| | | | | |
|---|---|---|---|---|
| 2. | Derivative | 35 | 2/12/2019 | |
| 3. | Application of Derivative | 38 | 9/12/2019 | AV 16/12/19 |
| 4. | Application of Derivative by Newton's Method | 41 | 16/12/2019 | AV 23/12/19 |
| 5 | Integration. | 45 | 23/12/2019 | AV 06/01/2020 |
| 6. | Application of Integration & Numerical Integration | 48 | 5/1/20 | |
| 7. | Differential Equations | 51 | 9/1/20 | AV 21/01/2020 |
| 8. | Euler's Method | 53 | 20/1/20 | |
| 9. | Limit & Partial Order Derivative | 55 | 27/1/20 | AV |

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# Practical - I

Topic :- Limit and Continuity

Q.1]
$$\lim_{x\to a}\left[\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$$

$$\lim_{x\to a}\left[\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\times\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}\times\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}\right]$$

$$\lim_{x\to a}\frac{(a+2x-3x)\,(\sqrt{3a+x}+2\sqrt{x})}{(3a+x-4x)\,(\sqrt{a+2x}+\sqrt{3x})}$$

$$\frac{1}{3}\lim_{x\to a}\frac{\cancel{(a-x)}\,(\sqrt{3a+x}+2\sqrt{x})}{\cancel{(a-x)}\,(\sqrt{a+2x}+\sqrt{3x})}$$

$$\frac{1}{3}\times\frac{\sqrt{3a+a}+2\sqrt{a}}{\sqrt{a+2a}+\sqrt{3a}}$$

$$\frac{1}{3}\times\frac{\sqrt{4a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$$

$$\frac{1}{3}\times\frac{2\sqrt{a}+2\sqrt{a}}{2\sqrt{3a}}$$

$$\frac{1}{3}\times\frac{{}^{2}\cancel{4}\sqrt{a}}{\cancel{2}\sqrt{3a}}$$

$$\frac{2}{3}\times\frac{1}{\sqrt{3}}=\underline{\underline{\frac{2}{3\sqrt{3}}}}$$

2] $\lim_{y \to 0} \left[ \frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

$\lim_{y \to 0} \left[ \frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+y} + \sqrt{a}}{\sqrt{a+y} + \sqrt{a}} \right]$

$\lim_{y \to 0} \frac{a + y - a}{y\sqrt{a+y}\left(\sqrt{a+y} + \sqrt{a}\right)}$

$\lim_{y \to 0} \frac{\cancel{y}}{\cancel{y}\sqrt{a+y}\left(\sqrt{a+y} + \sqrt{a}\right)}$

$\frac{1}{\sqrt{a+0}\left(\sqrt{a+0} + \sqrt{a}\right)}$

$= \frac{1}{\sqrt{a}\left(2\sqrt{a}\right)}$

$= \frac{1}{2a}$

3] $\lim_{x \to \pi/6} \frac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$

Taking $x - \frac{\pi}{6} = h$

$\therefore \quad x = h + \pi/6$

where $h \to 0$

$$\lim_{h\to 0} \frac{\cos\left(h+\pi/6\right) - \sqrt{3}\,\sin\left(h+\pi/6\right)}{\pi - 6\left(h+\frac{\pi}{6}\right)}$$

Using $\cos(A+B) = \cos A\cos B - \sin A\cdot\sin B$

& $\sin(A+B) = \sin A\cos B + \cos A\cdot\sin B.$

$$\lim_{h\to 0} \frac{\cos h\cdot\cos\pi/6 - \sin h\cdot\sin\pi/6 - \sqrt{3}\left(\sin h\cdot\cos\pi/6 + \cos h\,\sin\pi/6\right)}{\pi - \cancel{6}\left(\frac{6h+\pi}{\cancel{6}}\right)}$$

$$\lim_{h\to 0} \frac{\cos\cdot\frac{\sqrt{3}}{2}h - \sin\frac{h}{2} - \sqrt{3}\left(\sin\frac{\sqrt{3}}{2}h + \cos h\cdot\frac{1}{2}\right)}{-6h}$$

$$\lim_{h\to 0} \frac{\cancel{7}\sin\frac{4h}{2}}{\cancel{+}6h}$$

$$\lim_{h\to 0} \frac{\sin 4h}{12h}$$

$$\frac{1}{3}\lim_{h\to 0} \frac{\sin h}{h} = \frac{1}{3}\times 1$$

$$= \frac{1}{3}$$

4] $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}}$

By Rationalising Numerator & Denominator.

$$\lim_{x\to\infty} \left[ \frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \times \frac{\sqrt{x^2+5} + \sqrt{x^2-3}}{\sqrt{x^2+5} + \sqrt{x^2-3}} \times \frac{\sqrt{x^2+3} + \sqrt{x^2+1}}{\sqrt{x^2+3} + \sqrt{x^2+1}} \right]$$

$$\lim_{x\to\infty} \left[ \frac{(x^2+5-x^2+3\ (\sqrt{x^2+3} - \sqrt{x^2+1}}{(x^2+3-x^2-1)\ (\sqrt{x^2+5} + \sqrt{x^2-3}} \right]$$

$$\lim_{x\to\infty} \left[ \frac{\overset{4}{\cancel{8}}\ \sqrt{x^2+3} + \sqrt{x^2+1}}{\cancel{2}\ (\sqrt{x^2+5} + \sqrt{x^2-3}} \right]$$

$$4 \lim_{x\to\infty} \frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)} + \sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{\sqrt{x^2\left(1+\frac{5}{x^2}\right)} + \sqrt{x^2\left(1-\frac{3}{x^2}\right)}}$$

After Applying Limit we get

$= 4$

5.] Examine the Continuity :-

i] $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$ for $0 < x \leq \dfrac{\pi}{2}$

$\Rightarrow f\left(\frac{\pi}{2}\right) = \dfrac{\sin \cancel{2} \frac{\pi}{\cancel{2}}}{\sqrt{1-\cos \cancel{2}\frac{\pi}{2}}}$

$f\left(\frac{\pi}{2}\right) = 0$

$f$ at $\pi/2$ is defined.

ii] $\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\pi - 2x}$

Put $x - \frac{\pi}{2} = h$

$\therefore \quad x = \frac{\pi}{2} + h$

As $x \to 0 \quad \therefore h \to 0$

$$\lim_{h \to 0} \frac{\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)}$$

$$\lim_{h \to 0} \frac{\cos h}{\pi - 2\left(\frac{\pi + 2h}{2}\right)}$$

$$\lim_{h \to 0} \frac{\cos h}{-2h}$$

$$= -\frac{1}{2} \lim_{h \to 0} \frac{\cos h}{h}$$

Applying the limit

$$-\frac{1}{2} \times 1$$

$$= -\frac{1}{2}$$

Function is continuous at [illegible]

ii.

iii] $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & 0 < x < 3 \\ x+3 & 3 \le x < 6 \\ \dfrac{x^2-9}{x+3} & 6 \le x < 9 \end{cases}$

$f(x) = \dfrac{x^2-9}{x-3} = 0$

$f$ at $x = 3$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x+3$

$f(3) = x+3 = 6$

$f$ is defined at $x = 3$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x+3) = 6$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \dfrac{x^2-9}{x-3}$

$= \dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$

$= x+3$

L.H.L = R.H.L

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$

for $x = 6$

$f(6) = \dfrac{x^2-9}{x+3} = 3$

$\lim_{x \to 6^+} \dfrac{x^2-9}{x+3} = \lim_{x \to 6^+} \dfrac{(x-3)(x+3)}{(x+3)}$

$$\lim_{x \to 6^-} x + 3 = 9.$$

L.H.S $\neq$ R.H.S

Function is not continuous at 6.

6. Find the value of k that function f(x) is continuous at indicated point.

i] $f(x) = \frac{1-\cos 4x}{x^2}$, $x < 0$
$= k$, $x = 0$ } at $x = 0$

$\because$ f is continuous at $x = 0$

$\therefore \lim_{x \to 0} f(x) = f(0)$

$$\lim_{x \to 0} \frac{1-\cos 4x}{x^2} = k.$$

$$2 \lim_{x \to 0} \frac{2\sin^2 2x}{x^2} = k$$

Applying Limit

$$2(2)^2 = k$$

$$k = 8$$

ii] $f(x) = (\sec^2 x)^{\cot^2 x}$, $x \neq 0$
$= k$, $x = 0$ } at $x = 0$.

$$\lim_{x \to 0} (\sec^2 x)^{\cot^2 x} = k$$

$$\lim_{x \to 0} (1 + \tan^2 x)^{\cot^2 x} = k$$

SE

$$\lim_{x\to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}} = k.$$

Applying limit

$e = k.$

iii] $f(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$ , $x \neq \frac{\pi}{3}$ $\Big\}$ at $x = \pi/3$

$= k$ , $x = \frac{\pi}{3}$

$$\lim_{x\to \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x}$$

Let,

$x - \pi/3 = h.$

$x = \frac{\pi}{3} + h$

As $x \to 0$ , $h \to 0$

$$\lim_{h\to 0} \frac{\sqrt{3} - \tan\left(\frac{\pi}{3} + h\right)}{\pi - 3\left(\frac{\pi}{3} + h\right)}$$

$$\lim_{h\to 0} \frac{\sqrt{3} - \tan(\pi/3) + \tan h}{1 - \tan \pi/3 \cdot \tan h}.$$

$$\lim_{h\to 0} \sqrt{3} - \sqrt{3} \times \sqrt{3} \tan h - \sqrt{3} - \tan h$$

$\lim_{h\to 0}$

$\lim_{h\to 0}$

$\frac{4}{3}$

li

h

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Rede

cont

] f

$$\lim_{h\to 0} \frac{\frac{-4\tan h}{1-\sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{-4\tan h}{-3\tan(1-\sqrt{3}\tan h)}$$

$$\frac{4}{3} \lim_{h\to 0} \frac{\tan h}{h}$$

$$\lim_{h\to 0} \frac{1}{1-\sqrt{3}\tan h}$$

$$\rightarrow \underline{\underline{\frac{4}{3}}}$$

7. Discuss the continuity of the following function which of these function have removable discontinuity Redefine the function so as to make it continuous.

$$f(x) = \begin{cases} \frac{1-\cos 3x}{x\tan x} & x \neq 0 \\ 9 & x = 0 \end{cases} \quad \text{at } x = 0$$

$$\lim_{x\to 0} f(x) = \frac{1-\cos 3x}{x\tan x} \quad \lim_{x\to 0} \frac{9}{2} \neq 9 = f(0)$$

$f$ is not continuous at $x = 0$

$$f(x) = \frac{1-\cos 3x}{x\tan x}$$

$$= \frac{2\sin^2 \frac{3}{2}x}{x\tan x} \qquad x^2$$

$$= \frac{\frac{2\sin^2 \frac{3}{2}x}{x^2}}{\frac{x\tan x}{x^2}} \cdot x^2$$

$$= 2 \lim_{x\to 0} \frac{(3/2)^2}{1}$$

$$= \frac{9}{2}$$

$$\lim_{x\to 0} f(x) = \frac{9}{2} = f(0)$$

$f$ is not continuous at $x=0$

Redefine function,

$$f(x) = \frac{1-\cos 3x}{x\tan x}, \quad x \neq 0$$

$$= \frac{9}{2}, \quad x = 0$$

Now, $\lim_{x\to 0} f(x) = f(0)$

$\therefore$ $f$ has removable discontinuity at $x=0$.

ii] $$f(x) = \frac{(e^{3x}-1)\sin x}{x^2}, \quad x \neq 0$$

$$= \frac{\pi}{6}, \quad x = 0$$

at $x=0$

$$\lim_{x\to 0} \frac{(e^{3x}-1)\sin\left(\frac{\pi x}{180}\right)}{x^2}$$

$$3 \lim_{x\to 0} \frac{(e^{3x}-1)}{3x} \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}.$$

Applying limit

$$3 \log e \frac{\pi}{180} = \frac{\pi}{60} = f(0)$$

f is continuous at $x=0$

8] If $f(x) = \frac{e^{x^2} - \cos x}{x^2}$

for $x \neq 0$ is continuous at $x=0$.

Find $f(0)$

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} = f(0)$$

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x - 1 + 1}{x^2}$$

$$\lim_{x\to 0} \frac{\cancel{(e^{x^2}-1)}\ (1+\cos x)}{x^2}$$

$$\lim_{x\to 0} \frac{e^{x^2}-1}{x^2} + \lim_{x\to 0} \frac{2\sin^2 x/2}{\frac{x^2}{2}} \times 2$$

Applying Limit

$$2 \log e \ 1 \quad 2 \frac{1}{4}$$

$$= 2 \ 1 \quad 2 \times \frac{1}{4}$$

18

7] If $f(x) = \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$ for $x \neq \pi/2$ is continuous at $x = \pi/2$, find $f(\pi/2)$

$$\lim_{x\to 0} \left[ \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1+\sin x}}{\sqrt{2} + \sqrt{1+\sin x}} \right]$$

$$\lim_{x\to 0} \left[ \frac{2-1+\sin x}{\cos^2 x(\sqrt{2}+\sqrt{1+\sin x}} \right]$$

$$\lim_{x\to 0} \frac{1+\sin x}{1-\sin^2 x(\sqrt{2}+\sqrt{1+\sin x}})$$

$$\lim_{x\to 0} \frac{(\cancel{1+\sin x})}{(\cancel{1+\sin x})(1-\sin x)(\sqrt{2}+\sqrt{1+\sin x}})$$

$$\lim_{x\to 0} \frac{1}{(1-\sin x)(\sqrt{2}+\sqrt{1+\sin x})}$$

Applying limit

$$= \frac{1}{2(\sqrt{2}+\sqrt{2})}$$

$$= \frac{1}{2\times 2\sqrt{2}}$$

$$= \frac{1}{4\sqrt{2}}$$

$$f(\pi/2) = \frac{1}{4\sqrt{2}}$$

2/12/19

# Practical - II

Topic :- Derivative

Q1. Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are differentiable

1] $\cot x$

$f(x) = \cot x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\tan x} - \frac{1}{\tan a}}{x - a}$$

$$= \lim_{x \to a} \frac{\tan a - \tan x}{(x - a) \tan a \tan x}$$

Put $x - a = h$

$x = a + h$

As $x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(a+h-a) \tan(a+h) \tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{\tan(a+h) \tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \left[\frac{\tan a + \tan h}{1 - \tan a \tan h}\right]}{h(\tan(a+h))(\tan a)}$$

$$\lim_{h \to 0} \frac{\tan a - \tan^2 a \tan h - [\tan a + \tan h]}{h (\tan(a+h) \times \tan a] \times (1 - \tan a \tan h)}$$

$$\lim_{h \to 0} \frac{\tan^2 a - 0 - [1 + 0]}{h (\tan a \cdot \tan a) \times (1 - 0]}$$

$$\lim_{h \to 0} (-1) \times \frac{\tan^2 a + 1}{h (\tan^2 a)}$$

$$= \lim_{h \to 0} \frac{\sec^2 a}{\tan^2 a}$$

$$= -\cos^2 a$$

$$Df(a) = -\cos^2 a$$

$\therefore$ f is differentiable $\forall\ a \in \mathbb{R}$

c) cosec x

$$f(x) = \operatorname{cosec} x$$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x - a}$$

$$= \lim_{x \to a} \frac{\sin a - \sin x}{\sin a \cdot \sin x \, (x - a)}$$

Let $x - a = h$

$x = a + h$

$$\lim_{h\to 0} \frac{\sin a - \sin(a+h)}{\sin a \cdot \sin(h)}$$

$$= \lim_{h\to 0} \frac{2\cos\left(a + \frac{a+h}{2}\right) \sin\left(a - \frac{a+h}{2}\right)}{\sin a \cdot \sin h}$$

$$= \lim_{h\to 0} \frac{2\cos\left(a+\frac{h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\sin a \cdot \sin(a+h)(h)}$$

$$= \lim_{h\to 0} \frac{2\cos\left(a+\frac{h}{2}\right) \sin\left(-\frac{h}{2}\right) \times -\frac{1}{2}}{\sin a \cdot \sin(a+h)\left(-\frac{h}{2}\right)}$$

$$= \frac{-\cos(a)\ (1)}{\sin^2 a}$$

$$= \frac{-\cos a}{\sin^2 a}$$

$$= -\cot a \cdot \operatorname{cosec} a$$

3] $\sec^2 x$

$f(x) = \sec^2 x$

$$Df(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\sec x - \sec a}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x-a}$$

$$= \lim_{x\to a} \frac{\cos a - \cos x}{(\cos x)(\cos a)(x-a)}$$

Put $x = a+h$

As $x\to a$, $h\to 0$

36

$$\lim_{h\to 0} \frac{\cos a - \cos(a+h)}{\cos(a+h)\cos a \cdot h}$$

$$= \lim_{h\to 0} \frac{2\sin\left(a+\frac{a+h}{2}\right)\sin\left(\frac{a+h-a}{2}\right)}{\cos(a+h)\cos a \cdot \times h}$$

$$= \lim_{h\to 0} \frac{2\sin\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\times\frac{1}{2}}{\cos(a+h)\cdot\cos\frac{h}{2}}$$

$$= \frac{\sin(a+0)\ (1)}{\cos(a+0)\cos a}$$

$$= \frac{\sin a}{\cos^2 a}$$

$$= \tan a \cdot \sec a.$$

Q. 11] If $f(x) = 4x+1$, $x = 2$ $\}$ at $x = 2$
$= x^2+5$ $\quad x > 0$

then find $f$ is differentiable or not ?

$\Rightarrow$ L.H.D.

$$Df(2^-) = \lim_{x\to 2^-} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x\to 2} \frac{4x+1-(8+1)}{x-2}$$

$$= \lim_{x\to 2} \frac{4x+1-9}{x-2}$$

$$= \lim_{x\to 2} \frac{4x-8}{x-2}$$

R.H.D :-

$Df(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$

$= \lim_{x \to 2} \frac{x^2 + 5 - 9}{x - 2}$

$= \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$

$= 2 + 2$

$= 4$

$\therefore$ R.H.D = L.H.D

Q. If $f(x) = 4x + 7, \quad x < 3$

$= x^2 + 3x + 1, \quad x \geq 3$ } at $x = 3$

then find $f$ is differentiable or not

→ L.H.D.

$Df(3^-) = \lim_{x \to 3^-} \frac{4x + 7 - (12 + 7)}{x - 3}$

$= \lim_{x \to 3^-} \frac{4x - 12}{x - 3}$

$= \lim_{x \to 3^-} \frac{4\cancel{(x-3)}}{\cancel{(x-3)}}$

$= 4$

R.H.D. $Df(3^+) = \lim_{x \to 3^+} \frac{x^2 + 3x + 1 - 19}{x - 3}$

$= \lim_{x \to 3^+} \frac{x^2 + 3x - 18}{x - 3}$

$$\Rightarrow \lim_{x\to 3^+} \frac{(x+6)\,\cancel{(x-3)}}{\cancel{(x-3)}}$$

$= 3+6$

$= 9$.

L.H.D $\neq$ R.H.D.

$\therefore$ f is not differentiable at $x = 3$.

Q.IV]. If $f(x) = 8x - 5$. $\quad x \leq 2$

$= 3x^2 - 4x + 7 \quad x > 2$ $\Big\}$ at $x = 2$

then find f is differentiable or not

$\Rightarrow$ L.H.D :-

$$Df(2^-) = \lim_{x\to 2^-} \frac{8x - 5 - 11}{x-2}$$

$$= \lim_{x\to 2^-} \frac{8x - 16}{x-2}$$

$$= \lim_{x\to 2^-} \frac{8\,\cancel{(x-2)}}{\cancel{(x-2)}}$$

$= 8$.

$$\text{R.H.D} = Df(2^+) = \lim_{x\to 2^+} \frac{3x^2 - 4x + 7 - 11}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x^2 - 4x - 4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x^2 - 6x + 2x - 4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{3x(x-2) + 2(x-2)}{x-2}$$

$$= \lim_{x\to 2} \frac{3(x+2)\,\cancel{(x-2)}}{\cancel{x-2}} = 3(2+2) = $$

P.I 16/12/19

Topic : Application Of Derivative

1] Find the intervals in which function is increasing or decreasing.

i] $f(x) = x^3 - 5x - 11$

ii] $f(x) = x^2 - 4x$

iii] $f(x) = 2x^3 + x^2 - 20x + 4$

iv] $f(x) = x^3 - 27x + 5$

v] $f(x) = 69 - 24x - 9x^2 + 2x^3$

2] Find the intervals in which function is concave upwards

i] $y = 3x^2 - 2x^3$

ii] $y = x^4 - 6x^3 + 10x^2 + 5x + 7$

iii] $y = x^3 - 27x + 5$

iv] $y = 69 - 24x - 9x^2 + 2x^3$

v] $y = 2x^3 + x^2 - 20x + 4.$

Solution :-

$\Rightarrow$ 1. i] $f(x) = x^3 - 5x - 11$

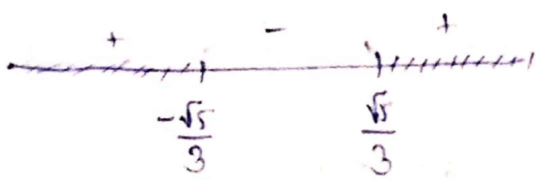
$\therefore f'(x) = 3x^2 - 5$

$\because$ f is increasing iff $f(x) > 0$

$3x^2 - 5 > 0$

$3(x^2 - 5/3) > 0$

$\left(x - \frac{\sqrt{5}}{3}\right)\left(x + \frac{\sqrt{5}}{3}\right) > 0$




$x \in (-\infty, \sqrt{5}/3) \cup (\sqrt{5}/3, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$\therefore 3(x^2 - 5/3) < 0$

$\therefore (x - \sqrt{5}/3)(x + \sqrt{5}/3) < 0$ $\qquad x \in (-\frac{\sqrt{5}}{3}, \sqrt{5}/3)$

$+ \quad - \quad +$

$-\sqrt{5}/3 \qquad \sqrt{5}/3$

2] $f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

$\because f$ is increasing iff $f'(x) > 0$

$\therefore 2x - 4 > 0$

$\therefore 2(x-2) > 0$

$\therefore x - 2 > 0$

$x \in (2, \infty)$

and $f$ is decreasing iff $f'(x) < 0$

$\therefore 2x - 4 < 0$

$\therefore 2(x-2) < 0$

$\therefore x - 2 < 0$

$x \in (-\infty, 2)$

3] $f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\because f$ is increasing iff $f'(x) > 0$

$\therefore 6x^2 + 2x - 20 > 0$

$\therefore 2(3x^2 + x - 10) > 0$

$\therefore 3x^2 + x - 10 > 0$

$\therefore 3x^2 + 6x - 5x - 10 > 0$

$\therefore 3x(x+2) - 5(x+2) > 0$

$\therefore (x+2)(3x-5) > 0$

16

$+ \quad - \quad +$

$-2 \qquad 5/3$ $\qquad x \in (-\infty, -2) \cup (5/3, \infty)$

and $f$ is decreasing iff $f'(x) < 0$

$\therefore 6x^2 + 2x - 20 < 0$

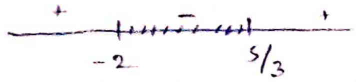
$\therefore 2(3x^2 + x - 10) < 0$

$\therefore 3x^2 + x - 10 < 0$

$\therefore 3x^2 + 6x - 5x - 10 < 0$

$\therefore 3x(x+2) - 5(x+2) < 0$

$\therefore (x+2)(3x-5) < 0$




$x \in (-2, 5/3)$

4] $f(x) = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$\because$ $f$ is increasing iff $f'(x) > 0$

$\therefore 3(x^2 - 9) > 0$

$\therefore (x-3)(x+3) > 0$

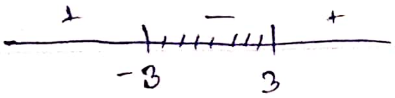

$x \in (-\infty, -3) \cup (3, \infty)$

and $f$ is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 27 < 0$

$\therefore 3(x^2 - 9) < 0$

$\therefore (x-3)(x+3) < 0$




$\therefore x \in (-3, 3)$

$f(x) = 2x^3 - 9x^2 - 24x + 69$

$f'(x) = 6x^2 - 18x - 24$

$\because$ $f$ is increasing iff $f'(x) > 0$

$\therefore 6x^2 - 18x - 24 > 0$

$\therefore 6(x^2 - 3x - 4) > 0$

$\therefore x^2 - 4x + x - 4 > 0$

$\therefore (x+1)(x-4) > 0$


$x \in (-\infty, -1) \cup (4, \infty)$

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and $f$ is decreasing iff $f'(x) < 0$

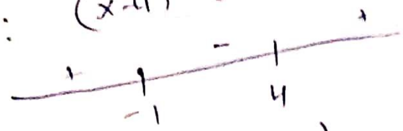
$6x^2 - 18x - 24 < 0$

$\therefore 6(x^2 - 3x - 4) < 0$

$\therefore x^2 - 4x + x - 4 < 0$

$\therefore (x-4)(x+1) < 0$

$\therefore$



$\therefore x \in (-1, 4)$

2.] $y = 3x^2 - 2x^3$

$\therefore f(x) = 3x^2 - 2x^3$

$\therefore f'(x) = 6x - 6x^2$

$\therefore f''(x) = 6 - 12x$

$f$ is concave upward if $f''(x) > 0$

$\therefore (6-12x) > 0$

$\therefore 12\left(\frac{1}{2} - x\right) > 0$

$\therefore x - \frac{1}{2} > 0$

$\therefore x > \frac{1}{2}$

$\therefore f''(x) > 0$

$\therefore x \in \left(\frac{1}{2}, \infty\right)$

2] $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$f'(x) = 4x^3 - 18x^2 + 24x + 5$

$f''(x) = 12x^2 - 36x + 24$.

$f$ is concave upward if $f''(x) > 0$

$\therefore 12x^2 - 36x + 24 > 0$

$\therefore 12(x^2 - 3x + 2) > 0$

$\therefore x^2 - 2x - x + 2 > 0$

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$(x-2)\quad(x-1)>0$

$x\in(-\infty,1)\cup(2,\infty)$

1 2

3] $y = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

f is concave upward iff $f''(x) > 0$

$\therefore\ 6x > 0$

$\therefore\ x > 0$

$\therefore\ x \in (0,\infty)$

4] $y = 69 - 24x - 9x^2 + 2x^3$

$f(x) = 69 - 24x - 9x^2 + 2x^3$

$f'(x) = 6x^2 - 18x - 24$

$f''(x) = 12x - 18$

f is concave upward iff $f''(x) > 0$

$\therefore\ 12x - 18 > 0$

$\therefore\ 12(x - 18/12) > 0$

$\therefore\ x - 3/2 > 0$

$\therefore\ x > 3/2$

$\therefore\ x \in (3/2, \infty)$

16/12/19

5] $y = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

f is concave upward iff $f''(x) > 0$

$\therefore\ f''(x) > 0$

$\therefore\ 12x + 2 > 0$

$\therefore\ x + 1/6 > 0$

$\therefore\ x < -1/6$

$\therefore\ f''(x) \not> 0$

$\therefore$ There exist no interval

08

## Practical No. 4

Topic :- Application of Derivatives and Newton Method

Q.I] Find Maximum and Minimum value of following functions :-

i] $f(x) = x^2 + \frac{16}{x^2}$

ii] $f(x) = 3 - 5x^3 + 3x^5$

iii] $f(x) = x^3 - 3x^2 + 1$ in $\left[-\frac{1}{2}, 4\right]$

iv] $f(x) = 2x^3 - 3x^2 - 12x + 1$ in $[-2, 3]$

Q.II] Find the real of following equation by Newton Method where $x_0 = 0$

1] $f(x) = x^3 - 3x^2 - 55x + 95$

2] $f(x) = x^3 - 4x - 9$ in $[2, 3]$

3] $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1, 2]$

Solution:—

1. i] $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$

Now Consider,

$f'(x) = 0$

$\therefore 2x - \frac{32}{x^3} = 0$

$\therefore 2x = \frac{32}{x^3}$

$\therefore x^4 = 16$

$\therefore x = \pm 2$

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = 2 + \frac{96}{24}$

$= 2 + 6$

$\therefore$ f has minimum value at $x=2$

$f''(2) = 2 + \frac{96}{-24}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has minimum value at $x = 2$

$\therefore$ The function reaches minimum value at $x = 2$ & $x = 4$

ii] $f(x) = 3 - 5x^3 + 3x^5$

$\therefore f'(x) = -15x^2 + 15x^4$

consider, $f'(x) = 0$

$-15x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x^2 = 1$

$x = \pm 1$

$f''(x) = -30x + 60x^3$

$f''(x) = -30 + 60$

$= 30 > 0$

$\therefore$ f has minimum value at $x = 1$

$\therefore f(1) = 3 - 5(1)^3 \cancel{+ 3(1)^5}$

$f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 - 3$

$= 5$

$\therefore$ f has the maximum value at $x = -1$ and has the minimum value at $x = 1$

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$f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x$

consider,

$\therefore f'(x) = 0$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$3x = 0$ or $x - 2 = 0$

$x = 0$ or $x = 2$

$f''(x) = 6x - 6$

$f''(0) = 6(0) - 6$

$= -6 < 0$

$\therefore f$ has maximum value at $x = 0$

$\therefore f(0) = (0)^3 - 3(0)^2 + 1 = 1$

$f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore f$ has minimum value at $x = 2$

$f(2) = (2)^3 - 3(2)^3 + 1$

$= 8 - 3(4) + 1$

$= -3$

$\therefore f$ has maximum value at $x = 0$ and

$f$ has ~~maximum~~ value at $x = 2$

$f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

consider,

$f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

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$x^2 + x - 2x - 2 = 0$

$x(x+1) - 2(x+1) = 0$

$(x-2) = 0$ or $(x+1) = 0$

$x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$

$f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18 > 0$

f has minimum value at $x = 2$

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 24 + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$f''(-1) = -12(-1) - 6$

$= -12 - 6$

$= -18 < 0$

$\therefore$ f has maximum value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 \cancel{+12} + 1$

$= 8$

$\therefore$ f has maximum value at $x = -1$ and
f has minimum value at $x = 2$

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58

2. i] $f(x) = x^3 - 3x^2 - 55x + 9.5$ $(x_0 : 0 \rightarrow \text{Given})$

$f'(x) = 3x^2 - 6x - 55$

By Newton's Method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 + \frac{9.5}{55}$$

$$x_1 = 0.1727$$

$f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$

$= 0.0051 - 0.0895 - 9.4985 + 9.5$

$= -0.0829$

$f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$

$= 0.0895 - 1.0362 - 55$

$= -55.9467$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.1727 - \frac{0.0829}{55.9467}$$

$= 0.1712$

$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$

$= 0.0050 - 0.0879 - 9.416 + 9.5$

$= 0.0011$

$f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$

$= 0.0879 - 1.0272 - 55$

$= -55.9393$

$f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$

$= 0.0879 \quad 1.0272 \quad -55$

$= -55.9393$

$\therefore x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 0.1712 + \dfrac{0.0011}{55.9393}$

$= 0.1712$

The root of the equation is 0.1712.

ii] $f(x) = x^2 - 4x - 9 \qquad [2,3]$

$f'(x) = 3x^2 - 4$

$f(2) = 2^3 - 4(2) - 9$

$= 8 - 8 - 9$

$= -9$

$f(3) = 3^3 (4) 3 - 9$

$= 27 - 12 - 9$

$= 6$

Let $x_0 = 3$ be the initial approximation,

By Newton's Method,

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$= 3 \quad \dfrac{6}{23}$

$= 2.7392$

$f(x_1) = (2.7392)^3 - 4(2.7392) - 9$

$= 20.5578 - 10.9168 - 9$

$= 0.196$

$$f'(x_1) = 3(2.7392)^2 - 4$$
$$= 22.5096 - 4$$
$$= 18.5096$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.7392 - \frac{0.596}{18.5096}$$
$$= 2.7071$$
$$f(x_2) = (2.7071)^3 - 4(2.7071) - 9$$
$$= 19.8386 - 10.8284 - 9$$
$$= 0.0102$$
$$f'(x_2) = 3(2.7071)^2 - 4$$
$$= 21.9851 - 4$$
$$= 17.9851$$
$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 2.7071 - \frac{0.0102}{17.9851}$$
$$= 2.7071 - 0.0056$$
$$= 2.7015$$
$$f(x_3) = (2.7015)^3 - 4(2.7015) - 9$$
$$= 19.7158 \quad 10.806 - 9$$
$$= -0.0901$$
$$f'(x_3) = 3(2.7015)^2 - 4$$
$$= 21.8943 - 4$$
$$= 17.8943$$
$$x_4 = 2.7015 + \frac{0.0901}{17.8943}$$
$$= 2.7015 + 0.0050$$

iii] $f(x) = x^3 - 1.8x^2 - 10x + 17$ $[1,2]$

$f'(x) = 3x^2 - 3.6x - 10$

$f(1) = (1)^3 - 1.8(1)^2 - 10(1) + 17$

$= 1 - 1.8 - 10 + 17$

$= 6.2$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$

$= 8 - 7.2 - 20 + 17$

$= -2.2$

Let $x_0 = 2$ be initial approximation,

By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2 - \frac{2.2}{5.2}$$

$= 2 - 0.4230$

$= 1.577$

$f(x_1)' = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$

$= 3.9219 - \cancel{0.4764} - 15.77 + 17$

$= \cancel{0.6755}$

$f'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10$

$= 7.4608 - 5.6772 - 10$

$= -8.2164$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.577 + \frac{0.6755}{8.2164}$$

$f(x_2) = (1.6992)^3 - 1.8(1.6992)^2 - 10(1.6992) + 17$

$= 4.1677 - 5.1972 - 16.992 + 17$

$= 0.0204$

$f'(x_2) = 3(1.6992)^2 - 3.6(1.6992) - 10$

$= 8.3188 - 5.97312 - 10$

$= -7.7143$

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 1.69592 + \dfrac{0.0204}{7.7143}$

$= 1.6992 + 0.0026$

$= 1.6618$

$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$

$= 4.5892 - 4.9708 - 16.618 + 17$

$= 0.0004$

$f'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$

$= 8.2847 - 5.9824 - 10$

$= -7.6977$

$x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)}$

$= 1.8618 + \dfrac{0.0004}{7.6977}$

$= 1.6618$

The root of equation is 1.6618

22/2/19

6]

$$\int x^2 + \frac{2x+4}{\sqrt{x}}\, dx$$

$$I = \int \frac{x^2 + 2x + 4}{\sqrt{x}}\, dx$$

$$= \int \left( \frac{x^2}{x^{1/2}} + \frac{2x}{x^{1/2}} + \frac{4}{x^{1/2}} \right) dx$$

$$= \int x^{3/2}\, dx + 3 \int x^{-1/2}\, dx + 4 \int x^{-1/2}\, dx$$

$$= \frac{2}{5} x^{5/2} + 2 x^{3/2} + 8\sqrt{x} + c$$

3]

$$\int t^7 \sin(2t^4)\, dx$$

$$I = \int t^7 \sin(2t^4)\, dx$$

Let $t^4 = x$

$4t^3\, dt = dx$

$$I = \frac{1}{4} \int 4t^3 \cdot t^4 \sin(2t^4)\, dt$$

$$= \frac{1}{4} \int x \cdot \sin 2x\, dx$$

$$= \frac{1}{4} \left[ x \int \sin 2x - \int \left[ \int \sin 2x \cdot \frac{d}{dx} x \right] \right]$$

$$= \frac{1}{4} \left[ -x \frac{\cos 2x}{2} + \frac{1}{2} \int \cos 2x \cdot 1 \right]$$

$$= \frac{1}{4} \left[ -x \frac{\cos 2x}{2} + \frac{1}{4} \sin 2x \right] + c$$

$$= -\frac{1}{8} x \cos 2x + \frac{1}{16} \sin 2x + c$$

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$$= -\frac{1}{8} x^4 \cos(2x^4) + \frac{1}{16} \sin(2x^4) + c$$

6] $\int \sqrt{x}\,(x^3 - 1)\,dx$

$$I = \int \sqrt{x}\,(x^3 - 1)\,dx$$

$$= \int (\sqrt{x}\,x^3 - \sqrt{x})\,dx$$

$$= \int (x^{7/2} - \sqrt{x})\,dx$$

$$= \frac{2}{9} x^{9/2} - \frac{2}{3} x^{3/2} + c$$

7] $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$$I = \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$$

Let $\frac{1}{x^2} = t$

$$x^{-2} = t$$

$$-\frac{2}{x^3}\,dx = dt$$

$$I = -\frac{1}{2} \int -\frac{2}{x^3} \sin\left(\frac{1}{x^2}\right) dx$$

$$= -\frac{1}{2} \cancel{\int \sin t}$$

$$= -\frac{1}{2} (-\cos t) + c$$

$$= \frac{1}{2} \cos t + c$$

Resubstitution $t = 1/x^2$

$$I = \frac{1}{2} \cos\left(\frac{1}{x^2}\right) + c$$

8] $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}$

$I = \int \frac{\cos x}{\sqrt[3]{\sin^2 x}}$

Let $\sin x = t$

$\cos x \, dx = dt$

$I = \int \frac{dt}{\sqrt[3]{t^2}}$

$= \int \frac{dt}{t^{2/3}}$

$= \int t^{-2/3} \, dt$

$= 3t^{1/3} + c$

$= 3(\sin x)^{1/3} + c$

$= 3\sqrt[3]{\sin x} + c$

9] $\int e^{\cos^2 x} \cdot \sin^2 x \, dx$

$I = \int e^{\cos^2 x} \cdot \sin^2 x \, dx$

Let $\cos^2 x = t$

$-2\cos x \sin x \cdot dx = dt$

$-\sin 2x \, dx = dt$

$I = \int \cdot \sin 2x \cos 2x \, dx$

$= -\int e^t dt$

$= e^t + c$

Resubstitution $t = \cos 2x$

$I = \int e^{t}$

$\cos x$

1) $\int \left( \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

$I = \int \left( \frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

Let $x^3 - 3x^2 + 1 = t$

$3(x^2 - 2x)\, dx = dt$

$(x^2 - 2x)\, dx = dt/3$

$I = \int \frac{1}{t}\, dt$

$= \frac{1}{3} \int dt/t$

$= \frac{1}{3} \log t + c$

$= \frac{1}{3} \log (x^3 - 3x^2 + 1) + c$

06/01/2020

58

# Practical - 6

Aim :- Application of Integration & Numerical Integration

Q.1 Find the length of the following curve.

i] $x = t\sin t \quad y = t - \cos t$ where $t \in [0, 2\pi]$

$\Rightarrow \quad L = \int_0^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$x = t - \sin t$

$\frac{dx}{dt} = t - \cos t$

$y = 1 - \cos t$

$\frac{dy}{dt} = 0 - (-\sin t) = \sin t$

$L = \int_0^{2\pi} \sqrt{(t - \cos t + \sin t)^2 + (\sin t)^2}\, dt$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\, dt$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\, dt$

$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$

$= \int_0^{2\pi} 2\left|\sin \frac{t}{2}\right| dt \qquad \sin^2 \frac{t}{2} \cdot \frac{1-\cos t}{2}$

$= \int_0^{2\pi} 2 \sin \frac{t}{2}\, dt$

$$\left(-4\cos\left(\frac{t}{2}\right)\right)_0^{2\pi} = (-4\cos\pi) - (-4\cos 0)$$

$$= 4+4 = \underline{\underline{8}}$$

2] $y = \sqrt{4-x^2}$ $\quad x \in [-2, 2]$

$$\Rightarrow L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dy}{dx} = \int_0^b \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx$$

$$= 2\int_0^2 \sqrt{1 + \frac{x^2}{4-x^2}}\, dx$$

$$= 4\int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx$$

$$= 4\left(\sin^{-1}\left(\frac{x}{2}\right)\right)_0^2$$

$$= 2\pi$$

3] $Y = x^{3/2}$ in $[0, 4]$

$$\Rightarrow f'(x) = \frac{3}{2} x^{1/2}$$

$$[f'(x)]^2 = \frac{9}{4} x$$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_0^x \sqrt{1 + \frac{9}{4}x}\, dx$$

Put $u = 1 + \frac{9}{4}x$ $\quad du = \frac{9}{4}dx$

3)

$$L = \int_1^{1+\frac{9}{4}x} \frac{4}{9}\sqrt{u}\, du = \left[ \frac{4}{9} \cdot \frac{2}{3} \left(u^{3/2}\right) \right]_1^{1+\frac{9}{4}x}$$

$$= \frac{8}{27}\left[ \left(1 + \frac{9x}{4}\right) - 1 \right]$$

4) $x = 3\sin t \qquad y = 3\cos t$

$$\Rightarrow \frac{dx}{dt} = 3\cos t$$

$$\frac{dy}{dt} = -3\sin t$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{2\pi} 3\sqrt{1}\, dt$$

$$= 3\int_0^{2\pi} x\, dt$$

$$= 3\left[x\right]_0^{2\pi}$$

$$= 3\left[2\pi - 0\right]$$

$$= 6\pi$$

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5] $x = \frac{1}{6} y^3 + \frac{1}{2y}$ on $y = [1, 2]$

$\Rightarrow \frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$

$\frac{dx}{dy} = \frac{y^2 - 1}{2y^2}$

$L = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y)^2}}\, dy$

$= \int_1^2 \frac{y^4+1}{2y^2}\, dy$

$= \frac{1}{2} \int_1^2 y^2\, dy + \frac{1}{2} \int_1^2 y^{-2}\, dy$

$= \frac{1}{2} \int_1^2 y^2\, dy + \frac{1}{2} \int_1^2 y^{-2}\, dy$

$= \frac{1}{2} \left[ \frac{y^3}{3} - \frac{y^{-2}}{1} \right]_1^2$

$= \frac{1}{2} \left[ \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 \right]$

$= \frac{1}{2} \left[ \frac{7}{3} + \frac{1}{2} \right]$

$= \frac{17}{12}$ unit

Q.2 Solve the following questions:-

1] $\int_0^2 e^{x^2}\,dx$ with $n=4$

$\Rightarrow$ $\int_0^2 e^{x^2}\,dx = 16.04526$

In each case, the width of the sub interval be

$\Delta x = \frac{2-0}{4} = \frac{1}{2}$

and so the sub intervals will be $[0, 0.5]$ $[0.5, 1]$ $[1, 1.5]$ $[1.5, 2]$

By simpson rule,

$$\int_0^2 e^{x^2}\,dx = \frac{1/2}{3}\left(\frac{1}{0} + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$\approx \frac{1/2}{3}\left(e^{0^2} + 4e^{(0.5)^2} + 2e^{(1)^2} + 4e^{(1.5)^2} \ldots\right.$$

$$\approx 17.35636$$

2] $\int_1^4 x^2\,dx$ $n=4$

$\Delta x = \cancel{\frac{4-0}{4}} = 1$

$$\int_a^b f(x)\,dx = \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right]$$

$$= \frac{1}{3}\left[y(0) + 4(1)^2 + 2(2)^2 + 4(3)^2 + 4^2\right)$$

$$= \frac{64}{3}$$

3] $\int_0^{\pi/3} \sqrt{\sin x}\, dx \qquad n = 6$

$\Rightarrow \quad \Delta x = \dfrac{b-a}{n} = \dfrac{\pi/3 - 0}{6} = \dfrac{\pi}{18}$

| | | | | | | |
|---|---|---|---|---|---|---|
| x | 0 | π/18 | 2π/18 | 3π/18 | 4π/18 | 5π/18 |
| y | 0 | 0.4167 | 0.584 | 0.707 | 0.801 | 0.87 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |

$$\int_0^{\pi/3} \sqrt{\sin x}\, dx \approx \frac{\Delta x}{3}\left(y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + \right.$$

$$= \frac{\pi/18}{3}\left(0 + 4(0.4167 + 0.707 + 0.875) + 0.930\right.$$

$$\approx 2(0.584 + 0.801) + 0.930$$

$$\approx 0.681$$

## Practical - 1

Aim:- Differential Equations

Q.1] Calculate the following differential equations:-

1] $x\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$

$p(x) = \frac{1}{x}$ $q(x) = \frac{e^x}{x}$

$If = e^{\int \frac{1}{x} dx} = x$

$y(IF) = \int Q(x)\,(IF)\,dx + c$

$= \int \frac{e^x}{x} x \cdot dx + c$

$= \int e^x dx + c$

$xy = e^x + c$

2] $e^x \frac{dy}{dx} + 2e^x y = 1$

$\Rightarrow \frac{dy}{dx} + \frac{2e^x}{e^x} = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = e^{-x}$

$P(x) = 2$ $Q(x) = e^{-x}$

$\int p(x) = dx$

$IF = e^{\int 2 dx}$

$= e^{2x}$

$y(IF) = \int Q(x)\,(IF)\,dx + c$

$= \int e^x dx + c.$

$= y \cdot e^{2x}.$

$= e^x + c.$

3] $x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$\Rightarrow x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$\frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x}$

$P(x) = 2(x) \qquad Q(x) = \frac{\cos x}{x^2}$

$IF = e^{\int P(x)\,dx}$

$= e^{\int 2x\,dx}$

$y\,(IF) = \int Q(x)\,(IF)\,dx + C$

$= \int \frac{\cos x}{x^2} - x^2\,dx + C$

$= \int \cos x + C$

$\therefore x^2 y = \sin x + C$

4] $x \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$\Rightarrow \frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$

$P(x) = 3/x \qquad Q(x) = \sin x / x^3$

$P(x) = \int 3/x\,dx$

$= x^3$

$IF = e^{\int P(x)\,dx}$

$= x^3$

$y\,(IF) = \int Q(x)\,(IF)\,dx + C$

$= \int \frac{\sin x}{x^3} \cdot x^3\,dx + C$

$= \int \sin x + C = -\cos x + C$

5] $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

$\rightarrow \frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$

$P(x) = 2$

$Q(x) = \frac{2x}{e} = 2xe^{-2x}$

$IF = e^{\int p(x)\,dx}$

$= e^{\int 2\,dx}$

$= e^{2x}$

$y(IF) = \int Q(x)\,(IF)\,dx + C$

$= \int 2x\,e^{-2x}\,e^{2x} + c$

$y.e^{2x} = \int 2x + c$

$= x^2 + C$

6] $\sec^2 x \cdot \tan y\,dx + \sec^2 y\,\tan x\,dy = 0$

$\rightarrow \sec^2 x \cdot \tan y\,dx = -\sec^2 y\,\tan x\,dy$

$\frac{\sec^2 x}{\tan x}\,dx = -\frac{\sec^2 x}{\tan y}\,dy$

$\int \frac{\sec^2 x}{\tan x}\,dx = -\int \frac{\sec^2 x}{\tan y}\,dy$

$\int \frac{\sec^2 x}{\tan x}\,dx = -\int \frac{\sec^2 x}{\tan y}\,dy$

$\log|\tan x| = -\log|\tan y| + c$

$\log|\tan x + \tan y| = c$

$\tan x + \tan y = e^c.$

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7] $\frac{dy}{dx} = \sin^2(x-y+1)$

Put $x-y+1 = v$

$x-y+1 = v$

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\frac{1-dv}{dx} = \frac{dy}{dx}$

$1 - \frac{dv}{dx} = \sin^2 v$

$\frac{dv}{dx} = \cos^2 x$

$\frac{dv}{\cos^2 x} = dx$

$\int \sec^2 x \, dv = \int dx$

$\tan v = x + c$

$\tan(x-y+1) = x + c$

8] $\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$

Put $2x+3y = u$

$2 + 3\frac{dy}{dx} = \frac{du}{dx}$

$\frac{du}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$= \frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\left(\frac{v-1}{v+2}\right)$

$\frac{dv}{dx} = \frac{v-1}{v+2} + 2$

$$\frac{dv}{dx} = \frac{v-1+2v+v}{v+2}$$

$$= \frac{3v+3}{v+2}$$

$$= \frac{3(v+1)}{(v+2)}$$

$$= \int \frac{v+2}{v+1}\,dv$$

$$= 3\,dx$$

$$= \int \frac{v+1}{v}\,dx + \int \frac{1}{v+1}\,dv = \int 3\,dx$$

$$v \log|v| = 3x + c$$

$$2x+3y+\log|2x+3y+1| = 3x+c$$

$$3y = x - \log|2x+3y+1| + c$$

## Practical - 8

Aim: Euler's Method.

Q1) $\frac{dy}{dx} = y + e^x - 2$ $\quad y(0) = 2 \quad h = 0.5$ find $y(2) = ?$

→ $f(x) = y + e^x - 2 \quad x_0 = 0 \quad y(0) = 2 \quad h = 0.5$

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $y_{n+1}$ |
|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 2.5 |
| 1 | 0.5 | 2.5 | 2.1487 | 3.5743 |
| 2 | 1 | 3.5743 | 4.2925 | 5.7201 |
| 3 | 1.5 | 5.7201 | 8.2011 | 9.8215 |
| 4 | 2 | 9.8215 | | |

∴ $y(2) = 9.8215$

Q2) $\frac{dy}{dx} = 1 + y^2$, $\quad y(0) = 0 \quad h = 0.2$ find $y(1) = ?$

$x_0 = 0 \quad y_0 = 0 \quad h = 0.2$

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $y_{n+1}$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0.2 |
| 1 | 0.2 | 0.2 | 1.04 | 0.408 |
| 2 | 0.4 | 0.408 | 1.1664 | 0.6412 |
| 3 | 0.6 | 0.6412 | 1.4111 | 0.9234 |
| 4 | 0.8 | 0.9234 | 1.8526 | 1.2939 |
| 5 | 1 | 1.2939 | | |

$y(1) = 1.2939$

82

Q 3] $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $y(0) = 1$ [illegible]

$x_0 = 0$ $y(0) = 1$ [illegible]

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | |
| 1 | 0.2 | 1 | [illegible] | [illegible] |
| 2 | 0.4 | 1.0894 | [illegible] | [illegible] |
| 3 | 0.6 | 1.210 | [illegible] | [illegible] |
| 4 | 0.8 | 1.3513 | [illegible] | [illegible] |
| 5 | 1 | 1.5051 | | |

$y(1) = 1.5051$

Q 4] $\frac{dy}{dx} = 3x^2 + 1$ $y(1) = 2$ find $y(2)$ [illegible]

$y_0 = 2$ $x_0 = 1$ $h = 0.5$

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $y_{n+1}$ |
|---|---|---|---|---|
| 0 | 1 | 2 | 4 | 4 |
| 1 | 1.5 | 4 | [illegible] | [illegible] |
| 2 | 2 | 7.875 | | |

$y(2) = 7.875$

ii] $y_0 = 2$ $x_0 = 1$ $h = 0.25$

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $y_{n+1}$ |
|---|---|---|---|---|
| 0 | 1 | 2 | 4 | 3 |
| 1 | 1.25 | 3 | 5.6875 | [illegible] |
| 2 | 1.5 | 4.4218 | 59.6563 | [illegible] |
| 3 | 1.75 | 19.3360 | 1102.6[illegible] | [illegible] |
| 4 | 2 | 299.9960 | | |

$y(2) = 299.9960$

Q3] $\frac{dy}{dx} = \sqrt{xy} + 2 \qquad y(1) = 1 \qquad h = 0.2$

$x_0 = 1 \qquad y_0 = 1 \qquad h = 0.2$

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $y_{n+1}$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 3 | 3.6 |
| 1 | 1.2 | 3.6 | | |

$y(1.2) = 3.6$

20/01/2020

22

# PRACTICAL - 9

Aim:- Limits & Partial order Derivatives

Q.1. Answer the following (Solve):-

1] $\lim_{(x,y)\to(-4,-1)} \dfrac{x^3-3y+y^2-1}{xy+5}$

At $(-4,-1)$, Denomination $\neq 0$

$\therefore$ By applying limit

$$= \frac{(-4)^3 - 3(-1) + (-1)^2 - 1}{(-4)(-1)+5}$$

$$= \frac{-64+3+1-1}{4+5}$$

$$= \frac{-61}{9}$$

2] $\lim_{(x,y)\to(2,0)} \dfrac{(y+1)(x^2+y^2-4x)}{x+2y}$

$\therefore$ By Applying Limit

$$= \frac{(0+1)((2)^2+0-4(2))}{2+0}$$

$$= \frac{1(4+0-8)}{2}$$

$$= \frac{-4}{2} = -\frac{1}{2}$$

3] $\lim_{(x,y,z)\to(1,1,1)} \dfrac{x^2-y^2z^2}{x^3-x^2yz^2}$

$\lim_{(x,y,z)\to(1,1,1)} \dfrac{x^2-y^2z^2}{x^3(-x^2yz)}$

$\lim_{(x,y,z)\to(1,1,1)} \dfrac{\cancel{(x-yz)}(x+yz)}{x^2\cancel{(x-yz)}}$

$$\lim_{(x,y,z)\to(1,1,1)} \frac{x+y^2}{x^2}$$

On Applying Limit

$$= \frac{1+1(1)}{(1)^2} = \underline{\underline{2}}$$

Solve the following:-

Q) $f(x,y) = xye^{x^2+y^2}$

$\therefore f_x = \frac{d}{dx}(f(x,y))$

$= \frac{d}{dx}(xye^{x^2+y^2})$

$= y\, e^{x^2+y^2}(2x)$

$\therefore f_x = 2xye^{x^2+y^2}$

$\therefore f_y = \frac{d}{dy}(f(x,y))$

$= \frac{d}{dy}(xye^{x^2+y^2})$

$= xe^{x^2+y^2}(2y)$

$\therefore f_y = 2yxe^{x^2+y^2}$

2) $f(x,y) = e^x \cdot \cos y$

$f_x = \frac{d}{dx}(f(x,y))$

$= \frac{d}{dx}(e^x \cdot \cos y)$

$\therefore f_x = e^x . \cos y .$

$f_y = \frac{d}{dy} (f(x,y))$

$= \frac{d}{dy} e^x . \cos y$

$\therefore f_y = -e^x . \sin y$

5] $f(x,y) = x^3 y^2 - 3x^2 y + y^3 + 1$

$f_x = \frac{d}{dx} (f(x,y))$

$= \frac{d}{dx} (x^3 y^2 - 3x^2 y + y^3 + 1)$

$\therefore f_x = 3x^2 y^2 - 6xy$

$f_y = \frac{d}{dy} (f(x,y))$

$= \frac{d}{dy} (x^3 - y^2 - 3x^2 y + y^3 + 1)$

$f_y = 2x^3 y - 3x^2 + 3y^2$

G III Solve :-

1] $f(x,y) = \frac{2x}{1+y^2}$

$f_x = \frac{d}{dx} \left( \frac{2x}{1+y^2} \right)$

$= \frac{1+y^2 \frac{d}{dx}(2x) - 2x \frac{d}{dx}(1+y^2)}{(1+y^2)^2}$

$$= \frac{2+2y^2-0}{(1+y^2)^2}$$

$$= \frac{2(\cancel{1+y^2})}{(1+y^2)\cancel{(1+y^2)}}$$

$$= \frac{2}{1+y^2}$$

At $(0,0)$ $= \frac{2}{1+0} = \underline{\underline{2}}$

$\therefore f_y = \frac{d}{dy}\left(\frac{2x}{1+y^2}\right)$

$$= \frac{1+y^2 \frac{d}{dx}(2x) - 2x\frac{d}{dx}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{1+y^2(0) - 2x(2y)}{(1+y^2)^2}$$

$$= \frac{-4xy}{(1+y^2)^2}$$

At $(0,0)$ $= \frac{-4(0)(0)}{(40)^2} = 0$

Q.I Solve :-

1] $f(x,y) = \frac{y^2 - xy}{x^2}$

$$f_x = \frac{x^2 \frac{d}{dx}(y^2 - xy) - (y^2 - xy)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2(-y) - (y^2 - xy)(2x)}{x^4}$$

$$= \frac{-x^2y - 2x(y^2 - xy)}{x^4}$$

$$F_y = \frac{2y - x}{x^2}$$

ii)

$$f_{xx} = \frac{d}{dx}\left(\frac{-x^2y - 2x(2y^2 - xy)}{x^4}\right)$$

$$= \frac{x^4\left(\frac{d}{dx}(-x^2y - 2xy^2 + 2x^2y)\right) - (-x^2y - 2xy^2 + 2x^2y)\frac{d}{dx}(x^4)}{(x^4)^2}$$

$$= \frac{x^4(-2xy - 2y^2 + 4xy) - 4x^3(-x^2y - 2xy + 2x^2y)}{x^8} \quad \text{— (i)}$$

$$f_{yy} = \frac{d}{dy}\left(\frac{2y-x}{x^2}\right)$$

$$= \frac{2-0}{x^2} = \frac{2}{x^2} \quad \text{— (ii)}$$

$$f_{xy} = \frac{d}{dy}\left(\frac{-x^2y - 2xy^2 + 2x^2y}{x^4}\right)$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4} \quad \text{— (iii)}$$

$$f_{yx} = \frac{d}{dx}\left(\frac{2y-x}{x^2}\right)$$

$$= \frac{x^2\frac{d}{dx}(2y-x) - (2y-x)\frac{d}{dx}(x^2)}{x^2}$$

$$= -\frac{x^2 - 4xy + 2x^2}{x^4} \quad \text{— (iv)}$$

From (iii) & (iv)

$$f_{yx} = f_{xy}$$

iii] $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$$f_x = \frac{d}{dx}\left(x^3 + 3x^2y^2 - \log(x^2+1)\right)$$

$$= 3x^2 + 6xy^2 - 2x / x^2+1$$

$$f_{xx} = 6x + 6y^2 - \left(\frac{x^2+1\,\frac{d}{dx}(2x) - 2x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}\right)$$

$$= 6x + 6y^2 - \left(\frac{2(x^2+1) - 4x^2}{(x^2+1)^2}\right) \quad \text{— (i)}$$

$f_y = \frac{d}{dy}\left(x^3 + 3x^2y^2 - \log(x^2+1)\right)$

$= 0 + 6x^2y - 0$

$= 6x^2y$

$f_{yy} = \frac{d}{dx}(6x^2y)$

$= 6x^2$ —— (ii)

$f_{xy} = \frac{d}{dx}\left(3x^2 + 6xy^2 - 2x/x^2+1\right)$

$= 0 + 12xy - 0$

$= 12xy$ —— (iii)

$f_{yx} = \frac{d}{dy}\left(3x^2 + 6y^2x - 2x/x^2+1\right)$

$= 0 + 12xy - 0$

$= 12xy$ —— (iv)

From (iii) & (iv)

$\therefore f_{xy} = f_{yx}$.

3) $f(x,y) = \sin(xy) + e^{x+y}$

$f_x = y\cos(xy) + e^{x+y}(1)$

$= y\cos(xy) + e^{x+y}$

$f_{xx} = \frac{d}{dx}\left(y\cos(xy) + e^{x+y}\right)$

$= -y\sin(xy)(y) + e^{x+y}(1)$

$= -y^2\sin(xy) + e^{x+y}$ —— (i)

$f_y = x\cos(xy) + e^{x+y}(1)$

$= x\cos(xy) + e^{x+y}$.

$f_{yy} = \frac{d}{dy}\left(x\cos(xy) + e^{x+y}\right)$

$= -x\sin(xy)\,x + e^{x+y}(1)$

$= -x^2\sin(xy) + e^{x+y}$ — (ii)

$f_{xy} = \frac{d}{dx}\left(y\cos(xy) + e^{x+y}\right)$

$= -y^2\sin(xy) + \cos(xy) + e^{x+y}$ — (iii)

$f_{yx} = \frac{d}{dx}\left(x\cos(xy) + e^{x+y}\right)$

$= -x^2\sin(xy) + \cos(xy) + e^{x+y}$ — (iv)

From (iii) & (iv)

$f_{xy} \neq f_{yx}$.

Q.5 Solve :-

1] $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$f(1,1) = \sqrt{(1)^2+(1)^2}$

$= \sqrt{1+1} = \sqrt{2}$

$f_x = \frac{1}{2\sqrt{x^2+y^2}}(2x)$

$= \frac{x}{\sqrt{x^2+y^2}}$

1. at $(1,1) = \frac{1}{\sqrt{2}}$

$f_y = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} (\cancel{2}y)$

$= \frac{y}{\sqrt{x^2+y^2}}$

$f_y$ at $(1,1) = \frac{1}{\sqrt{2}}$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1+y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x+y-2)$

$= \cancel{\sqrt{2}} + \frac{1x}{\sqrt{2}} - \frac{1}{\sqrt{2}}y - \cancel{\frac{2}{\sqrt{2}}}\cancel{\sqrt{2}}$

$= \frac{x+y}{\sqrt{2}}$

c] $f(x,y) = 1-x+y\sin x$ at $(\pi/2, 0)$

$f(\pi/2, 0) = 1-\pi/2+0 = 1-\pi/2$

$f_x = 0-1+y\cos x$

$f_x$ at $(\pi/2, 0) = -1+0$

$= -1$

$f_y$ at $(\pi/2, 0) = 0-0+\sin x$

$= \sin \pi/2$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 1-\pi/2 + (-1)(x-\pi/2) + (y-0)$

$= 1-\cancel{\pi/2} - x + \cancel{\pi/2} + y$

$= 1-x+y$

Q2.

3] $f(x,y) = \log x + \log y$ at $(1,1)$

$f(1,1) = \log(1) + \log(1) = 0$

$f_x = \frac{1}{x} + 0$ $\qquad$ $f_y = 0 + \frac{1}{y}$

$f_x$ at $(1,1) = 1$ $\qquad$ $f_y$ at $(1,1) = 1$

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$= 0 + 1(x-1) + 1(y-1)$$
$$= x - 1 + y - 1$$
$$= \underline{\underline{x + y - 2}}$$

# Practical - 10

Aim: Directional Derivative of Given Vectors

Q1) Find the directional derivative of the following function at given point and in the direction of given vector.

1) $f(x,y) = x+2y-3 \qquad a=(1,-1) \qquad u = 3i-j$

Here, $u = 3i-j$ is not a unit vector

$|\bar{u}| = \sqrt{(3)^2+(-1)^2} = \sqrt{9+1} = \sqrt{10}$

Unit Vector along $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a) = f(1,-1) = (1) + 2(-1) - 3$

$= 1-2-3 = -4$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= \left(1+\frac{3}{\sqrt{10}}, -1-\frac{h}{\sqrt{10}}\right)$

$f(a+hu) = \left(1+\frac{3}{\sqrt{10}}, -1\frac{-h}{\sqrt{10}}\right) -3$

$= 1+\frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = -4+\frac{h}{\sqrt{10}}$

$$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu)-f(a)}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{y}+\cancel{h}/\sqrt{10}+\cancel{y}}{\cancel{h}}$$

$$D_u f(a) = \frac{1}{\sqrt{10}}$$

2] $f(x) = y^2 - 4x + 1$ $\quad a = (3,4)$ $\quad u = i+5j$

$\Rightarrow$ Here $u = i+5j$ is not a unit vector.

$$|\bar{u}| = \sqrt{(1)^2+(5)^2} = \sqrt{1+25} = \sqrt{26}$$

Unit vector along $u$ is $\frac{u}{|u|} = \frac{1}{\sqrt{26}}(1,5)$

$$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$$

$$f(a) = f(3,4) = (4)^2 - 4(3) + 1 = 5$$

$$f(a+hu) = f(3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$$

$$= f\left(3+\frac{h}{\sqrt{26}}, 4+\frac{5h}{\sqrt{26}}\right)$$

$$f_{x,y}(a+hu) = \left(4+\frac{5h}{\sqrt{26}}\right)^2 - 4\left(3+\frac{h}{\sqrt{26}}\right) + 1$$

$$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$$

$$= \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - \frac{4h}{\sqrt{20}} + 5$$

$$= \frac{25h^2}{26} + \frac{40h-4h}{\sqrt{20}} + 5$$

$$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5.$$

$D_u f(a) = \lim_{h \to 0} \dfrac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + \cancel{8} - \cancel{8}}{h}$

$= \dfrac{\cancel{h}\left(\frac{25h}{26} + \frac{36h}{\sqrt{26}}\right)}{\cancel{h}}$

$D_u f(0) = \dfrac{25h}{20} + \dfrac{36}{\sqrt{26}}$

5] $2x+3y$ $\quad a = (1,2)$ $\quad u = 3i + 4j$

$f(x,y) = x^y + y^x$ $\quad a(=(1,1)$

$f_x = y \cdot x^{y-1} + y^x \log y$

$f_y = x^y \log x + xy^{x-1}$

$Df(x,y) = (f_x, f_y)$

$= (yx^{y-1} + y^x \log y,\ x^y \log x + xy^{x-1})$

$f(1,1) = (1+0,\ 1+0)$

$= (1,1)$

$\Rightarrow$ Here $u = 3i + 4j$ is not a unit vector

$|\bar{u}| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$

Unit vector along ~~u is~~ $\dfrac{U}{|U|} = \dfrac{1}{5}(3,4)$

$= \cancel{\left(\frac{3}{5}, \frac{4}{5}\right)}$

$f(a) = f(3,2) = 2(1) + 3(2) = 8.$

$f(a+hu) = f(1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right)$

$= f\left(1 + \frac{3h}{5},\ 2 + \frac{h}{5}\right)$

$f(a+hu) = 2\left(1 + \frac{3h}{5}\right) + 3\left(2 + \frac{4h}{5}\right)$

$$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$$

$$= \frac{18h}{5} + 8.$$

$$D_u f(a) = \lim_{h \to 0} \frac{\frac{18h}{5} + \cancel{8} - \cancel{8}}{h}$$

$$= \frac{18}{5}.$$

Q. 2. Find gradient vector for the following function

ii] $f(x,y) = (\tan^{-1}x) \cdot y^2 \qquad a = (1,-1)$

$$f_x = \frac{1}{1+x^2} \cdot y^2$$

$$f_y = 2y \cdot \tan^{-1}x$$

$$\Delta f(x,y) = (f_x, f_y)$$

$$= \left(\frac{y^2}{1+x^2},\ 2y \tan^{-1}x\right)$$

$$f(1,-1) = \left(\frac{1}{2},\ \tan^{-1}(1)\ (-2)\right)$$

$$= \left(\frac{1}{2},\ \frac{\pi}{4}(-2)\right)$$

$$= \left(\frac{1}{2}, -\frac{\pi}{2}\right)$$

iii] $f(x,y,z) = xyz - e^{x+y+z} \qquad a = (1,-1,0)$

$$f_x = yz - e^{x+y+z}$$

$$f_y = xz - e^{x+y+z}$$

$$f_z = xy - e^{x+y+z}$$

$$\Delta f(x,y,z) = f_x, f_y, f_z$$

$$= yz - e^{x+y+z},\ xz - e^{x+y+z},\ xy - e^{x+y+z}$$

$$f(1,-1,0) = \Big((-1)(0) - e^{(1+(-1)+0)},\ (1)(0) - e^{1+(-1)+0},\ (1)(-1) - e^{1+(-1)+0}\Big)$$

$$= (0 - e^0,\ 0 - e^0,\ -1 - e^0)$$

$$= (-1, -1, -2)$$

Find the equation of tangent of normal to each of the following using curves at given points.

$x^2 \cos y + e^{xy} = 2$ at $(1,0)$

$f_x = \cos y \cdot 2x + e^{xy} \cdot y$

$f_y = x^2(-\sin y) + e^{xy} \cdot x$

$(x_0, y_0) = (1,0) = x_0 = 1 \quad y_0 = 0$

Eqn of tangent

$f_x(x - x_0) + f_y(y - y_0) = 0$

$f_x(x_0, y_0) = \cos 0 \cdot 2(1) + e^0 \cdot 0$

$= 1(2) + 0$

$= 2$

$f_y(x_0, y_0) = (1)^2(-\sin 0) + e^0 \cdot 1$

$= 0 + 1 \cdot 1$

$= 1$

$2(x-1) + 1(y-0) = 0$

$2x - 2 + y = 0$

$2x + y - 2 = 0$

It is the required equation of tangent. eqn of Normal,

$= ax + by + c = 0$

$\therefore \; bx + ay + d = 0$

$(1)\; 1 + 2(y) + d = 0$

$1 + 2y + d = 0$ at $(1,0)$

$1 + 2(0) + d = 0$

$d + 1 = 0$

$d = -1$

ii] $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2,2)$

$f_x = 2x + 0 - 2 + 0 + 0$

$= 2x - 2$

$f_y = 0 + 2y - 0 + 3 + 0$

$= 2y + 3.$

$(x_0, y_0) = (2, -2)$

$\therefore x = 2 \quad , \quad y_0 = -2$

$f_x(x_0, y_0) = 2(2) - 2 = 2$

$f_y(x_0, y_0) = 2(-2) + 3 = -1$

Eqn of tangent

$f_x(x - x_0) + f_y(y - y_0) = 0$

$2(x-2) + (-1(y+2)) = 0$

$2x - 2 - y - 2 = 0$

$2x - y - 4 = 0$

It is the required equation of tangent of normal

$ax + by + c = 0$

$bx + ay + d = 0$

$\therefore -1(x) + 2(y) + d = 0$

$-x + 2y + d = 0$ at $(2, -2)$

$-2 + 2(-2) + d = 0$

$-2 - 4 + d = 0$

$-6 + d = 0$

$d = 6$

Q.IV Find the eqn of tangent & normal line to each:

1] $x^2 - 2yz + 3y + xz = 7$ at $(2,1,0)$

$f_x = 2x - 0 + 0 + z$

$f_x = 2x + z$

$f_y = 0 - 2z + 3 + 0$
$= 2z + 3$

$f_z = 0 - 2y + 0 + x$
$= x - 2y.$

$(x_0, y_0, z_0) = (2, 1, 0)$
$\therefore x_0 = 2, \quad y_0 = 1, \quad z_0 = 0.$

$f_x(x_0, y_0, z_0) = 2(2) + 0 = 4$
$f_y(x_0, y_0, z_0) = 2(0) + 3 = 3.$
$f_z(x_0, y_0, z_0) = -2(1) + 2 = 0$

Equation of tangent,

$f_x(x_0 - x_0) + f_y(y_0 - y_0) + f_z(z_0 - z_0) = 0$
$4(x - 2) + 3(y - 1) + 0(z - 0) = 0$
$4x - 8 + 3y - 3 = 0$
$4x + 3y - 11 = 0$

This is required equation of tangent equation of normal at $(4, 3, -11)$

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$\therefore \frac{x - 2}{4} = \frac{y - 1}{3} = \frac{z + 11}{0}.$$

ii] $3xyz - x - y + z = -4$ at $(1, -1, 2)$

$3xyz - x - y + z + 4 = 0$

$f_x = 3yz - 1 - 0 + 0 + 0$
$= 3yz - 1$
$f_y = 3xz - 0 - 1 + 0 + 0$
$= 3xz - 1$
$f_z = 3xy - 0 - 0 + 1 + 0$
$3xy + 1$

$\therefore\ x_0 = 1, \quad y_0 = -1, \quad z_0 = 2$

$f_x(x_0, y_0, z_0) = 3(-1)(2) - 1 = -7$

$f_y(x_0, y_0, z_0) = 3(+1)(2) - 1 = 5$

$F_z(x_0, y_0, z_0) = 3(1)(-1) + 1 = -2$

Equation of tangent

$-7(x-1) + 5(y+1) - 2(z-2) = 0$

$-7x + 7 + 5y + 5 - 2z + 4 = 0$

$-7x + 5y - 2z + 16 = 0$

This is required equation of tantant.

Equation of Normal at $(-7, 5, -2)$

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$\therefore \frac{x-1}{-7} = \frac{y+1}{1} = \frac{z-2}{-2}$$

Q.V] Find the local maxima & minima for following:-

i] $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$

$f_x = 6x + 0 - 3y + 6 - 0$

$= 6x - 3y + 6$

$f_y = 0 + 2y - 3x + 0 - 4$

$= 2y - 3x - 4$

$f_x = 0$

$6x - 3y + 6 = 0$

$3(2x - y + 2) = 0$

$2x - y + 2 = 0$

$2x - y = -2 \quad \text{—— (1)}$

$f_y = 0$

$2y - 3x - 4 = 0$

$2y - 3x = 4 \quad \text{—— (ii)}$

Multiply eq. (i) with eq. (ii).

$4x - 2y = -4$

$2y - 3x = 4$

$x = 0$

Substitute Value of x in eqn (i).

$2(0) - y = -2$

$-y = -2$

$\therefore y = 2$

Critical point are (0, 2)

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$

Here $r > 0$

$= rt - s^2$

$= 6(2) - (-3)^2$

$= 12 - 9$

$= 3 > 0$

$\therefore$ f has maximum at (0, 2)

$3x^2 + y^2 - 3xy + 6x - 4y$ at (0, 2)

$3(0)^2 + 2^2 - 3(0)(2) + 6(0) - 4(2)$

$0 + 4 - 0 + 0 - 8 = -4$

$f(x, y) = 2x^4 + 3x^2y - y^2$

$f_x = 8x^3 + 6xy$

$f_y = 3x^2 - 2y$

$f_x = 0$

$8x^3 + 6xy = 0$

$4x^2+3y=0 \rightarrow$ (i)

$f_y=0$

$3x^2-2y=0$ —— (ii)

Multiply eq (i) with eq 3 and eq (2) with 4.

$$12x^{2}+9y=0$$
$$-12x^2-8y=0$$
$$\overline{\quad y=0 \quad}$$

Subititute value of $y$ in eqn (1).

$4x^2+3(0)=0$

$4x^2=0$

$x=0$

Critical point is $(0,0)$

$r = f_{xx} = 24x^2+6x$

$t = f_{yy} = 0-2=-2$

$s = F_{xy} = 6x-0 = 6x = 6(0)=0$

$r$ at $(0,0)$

$=24(0)+6(0)=0$

$\therefore r=0$

$rt-s^2 = 0(-2)-(s)^2$

$=0-0=0$

$r=0$ & $rt-s^2=0$

Nothing to say.

$f(x,y)$ at $(0,0)$

$2(0)^4+3(0)^2(0)-0$

$=0+0-0.$

$=\underline{\underline{0}}$