CS2DB3 A5

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1.1 P1.1

To prove that if $X \to Y$ and $WY \to Z$, then $XW \to Z$ is sound. We start by assuming that $X \to Y$ and $WY \to Z$ and then we prove that we can get $XW \to Z$ using only the Armstrong Axioms.

$$\begin{aligned}
X &\to Y & & \\
WX &\to WY & & \text{by using Augmentation} \\
WX &\to Z & & \text{by using Transitivity } WY \to Z \\
XW &\to Z & & \text{by using Reflexivity } WX \to XW
\end{aligned}$$

Since $XW \to Z$ is proved.
$\therefore$ The first inference rule is sound.

1.2 P1.2

To prove that if $X \to YZ$ and $Z \to WV$, then $X \to YZV$ is sound. We start by assuming that we have rows $r_1, r_2 \in I$ such that $r_1[A] = r_2[A]$. Also like last time we assume that $X \to YZ$ and $Z \to WV$ and then prove $X \to YZV$.

- Using $X \to YZ$ and $r_1[X] = r_2[X]$, we conclude $r_1[YZ] = r_2[YZ]$
- By $r_1[YZ] = r_2[YZ]$, we have $r_1[Y] = r_2[Y]$ and $r_1[Z] = r_2[Z]$
- Using $Z \to WV$ and $r_1[Z] = r_2[Z]$, we conclude $r_1[WV] = r_2[WV]$
- By $r_1[WV] = r_2[WV]$, we have $r_1[W] = r_2[W]$ and $r_1[V] = r_2[V]$
- By $r_1[Y] = r_2[Y]$, $r_1[Z] = r_2[Z]$ and $r_1[V] = r_2[V]$, we have $r_1[YZV] = r_2[YZV]$

Since $X \to YZV$ is proved.
$\therefore$ The second inference rule is sound.

1.3 P1.3

To prove that if $X \to Y$ and $Z \subseteq Y$, then $X \to Z$ is sound. We start by assuming that $X \to Y$ and $Z \subseteq Y$ and then we prove $X \to Z$.

$X \to Y$	
$Y \to Z$	by using Reflexivity $Z \subseteq Y$
$X \to Z$	by using Transitivity $X \to Y$ and $Y \to Z$

Since $X \to Z$ is proved.
$\therefore$ The third inference rule is sound.

1.4 P1.4

The inference rules are not complete, since the axiom of reflexivity can not be proven.

1.5 P1.5

To show that the rules are independent, we only expect an indication of which functional dependencies cannot be derived without a particular rule.

- **Rule 1**: Without this rule, we can not derive $XW \to Z$, as the other two rules are not able to add attributes on the LHS of a functional dependency.
- **Rule 2**: Without this rule, we can not derive $X \to YZV$ as the other two rules are not able to combine functional dependencies on RHS
- **Rule 3**: Without this rule, we can not derive $X \to Z$ as the other two rules are not able to take the subset of an attribute

Since we have shown that some functional dependencies cannot be derived without a particular rule. Therefore the rules are independent.

1.6 P2.1

For Inference Rule 1, since $Y \subseteq Y$, therefore by reflexivity we can get $Y \to Y$. Therefore the inference rule is sound.

For Inference Rule 2, let's start by assuming $X \to Y$, $Y \to Z$, and $\alpha \subseteq \beta$ are true then we prove that $X\beta \to Z\alpha$ holds.

$$\begin{array}{ll}
\beta \to \alpha & \text{by using Reflexivity } \alpha \subseteq \beta \\
\beta Z \to \alpha Z & \text{by using Augmentation } \beta \to \alpha \\
Z\beta \to Z\alpha & \text{by using Reflexivity } \beta Z \to Z\beta \text{ and } \alpha Z \to Z\alpha \\
\\
X \to Z & \text{by using Transitivity } X \to Y \text{ and } Y \to Z \\
X\beta \to Z\beta & \text{by using Augmentation } X \to Z \\
X\beta \to Z\alpha & \text{by using Transitivity } X\beta \to Z\beta \text{ and } Z\beta \to Z\alpha
\end{array}$$

Since $X\beta \to Z\alpha$ is proved.
$\therefore$ The inference rule is sound.

1.7 P2.2

To prove that all inference rules are complete. We start by assuming that all inference rules are true and show that we can derive the Armstrong Axioms from them. As we already know that Armstrong Axioms are always complete.

Prove Reflexivity:
To prove reflexivity, let's assume $Y \subseteq X$ is true. Then let's take two empty sets which can be $\emptyset \to \emptyset$ and $\emptyset \to \emptyset$ by using the Inference rule 1. Now if we take all of that and apply inference rule 2, we get $X \to Y$ which is true and therefore proves reflexivity

Prove Transitivity:
To prove transitivity, let's assume $X \to Y$ and $Y \to Z$ are true and we know that $\emptyset \subseteq \emptyset$. So by applying inference rule 2, we get $X \to Z$ which is true and therefore proves transitivity

Prove Augmentation:
To prove augmentation, let's assume $X \to Y$ is true, and we know any set can derive itself so, let's take $X \to X$. Lastly, let's take an arbitrary set, W for which we know that $W \subseteq W$. So by applying inference rule 2, we get $XW \to YW$ which is true and therefore proves augmentation

1.8 P2.3

To show that the rules are independent, we only expect an indication of which functional dependencies cannot be derived without a particular rule.

- **Rule 1**: Without this rule, we can not derive $Y \to Y$, as the other rule requires functional dependencies to start with.
- **Rule 2**: Without this rule, we can not derive $X \to Z$ where $X \neq Z$, as the inference rule 1 can not get a functional dependency such that $X \neq Z$

Since we have shown that some functional dependencies cannot be derived without a particular rule. Therefore the rules are independent.

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2.1 P3.1

Before we do anything, we first make the $\mathfrak{S}$ minimalist by decomposing functional dependencies to make it easier to get the closure. So we get:

$\mathfrak{S} = \{A \to C, A \to D, A \to E, BD \to E, B \to D, C \to D, D \to A, D \to B, E \to B\}$

To get C^+:

- We start by adding C as any attribute is a subset of itself. {C}
- We add D as we have $C \to D$. {C, D}
- We add A as we have $D \to A$. {C, D, A}
- We add B as we have $D \to B$. {C, D, A, B}
- We add C as we have $A \to C$. {C, D, A, B, C}

To get EA^+:

- We start by adding E, A as any attribute is a subset of itself. {E, A}
- We add C as we have $A \to C$. {E, A, C}
- We add D as we have $C \to D$. {E, A, C, D}
- We add B as we have $D \to B$. {E, A, C, D, B}

2.2 P3.2

To get $\mathfrak{S}^+$:

- We start by taking all the attribute's closure $ABCDE^+ = \{A, B, C, D, E\}$ so this is our super key
- Next we try to take out attributes that can be derived, so we can remove $A \to C, A \to D, A \to E$ which will still get us the super key $AB^+ = \{A, B, C, D, E\}$
- Lastly we can remove B since A can derive D and D can derive A. $A^+ = \{A, B, C, D, E\}$. This is a key

- Now we look at the RHS of $\mathfrak{S}$ for A and we find $D \to A$ which means that $D^+ = \{A, B, C, D, E\}$ is also key since its proper subset is not a super key.
- Now again we look at the RHS of $\mathfrak{S}$ for D and we find $C \to D$ which means that $C^+ = \{A, B, C, D, E\}$ is also key since its proper subset is not a super key.
- We keep doing this, so look at the RHS of $\mathfrak{S}$ for D and we find $B \to D$ which means that $B^+ = \{A, B, C, D, E\}$ is also key since its proper subset is not a super key.
- Lastly, we look at the RHS of $\mathfrak{S}$ for B and we find $E \to B$ which means that $E^+ = \{A, B, C, D, E\}$ is also key since its proper subset is not a super key.

So in the end, we have that the $\mathfrak{S}^+$ is $\{A \to ABCDE, B \to ABCDE, C \to ABCDE, D \to ABCDE, E \to ABCDE\}$

The keys, as described above are:

- $A^+ = \{A, B, C, D, E\}$ since its proper subset is not a super key.
- $B^+ = \{A, B, C, D, E\}$ since its proper subset is not a super key.
- $C^+ = \{A, B, C, D, E\}$ since its proper subset is not a super key.
- $D^+ = \{A, B, C, D, E\}$ since its proper subset is not a super key.
- $E^+ = \{A, B, C, D, E\}$ since its proper subset is not a super key.

The super keys are going to be a combination of the keys but since all the attributes are the keys. Therefore any combination of the attributes will be a super key. To prove this let's say we take Q to be some combination of attributes. Then, for some $V_1, \ldots, V_n \in \{A, B, C, D, E\}$ where $1 \leq n \leq 5$, we have that $Q = V_1 \cdots V_n$ with $V_1, \ldots, V_n$ keys. Then, Closure $(\mathfrak{S}, Q)$ will always be a subset of $\{A, B, C, D, E\}$. Therefore proving that any combination of the attributes will always be a super key.

2.3 P3.3

To find the minimal cover of $\mathfrak{S}^+$, we determine whether any functional dependencies can be removed while maintaining the keys.

$\mathfrak{S} = \{A \to C, A \to D, A \to E, BD \to E, B \to D, C \to D, D \to A, D \to B, E \to B\}$

- If we remove $A \to C$ then the $A^+ = \{A, D, E, B\}$. Since it is not the same that means $A \to C$ can not be removed

- If we remove $A \to E$ then the $A^+ = \{A, E, B, D, C\}$. Since it is the same that means $A \to E$ can be removed
- If we remove $A \to D$ then the $A^+ = \{A, C, D, B, E\}$. Since it is the same that means $A \to D$ can be removed
- If we remove $BD \to E$ then the $BD^+ = \{B, D, A, C\}$. Since it is not the same that means $BD \to E$ can not be removed
- If we remove $B \to D$ then the $B^+ = \{B\}$. Since it is not the same that means $B \to D$ can not be removed
- If we remove $C \to D$ then the $C^+ = \{C\}$. Since it is not the same that means $C \to D$ can not be removed
- If we remove $D \to A$ then the $D^+ = \{D, B, E\}$. Since it is not the same that means $D \to A$ can not be removed
- If we remove $D \to B$ then the $D^+ = \{D, A, C\}$. Since it is not the same that means $D \to B$ can not be removed
- If we remove $E \to B$ then the $E^+ = \{E\}$. Since it is not the same that means $E \to B$ can not be removed

Therefore the minimal cover of $\mathfrak{G} = \{A \to C, BD \to E, B \to D, C \to D, D \to A, D \to B, E \to B\}$

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3.1 P4.1

The inference rule 1 is not sound. Here is a counterexample for if $A \twoheadrightarrow BC$, then $A \twoheadrightarrow B$:

A	B	C	D
1	X	W	Y
1	R	S	T
1	X	W	T
1	R	S	Y

In the above table, $A \twoheadrightarrow B$ can not hold as we have $r_1[A] = r_2[A]$ but none of the rows have $r_1[AB] = r_2[AB]$. Therefore $A \twoheadrightarrow B$ does not hold.

The inference rule 2 is sound. So let's start by making the R as a relation schema with attributes A, B, C, and D, and suppose $A \twoheadrightarrow BCD$ and $A \to B$ are true. Then, let r_1 and r_2 be tuples in R such that $r_1[A] = r_2[A]$. Since $A \to B$, we have $r_1[B] = r_2[B]$. Since $A \twoheadrightarrow BCD$, we have $r_1[BCD] = r_2[BCD]$. By using the definition of functional dependency, we get that $r_1[C] = r_2[C]$ and $r_1[D] = r_2[D]$. Therefore, $A \twoheadrightarrow CD$ holds in R.

The inference rule 3 is sound. So let's start by making the R as a relation schema with attributes A and B, and S as a relation schema with attributes C and C. Suppose $R[AB] \subseteq S[CC]$. Let r be a tuple in R and s be a tuple in S. Since $R[AB] \subseteq S[CC]$, we have $r[A] = s[C]$ and $r[B] = s[C]$ for some tuple s in S. Therefore, $r[A] = s[C]$, which implies that $t[A] \in S[C]$. Since $S[C] \subseteq S[CC]$, we have $r[A] \in S[CC]$. Hence, $r \in S$ and $r[A] = s[C] = s[C] = s[B]$. Therefore, $r[A] = s[B]$, which implies that $r[A] \in S[B]$. Therefore, $R[A] \subseteq S[B]$ holds.

The inference rule 4 is sound. So let's start by making the R as a relation schema with attributes A and B, and S as a relation schema with attributes C and C. Suppose $R[AB] \subseteq S[CC]$. Let r_1 and r_2 be tuples in R such that $r_1[A] = r_2[A]$. Since we know $R[AB] \subseteq S[CC]$, we have $r_1[A] = r_2[A] = s[C]$ and $r_1[B] = r_2[B] = s[C]$ for some tuple s in S. Therefore, $r_1[B] = s[C]$, which implies that $r_1[B] \in S[C]$. Therefore proving that $A \to B$ is in R.