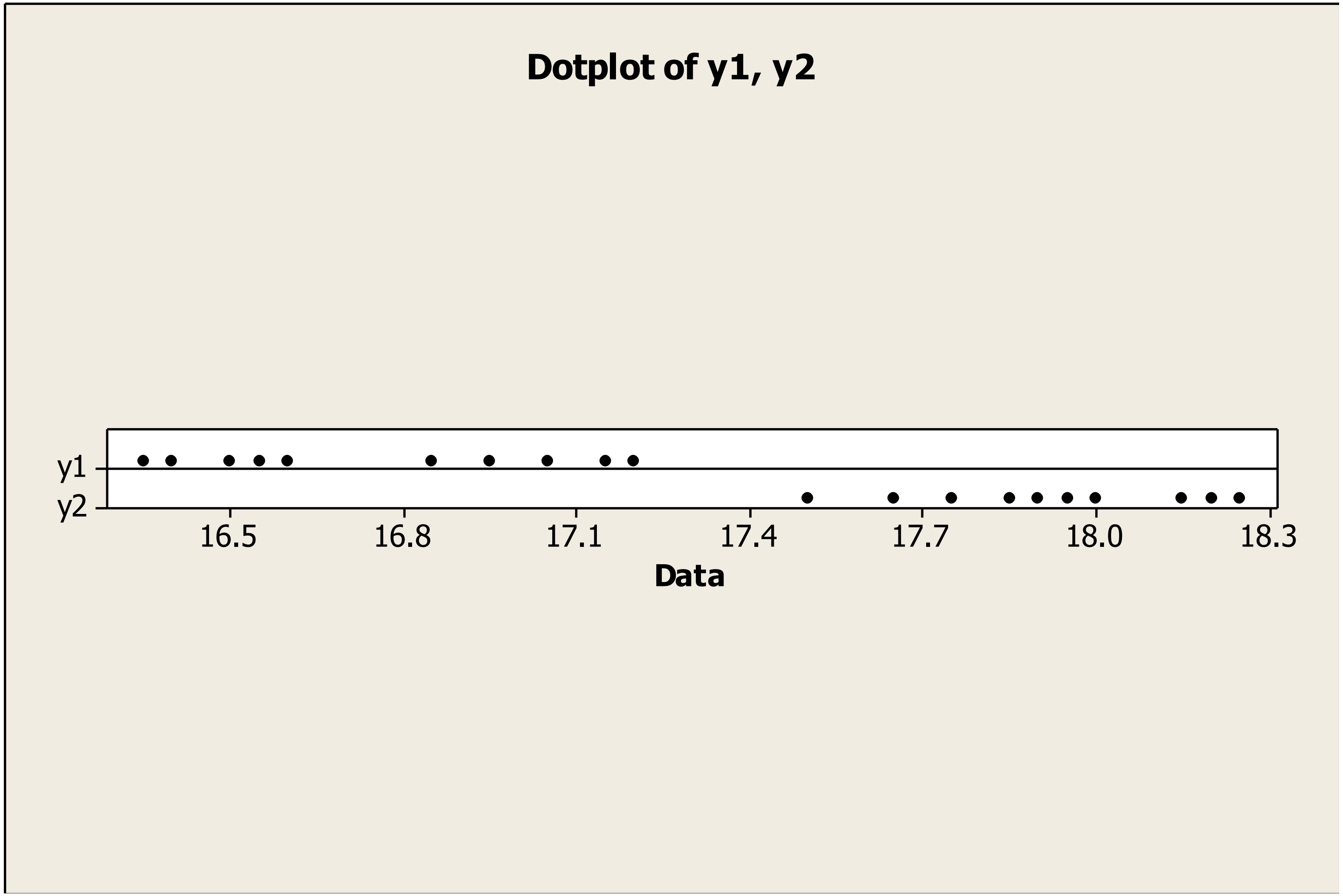
Basic Statistical Concepts

- Simple **comparative** experiments
  - The hypothesis testing framework
  - The two-sample *t*-test
  - Checking assumptions, validity
- Comparing more that two factor levels…**the analysis of variance**
  - ANOVA decomposition of total variability
  - Statistical testing & analysis
  - Checking assumptions, model validity
  - Post-ANOVA testing of means
- **Sample size** determination

Portland Cement Formulation

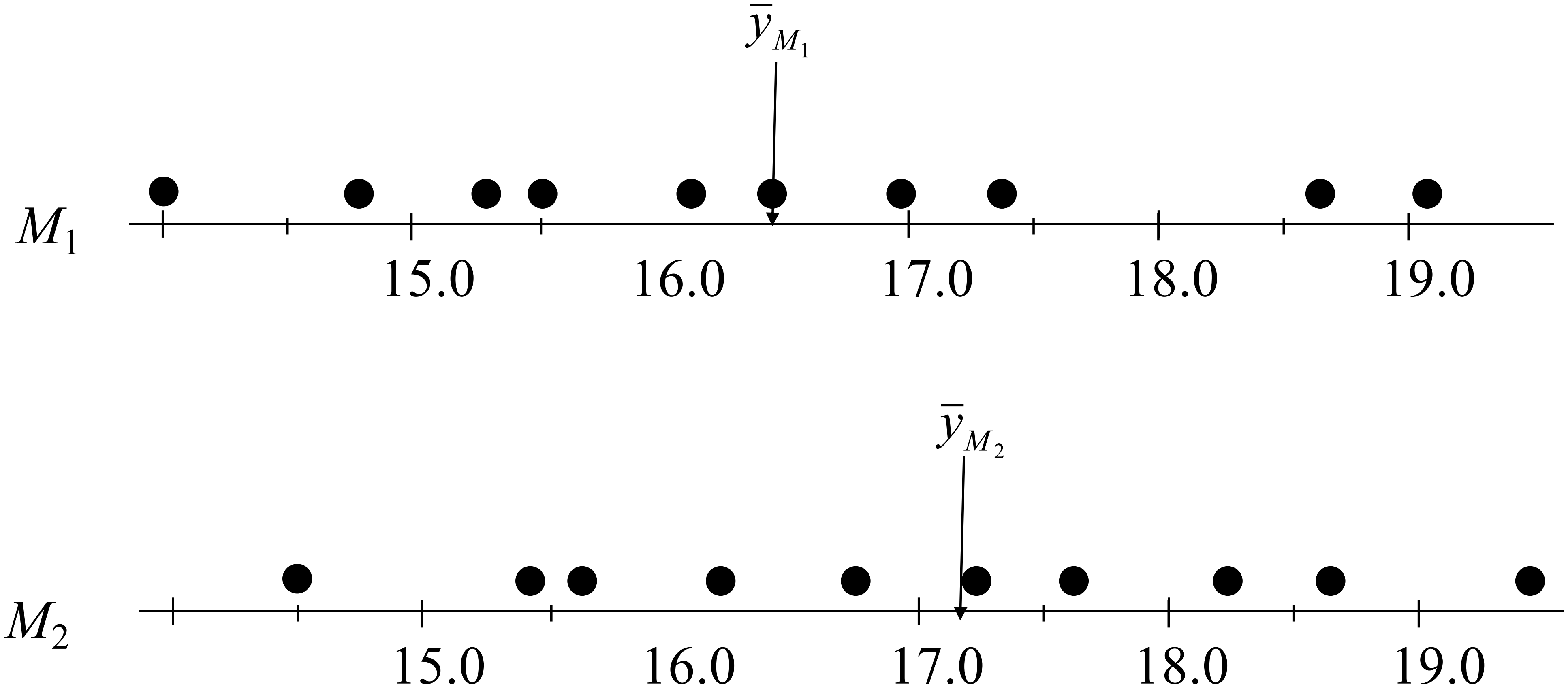
j	Modified Mortar y_{1j}	Unmodified Mortar y_{2j}
1	16.85	17.50
2	16.40	17.63
3	17.21	18.25
4	16.35	18.00
5	16.52	17.86
6	17.04	17.75
7	16.96	18.22
8	17.15	17.90
9	16.59	17.96
10	16.57	18.15

Graphical View of the Data

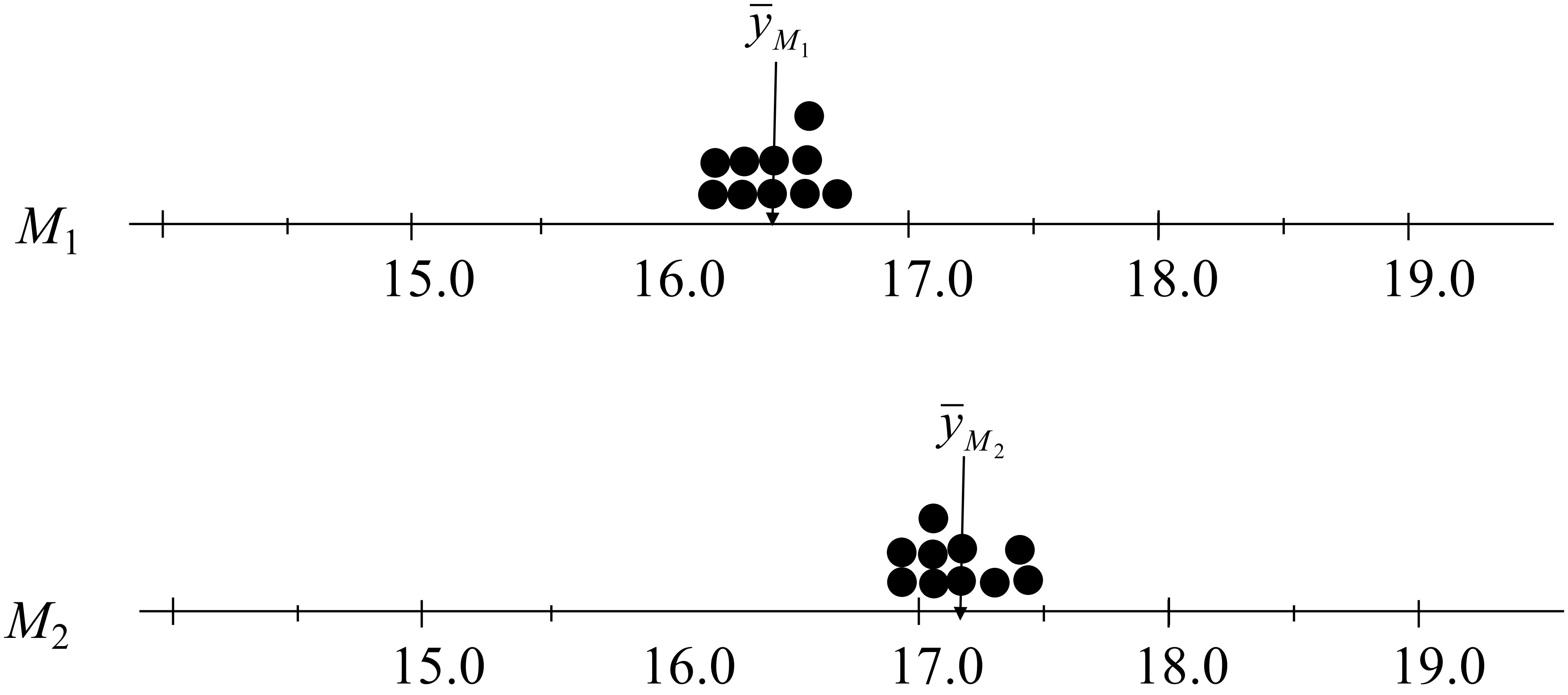


Consider the following two cases

Case 1



Case 2

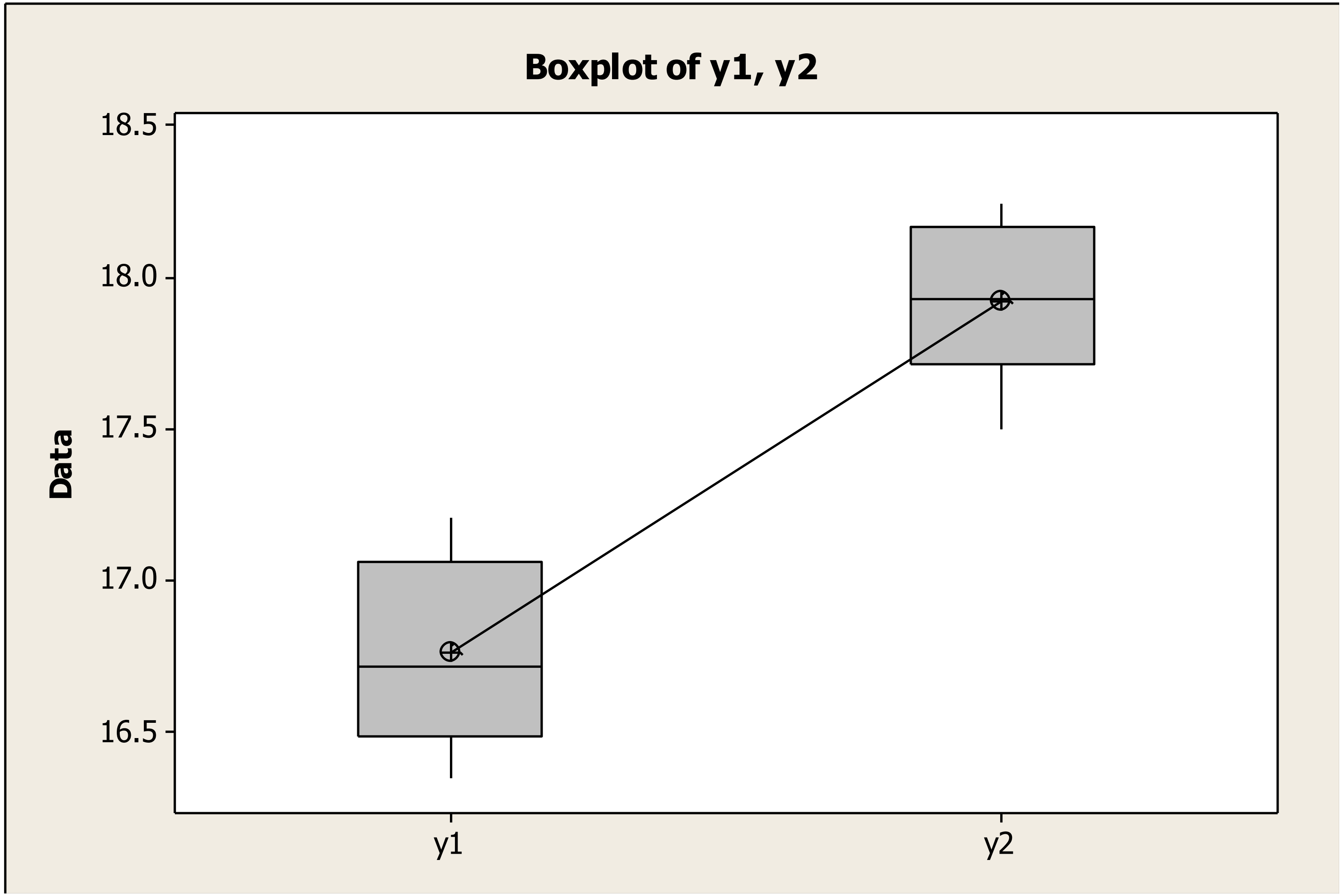


In which case are you more comfortable in comparing and possibly concluding that the two mortars give different results?

Clearly considering only the averages is not enough.

We also have to consider the spread.

Box Plot



The Hypothesis Testing Framework

- **Statistical hypothesis testing** is a useful framework for many experimental situations

- Origins of the methodology date from the early 1900s

- We will use a procedure known as the **two-sample *t*-test**

The Hypothesis Testing Framework

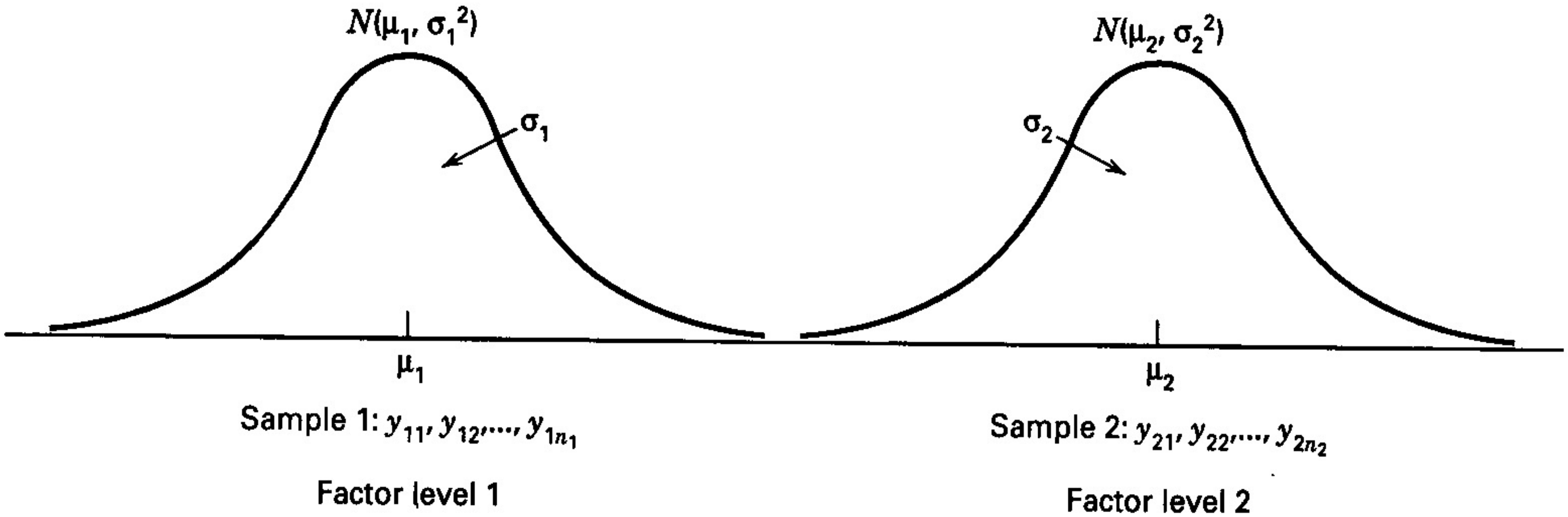


Figure 2-9 The sampling situation for the two-sample *t*-test.

- Sampling from a **normal** distribution
- Statistical hypotheses:

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

Estimation of Parameters

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \text{ estimates the population mean } \mu$$

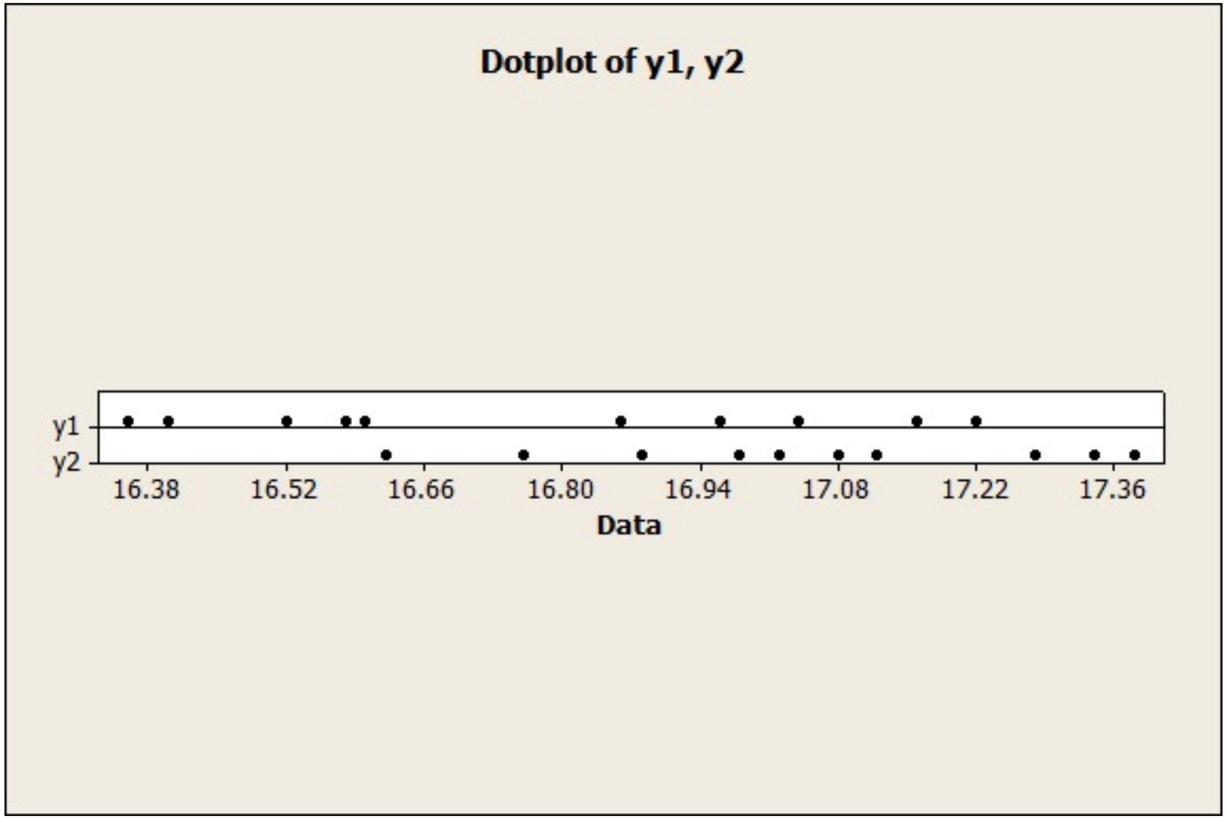
$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 \text{ estimates the variance } \sigma^2$$

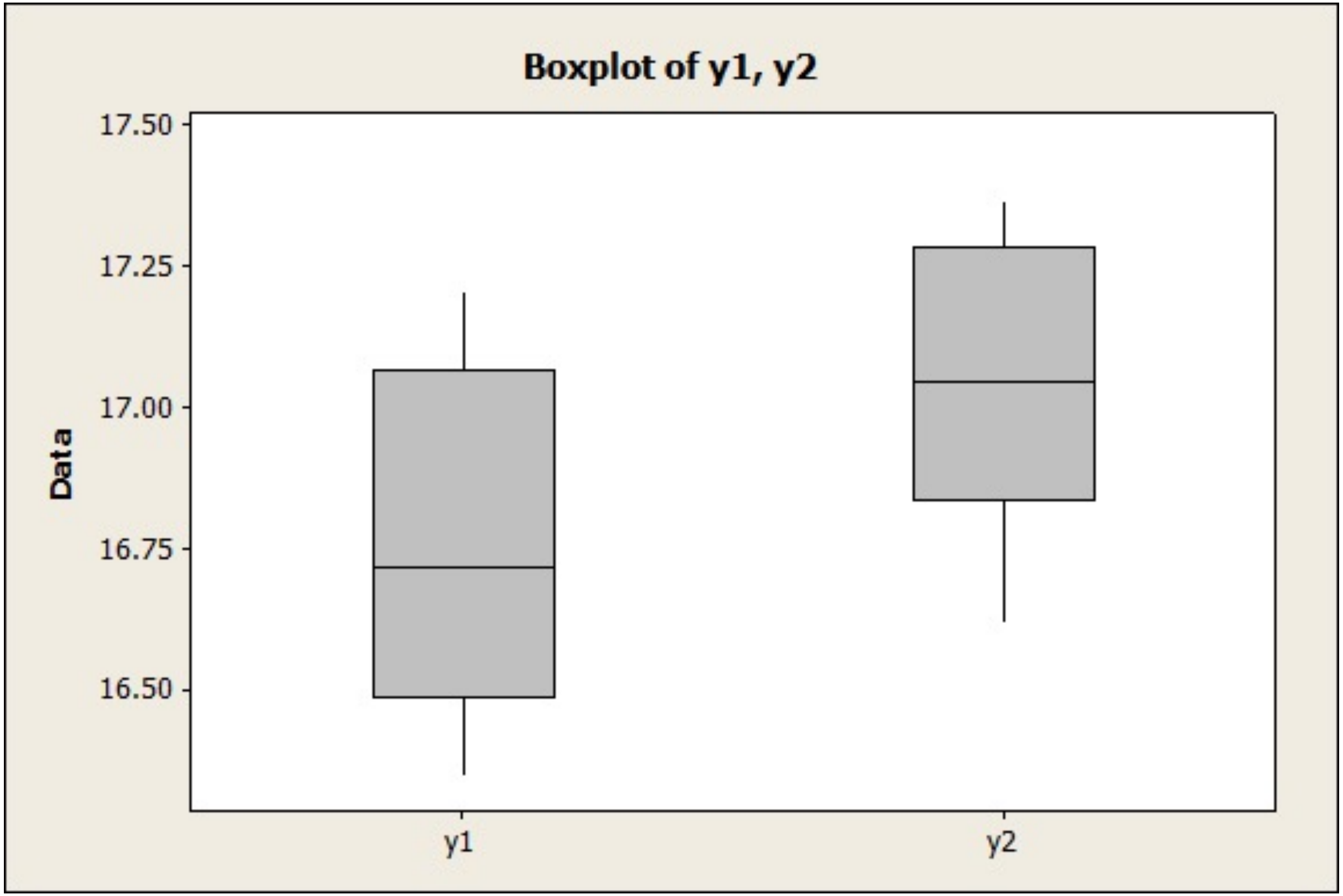
We will be covering the two-sample t-test in lecture today.
You are responsible for all hypothesis tests in Chapter 2.

See tables 2-4, 2-5, 2-8

New Example

	Recipe 1	Recipe 2
j	y_{1j}	y_{2j}
1	16.85	16.62
2	16.4	16.75
3	17.21	17.37
4	16.35	17.12
5	16.52	16.98
6	17.04	16.87
7	16.96	17.34
8	17.15	17.02
9	16.59	17.08
10	16.57	17.27


Dotplot of y1, y2
y1
y2
16.38
16.52
16.66
16.80
16.94
17.08
17.22
17.36
Data


Boxplot of y1, y2
17.50
17.25
17.00
16.75
16.50
Data
y1
y2

Summary Statistics

Formulation 1

"New recipe"

$$\bar{y}_1 = 16.76$$

$$S_1^2 = 0.100$$

$$S_1 = 0.316$$

$$n_1 = 10$$

Formulation 2

"Original recipe"

$$\bar{y}_2 = 17.04$$

$$S_2^2 = 0.061$$

$$S_2 = 0.248$$

$$n_2 = 10$$

How the Two-Sample *t*-Test Works:

Use the sample means to draw inferences about the population means

$\bar{y}_1 - \bar{y}_2 = 16.76 - 17.04 = -0.28$

$$\frac{\text{Difference in sample means}}{\text{Standard deviation of the difference in sample means}}$$

$$\sigma_{\bar{y}}^2 = \frac{\sigma^2}{n}, \quad \text{and } \sigma_{\bar{y}_1 - \bar{y}_2}^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}, \bar{y}_1 \text{ and } \bar{y}_2 \text{ independent}$$

This suggests a statistic:

$$Z_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

If the variances were known we could use the normal distribution as the basis of a test

Z_0 has a N(0,1) distribution if the two population means are equal

If we knew the two variances how would we use Z_0 to test H_0?

Suppose that $\sigma_1 = \sigma_2 = 0.30$. Then we can calculate

$$Z_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{-0.28}{\sqrt{\frac{0.3^2}{10} + \frac{0.3^2}{10}}} = \frac{-0.28}{0.1342} = -2.09$$

How "unusual" is the value Z_0 = -2.09 if the two population means are equal?

It turns out that 95% of the area under the standard normal curve (probability) falls between the values $Z_{0.025}$ = 1.96 and - $Z_{0.025}$ = -1.96.

So the value Z_0 = -2.09 is pretty unusual in that it would happen less that 5% of the time if the population means were equal

Standard Normal Table (see appendix)

I Cumulative Standard Normal Distribution[a]

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

z	0.00	0.01	0.02	0.03	0.04	z
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.0
0.1	0.53983	0.54379	0.54776	0.55172	0.55567	0.1
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.2
0.3	0.61791	0.62172	0.62551	0.62930	0.63307	0.3
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.4
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.5
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.6
0.7	0.75803	0.76115	0.76424	0.76730	0.77035	0.7
0.8	0.78814	0.79103	0.79389	0.79673	0.79954	0.8
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.9
1.0	0.84134	0.84375	0.84613	0.84849	0.85083	1.0
1.1	0.86433	0.86650	0.86864	0.87076	0.87285	1.1
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	1.2
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	1.3
1.4	0.91924	0.92073	0.92219	0.92364	0.92506	1.4
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	1.5
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	1.6
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	1.7
1.8	0.96407	0.96485	0.96562	0.96637	0.96711	1.8
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	1.9
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	2.0
2.1	0.98214	0.98257	0.98300	0.98341	0.93882	2.1
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	2.2
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	2.3
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	2.4
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	2.5
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	2.6
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	2.7
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	2.8
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	2.9
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	3.0
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	3.1
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	3.2
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	3.3
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	3.4
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	3.5
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	3.6
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	3.7
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	3.8
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	3.9

[a]Reproduced with permission from *Probability and Statistics in Engineering and Management Science*, 3rd edition, by W. W. Hines and D. C. Montgomery, Wiley, New York, 1990.

I Cumulative Standard Normal Distribution (*Continued*)

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

$Z_{0.025} = 1.96$

z	0.05	0.06	0.07	0.08	0.09	z
0.0	0.51994	0.52392	0.52790	0.53188	0.53586	0.0
0.1	0.55962	0.56356	0.56749	0.57142	0.57534	0.1
0.2	0.59871	0.60257	0.60642	0.61026	0.61409	0.2
0.3	0.63683	0.64058	0.64431	0.64803	0.65173	0.3
0.4	0.67364	0.67724	0.68082	0.68438	0.68793	0.4
0.5	0.70884	0.71226	0.71566	0.71904	0.72240	0.5
0.6	0.74215	0.74537	0.74857	0.75175	0.75490	0.6
0.7	0.77337	0.77637	0.77935	0.78230	0.78523	0.7
0.8	0.80234	0.80510	0.80785	0.81057	0.81327	0.8
0.9	0.82894	0.83147	0.83397	0.83646	0.83891	0.9
1.0	0.85314	0.85543	0.85769	0.85993	0.86214	1.0
1.1	0.87493	0.87697	0.87900	0.88100	0.88297	1.1
1.2	0.89435	0.89616	0.89796	0.89973	0.90147	1.2
1.3	0.91149	0.91308	0.91465	0.91621	0.91773	1.3
1.4	0.92647	0.92785	0.92922	0.93056	0.93189	1.4
1.5	0.93943	0.90462	0.94179	0.94295	0.94408	1.5
1.6	0.95053	0.95154	0.95254	0.95352	0.95448	1.6
1.7	0.95994	0.96080	0.96164	0.96246	0.96327	1.7
1.8	0.96784	0.96856	0.96926	0.96995	0.97062	1.8
1.9	0.97441	0.97500	0.97558	0.97615	0.97670	1.9
2.0	0.97982	0.98030	0.98077	0.98124	0.98169	2.0
2.1	0.98422	0.98461	0.98500	0.98537	0.98574	2.1
2.2	0.98778	0.98809	0.98840	0.98870	0.98899	2.2
2.3	0.99061	0.99086	0.99111	0.99134	0.99158	2.3
2.4	0.99286	0.99305	0.99324	0.99343	0.99361	2.4
2.5	0.99461	0.99477	0.99492	0.99506	0.99520	2.5
2.6	0.99598	0.99609	0.99621	0.99632	0.99643	2.6
2.7	0.99702	0.99711	0.99720	0.99728	0.99736	2.7
2.8	0.99781	0.99788	0.99795	0.99801	0.99807	2.8
2.9	0.99841	0.99846	0.99851	0.99856	0.99861	2.9
3.0	0.99886	0.99889	0.99893	0.99897	0.99900	3.0
3.1	0.99918	0.99921	0.99924	0.99926	0.99929	3.1
3.2	0.99942	0.99944	0.99946	0.99948	0.99950	3.2
3.3	0.99960	0.99961	0.99962	0.99964	0.99965	3.3
3.4	0.99972	0.99973	0.99974	0.99975	0.99976	3.4
3.5	0.99981	0.99981	0.99982	0.99983	0.99983	3.5
3.6	0.99987	0.99987	0.99988	0.99988	0.99989	3.6
3.7	0.99991	0.99992	0.99992	0.99992	0.99992	3.7
3.8	0.99994	0.99994	0.99995	0.99995	0.99995	3.8
3.9	0.99996	0.99996	0.99996	0.99997	0.99997	3.9

So if the variances were known we would conclude that we should reject the null hypothesis at the 5% level of significance

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

and conclude that the alternative hypothesis is true.

This is called a fixed significance level test, because we compare the value of the test statistic to a critical value (1.96) that we selected in advance before running the experiment.

The standard normal distribution is the reference distribution for the test.

Another way to do this that is very popular is to use the *P*-value approach. The *P*-value can be thought of as the observed significance level.

For the Z-test it is easy to find the *P*-value.

Normal Table

I Cumulative Standard Normal Distribution (*Continued*)

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

z	0.05	0.06	0.07	0.08	0.09	z
0.0	0.51994	0.52392	0.52790	0.53188	0.53586	0.0
0.1	0.55962	0.56356	0.56749	0.57142	0.57534	0.1
0.2	0.59871	0.60257	0.60642	0.61026	0.61409	0.2
0.3	0.63683	0.64058	0.64431	0.64803	0.65173	0.3
0.4	0.67364	0.67724	0.68082	0.68438	0.68793	0.4
0.5	0.70884	0.71226	0.71566	0.71904	0.72240	0.5
0.6	0.74215	0.74537	0.74857	0.75175	0.75490	0.6
0.7	0.77337	0.77637	0.77935	0.78230	0.78523	0.7
0.8	0.80234	0.80510	0.80785	0.81057	0.81327	0.8
0.9	0.82894	0.83147	0.83397	0.83646	0.83891	0.9
1.0	0.85314	0.85543	0.85769	0.85993	0.86214	1.0
1.1	0.87493	0.87697	0.87900	0.88100	0.88297	1.1
1.2	0.89435	0.89616	0.89796	0.89973	0.90147	1.2
1.3	0.91149	0.91308	0.91465	0.91621	0.91773	1.3
1.4	0.92647	0.92785	0.92922	0.93056	0.93189	1.4
1.5	0.93943	0.90462	0.94179	0.94295	0.94408	1.5
1.6	0.95053	0.95154	0.95254	0.95352	0.95448	1.6
1.7	0.95994	0.96080	0.96164	0.96246	0.96327	1.7
1.8	0.96784	0.96856	0.96926	0.96995	0.97062	1.8
1.9	0.97441	0.97500	0.97558	0.97615	0.97670	1.9
2.0	0.97982	0.98030	0.98077	0.98124	0.98169	2.0
2.1	0.98422	0.98461	0.98500	0.98537	0.98574	2.1
2.2	0.98778	0.98809	0.98840	0.98870	0.98899	2.2
2.3	0.99061	0.99086	0.99111	0.99134	0.99158	2.3
2.4	0.99286	0.99305	0.99324	0.99343	0.99361	2.4
2.5	0.99461	0.99477	0.99492	0.99506	0.99520	2.5
2.6	0.99598	0.99609	0.99621	0.99632	0.99643	2.6
2.7	0.99702	0.99711	0.99720	0.99728	0.99736	2.7
2.8	0.99781	0.99788	0.99795	0.99801	0.99807	2.8
2.9	0.99841	0.99846	0.99851	0.99856	0.99861	2.9
3.0	0.99886	0.99889	0.99893	0.99897	0.99900	3.0
3.1	0.99918	0.99921	0.99924	0.99926	0.99929	3.1
3.2	0.99942	0.99944	0.99946	0.99948	0.99950	3.2
3.3	0.99960	0.99961	0.99962	0.99964	0.99965	3.3
3.4	0.99972	0.99973	0.99974	0.99975	0.99976	3.4
3.5	0.99981	0.99981	0.99982	0.99983	0.99983	3.5
3.6	0.99987	0.99987	0.99988	0.99988	0.99989	3.6
3.7	0.99991	0.99992	0.99992	0.99992	0.99992	3.7
3.8	0.99994	0.99994	0.99995	0.99995	0.99995	3.8
3.9	0.99996	0.99996	0.99996	0.99997	0.99997	3.9

Find the probability above Z_0 = -2.09 from the table.

This is 1 – 0.98169 = 0.01832

The *P*-value is twice this probability, or 0.03662.

So we would reject the null hypothesis at any level of significance that is less than or equal to 0.03662.

Typically 0.05 is used as the cutoff.

The *t* -Test

- The Z-test just described would work perfectly if we knew the two population variances.
- Since we usually don't know the true population variances, what would happen if we just plugged in the sample variances?
- The answer is that if the sample sizes were large enough (say both $n > 30$ or 40) the Z-test would work just fine. It is a good **large-sample test** for the difference in means.
- But many times that isn't possible (as Gosset wrote in 1908, "…but what if the sample size is small…?).
- It turns out that if the sample size is small we can no longer use the N(0,1) distribution as the reference distribution for the test.

How the Two-Sample *t*-Test Works:

Use S_1^2 and S_2^2 to estimate σ_1^2 and σ_2^2

The previous ratio becomes $\dfrac{\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$

However, we have the case where $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Pool the individual sample variances:

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

How the Two-Sample *t*-Test Works:

The test statistic is

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

- Values of t_0 that are near zero are consistent with the null hypothesis
- Values of t_0 that are very different from zero are consistent with the alternative hypothesis
- t_0 is a "distance" measure-how far apart the averages are expressed in standard deviation units
- Notice the interpretation of t_0 as a **signal-to-noise** ratio

The Two-Sample (Pooled) *t*-Test

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{9(0.100) + 9(0.061)}{10 + 10 - 2} = 0.081$$

$$S_p = 0.284$$

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{16.76 - 17.04}{0.284\sqrt{\frac{1}{10} + \frac{1}{10}}} = -2.20$$

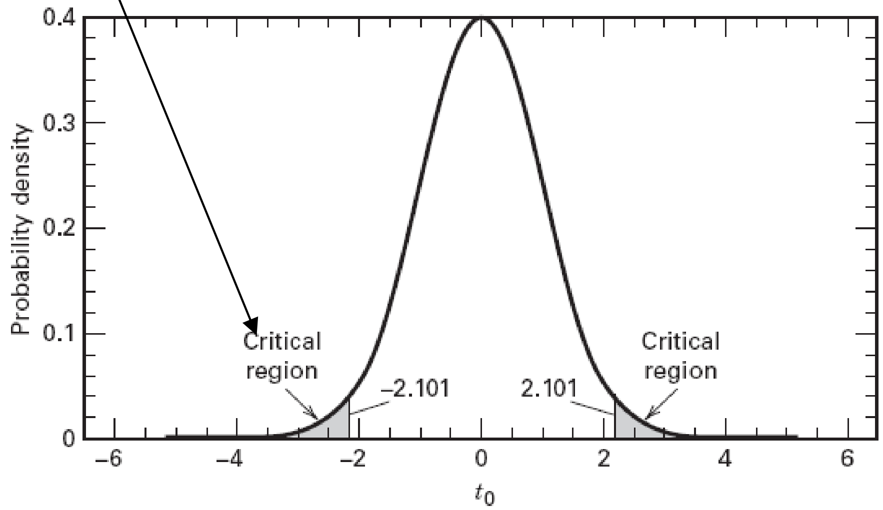
The two sample means are a little over two standard deviations apart

Is this a "large" difference?

The Two-Sample (Pooled) *t*-Test

- We need an **objective** basis for deciding how large the test statistic t_0 really is
- In 1908, W. S. Gosset derived the **reference distribution** for t_0 … called the *t* distribution
- Tables of the *t* distribution – see textbook appendix page 614

$t_0 = -2.20$



■ **FIGURE 2.10** The t distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$

II Percentage Points of the t Distribution[a]

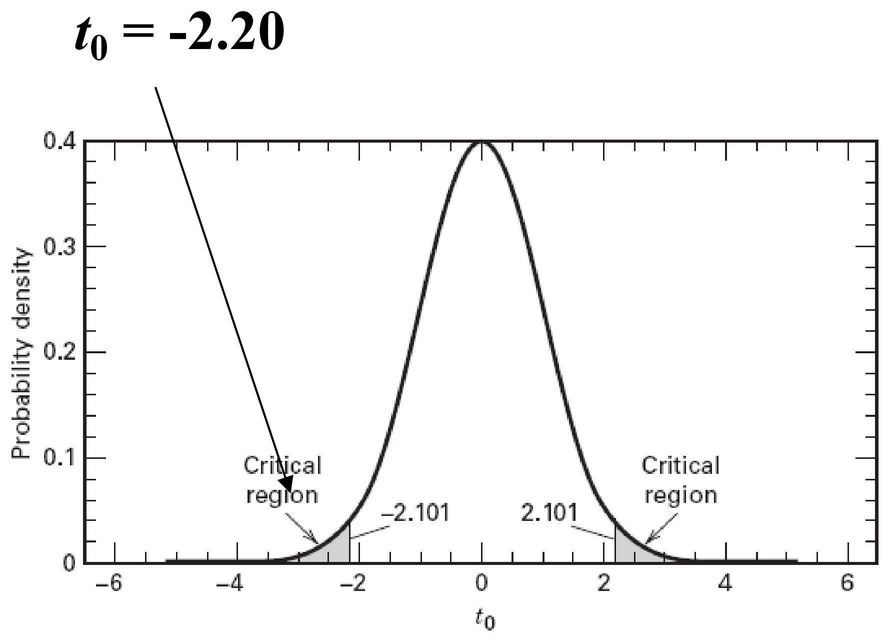
ν \ α	**0.40**	**0.25**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.0025**	**0.001**	**0.0005**
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326	31.598
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.265	0.727	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.499	4.019	4.785	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.260	0.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.258	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.258	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.257	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.257	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.257	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.257	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.257	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.256	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.256	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	0.256	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.256	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.256	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.256	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.256	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.256	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.256	0.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.255	0.681	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
60	0.254	0.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	0.254	0.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.253	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

ν = Degrees of freedom.

[a]Adapted with permission from *Biometrika Tables for Statisticians*, Vol. 1, 3rd edition, by E. S. Pearson and H. O. Hartley, Cambridge University Press, Cambridge, 1966.

The Two-Sample (Pooled) *t*-Test

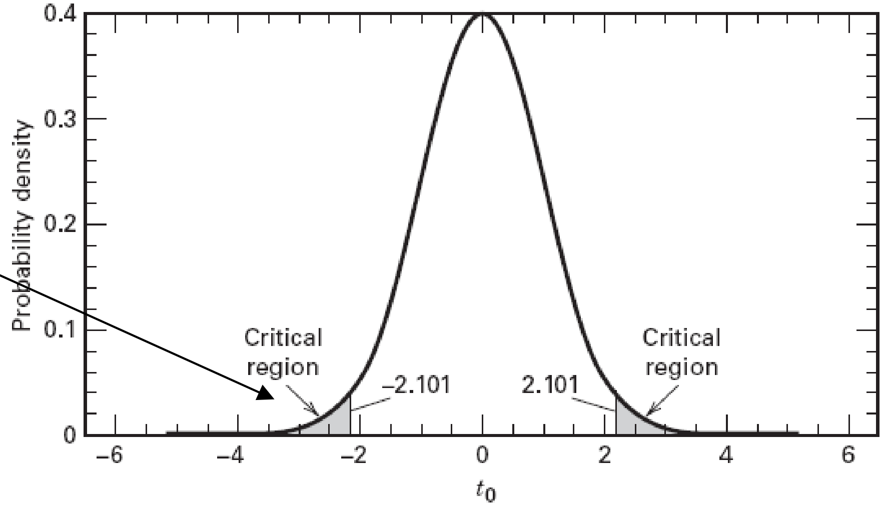
- A value of t_0 between –2.101 and 2.101 is consistent with equality of means
- It is possible for the means to be equal and t_0 to exceed either 2.101 or –2.101, but it would be a "**rare event**" … leads to the conclusion that the means are different
- Could also use the ***P*-value** approach



■ FIGURE 2.10 The t distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$

The Two-Sample (Pooled) *t*-Test

$t_0 = -2.20$



■ **FIGURE 2.10** The t distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$

- The ***P*-value** is the area (probability) in the tails of the *t*-distribution beyond -2.20 + the probability beyond +2.20 (it's a two-sided test)
- The *P*-value is a measure of how unusual the value of the test statistic is given that the null hypothesis is true
- The *P*-value the risk of **wrongly rejecting** the null hypothesis of equal means (it measures rareness of the event)
- The exact *P*-value in our problem is $P = 0.042$ (found from a computer)

Approximating the *P*-value

Our *t*-table only gives probabilities greater than positive values of *t*. So take the absolute value of t_0 = -2.20 or $|t_0|$= 2.20.

Now with 18 degrees of freedom, find the values of *t* in the table that bracket this value.

These are 2.101 < $|t_0|$= 2.20 < 2.552. The right-tail probability for *t* = 2.101 is 0.025 and for *t* = 2.552 is 0.01. Double these probabilities because this is a two-sided test.

Therefore the *P*-value must lie between these two probabilities, or

$$0.05 < P\text{-value} < 0.02$$

These are upper and lower bounds on the P-value.

We know that the actual *P*-value is 0.042.

Computer Two-Sample *t*-Test Results

■ TABLE 2.2
Computer Output for the Two-Sample *t*-Test

Minitab

Two-sample T for Modified vs Unmodified

	N	Mean	Std. Dev.	SE Mean
Modified	10	16.764	0.316	0.10
Unmodified	10	17.042	0.248	0.078

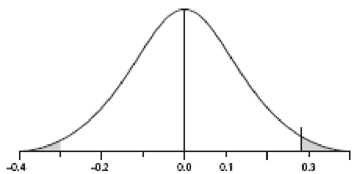
Difference = mu (Modified) − mu (Unmodified)
Estimate for difference: −0.278000
95% CI for difference: (−0.545073, −0.010927)
T-Test of difference = 0 (vs not =): T-Value = −2.19
P-Value = 0.042 DF = 18
Both use Pooled Std. Dev. = 0.2843

JMP t-test

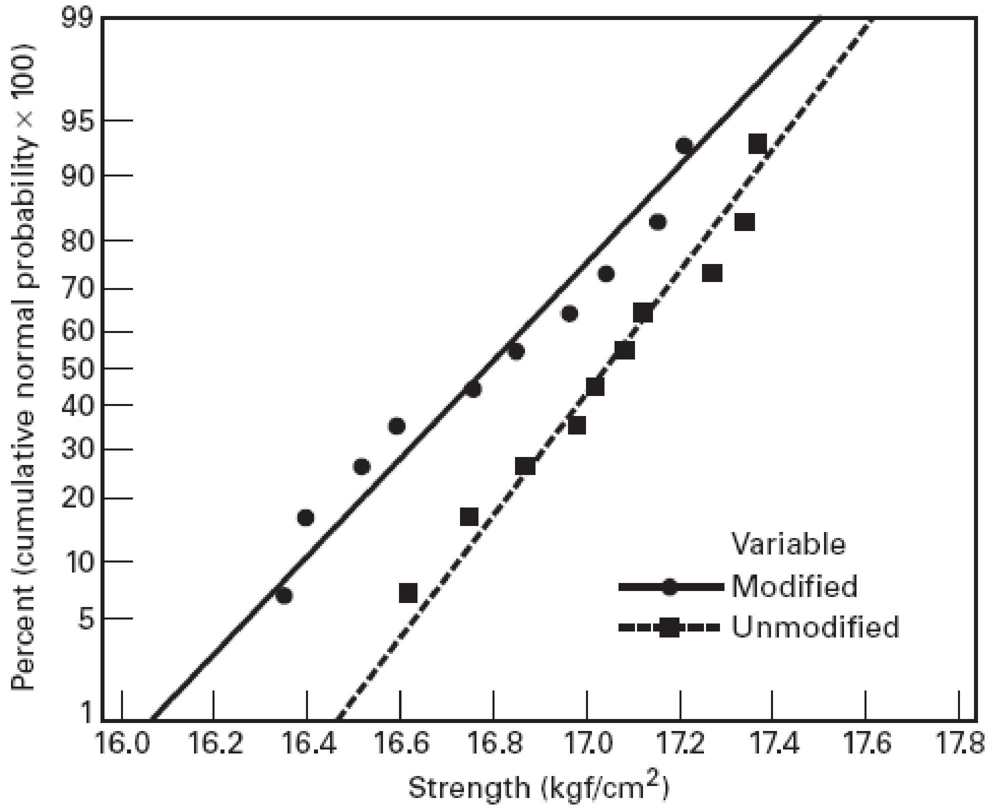
Unmodified-Modified

Assuming equal variances

Difference	0.278000	t Ratio	2.186876
Std Err Dif	0.127122	DF	18
Upper CL Dif	0.545073	Prob > |t|	0.0422
Lower CL Dif	0.010927	Prob > t	0.0211
Confidence	0.95	Prob < t	0.9789



Checking Assumptions – The Normal Probability Plot



■ **FIGURE 2.11** Normal probability plots of tension bond strength in the Portland cement experiment

Importance of the *t*-Test

- Provides an **objective** framework for simple comparative experiments

- Could be used to test all relevant hypotheses in a two-level factorial design, because all of these hypotheses involve the mean response at one "side" of the cube versus the mean response at the opposite "side" of the cube

Confidence Intervals (See pg. 43)

- Hypothesis testing gives an objective statement concerning the difference in means, but it doesn't specify "how different" they are
- **General form** of a confidence interval

$$L \le \theta \le U \text{ where } P(L \le \theta \le U) = 1 - \alpha$$

- The 100(1- α)% **confidence interval** on the difference in two means:

$$\bar{y}_1 - \bar{y}_2 - t_{\alpha/2, n_1+n_2-2} S_p \sqrt{(1/n_1) + (1/n_2)} \le \mu_1 - \mu_2 \le$$

$$\bar{y}_1 - \bar{y}_2 + t_{\alpha/2, n_1+n_2-2} S_p \sqrt{(1/n_1) + (1/n_2)}$$

The actual 95 percent confidence interval estimate for the difference in mean tension bond strength for the formulations of Portland cement mortar is found by substituting in Equation 2.30 as follows:

$$16.76 - 17.04 - (2.101)0.284\sqrt{\tfrac{1}{10} + \tfrac{1}{10}} \leq \mu_1 - \mu_2$$

$$\leq 16.76 - 17.04 + (2.101)0.284\sqrt{\tfrac{1}{10} + \tfrac{1}{10}}$$

$$-0.28 - 0.27 \leq \mu_1 - \mu_2 \leq -0.28 + 0.27$$

$$-0.55 \leq \mu_1 - \mu_2 \leq -0.01$$

Thus, the 95 percent confidence interval estimate on the difference in means extends from -0.55 to -0.01 kgf/cm^2. Put another way, the confidence interval is $\mu_1 - \mu_2 = -0.28 \pm 0.27$ kgf/cm^2, or the difference in mean strengths is -0.28 kgf/cm^2, and the accuracy of this estimate is ± 0.27 kgf/cm^2. Note that because $\mu_1 - \mu_2 = 0$ is *not* included in this interval, the data do not support the hypothesis that $\mu_1 = \mu_2$ at the 5 percent level of significance (recall that the P-value for the two-sample t-test was 0.042, just slightly less than 0.05). It is likely that the mean strength of the unmodified formulation exceeds the mean strength of the modified formulation. Notice from Table 2.2 that both Minitab and JMP reported this confidence interval when the hypothesis testing procedure was conducted.

What if the Two Variances are Different?

EXAMPLE 2.1

Nerve preservation is important in surgery because accidental injury to the nerve can lead to post-surgical problems such as numbness, pain, or paralysis. Nerves are usually identified by their appearance and relationship to nearby structures or detected by local electrical stimulation (electromyography), but it is relatively easy to overlook them. An article in *Nature Biotechnology* ("Fluorescent Peptides Highlight Peripheral Nerves During Surgery in Mice," Vol. 29, 2011) describes the use of a fluorescently labeled peptide that binds to nerves to assist in identification. Table 2.3 shows the normalized fluorescence after two hours for nerve and muscle tissue for 12 mice (the data were read from a graph in the paper).

We would like to test the hypothesis that the mean normalized fluorescence after two hours is greater for nerve tissue then for muscle tissue. That is, if μ_8 is the mean normalized fluorescence for nerve tissue and is the mean normalized fluorescence for muscle tissue, we want to test

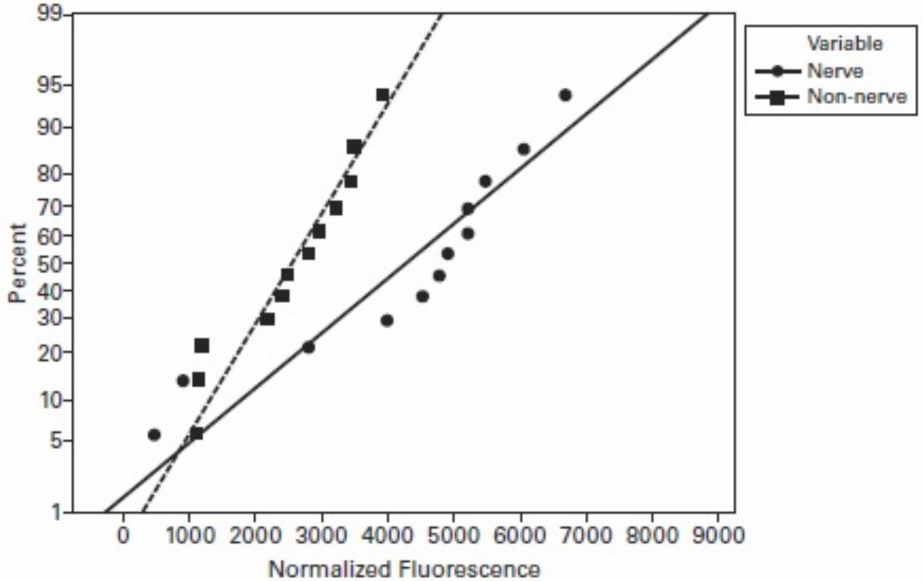
$$H_0{:}\mu_1 = \mu_2$$
$$H_1{:}\mu_1 > \mu_2$$

TABLE 2.3
Normalized Fluorescence After Two Hours

Observation	Nerve	Muscle
1	6625	3900
2	6000	3500
3	5450	3450
4	5200	3200
5	5175	2980
6	4900	2800
7	4750	2500
8	4500	2400
9	3985	2200
10	900	1200
11	450	1150
12	2800	1130

The descriptive statistics output from Minitab is shown below:

```
Variable      N   Mean   StDev   Minimum   Median   Maximum
Nerve        12   4228    1918       450     4825      6625
Non-nerve    12   2534     961      1130     2650      3900
```



■ **FIGURE 2.14** **Normalized Fluorescence Data from Table 2.3**

If we are testing

$$H_0: \mu_1 = \mu_2$$
$$H_1: \mu_1 \neq \mu_2$$

and cannot reasonably assume that the variances σ_1^2 and σ_2^2 are equal, then the two-sample t-test must be modified slightly. The test statistic becomes

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} \tag{2.31}$$

This statistic is not distributed exactly as t. However, the distribution of t_0 is well approximated by t if we use

$$v = \frac{\left(\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}\right)^2}{\dfrac{(S_1^2/n_1)^2}{n_1 - 1} + \dfrac{(S_2^2/n_2)^2}{n_2 - 1}} \tag{2.32}$$

as the number of degrees of freedom. A strong indication of unequal variances on a normal probability plot would be a situation calling for this version of the t-test. You should be able to develop an equation for finding that confidence interval on the difference in mean for the unequal variances case easily.

$$\boxed{t_0 = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} = \frac{4228 - 2534}{\sqrt{\dfrac{(1918)^2}{12} + \dfrac{(961)^2}{12}}} = 2.7354}$$

The number of degrees of freedom are calculated from Equation 2.32:

$$v = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{(S_1^2 / n_1)^2}{n_1 - 1} + \frac{(S_2^2 / n_2)^2}{n_2 - 1}} = \frac{\left(\frac{(1918)^2}{12} + \frac{(961)^2}{12}\right)^2}{\frac{[(1918)^2 / 12]^2}{11} + \frac{[(961)^2 / 12]^2}{11}} = 16.1955$$

If we are going to find a P-value from a table of the t-distribution, we should round the degrees of freedom down to 16. Most computer programs interpolate to determine the P-value. The Minitab output for the two-sample t-test is shown below. Since the P-value reported is small (0.015), we would reject the null hypothesis and conclude that the mean normalized fluorescence for nerve tissue is greater than the mean normalized fluorescence for muscle tissue.

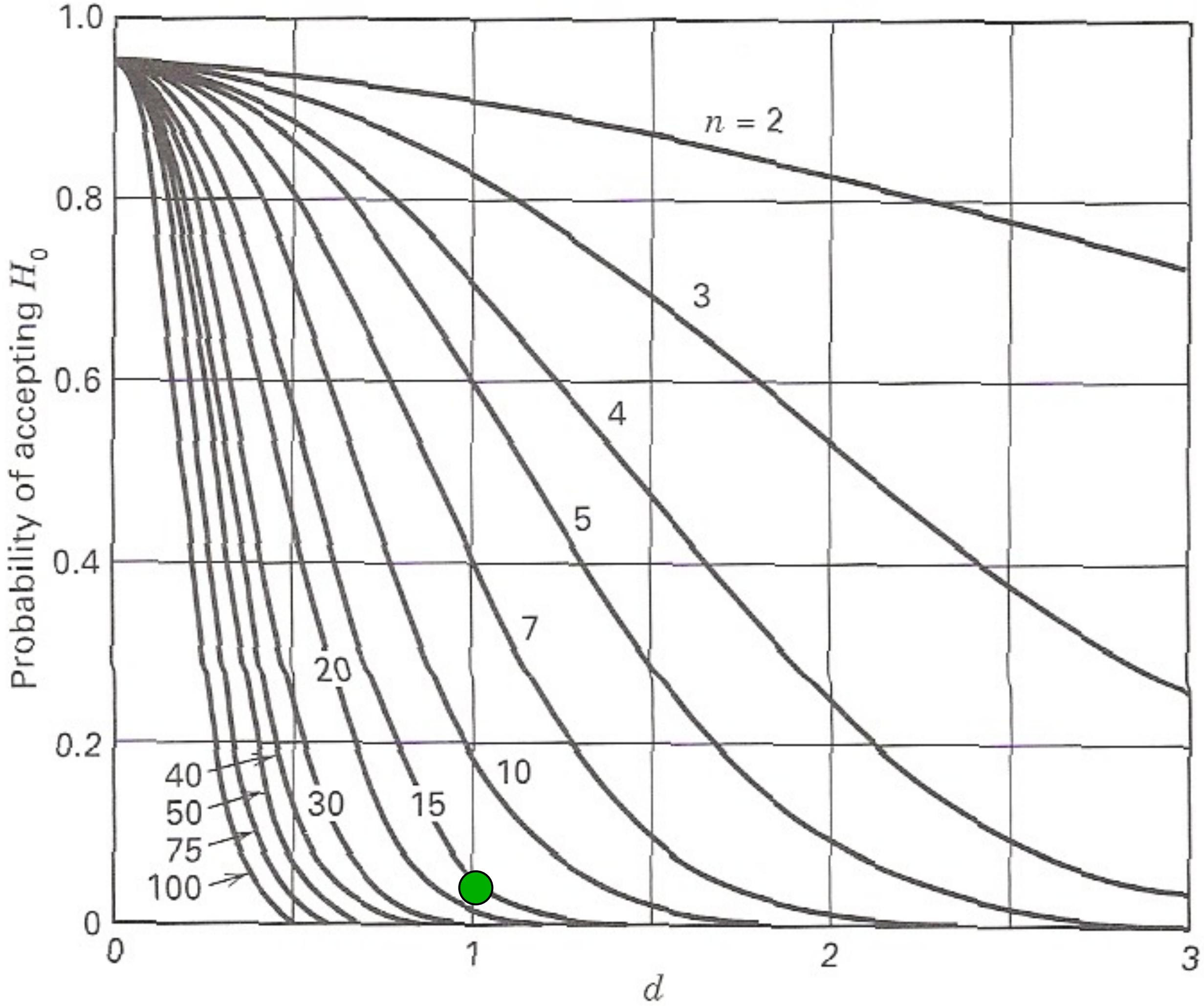
```
Difference = mu (Nerve) - mu (Non-nerve)
Estimate for difference:  1694
95% lower bound for difference:  613
T-Test of difference = 0 (vs >): T-Value = 2.74  P-Value = 0.007  DF = 16
```

Choosing Sample Size

- Can choose the sample size to detect a specific difference in means and achieve desired values of **type I and type II errors**
  - Type I error : reject H_0 when it is true (α)
  - Type II error : fail to reject H_0 when it is false (β)
  - **Power** = 1 – β : reject H_0 when it is false
- **Operating characteristic curves** plot β against a parameter d where

$$d = \frac{|\mu_1 - \mu_2|}{2\sigma}$$

OC Curve for two-sided *t*-test

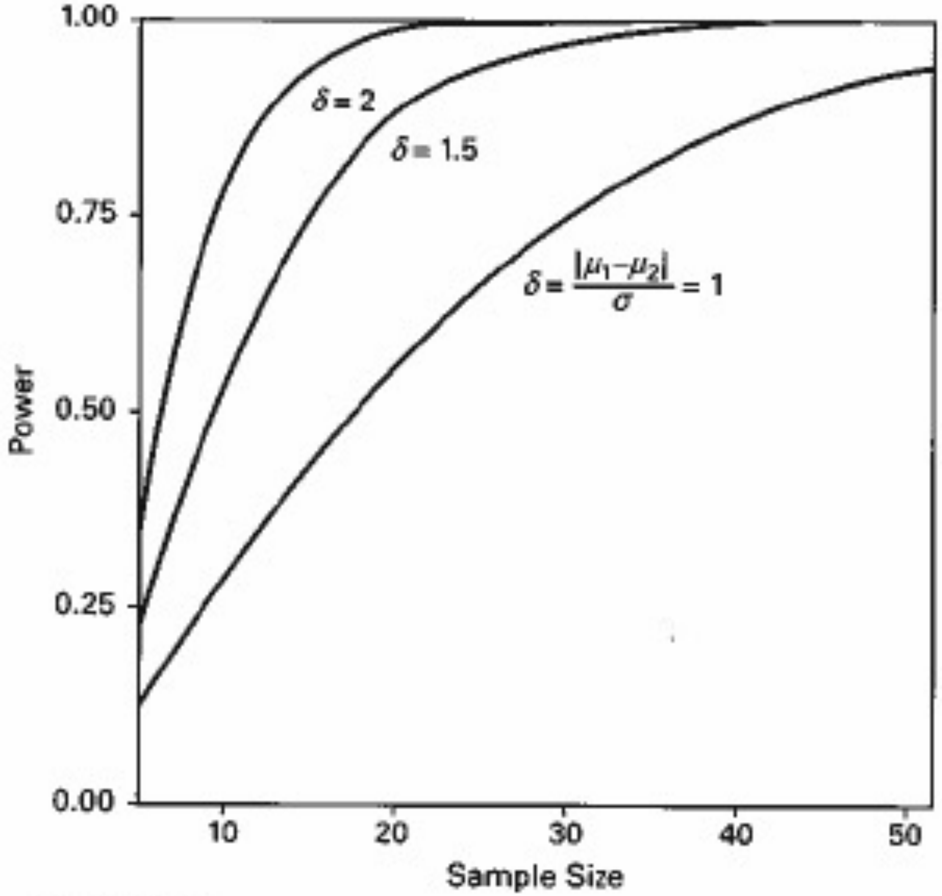


$$n^* = 2n - 1$$

Assume we would like to detect a critical difference of 0.5 and we estimate σ to be 0.25

$$d = \frac{0.5}{2 \times 0.25} = 1$$

Power Curves



■ **FIGURE 2.13** Power Curves (from JMP) for the Two-Sample t-Test Assuming Equal Varianes and $\alpha = 0.05$. The Sample Size on the Horizontal Axis is the Total sample Size, so the sample Size in Each population is n = sample size from graph/2.

Another Example

- The following are the burning times of chemical flares of two different formulations. The design engineers are interested in both the means and variances of the burning times.

Type 1	
65	82
81	67
57	59
66	75
82	70

Type II	
64	56
71	69
83	74
59	82
65	79

Another Example

- Test the hypothesis that the two variances are equal

$$H_0 : \sigma_1 = \sigma_2$$

$$H_1 : \sigma_1 \neq \sigma_2$$

- This requires a two sample F test (p. 57)

$$Fo = \frac{S_1^2}{S_2^2}$$

Reject Ho if $F_o > F_{\alpha/2, n_1 - 1, n_2 - 1}$ or $F_0 < F_{1-\alpha/2, n_1 - 1, n_2 - 1}$

$$\text{Re}\,m : F_{1-\alpha, \upsilon_1, \upsilon_2} = \frac{1}{F_{\alpha, \upsilon_2, \upsilon_1}}$$

Sample Problem 2-25

$$F_0 = \frac{S_1^2}{S_2^2} = \frac{9.264^2}{9.367^2} = 0.98$$

$$\mathrm{F}_{\alpha/2,\mathrm{n}_1-1,\mathrm{n}_2-1} = \mathrm{F}_{0.25,9,9} = 4.03$$

$$\mathrm{F}_{1-\alpha/2,\mathrm{n}_1-1,\mathrm{n}_2-1} = \mathrm{F}_{0.975,9,9} = \frac{1}{F_{0.025,9,9}} = \frac{1}{4.03} = 0.248$$

Reject Ho if $\mathrm{F}_\mathrm{o} > \mathrm{F}_{\alpha/2,\mathrm{n}_1-1,\mathrm{n}_2-1}$ or $\mathrm{F}_0 < \mathrm{F}_{1-\alpha/2,\mathrm{n}_1-1,\mathrm{n}_2-1}$

Fail to Reject Ho.. Insufficient Evidence to Prove Variances are not Equal

Sample Problem 2-25

- Using the results of a) test the hypothesis that the mean burning times are equal. Use $\alpha = 0.05$.

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2$$

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Reject if $|t_o| > t_{\alpha/2, n_1 + n_2 - 2}$

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Sample Problem 2-25

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{9(85.82) + 9(87.73)}{10 + 10 - 2} = 86.775$$

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{70.4 - 70.2}{9.32\sqrt{\frac{1}{10} + \frac{1}{10}}} = 0.048$$

$$\text{Reject if } |t_o| > 2.101$$

Fail to Reject H_0.. Insufficient Evidence to Prove Means are not Equal

Sample Problem 2-25

- What is the role of the normality assumption in this problem. Check assumption of normality for both.