Inductance, Capacitance and Maxwell's Equations

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***Abstract*—This note will describe the circuit component properties of inductance and capacitance and their relation to the fundamental laws of electrodynamics — Maxwell's equations. It will also show the derivation of the formula for the inductance of an air core solenoid and a parallel plate capacitor.**

***Index Terms*—inductance, capacitance, self inductance, solenoid, Maxwell's equation.**

I. Introduction

THE term inductance was coined by Oliver Heaviside in May 1884 as a shorthand for "coefficient of self-induction".

Engineers, technicians and radio amateurs are likely to perceive inductance as the circuit property that opposes a change in current, and by its characteristic to store energy in a magnetic field. Similarly, the property of capacitance opposes a change in voltage and has the characteristic of storing energy in an electric field.

While true, this perspective emphasises the effects of inductance and capacitance rather than the causal mechanism of the properties. In our day to day work we do not give much thought to the relationship of inductance and capacitance to basic physical law but an understanding of this can shed light on many aspects of circuit behavior.

II. Maxwell's Equations

In the 18^{th} and 19^{th} centuries, natural philosophers described their experimental observations of the interactions of electrically charged and magnetic objects. In 1820 Danish physicist Hans Christian Ørsted discovered that a compass needle (magnet) is deflected at right angle to the flow of an electric current. In 1830 English Physicist Michael Faraday, inspired by Ørsted's findings, discovered that a changing magnetic field can induce an electric field and thus a current in an electrical circuit. Critically, Faraday proposed the concept of the electric and magnetic field to explain how forces acted and energy was carried, amongst charged and magnetic bodies. Up to that point, the prevailing opinion was that electric and magnetic forces act instantly across space; a concept known as "action at a distance".

In the mid 19^{th} century, Scottish mathematician James Clerk Maxwell developed a rigorous mathematical description of the electrostatic and electromagnetic discoveries known to date. Maxwell expressed, in mathematical form, Faraday's experimental discoveries of induction and electrical charge. He also adapted and extended André-Marie Ampère's formalization of Ørsted's discovery to apply it to both steady state and changing currents.

Maxwell was able to mathematically express the known magnetic and electrical phenomena into a comprehensive set of twenty equations, uniting electricity and magnetism. Using Faraday's concept of fields, Maxwell showed, with his equations, that magnetism and electricity are different manifestations of a single electromagnetic phenomena, and that it propagates as waves at the speed of light.

Maxwell's work was enthusiastically embraced by Oliver Heaviside who was one of the few that perceived the significance of this great achievement. Heaviside developed vector calculus and used it to consolidate and reformulate Maxwell's twenty equations into the four concise equations we know today:

Gauss's law $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$	Electric field lines connect positive electric charges to negative charges or are closed loops
Gauss's law for magnetism $\nabla \cdot \vec{E} = 0$	Magnetic field lines only exist as closed loops
Faraday's law of induction $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	A time varying magnetic field is proportional to an accompanying curling electric field
Ampère–Maxwell law $\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$	A current or a time varying electric field is proportional to an accompanying curling magnetic field

The vector notation and the vector operators (del "$\nabla\cdot$" and curl "$\nabla\times$") may seem intimidating; however, they are merely a shorthand to simplify the equations and to remove verboseness. The equations above are given in their differential form and express fields and charge density at each point in space and time. For our purposes, the integral forms are more practical and can be used to evaluate electromagnetic effects over lines, curves or surface elements. The two forms are related to, and can be transformed to each other aided by two mathematical theorems[1].

In this paper we will use the integral form of Faraday's law

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot \hat{n}\, da \tag{1}$$

and the integral form of Ampère–Maxwell's law

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \left(I_{enc} + \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot \hat{n}\, da \right) \tag{2}$$

to explore inductance, and the integral form of Gauss's law to explore capacitance.

$$\oint_S \vec{E} \cdot \hat{n}\, da = \frac{q_{enc}}{\mathcal{E}_0} \tag{3}$$

[1]Divergence theorem and Stokes' theorem.

These equations contain the vector dot product operator ($\vec{A} \bullet \vec{B}$). This is an operation that takes two vectors and returns a single number (a scalar) representing the product of their magnitudes and the cosine of the angle between them.

If the two vectors are perfectly aligned (at $0°$), then the result is simply the product of their individual magnitudes. It they are at right angles to each other, the result of the dot product is zero.

III. What is Inductance and Capacitance?

A. Inductance

In the integral form of Faraday's law (1), the contour integral of the electric field $\vec{E}$ is the EMF $\mathcal{E}$ and the surface integral of the magnetic flux density component $\vec{B}$ normal to a surface bounded by the contour is the magnetic flux Φ_M.

$$\mathcal{E} = -\frac{d\Phi_M}{dt} \tag{4}$$

The Ampère–Maxwell law (2) relates the contour integral of the the magnetic flux density to the associated current that the contour encloses. The ratio of the total magnetic flux to its associated current is inductance.

$$L = \frac{\Phi_M}{I} \tag{5}$$

If we use this relationship to replace Φ_M in equation (4), then the coefficient that relates the time rate of change of current to the induced voltage will be inductance.

$$\mathcal{E} = -L\frac{dI}{dt} \tag{6}$$

To determine the inductance of a circuit element it is necessary to determine the magnetic field density at every point in space that accompanies the current flowing through it.

The Ampère–Maxwell law can be used for determining this; however, the calculation may be quite complicated. Fortunately we can often make use of the element's physical symmetry to make this easier. Frequently it is also appropriate to make simplifying assumptions to ease the computational burden so long at they do not unduly diminish the accuracy.

B. Capacitance

Self-capacitance is the ability of an isolated conductor to store electrical energy when a potential difference is applied, measured by the amount of charge needed to raise its potential by one volt. Voltage is a potential difference, so in this instance the potential difference is between the conductor and a virtual ground (like a conducting hollow sphere of infinite radius).

Mutual capacitance is the capacitance between two separate conductors and this is what we usually understand as the capacitance of the discrete circuit element.

Like charges repel, so it becomes progressively harder to add more and more charge to an object, requiring progressively move voltage. The higher the capacitance of the object, the less work (voltage) is required to add additional charge to the same total charge.

IV. Fields

A. Current and Magnetic Flux

The magnetic flux density $\vec{B}$ at at point in space, accompanying a current I in a circuit element, is expressed by the Ampère–Maxwell law. The total magnetic flux Φ_M passing through a surface is determined by integrating the flux density component that is normal (at $90°$) to the surface.[2]

B. Voltage and Electric Field Intensity

Electric field intensity $\vec{E}$, is a vector field expressed in $\frac{V}{m}$, that quantifies the force experienced by a positive test charge due to the influence of other charges.

The contour integral of the electric field vector $\vec{E}$ over a path is the voltage difference V between the end points.

The characteristics of the electric field intensity vector $\vec{E}$ at a point in space, accompanying a charge q, are expressed by Gauss's law.

[2]The dot product notation $\vec{B} \bullet \hat{n}$, is the calculation to determine the component of magnetic flux vector $\vec{B}$ that is piercing the surface at $90°$. This is also called the normal component. The vector $\hat{n}$ is of length 1 in the direction at $90°$ to the surface at the location of the $\vec{B}$ vector. The result of $\vec{B} \bullet \hat{n}$ at a point is $\|\vec{B}\|\|\vec{n}\|\cos\theta$ where θ is the angle between $\vec{B}$ and $\hat{n}$.

TABLE I
DEFINITIONS

Property	Symbol	SI Unit Name	Unit Symbol	Derived Unit
Magnetic flux density	$\vec{B}$	Tesla	T	$\frac{\mathrm{Wb}}{\mathrm{m}^2}$
Magnetic flux	Φ_M	Weber	Wb	$\mathrm{V} \cdot \mathrm{s}$
Inductance	L	Henry	H	$\frac{\mathrm{Wb}}{\mathrm{A}}$
Electric field strength	$\vec{E}$	Volt per meter		$\frac{\mathrm{V}}{\mathrm{m}}$
Electric flux	Φ_E	Volt-meter		$\mathrm{V} \cdot \mathrm{m}$
Capacitance	C	Farad	F	$\frac{\mathrm{C}}{\mathrm{V}}$
Electromotive force	$\mathcal{E}$	Volt	V	
Current density	$\vec{J}$	Ampere per sq. meter		$\frac{\mathrm{A}}{\mathrm{m}^2}$
Charge	q	Coulomb	C	$\mathrm{A} \cdot \mathrm{s}$
Vacuum electric permittivity	ε_0			$8.854187 \times 10^{-12}\,\frac{F}{m}$
Vacuum magnetic permeability	μ_0			$4\pi \times 10^{-7}\,\frac{H}{m}$

C. *Magnetic flux lines*

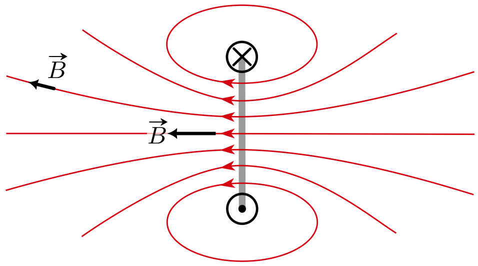


Fig. 1. Magnetic flux lines around a loop of current

The magnetic flux density $\vec{B}$ represents the amount of magnetic flux per unit area. As a vector, it has a magnitude and direction. It is represented on figure 1 by the density of the magnetic flux lines per area of surface; the more lines in a given area, the higher the flux density.

The magnetic flux Φ_M is a scalar value determined by the size and orientation of the magnetic field density vector, and the surface area and shape over which the the integral (summation) is performed.

The SI unit of magnetic flux is Weber (Wb) and magnetic flux density is Tesla (T) (which is $\frac{\mathrm{Wb}}{\mathrm{m}^2}$).

Since the total amount of flux is proportional to the current in the circuit, the magnetic flux lines depicted in a diagram, such as figure 1 represents the relative flux level passing through the space surrounding them, with each line representing a proportion of the total magnetic flux.

A magnet would feel a force to align itself along the field lines with the north pole attracted in the direction of the field line.

D. *Electric flux lines*

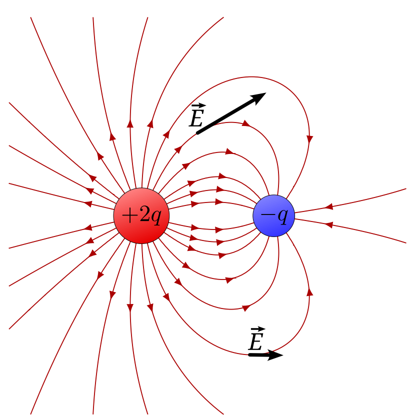


Fig. 2. Electric flux lines between opposite charges of differing values

The electric field intensity $\vec{E}$ represents the strength of electric field in time and space. As a vector, it has a magnitude and direction. It is represented on figure 2 by the density of the electric field lines per area of surface; the more lines in a given area, the higher the field strength. A line tangent to a field line indicates the direction of the electric field at that point.

The SI unit of electric field intensity is $\frac{\mathrm{V}}{\mathrm{m}}$.

A positive electrical charge would feel a force in the direction of the the field lines proportional to the density of lines.

The electric flux Φ_E is a scalar value determined by the size and orientation of the electric field intensity vector, and the surface area and shape over which the the integral (summation) is performed.

Electric field lines also exist as closed loops without terminating on charges if they were generated by accelerating charges. In this case they would propagate with a magnetic field as an electromagnetic wave.

V. The Solenoid

A. *Ampère–Maxwell's Law in a Solenoid*

In Integral form, Ampère–Maxwell's law says that an electric current or a changing electric flux through a surface is accompanied by a circulating magnetic field around any path that bounds that surface. When considering the solenoidal inductor, there is only conduction current and Maxwell's displacement current term is zero and it can be discarded.

Reminder that the magnetic flux density is a vector

Dot product tells you to find the part of $\vec{B}$ parallel to $d\vec{l}$ (along path C)

An incremental segment of path C

The electric current in amperes

$$\oint_C \vec{B} \circ d\vec{l} = \mu_0 I_{\text{enc}} \tag{7}$$

The magnetic flux density in teslas

Reminder that only the enclosed current contributes

Tells you to sum up the contributions from each portion of closed path C in a direction given by the right-hand rule

The magnetic permeability of free space

Ampère–Maxwell's law (with only a conduction current) [1]

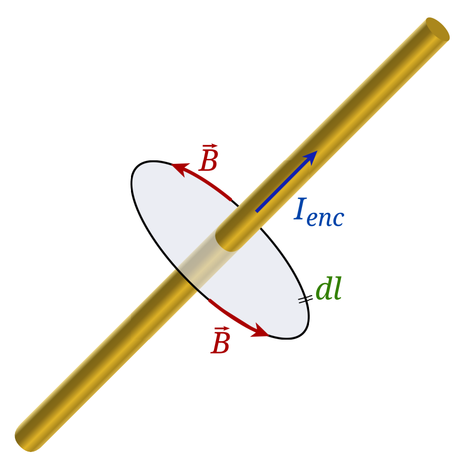


Fig. 3. An Illustration of terms Ampère–Maxwell's law

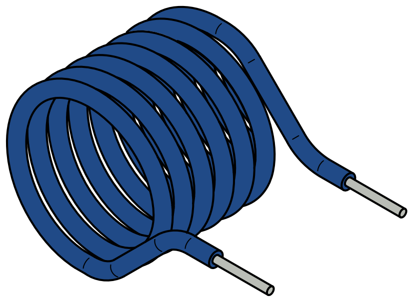


Fig. 4. Example solenoidal inductor

The left-hand side of equation 7 describes a closed loop integration of the magnetic flux density over any arbitrary path that encloses a current. The amperian loop we choose to evaluate, shown in figure 5, encloses the current of one turn. We could choose any shape or any path for this loop but as we will see, we can choose one that simplifies the integral calculation. For each segment along the closed path (a→b→c→d→a), we sum the product of segment length with the component of magnetic flux density aligned with the path segment *(i.e the dot product).*

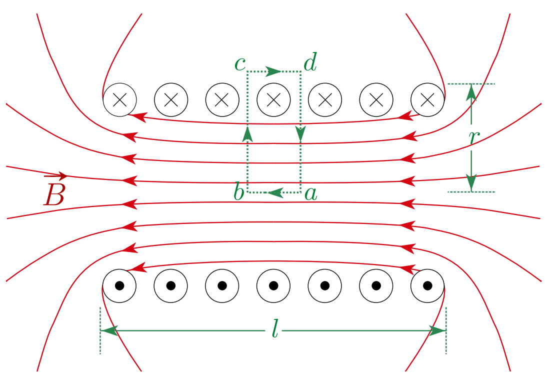


Fig. 5. Lines of magnetic flux and an amperian loop in a solenoid.

The right hand side of formula 7 is a product of the magnetic permeability constant and the current in the turn enclosed by the path.

In an ideal tightly wound infinite solenoid, the magnetic flux density is zero outside the solenoid (B_{ext}) and is even and parallel to its axis inside (B_{int}). A practical solenoid with a length exceeding several times that of its diameter has similar characteristics except at the ends; however, since Gauss's law for magnetism only permits magnetic flux as closed loops, the returning flux lines are present outside the solenoid, albeit with a low flux density.

Expressing the four components of our amperian loop we have:

$$\int_a^b B_{int} \cdot dl + \int_b^c B_{int} \cdot dl + \int_c^d B_{ext} \cdot dl + \int_d^a B_{int} \cdot dl = \mu_0 I \tag{8}$$

The contribution of the c→d arm is zero, because the magnetic flux density B_{ext} is zero outside the solenoid. The b→c and d→a contributions are also zero because those paths are at 90° to the magnetic flux lines. The length of path b→a is the solenoid length l divided by the number of turns N; the turn to turn spacing. Therefore:

$$B_{int}\frac{l}{N} + 0 + 0 + 0 = \mu_0 I \tag{9}$$

$$B_{int} = \mu_0 \frac{N}{l} I \tag{10}$$

The magnetic flux density inside the solenoid is proportional to the number of turns N, the current I and inversely proportional to the length l of the solenoid.

The magnetic flux Φ_M through a surface is the integral of the magnetic flux density component at 90° to that surface.

$$\Phi_M = \oint_S B \cdot \hat{n}\, da \tag{11}$$

The magnetic flux through each turn of the solenoid is the magnetic flux density B_{int}, from equation 10, times the solenoid cross sectional disc area (πr^2) (since the disc is at 90° to the magnetic flux). Integrating B_{int} over the internal area of the solenoid will capture the total flux associated with the current[3].

$$\Phi_M = \mu_0 \frac{N}{l} \pi r^2 I \tag{12}$$

B. *Faraday's law*

Faraday's law equates the electromotive force induced in a closed path to the time varying magnetic flux it bounds.

Reminder that the electric field is a vector

Dot product tells you to find the part of $\vec{E}$ parallel to $d\vec{l}$ (along path C)

The magnetic flux through any surface bounded by C

An incremental segment of path C

$$\oint_C \vec{E} \circ d\vec{l} = -\frac{d}{dt} \overbrace{\int_S \vec{B} \circ \hat{n}\, da} \tag{13}$$

The electric field in V/m

The rate of change with time

Tells you to sum up the contributions from each portion of the closed path C in a direction given by the right-hand rule

Reminder that this is a line integral (not a surface or a volume integral)

Integral form of Faraday's law) [1]

[3]To determine the total magnetic flux, each closed loop magnetic flux line must pass through the area of integration only once. *Note that all the flux lines outside of a finite length solenoid are returning.*

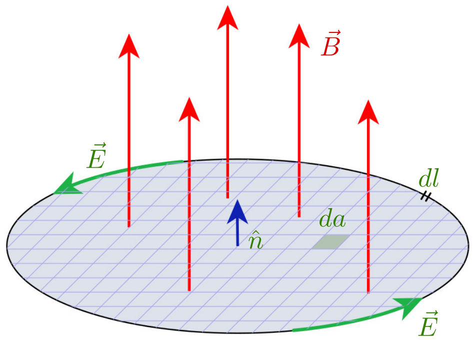


Fig. 6. An Illustration of the terms in Faraday's law

Using the value of magnetic flux we calculated for the solenoid in equation (12) to replace $\int_S B \bullet \hat{n}\, da$ in Faraday's law.

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt}\left(\mu_0 \frac{N}{l} \pi r^2 I\right) \tag{14}$$

Only the current varies with time so we can pull the constants out of the derivative.

$$\oint_C \vec{E} \bullet d\vec{l} = -\left(\mu_0 \frac{N}{l} \pi r^2\right) \frac{dI}{dt} \tag{15}$$

As we noted in equation (4), the integration of the electric field over a path is the EMF.

(n.b. the electric field is measured in $\frac{V}{m}$, so multiplying the tangential electric field component by a length in meters will give a result in volts.)

$$\mathcal{E} = -\left(\mu_0 \frac{N}{l} \pi r^2\right) \frac{dI}{dt} \tag{16}$$

Referring to figure 6 we can see that the value for EMF given in (16) is for a single turn *(our closed loop of integration encompasses the magnetic flux once)*. The solenoid is in practice N turns connected in series and so the EMF is N times that of a single turn.

$$\mathcal{E}_{solenoid} = -\left(\mu_0 \frac{N^2}{l} \pi r^2\right) \frac{dI}{dt} \tag{17}$$

The definition of the unit of electric inductance given by Bureau International des Poids et Mesures (BIPM) for the SI system of units [2] is:

"The henry is the inductance of a closed circuit in which an electromotive force of 1 volt is produced when the electric current in the circuit varies uniformly at the rate of 1 ampere per second."

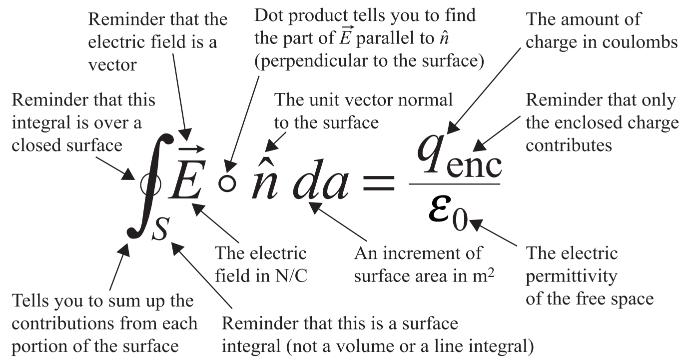
Using this definition and equation (17), the inductance of a solenoid is:

$$L = \left(\mu_0 \frac{N^2}{l} \pi r^2\right) \tag{18}$$

VI. The Parallel Plate Capacitor

A. Gauss's Law in a capacitor

The integral form of Gauss's law states that total electric flux (summation of all the normal component of electric flux lines) piercing a bounding surface is proportional to the enclosed electric charge.



(19)

Gauss's law (integral form) [1]

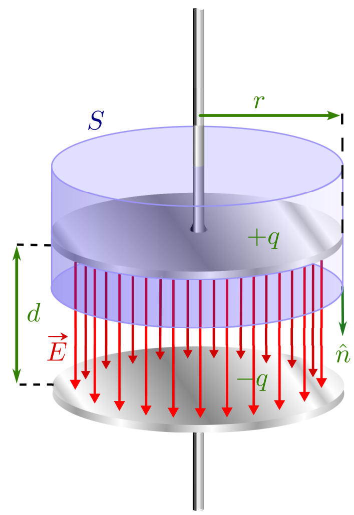


Fig. 7. An Illustration of the terms in Gauss's law for a capacitor

A parallel plate capacitor stores charge on one plate resulting in an opposite and equal charge on the other. The left hand side of Gauss's law (19) is a closed surface integral of the electric field intensity, or electric flux. In our analysis we shall use a cylinder, to enclose the upper plate, as the closed surface of integration. *(see figure 7)*. Apart from the requirement to enclose the charge, we are free to choose any shape and size for the integration surface. Since the capacitor plates are parallel, the electric field from one plate to the other will be of uniform intensity over the bottom surface of the top plate, zero everywhere else, and be at 90° to the plate. *In practice there are fringing fields at the plate edge which will not be exactly 90° to the surface but this discrepancy is minor.*

The area of the bottom of the cylinder is πr^2 and the electric field through all other surfaces are zero.

$$E\pi r^2 = \frac{q}{\mathcal{E}_0} \tag{20}$$

Since the electric field intensity is constant between the plates (the density of field lines is the same), the integration of E across the plates (i.e. voltage) is simply Ed where d is the distance between the plates; therefore, $E = \frac{V}{d}$

$$\frac{V}{d}\pi r^2 = \frac{q}{\mathcal{E}_0} \tag{21}$$

$$\mathcal{E}_0 \frac{\pi r^2}{d} = \frac{q}{V} \tag{22}$$

The definition of the unit of capacitance given by BIPM for the SI system of units [2] is:

"The farad is the capacitance of a capacitor between the plates of which there appears a potential difference of 1 volt when it is charged by a quantity of electricity of 1 coulomb."

therefore

$$C = \mathcal{E}_0 \frac{\pi r^2}{d} \tag{23}$$

or more generally

$$C = \mathcal{E}_0 \frac{A}{d} \tag{24}$$

where A is the plate area.

The capacitance of a parallel plate capacitor is proportional to the plate area and inversely proportional to the distance.

VII. Conclusions and observations

This note has shown the relationship of inductance and capacitance to Maxwell's equations and how inductance is related to the current and magnetic flux, and capacitance to charge and electric flux. Inductance and capacitance in a circuit are an inescapable consequence of Maxwell's equations. The Ampère–Maxwell law tells us that with a current there will be magnetic flux and thus inductance. Gauss's law tells us that in a circuit with a voltage (electric field) and charge and there will be capacitance.

The inductance of a solenoid is proportional to the number of turns, the density of turns (turns/meter) and on the area of the solenoid disc. The density of turns will influence the flux density while the number of turns will act as a multiplier for the inductance of a single turn.

Practical solenoidal inductors will not have the idealized magnetic flux distribution used in the example and the deviation will be most apparent at the ends. The example inductors described here are air cored and so the magnetic permeability is assumed to be μ_0. This constant would be replaced by $\mu_0\mu_r$ if a magnetic material is used for the core. The relative permeability of a real core will be complex, causing the inductor to have loss and it may change with frequency.

In our capacitor example we have assumed an air dielectric. If another material were used we would substitute $\mathcal{E}_0\mathcal{E}_r$ for $\mathcal{E}_0$. This relative dielectric may be complex (in which case there will be loss) and its dielectric constant may vary with frequency.

Maxwell's classical electrodynamics was superseded in the mid twentieth century by quantum electrodynamics to provide a more precise description of light and matter at atomic scale; however, Maxwell's equations continue as the engineer's language to describe electromagnetism in the twenty-first century.

For more information on the historical development of classical electrodynamics, the book by Nancy Forbes, "Faraday, Maxwell and the Electromagnetic Field: How Two Men Revolutionized Physics" is a good resource [3]. To go further into the meaning and applications of Maxwell's equations, Daniel Fleish's book "A Student's Guide to Maxwell's Equations" [1] is an easy to understand starting point.

References

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