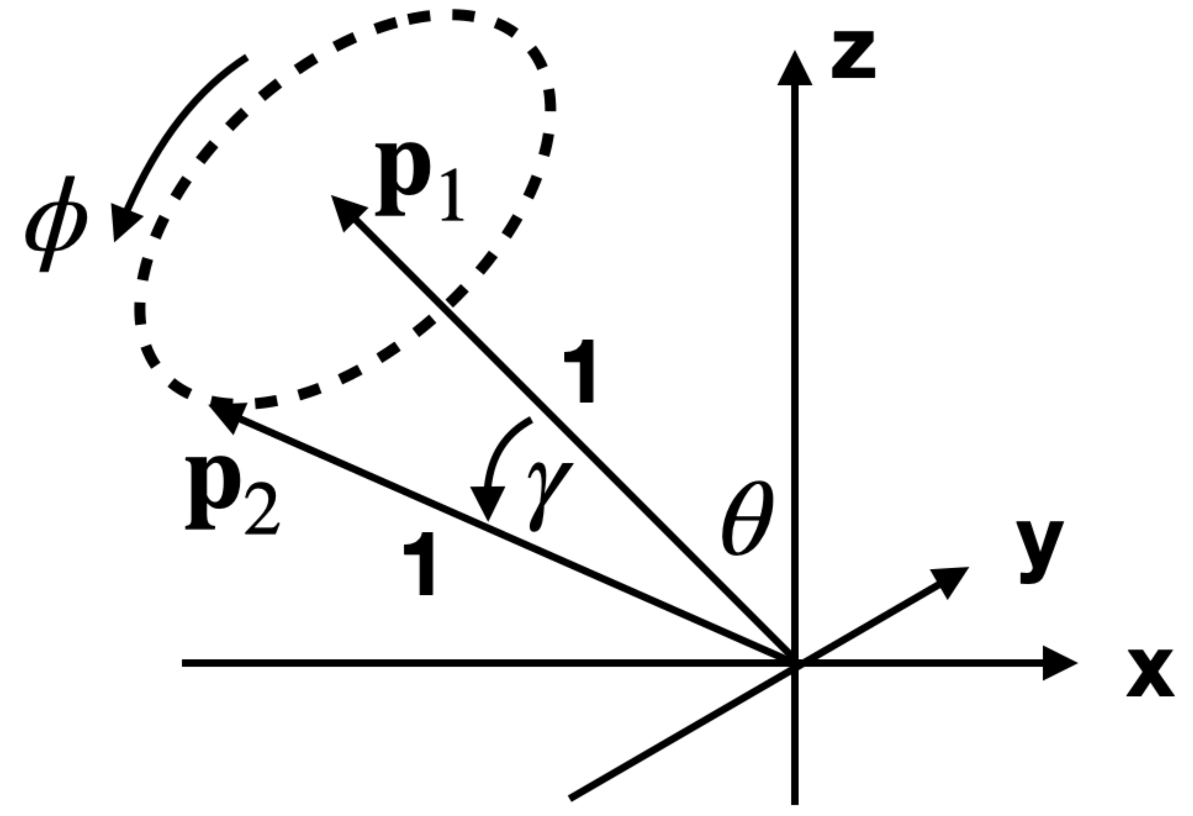
- Normalizing to 0.5 for zero angular separation:

$$\chi_{ab} = \frac{3}{2}\left(\frac{1-\cos\gamma_{ab}}{2}\right)\left[\ln\left(\frac{1-\cos\gamma_{ab}}{2}\right)-\frac{1}{6}\right]+\frac{1}{2}+\frac{1}{2}\delta_{ab}$$



Another way to arrive at the HD correlation (Cornish and Sesana, 2013)

- Rather than average over an isotropic distribution of sources, fix a **single point source** on the sky and average over all pulsar pairs separated by the same angle γ



[Credit: Allen, 2022]

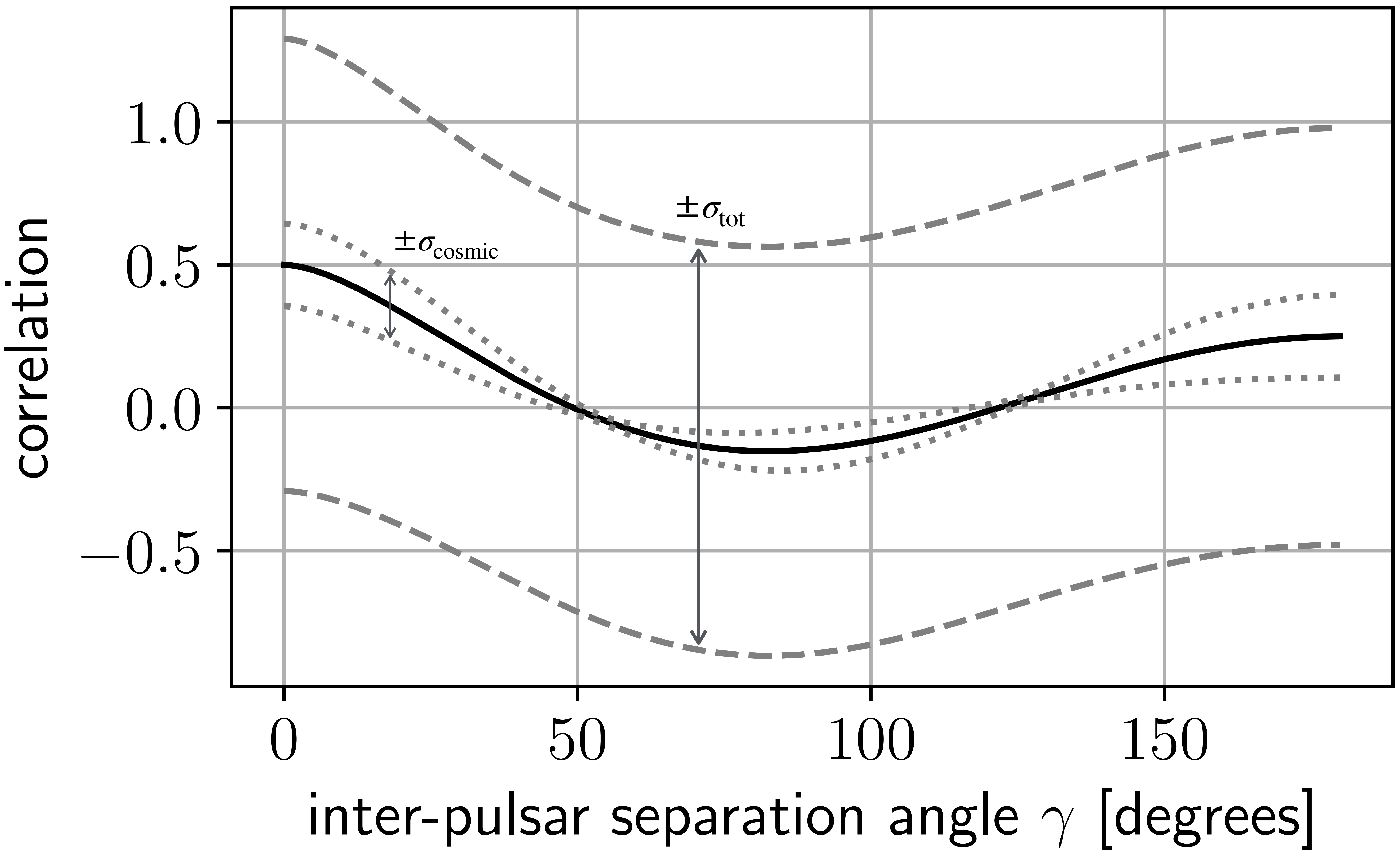
- This **pulsar averaging** leads to the same average correlation as Hellings and Downs found!!

But even for an infinite number of pulsars and no noise, the pulsar-averaged correlation will not agree exactly with the expected Hellings and Downs correlation for a universe consisting of interfering GW sources. WHY NOT??

Variance in HD correlation (Allen, 2022)

- **Pulsar variance**:
  - the **change in correlation** that results from **changing the directions to a pair pulsars** (keeping the same angular separation) for a fixed set of GW sources
  - **can be eliminated** by averaging over a large number of pulsar pairs with the same angular separation
- **Pulsar-averaged correlation:**
  - **can be calculated in our universe** or any other universe; it depends only on the angular separation between pairs of pulsars
  - the pulsar-averaged correlation **may or may not agree with the HD correlation**
- **Cosmic variance:**
  - the potential **difference between the pulsar-averaged correlation** (in one universe) and the **HD correlation** (obtained via an ensemble average)
  - there is **no cosmic variance** if the ensemble of universes consists of **non-interfering GW sources**
  - there is **non-zero cosmic variance** if our universe is one of an ensemble of universes containing **interfering GW sources** (e.g., confusion-noise limit model)

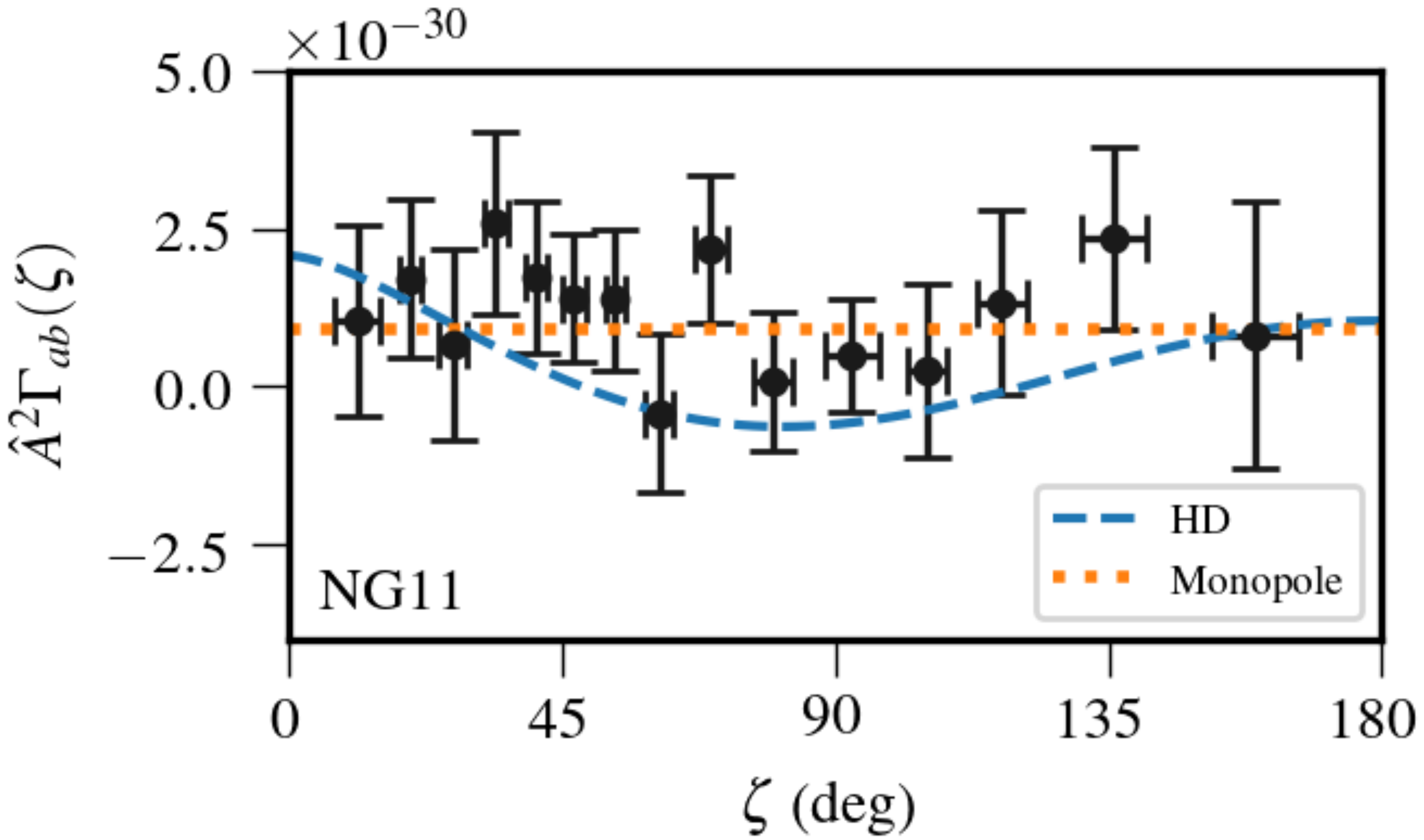
Variance in noise-free correlation measurements



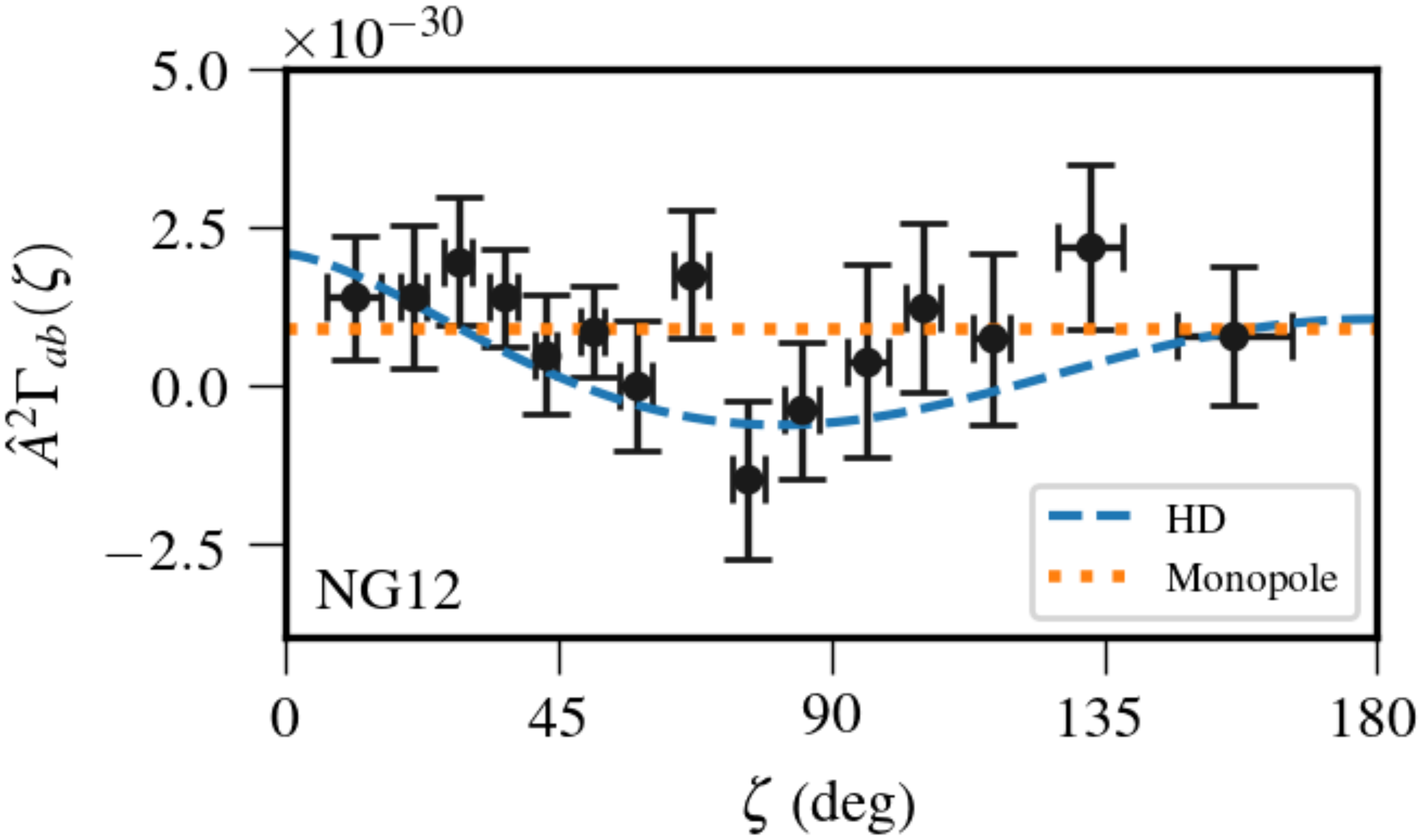
Current status

- Finite # of pulsars (67 for NANOGrav 15-yr data set)
- Contend with noise (measurement noise, intrinsic pulsar noise, etc.)
- Measured correlations in angular separation bins are combined to reduce the variance of the recovery

NANOGrav 11-yr (45 pulsars)



NANOGrav 12.5-yr (47 pulsars)



NANOGrav 15-yr (67 pulsars)

?? data is currently being analyzed

HW problem 1

- Fill in the missing steps leading from

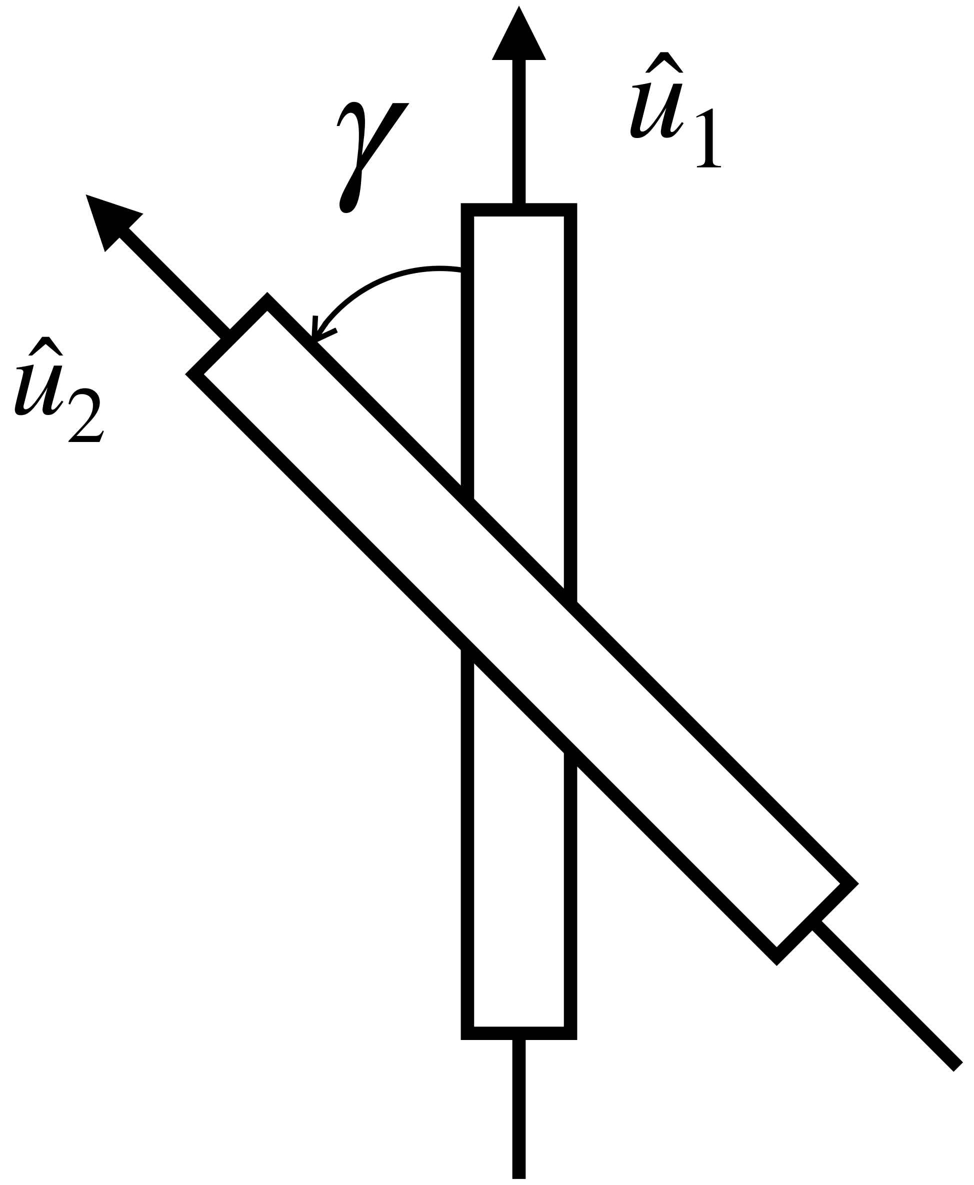
$$C_{ab}(f) = \Gamma_{ab}(f) S_h(f) \quad \text{to} \quad \Gamma_{ab}(f) = \frac{1}{8\pi} \int \mathrm{d}^2\Omega_{\hat{k}} \sum_A R_a^A(f,\hat{k}) R_b^{A\star}(f,\hat{k})$$

- Fill in the missing steps leading from

$$R_a^A(f,\hat{k}) = G\, \mathrm{e}^{-i2\pi f \hat{k}\cdot\vec{x}_a/c} \quad \text{to} \quad \Gamma_{ab}(f) = G^2\, \mathrm{sinc}\left(\frac{2\pi f D_{ab}}{c}\right)$$

HW problem 2 (Jenet and Romano, 2015)

- Calculate the value of the ORF for a pair of short electric dipole antennae pointing in directions $\hat{u}_1$ and $\hat{u}_2$ for an unpolarized, isotropic electromagnetic field



$$r_a(t) = \hat{u}_a \cdot \vec{E}(t, \vec{x} = \vec{0}) \quad \text{for} \quad a = 1,2$$

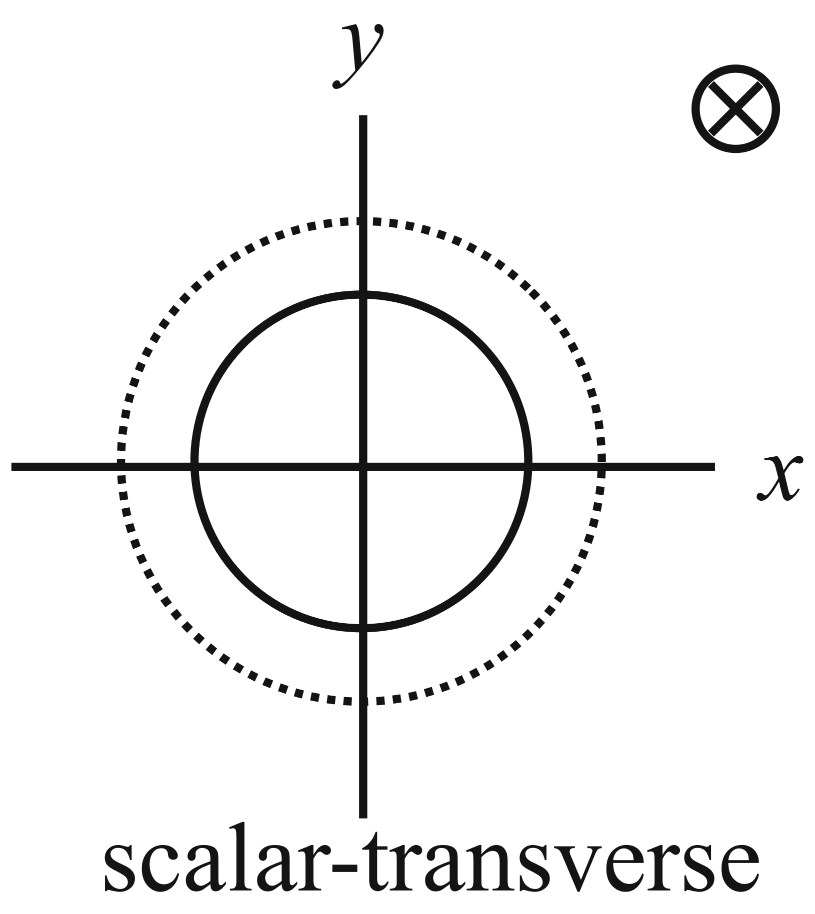
$$\vec{E}(t, \vec{x}) = \int_{-\infty}^{\infty} \mathrm{d}f \int \mathrm{d}^2\Omega_{\hat{k}} \sum_{\alpha=1,2} \tilde{E}_\alpha(f, \hat{k}) \hat{\epsilon}_\alpha(\hat{k})\, \mathrm{e}^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}$$

$$\hat{\epsilon}_1(\hat{k}) = -\hat{\phi}\,, \quad \hat{\epsilon}_2(\hat{k}) = -\hat{\theta}$$

- The final result should depend very simply on γ !

HW problem 3 (alternative polarization)

- Calculate the value of the ORF for a pair of pulsars 1 and 2 separated by angle γ subject to the (non-GR) ``scalar-transverse” or ``breathing” polarization mode:



$$e^{\mathrm{B}}_{ij}(\hat{k}) \equiv \hat{l}_i \hat{l}_j + \hat{m}_i \hat{m}_j$$

$$\hat{k} = -\hat{r}\,, \quad \hat{l} = -\hat{\phi}\,, \quad \hat{m} = -\hat{\theta}$$

- Again the final result should depend very simply on γ !

Questions??

Extra slides

Plane wave expansion, ensemble average

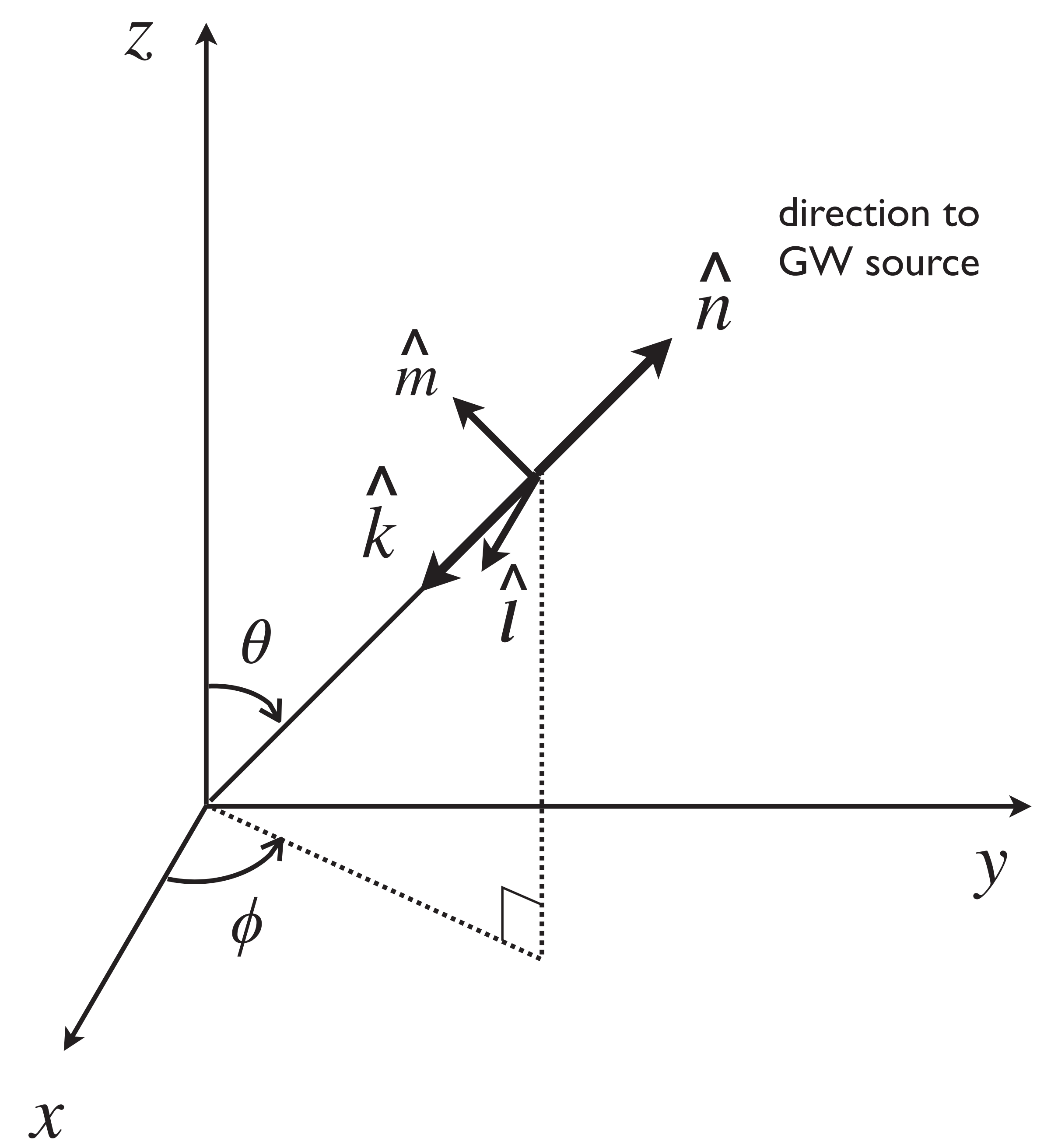
- Plane wave expansion:

$$h_{ij}(t, \vec{x}) = \int_{-\infty}^{\infty} \mathrm{d}f \int \mathrm{d}^2\Omega_{\hat{k}} \sum_{A=+,\times} h_A(f, \hat{k}) e_{ij}^A(\hat{k})\, \mathrm{e}^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}$$

- Polarization tensors:

$$e_{ij}^{+}(\hat{k}) = \hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j\,, \qquad e_{ij}^{\times}(\hat{k}) = \hat{l}_i\hat{m}_j + \hat{m}_i\hat{l}_j$$

$$\hat{k} = -\,\hat{r}\,, \quad \hat{l} = -\,\hat{\phi}\,, \quad \hat{m} = -\,\hat{\theta}$$



- Ensemble average (for an unpolarized, isotropic, stationary-Gaussian GWB):

$$\langle h_A(f, \hat{k}) \rangle = 0$$

$$\langle h_A(f, \hat{k}) h_{A'}^{\star}(f', \hat{k}') \rangle = \frac{1}{16\pi} S_h(f)\delta(f - f')\delta_{AA'}\,\delta^2(\hat{k}, \hat{k}')$$

Strength of GWB

- Strain power spectral density and energy density spectrum:

$$S_h(f) = \frac{3H_0^2}{2\pi^2}\frac{\Omega_{\rm gw}(f)}{f^3} \qquad \Omega_{\rm gw}(f) \equiv \frac{1}{\rho_{\rm crit}}\frac{{\rm d}\rho_{\rm gw}}{{\rm d}\ln f} \qquad \rho_{\rm crit} \equiv \frac{3H_0^2c^2}{8\pi G}$$

- Characteristic strain:

$$h_{\rm c}(f) \equiv \sqrt{fS_h(f)} = A_{\rm gw}\left(f/f_{\rm ref}\right)^{\alpha}$$

$\alpha = -2/3$ (for binary inspiral)

Detector response

- Response of detector a to a GW background signal:

$$r_a(t) = \int_{-\infty}^{\infty} \mathrm{d}f \int \mathrm{d}^2\Omega_{\hat{k}} \sum_{A=+,\times} h_A(f,\hat{k}) R_a^A(f,\hat{k})\, \mathrm{e}^{i2\pi f t}$$

$$\tilde{r}_a(f) = \int \mathrm{d}^2\Omega_{\hat{k}} \sum_{A=+,\times} h_A(f,\hat{k}) R_a^A(f,\hat{k})$$