Quantisation and Compensation in Sampled Interleaved Multichannel Systems

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Abstract

We consider a interleaved, multichannel sampling structure and perform the usual Nyquist- Shannon sampling for a given signal. We show that overall SQNR ratio of the signal can be increased by varying the quantizer step size, changing the relative time-delays between different channels.

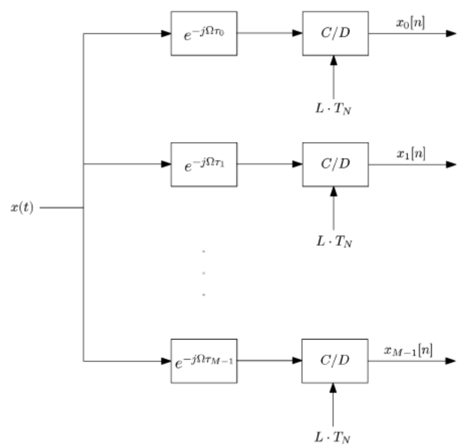
1 Introduction

Nyquist-Shannon theorem states that any bandlimited signal can be perfectly reconstructed from it's samples only when sampling is done at or above the nyquist rate.So a high bandwidth signal requires to be sampled at very large frequencies.To overcome this problem the solution suggested was multichannel sampling.Multichannel sampling allows us to sample at frequencies lower than the nyquist rate.

In multichannel sampling each channel samples a shifted version of the original signal and the interleaving of these samples results in uniform or recurrent non-uniform sampling.Many sensor networks use interleaved sampling where each sensor samples the shifted versionn of the signal and is then the samples are interleaved and processed.In multichannel sampling the relative timings between channels and the step size of quantizers of each channel can be uniform or different.This allows a lot of options to be explored while sampling and their effect on the reconstructed signal.

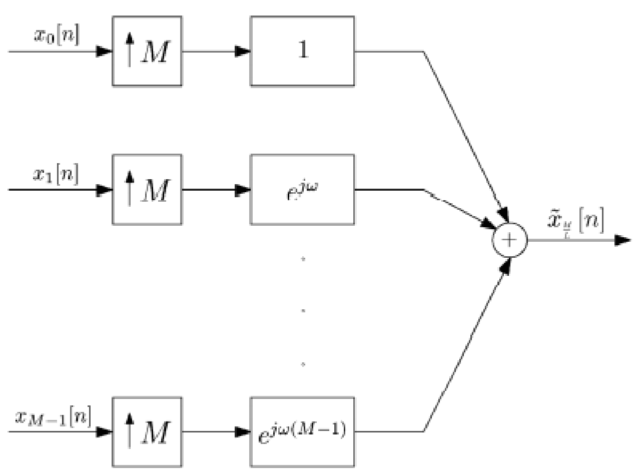
1.1 Multichannel Sampling

In multichannel sampling we use M channels each sampling the signal at a frequency $\frac{1}{LT_N}$ where $\frac{1}{T_N}$ is the nyquist frequency such that $M \geq L$.Since $M \geq L$ the ratio $\frac{M}{L}$ will be $\geq$1 resulting in the effective sampling rate (which is $(\frac{M}{L})\frac{1}{T_N}$) to be $\geq$ the nyquist rate.
In the figure we can see X(w) is being convolved with $e^{-j\Omega\tau_m}$ where τ_m represents the time delay of the mth channel,by fourier transform properties this effectively represents shift of the signal by τ_m in time domian.The C/D converters convert the continous time domain signal to discrete samples of the signal.The output of mth channel is represented by



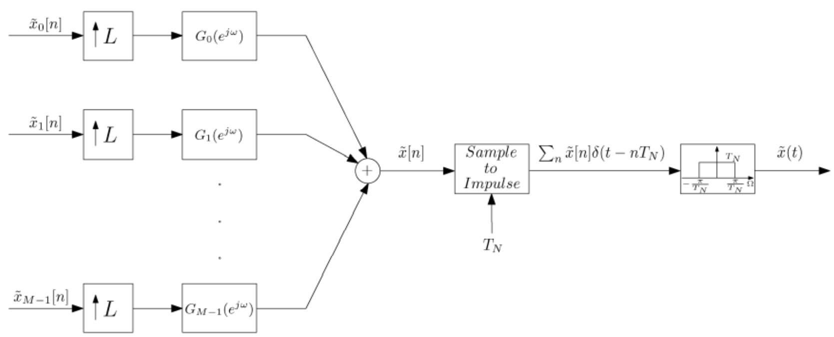
$$x_m[n] = x(n(LT_N) - \tau_m) \quad (1)$$

If we assume τ_m of the mth channel to be equal to $\frac{m}{M}LT_N$ then the timedelays



between the channels will be uniform.So the reconstruction filter of mth channel will be equal to e^{jwm} this will shift the signal.So as a result when these samples are interleaved the result is uniform samples of x(t).

1.2 Reconstruction



When the effective sampling rate of multichannel sampling is greater than or equal to nyquist rate then perfect reconstruction of the signal is possible.The above figure show reconstruction filters $G_m(e^{jw})$ for each channel these reconstruction filters are chosen such that after interleaving the resulting samples are either uniform samples of x(t) i.e $x[nT_N]$.After interleaving the samples are multiplied with an impulse train which in frequency domain would generate shifted copies of X(w),Passing this signal through a low pass filter gives x(t).
So $\Sigma_{m=0}^{M-1} G_m(e^{jw})X_m(e^{j}wL)$ has to be equal to $= \frac{1}{T_N}X(\frac{\omega}{T_N})$ This can be written as

$$\Sigma_{m=0}^{M-1} G_m(e^{jw})[\frac{1}{T}\Sigma_{k=-(L-1)}^{L-1} X(\frac{\omega - \frac{2\pi}{L}k}{T_N}).e^{-j(\omega - \frac{2\pi}{L}k)\frac{\tau_m}{T_N}}] = \frac{1}{T_N}X(\frac{\omega}{T_N}) \tag{2}$$

Rearranging the terms in the above eqn we get a condition that

$$\Sigma_{m=0}^{M-1} G_m(e^{jw})e^{-j(\omega - \frac{2\pi}{L}k)\frac{\tau_m}{T_N}} = L\delta(k) \tag{3}$$

If M=L then there will be only 1solution for the equation above but if M>L there will be M-L solutions of $G_m(e^{jw})$ which satisy the above equation.

2 Problem Statement

We try to find if it is possible to build a system such that it's perfomance gain is better than the performance gain obtained by a uniform step size or/and a uniform time delay system

3 Solution Approach

3.1 Sampling and reconstruction in presence of noise

The process of sampling and reconstruction of signals using multichannel sampling has been stated till now .But we haven't considered the noise associated with the signal. We model the noise associated with the signal to be white additive noise $q_m[n]$ that is independent of the signal,So after quantisation the signal will be represented as

$$\hat{x_m} = x_m[n] + q_m[n] \tag{4}$$

$q_m[n]$ varies from $[\frac{-\Delta_m}{2}, \frac{\Delta_m}{2}]$ where Δ_m represents the quantizer stepsize.

$q_m[n]$ represents the error in one sample so we denote e(t) as the total noise in the signal.We assume e(t) to be a zero mean gaussian random variable. The auto-correlation function of e(t) is periodic with period LT_N so e(t) is a wide sense cyclo stattionary process.For a better understanding of the effect of noise on the signal the ensemble average power of noise is then equal to $E(e^2(t))$ we then find the average over time of $E(e^2(t)$it is denoted by ${\sigma_e}^2$

$${\sigma_e}^2 = \frac{1}{T}\int_0^T E(e^2(t))dt \tag{5}$$

since e(t) is a wide sense cyclo stationary process this can be written as

$${\sigma_e}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma_{m=0}^{M-1}(\frac{{\sigma_m}^2}{L})|G_m(e^{j\omega})|^2 d\omega \tag{6}$$

It can be understood that if the value of ${\sigma_e}^2$ is less then the noise in the signal is less.So we try to find a lowerbound on the value of ${\sigma_e}^2$.Since ${\sigma_m}^2$ depends on the quantiser step size lets us assume it to be some constant now we can find the minimum of ${\sigma_e}^2$ by finding the appropriate $G_m(e^{j\omega})$. We have a constraint equation in (3) and the inequality equation in(5).So we apply lagrangian using these two equations and find the value of $G_m(e^{j\omega})$ which is equal to:

$$G_m(e^{j\omega}) = \frac{1}{{\sigma_m}^2}.e^{j\omega\tau_m/T_N}(\Sigma_{l=-i}^{L-i-1}({\lambda_l}^{(i)}e^{-j2\pi(\tau_m/LT_N)l})) \tag{7}$$

Here ${\lambda_l}^{(i)}{'s}$ are the lagrangian multiplers. We can see that the innermost term is the DTFT of ${\lambda_l}^{(i)}$ So we can write the equation as

$$G_m(e^{j\omega}) = \frac{1}{{\sigma_m}^2}.e^{j\omega\tau_m/T_N}\Lambda^{(i)}(e^{j\omega_m}) \tag{8}$$

Here

$$\omega_m = 2\pi\tau_m/LT_N \tag{9}$$

Substituting (7) in (6) we get

$${\sigma_{emin}}^2 = \frac{1}{L}\Sigma_{i=0}^{L-1}(\frac{1}{L}\Sigma_{m=0}^{M-1}|\Lambda^{(i)}(e^{j\omega_m}|^2)/{\sigma_m}^2) \tag{10}$$

For computational purposes a vectorised equation and the final result will be

$${\sigma_{emin}}^2 = tr(A_m)^{-1} \tag{11}$$

where

$$A_m = \Sigma_{m=0}^{M-1}(\nu_m.\nu_m{}^H) \tag{12}$$

and

$$\nu_m{}^H = [1, e^{-j2\pi\frac{\tau_m}{LT_N}}, \ldots\ldots, e^{-j2\pi\frac{\tau_m(L-1)}{LT_N}}] \tag{13}$$

From (10), where lagrangian multipliers were used to get an expression for σ^2_{emin}, we also get an expression in terms of i where i $\in$ 0,1,...,L-1

$$\sum_{m=0}^{M-1}\frac{1}{\sigma^2_m}\Lambda^{(i)}(e^{j\omega_m}) = L \tag{14}$$

Now, when we apply the Cauchy-Schwartz inequality to the above expression, we get

$$\sum_{n=0}^{M-1}1/\sigma^2_m\sum_{m=0}^{M-1}|\Lambda^{(i)}(e^{j\omega_m})|/\sigma^2_m \geq L^2 \tag{15}$$

for any i = 0,1,..,L-1.
From (10) and (15), we get

$$\sigma^2_{emin} \geq \frac{L}{\sum_{m=0}^{M-1}1/\sigma^2_m} \tag{16}$$

Now from the inequality, equality holds only if

$$\sum_{m=0}^{M-1}1/\sigma^2_m e^{j\omega_m l} = 0 \ \forall \ l = 1, 2, ..., L-1 \tag{17}$$

Now consider the case where the quantier step sizes are same for all channels of our system. Then the above expression becomes

$$\sum_{m=0}^{M-1}e^{j\omega_m l} = 0 \ \forall \ l = 1, 2, .., L-1 \tag{18}$$

One set of solutions for $\sum_{m=0}^{M-1}e^{j\omega_m} = 0$ can be easily found out i.e the M roots of unity. All the solutions we get are equally spaced when plotted on the unit circle. This solution correspond to uniform sampling. Also other possible solutions may exist for the above equation.
Also from the solutions obtained from (16) and (18) we get

$$\sigma^2_{emin} = \frac{L}{M}\sigma^2 \tag{19}$$

Now let quantization noise power averaged over all channels be σ^2 i.e

$$\frac{1}{M}\sum_{m=0}^{M-1}\sigma_m^2 = \sigma^2 \tag{20}$$

Using the Cauchy-Schwartz inequality for the relation $\sum_{m=0}^{M-1}\sigma_m \frac{1}{\sigma_m} = M$,

$$\sum_{m_1=0}^{M-1}\sigma_{m_1}^2 \sum_{m_2=0}^{M-1}\frac{1}{\sigma_{m_2}^2} \geq M^2 \tag{21}$$

This can be written as

$$\frac{L}{\sum_{m=0}^{M-1}\frac{1}{\sigma_m^2}} \leq (L/M)\sigma^2 \tag{22}$$

From equations (16) and (22) , we can say that by having different quantizer step sizes, we **might** achieve a better SQNR for a given signal.

From the above observations that we can achieve better SQNR by tuning the quantizer step sizes and time-delays in reconstruction filters, we could sometimes deliberately leave the error in time-delays of reconstruction filters and instead tune the step sizes such that resulting SQNR is better than it's previous value.This can be done in cases where it's difficult to set the time-delays of reconstruction filters to the desired values.

4 Simulations

In this section, we consider a multichannel sampling system with M=3 and L=2. We consider three cases and perform required simulations:

1. Consider equal values of step sizes for all the quantizers and find optimal time delays for each channel.
2. Consider a set of time delays for each channel and decide optimal combination of step-sizes for each channel.
3. Consider different values of step sizes for the quantizers and find optimal time delays for each channel.

4.1 Same step-sizes for all channels

We consider the step-size value for all three channels (0,1,2) as 4. We'll define a quantity $\gamma = \frac{\sigma^2}{\sigma_{e_{min}}^2}$. Without loss of generality let τ_0=0.
Figure 1 shows the γ plot for values of τ_1 and τ_2.
From the figure we can note that we achieve minimum value of σ_{emin}^2 for $\tau_1 = -\tau_2 = \pm 0.67$(approx) and $\gamma = 1.48$.

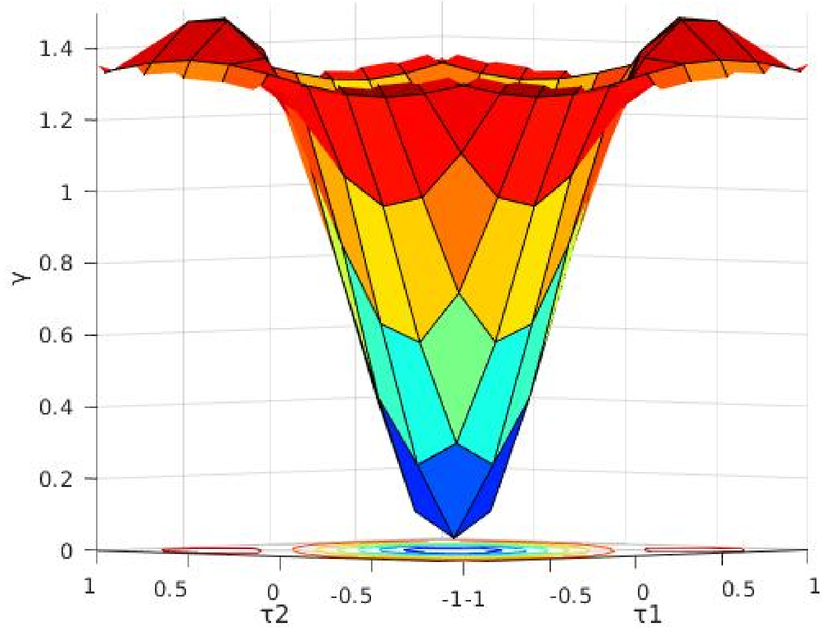


Figure 1: γ plot for values of τ_1 and τ_2.

To verify this theoretically, from (18) we get

$$\tau_m = [0, 2/3T_N, -2/3T_N] \text{ and } \sigma^2_{emin} = (2/3)\sigma^2. \tag{23}$$

We consider another set of values for $\tau_m = [0, 1/8T_N, -3/4T_N]$ for our system. Then we observed a gain in output average noise power by 20%

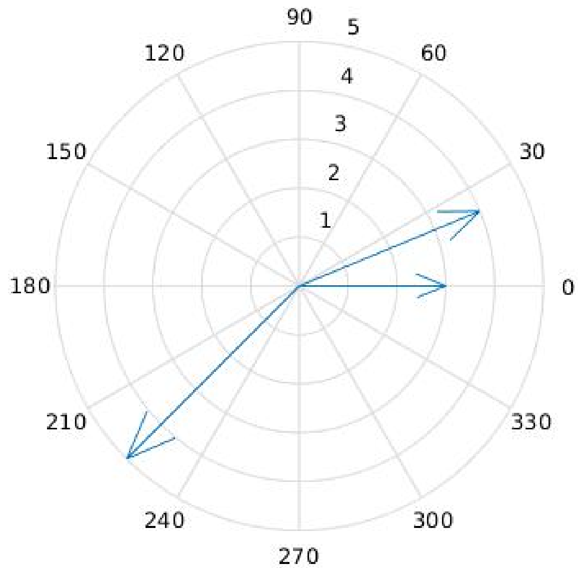


Figure 2: Each vector demonstrates a channel.

4.2 Fixed time-delays for all channels

We consider the time-delays for the channels as $\tau_m = [0, 1/8T_N, -3/4T_N]$ and also consider that only 3 quantizers of step-sizes 3,4 and 5 each are available.

We now find the performance gain for possible 6 combinations of quantizers.

Performance gain for different bit allocation rounded to 2 decimals			
N_0	N_1	N_2	Performance Gain
3	4	5	1.45
4	3	5	1.36
3	5	4	1.26
5	3	4	1.14
4	5	3	0.41
5	4	3	0.38

Performance gain values obtained for different combinations are almost same as those given in the paper.
To better understand why the combination [3,4,5] is better as step-sizes, Consider Figure 2

We can see that for vector of length 5, it's more seperated from other vec-

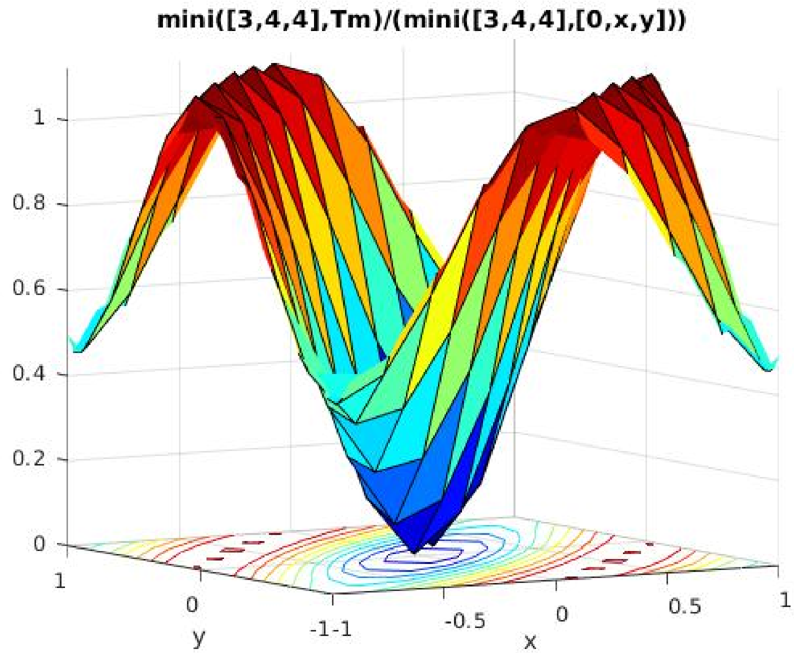


Figure 3: Each vector demonstrates a channel.

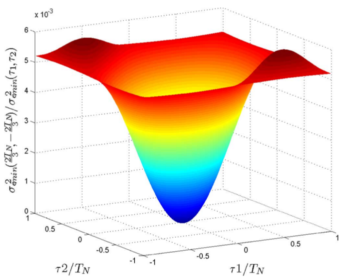
tors relatively. So since the sampling instants are more for that channel we can expect more errors in it. So we can cover error to some extent by using a quantiser of larger bits .

4.3 Different step-sizes for channels

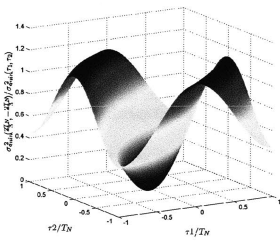
Consider channel 0 is allocated a 3 bit quantizer while channel 1 and 2 are allocated 4-bit quantizers.

Theoretically, from eqn (24) we get the relation $64 + 256e^{j\omega_1} + 256e^{j\omega_2} = 0$, where $\omega_1 = -\omega_2 = \pm 0.5402\pi$ are the solutions. From Figure 3 we get the same results. Also the results obtained were intuitive as we know channel 1 and channel 2 were slightly moved away from channel 0 as comapared to the case with uniform sampling. So we can say that channel1 and channel 2 must be provided with higher bits compared to channel 0 to compensate the low accuracy.

Note: Initially we were getting a wrong plot for this case when compared to the plot given in our refernce paper. On referring to other sources, we found a PhD thesis paper by the same author of this paper who has worked on similar work and the graph given in that matched with ours. We tried checking if the author is trying to plot for any different thing but couldnt find any.



(a) Plot given in the reference paper.



(b) Plot given in the PhD thesis paper

5 Conclusions

We have shown that a better performance gain can be obtained by using non-uniform quantization and/or non-uniform time delays compared to the case where uniform quantization step-sizes and time delays were use.

6 References

- Section 7.1 , Systems and Signals, ALAN V. OPPENHEIM ,ALAN S. WILLSKY
- https://dspace.mit.edu/bitstream/handle/1721.1/66016/751980825-MIT.pdf?sequence=2 - Sampling and Quantization for Optimal Reconstruction Shay Maymon (PhD Thesis paper)
- https://www.rle.mit.edu/dspg/documents/ShayICASSP.pdf
- https://brilliant.org/wiki/cauchy-schwarz-inequality/
- https://www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/lagrange-multipliers-and-constrained-optimization/v/lagrange-multiplier-example-part-1