

Complex Analysis Final Study Guide

Definitions + Facts

$$z = r e^{i\theta} = r(\cos \theta + i \sin \theta)$$

$$\arg(z) = \theta$$

$$|e^{i\theta}| = 1$$

$$\log z = \log |z| + i \arg(z)$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

Cauchy Riemann Equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Harmonic Functions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \leftarrow \text{All analytic functions are harmonic via CR equations}$$

Harmonic Conjugate:

Suppose u is harmonic, v is its harmonic conjugate if $h = u + iv$ is analytic and v is harmonic

Closed path:

starts and ends at the same point

Exact differentials:

For some holomorphic h , $dh = P dx + Q dy$

Closed differentials:

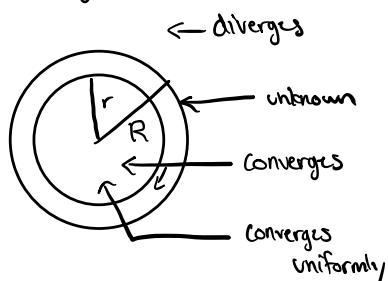
$$\frac{dP}{dy} = \frac{dQ}{dx}$$

Path independence, closed, and exact differentials

Path independent \Leftrightarrow Exact \Rightarrow closed

Star-shaped domains
path independent \Leftrightarrow Exact \Leftrightarrow closed

Radius of Convergence:



Power Series Expansion:

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k, \quad |z-z_0| < r \quad ; \quad \text{when } R \geq r$$

$$a_k = \frac{f^{(k)}(z_0)}{k!} \quad ; \quad a_k = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(s)}{(s-z_0)^{k+1}} ds \quad ; \quad 0 \leq r \leq R$$

Isolated Singularities:

z_0 is an isolated singularity if $f(z)$ is analytic in some punctured disk around z_0 .

Removable Singularity:

z_0 is a removable singularity if $a_k=0$ for all $k < 0$

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \quad 0 < |z-z_0| < r$$

Pole Singularity:

An isolated singularity is a pole if there exists $N > 0$ s.t. $a_{-N} \neq 0$ but $a_k=0$ for all $k < -N$. N is the order of the pole

$$f(z) = \sum_{k=-N}^{\infty} a_k (z-z_0)^k$$

Meromorphic Functions:

$f(z)$ is analytic on D except for possible isolated singularities which are poles

Essential Singularity:

$a_k \neq 0$ for infinitely many $k < 0$

Isolated Singularities at ∞ :

f has isolated singularity if $g(w) = f(1/w)$ has an isolated singularity at $w=0$

$$f(z) = \sum_{k=-\infty}^{\infty} b_k z^k \quad |z| > R$$

Removable: $b_k = 0$ for all $k > 0 \rightarrow f(z)$ is analytic at ∞

Essential: $b_k \neq 0$ for infinitely many $k > 0$

Pole: Pole of order N at ∞ if $b_N \neq 0$ while $b_k=0$ for $k > N$

Principle Part of $f(z)$

Negative powers; positive powers for ∞

Residue:

The residue of $f(z)$ at z_0 is a_{-1} or the coefficient of $\frac{1}{z-z_0}$

Residue Calculation Rules

Rule 1: $f(z)$ has a simple pole at z_0

$$\text{Res}[f(z), z_0] = \lim_{z \rightarrow z_0} (z-z_0) f(z)$$

Rule 2: $f(z)$ has a double pole at z_0

$$\text{Res}[f(z), z_0] = \lim_{z \rightarrow z_0} \frac{d}{dz} [(z-z_0)^2 f(z)]$$

Rule 3: $f(z)$ and $g(z)$ are analytic at z_0 ; $g(z)$ has a simple zero at z_0

$$\text{Res}\left[\frac{f(z)}{g(z)}, z_0\right] = \frac{f(z_0)}{g'(z_0)}$$

Rule 4: $g(z)$ is analytic and has a simple zero at z_0

$$\text{Res}\left[\frac{1}{g(z)}, z_0\right] = \frac{1}{g'(z_0)}$$

Principal Value:

$$\text{PV} \int_a^b f(x) dx = \lim_{\epsilon \rightarrow 0} \left(\int_a^{x_0-\epsilon} + \int_{x_0+\epsilon}^b \right) f(x) dx$$

Univalent: A function on domain D that is analytic and one-to-one

Winding Number:

$$W(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0} = \frac{1}{2\pi} \int_{\gamma} d\arg(z - z_0)$$

$W(\gamma, s) = 0$ for all s in the unbound component

Winding number increases as s crosses ∂D to D

Proofs/Theorems:

Green's Theorem

Let P and Q be continuously differentiable functions on $D \cup \partial D$.

$$\int_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

Path Independence and Exact differentials

$\int P dx + Q dy$ is independent of path in D iff $P dx + Q dy$ is exact, $dh = P dx + Q dy$

Proof: ← Proof incomplete

$\int P dx + Q dy$ is path independent \Rightarrow $P dx + Q dy$ is exact

$$\int_a^b P dx + Q dy$$

Exact differentials are closed

Exact differentials: $dh = P dx + Q dy$

$$P = \frac{dh}{dx} \quad Q = \frac{dh}{dy}$$

$$\frac{\partial P}{\partial y} = \frac{\partial h}{\partial x \partial y} = \frac{\partial Q}{\partial x} \quad \rightarrow \text{closed } \checkmark$$

Closed differentials are exact on star-shaped domains

Proof: define $h(B) = \int_A^B P dx + Q dy$

Let $B = (x_0, y_0)$ and $C = (x_1, y_1)$

$$\int_A^B + \int_B^C + \int_C^A (P dx + Q dy) = 0$$

$$\int_A^B - \int_A^C P dx + Q dy = \int_C^B P dx + Q dy$$

\downarrow Parameterizing path

$$h(B) - h(C) = \int_{x_1}^{x_0} P(t, y_0) dt$$

From FTC:

$$\frac{\partial h}{\partial x}(x_0, y_0) = P(x_0, y_0) \quad \frac{\partial h}{\partial y}(x_0, y_0) = Q(x_0, y_0)$$

Analogous proof

$Pdx + Qdy$ is exact

Harmonic Conjugate on a star-shaped domain

$$V(B) = \int_A^B -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

Proof: ← Proof Incomplete

Mean Value Property

Let $h(z)$ be a continuous real valued function on a domain D . Let $z_0 \in D$ and suppose D contains the disk $\{|z-z_0| < r\}$

Average value of $h(z)$ on the circle $\{|z-z_0|=r\}$

$$A(r) = \frac{1}{2\pi} \int_0^{2\pi} h(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} \quad 0 < r < p$$

As $r \rightarrow 0$ $A(r) \rightarrow h(z_0)$

If $u(z)$ is a harmonic function on a domain D and if the disk $\{|z-z_0| < r\}$ is contained in D , then

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} \quad 0 < r < p$$

Average value at the boundary is its value at the center

$$0 = \oint_{\{|z-z_0|=r\}} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

closed differential = 0 via green's theorem

Parameterizing on a circle
 $x = r \cos \theta \quad dx = -r \sin \theta$
 $y = r \sin \theta \quad dy = r \cos \theta$

$$0 = r \int_0^{2\pi} \left[\frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \right] d\theta = r \int_0^{2\pi} \frac{\partial u}{\partial r} (z_0 + re^{i\theta}) d\theta$$

Interchange differentiation and integration since u is smooth

$$0 = \frac{\partial}{\partial r} r \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta \stackrel{\text{divide by } \frac{2\pi r}}{\Rightarrow} 0 = \frac{\partial}{\partial r} \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi}$$

For $0 < r < p$

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta \text{ is constant so as } r \rightarrow 0 \quad u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

Maximum Principle

Strict Maximum Principle:

Real Version: Let $u(z)$ be a real valued harmonic function on a domain D s.t. $u(z) \leq M \quad \forall z \in D$.

If $u(z_0) = M$ for some $z_0 \in D$ then $u(z) = M \quad \forall z \in D$.

Suppose $u(z_0) = M$

$$0 = \int_0^{2\pi} u(z_0) - u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} \quad 0 < r < p$$

Since the integrand is non-negative and continuous this can only be true if the integrand is 0

$$u(z_0) = u(z_0 + re^{i\theta}) = M$$

Therefore there exists a disk centered around each point in the set $\{u(z) = M\}$. Therefore we can determine that it is an open set.

Since D is either $\{u(z) < M\}$ or $\{u(z) = M\}$ we know that one must be empty if the other is open. ■

Complex Version:

Let h be a bounded complex-valued harmonic function on a domain D . If $|h(z)| \leq M$ for all $z \in D$, and $|h(z_0)| = M$ for some $z_0 \in D$ then $h(z)$ is constant on D .

Suppose $|h(z_0)| = M$. Let λ be a constant s.t. $\lambda h(z) = M$. λ is also unimodular

Let $u(z) = \operatorname{Re}(\lambda h(z))$ where u is definitionally harmonic and real-valued.

$u(z_0) = M$ and by the real version of the strict maximum principle we find that

$$u(z) = M.$$

Since $|\lambda h(z)| \leq M$ and $u(z) = M$ we conclude that $\operatorname{Im}(\lambda h(z)) = 0$.

Therefore $\lambda h(z)$ is constant and by extension $h(z)$ as well.

Maximum Principle:

Let $h(z)$ be a complex valued harmonic function on a bounded domain D such that $h(z)$ extends continuously to the boundary ∂D . If $|h(z)| \leq M \quad \forall z \in \partial D$, then $|h(z)| \leq M$ for all $z \in D$

Fundamental Theorem of Calculus for Analytic Functions

FTC Part 1: If $f(z)$ is continuous on domain D , and if $F(z)$ is primitive for $f(z)$

$$\int_A^B f(z) dz = F(B) - F(A)$$

$$F'(z) = \frac{\partial F}{\partial x} + i \frac{\partial F}{\partial y}$$

$$F(B) - F(A) = \int_A^B dF = \int_A^B \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = \int_A^B F'(z) (dx + idy) = \int_A^B F'(z) dz$$

FTC Part 2: Let D be a star-shaped domain and let $f(z)$ be analytic on D . Then $f(z)$ has a primitive on D

and the primitive is unique up to adding a constant. The primitive of $f(z)$:

$$F(z) = \int_{z_0}^z f(\bar{z}) d\bar{z} \quad z \in D \quad \text{for a fixed point } z_0$$

Let $f(z) = u(z) + iv(z)$.

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{from C-R equations}$$

Therefore $u dx - v dy$ is closed and exact.

Since $u dx - v dy$ is exact we know $dU = u dx - v dy$ for a continuously differentiable function U .

U is also harmonic

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \quad \text{via C-R equations}$$

For a harmonic function on a star-shaped domain we know there exists a harmonic conjugate V

$$G = U + iV$$

$$G' = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y} = u + iv = f$$

$\xrightarrow{\text{C-R equations}}$

Cauchy's Theorem

Lemma: A continuously differentiable function $f(z)$ on D is analytic iff the differential $f(z)dz$ is closed

$$f(z) = u + iv$$

$$f(z)dz = (u+iv)(dx+idy) = (u+iv)dx + (-v+iu)dy$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ from CR equations}$$

$$\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} + -\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} = 0 \Rightarrow f(z)dz \text{ is closed}$$

Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is analytic on D that extends smoothly to ∂D , then

$$\int_D f(z)dz = 0$$

$$f(z) \text{ is analytic} \Rightarrow f(z)dz \text{ is closed} \Rightarrow \int_D f(z)dz = 0$$

Lemma Green's
Theorem

Cauchy Integral Formula

Let D be a bounded domain with a piecewise smooth boundary. If $f(z)$ is analytic on D and $f(z)$ extends smoothly to the boundary of D , then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{z-w} dw, \quad z \in D$$

For a fixed point z in D , let $D_\epsilon = D \setminus \{w : |w-z| \leq \epsilon\}$ for small $\epsilon > 0$

∂D_ϵ is then the union of ∂D and $\{w : |w-z| = \epsilon\}$

Since $z \notin D_\epsilon$, $\frac{f(w)}{w-z}$ is analytic on the domain

Applying Cauchy Formula:

$$\begin{aligned} \int_{\partial D_\epsilon} \frac{f(w)}{w-z} dw &= 0 \\ \int_{\partial D_\epsilon} \frac{f(w)}{w-z} dw &= \int_{\partial D} \frac{f(w)}{w-z} dw - \int_{|w-z|=\epsilon} \frac{f(w)}{w-z} dw \quad \Rightarrow \quad \int_{\partial D} \frac{f(w)}{w-z} dw = \int_{|w-z|=\epsilon} \frac{f(w)}{w-z} dw \end{aligned}$$

Change of Variable

$$w = z + \epsilon e^{i\theta} \quad dw = \epsilon i e^{i\theta} d\theta$$

$$\int_0^{2\pi} \frac{f(z + \epsilon e^{i\theta})}{\cancel{\epsilon e^{i\theta}}} \cdot \cancel{\epsilon i e^{i\theta}} d\theta$$

Mean value property:

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) d\theta$$

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} dw$$

Cauchy Integral Formula for Derivatives

Let D be a bounded domain with piecewise smooth boundary. If $f(z)$ is an analytic function on D that extends smoothly to the boundary of D , then $f(z)$ has complex derivatives of all orders on D .

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}} dw \quad z \in D, m \geq 0$$

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{\Delta z} \cdot \frac{1}{2\pi i} \left[\int_{\partial D} \frac{f(w)}{w-z-\Delta z} dw - \int_{\partial D} \frac{f(w)}{w-z} dw \right]$$

$$\frac{1}{2\pi i} \int_{\partial D} f(w) \cdot \frac{1}{(w-z-\Delta z)(w-z)} dw$$

As $\Delta z \rightarrow 0$

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^2} dw$$

Use induction and binomial expansion for higher order derivatives

Liouville's Theorem

Cauchy Estimate:

Suppose $f(z)$ is analytic for $|z - z_0| \leq p$. If $|f(z)| \leq M$ for $|z - z_0| = p$

then

$$|f^{(m)}(z_0)| \leq \frac{m!}{p^m} M$$

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{|z-z_0|=p} \frac{f(z)}{(z-z_0)^{m+1}} dz$$

Parameterize z

$$z = z_0 + pe^{i\theta} \quad dz = p \cdot e^{i\theta} i \cdot d\theta$$

$$\begin{aligned} f^{(m)}(z_0) &= \frac{m!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + pe^{i\theta})}{p^{m+1} e^{im\theta}} pe^{i\theta} i \cdot d\theta \\ &= \frac{m!}{2\pi} \cdot \int_0^{2\pi} \frac{f(z_0 + pe^{i\theta})}{p^m e^{im\theta}} \frac{d\theta}{2\pi} \end{aligned}$$

$\leq M$

Let $f(z)$ be an analytic function on the complex plane. If $|f(z)|$ is bounded, then it is constant.

Suppose $|f(z)| \leq M \quad \forall z \in \mathbb{C}$

From the Cauchy estimate:

$$|f'(z_0)| \leq \frac{M}{r} \quad \text{for arbitrary disk size } r \text{ and } z_0$$

As $r \rightarrow \infty$, $|f'(z)| = 0$ so $f'(z) = 0$ and f is constant.

Morera's Theorem

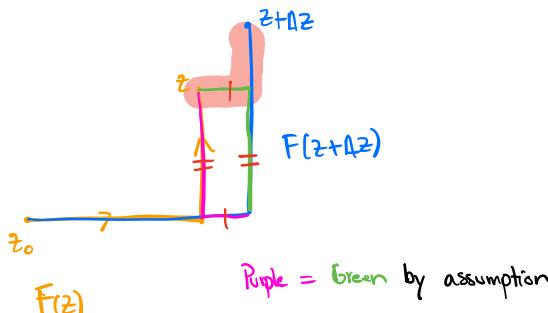
Let $f(z)$ be a continuous function on a domain D . If $\int_R f(z) dz = 0$ for every closed rectangle R contained in D with sides parallel to the coordinate axis's, then $f(z)$ is analytic on D .

Assume D is a disk with center at z_0

$$\text{Let } F(z) = \int_{z_0}^z f(s) ds \quad z \in D \quad \text{where the integration path is horizontal and then vertical}$$

$$F(z + \Delta z) - F(z) = \int_z^{z + \Delta z} f(s) ds$$

\underbrace{z}_z
red portion



Add constant $f(z)$ and subtract $f(z)$ to evaluate RHS

$$\begin{aligned} F(z + \Delta z) - F(z) &= f(z) \int_z^{z + \Delta z} ds + \int_z^{z + \Delta z} f(s) - f(z) ds \\ &= f(z) \Delta z + \int_z^{z + \Delta z} f(s) - f(z) ds \end{aligned}$$

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_z^{z + \Delta z} f(s) - f(z) ds$$

Applying ML-estimates

$$\int_z^{z + \Delta z} f(s) - f(z) ds \leq 2|\Delta z| \cdot M_\epsilon \quad \text{where } M_\epsilon \text{ is maximum of } |f(s) - f(z)|$$

Therefore, $\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| \leq 2M_\epsilon$ which approaches 0 as $\Delta z \rightarrow 0$.

$f(z)$ is therefore analytic since $f(z)$ is continuous, $F(z)$ is analytic and $F'(z) = f(z)$.

Goursat's Theorem

If $f(z)$ is a complex valued function on a domain D such that $f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists at each point of D then $f(z)$ is analytic on D .

"Goursat's Theorem is as useless as it is aesthetically pleasing"

Let R be a closed rectangle in D . Divide R into 4 sub rectangles

$$\int_{\partial R} f(z) dz = \int_{\partial R_1} + \int_{\partial R_2} + \int_{\partial R_3} + \int_{\partial R_4} f(z) dz$$

so we can say for at least one sub rectangle R_i

$$\left| \int_{\partial R_i} f(z) dz \right| \geq \frac{1}{4} \left| \int_{\partial R} f(z) dz \right|$$

Repeating this procedure n times we find

$$\left| \int_{\partial R_n} f(z) dz \right| = \frac{1}{4} \left| \int_{\partial R_m} f(z) dz \right| = \dots = \frac{1}{4^n} \left| \int_{\partial R} f(z) dz \right|$$

Eventually R_n approaches to some point in z_0 which we know to be differentiable

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \leq \epsilon_n \quad z \in R_n$$

$\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$

If L is the length of ∂R then $\partial R_n = \frac{L}{2^n}$

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon_n |z - z_0| \leq \frac{2\epsilon_n L}{2^n}$$

ML-estimate

$$\left| \int_{\partial R_n} f(z) dz \right| = \left| \int_{\partial R_n} [f(z) - f(z_0) - f'(z_0)(z - z_0)] dz \right| \leq \frac{2\epsilon_n L}{2^n} \cdot \left(\frac{L}{2^n} \right) = \frac{2L^2}{4^n} \epsilon_n$$

$$\left| \int_{\partial R} f(z) dz \right| \leq 4^n \left| \int_{\partial R_n} f(z) dz \right| \leq 2L^2 \epsilon_n \quad \text{but as } \epsilon_n \rightarrow 0 \text{ with } n \rightarrow \infty \text{ we find } \int_{\partial R} f(z) dz = 0$$

Now apply Morera's Theorem to finalize $f(z)$ is analytic

Weierstrass M-Test

Suppose $M_k \geq 0$ and $\sum M_k$ converges. If $g_k(x)$ are complex valued functions on a set E such that $|g_k(x)| \leq M_k$ for all $x \in E$, then $\sum g_k(x)$ converges uniformly on E .

For each fixed x we have that $\sum g_k(x)$ is absolutely convergent and $\sum |g_k(x)| \leq \sum M_k$
 $\sum g_k(x)$ converges to some $g(x)$ s.t. $|g(x)| \leq \sum |g_k(x)| \leq \sum M_k$

Considering the tail of the series

$$|g(x) - S_n(x)| = \sum_{k=n+1}^{\infty} g_k(x) \leq \sum_{k=n+1}^{\infty} M_k$$

If $\epsilon_n = \sum_{k=n+1}^{\infty} M_k$ as $n \rightarrow \infty$ $\epsilon_n \rightarrow 0$ and we find $S_n(x)$ converges uniformly to $g(x)$

Ratio Test

If $\left| \frac{a_k}{a_{k+1}} \right|$ has a limit as $k \rightarrow \infty$, either finite or $+\infty$, then the limit is the radius of convergence R of $\sum a_k z^k$

$$R = \lim_{k \rightarrow \infty} \left| \frac{a_k}{a_{k+1}} \right|$$

Let $L = \lim_{k \rightarrow \infty} \left| \frac{a_k}{a_{k+1}} \right|$. If $r < L$ then $\left| \frac{a_k}{a_{k+1}} \right| > r$ for $k \geq N$.

Then $|a_k| > r |a_{k+1}|$ for $k \geq N$

$$|a_N|r^N \geq |a_{N+1}|r^{N+1} \geq |a_{N+2}|r^{N+2} \geq \dots \quad |a_k|r^k \text{ is bounded}$$

By definition of R we have that for $r \geq R$ and since $r < L$ is arbitrary, $L \leq R$

Suppose $s > L$. Then $\left| \frac{a_k}{a_{k+1}} \right| < s$ and eventually for $k \geq N$, $|a_k| < s |a_{k+1}|$

$$|a_N|s^N \leq |a_{N+1}|s^{N+1} \leq |a_{N+2}|s^{N+2} \leq \dots \quad \text{and } a_k z^k \text{ does not converge to 0 for } |z| \geq s$$

Since $s \geq R$ and $s > L$ is arbitrary we conclude $L \geq R$ and finally $L = R$

Cauchy-Hammond Formula

If $\sqrt[k]{|a_k|}$ has a limit as $k \rightarrow \infty$, then the radius of convergence of $\sum a_k z^k$ is given by

$$R = \frac{1}{\limsup \sqrt[k]{|a_k|}}$$

\limsup is the value at which there are finitely many terms greater than t and infinitely < t

Proof of simplified version : $R = \frac{1}{\lim \sqrt[k]{|a_k|}}$

If $r > \frac{1}{\lim \sqrt[k]{|a_k|}}$, then $\sqrt[k]{|a_k|} r > 1$ eventually.

$|a_n|r^n = 1$ and terms of $\sum a_n z^n$ do not converge to 0 for $|z|=r$ and $r \geq R$

If $r < \frac{1}{\lim \sqrt[k]{|a_k|}}$ we see that $|a_n|r^n < 1$ is bounded and we see that $R = \frac{1}{\lim \sqrt[k]{|a_k|}}$

Identity Principle

If D is a domain and $f(z)$ is an analytic function on D that is not identically zero, then the zeros of $f(z)$ are isolated.

Let U be the set of all $z \in D$ such that $f^{(m)}(z) = 0$ for all $m \geq 0$.

If $z_0 \in U$, then the power series expansion $f(z) = \sum a_k(z-z_0)^k$ has $a_k = \frac{f^{(k)}(z_0)}{k!} = 0$

Therefore, $f(z) = 0$ for z belonging to a disk centered at z_0

\hookrightarrow points in the disk also exist in U

$\hookrightarrow U$ is an open set

If $z_0 \in D \setminus U$ then $f^{(k)}(z_0) \neq 0$ and $f^{(k)}(z) \neq 0$ for some disk around z_0 showing that $D \setminus U$ is also open.

Since D is connected, either $U=D$ or U is empty.

If $U=D$, we find that $f(z)=0$ on D which contradicts our hypothesis so U must be empty.

Suppose z_0 is a zero of $f(z)$ with finite order N .

$f(z) = (z-z_0)^N h(z)$ where $h(z)$ is analytic at z_0 and $h(z_0) \neq 0$

For small $p > 0$ we have $h(z) \neq 0$ for $|z-z_0| < p$, thus each zero of $f(z)$ is isolated.

Uniqueness Principle

If $f(z)$ and $g(z)$ are analytic on a domain D , and if $f(z)=g(z)$ for z belonging to a set that has a nonisolated point then $f(z)=g(z)$ for all $z \in D$.

$f(z)-g(z)$ has non isolated zero so $f(z)-g(z)$ is identically 0.

Final Exam

Laurent Decomposition

Suppose $0 \leq p < \sigma \leq +\infty$ and suppose $f(z)$ is analytic for $p < |z-z_0| < \sigma$. Then $f(z)$ can be decomposed as a sum

$$f(z) = f_0(z) + f_1(z)$$

where $f_0(z)$ is analytic for $|z-z_0| < \sigma$ and $f_1(z)$ is analytic for $|z-z_0| > p$ and at ∞ . If $f_1(\infty) = 0$ then the decomposition is unique.

Proof:

Suppose $f(z)$ is analytic for $|z-z_0| < \sigma$, then $f(z) = f_0(z)$ and $f_1(z) = 0$

Similarly if $f(z)$ is analytic for $|z-z_0| > p$ and vanishes at ∞ , then $f(z) = f_1(z)$ with $f_0(z) = 0$

Uniqueness Argument:

Suppose $f(z) = g_0(z) + g_1(z)$ is another decomposition

$$f(z) - f(z) = f_0(z) + f_1(z) - g_0(z) - g_1(z) = 0$$

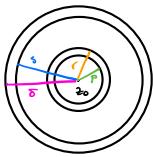
$$g_0(z) - f_0(z) = f_1(z) - g_1(z) \quad p < |z-z_0| < \sigma$$

If $h(z) = g_0(z) - f_0(z)$ for $|z-z_0| < \sigma$ and $h(z) = f_1(z) - g_1(z)$ for $|z-z_0| > p$ we define $h(z)$ as an entire function where as $z \rightarrow \infty$ $h(z) \rightarrow 0$.

By Liouville's theorem we find $h(z) = 0$ and as a consequence, uniqueness of decomposition

Finding the decomposition

Choose r and s such that $p < r < s < \sigma$



Applying Cauchy Integral Formula

$$f(z) = \frac{1}{2\pi i} \oint_{|w-z_0|=s} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{|w-z_0|=r} \frac{f(w)}{w-z} dw \quad r < |z-z_0| < s$$

Splitting $f(z)$ by components on the annulus

$$f_0(z) = \frac{1}{2\pi i} \oint_{|w-z_0|=s} \frac{f(w)}{w-z} dw \quad |w-z_0| < s$$

$$f_1(z) = - \frac{1}{2\pi i} \oint_{|w-z_0|=r} \frac{f(w)}{w-z} dw \quad |w-z_0| > r$$

By uniqueness argument we can assert $f_0(z)$ and $f_1(z)$ defined for $r < |z-z_0| < s$ apply to $p < r < s < \sigma$

Laurent Series Expansion

Suppose $0 \leq p < \sigma \leq \infty$ and suppose $f(z)$ is analytic for $p < |z-z_0| < \sigma$. Then $f(z)$ has a Laurent series expansion that converges absolutely at each point of the annulus and that converges uniformly on each subannulus $r \leq |z-z_0| \leq s$ where $p < r < s < \sigma$. The coefficients are uniquely determined by $f(z)$ for any fixed r , $p < r < \sigma$

$$\text{Expansion: } f(z) = \sum_{-\infty}^{\infty} a_k (z-z_0)^k \quad p < |z-z_0| < \sigma$$

$$\text{Coefficients: } a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz \quad -\infty < n < \infty$$

Proof:

$$\text{Suppose } f(z) = f_0(z) + f_1(z) \quad \text{for } p < |z-z_0| < \sigma$$

$$f_0(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k \quad |z-z_0| < \sigma$$

Series converges uniformly for $|z-z_0| \leq s$

$$f_1(z) = \sum_{k=-\infty}^{-1} a_k (z-z_0)^k \quad |z-z_0| > p$$

Satisfies $f_1(z) \rightarrow 0$ as $z \rightarrow \infty$

Converges uniformly for $|z-z_0| \geq r$

Therefore,

$$f(z) = f_0(z) + f_1(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \quad p < |z-z_0| < \sigma \quad \leftarrow \text{Laurent Expansion}$$

Consider $\frac{f(z)}{(z-z_0)^{n+1}}$ integrated over $|z-z_0|=r$

$$\oint_{|z-z_0|=r} \frac{1}{(z-z_0)^{n+1}} f(z) dz = \oint_{|z-z_0|=r} \frac{1}{(z-z_0)^{n+1}} \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k dz = \sum_{k=-\infty}^{\infty} a_k \oint_{|z-z_0|=r} (z-z_0)^{k-n-1} dz$$

Swap since
Series converges uniformly

Integral of $(z-z_0)^m = 2\pi i$ when $m=-1$ and 0 otherwise

Series term becomes $2\pi i a_n$

$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz \quad -\infty < n < \infty$$

Term of Laurent decomposition with positive powers converges on the largest open disk centered at z_0 to which $f(z)$ extends to be analytic

Term of Laurent decomposition with negative powers converges on the largest exterior domain to which $f(z)$ extends analytically

Riemann's Theorem on Removable Singularities

Let z_0 be an isolated singularity of $f(z)$. If $f(z)$ is bounded near z_0 , then $f(z)$ has a removable singularity at z_0 .

Suppose $|f(z)| \leq M$ for z near z_0 and small $r > 0$

Applying ML estimates

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$|a_n| \leq \frac{1}{2\pi r} \cdot 2\pi r \cdot \frac{M}{r^{n+1}} = \frac{M}{r^n}$$

If $n < 0$, as $r \rightarrow 0$ $\frac{M}{r^n}$ approaches 0. Therefore $a_n = 0$ for $n < 0$ and therefore it is removable.

Poles and Limits

Let z_0 be an isolated singularity of $f(z)$. Then z_0 is a pole if and only if $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$

\Rightarrow suppose f has a pole of order N at z_0

$g(z) = (z-z_0)^N f(z)$ is analytic at non zero at z_0

$$|f(z)| = |(z-z_0)^{-N} g(z)| \rightarrow \infty \text{ as } z \rightarrow z_0$$

\Leftarrow Suppose $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$

Then $f(z) \neq 0$ for z near z_0 , therefore $h(z) = \frac{1}{f(z)}$ is analytic on a punctured disk around z_0 .

$h(z) \rightarrow 0$ as $z \rightarrow z_0$ and by Riemann's Theorem $h(z)$ is analytic at z_0 where $h(z_0) = 0$

N is the order of the zero of $h(z)$ at z_0 , so $f(z) = \frac{1}{h(z)}$ has a pole of order N at z_0 .

Casorati-Weierstrass Theorem

Suppose z_0 is an essential isolated singularity of $f(z)$. Then for every complex number w_0 , there is a sequence $z_n \rightarrow z_0$ such that $f(z_n) \rightarrow w_0$.

Proof of contrapositive: Suppose there is some complex number w_0 that is not the limit of $f(z)$ as $z \rightarrow z_0$.

Then there exists some $\epsilon > 0$ s.t. $|f(z) - w_0| > \epsilon$ for all z near z_0 . Therefore, $h(z) = \frac{1}{f(z)-w_0}$ is bounded for z near z_0 . Applying Riemann's theorem we find that $f(z)$ has a removable singularity at z_0 .

$h(z) = (z-z_0)^N g(z)$ for $N \geq 0$ and some analytical function $g(z)$ where $g(z_0) \neq 0$

$$\frac{1}{f(z)-w_0} = (z-z_0)^N g(z)$$

$$f(z) = (z-z_0)^{-N} \cdot \frac{1}{g(z)} + w_0$$

If $N=0$ $f(z)$ is analytic at z_0 , otherwise if $N > 0$ $f(z)$ has a pole of order N at z_0 .

Meromorphic Functions on \mathbb{C}^*

A meromorphic function on the extended complex plane is rational

Meromorphic function must have a finite number of poles, otherwise they would not be isolated

Define $P_\infty(z)$ to be $f(\infty)$ if f is analytic at ∞ , otherwise $P_\infty(z)$ is the principle part of $f(z)$ at ∞

$$f(z) - P_\infty(z) \rightarrow 0 \text{ as } z \rightarrow \infty$$

For poles z_1, \dots, z_m

$$P_K(z) = \frac{a_1}{z-z_1} + \frac{a_2}{(z-z_2)^2} + \dots + \frac{a_n}{(z-z_n)^n} \quad \leftarrow \text{Principle part of } f(z) \text{ at } z_K$$

Consider

$$g(z) = f(z) - P_\infty(z) - \sum_{j=1}^m P_j(z)$$

$g(z)$ is an entire function since all trouble areas (singularities) are accounted for

$g(z) \rightarrow 0$ as $z \rightarrow \infty$ so by Liouville's Theorem we get that $g(z) = 0$

$$f(z) = P_\infty(z) + \sum_{j=1}^m P_j(z) \quad \leftarrow \text{Partial fraction decomposition}$$

Residue Theorem

Let D be a bounded domain in the complex plane with piecewise smooth boundary. Suppose that $f(z)$ is analytic on $D \cup \partial D$ except for a finite number of isolated singularities z_1, \dots, z_m in D . Then

$$\int_{\partial D} f(z) dz = 2\pi i \sum_{j=1}^m \operatorname{Res}[f(z), z_j]$$



Let D_ϵ be the domain obtained by punching out small disks around each singularity

By definition we find that for the integral around each disk \mathcal{U}_j

$$\int_{\partial \mathcal{U}_j} f(z) dz = 2\pi i \operatorname{Res}[f(z), z_j]$$

From Cauchy's Theorem we have

$$0 = \int_{\partial D_\epsilon} f(z) dz = \int_{\partial D} f(z) dz - \sum_{j=1}^m \int_{\partial \mathcal{U}_j} f(z) dz = \int_{\partial D} f(z) dz - 2\pi i \sum_{j=1}^m \operatorname{Res}[f(z), z_j]$$

$$\int_{\partial D} f(z) dz = 2\pi i \sum_{j=1}^m \operatorname{Res}[f(z), z_j]$$

Fractional Residue Theorem

If z_0 is a simple pole of $f(z)$, and γ_ϵ is an arc of the circle $|z-z_0|=\epsilon$ of angle α , then

$$\lim_{\epsilon \rightarrow 0} \int_{\gamma_\epsilon} f(z) dz = \alpha i \operatorname{Res}[f(z), z_0]$$

Write $f(z) = \frac{A}{z-z_0} + g(z)$ where A is $\operatorname{Res}[f(z), z_0]$ and $g(z)$ is analytic at z_0

$$\int_{\gamma_\epsilon} \frac{A}{z-z_0} + g(z) dz = \int_{\theta_0}^{\theta_0+\alpha} \left(\frac{A}{\epsilon e^{i\theta}} + g(z_0 + \epsilon e^{i\theta}) \right) \cdot \epsilon i e^{i\theta} d\theta = iA \int_{\theta_0}^{\theta_0+\alpha} d\theta + \underbrace{\int_{\theta_0}^{\theta_0+\alpha} g(z_0 + \epsilon e^{i\theta}) \cdot \epsilon i e^{i\theta} d\theta}_{\text{ML-estimate: } g \text{ is bounded near } z_0} = iA \int_0^{\alpha} d\theta = \alpha i A = \alpha i \operatorname{Res}[f(z), z_0]$$

parametrize $z = z_0 + \epsilon e^{i\theta}$
 $dz = \epsilon i e^{i\theta} d\theta$

Jordan's Lemma

If T_R is a semicircular contour $z(\theta) = R e^{i\theta}$, $0 \leq \theta \leq \pi$, in the upper half-plane, then

$$\int_{T_R} |e^{iz}| |dz| < \pi$$

$$z(\theta) = R e^{i\theta}, |e^{iz}| = |e^{iR(\cos \theta + i \sin \theta)}| = e^{-R \sin \theta}$$

$$|dz| = R$$

$$\int_{T_R} |e^{iz}| |dz| = \int_0^\pi e^{-R \sin \theta} \cdot R d\theta < \pi \Rightarrow \int_0^\pi e^{-R \sin \theta} d\theta < \frac{\pi}{R}$$

Proving Estimate:

$$\sin \theta \geq \frac{2\theta}{\pi}, \quad 0 \leq \theta \leq \pi/2$$

$$\int_0^{\pi/2} e^{-R \sin \theta} d\theta = 2 \int_0^{\pi/2} e^{-R \sin \theta} d\theta \leq 2 \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{\pi}{R} \int_0^{\infty} e^{-t} dt < \frac{\pi}{R} \int_0^{\infty} e^{-t} dt = \frac{\pi}{R}$$

$$t = \frac{2R\theta}{\pi}$$

$$dt = \frac{2R}{\pi} d\theta$$

Residue Theorem for Exterior Domains

Let D be an exterior domain with a piecewise smooth boundary. Suppose $f(z)$ is analytic on $D \cup \partial D$ except for a finite number of isolated singularities z_1, \dots, z_m in D . Let a_{-1} be the coefficient of $\frac{1}{z}$ in the Laurent expansion $f(z) = \sum a_n z^n$ that converges for $|z| > R$. Then

$$\int_{\partial D} f(z) dz = -2\pi i a_{-1} + 2\pi i \sum_{j=1}^m \operatorname{Res}[f(z), z_j]$$



Apply residue theorem to the bounded domain D_R where $z \in D$ s.t. $|z| < R$

$$\int_{\partial D} f(z) dz + \int_{|z|=R} f(z) dz = \int_{\partial D_R} f(z) dz = 2\pi i \sum_{j=1}^m \operatorname{Res}[f(z), z_j]$$

evaluates to

$2\pi i a_{-1}$ when
integrated term
by term

$$\int_{\partial D} f(z) dz = -2\pi i a_{-1} + 2\pi i \sum_{j=1}^m \operatorname{Res}[f(z), z_j]$$

Argument Principle

Let D be a bounded domain with piecewise smooth boundary ∂D and let $f(z)$ be a meromorphic function on D that extends to be analytic on ∂D , such that $f(z) \neq 0$ on ∂D . Then

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz = N_o - N_{\infty}$$

where N_o is the number of zeros of $f(z)$ in D and N_{∞} is the number of poles of $f(z)$ in D , counting multiplicities.

For a pole of $f(z)$ at z_0

$$f(z) = (z - z_0)^N g(z)$$

$$\frac{f'(z)}{f(z)} = \frac{N(z - z_0)^{N-1} g(z) + (z - z_0)^N g'(z)}{(z - z_0)^N g(z)} = \frac{N}{z - z_0} + \frac{g'(z)}{g(z)}$$

Analytic function

$\frac{f'(z)}{f(z)}$ has a simple pole at z_0 with residue N

Residue theorem on $\frac{f'(z)}{f(z)}$ is the sum of residues which are $+N$ for each zero and $-N$ for each pole.

$$\text{Thus, } \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz = N_o - N_{\infty}$$

Consider the logarithmic integral

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{\gamma} d \log |f(z)| = \frac{1}{2\pi i} \int_{\gamma} d \log |f(z)| + \frac{1}{2\pi i} \int_{\gamma} d \arg(f(z))$$

exact

parameterizing $\gamma(t) = x(t) + iy(t)$ for $a \leq t \leq b$

$$\int_{\gamma} d \log |f(z)| = \log |f(\gamma(b))| - \log |f(\gamma(a))| \leftarrow 0 \text{ for closed curve}$$

$$\int_{\gamma} d \arg(f(z)) = \arg f(\gamma(b)) - \arg f(\gamma(a))$$

Rouche's Theorem

Let D be a bounded domain with piecewise smooth boundary ∂D . Let $f(z)$ and $h(z)$ be analytic on $D \cup \partial D$. If $|h(z)| < |f(z)|$ for $z \in \partial D$, then $f(z)$ and $f(z) + h(z)$ have the same number of zeros in D , counting multiplicities.

$f(z)$ and $f(z) + h(z) \neq 0$ on ∂D

$$f(z) + h(z) = f(z) \left[1 + \frac{h(z)}{f(z)} \right]$$

$$\arg(f(z) + h(z)) = \arg(f(z)) + \arg\left(1 + \frac{h(z)}{f(z)}\right)$$

Since $|h(z)| < |f(z)|$, $1 + \frac{h(z)}{f(z)}$ must exist in the right half-plane $\Rightarrow \arg\left(1 + \frac{h(z)}{f(z)}\right) = 0$

$$\text{Therefore } \arg(f(z) + h(z)) = \arg(f(z))$$

Thus by the argument principle they have the same number of zeros

Hurwitz's Theorem

Suppose $\{f_k(z)\}$ is a sequence of analytic functions on a domain D that converges normally on D to $f(z)$ and suppose that $f(z)$ has zero of order N at z_0 . Then there exists $p > 0$ such that for k large, $f_k(z)$ has exactly N zeros in the disk $\{|z - z_0| \leq p\}$ counting multiplicity, and these zeros converge to z_0 as $k \rightarrow \infty$.

Let $p > 0$ be sufficiently small st. $\{|z - z_0| \leq p\}$ is contained in D and so $f(z) \neq 0$ for $0 < |z - z_0| \leq p$.

Choose $\delta > 0$ st. $|f(z)| \geq \delta$ on the circle $|z - z_0| = p$. Since $f_k(z)$ converges uniformly to $f(z)$ for $|z - z_0| \leq p$, for k large we have $|f_k(z)| \geq \delta/2$.

for $|z - z_0| = p$ and $\frac{f'_k(z)}{f_k(z)}$ converges uniformly to $\frac{f'(z)}{f(z)}$ for $|z - z_0| = p$

$$\underbrace{\frac{1}{2\pi i} \int_{|z-z_0|=p} \frac{f'_k(z)}{f_k(z)} dz}_{N_k} \rightarrow \underbrace{\frac{1}{2\pi i} \int_{|z-z_0|=p} \frac{f'(z)}{f(z)} dz}_N \Rightarrow N_k \rightarrow N$$

Open Mapping for Analytic Functions

If $f(z)$ is analytic on a domain D and $f(z)$ is not constant, then $f(z)$ maps open sets to open sets, that is, $f(U)$ is open for each open subset U of D .

Let $w_0 \in f(U)$ say $w_0 = f(z_0)$.

Consider $f(z_0) - w_0$ which has a zero of order m at z_0 . Since the zeros are isolated we can construct a disk of radius p so that $f(z) - w_0 \neq 0$ for $0 < |z - z_0| \leq p$.

Inverse Function Theorem

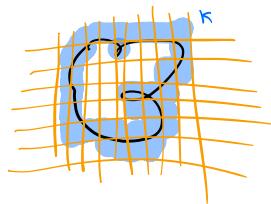
Suppose $f(z)$ is analytic for $|z - z_0| \leq p$ and satisfies $f(z_0) = w_0$, $f'(z_0) = 0$ and $f(z) \neq w_0$ for $0 < |z - z_0| \leq p$. Let $\delta > 0$ be chosen st. $|f(z) - w_0| \geq \delta$ for $|z - z_0| = p$. Then for w st. $|w - w_0| < \delta$ there is a unique z satisfying $|z - z_0| < p$ and $f(z) = w$. $z = f^{-1}(w)$

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{|z-z_0|=p} \frac{df(s)}{f(s) - w} ds \quad (w - w_0) < \delta$$

Generalized Cauchy Formula

If $f(z)$ is analytic on a domain D , then $\int f(z) dz = 0$ for each closed path γ in D s.t. $W(\gamma, s) = 0$ for all $s \in \mathbb{C} \setminus D$

Choose $\delta > 0$ small s.t. every point of T has distance at least 4δ from any point of $\mathbb{C} \setminus D$. Then divide D into a grid of squares of side length δ .



Let K be the union of all squares in the grid with a point less than δ away from T . $U = K \cap \mathbb{C}$ so $\partial U = \partial K$

Using Cauchy integral formula

$$z \in T \quad f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(s)}{s-z} ds$$

$$\int_{\gamma} f(z) dz = \frac{1}{2\pi i} \int_{\partial U} \left[\int_{\gamma} \frac{1}{s-z} dz \right] f(s) ds = - \int_{\partial U} W(\gamma, s) f(s) ds$$

$$\text{Winding number is } 0 \quad \text{so} \quad \int_{\gamma} f(z) dz = 0$$

Generalized Cauchy Integral Formula

Let $f(z)$ be analytic on a domain D and let γ be a closed path in D with trace $T = \gamma([a, b])$. If $W(\gamma, s) = 0$ for all $s \in \mathbb{C} \setminus D$, then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-z_0} dz = W(\gamma, z_0) f(z_0) \quad z_0 \in D \setminus T$$

$$\text{Let } g(z) = \frac{f(z) - f(z_0)}{z - z_0}. \quad \int_{\gamma} g(z) dz = 0$$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-z_0} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z_0)}{z-z_0} dz = W(\gamma, z_0) f(z_0)$$

Schwarz Lemma

Let $f(z)$ be analytic for $|z| < 1$. Suppose $|f(z)| \leq 1$ for all $|z| < 1$ and $f(0) = 0$. Then

$$|f(z)| \leq |z|, \quad |z| < 1$$

If the equality holds for some $z, z_0 \neq 0$ then $f(z) = \lambda z$ for some constant λ of unit modulus

Write $f(z) = zg(z)$ where $g(z)$ is analytic.

Let $r < 1$. If $|z| = r$ then $|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}$. By the maximum principle $|g(z)| \leq \frac{1}{r}$ for all z satisfying $|z| \leq r$

As $r \rightarrow 1$, $|g(z)| \leq 1$ for all $|z| < 1$.

If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$ then $|g(z_0)| = 1$ and by strict maximum principle $g(z)$ is constant.

$$g(z) = \lambda \quad \text{and} \quad f(z) = \lambda z$$

Theorem can easily be extended to other disks centered at other locations

* Part 2: Let $f(z)$ be analytic for $|z| < 1$. If $|f(z)| \leq 1$ for $|z| < 1$, and $f(0) = 0$, then $|f'(0)| \leq 1$ with equality if and only if $f(z) = \lambda z$ for some constant λ with $|\lambda| = 1$

$$\int_0^{\pi} \frac{\sin^2 \theta}{a + \cos \theta} d\theta$$

