Griffiths Electrodynamics

Chapter 1: Vector Analysis

1.1 Vector Algebra

Vectors have direction and magnitude but no location

Vector Addition

$$\vec{A}+\vec{B} = \vec{B}+\vec{A}$$

Scalar Multiplication

$$a(\vec{A}+\vec{B}) = a\vec{A} + a\vec{B}$$

Dot Product (Scalar Product)

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$ ← Angle between vectors

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

A times projection of B along A

$$\vec{A}\cdot\vec{A} = A^2 \qquad A\perp B \quad \vec{A}\cdot\vec{B} = 0$$

Cross Product (Vector Product)

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$$ ← unit vector perpendicular to $\vec{A}\vec{B}$ plane

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

But! $\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$

$\vec{A}\parallel\vec{B} \quad \vec{A}\times\vec{B} = \vec{0}$

Vectors can be written in terms of components

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Vector triple products

i) Scalar Triple Product $\vec{A}\cdot(\vec{B}\times\vec{C})$

Area of parallelipiped generated by $\vec{A}$, $\vec{B}$, and $\vec{C}$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

ii) Vector Triple Product $\vec{A}\times(\vec{B}\times\vec{C})$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

All expressions can be simplified to a single cross product

Vectors must undergo coordinate transformations like a vector

$$\begin{pmatrix}\bar{A}_y\\ \bar{A}_z\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}A_y\\ A_z\end{pmatrix}$$

$$\begin{pmatrix}\bar{A}_x\\ \bar{A}_y\\ \bar{A}_z\end{pmatrix} = \begin{pmatrix}R_{xx} & R_{xy} & R_{xz}\\ R_{yx} & R_{yy} & R_{yz}\\ R_{zx} & R_{zy} & R_{zz}\end{pmatrix}\begin{pmatrix}A_x\\ A_y\\ A_z\end{pmatrix} \quad \text{or} \quad \bar{A}_i = \sum_{j=1}^{3} R_{ij}A_j$$

n-th rank tensor has n indicies and 3^n components and transforms with n-factors of R

Vectors are tensors of rank 1

Scalars are tensors of rank 0

1.2 Differential Calculus

Gradient: Generalization of derivative to higher dimensions

$$\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

akin to a dot product

$$dT = \nabla T\cdot d\ell = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)\cdot\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$= |\nabla T|\,|d\ell|\cos\theta$$

dT is maximized when $\cos\theta = 1$ or $\theta = 0$

In other words, ∇T is the direction of maximum increase

$|\nabla T|$ gives the slope along this direction

Del is a vector operator

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

cute little notational shorthand

1. On a scalar function

   ∇T is the gradient

2. On a vector function via dot product

   $\nabla\cdot T$ is the divergence

3. On a vector function via cross product

   $\nabla\times T$ is the curl

Divergence is a measure of how much a vector $\vec{v}$ spreads out from the point

$$\nabla \cdot \vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z}\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad \leftarrow \text{Scalar!}$$

Positive divergence for outward spread

Curl is a measure of how much the vector $\vec{v}$ swirls around the point

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Product Rules for Vector Derivatives

Gradient:

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A}$$

Divergence:

$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$$

$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A(\nabla\times B)$$

Curl:

$$\nabla\times(f\vec{A}) = f(\nabla\times A) - A\times(\nabla f)$$

$$\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A})$$

Vector Second Derivatives

1) Divergence of Gradient

$$\nabla\cdot(\nabla T) = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z} \quad \leftarrow \text{Laplacian}$$

$$\nabla^2 r = (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z}$$

2) Curl of Gradient

$$\nabla\times(\nabla T) = 0 \quad \leftarrow \text{always.}$$

$$\left(\frac{\partial}{\partial x}\hat{x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) = \hat{x}\left(\frac{\partial^2 T}{\partial y \partial z} - \frac{\partial^2 T}{\partial y \partial z}\right) - \hat{y}\left(\frac{\partial^2 T}{\partial x \partial z} - \frac{\partial^2 T}{\partial x \partial z}\right) + \hat{z}\left(\frac{\partial^2 T}{\partial y \partial x} - \frac{\partial^2 T}{\partial x \partial y}\right) = 0$$

3) Gradient of Divergence

$$\nabla(\nabla\cdot v)$$

Not very physically relevant

4) Divergence of Curl

$$\nabla\cdot(\nabla\times v) = 0 \quad \leftarrow \text{always.}$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

5) Curl of Curl

$$\nabla\times(\nabla\times v) = \underbrace{\nabla(\nabla\cdot v)}_{3} - \underbrace{\nabla^2 v}_{\text{Laplacian}}$$

1.3 Integral Calculus

Line Integrals

$$\int_a^b \vec{v}\cdot d\vec{l} \quad \text{over a path P from a to b}$$

For a closed loop $(a=b)$

$$\oint \vec{v}\cdot d\vec{l}$$

Surface Integrals

$$\int_S \vec{v}\cdot d\vec{a} \quad \leftarrow \text{sign is intrinsically ambiguous}$$

For a closed surface

$$\oint \vec{v}\cdot d\vec{a} \qquad \text{tradition dictates outwards as positive}$$

Volume Integrals

$$\int_V T\,d\tau \qquad d\tau = dx\,dy\,dz$$

Fundamental Theorem of Calculus for Gradients

For $T(x,y,z)$ starting at a and moving $d\vec{l}_1$

$$dT = (\nabla T) \cdot d\vec{l}_1$$

So, repeating this process

$$\int_a^b (\nabla T) \cdot d\vec{l} = T(b) - T(a)$$

Corollary 1: $\int_a^b (\nabla T) \cdot d\vec{l}$ is path independent

Corollary 2: $\oint (\nabla T) \cdot d\vec{l} = 0$

Fundamental Theorem for Divergences

$$\int_V (\nabla \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

Also known as Gauss' Theorem, Green's Theorem, or divergence Theorem

Fundamental Theorem for Curl

Stoke's Theorem:

$$\int_S (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_P \vec{v} \cdot d\vec{l}$$

total amount of swirl can be calculated by seeing what exits the boundary

Corollary 1: $\int (\nabla \times \vec{v}) \cdot d\vec{a}$ depends on boundary line

Corollary 2: $\oint (\nabla \times \vec{v}) \cdot d\vec{a} = 0$ for any closed surface

Integration by Parts

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\int_a^b \frac{d}{dx}(fg)\,dx = fg\Big|_a^b = \int_a^b f\left(\frac{dg}{dx}\right)dx + \int_a^b g\left(\frac{df}{dx}\right)dx$$

$$\Rightarrow \int_a^b f\left(\frac{dg}{dx}\right)dx = fg\Big|_a^b - \int_a^b g\left(\frac{df}{dx}\right)dx$$

For vectors

$$\nabla \cdot f\vec{A} = f(\nabla \cdot A) + A \cdot (\nabla f)$$

$$\int \nabla \cdot (fA)\, d\tau = \int f(\nabla \cdot \vec{A})\, d\tau + \int \vec{A} \cdot (\nabla f)\, d\tau = \oint f\vec{A} \cdot d\vec{a}$$

$$\int_V f(\nabla \cdot \vec{A})\, d\tau = -\int_V \vec{A} \cdot (\nabla f)\, d\tau + \oint_S f\vec{A} \cdot d\vec{a}$$

1.4 Curvilinear Coordinates

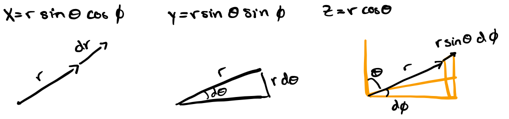
Spherical Coordinates

r: distance from origin

θ: polar angle - down from z-axis

ϕ: Azimuthal angle - around the x-axis

$x = r\sin\theta\cos\phi$ $\quad y = r\sin\theta\sin\phi$ $\quad z = r\cos\theta$



$$dl = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$d\tau = dl_r \, dl_\theta \, dl_\phi = r^2 \sin\theta \, dr \, d\theta \, d\phi$

infinitesimal volume change

Vector Derivatives in spherical coordinates

$$\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial T}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\right) + \frac{\partial T}{\partial \phi}\left(\frac{\partial \phi}{\partial x}\right)$$

Expanding via this method or a better one

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

Divergence:

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl:

$$\nabla \times \vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta \, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$$

Laplacian:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

Cylindrical Coordinates

$$x = s\cos\phi \qquad y = s\sin\phi \qquad z = z$$

$$dl_s = ds \qquad dl_\phi = s\,d\phi \qquad dl_z = dz$$

$$d\tau = s \, ds \, d\phi \, dz$$

Vector Derivatives

Gradient:

$$\nabla T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$

Divergence:

$$\nabla \cdot \vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl:

$$\nabla \times \vec{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{z}$$

Laplacian:

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

1.5 The Dirac Delta Function

Consider

$$v = \frac{1}{r^2}\hat{r}$$

directed radially outward at every point

Now calculate divergence

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0 \leftarrow \text{Funky!}$$

Consider a sphere of radius R centered at the origin

$$\oint \vec{v} \cdot d\vec{a} = \int \frac{1}{R^2}\hat{r} \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$$

$$= \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi = 4\pi \leftarrow \text{Even funkier}$$

Everything blows up at $r = 0$

One-dimensional Dirac Delta Function

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

Technically generalized function or distribution

Delta function "picks" out certain values

$$f(x)\,\delta(x) = f(0)\,\delta(x)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = \int_{-\infty}^{\infty} f(0)\,\delta(x)\,dx = f(0)\int_{-\infty}^{\infty}\delta(x)\,dx = f(0)$$

Shifting the peak

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(x)$$

Three-Dimensional Dirac Delta

$$\delta^3(r) = \delta(x)\,\delta(y)\,\delta(z)$$

$$\int \delta^3(r) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1$$

Generalized:

$$\int_{\text{all space}} f(r)\,\delta^3(r-a)\,d\tau = f(a)$$

1.6 The Theory of Vector Fields

Helmholtz Theorem guarantees that a field is uniquely determined by curl and divergence with the assumption the field goes to 0 at ∞

Suppose $\nabla\cdot F = D$, $\nabla\times F = C$ and $\nabla\cdot C = 0$ ← as always

Let $F = -\nabla U + \nabla\times W$

$$U(r) = \frac{1}{4\pi}\int \frac{D(r')}{\mathscr{r}}\,d\tau' \qquad \mathscr{r} = |r-r'|$$

$$W(r) = \frac{1}{4\pi}\int \frac{C(r')}{\mathscr{r}}\,d\tau'$$

Consider Divergence:

$$\nabla\cdot F = \underbrace{-\nabla^2 U + 0}_{\text{divergence of curl is always } 0} = -\frac{1}{4\pi}\int D\,\underbrace{\nabla^2\left(\frac{1}{\mathscr{r}}\right)}_{-4\pi\delta^3(\mathscr{r})}d\tau' = \int D(r')\,\underbrace{\delta^3(r-r')}_{\text{dirac delta}}\,d\tau' = D(r)$$

Now Consider Curl:

$$\nabla\times F = \nabla\times(-\nabla U + \nabla\times W) = \underbrace{\nabla\times -\nabla U}_{\text{Curl of gradient is always } 0} + \nabla\times(\nabla\times W) = \nabla(\nabla\cdot W) - W(\nabla\cdot\nabla) = -\nabla^2 W + \nabla(\nabla\cdot W)$$

$$-\nabla^2 W = -\frac{1}{4\pi}\int C\,\underbrace{\nabla^2\left(\frac{1}{\mathscr{r}}\right)}_{-4\pi\delta^3(\mathscr{r})}d\tau' = \int C(r')\,\underbrace{\delta^3(r-r')}_{\text{dirac delta}}\,d\tau' = C(r)$$

For $\nabla(\nabla\cdot W)$ Consider

$$\nabla\cdot W = \frac{1}{4\pi}\int C\cdot\underbrace{\nabla\left(\frac{1}{\mathscr{r}}\right)}_{\nabla = -\nabla'}d\tau' = -\frac{1}{4\pi}\int C\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)d\tau' \underset{\text{Integration by parts}}{=} \frac{1}{4\pi}\left[\int \frac{1}{\mathscr{r}}\nabla'\cdot C\,d\tau - \oint \frac{1}{\mathscr{r}}C\cdot d\vec{a}\right]$$

$$\int f(\nabla\cdot\vec{A})\,d\tau = -\int \vec{A}\cdot(\nabla f)\,d\tau + \oint f\vec{A}\cdot d\vec{a}$$

$$= 0$$

Divergence of C must be 0 and at infinity the surface integral vanishes as long as C approaches 0 sufficiently quickly

Formally,

Helmholtz Theorem: If the divergence $D(r)$ and the curl $C(r)$ of a vector function $F(r)$ are specified and if they both approach zero faster than $\frac{1}{r^2}$ as $r \to \infty$ and if $F(r) \to 0$ as $r \to \infty$, then F is defined uniquely

If the curl of a vector field vanishes everywhere, then F can be written as the gradient of a scalar potential

$$\nabla \times F = 0 \iff F = -\nabla V$$

Theorem: Curl-less or Irrotational fields

i) $\nabla \times F = 0$ everywhere

ii) $\int_a^b F \cdot dl$ is independent of path

iii) $\oint F \cdot dl = 0$ for any closed loop

iv) F is the gradient of some scalar function: $F = -\nabla V$

V is not unique since adding a constant doesn't impact $-\nabla V$

are all equivalent

Theorem: Divergence-less or Solenoidal Fields

i) $\nabla \cdot F = 0$ everywhere

ii) $\int F \cdot da$ is independent of surface for a given boundary

iii) $\oint F \cdot da = 0$ for any closed surfaces

iv) F is the curl of some vector function $F = \nabla \times A$

are all equivalent

Any vector field F can be represented as

$$F = -\nabla V + \nabla \times A$$

Chapter 2: Electrostatics

2.1 The Electric Field

Fundamental problem of Electrodynamics is to answer the question of what force a test charge Q will feel given a certain set of source charges as the system evolves through time

This problem is solved via superposition : interaction between charges isn't affected by the presence of other charges

Force on Q is sum of forces from each individual source charge

To simplify the study of this question we begin with Electrostatics : the study of stationary charges

Coulomb's Law

$$F = \frac{1}{4\pi\varepsilon_0} \frac{qQ}{\mathscr{r}^2} \hat{\mathscr{r}}$$ (determined experimentally)

$$\varepsilon_0 = 8.85 \cdot 10^{-12} \frac{C^2}{N \cdot m^2}$$ permittivity of free space

For multiple point charges, the force on Q

$$F = F_1 + F_2 + \dots = \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \dots\right) = Q\vec{E}$$

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i^2} \hat{\mathscr{r}}$$

E is known as the electric field

E is a function of r because $\mathscr{r}_i$ is dependent on the location of the field point, P

Essentially the force a point charge of Q would feel at a given point

For a continuous charge distribution

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} dq$$

For a line charge:

$dq = \lambda \, dl'$ so $$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda(r')}{\mathscr{r}^2} \hat{\mathscr{r}}$$

For a surface charge:

$dq = \sigma \, da'$ so $$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(r')}{\mathscr{r}^2} \hat{\mathscr{r}} \, da'$$

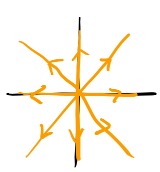
For a volume charge:

$dq = \rho \, d\tau'$ so $$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{\mathscr{r}^2} \hat{\mathscr{r}} \, d\tau'$$

2.2 Divergence and Curl of Electrostatic Fields

Field lines can be used to illustrate vector fields

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$



Field line density represents strength

Field lines begin on positive charges and end on negative ones

Field lines can also never intersect

Flux is a measure of field lines passing through a surface

$$\Phi_E = \int_S \vec{E} \cdot d\vec{a}$$

Gauss's Law

$$\oint \vec{E} \cdot d\vec{a} = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\right)\hat{r} \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}) = \frac{1}{\varepsilon_0} q$$

Flux through any surface enclosing charge is q/ε_0

Intuitively, positive charge field lines will either terminate within the surface if there is a negative charge within or exit the object contributing to flux.

Alternatively, charge outside the surface will pass through effectively cancelling its flux contribution

For multiple charges:

$$\oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^{n} \oint \vec{E}_i \cdot d\vec{a} = \sum_{i=1}^{n} \frac{1}{\varepsilon_0} q_i = \frac{Q_{enc}}{\varepsilon_0}$$

$$\boxed{\oint \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}}$$

Converting to differential form

Applying the divergence theorem

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{E}) \, d\tau$$

We know $Q_{enc} = \int_V \rho \, d\tau$

so, $$\int_V (\nabla \cdot \vec{E}) \, d\tau = \frac{Q_{enc}}{\varepsilon_0} = \int_V \rho/\varepsilon_0 \, d\tau$$

Since this equation is for an arbitrary V

$$\boxed{\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho}$$ Gauss's Law in differential form

Divergence of E

$$E(r) = \frac{1}{4\pi\varepsilon_0} \int_{\text{All space}} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(r') \, d\tau'$$

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0} \int \nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \rho(r') \, d\tau'$$

$$\nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi \, \delta^3(\mathscr{r})$$

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0} \int 4\pi \, \delta^3(r - r') \rho(r') \, d\tau' = \frac{1}{\varepsilon_0} \rho(r)$$

Curl of E

Initially consider a single point charge

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

Now consider an arbitrary line integral from a to b

$$\int_a^b \vec{E}\cdot d\vec{l}$$

Since $d\vec{l} = dr\,\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta\, d\phi\,\hat{\phi}$

$$\vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr$$

$$\int_a^b \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_a^b \frac{q}{r^2}\,dr = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

Thus we have $\oint \vec{E}\cdot d\vec{l} = 0$

Applying Stoke's Theorem we recover

$$\nabla \times E = 0$$

While this is proven for a point charge, we can extrapolate this to any static charge distribution via superposition

$$E = E_1 + E_2 + \dots$$

$$\nabla \times E = \nabla \times (E_1 + E_2 + \dots) = (\nabla \times E_1) + (\nabla \times E_2) + \dots = 0$$

2.3 Electric Potential

Electric field must obey $\nabla \times E = 0$ as proven above

This fact simplifies the goal of finding $\vec{E}$

$\nabla \times E = 0 \implies \oint E\cdot dl = 0$ for closed loops $\implies \int_a^b E\cdot dl$ is path independent

With that in mind let's consider

$$V(r) = -\int_{\mathcal{O}}^{r} \vec{E}\cdot d\vec{l} \quad \text{for some reference point } \mathcal{O}$$

$V(r)$ is the electric potential

$$V(b) - V(a) = -\int_{\mathcal{O}}^{b} \vec{E}\cdot d\vec{l} + \int_{\mathcal{O}}^{a} \vec{E}\cdot d\vec{l} = -\int_{\mathcal{O}}^{b} \vec{E}\cdot d\vec{l} - \int_{a}^{\mathcal{O}} \vec{E}\cdot d\vec{l} = -\int_a^b \vec{E}\cdot d\vec{l}$$

From Fundamental Theorem of Calculus for Gradients

$$V(b) - V(a) = \int_a^b \nabla V\cdot d\vec{l} = -\int_a^b \vec{E}\cdot d\vec{l}$$

Since this is true for arbitrary a, b we can extrapolate

$$E = -\nabla V$$

potential obeys the superposition principle

measured in $\frac{J}{C}$ or Volts

We know that $E = -\nabla V$, $\nabla\cdot E = \rho/\varepsilon_0$, and $\nabla \times E = 0$

putting them together $\nabla\cdot E = \nabla\cdot\nabla V = -\nabla^2 V$

$$-\nabla^2 V = \rho/\varepsilon_0 \quad \leftarrow \text{Poisson's Equation}$$

When there is no charge poisson's equation simplifies to Laplace's Equation

$$\nabla^2 V = 0$$

Once again let's consider a point charge

$$\vec{E}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr \quad \leftarrow \text{same calculation as above}$$

With the reference point at ∞

$$V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l} = \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{r} \frac{q}{r'^2}\,dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\Bigg|_{\infty}^{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

So, $V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}}$, $V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\,dq$

For volume

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{\mathscr{r}}\,d\tau'$$

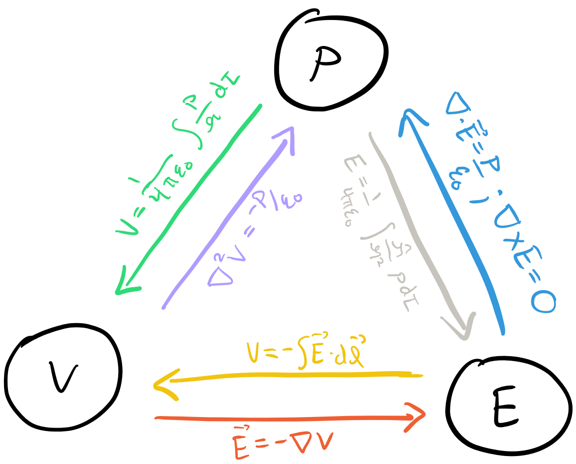
For a surface

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(r')}{\mathscr{r}}\,da'$$

For a line

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(r')}{\mathscr{r}}\,dl'$$

Relations between Fundamental Quantities of Electrostatics



Something about boundary conditions I didn't exactly follow

2.4 Work and Energy in Electrostatics

Work required to move a test charge Q from a to b

$$W = \int_a^b \vec{F}\cdot d\vec{l} = -Q\int_a^b \vec{E}\cdot d\vec{l} = Q\left[V(b) - V(a)\right]$$

Work is path independent "conservative force"

Energy required to move a collection of point charges is equal to sum of moving one point charge against the force of all preceeding point charges

Generalized: $W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j>i}\frac{q_i q_j}{\mathscr{r}_{ij}}$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{j\neq i}^{n}\frac{1}{4\pi\varepsilon_0}\,q_j/\mathscr{r}_{ij}\right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(r_i)$$

In the continuous case

$$W = \frac{1}{2}\int \rho V\,d\tau$$ but we know $\rho = \varepsilon_0\,\nabla\cdot\vec{E}$

$$W = \frac{1}{2}\int \varepsilon_0(\nabla\cdot\vec{E})V\,d\tau$$

via integration by parts

$$W = \frac{\varepsilon_0}{2}\left[-\int E\cdot(\nabla V)\,d\tau + \oint V E\cdot da\right]$$

$$\nabla V = -E$$

$$W = \frac{\varepsilon_0}{2}\left(\int_V E^2 d\tau + \oint_S VE \cdot da\right)$$

When integrating over all space

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau \qquad \text{since } \oint_S VE \cdot da \text{ approaches } 0$$

Energy doesn't obey the superposition principle since it is quadratic in fields

Technically has cross terms

2.5 Conductors

Insulators have electrons bound to atoms while in metallic conductors electrons are free to roam

$E = 0$ inside a conductor

For sake of argument suppose the internal $E \neq 0$. In that case, charges would migrate through the conductor piling up at opposite ends

These charge densities would create their own electric fields which would cancel the original E.

This theoretical process happens instantaneously ensuring $E = 0$

$\rho = 0$ inside a conductor

From Gauss' law

$\nabla \cdot E = \rho/\varepsilon_0$ but since $E = 0$, ρ must also be 0

As a consequence charges must be located on the surface

Conductors are equipotential

$$V(b) - V(a) = -\int_a^b E \cdot dl = 0 \quad \text{as shown above}$$

E is perpendicular to the surface of a conductor

Otherwise charge would flow and negate E tangential

A charge held near an inductor will create a charge inbalance in the conductor by bringing oppositely charged particles closer to itself

The electric field and potential immediately outside of a conductor can be represented by

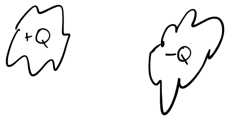
$$E = \frac{\sigma}{\varepsilon_0}\hat{n} \quad \text{and} \quad \sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$

Outward electrostatic pressure on the surface

$$f = \frac{1}{2\varepsilon_0}\sigma^2 \hat{n} \leftarrow \text{force per unit area}$$

$$P = \frac{\varepsilon_0}{2}E^2$$

Consider two capacitors with $+Q$ and $-Q$



Since the surfaces are equipotential, we can meaningfully speak of the potential difference between the conductors

We can't solve for E but we know E is proportional to Q. By extension V is proportional to Q as well.

Let C be the constant of proportionality

$$C \equiv \frac{Q}{V}$$

C is the capacitance measured in Farads

Chapter 3: Potentials

3.1 Laplace's Equation

potential allows us to calculate electric field more easily

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho(r')\,d\tau'$$

However even this expression can be difficult to solve

Using poisson's equation we get

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho$$

When we consider regions where $\rho = 0$ our expression simplifies to Laplace's Equation

$$\nabla^2 V = 0 \quad \text{or} \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Electrostatics can be considered the study of Laplace's equation

Functions that satisfy Laplace's Equation are classified as Harmonic Functions

Laplace's Equation in 1-Dimension

$$\nabla^2 V = 0 \implies \frac{\partial^2 V}{\partial x^2} = 0$$

General Solution: $V(x) = mx + b$

two undetermined constants like second-order ODEs

Properties

i) $V(x)$ is the average of $V(x+a)$ and $V(x-a)$ for any a

$$V(x) = \frac{1}{2}\left[V(x+a) + V(x-a)\right]$$

Laplaces equation is sort of an averaging equation; solution at x is average of values to left and right of x

ii) Laplace's equation doesn't tolerate local maxima or minima; extreme values must occur at endpoints

Consequence of i)

Laplace's Equation in two-dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$ ← No longer an ordinary differential equation, now a PDE

Properties

i) The value of V at (x,y) is the average around the point

$$V(x,y) = \frac{1}{2\pi R}\oint_{\text{circle}} V\,d\ell$$ ← Mean value property

ii) V has no local maxima or minima; all extrema are at the boundaries

Harmonic function in two dimensions minimizes surface area spanning the boundary line

Laplace's Equation in 3-dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Properties

i) The value V at a point r is the average value of V over a spherical surface of radius R centered at r

$$V(r) = \frac{1}{4\pi R^2}\oint_{\text{sphere}} V\,da$$

ii) V can have no local maxima or minima; extrema must occur at boundaries

In addition to Laplace's equation we need boundary conditions to determine V

First Uniqueness Theorem : The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface S

Proof: Suppose V_1 and V_2 are two solutions to Laplace's equation that are equivalent on the surface of a volume

Let $V = V_1 - V_2$

$\nabla^2 V = \nabla^2 V_1 - \nabla^2 V_2 = 0$ and is 0 at all boundaries

Since laplaces equation doesn't allow maxima/minima it must be identically 0 everywhere

Thus, $V_1 = V_2$

Uniqueness theorem tells us that if we have a solution that satisfies Laplace's equation and satisfies boundary conditions then it is correct

Corollary of First Uniqueness Theorem: The potential in a volume $\mathcal{V}$ is uniquely determined if the charge density and boundary conditions are specified/met

Second Uniqueness Theorem : In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given

Proof: Suppose E_1 and E_2 satisfy the conditions of the problem

$$\nabla \cdot E_1 = \frac{1}{\varepsilon_0}\rho \qquad \nabla \cdot E_2 = \frac{1}{\varepsilon_0}\rho$$

$$\oint_{i^{th}\text{ surface}} \vec{E}_1 \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_i \qquad \oint_{i^{th}\text{ surface}} \vec{E}_2 \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_i$$

$$\oint_{\text{Total boundary}} \vec{E}_1 \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{tot} \qquad \oint_{\text{Total boundary}} \vec{E}_2 \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{Tot}$$

Consider $\vec{E} = \vec{E}_1 - \vec{E}_2$

$$\nabla \cdot \vec{E}_3 = 0 \quad \text{and} \quad \oint E_3 \cdot da = 0$$

Now consider V_3 the potential over each conductor

$$\nabla \cdot (V_3 \vec{E}_3) = V_3(\nabla \cdot \vec{E}_3) + \vec{E}_3 \cdot (\nabla V_3) = -(E_3)^2$$

Now,

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \vec{E}_3)\, d\tau = \oint_S V_3 \vec{E}_3 \cdot d\vec{a} = -\int_{\mathcal{V}} (E_3)^2 d\tau$$

V_3 is constant over each surface so it can be pulled out and the remaining result is 0 ($V_3 = 0$ if outer bound is ∞)

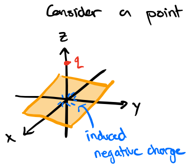
Therefore,

$$\int_{\mathcal{V}} (E_3^2)\, d\tau = 0$$ and this is only true if $E_3 = 0$

3.2 Method of Images

Classic Image problem

Consider a point charge q held above an infinite grounded conducting plane at a distance d



Difficult to solve since we don't know how much the induced charge contributes

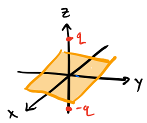
Boundary Conditions:

1) $V=0$ when $z=0$ ← grounded plane

2) $V \to 0$ far from the charge

First uniqueness theorem guarantees a singular solution to this situation

Trick! Consider a new situation with a second charge of $-q$ at $-d$



Since there are no inductive effects we can solve for $V(x,y,z)$

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Boundary Conditions

i) $V=0$ when $z=0$

ii) $V \to 0$ for $x^2+y^2+z^2 \gg d^2$

The only charge in the $z>0$ region is simply our original $+q$ at $(0,0,d)$

Second scenario reproduces conditions of first situation in the area of question

Once we have our potential we can solve for surface charge σ

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$

Energy is not equivalent in both systems

In general the method of images involves introducing mirror images to the system

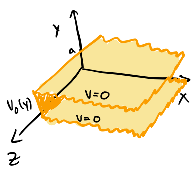
3.3 Separation of Variables

"physicist favorite tool for solving partial differential equations"

Applicable when potential or charge density is specified on boundaries of some region and we want to solve the potential on the interior

General solution is to look for solutions that are products of functions which each depend on only one of the coordinates

Cartesian Coordinate Example: Infinite Slot



Configuration is independent of z so the problem reduces to 2-dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Boundary Conditions:

i) $V=0$ when $y=0$

ii) $V=0$ when $y=a$

iii) $V=V_0(y)$ when $x=0$

iv) $V \to 0$ as $x \to 0$

First step: Look for solutions in the form

$$V(x,y) = X(x)Y(y)$$

↑ restricts our possible solutions to a very small subset

Second step: Applying Laplaces Equation

$$Y\frac{d^2X}{dx} + X\frac{dY}{dy} = 0$$

Third step: Divide through by V

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy} = 0$$

Fourth step: Rewrite as separate functions

$$f(x) + g(y) = 0$$

Notice that this expression holds for all x,y

Thus, $f(x)$ and $g(y)$ must be constant

$$\frac{1}{X}\frac{d^2X}{dx} = C_1 \qquad \frac{1}{Y}\frac{d^2Y}{dy} = C_2 \qquad C_1+C_2=0$$

$$\frac{d^2X}{dx^2} = k^2X \quad \text{and} \quad \frac{d^2Y}{dy} = -k^2Y$$ ← Assumes one is positive and the other is negative

Our partial differential equation is now just ODE's!

$$X(x) = Ae^{kx} + Be^{-kx} \qquad Y(y) = C\sin ky + D\cos ky$$

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C \sin ky + D \cos ky)$$

Now apply boundary conditions

iv) requires $A = 0$

We can now rewrite our expression as

$$V(x,y) = e^{-kx}(C \sin ky + D \cos ky)$$ ← Absorb B into C and D

i) requires $D = 0$

$$V(x,y) = Ce^{-kx} \sin ky$$

ii) tells us that $\sin ka = 0$

$$k = \frac{n\pi}{a} \quad \text{for } n = 1, 2, \ldots$$

We chose C_1 to be positive so we could kill terms as $x \to \infty$ which wouldn't be possible for sinusoidal functions

Similarly, Y wouldn't be able to vanish at 0 and a as a function of e^{kx}

Separation of variables gives us a family of functions of solutions that don't independently satisfy the final boundary condition, but we can combine them in such a way that they do

If an expression is linear and satisfies Laplace's equation then its components will as well

Our general solution:

$$V(x,y) = \sum_{i=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

We want to pick coefficients s.t.

$$V(0,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$ ← Fourier Series!

Recovering Coefficients:

Multiply both sides by $\sin\left(\frac{n'\pi y}{a}\right)$ and integrate from 0 to a

$$\sum_{i=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy$$

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \begin{cases} 0 & \text{if } n' \neq n \\ a/2 & \text{if } n' = n \end{cases}$$

Series terms drop out

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

A set of functions $f_n(y)$ is said to be complete if any other function $f(y)$ can be expressed as a linear combination of them

$$f(y) = \sum_{i=1}^{\infty} C_n f_n(y)$$

A set of functions is orthogonal if the integral of the product of any two different members of the set is 0

$$\int_0^a f_n(y) f_{n'}(y)\, dy = 0 \quad \text{for } n' \neq n$$

Spherical Coordinates

Laplaces equation in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Assume azimuthal symmetry (V is independent of ϕ)

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

Want solutions of the form

$$V(r,\theta) = R(r)\,\Theta(\theta)$$

Plugging into Laplace's Equation

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = 0$$

Once again since each term is independent we can write them as constants of the form $\ell(\ell+1)$ with differing signs

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1) \qquad \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)$$

ODE: $\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1)R \qquad \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)\Theta\sin\theta$

General Soln: $R(r) = Ar^{\ell} + \frac{B}{r^{\ell+1}}$

Not easily solved

Special class of functions called Legendre Polynomials

$\Theta(\theta) = P_\ell \cos(\theta)$

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \quad \leftarrow \text{Rodrigues Formula}$$

only has one solution since other solutions violate physical laws

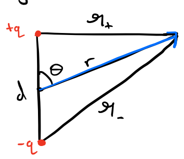
Singular Solution: $V(r,\theta) = \left(Ar^\ell + \frac{B}{r^{\ell+1}}\right) P_\ell \cos\theta$

Our general Solution: $\sum_{\ell=0}^{\infty}\left(Ar^\ell + \frac{B}{r^{\ell+1}}\right) P_\ell \cos\theta$

3.4 Multipole Expansion

Consider an two equal and opposite charges $\pm q$ separated by a distance d (Electric dipole).

Solving for potential for from the dipole



$$V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathfrak{r}_+} - \frac{q}{\mathfrak{r}_-}\right)$$

Applying the law of cosines

$$\mathfrak{r}_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

Since $r \gg d$

$$\frac{1}{\mathfrak{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

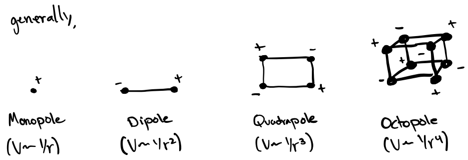
$$\frac{1}{\mathfrak{r}_+} - \frac{1}{\mathfrak{r}_-} \cong \frac{d}{r^2}\cos\theta$$

Finally,

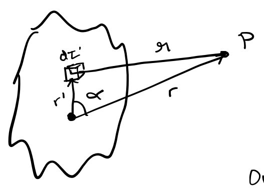
$$V(r) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$

Notice that the dipole potential decays at $1/r^2$ at large r

More generally,



Generalized charge distribution



$$V(r) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathfrak{r}}\rho(r')d\tau$$

Once again applying the law of cosines

$$\mathfrak{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$\mathfrak{r} = r\sqrt{1+\epsilon}$ where $\epsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$

$$\frac{1}{\mathfrak{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^2 + \cdots\right) = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \cdots\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \dots\right]$$ ← collecting powers of $\left(\frac{r'}{r}\right)$

The coefficients of $\left(\frac{r'}{r}\right)$ are simply Legendre polynomials!

$$\frac{1}{\eta} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\,\rho(r')\,d\tau'$$

↑ Multipole expansion

first term is monopole, second term is dipole, etc.

Ordinarily monopole term dominates at large r

If the total charge is 0 then the dipole contribution dominates

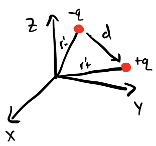
$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\int r'\cos\alpha\,\rho(\vec{r}')\,d\tau'$$

$$r'\cos\alpha = \hat{r}\cdot r'$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{r^2}\,\hat{r}\cdot\underbrace{\int r'\,\rho(r')\,d\tau'}_{\text{dipole moment}}$$

For physical dipole

$$p = qr_+' - qr_-' = q(r_+' - r_-') = qd$$



A perfect dipole is a point dipole which occurs as $d \to 0$ and $q \to \infty$

A point charge at the origin is a pure monopole

Moving the origin can change the multipole expansion

Monopole moment or total charge Q is constant

If the total charge is 0, then the dipole moment is independent of origin

Finding Electric field of a dipole

Choose coordinates such that p is at the origin and points in the z-direction

$$V_{dip}(r,\theta) = \frac{\hat{r}\cdot p}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$

$$E_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

Chapter 4: Electric Fields in Matter

4.1 Polarization

Electrons in dielectrics/insulators are bound to their atoms

Electric fields can distort the charge distribution in a diadectric object via stretching and rotating

Induced Dipoles

$\vec{E}$ acting on a neutral atom will pull the positive nucleus in the opposite direction of electrons

If $\vec{E}$ is strong enough it will ionize the atom

Usually an equilibrium is reached between $\vec{E}$ and attractive forces between electrons and the nucleus

Polarized state

Polarized atoms have small dipole moments pointing in the same direction as $\vec{E}$.

$$\vec{p} = \alpha \vec{E}$$

α is the atomic polarizability

Molecules are more complicated where they polarize more readily in certain directions

General Form:

$$P_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$$

$$P_y = \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z$$

$$P_z = \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z$$

→ coefficients form a polarizability tensor

can select principal axis that diagonalize the matrix

Some molecules have inherent dipole moments (i.e. Water)

When placed in an uniform electric field

Net Force on each end cancels to 0

$$F_+ = q\vec{E} \qquad F_- = -q\vec{E}$$

Torque, however, will exist

$$\vec{N} = (r_+ \times F_+) + (r_- \times F_-)$$

$$= \left[\frac{d}{2} \times qE\right] + \left[-\frac{d}{2} \times -qE\right] = qd \times E$$

$$= \vec{p} \times \vec{E}$$

Recall $p = qd$

N will work to allign $\vec{p}$ parallel with $\vec{E}$

If the field is nonuniform then there will be a net force

$$F = F_+ + F_- = q(E_+ - E_-) = q\,\Delta E$$

For small dipoles we can estimate

$$\Delta E = (d \cdot \nabla)E$$

$$F = (p \cdot \nabla)E$$

Diaelectric materials in an electric field will produce many small dipoles pointing in the direction of $\vec{E}$

Polarization is the measure of dipole moment per unit volume

4.2 The Field of a Polarized Object

Bound Charges

Consider a polarized material with P, the dipole moment per unit volume

Recall that for a single dipole

$$V(r) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}$$

Generalizing for a volume ($p = P\,d\tau'$)

$$V(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{P(r')\cdot\vec{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

Notice

$$\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \quad \leftarrow \text{from Chapter 1}$$

$$V = \frac{1}{4\pi\varepsilon_0} \int_V \mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

Integration by parts and integration tricks yield:

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_V \nabla'\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right) d\tau' - \int_V \frac{1}{\mathscr{r}}(\nabla'\cdot P)\, d\tau'\right]$$

Now using the divergence theorem

$$V = \underbrace{\frac{1}{4\pi\varepsilon_0} \oint_S \frac{1}{\mathscr{r}} \mathbf{P}\cdot d\vec{a}'}_{\text{Potential from surface charge}} - \underbrace{\frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{\mathscr{r}} (\nabla'\cdot\mathbf{P})\, d\tau'}_{\text{Potential from Volume charge}}$$

Denote

$$\sigma_b = \mathbf{P}\cdot\hat{n} \qquad \rho_b = -\nabla\cdot\mathbf{P}$$

Finally,

$$V(r) = \frac{1}{4\pi\varepsilon_0} \oint_S \frac{\sigma_b}{\mathscr{r}}\, da' + \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_b}{\mathscr{r}}\, d\tau' \quad \leftarrow \text{Potentials from Bound Charges}$$

4.3 The Electric Displacement

The effect of polarization is to create bound charge ρ_b within the dialectric and σ_b on the surface

Total charge density within a dialectric

$$\rho = \underbrace{\rho_b}_{\text{bound charge}} + \underbrace{\rho_f}_{\text{free charge}}$$

Gauss's Law:

$$\varepsilon_0\nabla\cdot\vec{E} = \rho = \rho_b + \rho_f = -\nabla\cdot P + \rho_f$$

$$\nabla\cdot(\varepsilon_0\vec{E} + \vec{\mathbf{P}}) = \rho_f$$

Denote

$$\varepsilon_0\vec{E} + \vec{\mathbf{P}} = \vec{D} \quad \leftarrow \text{Electric displacement}$$

Now,

$$\nabla\cdot\vec{D} = \rho_f$$

$$\oint \vec{D}\cdot d\vec{a} = Q_{f_{enc}}$$

While convinent, $\vec{D}$ is not analagous to $\vec{E}$

Colomb's Law doesn't hold for $\vec{D}$

Specifically, the curl of $\vec{D}$ is not necessarily 0

$$\nabla \times D = \nabla \times (\varepsilon_0 E + P) = \varepsilon_0 (\nabla \times E) + (\nabla \times P) = \nabla \times P$$

Furthermore, there is no potential for D

$\vec{D}$ at boundaries

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

$$D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$$

Analagous for $\vec{E}$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$$

$$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$$

4.4 Linear Dielectrics

Usually

$$P = \varepsilon_0 \chi_e \vec{E}$$

χ_e is the electric susceptibility of the medium

Materials that obey the above relation are linear dielectrics

In linear dielectrics

$$D = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 E + \varepsilon_0 \chi_e E = \varepsilon_0 (1 + \chi_e) E$$

D is proportional to E

$$D = \epsilon E \quad \text{where} \quad \epsilon = \varepsilon_0 (1 + \chi_e)$$

permittivity of the material

ε_0 is the permittivity of free space

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$ ← relative permittivity or dielectric constant

Linear dielectrics are still victim to the issues with $\nabla \times D \neq 0$ since the permittivity on either side of the boundary is not constant

However, if the space is entirely filled with a homogenous linear dielectric, $\nabla \times D = 0$

$$\nabla \times D = 0 \qquad \nabla \cdot D = \rho_f$$

$$D = \varepsilon_0 E_{vac}$$

$$E = \frac{1}{\epsilon} D = \frac{1}{\epsilon_r} E_{vac}$$ ← field is reduced by $\frac{1}{\epsilon_r}$

Boundary Value Problems

For a homogeneous isotropic linear dielectric

$$\rho_b = -\nabla \cdot P = -\nabla \cdot \left(\varepsilon_0 \frac{\chi_e}{\epsilon} D \right) = -\left(\frac{\chi_e}{1 + \chi_e} \right) \rho_f$$

Unless charge is embedded in the material, $\rho = 0$ so net charge must be at the surface

$$\epsilon_{above} E^{\perp}_{above} - \epsilon_{below} E^{\perp}_{below} = \sigma_f$$

$$\epsilon_{above} \frac{\partial V_{above}}{\partial n} - \epsilon_{below} \frac{\partial V_{below}}{\partial n} = -\sigma_f$$

$$V_{above} = V_{below}$$ ← V is continuous

A capacitor filled with a linear dielectric has capacitance

$$C = \epsilon_r C_{vac}$$

Since $W = \frac{1}{2} CV^2$ work is also increased by ϵ_r

Proof: For an addition of $\Delta \rho_f$ to the free charge

$$\Delta W = \int \Delta \rho_f V \, d\tau$$

Since $\nabla \cdot D = \rho_f$, $\Delta \rho_f = \nabla \cdot (\Delta D)$

$$\Delta W = \int [\nabla \cdot (\Delta D)] V \, d\tau$$

Now, $\nabla \cdot [(\Delta D) V] = [\nabla \cdot (\Delta D)] V + \Delta D \cdot (\nabla V)$

Integrating by parts:

$$\Delta W = \int \nabla \cdot [(\Delta D) V] \, d\tau + \int (\Delta D) \cdot E \, d\tau$$

divergence theorem turns this into a surface integral

$$\Delta W = \int (\Delta D) \cdot E \, d\tau$$

Because we have a linear dielectric $D = \epsilon E$

$$\frac{1}{2} \Delta (D \cdot E) = \frac{1}{2} \Delta (\epsilon E^2) = \epsilon (\Delta E) \cdot E = (\Delta D) \cdot E$$

$$\Delta W = \Delta \left(\frac{1}{2} \int D \cdot E \, d\tau \right)$$

Finally,

$$W = \frac{1}{2} \int \vec{D} \cdot \vec{E} \, d\tau$$

Chapter 5: Magnetostatics

5.1 The Lorentz Force Law

Moving charges generate a magnetic field, $\vec{B}$

Lorentz Force Law:

$$F_{mag} = Q(\vec{v} \times \vec{B})$$

magnetic force on a charge Q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$

In the presence of electric and magnetic fields

$$F = Q[\vec{E} + (\vec{v} \times \vec{B})]$$

Lorentz force law is axiomatic

Cyclotron Formula:

$$QvB = m\frac{v^2}{R} \quad \text{or} \quad p = QBR$$

Applicable to a charged particle moving counterclockwise with a velocity v in a circle of radius R and B field pointing into the page.

Can be applied to find the momentum of a particle (measuring radius of motion in a known B and a particle with known Q)

If the particle initially has $\perp$ velocity it will travel in a helical fashion

Magnetic Forces do no work

Suppose Q moves an amount $d\vec{l} = \vec{v}dt$

$$dW_{mag} = F_{mag} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v}dt = 0$$

$(\vec{v} \times \vec{B})$ is perpendicular to $\vec{v}$ so $(\vec{v} \times \vec{B}) \cdot \vec{v} = 0$

Magnetic force only alters the direction a particle may move

Current in a wire is charge per unit time passing a given point

Usually negative charges flow but by convention we study positive charges moving

Current is measured in amperes (A): $1A = 1\,C/s$

A line charge moving at velocity $\vec{v}$ constitutes a current

$$\vec{I} = \lambda\vec{v}$$

Magnetic force on a segment of current carrying wire

$$F_{mag} = \int (\vec{v} \times \vec{B})\, dq = \int (\vec{v} \times \vec{B})\lambda\, dl = \int (\vec{I} \times \vec{B})\, d\vec{l}$$

or simply,

$$F_{mag} = \int I(d\vec{l} \times \vec{B}) \quad \text{when } \vec{I} \text{ and } d\vec{l} \text{ point in the same direction}$$

When current is constant

$$F_{mag} = I\int (d\vec{l} \times \vec{B})$$

When charge flows over a surface we describe it as a surface charge density, $\vec{K}$

$$K = \frac{d\vec{I}}{dl_\perp} \quad \text{where } dl_\perp \text{ is an infinitesimal ribbon running parallel to current flow}$$

current per unit width

$$\vec{K} = \sigma\vec{v} \quad \text{for surface charge density } \sigma$$

$$F_{mag} = \int (\vec{v} \times \vec{B})\underbrace{\sigma\, da}_{dq} = \int (\vec{K} \times \vec{B})\, da$$

$\vec{B}$ is discontinuous at a surface current

Volume charge density, J refers to flow of charge through a 3-D region

$$J = \frac{d\vec{I}}{d\vec{a}_\perp}$$

For charge density ρ

$$F_{mag} = \int (\vec{v} \times \vec{B})\rho\, d\tau = \int (\vec{J} \times \vec{B})\, d\tau$$

Total current crossing a surface S can be written

$$I = \int_S J\, da_\perp = \int_S \vec{J} \cdot d\vec{a}$$

Charge leaving a volume

$$\oint_S \vec{J} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{J})\, d\tau$$

Since charge is conserved, what flows out of the volume must come from inside

$$\int_V (\nabla \cdot \vec{J})\, d\tau = -\frac{d}{dt} \int_V \rho\, d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

Since this holds for arbitrary volume V

Continuity Equation:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$ ← local statement of charge conservation

General dimensional correspondence

$$\sum_{i=1}^{n} (\,)\, q_i v_i \sim \int_{\text{line}} (\,)\, \vec{I}\, d\ell \sim \int_{\text{surface}} (\,)\, \vec{K}\, da \sim \int_{\text{volume}} (\,)\, \vec{J}\, d\tau$$

Analagous,

$$q \sim \lambda\, d\ell \sim \sigma\, da \sim \rho\, d\tau$$

5.2 The Biot-Savart Law

Steady currents create magnetic fields that are constant in time (magnetostatics)

Analagous to stationary charges creating constant electric fields (electrostatics)

Steady currents have a continuous flow

$$\frac{\partial \rho}{\partial t} = 0 \,, \quad \frac{\partial J}{\partial t} = 0$$

Applying the continuity equation to magnetostatics we find that

$$\nabla \cdot \vec{J} = \frac{-\partial \rho}{\partial t} = 0$$

Biot-Savart Law

$$B(r) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\ell' = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$ ← magnetic field of a steady current

$$\mu_0 = 4\pi \cdot 10^{-7}\ \frac{N}{A^2}$$ ← permeability of free space

B has units $\frac{N}{A \cdot m}$ = tesla (T)

Analagous to Coulomb's law in electrostatics

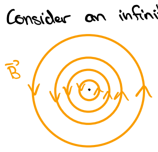
In higher dimensions,

$$B(r) = \frac{\mu_0}{4\pi} \int \frac{K(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da' \quad \text{and} \quad B(r) = \frac{\mu_0}{4\pi} \int \frac{J(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

Superposition also applies to magnetic fields

5.3 The Divergence and Curl of B

Consider an infinite straight wire with current flowing out of the page



There is very clearly a non-zero curl in $\vec{B}$

$$\oint \vec{B} \cdot d\vec{\ell} = \oint \frac{\mu_0 I}{2\pi s}\, d\ell = \frac{\mu_0 I}{2\pi s} \oint d\ell = \mu_0 I$$

Answer is independent of s since B decreases at the same rate that the circumference increases

More generally,

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$ ← total current enclosed in integration path

$$I_{enc} = \int \vec{J} \cdot d\vec{a}$$

Therefore,

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

Now, applying Stoke's Theorem

$$\int (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

Finally,

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$ ← However, this derivation is contingent on infinite straight wires

Generalized Divergence and curl of a magnetic field

$$B(r) = \frac{\mu_0}{4\pi} \int \frac{J(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$

Note: B is a function of (x, y, z)

J is a function of (x', y', z')

$\mathscr{r} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$

$d\tau' = dx'dy'dz'$

Notice that integration is over primed coordinates but curl of B is our unprimed coordinates

Applying Divergence Theorem

$$\nabla \cdot \vec{B} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

Recall: $\nabla \cdot \left(\vec{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \cdot (\nabla \times J) - J \cdot \left(\nabla \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right)$

We know $\nabla \times J = 0$ because J is independent of unprimed variables

Also, $\nabla \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0$

$$\nabla \cdot B = 0$$

Now, let's consider curl

$$\nabla \times B = \frac{\mu_0}{4\pi} \int \nabla \times \left(J \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

Recall: $\nabla \times \left(J \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = J \left(\nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) - (J \cdot \nabla) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

$\nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi \delta^3(\mathscr{r})$

Let's consider the second term

$$-(J \cdot \nabla) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (J \cdot \nabla') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

In the x - component,

$$(J \cdot \nabla') \left(\frac{x-x'}{\mathscr{r}^3} \right) = \nabla' \left(\frac{(x-x')}{\mathscr{r}^3} \vec{J} \right) - \left(\frac{x-x'}{\mathscr{r}^3} \right) (\nabla' \cdot J)$$

Recall that $\nabla' \cdot J = 0$

$$(J \cdot \nabla') \left(\frac{x-x'}{\mathscr{r}^3} \right) = \nabla' \left(\frac{(x-x')}{\mathscr{r}^3} \vec{J} \right)$$

Under integration,

$$\int_V \nabla' \left(\frac{(x-x')}{\mathscr{r}^3} J \right) d\tau' = \oint_S \frac{(x-x')}{\mathscr{r}^3} \vec{J} \cdot da'$$

If you make the boundary large enough, $J = 0$ and this term vanishes

$$\nabla \times B = \frac{\mu_0}{4\pi} \int J(r') 4\pi \delta^3(r-r') d\tau' = \mu_0 J(r)$$

Ampere's Law

$$\nabla \times B = \mu_0 J$$

Applying Stoke's theorem we can rewrite it as

$$\int (\nabla \times B) \cdot da = \oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a} = \mu_0 I_{enc}$$

Current enclosed by an Amperian loop

Direction of positive current is the direction your right thumb points when your fingers envelope the boundary

Ampere's Law is analogous to gauss's Law

Only useful for nice symmetries

1. Infinite straight lines (Ex. 5.7)
2. Infinite planes (Ex. 5.8)
3. Infinite Solenoids (Ex. 5.9)
4. Toroids (Ex. 5.10)

Magnetostatics vs. Electrostatics

Electrostatic Fields

Maxwell's Equations for Electrostatics
$$\begin{cases} \nabla \cdot E = \frac{1}{\varepsilon_0}\rho & \text{Gauss's Law} \\ \nabla \times E = 0 \end{cases}$$

Along w/ the boundary conditions $E \to 0$ far from all charges, Maxwell's equations fully describe the field

Magnetostatic Fields

Maxwell's Equations for magnetostatics
$$\begin{cases} \nabla \cdot B = 0 \\ \nabla \times B = \mu_0 J & \text{Ampere's Law} \end{cases}$$

Along w/ the boundary condition $B \to 0$ far from all currents, Maxwell's equations determine the magnetic field

Electric field diverges from positive charge and the magnetic field curls around a current

There are no point sources of B $(\nabla \cdot B = 0)$

B is divergenceless and there are no magnetic monopoles

5.4 Magnetic Vector Potential

Analogous to scalar potential in electrostatics, $\vec{E} = -\nabla V$, we can define a vector potential in magnetostatics

$$B = \nabla \times A$$

Ampere's Law

$$\nabla \times B = \nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A = \mu_0 \vec{J}$$

You can add any function to A whose curl vanishes (Analogous to scalar addition to potential)

Using this freedom we can define $\nabla \cdot A = 0$

Proof: Suppose A_0 has non-zero divergence

$$A = A_0 + \nabla\lambda$$

$$\nabla \cdot A = \nabla \cdot A_0 + \nabla^2 \lambda$$

$$\nabla^2 \lambda = -\nabla \cdot A_0 \quad \leftarrow \text{takes the form of poisson's equation}$$

$$\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot A_0}{\mathscr{r}}\, d\tau'$$

It is always possible to make the vector potential divergenceless

Ampere's Law

$$\nabla^2 A = -\mu_0 J$$

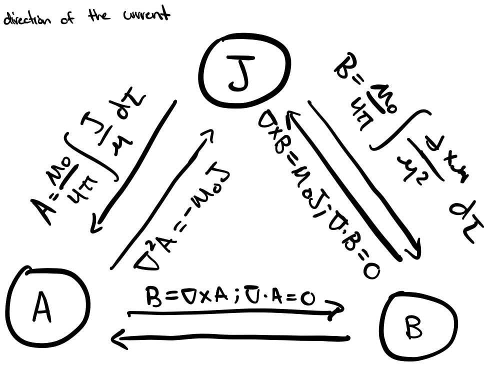
Assuming J goes to 0 at infinity

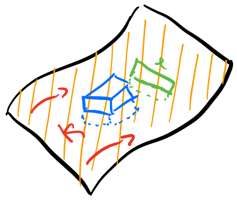
$$A(r) = \frac{\mu_0}{4\pi}\int \frac{J(r)}{\mathscr{r}}\, d\tau' \qquad \text{for } \underline{\text{line}}: \; A = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{\mathscr{r}}\, dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{r}}\, d\vec{l}' \qquad \underline{\text{Surface}}: \; A = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{\mathscr{r}}\, da'$$

If A doesn't go to 0 at infinity special considerations must be made

A isn't nearly as useful as V

Typically, the direction of A mimics the direction of the current





Consider a gaussian pill box on a surface

Applying $\oint B \cdot da = 0$

we find $B^{\perp}_{above} = B^{\perp}_{below}$

To find tangential components, consider an amperian loop perpendicular to current

$$\oint B \cdot d\vec{l} = \left(B^{\parallel}_{above} - B^{\parallel}_{below}\right) l = \mu_0 I_{enc} = \mu_0 K l$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$$

discontinuous by $\mu_0 K$

An amperian loop parallel to the current finds the components to be continuous

$$B_{above} - B_{below} = \mu_0 (K \times \hat{n})$$

Vector potential is continuous across any boundary

The derivative of A has the discontinuity of B

$$\frac{\partial A_{above}}{\partial n} - \frac{\partial A_{below}}{\partial n} = -\mu_0 \vec{K}$$

Using a multipole expansion,

$$A(r) = \frac{\mu_0 I}{4\pi} \oint \frac{1}{\mathscr{r}} d\vec{l}' = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\alpha)\, d\vec{l}'$$

Magnetic monopole term is always 0

$$\oint d\vec{l}' = 0$$

Assumption from $\nabla \cdot B = 0$

$$A_{dip} = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha\, d\vec{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot r')\, d\vec{l}'$$

Recall: $\oint (\hat{r} \cdot r')\, d\vec{l}' = -\hat{r} \times \int da'$

$$A_{dip}(r) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

$m = I \int da = I\vec{a}$ is the magnetic dipole moment

Magnetic dipole moment is independent of origin

Chapter 6: Magnetic Fields in Matter

6.1 Magnetization

Electron movement in an atom technically creates a magnetic field

These movements are small enough for us to consider them magnetic dipoles

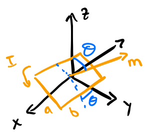
Generally, these magnetic contributions cancel eachother out, but when a magnetic field is applied, the magnetic dipoles allign.

This state of allignment is known as magnetization

These magnetic fields don't always allign with applied $\vec{B}$

- Paramagnets are parallel to $\vec{B}$
- Diamagnets are opposite to $\vec{B}$
- Ferromagnets retain magnetization after removal of the external field
  - Magnetization is determined by "history" of object

Consider the torque on a rectangular current loop in a uniform field $\vec{B}$



Loop centered at the origin and tilted by θ in the y-direction

Suppose $\vec{B}$ points in the z-direction

Forces on slanted edges cancel eachother out

The other edges generate a torque

$$\vec{N} = aF\sin\theta\,\hat{x}$$

$$F = IbB$$

$$N = IabB\sin\theta\,\hat{x} = mB\sin\theta\,\hat{x} = \underset{\text{magnetic dipole moment}}{\vec{m}} \times \vec{B}$$

Torque acts to allign the dipole parallel to the field

This torque is responsible for paramagnetism

Pauli's exclusion principle locks electrons into pairs with opposing spin which effectively cancel the effect of torque

Paramagnetis is therefore mainly seen in species with an odd number of electrons

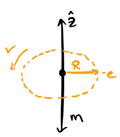
Net force on a current loop in an uniform field is 0

$$\vec{F} = I\oint(d\ell \times B) = I\oint d\ell \times B = 0$$

However, in a non-uniform field an infinitesimal loop has force

$$\vec{F} = \nabla(\vec{m}\cdot\vec{B})$$

Electrons also revolve around the nucleus



Technically orbital motion doesn't satisfy the conditions of a steady current, but it can be estimated as such

$$T = \frac{2\pi R}{v}$$

$$I = \frac{-e}{T} = \frac{-ev}{2\pi R}$$

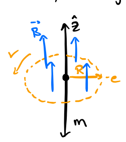
Corresponding dipole moment

$$m = I\pi R^2\hat{z} = -\frac{1}{2}evR\hat{z}$$

Similar to electron spin, this dipole moment will be subjected to a torque under $\vec{B}$

This torque contribution is minimal

Instead, the significant contribution is how $\vec{B}$ speeds up or slows down e^-



Suppose $\vec{B}$ is perpendicular to the plane of orbit

Centripetal Acceleration:

$$\underbrace{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2}}_{\text{electric forces}} + e\bar{v}B = m_e\frac{\bar{v}^2}{R}$$

$$e\bar{v}B = m_e\frac{\bar{v}^2}{R} - \underbrace{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2}}_{\text{usual contribution to } a_c\ (m_e\frac{v}{R})} = \frac{m_e}{R}(\bar{v}^2 - v^2)$$

For small $\Delta v = \bar{v} - v$

$$\Delta v = \frac{eRB}{2m_e} \leftarrow \text{derived from binomial expansion}$$

This in turn changes the dipole moment

$$\Delta m = -\frac{1}{2}e(\Delta v)R\hat{z} = -\frac{e^2R^2}{4m_e}\vec{B}$$

↳ Notice dipole opposes magnetic field

Ordinarily orbital dipole moments cancel eachother out, but under an external magnetic field each orbit picks up an antiparallel dipole moment

This antiparallel dipole moment is responsible for diamagnetism

Diamagnetism is weaker than paramagnetism so it is only detected in even

Regardless of its source, we discuss magnetic polarization it terms of magnetization

$M \equiv$ magnetic dipole moment per unit volume

In general, a paramagnet is attracted to a field and a diamagnet is repelled away

6.2 The Field of a Magnetized Object

Recall, for a single dipole

$$A(r) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

Extrapolating to a magnetized object with magnetization M

$$A(r) = \frac{\mu_0}{4\pi} \int \frac{M(r') \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, d\tau'$$

Recall $\nabla' \frac{1}{\mathscr{r}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

$$A(r) = \frac{\mu_0}{4\pi} \int \left[M(r') \times \left(\nabla' \frac{1}{\mathscr{r}} \right) \right] d\tau'$$

Integration by parts

$$A(r) = \frac{\mu_0}{4\pi} \left[\int \frac{1}{\mathscr{r}} \left(\nabla' \times M(r') \right) d\tau' - \int \nabla' \times \left[\frac{M(r')}{\mathscr{r}} \right] d\tau' \right]$$

Expressing as a surface integral

$$A(r) = \frac{\mu_0}{4\pi} \int \frac{1}{\mathscr{r}} \left[\nabla' \times M(r') \right] d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\mathscr{r}} \left[M(r') \times da' \right]$$

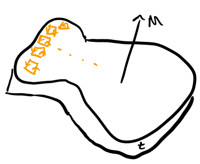
Potential of a Volume Current — $J_b = \nabla \times M$

Potential of a surface current — $K_b = M \times \hat{n}$

Finally,

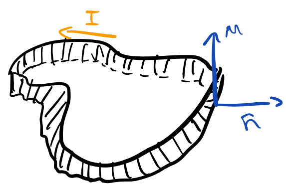
$$A(r) = \frac{\mu_0}{4\pi} \int_V \frac{J_b(r')}{\mathscr{r}} \, d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{K_b(r')}{\mathscr{r}} \, da'$$

Conceptually, consider a thin slab of uniform magnetization



At the edges there is no adjacent current to "cancel" out each mini current

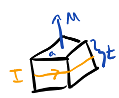
This edge current can be considered a contiguous current flowing around the boundary



For current loops of area a and thickness t, the dipole moment is given by

$$\vec{m} = M a t = I a$$

M · volume — Current · Area

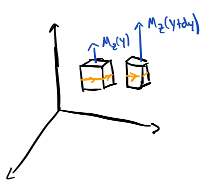


Therefore, $I = Mt$. Furthermore, $K_b = \frac{I}{t} = M$

Direction of $K_b = M \times \hat{n}$

No single charge makes the entire trip but it acts as a real current

For a non-uniform magnetic field the currents don't cancel



$$I_x = \left[M_z(y+dy) - M_z(y) \right] dz = \frac{\partial M_z}{\partial y} dy \, dz$$

$$J_{b_x} = \frac{\partial M_z}{\partial y}$$

In the y-direction we'd find the analagous

$$J_{b_x} = -\frac{\partial M_y}{\partial z}$$

$$J_b = \nabla \times M \longleftarrow \text{Generalized}$$

6.3 The Auxiliary Field H

Let's decompose the current flowing through our material into bound and free components

$$J = J_b + J_f \qquad J_b = \nabla \times M \quad \text{and} \quad K_b = M \times \hat{n}$$

Ampere's Law now takes the form

$$\frac{1}{\mu_0}(\nabla \times B) = J = J_f + J_b = J_f + (\nabla \times M)$$

$$\nabla \times \left(\frac{1}{\mu_0} B - M\right) = J_f$$

We can rewrite this as

$$\nabla \times H = J_f$$

$$H = \frac{1}{\mu_0} B - M$$

Alternatively,

$$\oint H \cdot d\vec{l} = I_{f_{enc}}$$

free current passing through the Amperian loop

Divergence of $H \neq 0$

$$\nabla \cdot H = -\nabla \cdot M$$

When we have symmetries we find that $\nabla \cdot H = 0$

Don't assume H is 0 since there is no free current in sight

Boundary Conditions

$$H^{\perp}_{above} - H^{\perp}_{below} = -\left(M^{\perp}_{above} - M^{\perp}_{below}\right)$$

$$H^{\parallel}_{above} - H^{\parallel}_{below} = K_f \times \hat{n}$$

6.4 Linear and Nonlinear Media

Magnetization is usually proportional to the field

$$M = \chi_m H$$

χ_m is the magnetic susceptibility

positive for paramagnets

negative for dia-magnets

Materials that abide by this relation are said to be linear media

$$B = \mu_0 (H + M) = \mu_0 (1 + \chi_m) H$$

$$\mu = \mu_0 (1 + \chi_m)$$

μ is the magnetic permeability

Ferromagnetism

Dipoles in ferromagnets like to point in the same direction as eachother

Explained by QM

Allignment occurs in patches called domains

Domains can expand into other domains under an external magnetic field

the domain with better allignment "wins"

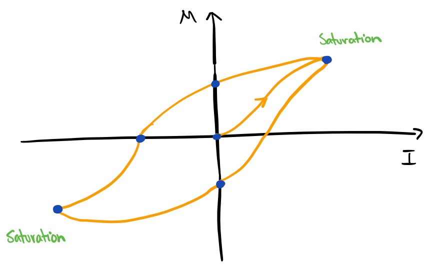
Material is said to be saturated when a single domain takes over entirely

After the external field is removed, some dipoles will revert their orientation but the majority remains in line

You can unmagnetize an object by reversing the direction of the applied field

be careful to not saturate in the other direction

Hysteresis Loop



Magnetic allignment competes with thermal motion

At the curie point a material stops being ferromagnetic (becomes paramagnetic)

7.1 Electromotive Force

Generally, current density is proportional to the force per unit charge

$$J = \sigma f$$

↑ conductivity of the medium

$\rho = 1/\sigma$ is the resistivity

Usually an electromagnetic force drives the currents

$$J = \sigma(E + vB)$$

However, since v is sufficiently small

$$J = \sigma E$$ ← Ohm's Law

Total current flowing between electrodes is proportional to the potential difference between them

$$V = IR$$ ← Ohm's Law once again

↑ Resistance

Power delivered by a current

$$P = VI = I^2R$$ ← Joule heating Law

Current is self-adjusting to keep it uniform

Build up of charge creates a field that counteracts the build up of charge

The force driving a current around the circuit can be broken up into the source and an electrostatic force which smooths out current flow

$$f = f_s + E$$

The Electromotive Force

$$\mathcal{E} = \oint f \cdot d\ell = \oint f_s \cdot d\ell$$

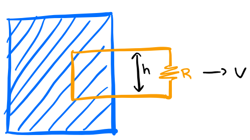
$$\oint E \cdot d\ell = 0$$

An ideal source (resistanceless battery) has a net force of 0, therefore, $E = -f_s$

$$V = -\int_a^b \vec{E} \cdot d\vec{\ell} = \int_a^b f_s \cdot d\ell = \oint f_s \cdot d\ell =$$

Battery establishes voltage difference equal to emf so that the electrostatic force can drive the current

Generators use wires moving through a magnetic field to create a source of emf



B into the page

v generates a force upwards which causes current to move in a clockwise direction due to a force with magnitude qvB

$$\mathcal{E} = \oint f_{mag} \cdot d\vec{\ell} = vBh$$

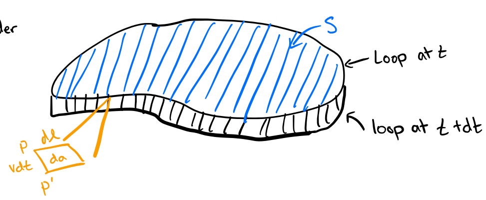
horizontal components don't contribute since they are perpendicular to the force

Work is done by the object pulling the loop, NOT magnetism

EMF can also be expressed in terms of flux

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Consider



$$d\Phi = \Phi(t+dt) - \Phi(t) = \Phi_{(ribbon)} = \int_{ribbon} B \cdot d$$

For velocity v of the wire and velocity u of charges along the wire, velocity of a charge is given by $w = u + v$

Notice: $da = (v \times d\ell)\, dt$

$$\frac{d\Phi}{dt} = \oint B \cdot (v \times d\ell) = \oint B \cdot (w \times d\ell) = -\oint (w \times B) \cdot d\ell = -\oint f_{mag} \cdot d\ell = -$$

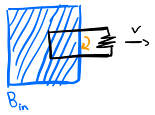
since u is parallel to $d\ell$

Recall: $B \cdot (w \times d\ell) = -(w \times B) \cdot d\ell$

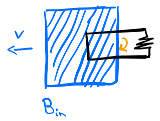
7.2 Electromagnetic Induction

Michael Faraday's Experiments

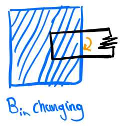
1) Pulling a wire loop through a magnetic field generated a current in the loop



2) Moving the magnetic field to the left also generated a current in the loop



3) Changing the strength of a stationary field and wire also generated a current in the loop



Changing magnetic fields generate an electric field (1)

Faraday's Law

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial B}{\partial t} \cdot da \quad \text{or} \quad \nabla \times E = -\frac{\partial B}{\partial t}$$

Changing magnetic flux produces a current in a wire (2)

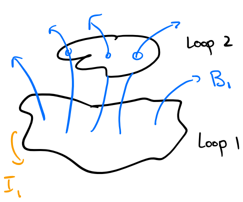
$$\mathcal{E} = -\frac{d\Phi}{dt}$$

(1) and (2) do not occur for the same reasons

This lead Einstein to develop special relativity

Lenz's Law: Nature abhors change in flux

Consider two loops of wire with current I_1 flowing through the first wire



Let Φ_2 represent the flux through loop 2 from B_1

$$B_1 = \frac{\mu_0}{4\pi} I_1 \int \frac{dl_1 \times \hat{\mathscr{r}}}{\mathscr{r}^2} \quad \text{and} \quad \Phi_2 = \int B_1 \cdot da_2$$

$\Phi_2 = M_{21} I_1$ where M_{21} is a constant of proportionality known as the mutual inductance

$$\Phi_2 = \int B_1 \cdot da_2 = \int (\nabla \times A) \cdot da_2 = \oint A_1 \cdot dl_2$$

Stoke's Thm

Recall: $A_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{dl_1}{\mathscr{r}}$

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{dl_1}{\mathscr{r}} \right) \cdot dl_2$$

Accordingly,

$$M_{21} = \frac{\mu_0}{4\pi} \oint\oint \frac{dl_1 \cdot dl_2}{\mathscr{r}} \quad \leftarrow \text{Neumann Formula}$$

Observations from mutual inductance formula

- Purely a geometric quantity
- mutual inductance is equivalent if you switch the roles of loop 1 and loop 2

$$M_{21} = M_{12}$$

If you were to vary the current in loop 1, the flux through loop 2 would vary accordingly

$$\mathcal{E}_2 = -\frac{d\Phi}{dt} = -M \frac{dI_1}{dt}$$

The flux through the wire itself will also change

$$\phi = LI$$

↑ self inductance of the loop (dependent on geometry)

$$\mathcal{E} = -L\frac{dI}{dt}$$

Inductance will oppose change in current acting like a back emf

Work to move a charge around the circuit is $-\mathcal{E}$ and rate of charge passing through the wire is given by I

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

Consequently,

$$W = \frac{1}{2}LI^2 = \frac{1}{2}I\oint A\cdot d\ell = \frac{1}{2}\oint (A\cdot I)\,d\ell$$

Recall: $LI = \Phi = \oint A\cdot d\ell$

Generalized,

Volume:

$$W = \frac{1}{2}\int_V (A\cdot J)\,d\tau$$

Once again recall $\nabla\times B = \mu_0 J$

$$W = \frac{1}{2\mu_0}\int_V A\cdot(\nabla\times B)\,d\tau$$

Applying integration by parts,

$$W = \frac{1}{2\mu_0}\left[\int_V B^2\,d\tau - \oint_S (A\times B)\cdot da\right]$$

Over all space,

$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

7.3 Maxwell's Equations

Before Maxwell, electrodynamics rules only applied to steady currents

i) $\nabla\cdot E = \frac{1}{\varepsilon_0}\rho$ Gauss's Law

ii) $\nabla\cdot B = 0$

iii) $\nabla\times E = -\frac{\partial B}{\partial t}$ Faraday's Law

iv) $\nabla\times B = \mu_0 J$ Ampère's Law

Recall the divergence of curl must be 0

$$\nabla\cdot(\nabla\times B) = 0 = \nabla\cdot(\mu_0 J) = \mu_0(\nabla\cdot J)$$

↳ only 0 when J is steady

Maxwell's correction involved adding a correction term to cancel the right hand side

$$\nabla\cdot J = -\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0\nabla\cdot E) = -\nabla\cdot\left(\varepsilon_0\frac{\partial E}{\partial t}\right)$$

So,

$\nabla\times B = \mu_0 J + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$ would abide by our $\nabla\cdot(\nabla\times B)$ condition

In magnetostatics the $\mu_0\varepsilon_0\frac{\partial E}{\partial t}$ term is 0 so $\nabla\times B = \mu_0 J$ remains

The $\mu_0\varepsilon_0\frac{\partial E}{\partial t}$ is small enough that it is only relevant in the propagation of E+M waves

$\varepsilon_0\frac{\partial E}{\partial t}$ is sometimes referred to as the displacement current

Maxwell's Equations

Force Law
$F = q(E + v\times B)$

i) $\nabla\cdot E = \frac{1}{\varepsilon_0}\rho$ Gauss's Law

ii) $\nabla\cdot B$

iii) $\nabla\times E = -\frac{\partial B}{\partial t}$ Faraday's Law

iv) $\nabla\times B = \mu_0 J + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$ Ampère's law + Maxwell correction

Electric fields are produced by charges or changing magnetic fields

Magnetic fields are produced by currents or changing electric fields

In materials we describe electric and magnetic fields with bound and free charges/currents

$$\rho_b = -\nabla \cdot P \qquad J_b = \nabla \times M$$

Any change in polarization leads to a movement of charge (i.e. a current), J_p

$$J_p = \frac{\partial P}{\partial t}$$ ← Polarization current

Charge Density:

$$\rho = \rho_b + \rho_f = \rho_f - \nabla \cdot P$$

Current Density

$$J = J_f + J_b + J_p = J_f + \nabla \times M + \frac{\partial P}{\partial t}$$

Maxwell's Equations now become

i) $\nabla \cdot D = \rho_f$ iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

ii) $\nabla \cdot B = 0$ iv) $\nabla \times H = J_f + \frac{\partial D}{\partial t}$

$$\underbrace{D = \epsilon_0 E + P \qquad H = \frac{1}{\mu_0} B - M}_{\text{Requires definitions of } D \text{ and } H \text{ in terms of } E \text{ and } B}$$

The fields E, B, D and H will be discontinuous at a boundary between two media or at a surface with charge σ or current K

Integral form of Maxwell's Equations

i) $\oint_S D \cdot da = Q_{f_{enc}}$ iii) $\oint_P E \cdot d\ell = -\frac{d}{dt} \int_S B \cdot da$

Closed surface S with boundary P

ii) $\oint_S B \cdot da = 0$ iv) $\oint_P H \cdot d\ell = I_{f_{enc}} + \frac{d}{dt} \int_S D \cdot da$

Corresponding Boundary Conditions (General Conditions and Linear Media)

i) $D_1^{\perp} - D_2^{\perp} = \sigma_f$

$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$

iii) $E_1^{\parallel} - E_2^{\parallel} = 0$

$E_1^{\parallel} - E_2^{\parallel} = 0$

ii) $B_1^{\perp} - B_2^{\perp} = 0$

$B_1^{\perp} - B_2^{\perp} = 0$

iv) $H_1^{\parallel} - H_2^{\parallel} = K_f \times \hat{n}$

$\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = K_f \times \hat{n}$

Chapter 8: Conservation Laws

8.1 Charge and Energy

Conservation of charge

$$Q(t) = \int_V \rho(r,t)\, d\tau$$ ← Total charge in a volume

$$\oint_S J \cdot da$$ ← Charge flow through a boundary

$$\frac{dQ}{dt} = -\oint_S J \cdot da$$ ← Conservation of charge

Combining the equations and applying the divergence theorem

$$\int_V \frac{d\rho}{dt}\, d\tau = -\int_V \nabla \cdot J\, d\tau$$ ← An arbitrary volume

Thus,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot J$$

Conservation of charge is baked into maxwell's equations

Let's consider a system at time t with E and B

For a change in time dt during which some work is done

$$F \cdot d\ell = q(E + v \times B) \cdot v dt = qE \cdot v dt$$

This generalizes with $q \to \rho d\tau$ and $\rho v \to J$

So work becomes

$$\frac{dW}{dt} = \int_V (E \cdot J)\, d\tau$$ Notice $E \cdot J$ is the work done per unit time per unit volume ie. power per unit volume

$$E \cdot J = \frac{1}{\mu_0} E \cdot (\nabla \times B) - \epsilon_0 E \cdot \frac{\partial E}{\partial t}$$ ← Applying Ampere-Maxwe

Recall: $\nabla \cdot (E \times B) = B \cdot (\nabla \times E) - E \cdot (\nabla \times B)$

$$E \cdot (\nabla \times B) = B \cdot (\nabla \times E) - \nabla \cdot (E \times B)$$

Faraday's Law : $\nabla \times B = -\frac{\partial B}{\partial t}$

$$E \cdot (\nabla \times B) = -B\frac{\partial B}{\partial t} - \nabla \cdot (E \times B)$$

Notice :

$$B\frac{\partial B}{\partial t} = \frac{1}{2}\frac{\partial (B^2)}{\partial t} \quad \text{and} \quad E\frac{\partial E}{\partial t} = \frac{1}{2}\frac{\partial (E^2)}{\partial t}$$

Putting it all together

$$E \cdot J = \frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) - \frac{1}{\mu_0}\nabla \cdot (E \times B)$$

$$\frac{dW}{dt} = -\frac{\partial}{\partial t}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau - \mu_0 \oint (E \times B) \cdot da$$ ← Poynting Theorem "Work Energy theorem" of electrodynamics

Work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface

Ponyting Vector

$$S = \frac{1}{\mu_0}(E \times B)$$ ← Energy per unit time per unit area transported by fields

Akin to energy flux density

Energy stored in electromagnetic fields per volume

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

We can now express Poynting's Theorem

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V u \, d\tau - \oint_S S \cdot da$$

When we are in empty space, $\frac{dW}{dt} = 0$

$$\int \frac{\partial u}{\partial t} d\tau = -\oint S \cdot da = -\int (\nabla \cdot S)\, d\tau$$

$$\frac{\partial u}{\partial t} = -\nabla \cdot S \quad \leftarrow \text{Continuity equation for energy}$$

8.2 Momentum

Momentum is stored in fields in electrodynamics; otherwise momentum is not conserved

Calculate total electromagnetic force on the charges in a volume V

$$F = \int_V (E + v \times B)\rho \, d\tau = \int_V (\underbrace{\rho E + J \times B}_{\text{force per unit volume}})\, d\tau$$

Express this relation in terms of Maxwell's equations

$$f = \rho E + J \times B = \epsilon_0 (\nabla \cdot E) E + \left(\frac{1}{\mu_0}\nabla \times B - \epsilon_0 \frac{\partial E}{\partial t}\right) \times B$$

Notice $\frac{\partial}{\partial t}(E \times B) = \frac{\partial E}{\partial t} \times B + E \times \frac{\partial B}{\partial t}$ and $\frac{\partial B}{\partial t} = -\nabla \times E$ from Faraday's Law

$$E \times \frac{\partial B}{\partial t} = \frac{\partial}{\partial t}(E \times B) + E \times (\nabla \times E)$$

$$f = \epsilon_0 \left[(\nabla \cdot E)E - E \times (\nabla \times E)\right] - \frac{1}{\mu_0}\left[B \times (\nabla \times B)\right] - \epsilon_0 \frac{\partial}{\partial t}(E \times B)$$

Recall: $\nabla(E^2) = 2(E \cdot \nabla)E + 2E \times (\nabla \times E)$

$$E \times (\nabla \times E) = \frac{1}{2}\nabla(E^2) - (E \cdot \nabla)E$$

$$B \times (\nabla \times B) = \frac{1}{2}\nabla(B^2) - (B \cdot \nabla)B$$

$$f = \epsilon_0\left[(\nabla \cdot E)E + (E \cdot \nabla)E\right] + \frac{1}{\mu_0}\left[(\nabla \cdot B)B + (B \cdot \nabla)B\right] - \frac{1}{2}\nabla\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \epsilon_0 \frac{\partial}{\partial t}(E \times B)$$

Maxwell stress tensor:

$$T_{ij} = \epsilon_0\left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

if $i=j$ $\delta_{ij} = 1$ else: $\delta_{ij} = 0$

Maxwell stress tensor can clean up the expression

$$f = \nabla \cdot \overleftrightarrow{T} - \epsilon_0 \mu_0 \frac{\partial S}{\partial t}$$

Note: S is the poynting vector

$$(\nabla \cdot \overleftrightarrow{T})_j = \epsilon_0\left[(\nabla \cdot E)E_j + (E \cdot \nabla)E_j - \frac{1}{2}\nabla_j E^2\right] + \frac{1}{\mu_0}\left[(\nabla \cdot B)B_j + (B \cdot \nabla)B_j - \frac{1}{2}\nabla_j B^2\right]$$

Total electromagnetic force can be given

$$F = \oint_S \overleftrightarrow{T} \cdot da - \epsilon_0 \mu_0 \frac{d}{dt}\int_V S \, d\tau$$

in the static case the second term drops to 0

$\overleftrightarrow{T}$ is the force per unit area (stress) acting on the surface

T_{xx}, T_{yy}, T_{zz} represent pressures while off-diagonal elements represent shears

Newton's second law tells us that

$$F = \frac{dP_{mech}}{dt}$$

Therefore we can write

$$\frac{dP_{mech}}{dt} = -\epsilon_0 \mu_0 \frac{d}{dt}\int_{V} S \, d\tau + \oint_{S} \overleftrightarrow{T} \cdot d\vec{a}$$

momentum stored in fields

momentum per unit time flowing through the surface

If momentum increases then either the field momentum decreased or momentum is carried into the volume

Momentum density

$$g = \mu_0 \epsilon_0 S = \epsilon_0 (E \times B)$$

$$\frac{\partial g}{\partial t} = \nabla \cdot \overleftrightarrow{T}$$ ← continuity equation for electromagnetic momentum

To recap, fields carry

energy, u $\quad u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$ and momentum, g $\quad g = \epsilon_0 (E \times B)$ and finally, angular momentum, ℓ $\quad \ell = r \times g = \epsilon_0 [r \times (E \times B)]$

8.3 Magnetic Forces do no work

If magnetic monopoles exist, magnetic fields can do work, but only on magnetic charges

9.1 Waves in one dimension

Roughly, a wave is a disturbance of a continuous medium that propogates with a fixed shape at a constant velocity

Many caveats to this general definition

Waves can mathematically be represented as a function, $g(z-vt)$ for a wave travelling in the z-direction with a velocity v

The general wave equation:

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$ ← admits all solutions to $f(z,t) = g(z-vt)$

← Only applies in a classical setting

This wave equation also admits functions of the form $g(z+vt)$

Solutions to the wave equation are linear

Most general wave equation: $f(z,t) = g(z-vt) + g(z+vt)$

Sinusoidal Waves:

$$f(z,t) = A\cos\left[k(z-vt) + \delta\right]$$

A = Amplitude

$k(z-vt)+\delta$ = phase

δ = phase constant

k = wave number

λ = wave length

$$\lambda = \frac{2\pi}{k}$$

Period: $T = \frac{2\pi}{kv}$

Frequency: $\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}$

Angular Frequency: $\omega = 2\pi\nu = kv$

Putting it all together,

$$f(z,t) = A\cos(kz + \omega t - \delta)$$

Complex Notation:

$$f(z,t) = \mathrm{Re}\left[Ae^{i(kz-\omega t+\delta)}\right]$$

$$= \tilde{A}e^{i(kz-\omega t)}$$

where $\tilde{A} = Ae^{i\delta}$

Any wave function can be expressed as a sum of sinusoidal waves

Suppose we have two strings tied together at $z=0$

A wave transmitted from $-$ side will travel along the first rope and interact with the knot at $k=0$

Specifically, there will be a reflected wave and a transmission wave

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z-\omega t)} + \tilde{A}_R e^{i(-k_1 z-\omega t)} & z<0 \\ \tilde{A}_T e^{i(k_2 z-\omega t)} & z>0 \end{cases}$$

At $z=0$ the two equations must be continuous and smooth if the knot is massless

With these conditions in place, we find

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T, \quad k_1(\tilde{A}_1 - \tilde{A}_R) = k_2\tilde{A}_T$$

Taking the real portions

$$A_R e^{i\delta_R}\left(\frac{v_2-v_1}{v_2+v_1}\right)A_I e^{i\delta_I}, \quad A_T e^{i\delta_T}\left(\frac{2v_2}{v_2+v_1}\right)A_I e^{i\delta_I}$$

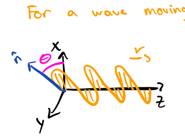
Depending on the mass differences between the strings we can determine the relation between δ_R, δ_T, and δ_I

Transverse waves have displacement perpendicular to the direction of propogation

Longitudinal waves are characterized by displacement along the direction of propogation

There are multiple directions of displacement that are perpendicular to the direction of propogation

For a wave moving in the $\hat{z}$ direction, propogation can be in any direction in the xy plane



We call $\hat{n}$ the polarization vector where $\hat{n}\cdot\hat{z} = 0$

$\hat{n} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$ for the polarization angle θ

We can consider such waves the superposition of two waves in the x and y plane respectively

9.2 Electromagnetic Waves in Vacuum

Recall Maxwell's Equations in empty space

i) $\nabla \cdot E = 0$ iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

ii) $\nabla \cdot B = 0$ iv) $\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$

If we take the curl of iii) and iv)

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = \nabla \times \left(-\frac{\partial B}{\partial t}\right)$$
$$= -\frac{\partial}{\partial t}(\nabla \times B) = -\mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\nabla \times (\nabla \times B) = \nabla(\nabla \cdot B) - \nabla^2 B = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial E}{\partial t}\right)$$
$$= \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times E) = -\mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}$$

Since the divergence of E and B is 0 we are left with

$$\nabla^2 E = \mu_0 \epsilon_0 \frac{\partial E^2}{\partial t^2} \quad \text{and} \quad \nabla^2 B = \mu_0 \epsilon_0 \frac{\partial B^2}{\partial t^2}$$ ← decoupled second order equations for E and B

Notice that this implies that each cartesian component of E and B satisfy the wave equation from earlier!

This further implies that a vacuum supports the propogation of an EM wave travelling at $V = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \cdot 10^8$ m/s

Perhaps light itself is an electromagnetic wave!

Waves travelling in the z-direction with no x or y dependence are called plane waves

$$\tilde{E}(z,t) = \hat{E}_0 e^{i(kz-\omega t)} \qquad \tilde{B}(z,t) = \tilde{B}_0 e^{i(kz-\omega t)}$$

Faraday's Law tells us

$$\nabla \times E = -\frac{\partial B}{\partial t} \longrightarrow \tilde{B}_0 = \frac{k}{\omega}(\hat{z} \times \tilde{E}_0)$$ ← take the curl and look at components

Amplitudes of E and B

$$B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0$$

We can further generalize our wave equations with the introduction of the propogation vector $\vec{k}$

$\vec{k}$ points in the direction of propogation and has magnitude equal to the wave number k

$\vec{k} \cdot \vec{r}$ is the generalization of kz

General Wave Equations

$$\tilde{E}(r,t) = \tilde{E}_0 e^{i(k \cdot r - \omega t)} \hat{n}$$

$$\tilde{B}(r,t) = \frac{1}{c} \tilde{E}_0 e^{i(k \cdot r - \omega t)} (\hat{k} \times \hat{n}) = \frac{1}{c}(\hat{k} \times \tilde{E})$$

$\hat{n} \cdot \hat{k} = 0$ since E is transverse

Energy and Momentum in Electromagnetic Waves

Recall our equation for energy per unit volume in electromagnetic fields

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

For plane waves we have the relation $B^2 = \frac{1}{c^2} E^2 = \mu_0 \epsilon_0 E^2$

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{\mu_0 \epsilon_0 E^2}{\mu_0}\right) = \epsilon_0 E^2 = \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

↳ electric and magnetic contributions are equal

The Poynting vector gives us energy flux density

$$S = \frac{1}{\mu_0}(E \times B)$$
$$= c \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\hat{z} = cu\hat{z}$$ ← plane wave

Momentum density of an EM field is given by

$$g = \frac{1}{c^2} S$$
$$= \frac{1}{c} \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)\hat{z} = \frac{1}{c} u \hat{z}$$

Generally, we are only concerned with averages

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2$$

$$\langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2 \hat{z}$$

$$\langle g \rangle = \frac{1}{2c}\epsilon_0 E_0^2 \hat{z}$$

We refer to intensity as the average power per unit area

$$I = \langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2 \hat{z}$$

Light can deliver its momentum when it falls on a perfect absorber

$$\Delta p = \langle g \rangle Ac\, \Delta t$$

Radiation pressure is the average force per unit area from light

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}$$

9.3 Electromagnetic Waves in Matter

Maxwell's equations in matter with no free current or free charge

i) $\nabla \cdot D = 0$ iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

ii) $\nabla \cdot B = 0$ iv) $\nabla \times H = \frac{\partial D}{\partial t}$

In linear media

$$D = \epsilon E \quad \text{and} \quad H = \frac{1}{\mu} B$$

If the material is homogenous (ϵ and μ are constant), our equations become

i) $\nabla \cdot E = 0$ iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

ii) $\nabla \cdot B = 0$ iv) $\nabla \times B = \mu\epsilon \frac{\partial E}{\partial t}$

Waves propogate through linear homogenous media at a speed

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n} \quad \text{where} \quad n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$

We refer to n as the index of refraction of the material

generally $\mu \approx \mu_0$ so $n \approx \sqrt{\epsilon_r}$ where ϵ_r is the dielectric constant

ϵ_r is almost always greater than 1 so light travels slower through mediums than a vacuum

All of the previous calculations for waves in a vacuum translate over with $\epsilon_0 \to \epsilon$, $\mu_0 \to \mu$, and $c \to v$

Reflection

Suppose a plane wave propogates along the $\hat{z}$ direction with polorization in the $\hat{x}$ direction and wave frequency ω

Further suppose the wave approaches the interface between two linear medias

Reflected Wave

$$\tilde{E}_R(z,t) = \tilde{E}_{0_R} e^{i(-k_1 z - \omega t)} \hat{x}$$

$$\tilde{B}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0_R} e^{i(-k_1 z - \omega t)} \hat{y}$$

To the left

$$\tilde{E}_L(z,t) = \tilde{E}_{0_L} e^{i(kz - \omega t)} \hat{x}$$

$$\tilde{B}_L(z,t) = \frac{1}{v_1}\tilde{E}_{0_L} e^{i(kz - \omega t)} \hat{y}$$

Transmitted Wave

$$\tilde{E}_T(z,t) = \tilde{E}_{0_T} e^{i(k_2 z - \omega t)} \hat{x}$$

$$\tilde{B}_T(z,t) = \frac{1}{v_2}\tilde{E}_{0_T} e^{i(k_2 z - \omega t)} \hat{y}$$

Our fields must align at the boundary $z=0$

By applying the boundary conditions

i) $\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$ iii) $E_1^{\parallel} = E_2^{\parallel}$

ii) $B_1^{\perp} = B_2^{\perp}$ iv) $\frac{1}{\mu_1}B_1^{\parallel} = \frac{1}{\mu_2}B_2^{\parallel}$

We find $\tilde{E}_{0_I} - \tilde{E}_{0_R} = \beta \tilde{E}_{0_T}$

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2}$$

$$E_{0_T} = \frac{2}{1+\beta} E_{0_I}$$

$$E_{0_R} = \left(\frac{1-\beta}{1+\beta}\right) E_{0_I}$$

Reflected Intensity

$$R = \frac{I_R}{I_I} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2$$

Transmitted Intensity

$$T = \frac{I_T}{I_I} = \frac{4 n_1 n_2}{(n_1+n_2)^2}$$

reflectional and transmission coefficients

$$R + T = 1$$

If we generalize further for oblique incidence (light hitting the medium at an angle)

Initial Wave:

$$\tilde{E}_I(r,t) = \tilde{E}_{0_I} e^{i(k_I \cdot r - \omega t)} \qquad \tilde{B}_I(r,t) = \frac{1}{v_1}(\hat{k}_I \times \tilde{E}_I)$$

Reflected Wave:

$$\tilde{E}_R(r,t) = \tilde{E}_{0_R} e^{i(k_R \cdot r - \omega t)} \qquad \tilde{B}_R(r,t) = \frac{1}{v_1}(\hat{k}_R \times \tilde{E}_R)$$

Transmitted Wave:

$$\tilde{E}_T(r,t) = \tilde{E}_{0_T} e^{i(k_T \cdot r - \omega t)} \qquad \tilde{B}_T(r,t) = \frac{1}{v_2}(\hat{k}_T \times \tilde{E}_T)$$

All waves have the same frequency ω

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega$$

$$k_I = k_R = \frac{v_2}{v_1} k_T$$

Once again the combined fields to the left must satisfy boundary conditions with the right

Conclusions:

First Law: The incident, reflected, and transmitted wave vectors form a plane which includes the normal vector to the surface

↳ Plane of incidence

Second Law: Angle of Incidence is equal to the angle of reflection

Third Law: Snell's Law

$$\frac{\sin \theta_T}{\sin \theta_I} = \frac{n_1}{n_2}$$

Fresnel's Equations

$$\tilde{E}_{0_R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0_I} \quad \text{and} \quad \tilde{E}_{0_T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0_I}$$

Applicable when polarization is in the plane of incidence

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} \qquad \beta = \frac{\mu_1 n_1}{\mu_2 n_2}$$

Some more information about angles and intensity

9.4 Absorption and Dispersion

The preceeding logic was based on the assumption that free charge density and free current density were 0

Ignoring that assumption, Maxwell's Equations become

i) $\nabla \cdot E = \frac{1}{\epsilon}\rho_f$ iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

ii) $\nabla \cdot B = 0$ iv) $\nabla \times B = \mu\sigma E + \mu\epsilon\frac{\partial E}{\partial t}$

Recall the continuity equation for free charge

$$\nabla \cdot J_f = -\frac{\partial \rho_f}{\partial t}$$

And Ohm's Law

$$J_f = \sigma E$$

$$\frac{\partial \rho_f}{\partial t} = -(\nabla \cdot J_f) = -(\nabla \cdot \sigma E) = -\sigma(\nabla \cdot E) = -\frac{\sigma}{\epsilon}\rho_f$$

We can solve this differential equation ...

$$\rho_f(t) = e^{-\frac{\sigma}{\epsilon}t}\rho_f(0)$$ ⟶ Charge dissipates in a characteristic time $\tau = \frac{\epsilon}{\sigma}$

In this situation our plane wave solutions have complex wave numbers

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z-\omega t)} \qquad \tilde{B}(z,t) = \tilde{B}_0 e^{i(\tilde{k}z-\omega t)}$$

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

Imaginary portion of $\tilde{k}$ is responsible for the attenuation of the wave (decreasing amplitude with increasing z)

$$\tilde{E}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \qquad \tilde{B}(z,t) = \tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)}$$

We refer to skin depth as the distance it takes to reduce amplitude by a factor of $\frac{1}{e}$ (about $1/3$)

$$d = \frac{1}{\kappa}$$

The complex term also introduces a lag between the electric and magnetic field

$$E(z,t) = E_0 e^{-\kappa z} \cos\left(kz - \omega t + \delta_E\right)\hat{x}$$

$$B(z,t) = B_0 e^{-\kappa z} \cos\left(kz - \omega t + \delta_E + \phi\right)\hat{y}$$

$$\phi = \tan^{-1}\left(\kappa/k\right)$$

When we have free charges and currents present we must apply more general boundary conditions

i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$ iii) $E_1^{\parallel} - E_2^{\parallel} = 0$

ii) $B_1^{\perp} - B_2^{\perp} = 0$ iv) $\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = K_f \times \hat{n}$

Consider the boundary between a non-conducting linear media and a conductor

Applying boundary conditions we find the following relations

$$\tilde{E}_{0_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0_I} \quad \text{and} \quad \tilde{E}_{0_T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0_I}$$

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$

For perfect conductors $\sigma = \infty \implies k_2 = \infty$

so we have

$$\tilde{E}_{0_R} = -\tilde{E}_{0_I} \quad \text{and} \quad \tilde{E}_{0_T} = 0$$ (wave is completely reflected)

A medium is said to be dispersive if the speed of the wave is dependent on its frequency

Therefore a wave form with different frequencies will change shape as it passes through the dispersive medium

Skipped part on the frequency dependence of permittivity

9.5 Guided Waves

Assume that the wave guide is a perfect conductor such that $E=0$ and $B=0$ inside the material

$$E^{\parallel} = 0 \quad \text{and} \quad B^{\perp} = 0$$

Suppose we have waves propogating along the inner of the wave guide

$$\left.\begin{aligned} \tilde{E}(x,y,z,t) &= \tilde{E}_0(x,y)e^{i(kz-\omega t)} \\ \tilde{B}(x,y,z,t) &= \tilde{B}_0(x,y)e^{i(kz-\omega t)} \end{aligned}\right\}$$ Subject to Maxwell's equations and boundary conditions of the wave guide

Confined waves are not generally purely transverse waves

More information on how to solve confined wave problems in section 9.5

Chapter 10: Potentials and Fields

10.1 The Potential Formulation

We can no longer define an electric potential in electrodynamics since $\nabla \times E$ is no longer 0

i) $\nabla \cdot E = \frac{1}{\epsilon_0} \rho$ iii) $\nabla \times E = -\frac{\partial B}{\partial t}$

ii) $\nabla \cdot B = 0$ iv) $\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$

However, since B is still divergence-less we can define a magnetic potential as before

$$B = \nabla \times A$$

In Faraday's law (iii) this yields

$$\nabla \times E = -\frac{\partial}{\partial t}(\nabla \times A) \quad \text{or} \quad \nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0$$

This quantity does have a 0 curl so we can find its potential

$$E + \frac{\partial A}{\partial t} = -\nabla V$$

$$E = -\nabla V - \frac{\partial A}{\partial t}$$

Substituting these expressions into our remaining Maxwell's equations

$$\nabla \cdot E = \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{1}{\epsilon_0}\rho$$

$$\nabla \times (\nabla \times A) = \mu_0 J - \mu_0 \epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) - \mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2}$$

$$\left(\nabla^2 A - \mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t^2}\right) - \nabla\left(\nabla \cdot A + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 J$$

Gauge freedom is the process of redefining fields without changing their physics

What freedoms does our potential formulation offer?

Suppose we have two potential sets describing the same electric and magnetic fields

$$A' = A + \alpha \qquad V' = V + \beta$$

Both A, A' yield the same B so

$$\nabla \times A' = B = \nabla \times A \quad \Rightarrow \quad \nabla \times \alpha = 0$$

Therefore we can write α as the gradient of a scalar

$$\alpha = \nabla \lambda$$

We also know that both potentials yield the same E

$$\cancel{-\nabla V} - \cancel{\frac{\partial A}{\partial t}} = E = -\nabla V' - \frac{\partial A'}{\partial t}$$

$$= \cancel{-\nabla V} - \nabla \beta - \cancel{\frac{\partial A}{\partial t}} - \frac{\partial \alpha}{\partial t}$$

$$\nabla \beta + \frac{\partial \alpha}{\partial t} = 0 \quad \text{or alternatively,} \quad \nabla\left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0$$

$$\beta = -\frac{\partial \lambda}{\partial t} + k(t)$$

We can absorb $k(t)$ into λ s.t.

$$A' = A + \nabla \lambda$$

$$V' = V - \frac{\partial \lambda}{\partial t}$$

We have the freedom to add the gradient of a scalar function to A provided we remove $\frac{\partial \lambda}{\partial t}$ from V

Coulomb Gauge

Pick $\nabla \cdot A = 0$ (akin to magnetostatics)

Simplifies $\nabla \cdot E = \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{1}{\varepsilon_0}\rho$ to $\nabla \cdot E = \nabla^2 V = -\frac{1}{\varepsilon_0}\rho$

Poisson's Equation!

Therefore,

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r',t)}{\eta}\,d\tau'$$

Scalar potential is determined by charge distribution right now

Scalar potential is simple to calculate but A is not

$$\nabla^2 A - \mu_0\epsilon_0 \frac{\partial^2 A}{\partial t^2} = \mu_0 J + \mu_0\epsilon_0 \nabla\left(\frac{\partial V}{\partial t}\right) \quad \leftarrow \text{ugly!}$$

Lorenz Gauge

Select $\nabla \cdot A = -\mu_0\epsilon_0 \frac{\partial V}{\partial t}$

$$\nabla^2 A - \mu_0\epsilon_0 \frac{\partial^2 A}{\partial t^2} = -\mu_0 J$$

$$\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho$$

Treats both V and A with the operator

$$\nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} = \Box^2$$

d'Alembertian

i) $\Box^2 V = -\frac{1}{\epsilon_0}\rho$ $\leftarrow$ Natural generalization of the Laplacian

ii) $\Box^2 A = -\mu_0 J$

Lorenz gauge sets V and A as solutions to the inhomogeneous wave equation with a source term

Reduces all of electrodynamics to solving the inhomogeneous wave equation for a specified source

Let's write Lorentz Force law in terms of potentials

$$F = \frac{dp}{dt} = q(E + v \times B) = q\left[-\nabla V - \frac{\partial A}{\partial t} + v \times (\nabla \times A)\right]$$

$p = mv$

Using the rule $\nabla(v \cdot A) = v \times (\nabla \times A) + (v \cdot \nabla)A$ we can rewrite

$$\frac{dp}{dt} = -q\left[\frac{\partial A}{\partial t} + (\vec{v}\cdot\nabla)A + \nabla(V - \vec{v}\cdot A)\right]$$

The term $\frac{\partial A}{\partial t} + (\vec{v}\cdot\nabla)A$ is called the convective derivative

represents the time rate of change of A at the moving location of the particle

$$dA = A(r + v\,dt, t + dt) - A(r,t)$$

$$= \left(\frac{\partial A}{\partial x}\right)v_x\,dt + \left(\frac{\partial A}{\partial y}\right)v_y\,dt + \left(\frac{\partial A}{\partial z}\right)v_z\,dt + \left(\frac{\partial A}{\partial t}\right)dt$$

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + (\vec{v}\cdot\nabla)A$$

two changes for position and time

Now we have

$$\frac{d}{dt}(p + qA) = -\nabla\left[q(V - \vec{v}\cdot A)\right]$$

We refer to $p + qA$ as the canonical momentum

Rate of change of a particle's energy

$$\frac{d}{dt}(T + qV) = \frac{\partial}{\partial t}\left[q(V - \vec{v}\cdot A)\right]$$

$T = \frac{1}{2}mv^2$ qV is potential energy

10.2 Continuous Distributions

In the static case our generalized equations simplify to the familiar versions of poisson's equation

$$\nabla^2 V = -\frac{1}{\varepsilon_0}\rho \qquad \nabla^2 A = -\mu_0 J$$

with solutions

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{\eta}\, d\tau' \qquad A(r) = \frac{\mu_0}{4\pi}\int \frac{J(r')}{\eta}\, d\tau'$$

Information is limited by the speed of light so we care about the status at a time t_r, retarded time

$$t_r = t - \frac{\eta}{c}$$

travel a distance η at speed of light

Generalization for non-static sources

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r',t_r)}{\eta}\, d\tau' \qquad A(r,t) = \frac{\mu_0}{4\pi}\int \frac{J(r',t_r)}{\eta}\, d\tau'$$ ⟵ Retarded Potentials

To prove these equations we need to show that they satisfy the inhomogeneous wave equation and satisfy the Lorenz equation

Let's take the Laplacian of $V(r,t)$

$$\nabla V = \frac{1}{4\pi\varepsilon_0}\int\left[(\nabla\rho)\frac{1}{\eta} + \rho\nabla\left(\frac{1}{\eta}\right)\right]d\tau'$$

Recall ρ implicitly depends on η since $t_r = t - \frac{\eta}{c}$

$$\nabla\rho = \frac{\partial\rho}{\partial t}\nabla t_r = -\frac{1}{c}\frac{\partial\rho}{\partial t}\nabla\eta$$

chain rule

$$\nabla\eta = \hat{\eta} \quad\text{and}\quad \nabla\left(\frac{1}{\eta}\right) = -\frac{\hat{\eta}}{\eta^2}$$

$$\nabla V = \frac{1}{4\pi\varepsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\eta}}{\eta} - \rho\frac{\hat{\eta}}{\eta^2}\right]d\tau'$$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\int -\frac{1}{c}\left[\frac{\hat{\eta}}{\eta}\cdot(\nabla\dot{\rho}) + \dot{\rho}\nabla\cdot\left(\frac{\hat{\eta}}{\eta}\right)\right] - \left[\frac{\hat{\eta}}{\eta^2}\cdot(\nabla\rho) + \rho\nabla\cdot\left(\frac{\hat{\eta}}{\eta^2}\right)\right] d\tau'$$

$$\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\nabla t_r = -\frac{1}{c}\ddot{\rho}\hat{\eta}$$

$$\nabla\cdot\left(\frac{\hat{\eta}}{\eta}\right) = \frac{1}{\eta^2}$$

$$\nabla\cdot\left(\frac{\hat{\eta}}{\eta^2}\right) = 4\pi\delta^3(\eta)$$

$$\nabla^2 V = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{c^2}\frac{\ddot{\rho}}{\eta} - 4\pi\rho\,\delta^3(\eta)\, d\tau' = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\varepsilon_0}\rho(r,t)$$ ⟵ satisfies inhomogeneous wave equatio

Recovering fields from potentials (Jefimenko's Equations

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r',t_r)}{\eta}\, d\tau' \qquad A(r,t) = \frac{\mu_0}{4\pi}\int \frac{J(r',t)}{\eta}\, d\tau'$$

$$E = -\nabla V - \frac{\partial A}{\partial t} \quad\text{and}\quad B = \nabla\times A$$

$$E(r,t) = \frac{1}{4\pi\varepsilon_0}\int\left[\frac{\rho(r',t_r)}{\eta^2}\hat{\eta} + \frac{\dot{\rho}(r',t_r)}{c\eta}\hat{\eta} - \frac{\dot{J}(r',t_r)}{c^2\eta}\hat{\eta}\right]d\tau'$$ ⟵ time dependent generalization of Culomb's Law

$$B(r,t) = \frac{\mu_0}{4\pi} \int \left[\frac{J(r',t_r)}{\eta^2} + \frac{\dot{J}(r',t_r)}{c\eta} \right] \times \hat{\eta} \, d\tau'$$ ← time dependent generalization of Biot-Savart Law

10.3 Point Charges

Consider the retarded potentials $V(r,t)$ and $A(r,t)$ for a point charge moving on a specified trajectory

$w(t)$ = position of q at time t

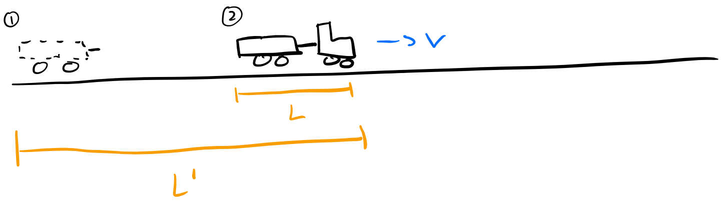
$$V(r,t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r',t_r)}{\eta} d\tau'$$

We can pull out the $1/\eta$ term but $\int \rho(r',t_r)\, d\tau'$ doesn't evaluate to q

In fact,

$$\int \rho(r',t_r)\, d\tau' = \frac{q}{1-\hat{\eta}\cdot\frac{v}{c}}$$ ← Purely geometric

Consider a moving train



Light from the back of the train ① reaches us at the same time as light from the front of the train ②

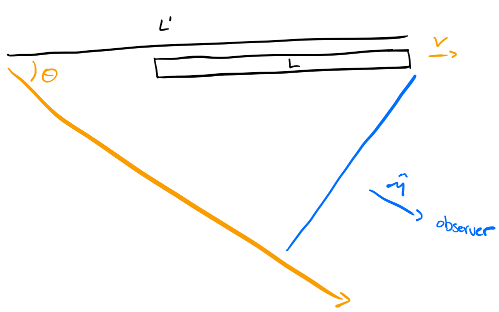
$$\frac{L'}{c} = \frac{L'-L}{v} \Rightarrow L' = \frac{L}{1-v/c}$$

How far the train moves in the time it takes light to make up the L' distance

Approaching train always looks longer by a factor of $\frac{1}{1-v/c}$

Leaving train looks shorter by a factor of $\frac{1}{1+v/c}$

Now consider looking at the train from a line of sight angle θ



$$\frac{L'\cos\theta}{c} = \frac{L'-L}{v} \Rightarrow L' = \frac{L}{1-\frac{v\cos\theta}{c}}$$

Volume appears to be shifted

$$\tau' = \frac{\tau}{1-\hat{\eta}\cdot v/c}$$

Volume of integration in the calculation of the retarded potential for a point charge is impacted by the same scaling factor

Only one retarded point contributes to potentials

$$V(r,t) = \frac{1}{4\pi\varepsilon_0} \frac{qc}{(\eta c - \boldsymbol{\eta}\cdot v)}$$

Our current density is ρv so

$$A(r,t) = \frac{\mu_0}{4\pi} \int \frac{\rho(r',t_r)\, v(t_r)}{\eta} d\tau' = \frac{\mu_0}{4\pi} \frac{v}{\eta} \int \rho(r',t_r)\, d\tau' = \frac{\mu_0}{4\pi} \frac{qcv}{(\eta c - \boldsymbol{\eta}\cdot v)} = \frac{v}{c^2} V(r,t)$$

Liénard-Wiechart Potentials

Recovering fields from Liénard-Wiechert potentials

Recall: $E = -\nabla V - \frac{\partial A}{\partial t}$, $B = \nabla \times A$

A FUCK TON OF MESSY ALGEBRA

velocity field

acceleration field

$$E(r,t) = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(\mathscr{r} \cdot u)^3} \left[(c^2 - v^2) u + \hat{\mathscr{r}} \times (u \times a) \right]$$

$$u = c\hat{\mathscr{r}} - v$$

$$B(r,t) = \frac{1}{c} \hat{\mathscr{r}} \times E(r,t)$$

Velocity field -> Generalized Coulomb Field

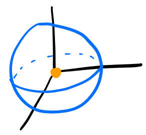
Acceleration field -> Radiation field

Chapter 11: Radiation

11.1 Dipole Radiation

When charges accelerate their fields can transport energy irreversibly out to infinity in a process called Radiation

Consider a localized charge set at the origin and a sphere of radius r



Power passing through surface of sphere

$$P(r,t) = \oint S \cdot da = \frac{1}{\mu_0} \oint (E \times B) \cdot da \quad \leftarrow \text{Integral of Poynting Vector}$$

Consider the time lag for electromagnetic information to reach the sphere

$$t_0 = t - r/c$$

$$P_{rad}(t_0) = \lim_{r \to \infty} (r, t_0 + r/c)$$

Notice that for large r, the surface area grows at $4\pi r^2$

In order for radiation to occur, the poynting vector must decrease at most by $1/r^2$ otherwise P would go $1/r^n$ implying $P_{rad} = 0$

Under static conditions from Coulomb's and Biot-Savart law we have E and B with $1/r^2$ dependence

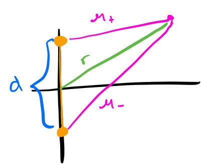
Therefore, the poynting vector, S goes $1/r^4$

Therefore static sources do not radiate

However, as shown in the Jefimenko's Equations, time-dependent fields go at $1/r$ in turn allowing radiation

Electric Dipole Radiation

Consider two metal spheres separated by a distance d and connected by a fine wire



At time t the upper sphere has charge $q(t)$ and the lower sphere has charge $-q(t)$

Consider a driven charge between each sphere

$$q(t) = q_0 \cos(\omega t)$$

$$\left.\begin{array}{l} p(t) = p_0 \cos(\omega t)\,\hat{z} \\ p_0 = q_0 d \end{array}\right\} \text{ oscillating electric dipole}$$

Associated retarded potential

$$V(r,t) = \frac{1}{4\pi\epsilon_0}\left\{ \frac{q_0 \cos(\omega[t - \frac{\mathscr{r}_+}{c}])}{\mathscr{r}_+} - \frac{q_0 \cos(\omega(t - \frac{\mathscr{r}_-}{c}))}{\mathscr{r}_-} \right\}$$

Applying law of cosines $\mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}$

Approximation 1: $d \ll r$

$$\mathscr{r}_\pm \cong r\left(1 \mp \frac{d}{2r}\cos\theta\right)$$

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\Rightarrow \cos\left[\omega\left(t - \frac{\mathscr{r}_\pm}{c}\right)\right] \cong \cos\left[\omega\left(t - \frac{r}{c}\right) \pm \frac{\omega d}{2c}\cos\theta\right]$$

$$= \cos[\omega(t - r/c)]\cos\left(\frac{\omega d}{2c}\cos\theta\right) \mp \sin[\omega(t - r/c)]\sin\left(\frac{\omega d}{2c}\cos\theta\right)$$

Approximation 2: $d \ll \frac{c}{\omega}$

$$\cos[\omega(t - \mathscr{r}_\pm/c)] \cong \cos[\omega(t - r/c)] \mp \frac{\omega d}{2c}\cos\theta \sin[\omega(t - r/c)]$$

Approximation 3: $r \gg \frac{c}{\omega}$

Fields that survive at large distances from the source (Radiation Zone)

$$V(r,\theta,t) = -\frac{p_0 \omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin[\omega(t - r/c)]$$

Vector potential is determined by current flowing in the wire

$$I(t) = \frac{dq}{dt}\hat{z} = -q_0\omega \sin(\omega t)\,\hat{z}$$

$$A(r,t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{-q_0\omega\sin(\omega[t - \mathscr{r}/c])\,\hat{z}}{\mathscr{r}}\,dz$$

Since the integration introduces a factor of d we can replace the integrand by its value at the center

$$A(r,\theta,t) = -\frac{\mu_0 P_0 \omega}{4\pi r} \sin[\omega(t - r/c)]\,\hat{z}$$

Recovering the fields

$$E = -\nabla V - \frac{\partial A}{\partial t} = -\frac{\mu_0 P_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\theta}$$

$$B = \nabla \times A = -\frac{\mu_0 P_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\phi}$$

represent spherical waves in free space

Energy Radiated by Oscillating Electric Dipole

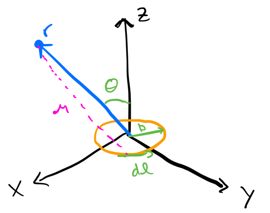
$$S(r,t) = \frac{1}{\mu_0}(E \times B) = \frac{\mu_0}{c}\left[\frac{P_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right]^2 \hat{r}$$

$$\langle S \rangle = \frac{\mu_0 P_0^2 \omega^4}{32\pi^2 c}\frac{\sin^2\theta}{r^2}\,\hat{r}$$

Total power radiated

$$\langle P \rangle = \int \langle S \rangle \cdot da = \frac{\mu_0 P_0^2 \omega^4}{12\pi c}$$

Magnetic Dipole Radiation



Consider a wire loop w/ an alternating current

$$I(t) = I_0\cos(\omega t)$$

Magnetic Dipole

$$m(t) = \pi b^2 I(t)\,\hat{z} = m_0\cos(\omega t)\,\hat{z}$$

$$m_0 = \pi b^2 I_0$$

Scalar potential is 0 since the loop is uncharged

$$A(r,t) = \frac{\mu_0}{4\pi}\int \frac{I_0\cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}}\,dl$$

$$= \frac{\mu_0 I_0 b}{4\pi}\,\hat{y}\int_0^{2\pi}\frac{\cos[\omega(t - \mathscr{r}/c)]}{\mathscr{r}}\cos\phi'\,d\phi'$$

Applying the above approximations

i) $b \ll r$

ii) $b \ll \frac{c}{\omega}$ $\Rightarrow$ $A(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin[\omega(t - r/c)]\,\hat{\phi}$

iii) $r \gg \frac{c}{\omega}$

Associated Fields

$$E = -\frac{\partial A}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\phi}$$

$$B = \nabla \times A = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\theta}$$

Akin to electric dipole but the directions are switched

Energy Flux

$$S(r,t) = \frac{1}{\mu_0}(E \times B) = \frac{\mu_0}{c}\left[\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right]^2 \hat{r}$$

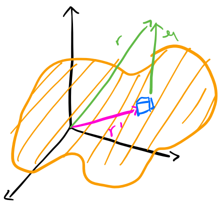
$$\langle S \rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}\,\hat{r}$$

$$\langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

Comparing this power against that from electrical radiation we see

$$\frac{P_{magnetic}}{P_{electric}} = \left(\frac{m_0}{P_0 c}\right)^2$$ ← Very small

Radiation from an Arbitrary Source



Consider a finite volume with an arbitrary configuration of charge and current

Retarded Scalar Potential

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r',t-\mathscr{r}/c)}{\mathscr{r}}\,d\tau'$$

$$\mathscr{r} = \sqrt{r^2+r'^2-2r\cdot r'}$$

Approximation 1: $r' \ll r$

$$\mathscr{r} \cong r\left(1-\frac{\vec{r}\cdot\vec{r}'}{r^2}\right) \quad \Rightarrow \quad \frac{1}{\mathscr{r}} \cong \frac{1}{r}\left(1+\frac{\vec{r}\cdot\vec{r}'}{r^2}\right)$$

Thus, $\rho(\vec{r}',t-\mathscr{r}/c) \cong \rho(\vec{r}', t-\frac{r}{c}+\frac{\hat{r}\cdot\vec{r}'}{c})$

Taylor Series Expansion of t about the origin

$$t_0 = t-\frac{r}{c}$$

$$\rho(r',t-r/c) \cong \rho(r',t_0) + \dot{\rho}(r',t_0)\left(\frac{\hat{r}\cdot r'}{c}\right) + \cdots$$

Approximation 2: $r' \ll \frac{c}{|\dot{\rho}/\rho|}, \frac{c}{|\ddot{\rho}/\rho|^{1/2}}, \frac{c}{|\dddot{\rho}/\rho|^{1/3}} \cdots$

Works to keep only the first order terms of the taylor expansion

$$V(r,t) \cong \frac{1}{4\pi\varepsilon_0 r}\left[\int \rho(r',t_0)\,d\tau' + \frac{\hat{r}}{r}\cdot\int r'\rho(r',t_0)\,d\tau' + \frac{\hat{r}}{c}\cdot\frac{d}{dt}\int r'\rho(r',t_0)\,d\tau'\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot p(t_0)}{r^2} + \frac{\hat{r}\cdot\dot{p}(t_0)}{rc}\right]$$

$$A(r,t) \cong \frac{\mu_0}{4\pi}\int\frac{J(r',t-\mathscr{r}/c)}{\mathscr{r}}\,d\tau' = \frac{\mu_0}{4\pi r}\int J(r',t_0)\,d\tau' = \frac{\mu_0}{4\pi}\frac{\dot{p}(t_0)}{r}$$

Approximation 3: discard $1/r^2$ terms in E and B

Only want terms that go $1/r$

We have $\nabla t_0 = -\frac{1}{c}\nabla r = -\frac{1}{c}\hat{r}$

$$\nabla V \cong \nabla\left[\frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\dot{p}(t_0)}{rc}\right] \cong \frac{1}{4\pi\varepsilon_0}\left[\frac{\hat{r}\cdot\ddot{p}(t_0)}{rc}\right]\nabla t_0 = -\frac{1}{4\pi\varepsilon_0 c^2}\frac{[\hat{r}\cdot\ddot{p}(t_0)]}{r}\hat{r}$$

$$\nabla\times A \cong -\frac{\mu_0}{4\pi rc}[\hat{r}\times\ddot{p}(t_0)]$$

$$\frac{\partial A}{\partial t} \cong \frac{\mu_0}{4\pi}\frac{\ddot{p}(t_0)}{r}$$

Therefore,

$$E(r,t) = \frac{\mu_0}{4\pi r}\left[(\hat{r}\cdot\ddot{p})\hat{r}-\ddot{p}\right] = \frac{\mu_0}{4\pi r}[\hat{r}\times(\hat{r}\times\ddot{p})]$$

$$B(r,t) = -\frac{\mu_0}{4\pi rc}[\hat{r}\times\ddot{p}]$$

$$\vec{S} = \frac{\mu_0}{16\pi^2 c}\left[\ddot{p}(t_0)\right]^2\left(\frac{\sin\theta}{r}\right)^2\hat{r}$$

$$P_{rad} = \frac{\mu_0}{6\pi c}\left[\ddot{p}(t_0)\right]^2$$ ← radiation is related to squared acceleration of the dipo

11.2 Point Charges

To calculate power radiated by the particle at time t_r, we draw a large sphere around the position of the particle at time t_r, and integrate the poynting vector along the sphere after $t - t_r = \frac{\mathscr{r}}{c}$

↑ time for radiation to reach the sphere

Since surface area of the sphere grows at r^2 we only need to consider terms of the poynting vector that go like $1/\mathscr{r}^2$

terms like $1/\mathscr{r}^3$ or $1/\mathscr{r}^4$ will approach 0 as $\mathscr{r} \to \infty$

Recall our expressions for a point charge in arbitrary motion

$$E(r,t) = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(\mathscr{r}\cdot u)^3}\left[(c^2 - v^2)u + \mathscr{r} \times (u \times a)\right] \qquad u = c\hat{\mathscr{r}} - v$$

$$B(r,t) = \frac{1}{c}\hat{\mathscr{r}} \times E(r,t)$$

$$S = \frac{1}{\mu_0}(E \times B) = \frac{1}{\mu_0 c}\left[E^2\hat{\mathscr{r}} - (\hat{\mathscr{r}}\cdot E)E\right]$$

Only the acceleration or radiation fields remain

$$E_{rad} = \frac{q}{4\pi\varepsilon_0}\frac{\mathscr{r}}{(\mathscr{r}\cdot u)^3}\left[\mathscr{r} \times (u \times a)\right] \leftarrow \text{perpendicular to } \hat{\mathscr{r}}$$

$$S_{rad} = \frac{1}{\mu_0 c} E_{rad}^2 \hat{\mathscr{r}}$$

If the charge is instantaneously at rest, $u = c\hat{\mathscr{r}}$

$$E_{rad} = \frac{q}{4\pi\varepsilon_0 c^2 \mathscr{r}}\left[\hat{\mathscr{r}} \times (\hat{\mathscr{r}} \times a)\right] = \frac{\mu_0 q}{4\pi\mathscr{r}}\left[(\hat{\mathscr{r}}\cdot a)\hat{\mathscr{r}} - a\right]$$

$$S_{rad} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi\mathscr{r}}\right)^2\left[a^2 - (\hat{\mathscr{r}}\cdot a)^2\right]\hat{\mathscr{r}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\mathscr{r}}$$

θ is the angle between $\mathscr{r}$ and a

Power radiated

$$P = \oint S_{rad}\cdot da = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\int \frac{\sin^2\theta}{\mathscr{r}^2}\mathscr{r}^2 \sin d\theta\, d\phi = \frac{\mu_0 q^2 a^2}{6\pi c}$$

Liénard's Generalization

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{v \times a}{c}\right|^2\right)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Since accelerating charge radiates and radiation carries off energy at the cost of the particle's K.E., an charged mass will accelerate less than a neutral one

We call this force a radiation reaction force

Radiation reaction is a misnomer and we must also consider contributions to K.E. from the velocity field

On average,

we radiate $P = \frac{\mu_0 q^2 a^2}{6\pi c}$ so we expect $\int_{t_1}^{t_2} F_{rad}\cdot v\, dt = -\frac{\mu_0 q^2}{6\pi c}\int_{t_1}^{t_2} a^2\, dt$ when the state of the system is identical at t_1 and t_2

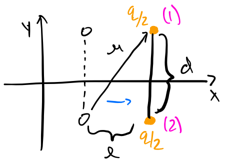
Via integration by parts, we find the Abraham-Lorentz formula

$$F_{rad} = \frac{\mu_0 q^2}{6\pi c}\dot{a} \leftarrow \text{simplest form of radiation reaction force consistent with the conservation of energy}$$

↑ has some very! ugly implications

The true mechanism behind radiation reaction is an imbalance in internal electromagnetic forces

Consider a dumbell with two ends of charge $q/2$ separated by a distance d



Assume the dumbell moves in the x-direction and is instantaneously at rest at the retarded time

E_{field} at 1 due to 2:

$$E_1 = \frac{(q/2)}{4\pi\varepsilon_0} \frac{r}{(r \cdot u)^3}\left[(c^2 + r \cdot a)u - (r \cdot u)a\right]$$

$$u = c\hat{r} \qquad r = l\hat{x} + d\hat{y}$$

We can only consider the X-component of E_1 since the y-components will cancel eachother out

$$u_x = \frac{cl}{r}$$

$$E_{1x} = \frac{q}{8\pi\varepsilon_0 c^2} \frac{(lc^2 - ad^2)}{(l^2 + d^2)^{3/2}}$$

By symmetry, $E_{2x} = E_{1x}$

$$F_{self} = \frac{q}{2}(E_1 + E_2) = \frac{q^2}{8\pi\varepsilon_0 c^2} \frac{(lc^2 - ad^2)}{(l^2+d^2)^{3/2}} \hat{x}$$

SOME FUCKED UP MATH

$$F_{self} = \frac{q^2}{4\pi\varepsilon_0}\left[-\frac{a(t)}{4c^2 d} + \frac{\dot{a}(t)}{3c^3} + (\ )d + \dots\right]\hat{x}$$

proportional to acceleration of charge

Under Newton's Second Law

$$m = 2m_0 + \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4dc^2}$$

mass of either end alone

← Alligns with relativistic model

Chapter 12: Electrodynamics and Relativity

12.1 The Special Theory of Relativity

Classical Mechanics obeys the principle of relativity: the same laws apply in any inertial reference frame

Electrodynamics on the other hand seems to be velocity dependent

Einstein's two postulates

1) The principle of relativity: The laws of physics apply in all inertial reference systems
2) The Universal Speed of Light: The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source

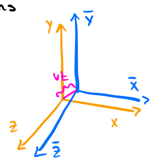
There is no ether

Einstein's velocity Addition rule:

$$V_{AC} = \frac{V_{AB} + V_{BC}}{1 + (V_{AB}V_{BC}/c^2)}$$

Galilean Transformations

i) $\bar{x} = x - vt$
ii) $\bar{y} = y$
iii) $\bar{z} = z$
iv) $\bar{t} = t$



Lorentz Transformations

i) $\bar{x} = \gamma(x - vt)$
ii) $\bar{y} = y$
iii) $\bar{z} = z$
iv) $\bar{t} = \gamma(t - \frac{v}{c^2}x)$

We can better represent Lorentz transformations by adopting four-vectors

$x^0 = ct$ ← represent time in meters

$\beta = \frac{v}{c}$

$x^1 = x$

$x^2 = y$

$x^3 = z$

$$\Longrightarrow \quad \bar{x}^0 = \gamma(x^0 - \beta x^1), \quad \bar{x}^1 = \gamma(x^1 - \beta x^0), \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = x^3$$

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

We can generalize this relation and define 4-vectors as any set of components that obey $\bar{a}^u = \sum_{v=0}^{3} \Lambda^u_v a^v$ where Λ is the Lorentz transformation matrix

The 4-dimensional scalar product is invariant under Lorentz transformations as the 3-dimensional version is invariant to rotations

$$-a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3$$

↑ many ways of representing the negative sign

Consider the 4-dimensional scalar product of a 4-vector with itself

$$a^u a_u = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2$$

If $a^u a_u > 0$, a^u is called spacelike

$a^u a_u < 0$, a^u is called timelike

$a^u a_u = 0$, a^u is called lightlike

For two events $A(x^0_A, x^1_A, x^2_A, x^3_A)$ and $B(x^0_B, x^1_B, x^2_B, x^3_B)$ we define $\Delta x^u = x_A^u - x_B^u$ as the displacement 4-vector

The scalar product of Δx^u with itself is called the invariant interval

$$I = -c^2t + d$$

↑ time difference ↑ spatial separation

I is constant across moving systems

If the displacement is timelike $(I<0)$, there exists an inertial system in which two events occur at the same point

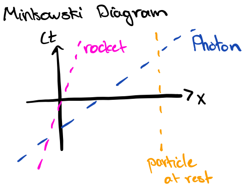
You can be present at A and B

If the displacement is space like $(I>0)$ there exists a system in which the two events occur at the same time

If the displacement is light like $(I=0)$ then the two events could be connected by a light signal

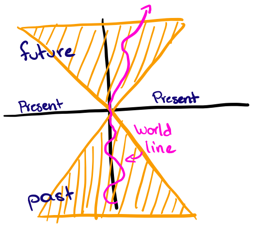
Convention dictates that time is plotted against distance. Therefore velocity is the reciprocal of the slope

Minkowski Diagram



Trajectory of a particle on a Minkowski diagram is called a world line

Suppose you are at the origin ($t=0$)



Since motion is restricted by the speed of light, your motion is bound in the orange triangular regions

We call the future the forward lightcone and the past the backwards light cone

The displacement between causally related events is always timelike and their temporal ordering is absolute

12.1 Relativistic Mechanics

Proper time is the time associated with a moving object

Proper velocity is the distance traveled over proper time

$$\eta = \frac{dl}{d\tau}$$

$$\eta = \frac{1}{\sqrt{1-u^2/c^2}}\, u$$

η is the spatial part of a 4-vector

η^{μ} is called the proper velocity 4-vector of the 4-velocity and transforms simply between inertial systems

Don't need to transform the denominator $d\tau$ since it is invariant

We define relativistic momentum via the proper velocity

$$p = m\eta = \frac{mu}{\sqrt{1-u^2/c^2}}$$

$$p^{\mu} = m\eta^{\mu}$$

The temporal component is referred to as relativistic energy (times c)

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1-u^2/c^2}}$$

$$E = p^0 c = \frac{mc^2}{\sqrt{1-u^2/c^2}}$$

We call p^{μ} the energy-momentum 4-vector

At rest, relativistic energy is referred to as rest energy

$$E_{rest} = mc^2$$

The rest is kinetic energy

$$E_{kin} = E - E_{rest} = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right)$$

⟵ Under classical conditions the square root term can be expanded

$$E_{kin} = \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \dots$$

In every closed system, the total relativistic energy and momentum are conserved

In terms of relativistic energy and momentum

$$E^2 - p^2c^2 = m^2c^4$$

A massless particle can have a momentum provided that it is travelling at the speed of light

$$E = pc$$

Newton's Second law holds under relativistic conditions given that we use relativistic momentum

$$F = \frac{dp}{dt}$$

Relativistic motion under a constant force takes the shape of a hyperbola

Newton's third law does not hold under relativistic conditions

Since F is the derivative of momentum relative to ordinary time, it does not transform well

The Minkowski force is the "proper" analog of traditional force

$$K^{\mu} = \frac{dp^{\mu}}{d\tau}$$

$$K = \left(\frac{dt}{d\tau}\right)\frac{dp}{dt} = \frac{1}{\sqrt{1-u^2/c^2}}\, F$$

The zeroth component is the proper power of the particle (ignoring $1/c$ term)

$$K^0 = \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau}$$

Center-of-mass analog under relativistic considerations is the center-of-energy

$$R_e = \frac{1}{E}\sum E_i r_i \quad \text{and} \quad M \to \frac{E}{c^2} \quad \text{so} \quad P = \frac{E}{c^2}\frac{dR_e}{dt}$$

12.3 Relativistic Electrodynamics

Consider a line of positive charges moving to the right with a line of negative charges moving to the left superimposed on top

From the perspective of these line currents a moving charge q would feel no electric force since the system is electrically neutral

However, from the POV of a moving charge with speed u, things change

Primarily, q is at rest

Under Einstein's velocity addition rules

$$V_{\pm} = \frac{v \mp u}{1 \mp vu/c^2}$$

Due to Lorentz contractions, we find that the wire carries a net negative charge

$$\lambda_{\pm} = \pm(\gamma_{\pm})\lambda_0$$

$$\gamma_{\pm} = \frac{1}{\sqrt{1 - v_{\pm}^2/c^2}}$$

Skipping some complicated algebra we find

$$\lambda_{tot} = \frac{-2\lambda u v}{c^2\sqrt{1 - u^2/c^2}}$$

This results in an electric field and associated force

$$E = \frac{\lambda_{tot}}{2\pi\epsilon_0 s} \qquad \bar{F} = qE = -\frac{\lambda v}{\pi\epsilon_0 c^2 s}\frac{qu}{\sqrt{1-u^2/c^2}}$$

frame of reference

Transforming this force back into our original frame of reference

$$F = \sqrt{1-u^2/c^2}\,\bar{F} = -\frac{\lambda v}{\pi\epsilon_0 c^2}\frac{qu}{s}$$

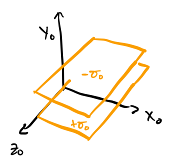
Electrostatics and relativity imply the existence of another force (Hint: magnetism)

Recall $c^2 = \frac{1}{\epsilon_0\mu_0}$ and $\lambda v = I$

$$F = -qu\left(\frac{\mu_0 I}{2\pi s}\right)$$

Charge is invariant

Consider a large parallel plate capacitor at rest in S_0 with surface charges $\pm\sigma_0$



$$E_0 = \frac{\sigma_0}{\epsilon_0}\hat{y}$$

If we consider this capacitor from a system moving to the right with a speed v_0 we will find the capacitor moving to the left

Now, E takes the form

$$E = \frac{\sigma}{\epsilon_0}\hat{y}$$ where σ has changed

While the charge and width are invariant, the length of the plate is subject to Lorentz contractions

$$\gamma_0 = \frac{1}{\sqrt{1 - v_0^2/c^2}}$$ so charge per unit area is increased $\sigma = \gamma_0\sigma_0$

Accordingly,

$$E^{\perp} = \gamma_0 E_0^{\perp}$$

If we consider contraction parallel to E we find that $E^{\parallel} = E_0^{\parallel}$

Generalized Field Transformation Rules

$$\boxed{\begin{array}{lll} \bar{E}_x = E_x & \bar{E}_y = \gamma(E_y - vB_z) & \bar{E}_z = \gamma(E_z + vB_y) \\ \bar{B}_x = B_x & \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) & \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right) \end{array}}$$

Case 1: $B = 0$ in S

$$\bar{B} = -\frac{1}{c^2}(v \times \bar{E})$$

Case 2: $E = 0$ in S

$$\bar{E} = v \times \bar{B}$$

The transformations of E and B can be described by an anti-symmetric, second rank tensor

A tensor in 4-dimensions has 16 components

A second rank tensor has two indices

Symmetric tensors have the property

$$t^{uv} = t^{vu}$$

Antisymmetric tensors obey the property

$$t^{uv} = -t^{vu}$$

Generally of the form

$$t^{uv} = \begin{Bmatrix} 0 & t^{01} & t^{02} & t^{03} \\ -t^{01} & 0 & t^{12} & t^{13} \\ -t^{02} & -t^{12} & 0 & t^{23} \\ -t^{03} & -t^{13} & -t^{23} & 0 \end{Bmatrix}$$

Using this form we can write the field tensor and its dual tensor

$$F^{uv} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ E_z/c & -B_y & B_x & 0 \end{Bmatrix} \qquad G = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & E_x/c \\ -B_z & -E_y/c & -E_x/c & 0 \end{Bmatrix}$$

$E/c \to B$

$B \to -E/c$

To reformulate the laws of electrodynamics with our new tools we need to express ρ and J and see how they transform

Specifically, we want ρ in terms of proper charge density ρ_0

$$\rho_0 = \frac{Q}{V_0}$$

Since V will be contracted via lorentz contraction

$$V = \sqrt{1-u^2/c^2}\, V_0$$

$$\rho = \frac{Q}{V} \to \rho = \rho_0 \frac{1}{\sqrt{1-u^2/c^2}} \quad \text{and} \quad J = \rho u \to J = \rho_0 \frac{u}{\sqrt{1-u^2/c^2}}$$

Notice

$$J^u = \rho_0 \eta^u = (c\rho, J_x, J_y, J_z) \leftarrow \text{current density 4-vector}$$

The continuity equation becomes

$$\nabla \cdot J = \frac{\partial \rho}{\partial t} \to \frac{\partial J^u}{\partial x^u} = 0 \leftarrow \text{current density 4-vector is divergenceless}$$

summation over u implied

Maxwell's Equations

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^u, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \leftarrow \text{summation over } \nu \text{ implied}$$

You can retrieve the original form of Maxwell's equations by plugging in various values of $u = \{0,1,2,3\}$

Minkowski Force on a charge q

$$K^u = q\eta_\nu F^{\mu\nu}$$

We can continue to use our potential formulation by combining V and A into a potential 4-vector

$$A^u = (V/c, A_x, A_y, A_z)$$

$$F^{u\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^u}{\partial x_\nu} \leftarrow \text{notice covariant vectors}$$