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INTRODUCTION TO COMPUTER VISION

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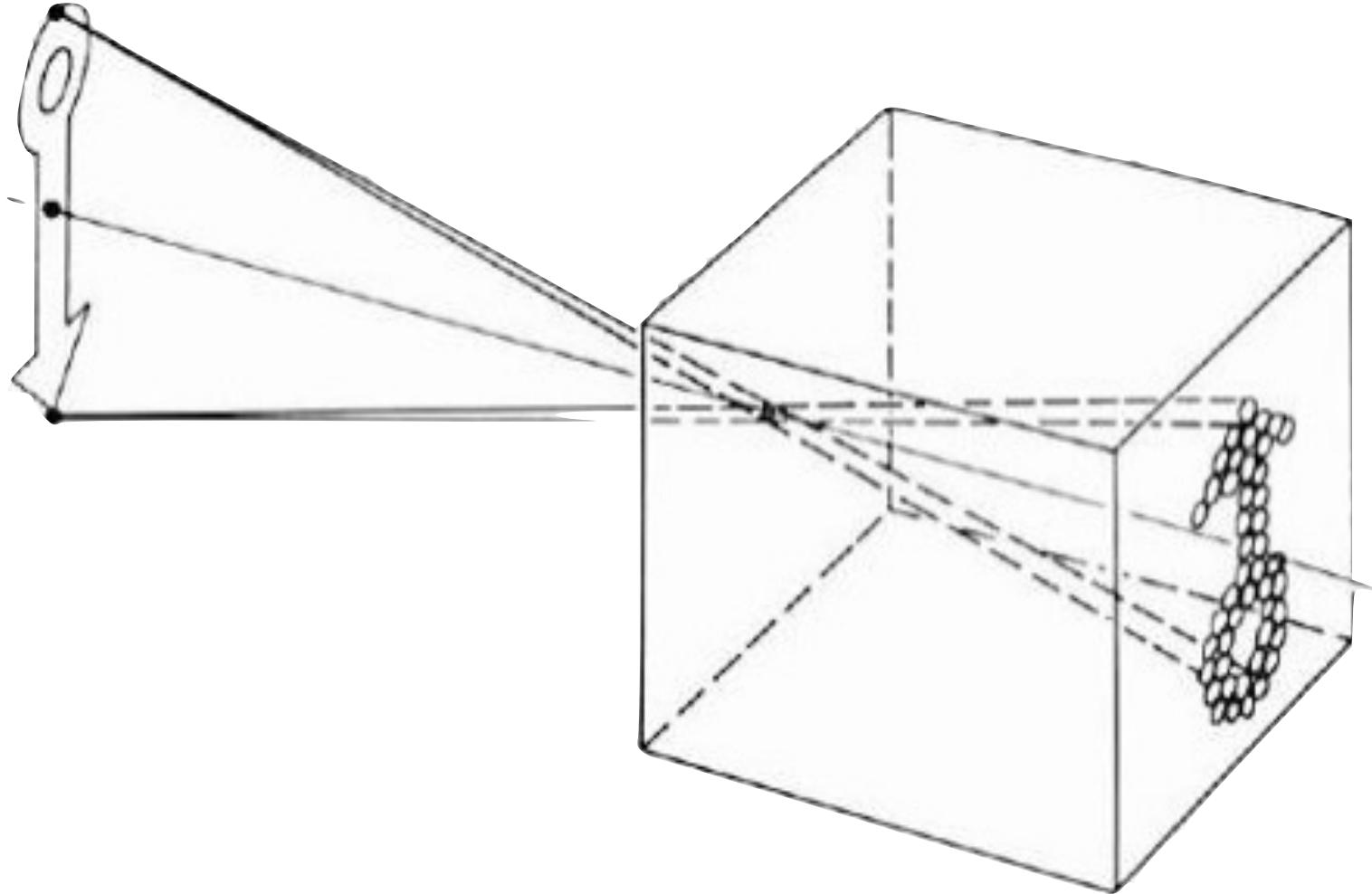
Many slides here were adapted from CMU 16-385

Camera Model

Pinhole and Lens



Pinhole camera a.k.a. camera obscura



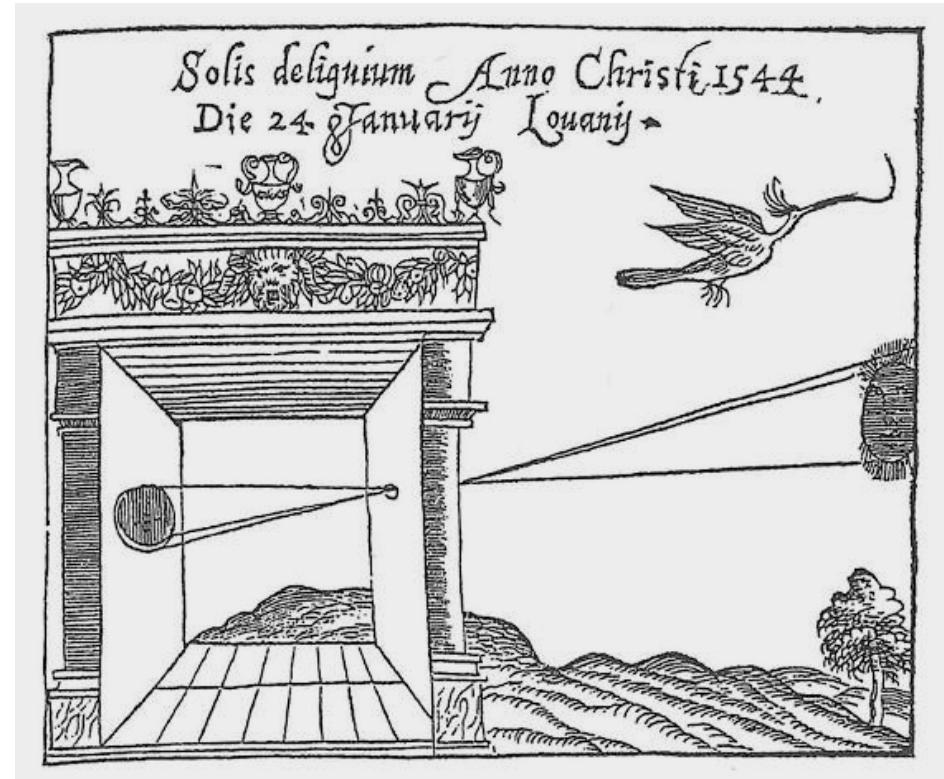
Pinhole camera a.k.a. camera obscura

First mention ...



Chinese philosopher Mozi
(470 to 390 BC)

First camera ...



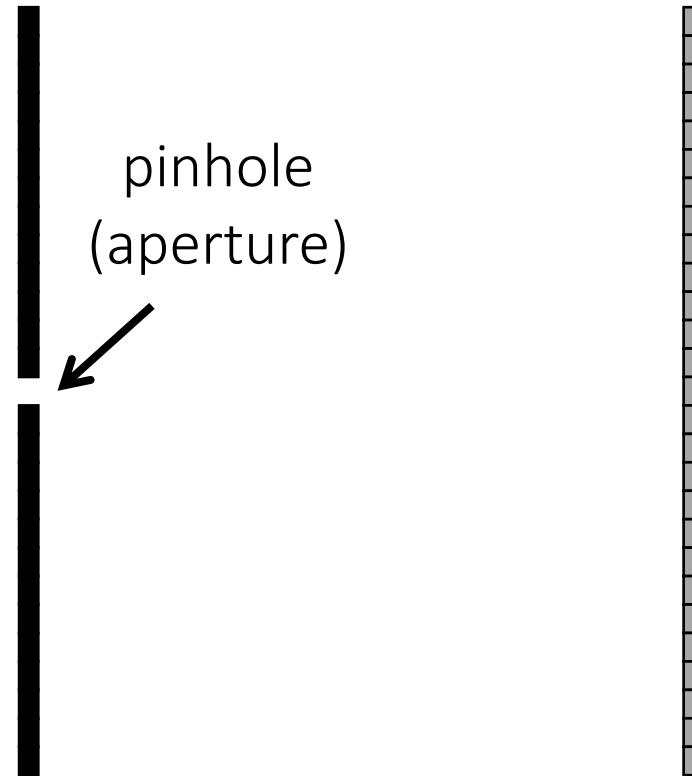
Greek philosopher Aristotle
(384 to 322 BC)

Pinhole camera terms

real-world
object



barrier (diaphragm)



digital sensor
(CCD or CMOS)

Pinhole camera terms

real-world
object



barrier (diaphragm)

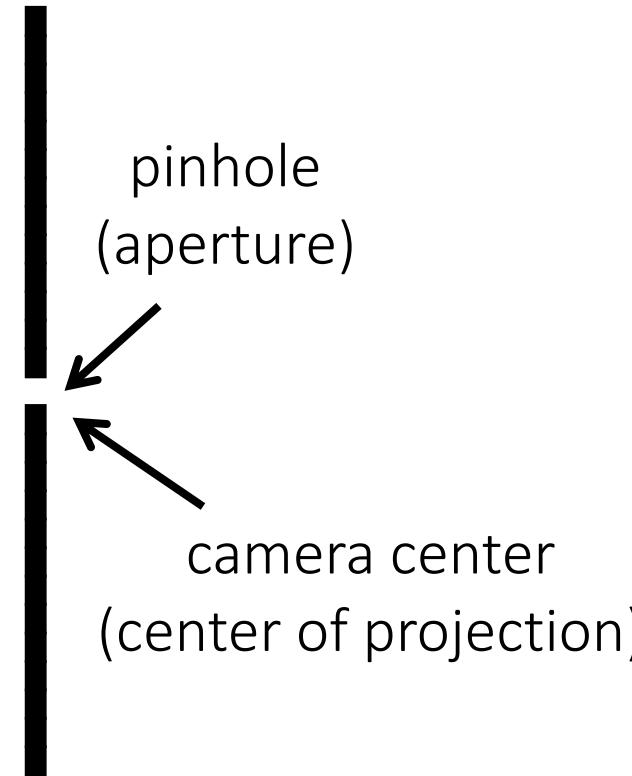
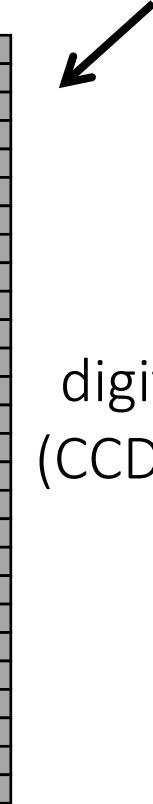


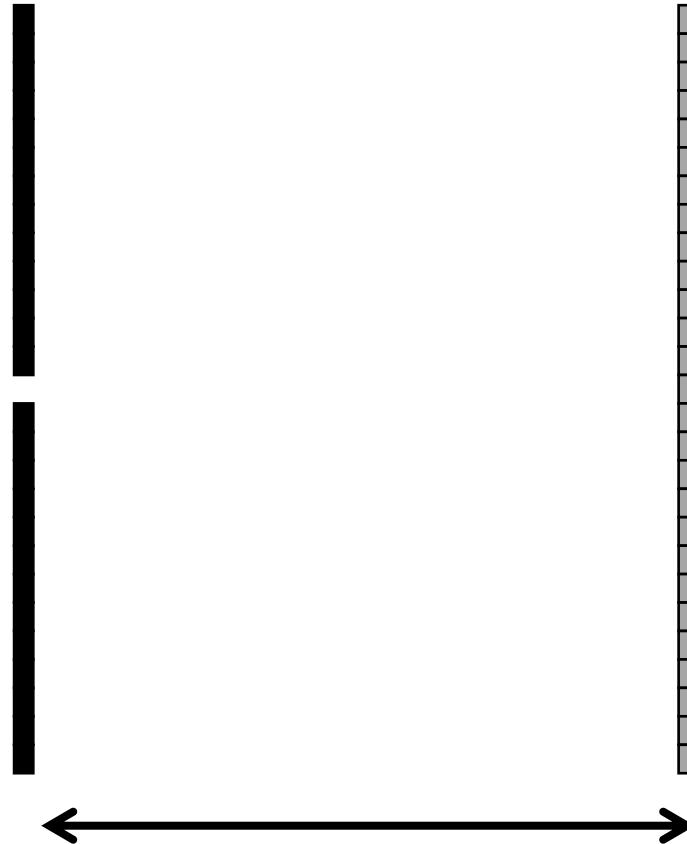
image plane

digital sensor
(CCD or CMOS)



Focal length

real-world
object

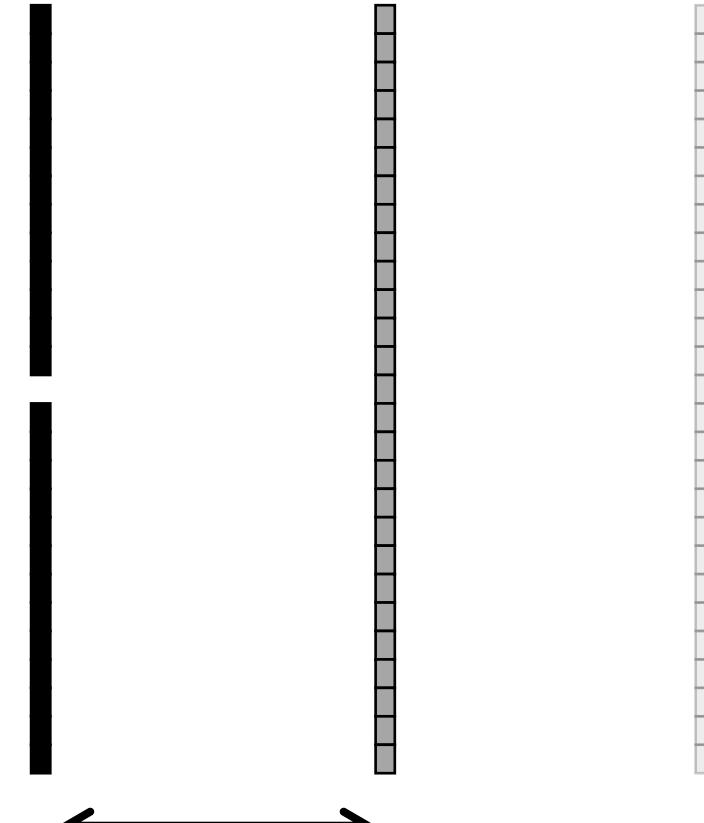


focal length f

Focal length

What happens as we change the focal length?

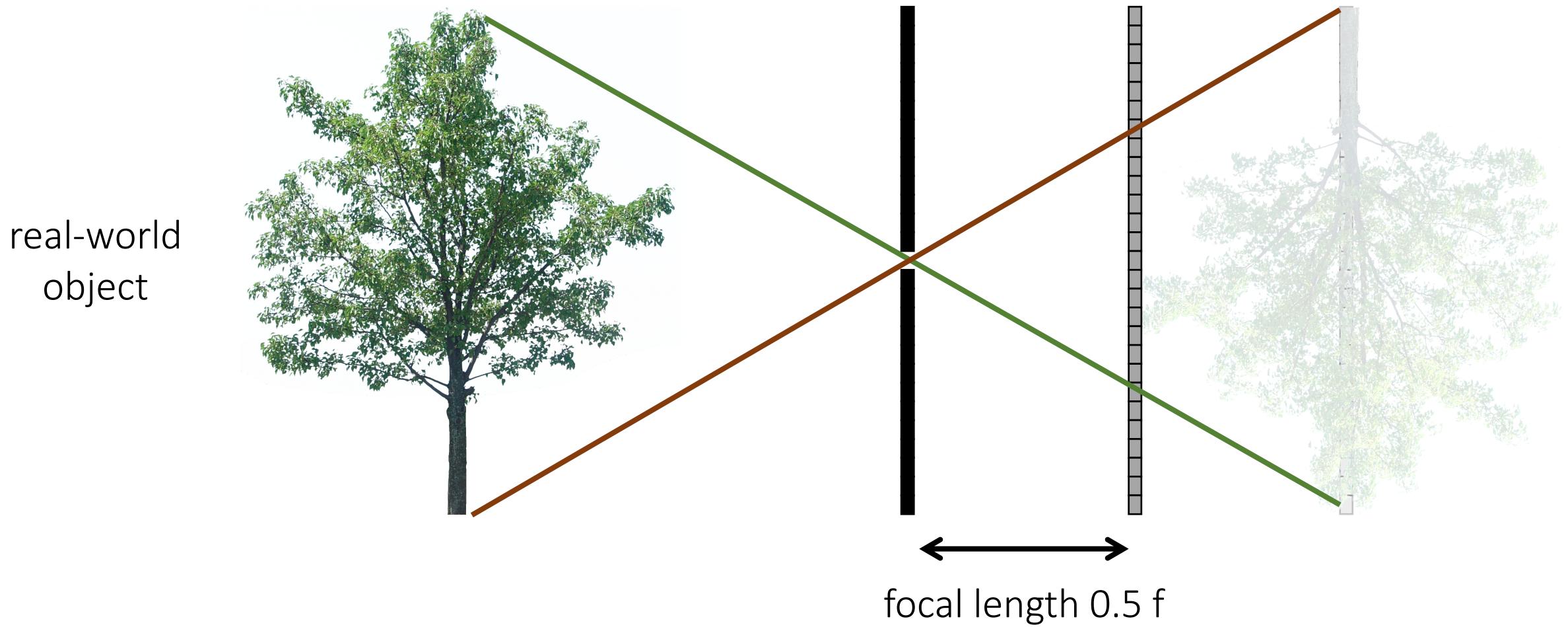
real-world
object



focal length $0.5 f$

Focal length

What happens as we change the focal length?

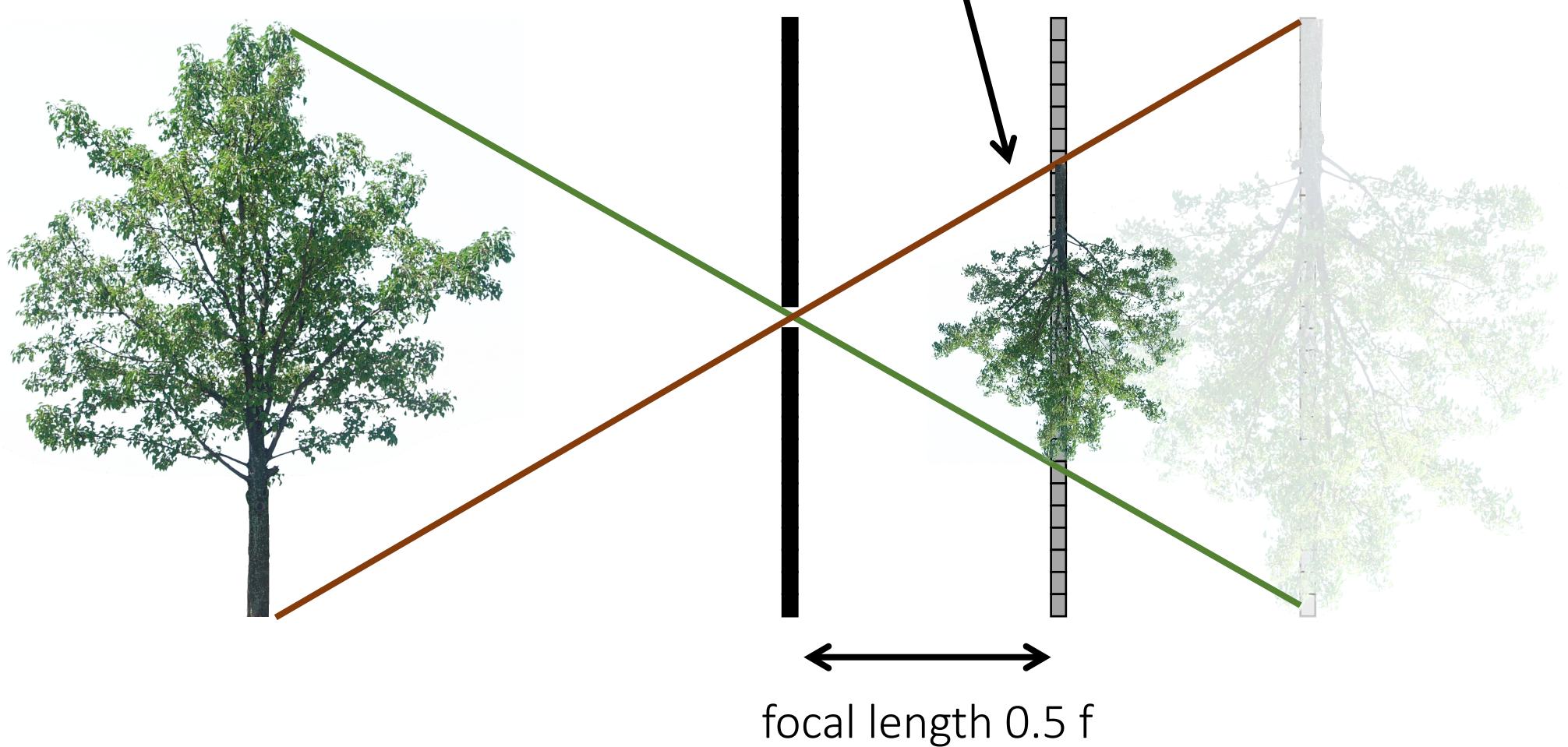


Focal length

What happens as we change the focal length?

object projection is half the size

real-world
object

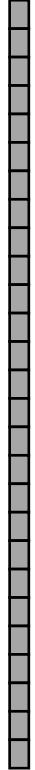


Pinhole size

real-world
object



pinhole
diameter



Ideal pinhole has infinitesimally small size

- In practice that is impossible.

Pinhole size

What happens as we change the pinhole diameter?

real-world
object



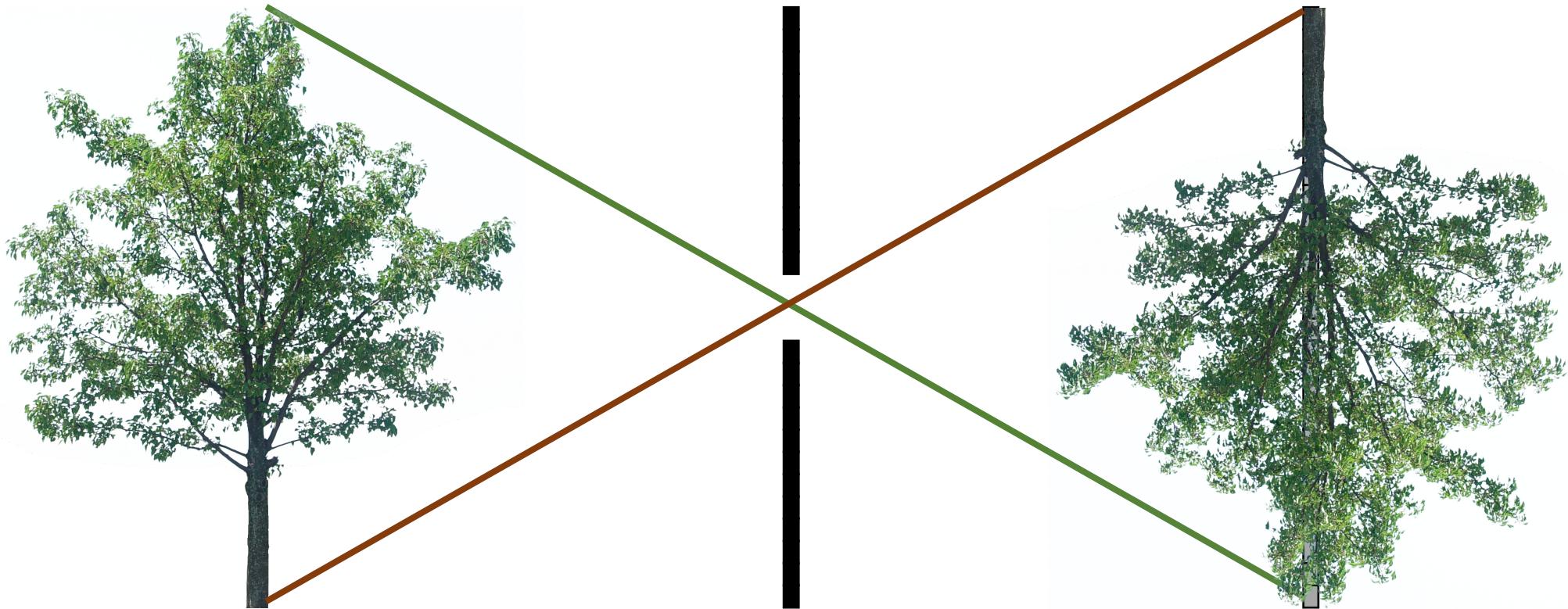
pinhole
diameter



Pinhole size

What happens as we change the pinhole diameter?

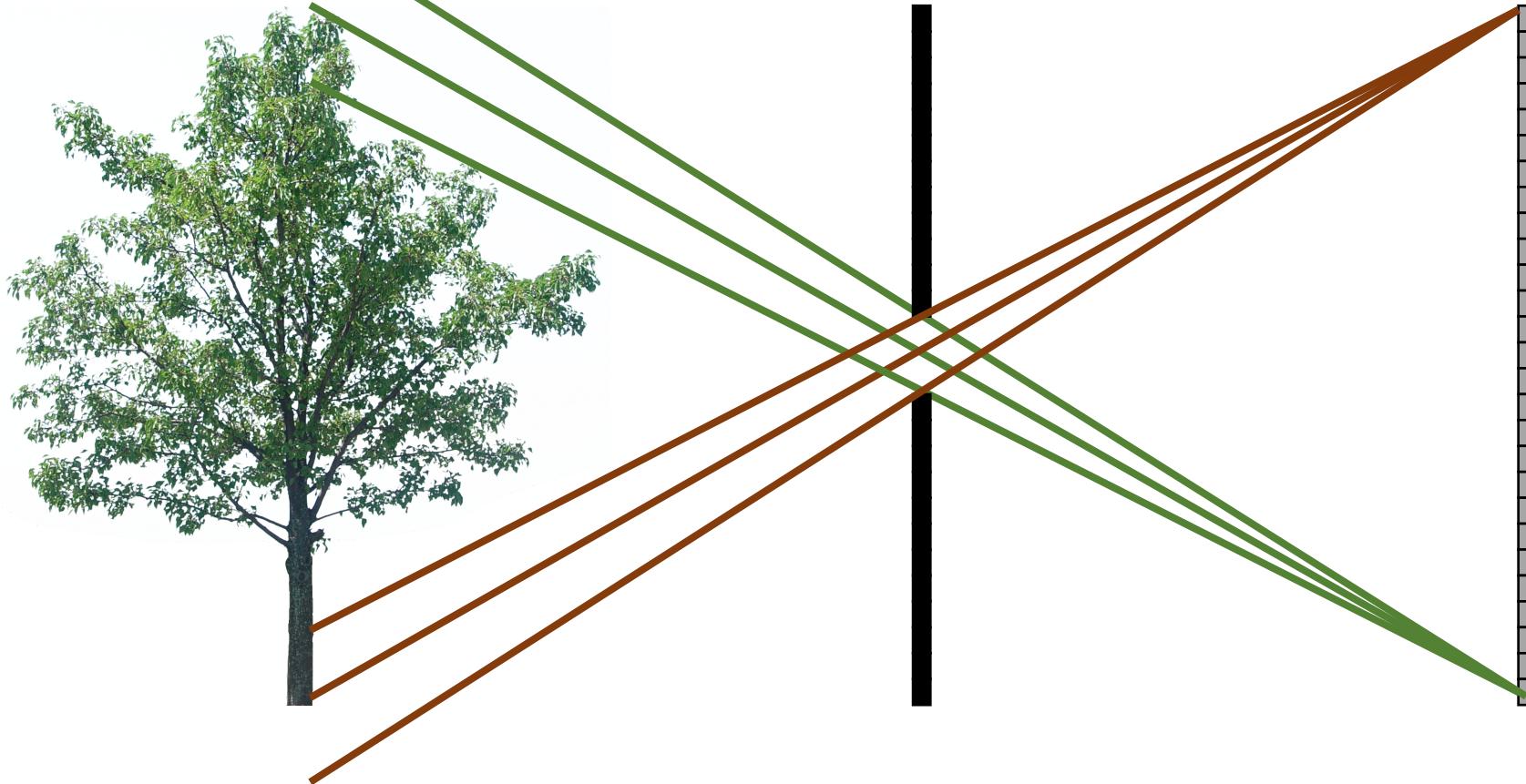
real-world
object



Pinhole size

What happens as we change the pinhole diameter?

real-world
object

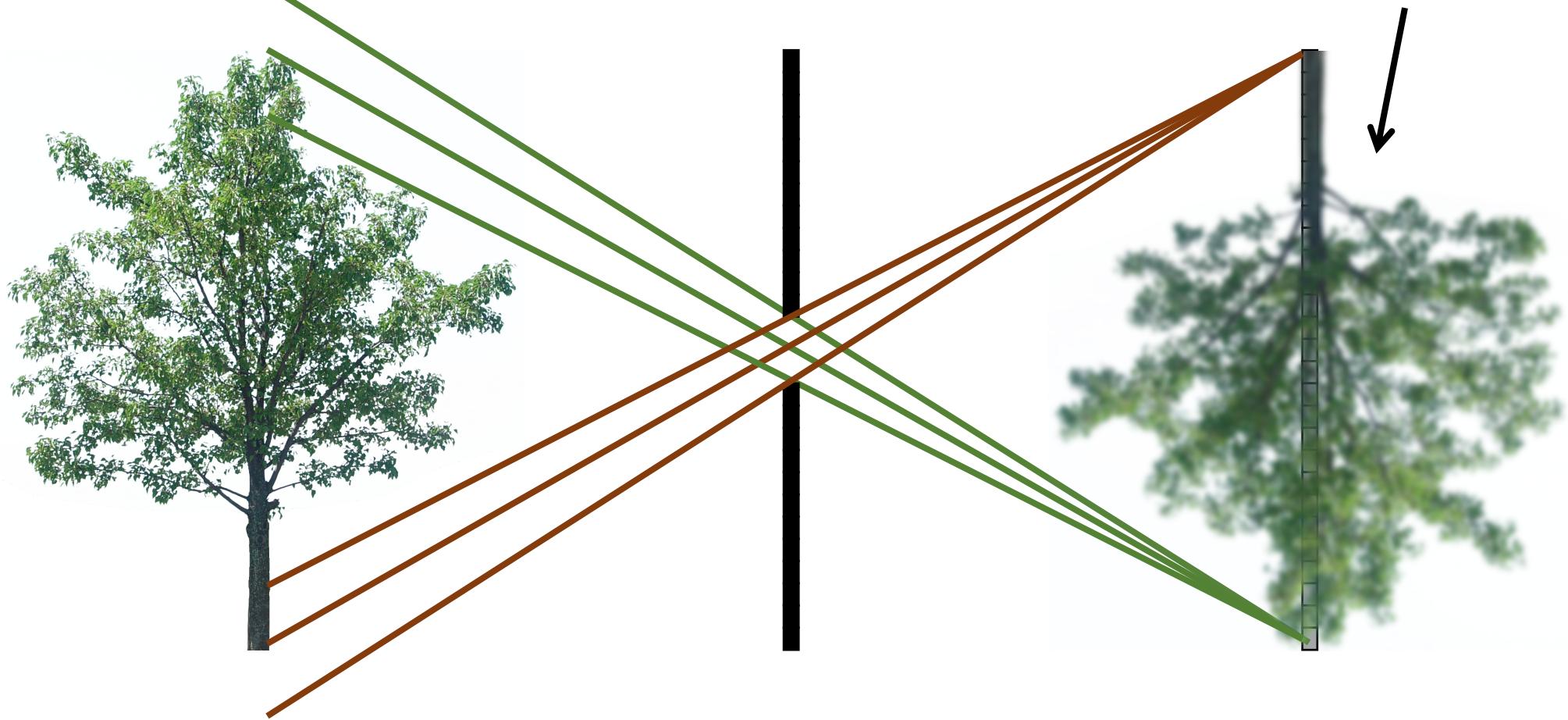


Pinhole size

What happens as we change the pinhole diameter?

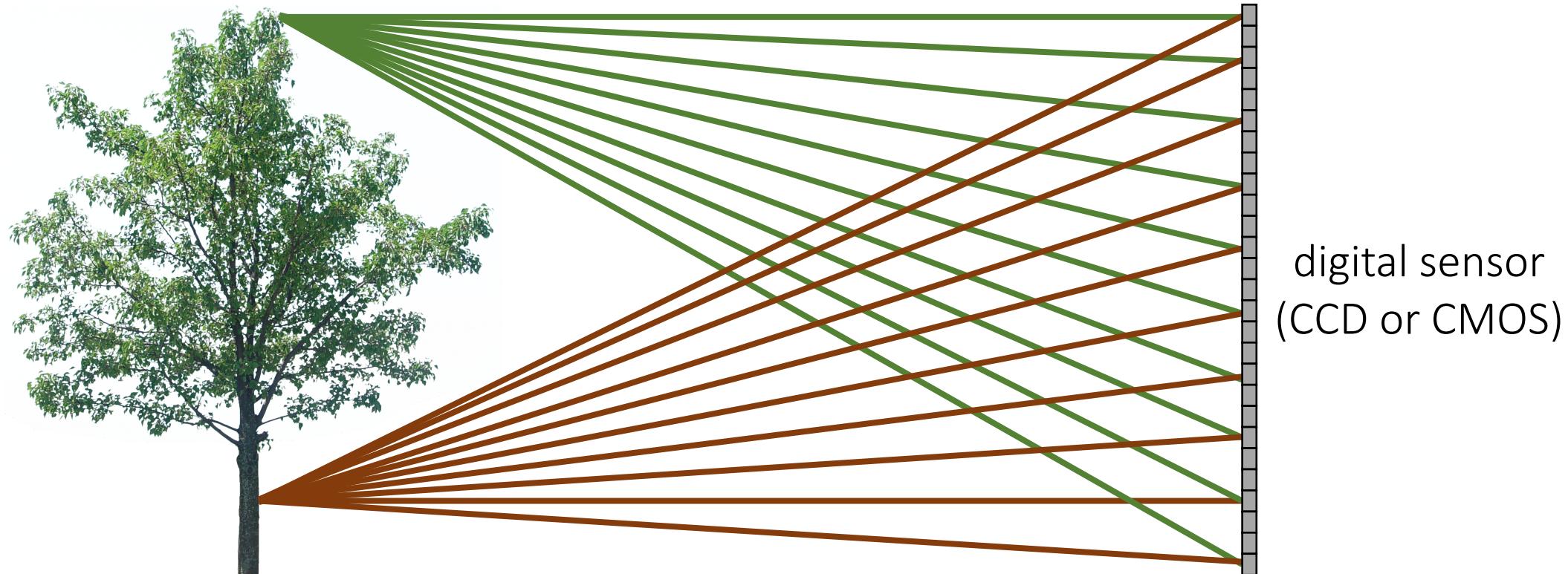
object projection becomes blurrier

real-world
object



Extreme Case: Bare-sensor imaging

real-world
object



All scene points contribute to all sensor pixels

What does the
image on the
sensor look like?

Extreme Case: Bare-sensor imaging



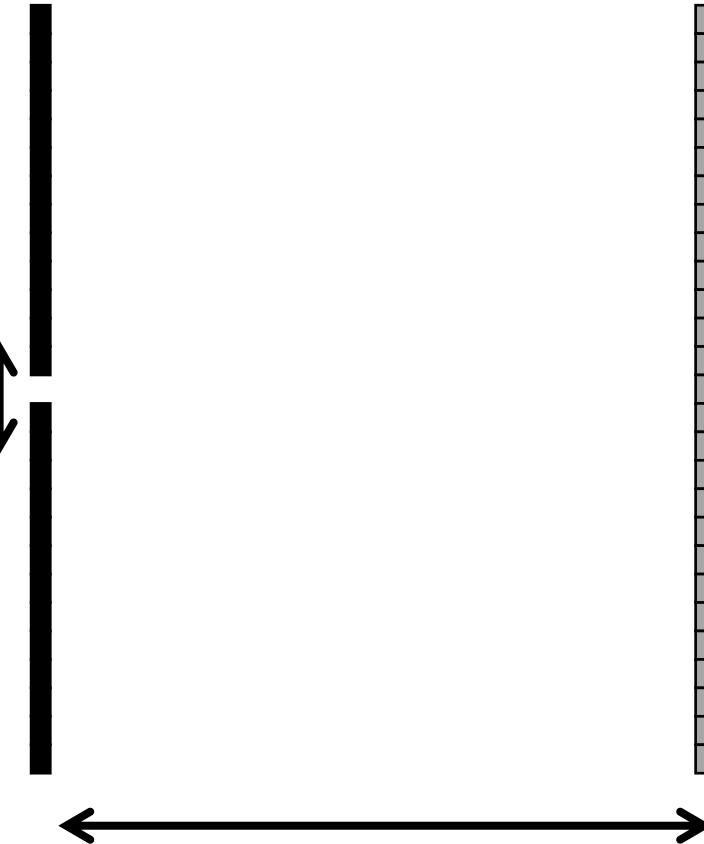
All scene points contribute to all sensor pixels

What about light efficiency?

real-world
object



pinhole
diameter



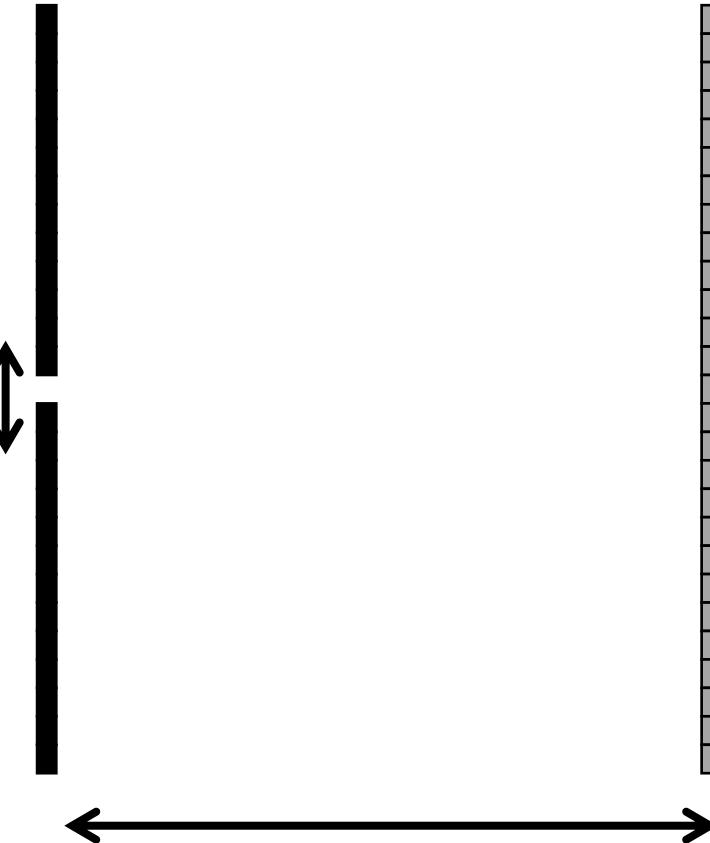
- What is the effect of doubling the pinhole diameter?
- What is the effect of doubling the focal length?

What about light efficiency?

real-world
object

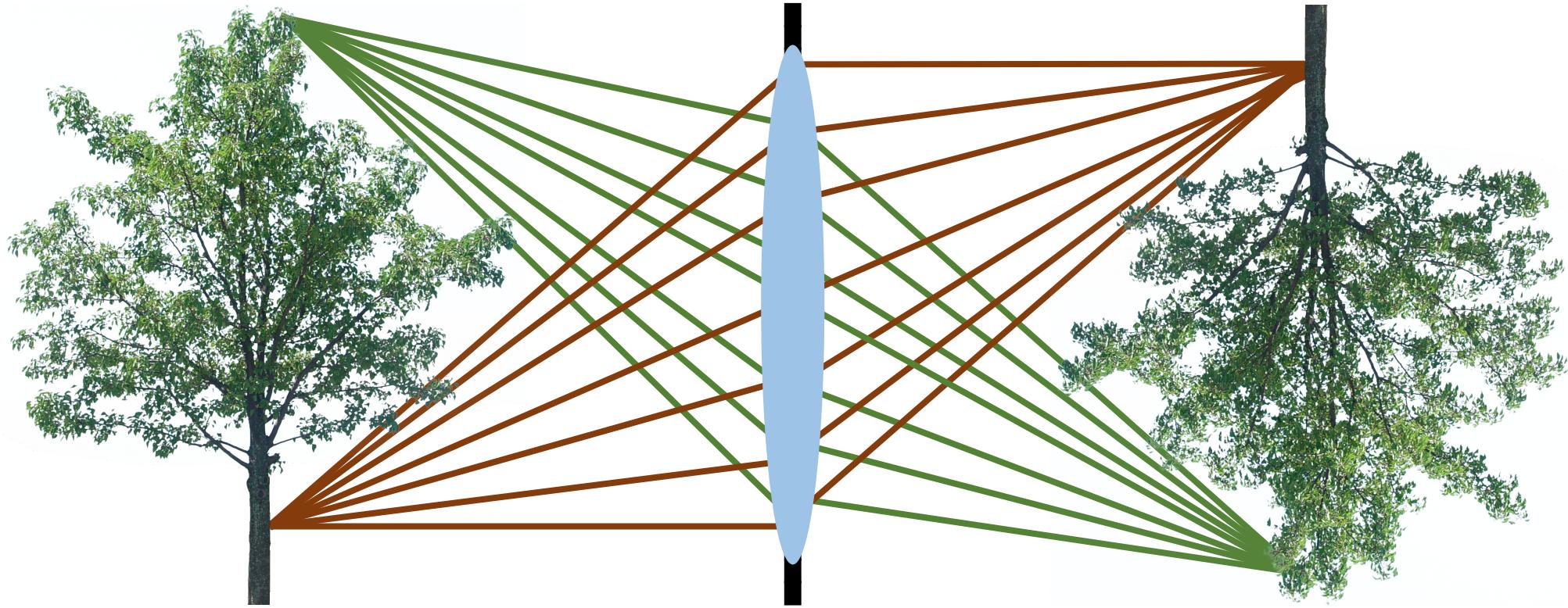


pinhole
diameter



- $2 \times$ pinhole diameter $\rightarrow 4 \times$ light
- $2 \times$ focal length $\rightarrow \frac{1}{4} \times$ light

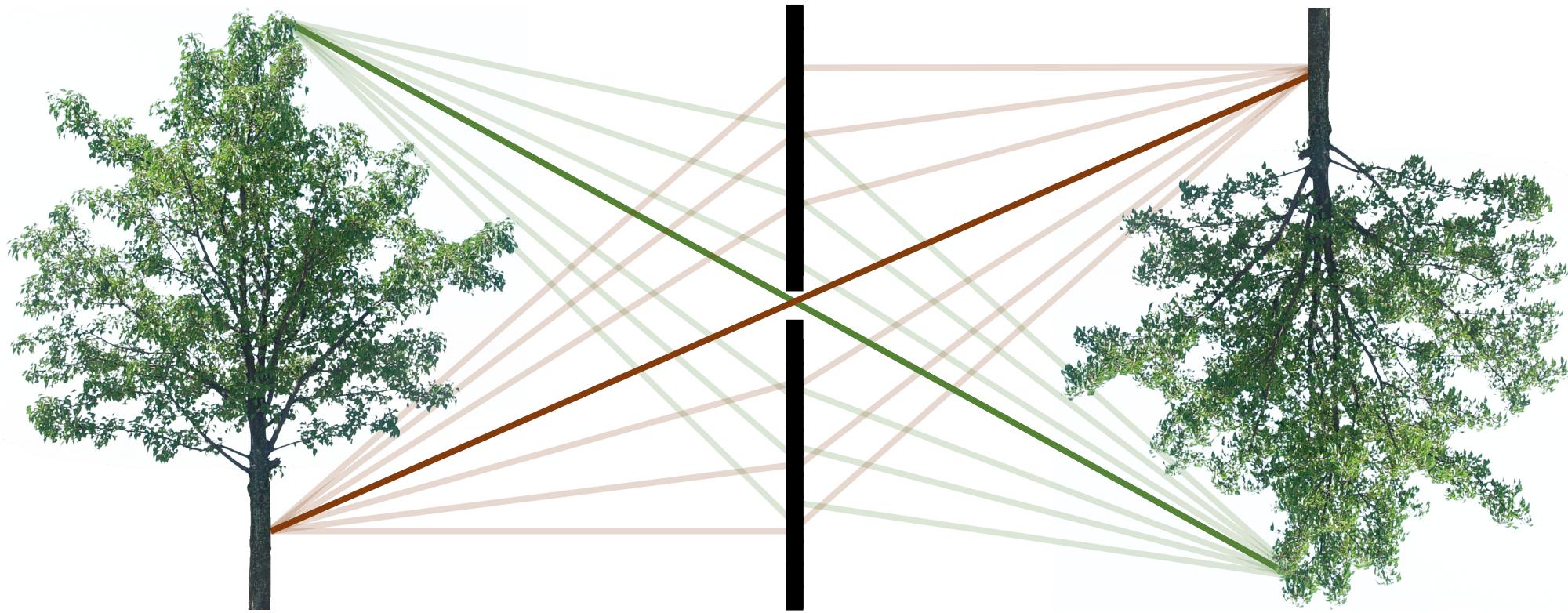
The lens camera



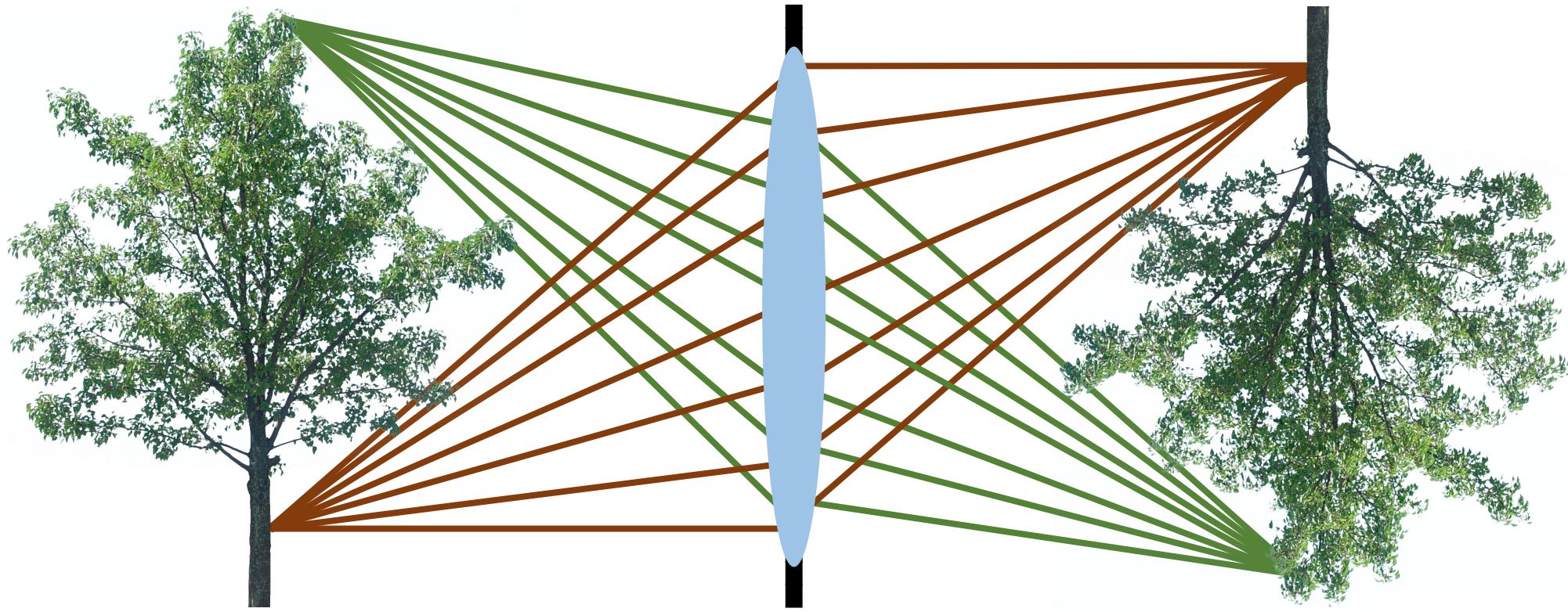
Lenses map “bundles” of rays from points on the scene to the sensor.

How does this mapping work exactly?

The pinhole camera

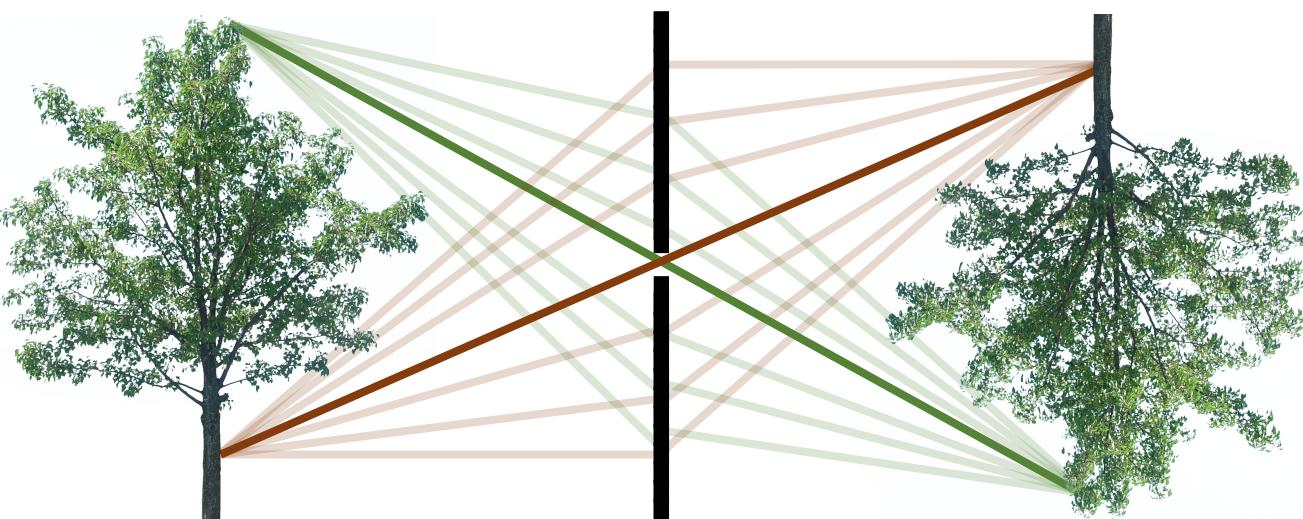
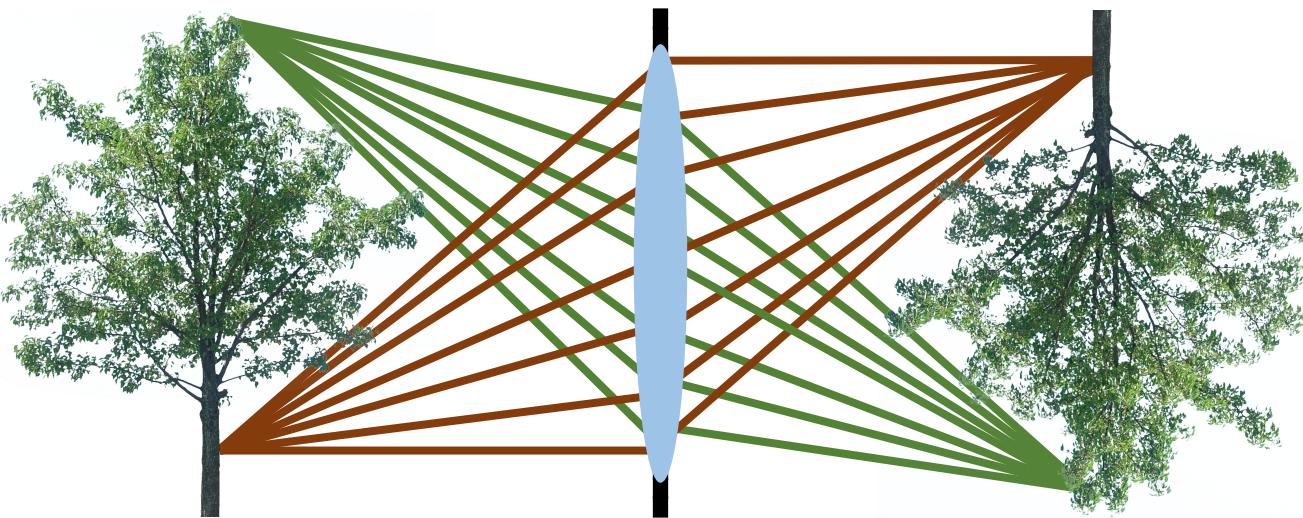


The lens camera



Central rays propagate in the same way for both models!

Describing both lens and pinhole cameras

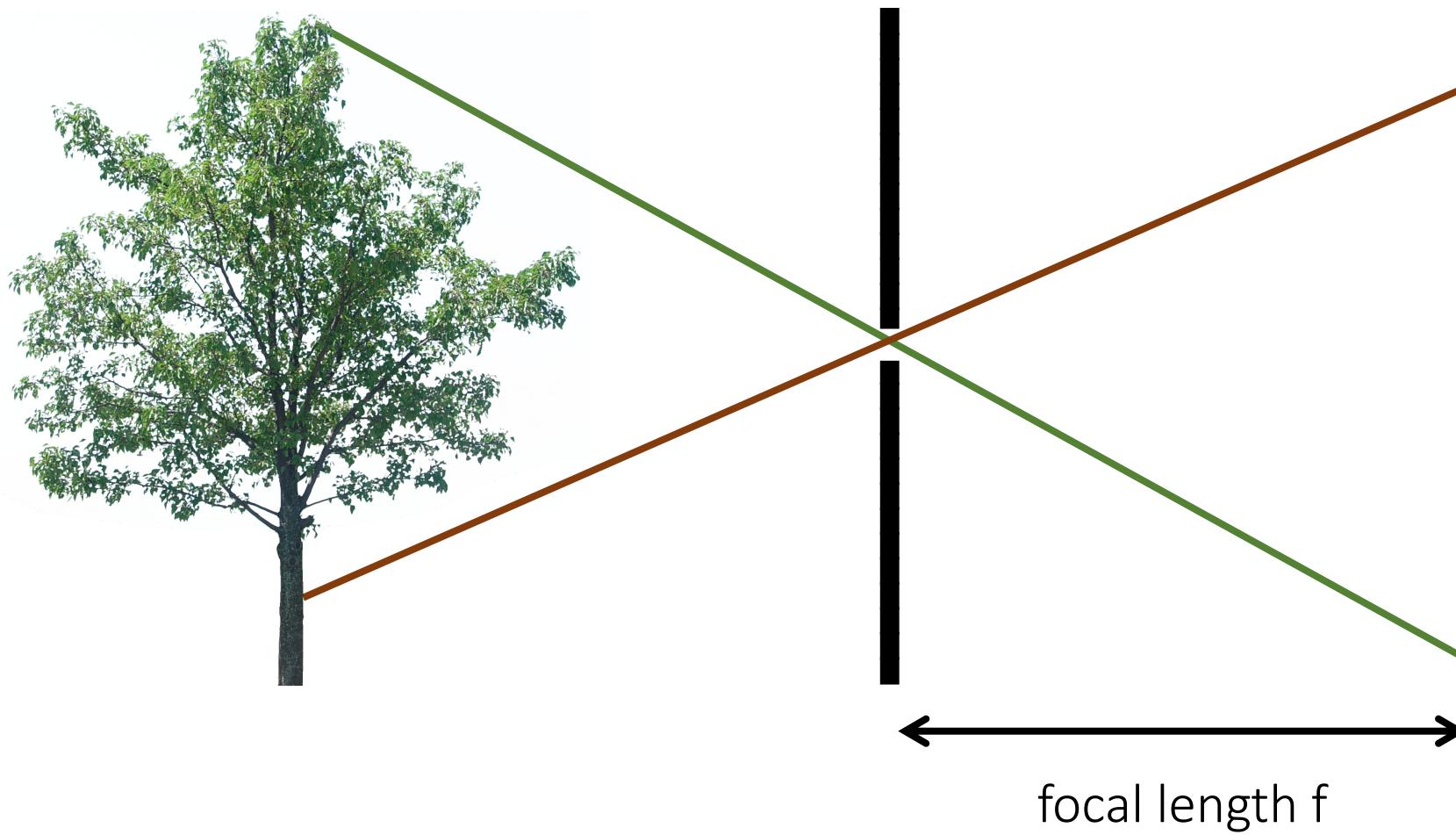


We can derive properties and descriptions that hold for both camera models if:

- We use only central rays.
- We assume the lens camera is in focus.

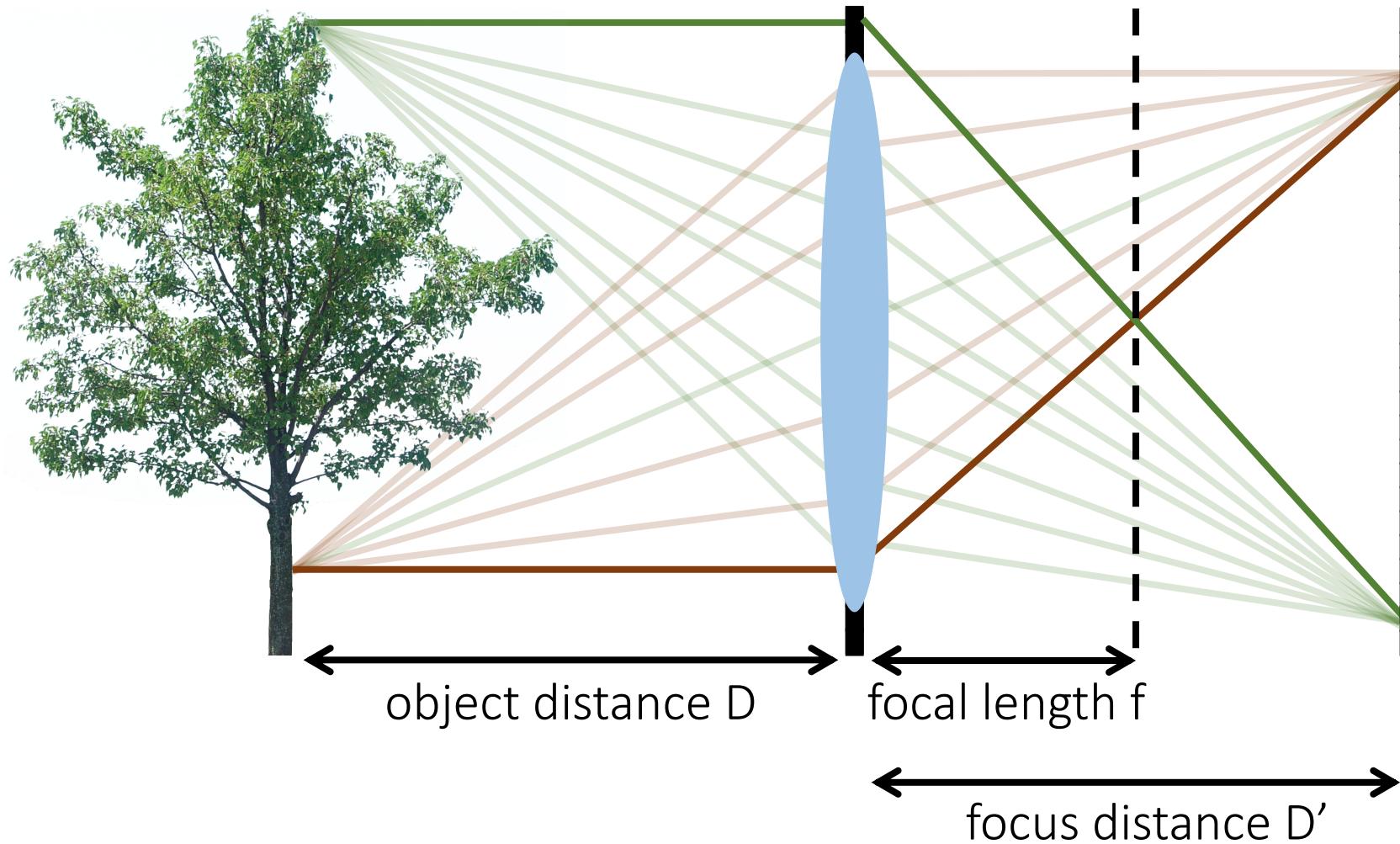
Important difference: focal length

In a pinhole camera, focal length is distance between aperture and sensor

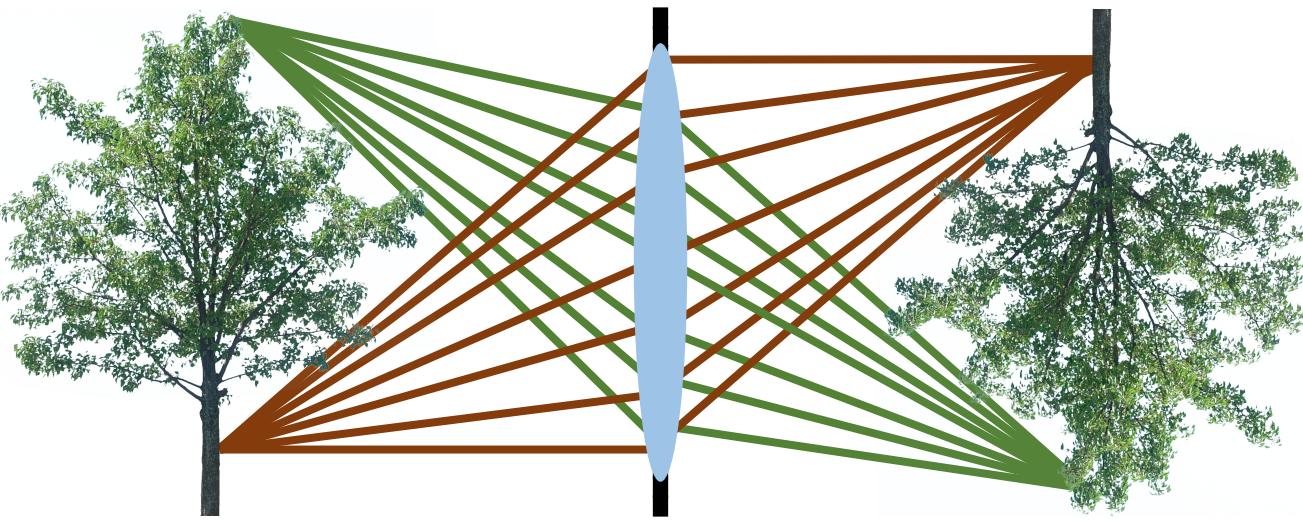


Important difference: focal length

In a lens camera, focal length is distance where parallel rays intersect

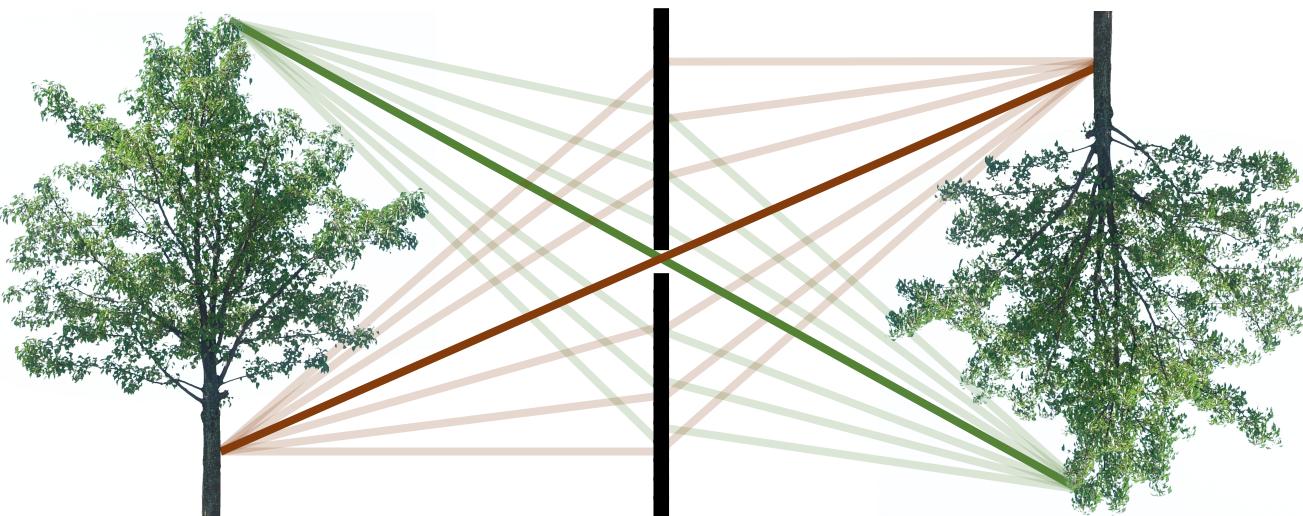


Describing both lens and pinhole cameras



We can derive properties and descriptions that hold for both camera models if:

- We use only central rays.
- We assume the lens camera is in focus.
- We assume that the focus distance of the lens camera is equal to the focal length of the pinhole camera.



Remember: *focal length f* refers to different things for lens and pinhole cameras.

- In this lecture, we use it to refer to the aperture-sensor distance, as in the pinhole camera case.

Camera Matrix

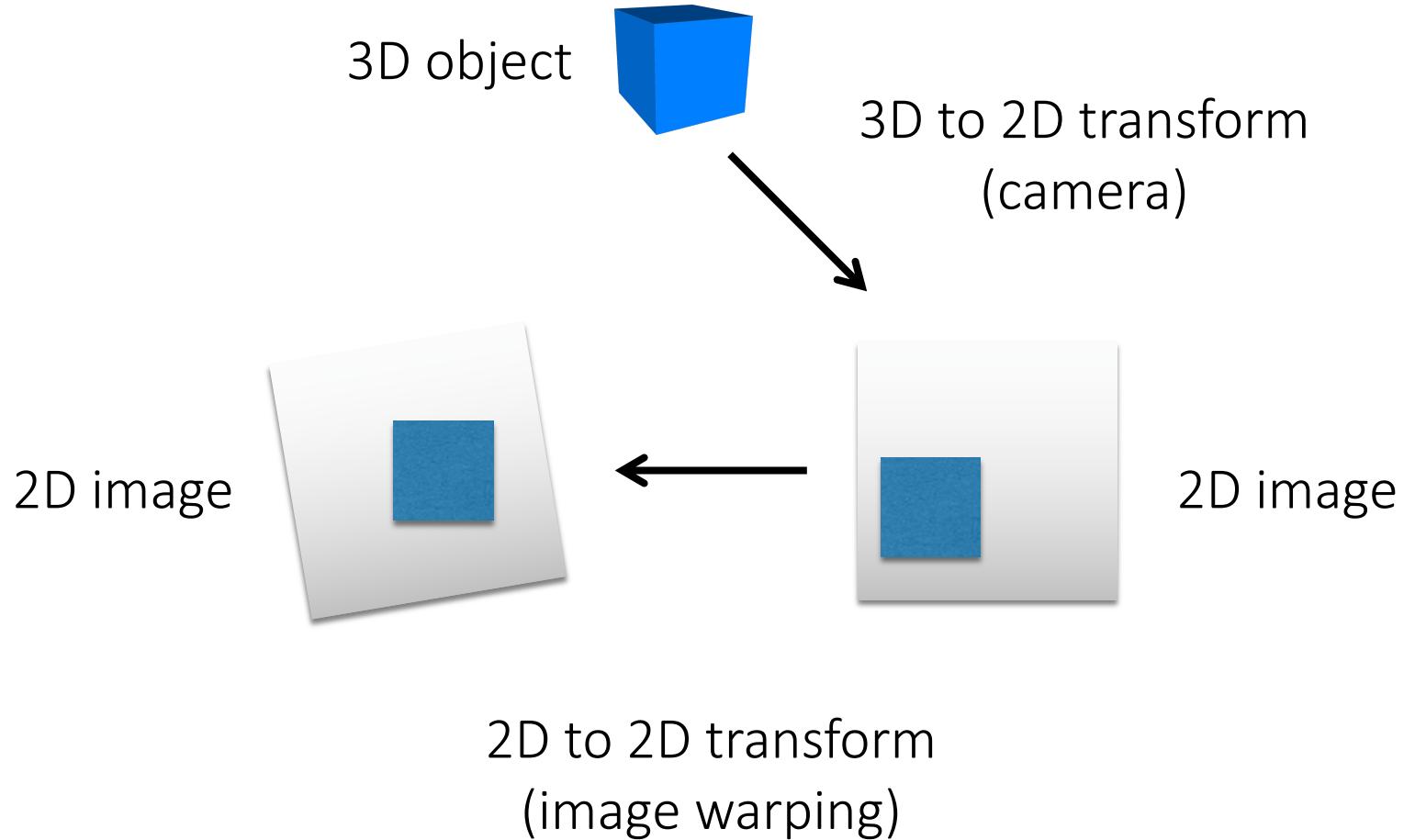
The camera as a coordinate transformation

The camera as a coordinate transformation

A camera is a mapping from:

the 3D world
to:

a 2D image



The camera as a coordinate transformation

A camera is a mapping from:

the 3D world

to:

a 2D image

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

homogeneous coordinates



2D image
point

camera
matrix 3D world
 point

What are the dimensions of each variable?

The camera as a coordinate transformation

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

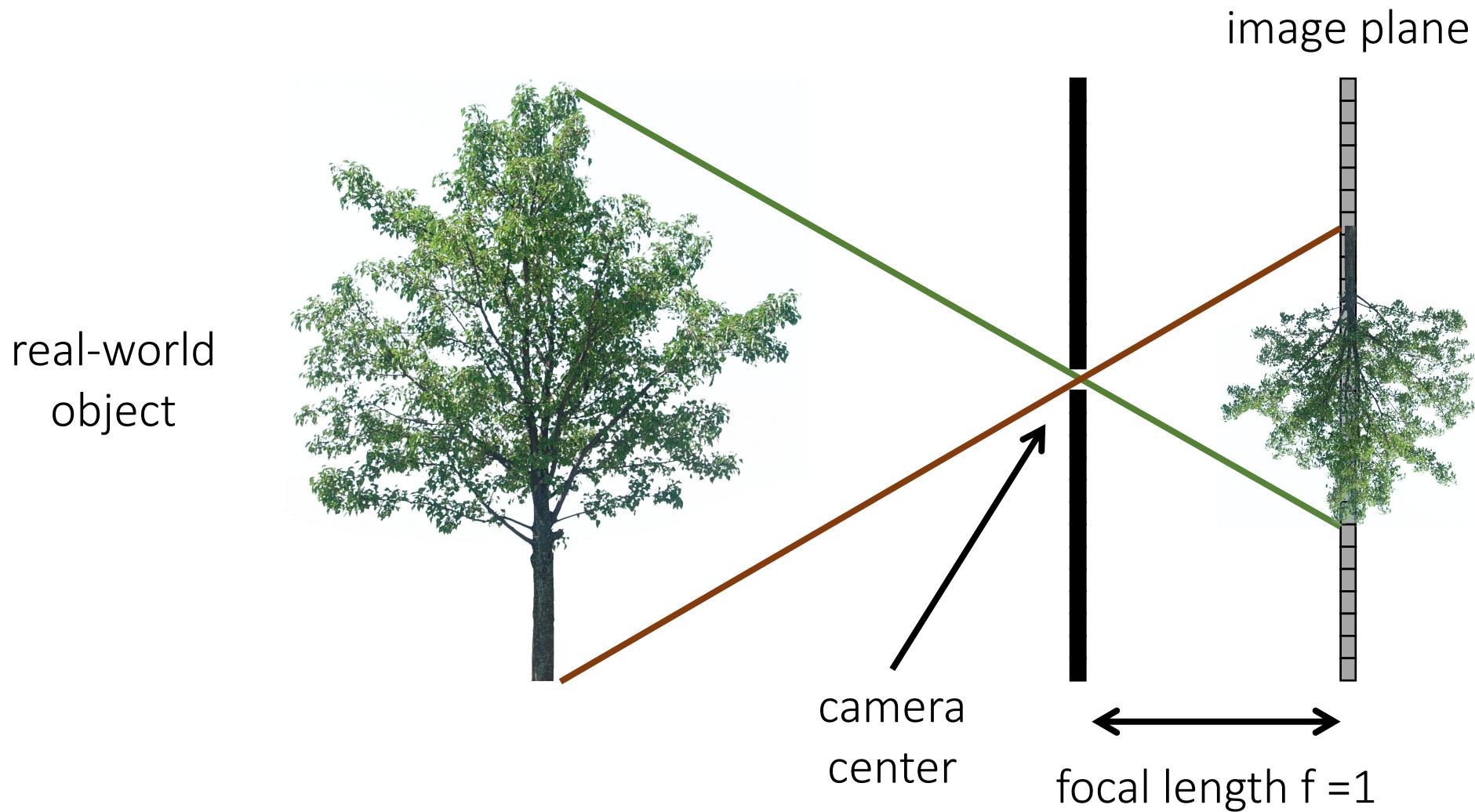
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

homogeneous
image coordinates
 3×1

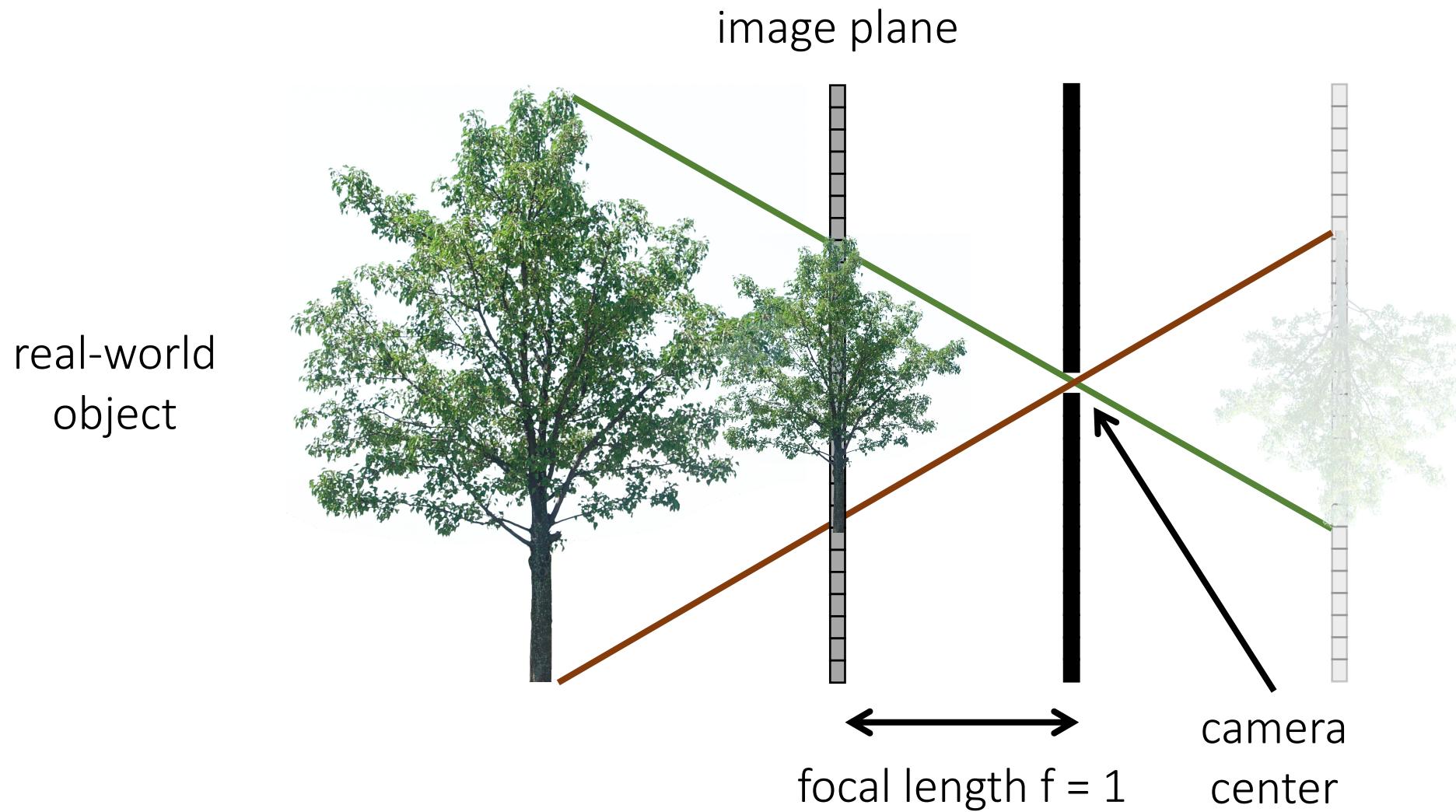
camera
matrix
 3×4

homogeneous
world coordinates
 4×1

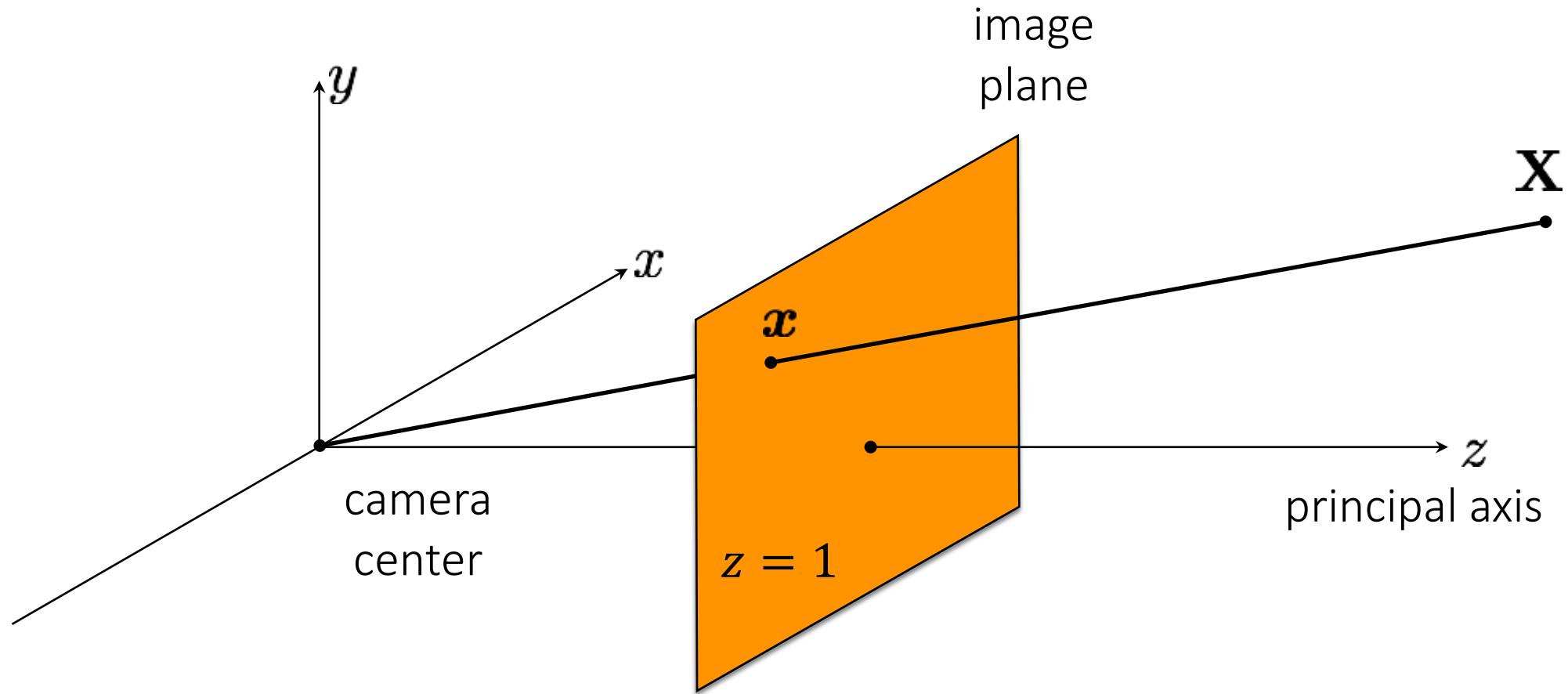
The pinhole camera



The (rearranged) pinhole camera

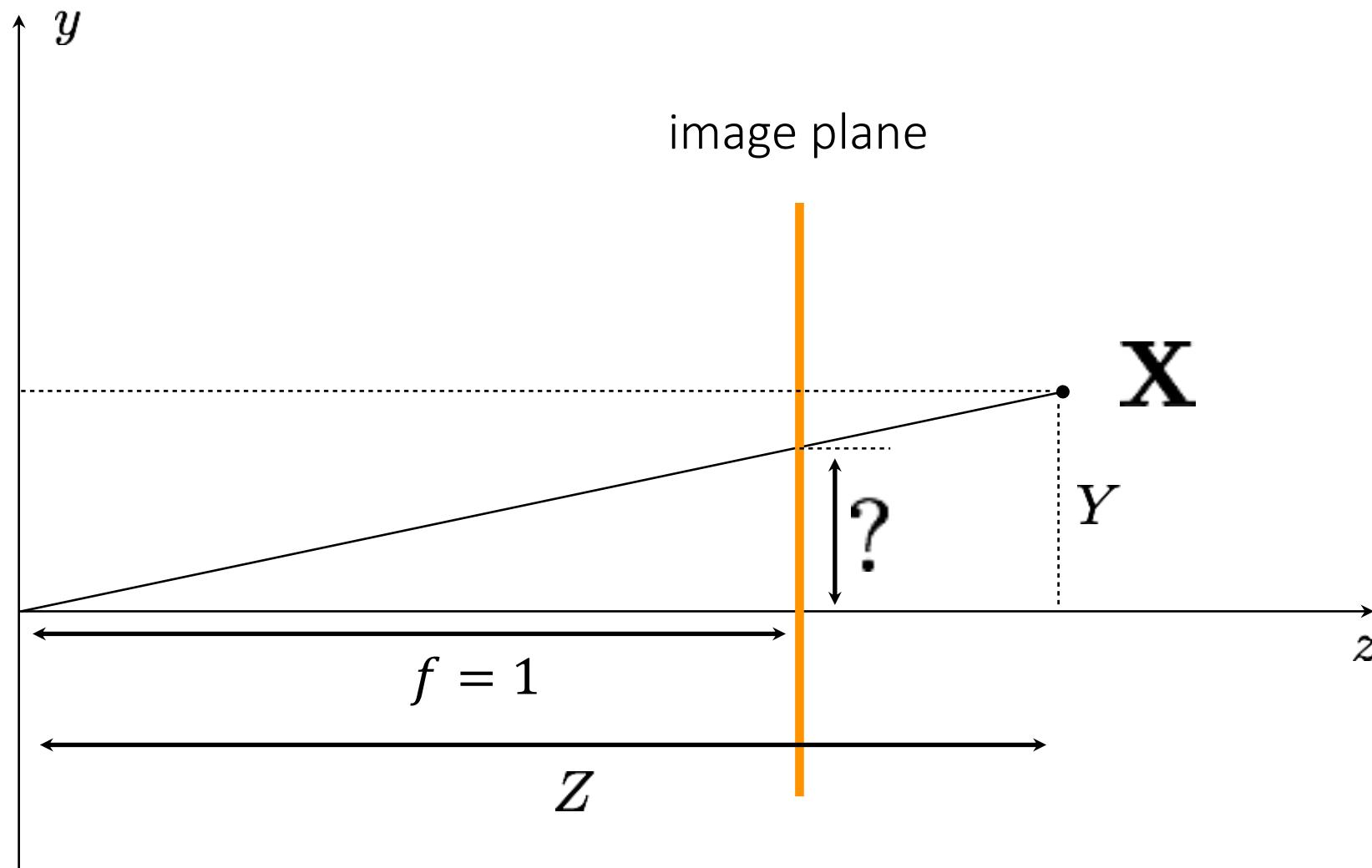


The (rearranged) pinhole camera



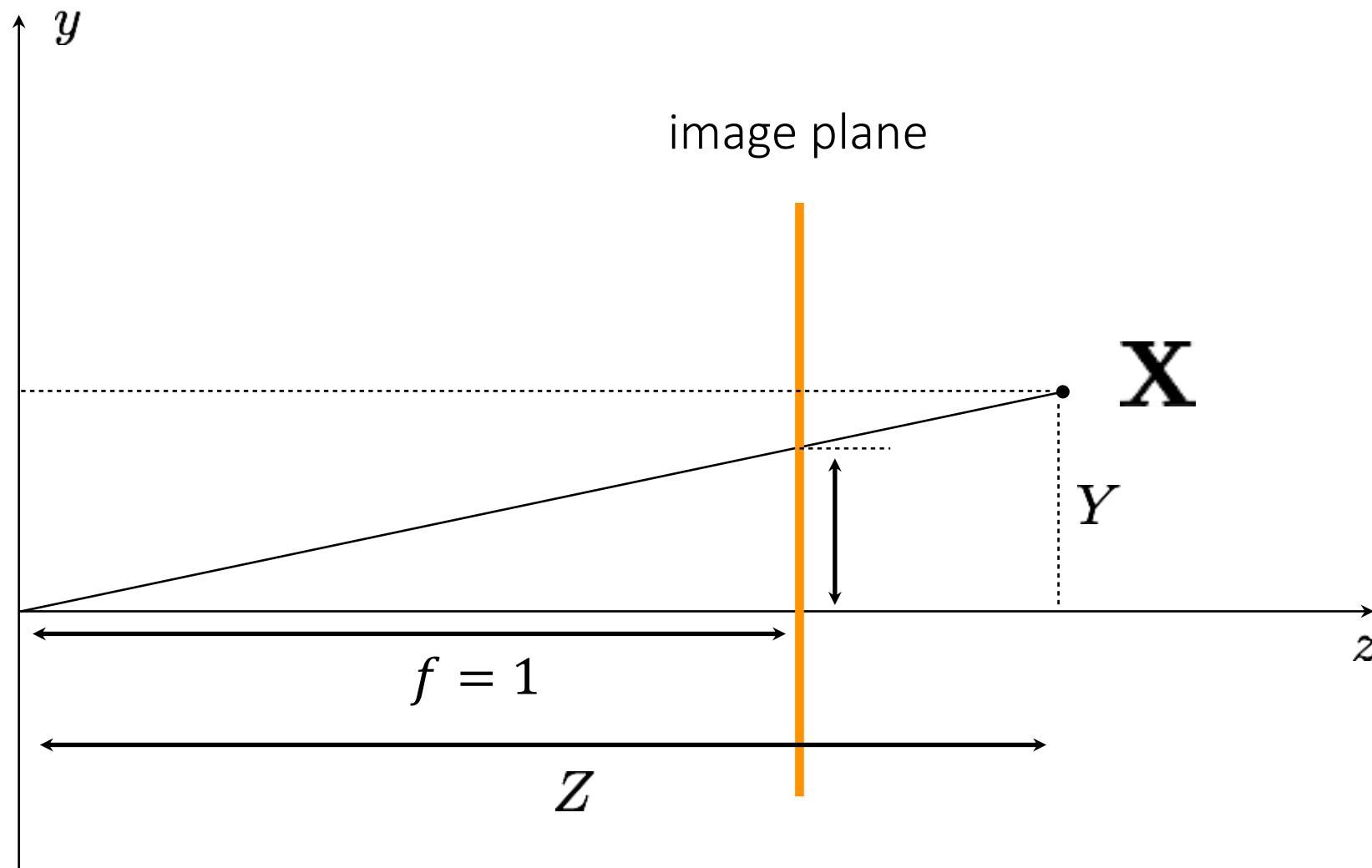
What is the equation for image coordinate \mathbf{x} in terms of X ?

The 2D view of the (rearranged) pinhole camera



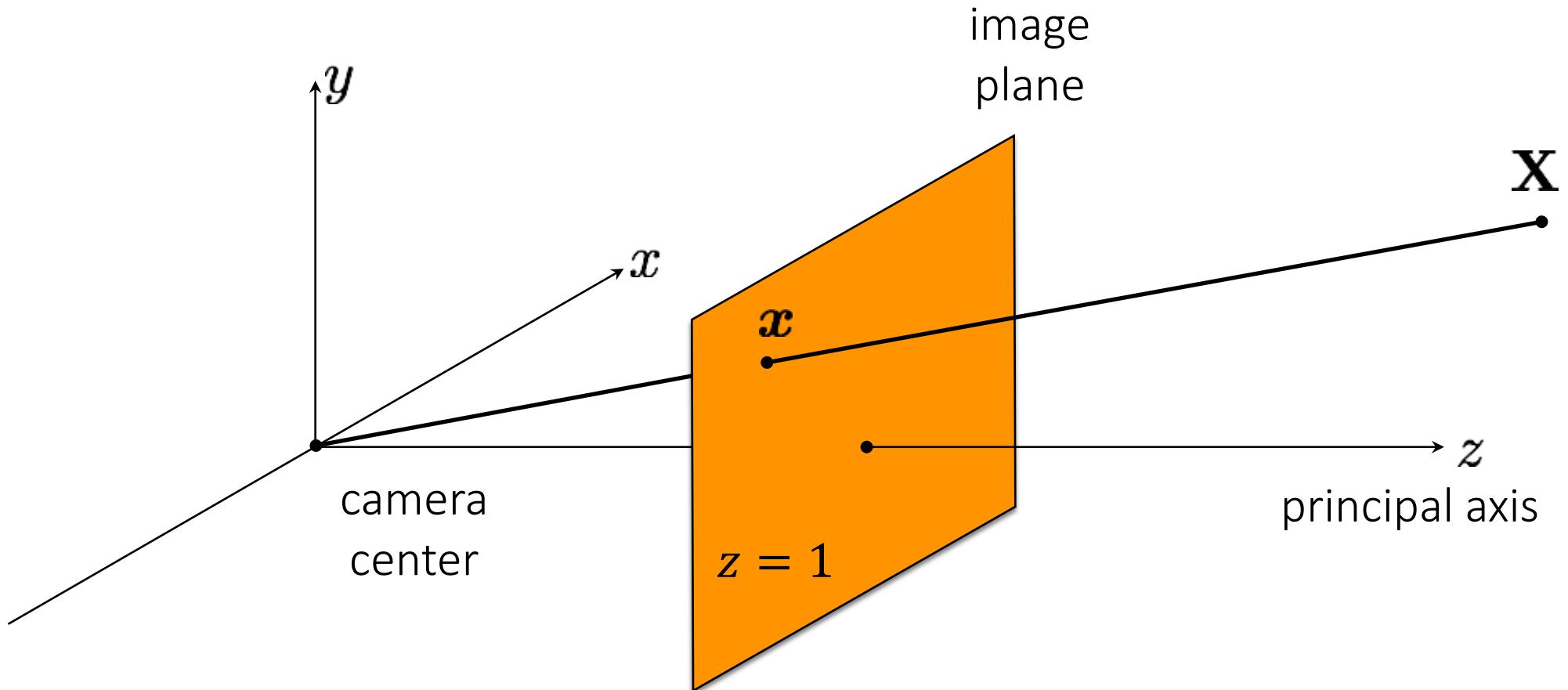
What is the equation for image coordinate x in terms of X ?

The 2D view of the (rearranged) pinhole camera



$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

The (rearranged) pinhole camera



What is the camera matrix \mathbf{P} for a pinhole camera?

$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

General camera model in *homogeneous coordinates*:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

$$\mathbf{P} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

General camera model *in homogeneous coordinates*:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

The perspective
projection matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

General camera model *in homogeneous coordinates*:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

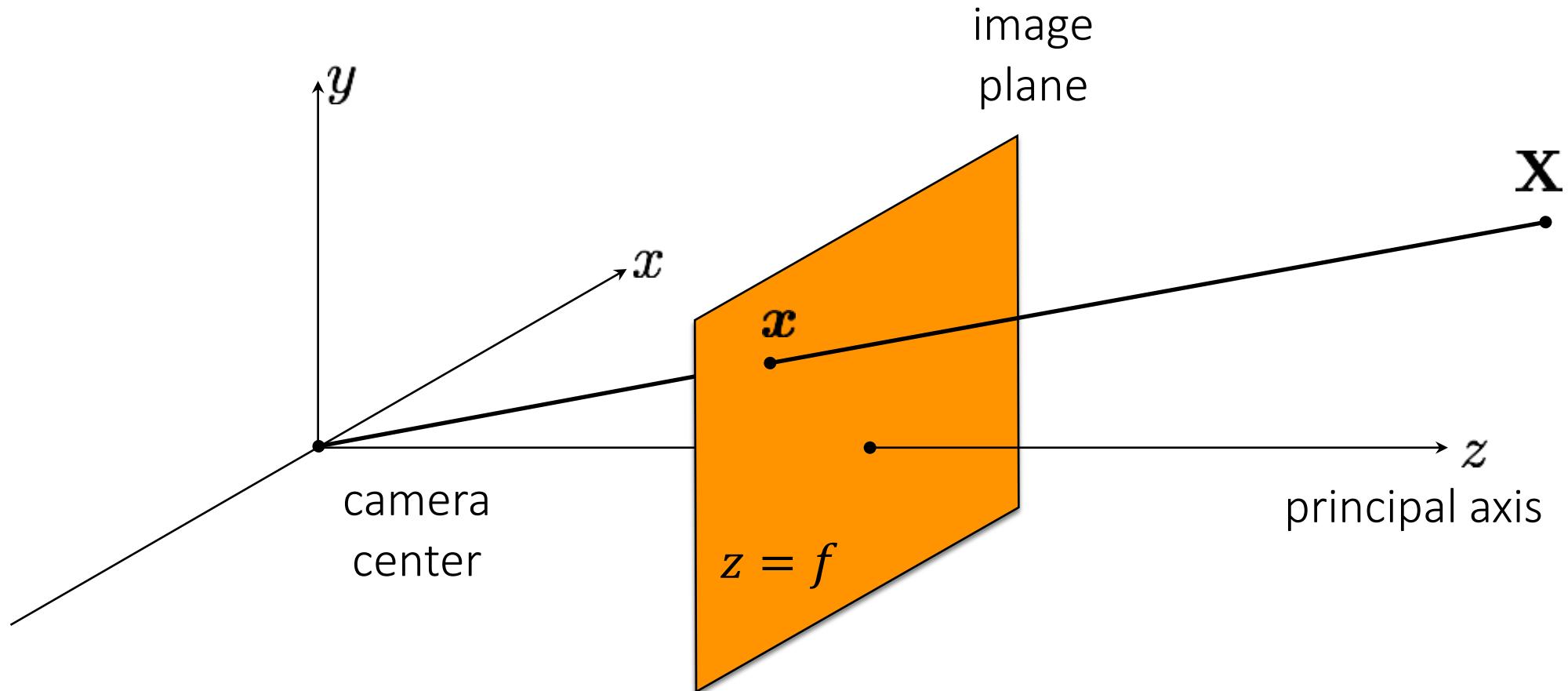
What does the pinhole camera projection look like?

The perspective
projection matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] = [\mathbf{I} \quad | \quad \mathbf{0}]$$

alternative way to write
the same thing

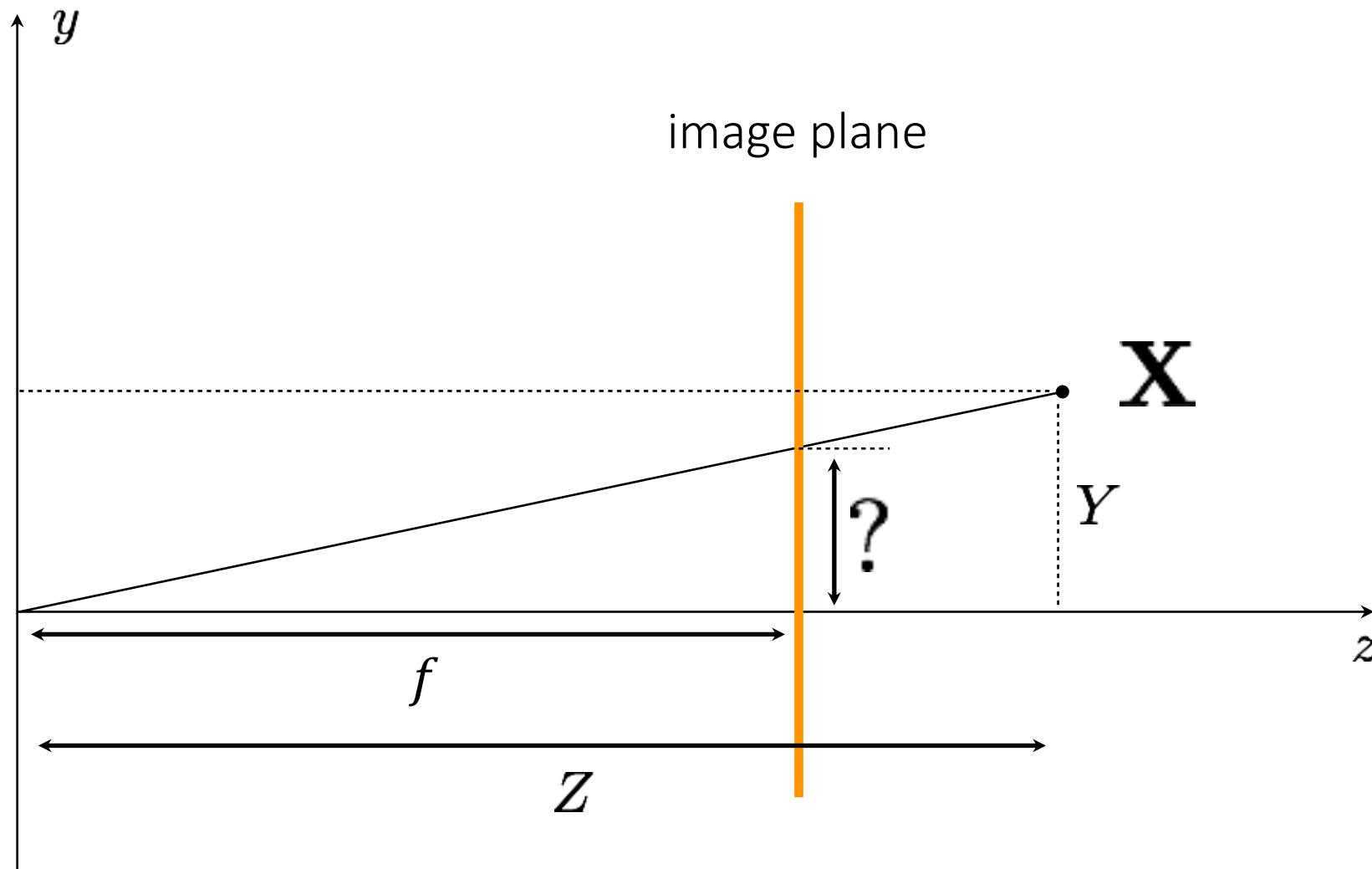
More general case: arbitrary focal length



What is the camera matrix \mathbf{P} for a pinhole camera?

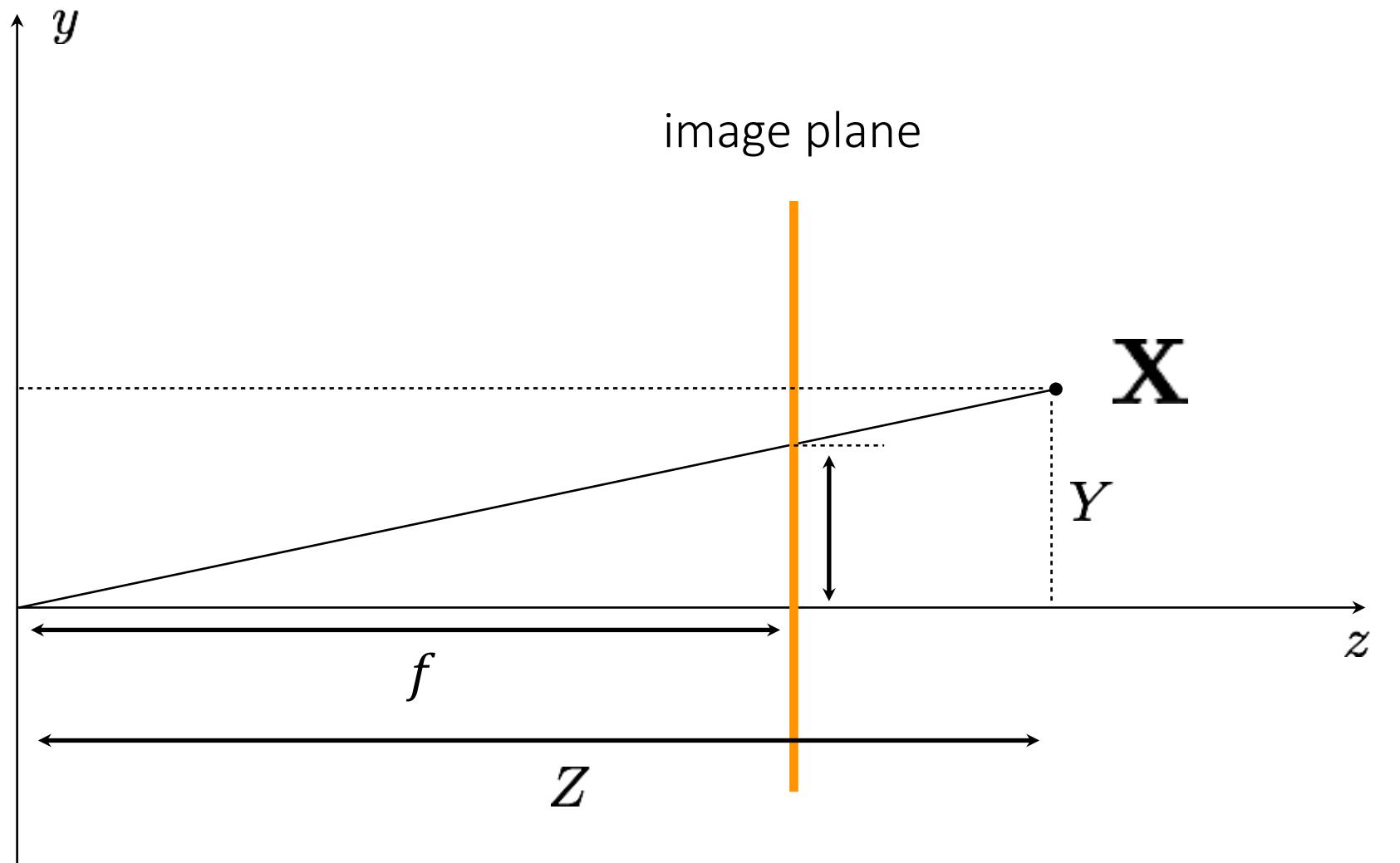
$$\mathbf{x} = \mathbf{P}\mathbf{X}$$

More general (2D) case: arbitrary focal length



What is the equation for image coordinate x in terms of X ?

More general (2D) case: arbitrary focal length



$$[X \ Y \ Z]^\top \mapsto [fX/Z \ fY/Z]^\top$$

The pinhole camera matrix for arbitrary focal length

Relationship from similar triangles:

$$[X \ Y \ Z]^\top \mapsto [fX/Z \ fY/Z]^\top$$

General camera model *in homogeneous coordinates*:

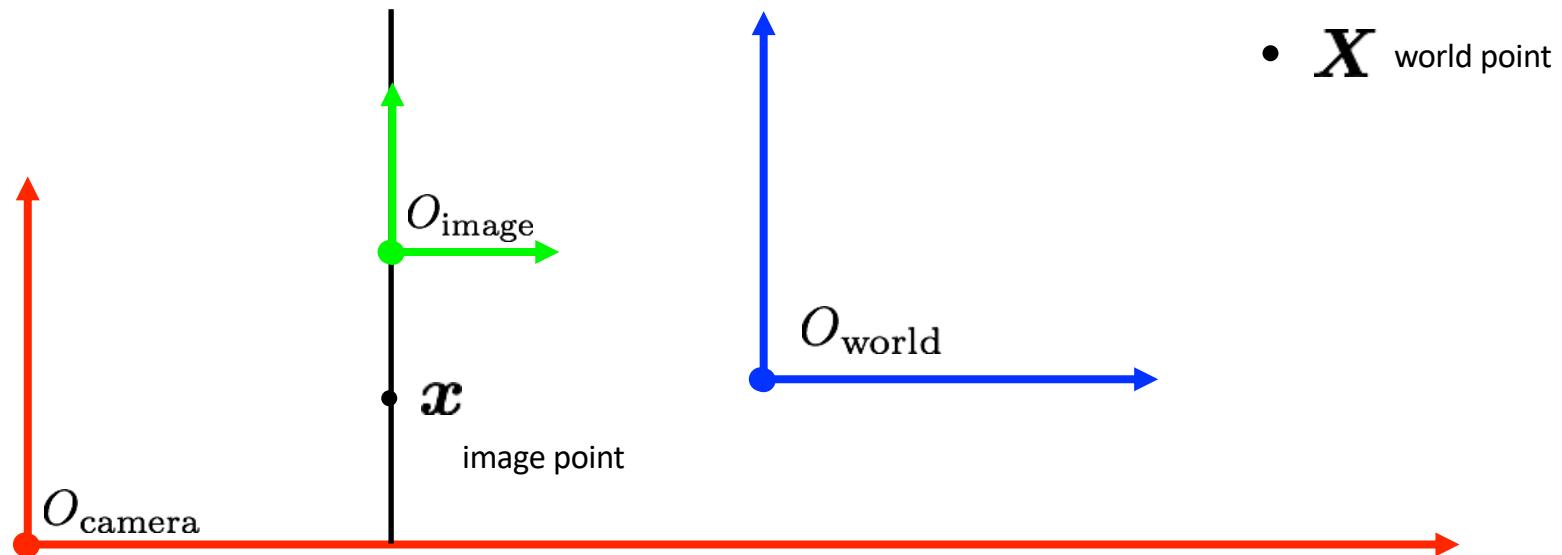
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

$$\mathbf{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

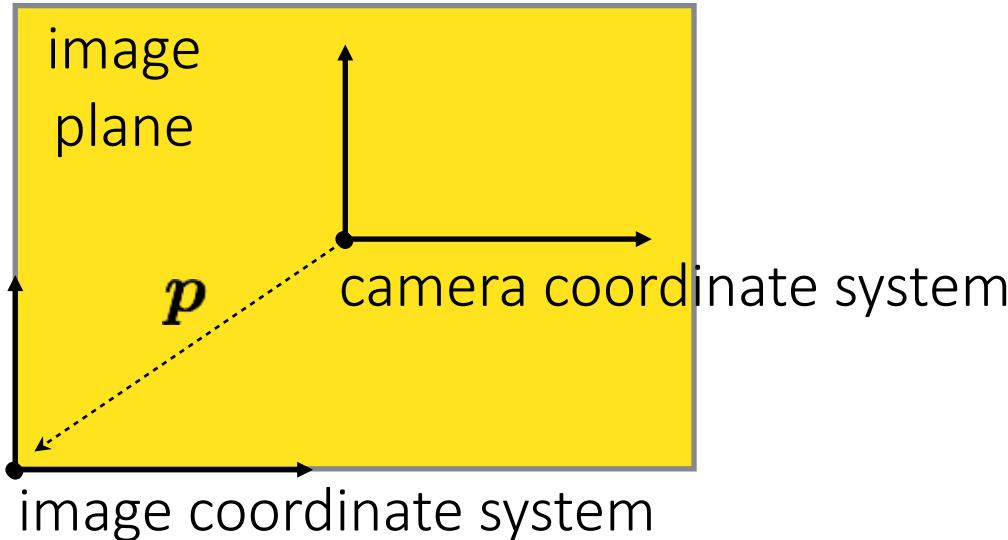
Generalizing the camera matrix

In general, the camera and image have *different* coordinate systems.



Generalizing the camera matrix

In particular, the camera origin and image origin may be different:

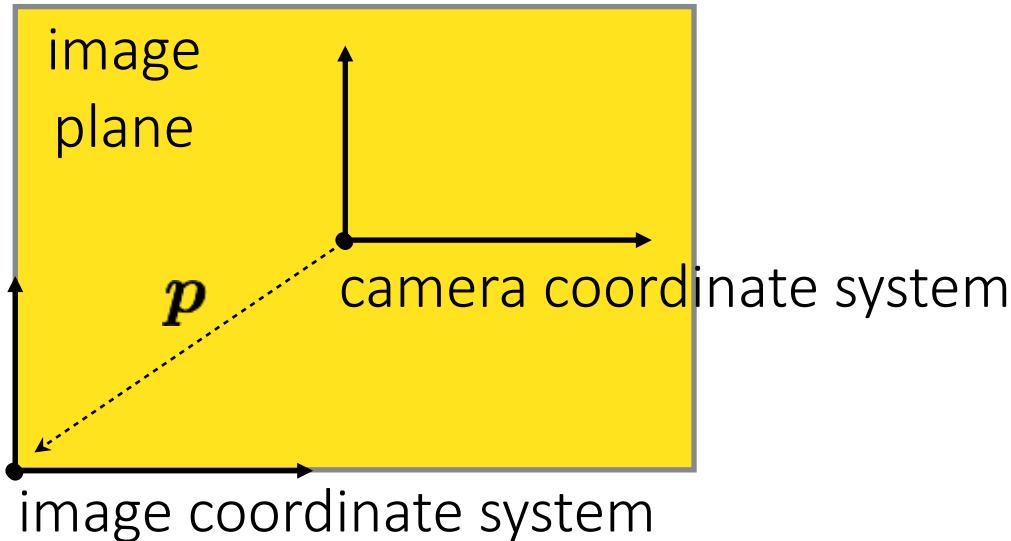


How does the camera matrix change?

$$\mathbf{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Generalizing the camera matrix

In particular, the camera origin and image origin may be different:



How does the camera matrix change?

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

shift vector
transforming
camera origin to
image origin

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

What does each part of the matrix represent?

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



(homogeneous) transformation
from 2D to 2D, accounting for non
unit focal length and origin shift

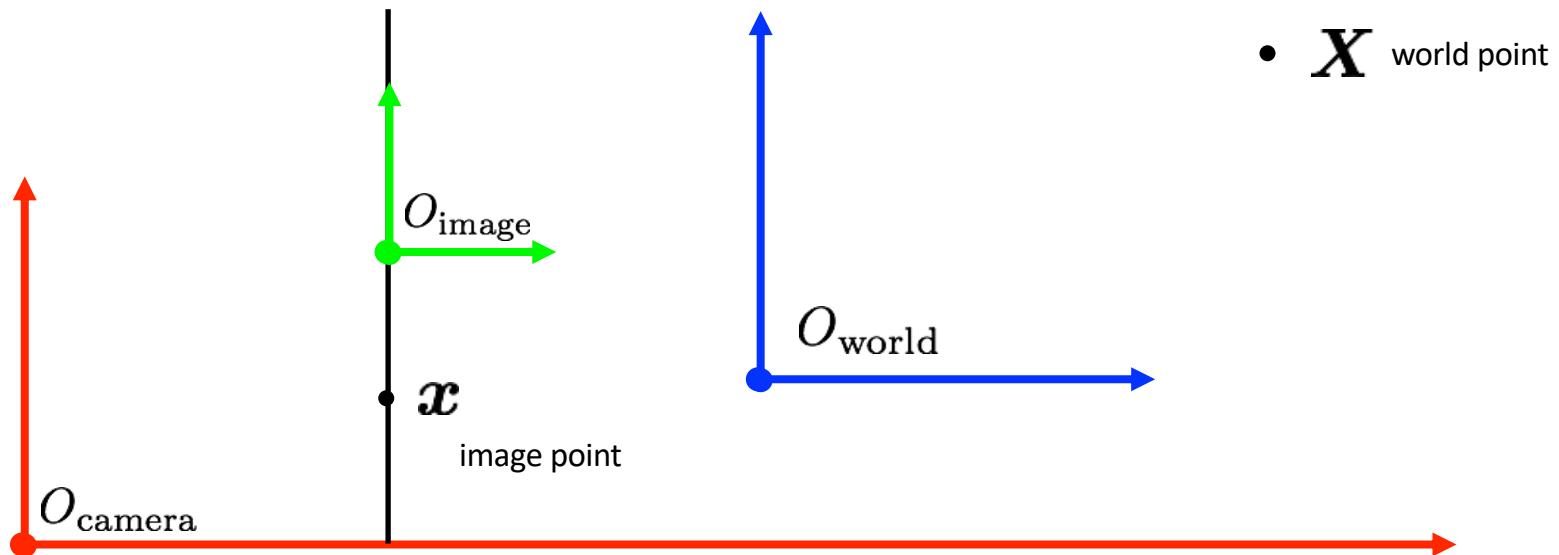
(homogeneous) perspective projection
from 3D to 2D, assuming image plane at
 $z = 1$ and shared camera/image origin

Also written as: $\mathbf{P} = \mathbf{K}[\mathbf{I}|0]$

where $\mathbf{K} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$

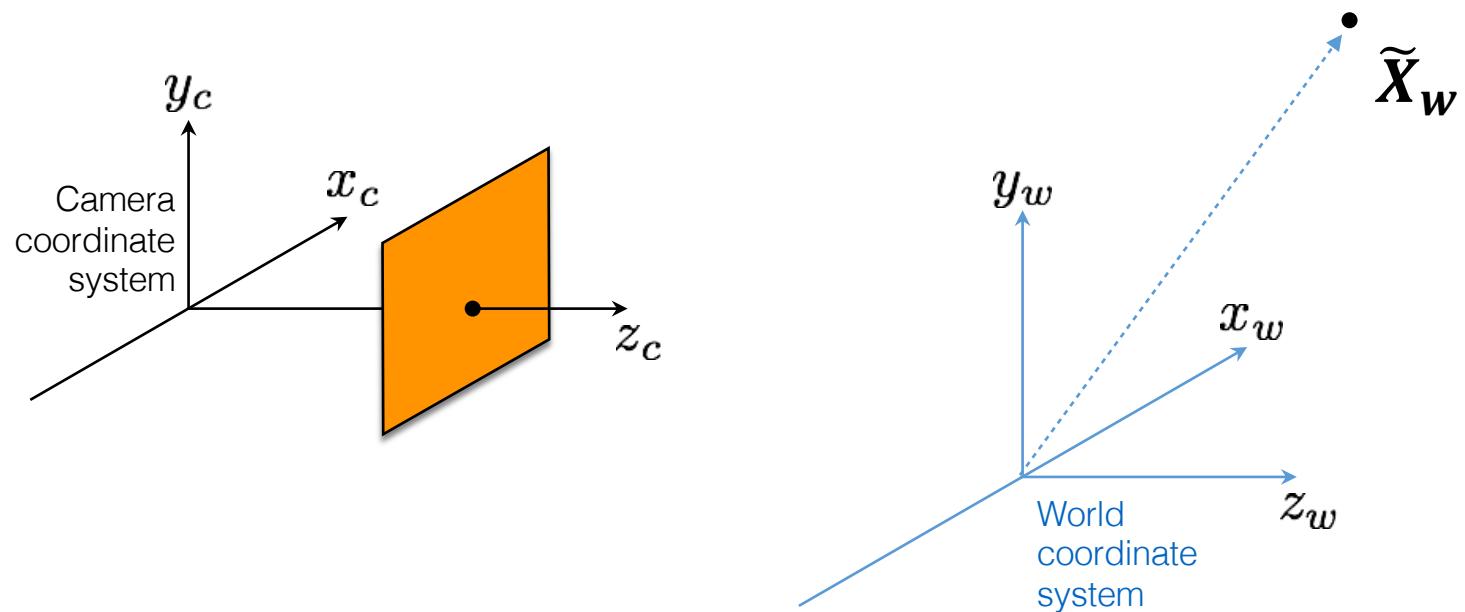
Generalizing the camera matrix

In general, there are *three*, generally different, coordinate systems.



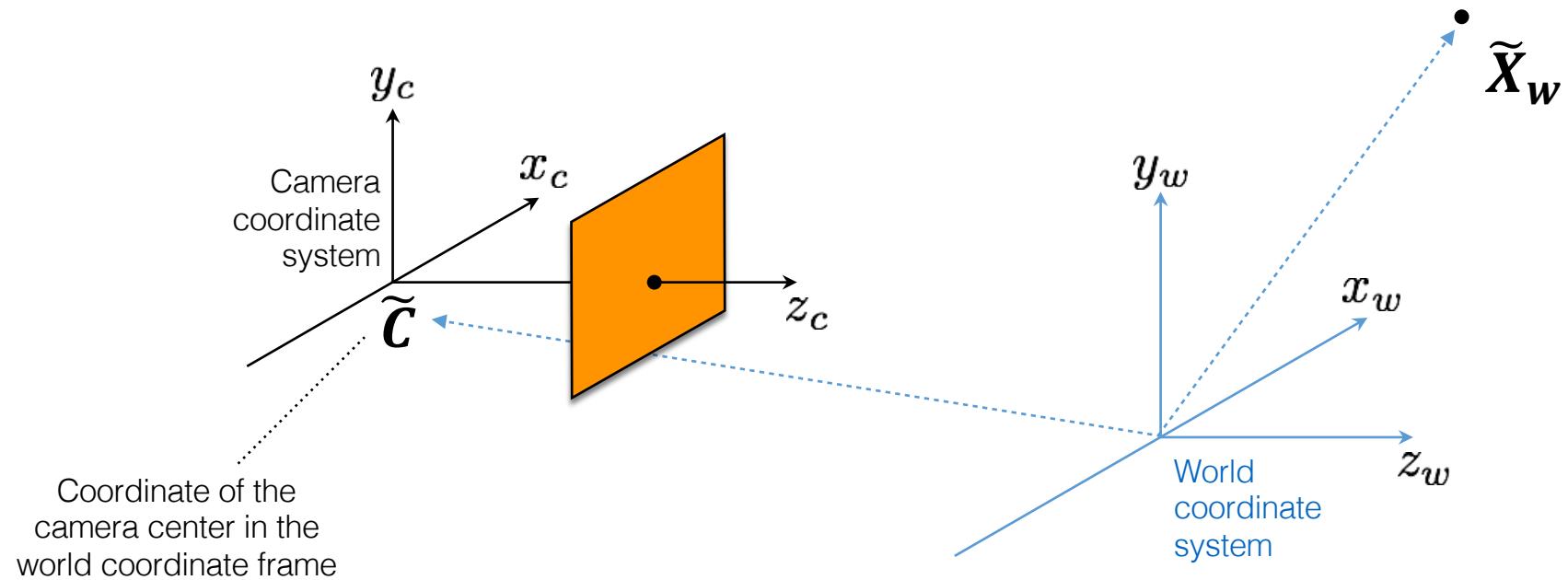
We need to know the transformations between them.

World-to-camera coordinate system transformation

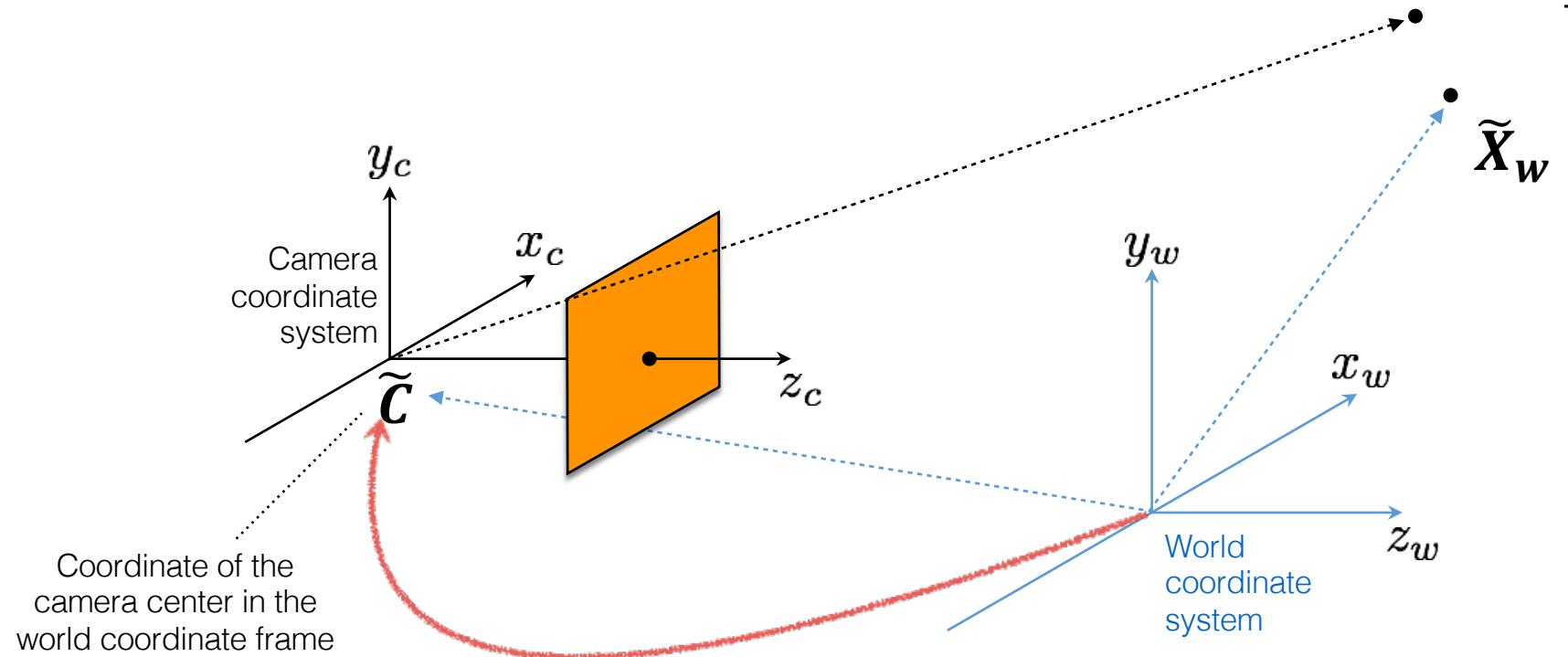


tilde means
heterogeneous
coordinates

World-to-camera coordinate system transformation



World-to-camera coordinate system transformation

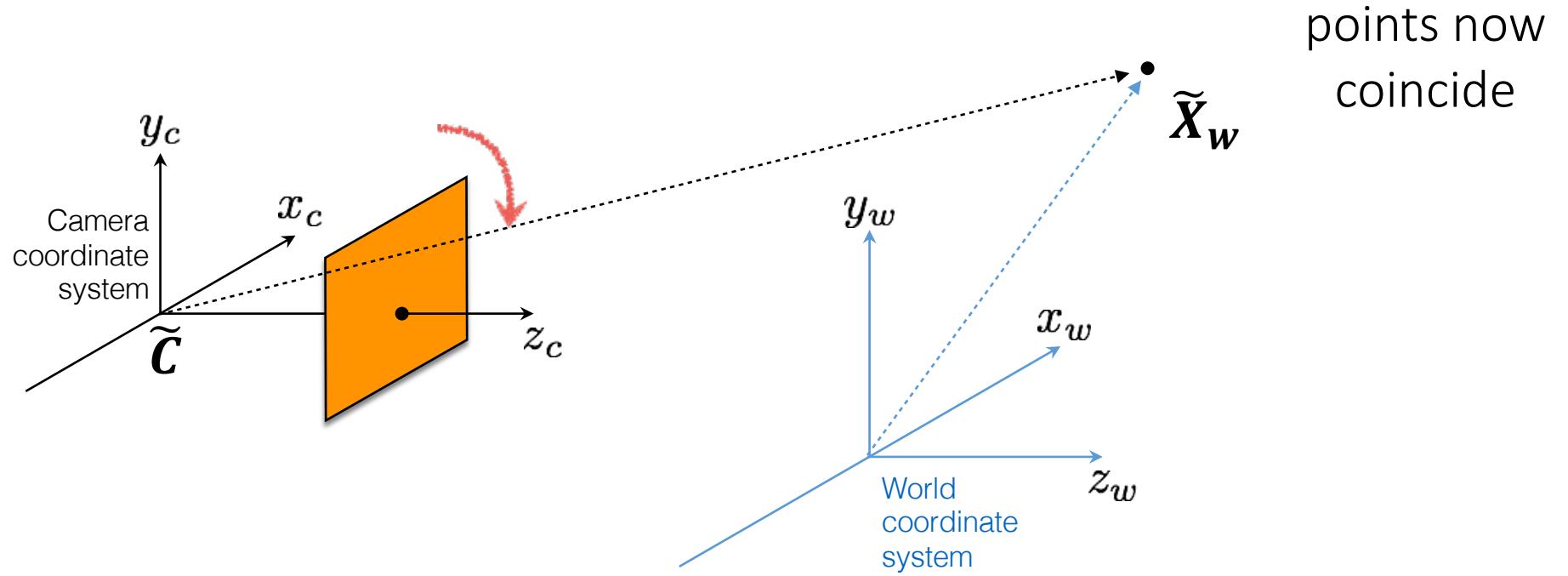


Why aren't
the points
aligned?

$$(\tilde{X}_w - \tilde{C})$$

translate

World-to-camera coordinate system transformation



$$R \cdot (\tilde{X}_w - \tilde{C})$$

rotate translate

Modeling the coordinate system transformation

In heterogeneous coordinates, we have:

$$\tilde{\mathbf{X}}_c = \mathbf{R} \cdot (\tilde{\mathbf{X}}_w - \tilde{\mathbf{C}})$$

How do we write this transformation in homogeneous coordinates?

Modeling the coordinate system transformation

In heterogeneous coordinates, we have:

$$\tilde{\mathbf{X}}_c = \mathbf{R} \cdot (\tilde{\mathbf{X}}_w - \tilde{\mathbf{C}})$$

In homogeneous coordinates, we have: (pay attention to R and C dimension!)

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} \quad \text{or} \quad \mathbf{x}_c = \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{x}_w$$

Incorporating the transform in the camera matrix

The previous camera matrix is for homogeneous 3D coordinates in camera coordinate system:

$$\mathbf{x} = \mathbf{P}\mathbf{X}_c = \mathbf{K}[\mathbf{I}|0]\mathbf{X}_c$$

We also just derived:

$$\mathbf{X}_c = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{X}_w$$

Putting it all together

We can write everything into a single projection:

$$\mathbf{x} = \mathbf{P}\mathbf{X}_w$$

The camera matrix now looks like:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} [\mathbf{I} \quad | \quad \mathbf{0}] \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{bmatrix}$$

intrinsic parameters (3×3):
correspond to camera
internals (image-to-image
transformation)

perspective projection (3×4):
maps 3D to 2D points
(camera-to-image
transformation)

extrinsic parameters (4×4):
correspond to camera
externals (world-to-camera
transformation)

Putting it all together

We can write everything into a single projection:

$$\mathbf{x} = \mathbf{P}\mathbf{X}_w$$

The camera matrix now looks like:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & | & -\mathbf{RC} \end{bmatrix}$$

intrinsic parameters (3×3):
correspond to camera internals
(sensor not at $f = 1$ and origin shift)

extrinsic parameters (3×4):
correspond to camera externals
(world-to-image transformation)

General pinhole camera matrix

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

$$\mathbf{P} = \left[\begin{array}{ccc} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc|c} r_1 & r_2 & r_3 & t_1 \\ r_4 & r_5 & r_6 & t_2 \\ r_7 & r_8 & r_9 & t_3 \end{array} \right]$$

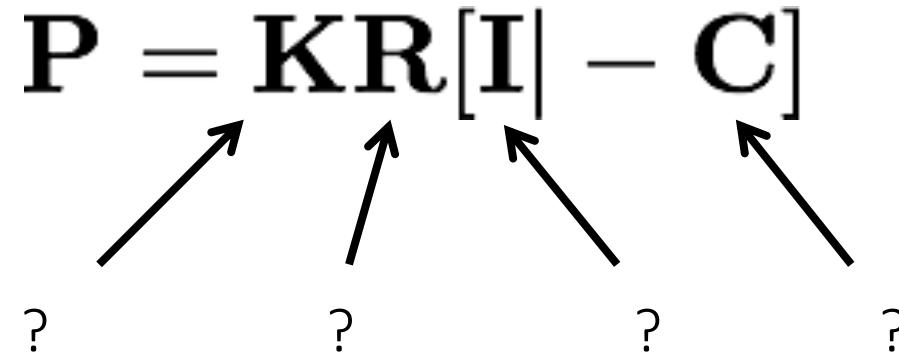
intrinsic extrinsic
parameters parameters

$$\mathbf{R} = \left[\begin{array}{ccc} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{array} \right] \quad \mathbf{t} = \left[\begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array} \right]$$

3D rotation 3D translation

Recap

What is the size and meaning of each term in the camera matrix?

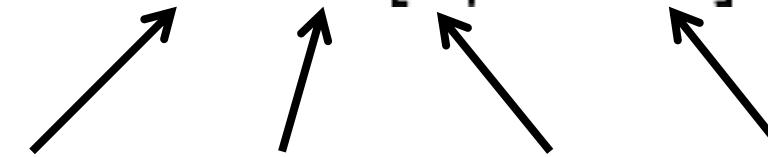
$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I}] - \mathbf{C}$$


The diagram consists of a mathematical equation $\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I}] - \mathbf{C}$ centered on the page. Below the equation, there are four question marks: one under \mathbf{K} , one under \mathbf{R} , one under $[\mathbf{I}]$, and one under \mathbf{C} . Four black arrows originate from these question marks and point diagonally upwards towards their respective terms in the equation above.

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I}] - \mathbf{C}$$



3x3

intrinsics

?

?

?

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I}] - \mathbf{C}$$

3x3
intrinsics 3x3
3D rotation ? ?

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I}] - \mathbf{C}$$

3x3 3x3 3x3 ?

intrinsics 3D rotation identity

Recap

What is the size and meaning of each term in the camera matrix?

$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I}] - \mathbf{C}$$

3x3

3x3

3x3

3x1

intrinsics

3D rotation

identity

3D translation

Quiz

The camera matrix relates what two quantities?

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homogeneous 3D points to 2D image points

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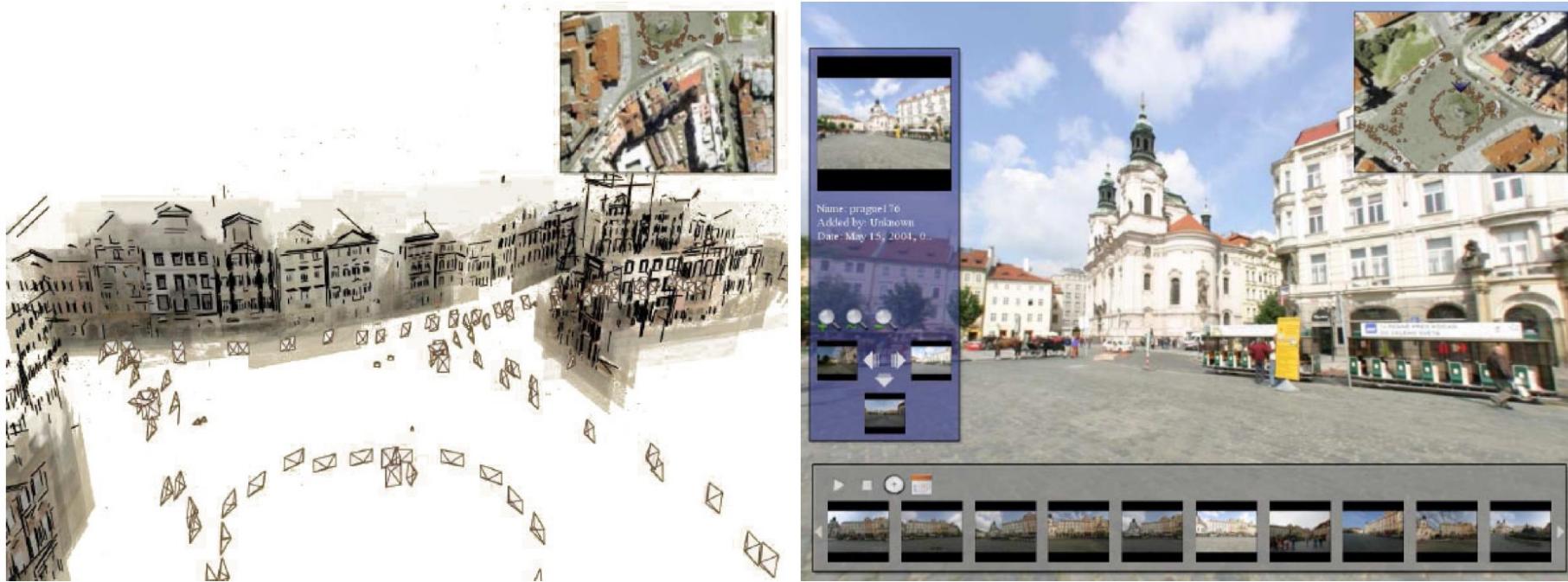
$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

intrinsic and extrinsic parameters



Geometric camera calibration (a.k.a. camera pose estimation)

Pose Estimation



Given a single image,
estimate the exact position of the photographer

Geometric camera calibration

Given a set of matched points

$$\{\mathbf{X}_i, \mathbf{x}_i\}$$

point in 3D
space point in the
image

and camera model

$$\mathbf{x} = \mathbf{f}(\mathbf{X}; \mathbf{p}) = \mathbf{P}\mathbf{X}$$

projection
model parameters Camera
matrix

Find the (pose) estimate of

$$\mathbf{P}$$

Mapping between 3D point and image points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What are the unknowns?

Mapping between 3D point and image points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \text{---} & \mathbf{p}_1^\top & \text{---} \\ \text{---} & \mathbf{p}_2^\top & \text{---} \\ \text{---} & \mathbf{p}_3^\top & \text{---} \end{bmatrix} \begin{bmatrix} | \\ \mathbf{X} \\ | \end{bmatrix}$$

Heterogeneous coordinates

$$x' = \frac{\mathbf{p}_1^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}} \quad y' = \frac{\mathbf{p}_2^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}}$$

(non-linear relation between coordinates)

How can we make these relations linear?

How can we make these relations linear?

$$x' = \frac{\mathbf{p}_1^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}} \quad y' = \frac{\mathbf{p}_2^\top \mathbf{X}}{\mathbf{p}_3^\top \mathbf{X}}$$

Make them linear with algebraic manipulation...

$$\mathbf{p}_2^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} y' = 0$$

$$\mathbf{p}_1^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} x' = 0$$

Now we can setup a system of linear equations
with multiple point correspondences

$$\mathbf{p}_2^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} y' = 0$$

$$\mathbf{p}_1^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} x' = 0$$

How do we proceed?

$$\mathbf{p}_2^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} y' = 0$$

$$\mathbf{p}_1^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} x' = 0$$

In matrix form ...

$$\begin{bmatrix} \mathbf{X}^\top & \mathbf{0} & -x' \mathbf{X}^\top \\ \mathbf{0} & \mathbf{X}^\top & -y' \mathbf{X}^\top \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{0}$$

How do we proceed?

$$\mathbf{p}_2^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} y' = 0$$

$$\mathbf{p}_1^\top \mathbf{X} - \mathbf{p}_3^\top \mathbf{X} x' = 0$$

In matrix form ...

$$\begin{bmatrix} \mathbf{X}^\top & \mathbf{0} & -x' \mathbf{X}^\top \\ \mathbf{0} & \mathbf{X}^\top & -y' \mathbf{X}^\top \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{0}$$

For N points ...

$$\begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -x' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -y' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -x' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -y' \mathbf{X}_N^\top \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{0}$$

*How do we solve
this system?*

Solve for camera matrix by

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -\mathbf{y}' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -\mathbf{y}' \mathbf{X}_N^\top \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

SVD!

Solve for camera matrix by

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -\mathbf{y}' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -\mathbf{y}' \mathbf{X}_N^\top \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Solution \mathbf{x} is the column of \mathbf{V}
corresponding to smallest singular
value of

$$\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^\top$$

Solve for camera matrix by

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}\|^2 \text{ subject to } \|\mathbf{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{X}_1^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_1^\top \\ \mathbf{0} & \mathbf{X}_1^\top & -\mathbf{y}' \mathbf{X}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{X}_N^\top & \mathbf{0} & -\mathbf{x}' \mathbf{X}_N^\top \\ \mathbf{0} & \mathbf{X}_N^\top & -\mathbf{y}' \mathbf{X}_N^\top \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Equivalently, solution \mathbf{x} is the
Eigenvector corresponding to
smallest Eigenvalue of

$$\mathbf{A}^\top \mathbf{A}$$

Now we have:

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix}$$

Are we done?

Almost there ...

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix}$$

How do you get the intrinsic and extrinsic parameters from the projection matrix?

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

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$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

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$$\begin{aligned}\mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}] - \mathbf{R}\mathbf{c} \\ &= [\mathbf{M}] - \mathbf{M}\mathbf{c}\end{aligned}$$

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$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

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Find the camera center **C**

What is the projection of the camera center?

Find intrinsic **K** and rotation **R**

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

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Find the camera center \mathbf{c}

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

How do we compute the camera center from this?

Find intrinsic \mathbf{K} and rotation \mathbf{R}

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\begin{aligned}\mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}] - \mathbf{R}\mathbf{c} \\ &= [\mathbf{M}] - \mathbf{M}\mathbf{c}\end{aligned}$$

Find the camera center \mathbf{c}

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of \mathbf{P} !

\mathbf{c} is the Eigenvector corresponding to
smallest Eigenvalue

Find intrinsic \mathbf{K} and rotation \mathbf{R}

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

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Find the camera center \mathbf{c}

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of P!

\mathbf{c} is the Eigenvector corresponding to
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Find intrinsic \mathbf{K} and rotation \mathbf{R}

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

*Any useful properties of K
and R we can use?*

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\begin{aligned}\mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}] - \mathbf{R}\mathbf{c} \\ &= [\mathbf{M}] - \mathbf{M}\mathbf{c}\end{aligned}$$

Find the camera center \mathbf{c}

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of P!

\mathbf{c} is the Eigenvector corresponding to
smallest Eigenvalue

Find intrinsic \mathbf{K} and rotation \mathbf{R}

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

right upper triangle orthogonal

*How do we find K
and R?*

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

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Find the camera center \mathbf{c}

$$\mathbf{P}\mathbf{c} = \mathbf{0}$$

SVD of P!

\mathbf{c} is the Eigenvector corresponding to
smallest Eigenvalue

Find intrinsic \mathbf{K} and rotation \mathbf{R}

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

QR decomposition

Geometric camera calibration

Given a set of matched points

$$\{\mathbf{X}_i, \mathbf{x}_i\}$$

point in 3D
space point in the
image

*Where do we get these
matched points from?*

and camera model

$$\mathbf{x} = \mathbf{f}(\mathbf{X}; \mathbf{p}) = \mathbf{P}\mathbf{X}$$

projection
model parameters Camera
matrix

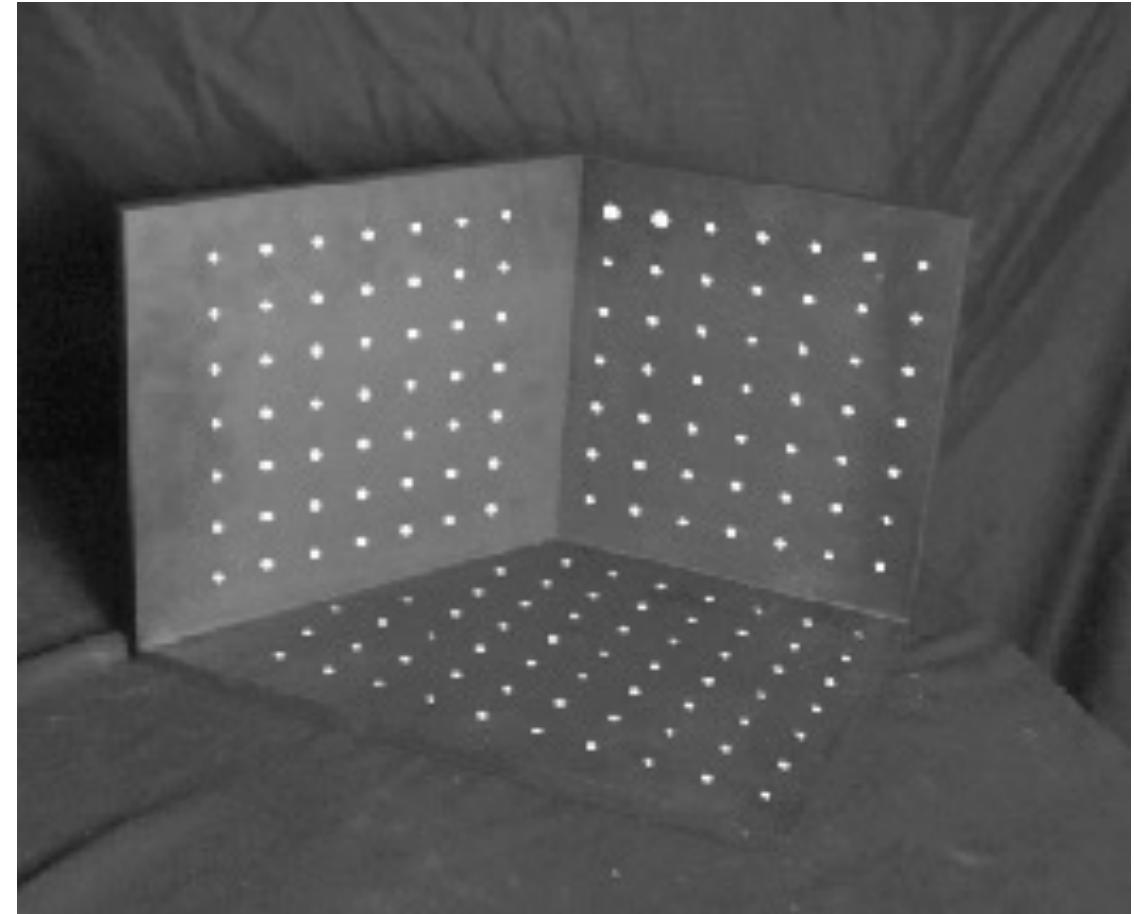
Find the (pose) estimate of

P

Calibration using a reference object

Place a known object in the scene:

- identify correspondences between image and scene
- compute mapping from scene to image



Issues:

- must know geometry very accurately
- must know 3D->2D correspondence

Geometric camera calibration

Advantages:

- Very simple to formulate.
- Analytical solution.

Disadvantages:

- Doesn't model radial distortion.
- Hard to impose constraints (e.g., known f).
- Doesn't minimize the correct error function.

For these reasons, *nonlinear methods* are preferred

- Define error function E between projected 3D points and image positions
 - E is nonlinear function of intrinsics, extrinsics, radial distortion
- Minimize E using nonlinear optimization techniques

What does it mean to “calibrate a camera”?

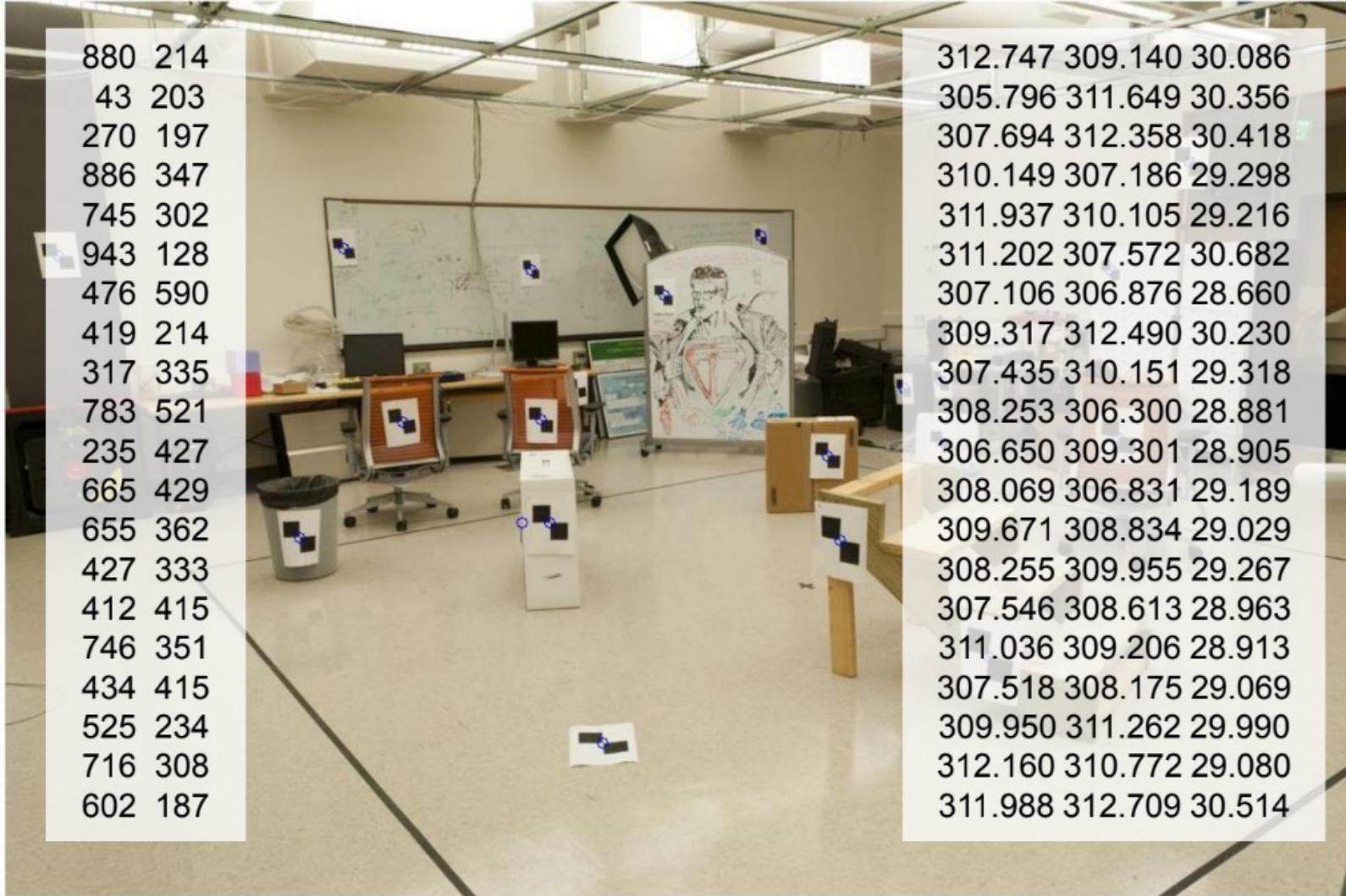
Many more different ways to calibrate a camera:

- Radiometric calibration.
 - straight lines will appear curved
- Color calibration.
- Geometric calibration.
- Noise calibration.
- Lens (or aberration) calibration.



Known 2d image coords

```
880 214
43 203
270 197
886 347
745 302
943 128
476 590
419 214
317 335
783 521
235 427
665 429
655 362
427 333
412 415
746 351
434 415
525 234
716 308
602 187
```



Known 3d world locations

```
312.747 309.140 30.086
305.796 311.649 30.356
307.694 312.358 30.418
310.149 307.186 29.298
311.937 310.105 29.216
311.202 307.572 30.682
307.106 306.876 28.660
309.317 312.490 30.230
307.435 310.151 29.318
308.253 306.300 28.881
306.650 309.301 28.905
308.069 306.831 29.189
309.671 308.834 29.029
308.255 309.955 29.267
307.546 308.613 28.963
311.036 309.206 28.913
307.518 308.175 29.069
309.950 311.262 29.990
312.160 310.772 29.080
311.988 312.709 30.514
```

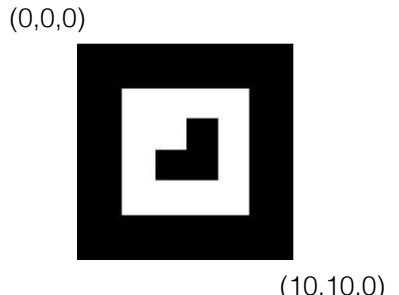
Known 2d
image cords
(px)

$$\begin{bmatrix} su \\ sv \\ s \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

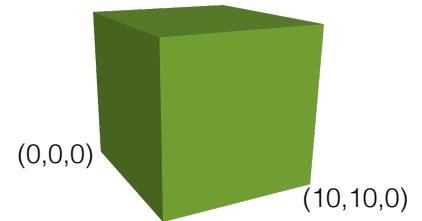
Unknown Camera Parameters



3D locations of planar marker features are known in advance



3D content prepared in advance



Simple AR program

1. Compute point correspondences (2D and AR tag)
2. Estimate the pose of the camera \mathbf{P}
3. Project 3D content to image plane using \mathbf{P}



<https://grandvisual.com/work/pepsi-max-bus-shelter/> (London, 2014)



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