Adaptive Control of Time-Varying Parameter Systems With Asymptotic Tracking

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1 ABSTRACT

In this report, we studied and implemented the Journal Paper on Adaptive Control of Time-Varying Parameter Systems With Asymptotic Tracking. We documented the results and derived all the equations given in the report along with the necessary explanation. In the paper the authors developed a continuous adaptive controller is developed for nonlinear dynamical systems with linearly parameterizable uncertainty involving time-varying uncertain parameters. Through a unique stability analysis strategy, a new adaptive feedforward term is developed along with specialized feedback terms, to yield an asymptotic tracking error convergence result by compensating for the time-varying nature of the uncertain parameters. A Lyapunov based stability analysis is shown for Euler-Lagrange systems, which ensures asymptotic tracking error convergence and boundedness of the closed-loop signals. Additionally, the time-varying uncertain function approximation error is shown to converge to zero. A simulation example of a two-link manipulator is provided to demonstrate the asymptotic tracking result.

Index Terms-Adaptive control, Lyapunov methods, Time varying systems.

2 INTRODUCTION

In this section, the concepts that are needed to build a proper understanding of the problem statement are introduced. We will introduce a system, its stability and control, Lyapunov analysis and the concept of UUB for a system. We will then review the techniques already developed in the field, and the current state of art in the field of adaptive control and we will explain the problem statement and why it is still an open challenge.

A system is any machine/problem/case which we want to study and can represent it as a mathematical model to be able to decide on its nature. Any system can be classified as linear or non-linear depending on the system behaviour and how its output depends on the inputs. If the system satisfies the superposition theorem (additive and homogeneity property) without restrictions (for all time and all inputs) only then the system is called a linear system, else it is a non-linear system.

Mathematically, for a continuous time system, given two arbitrary inputs:
$x_1(t)$ and $x_2(t)$ as well as their respective zero state outputs:
$y_1(t) = H\{x_1(t)\}$
$y_2(t) = H\{x_2(t)\}$
then a linear system must satisfy:

$$\alpha y_1(t) + \beta y_2(t) = H\{ax_1(t) + bx_2(t)\}$$

for any scalar values α and β, all given inputs $x_1(t)$ and $x_2(t)$, and for all time t.

Linear system exhibits properties which are much simpler than the non-linear systems. But most of the systems in nature are non-linear and in this report as well we are analyze a non-linear system.

When a system is modelled by equations, the terms that define the system behaviours are called parameters. A parameter can be described as a configuration variable that is intrinsic to the model. Models can be parametric or non-parametric and our interest for this report lies in parametric models. In parametric system models there can be many terms that affect the behaviour of the system. Parametric modelling techniques find the parameters for a mathematical model describing a system or process. These techniques use known information about the system to determine the model. For example, if we look at the following system:

$$y = x\beta + \alpha$$

here α and β are parameters of the system and X is the coefficient of the parameter

A system can have two types of stability, internal and external. Internal stability (or Asymptotic stability) means the zero input response decays to zero as time approached infinity for all possible initial conditions. While external stability defines how the system behaves when faced with the external disturbances. Asymptotic stability is also a very common used terms in control system stability analysis. It denotes the response of the system when the input of the system is very large or the time of observing is very large. Basically whether the system output is bounded when the input is very large and time tends to infinity (very large). This also in a way denotes internal stability of the system i.e. whether the system is able to stabilize itself when left for an infinite amount of time. Figure 1 shows an ideal case of asymptotic stability.

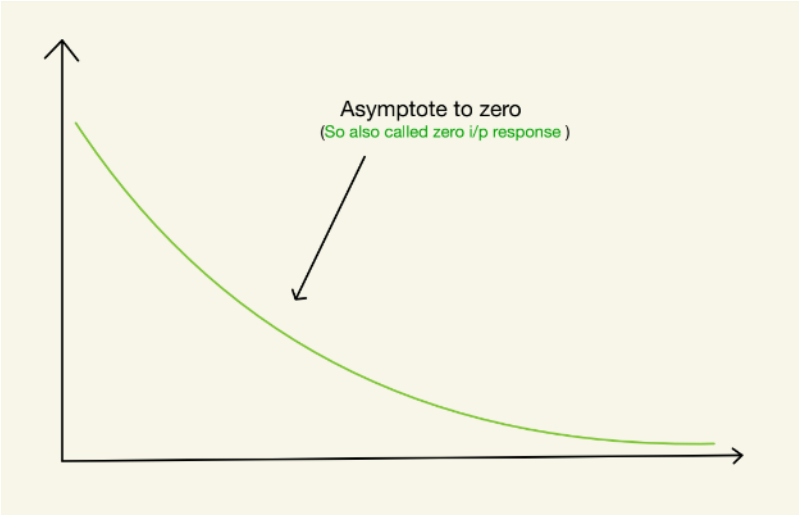


Figure 1: Asymptotic Behaviour

A given system can be stable or unstable. Usually when a system is designed for a specific task it can have certain degree of instability, and this is where the concept of control systems come in. Using the parameters of a system and by adding terms to the system model, we can control the states of the system in a way that it makes an unstable system stable. Two such terms are feedback and feed forward terms and are used separately or in sync with one other. Feedback terms introduce the input terms being adjusted based on the output terms so as to compensate for the error. To achieve this, the output of the system is fed back to the controller which compares the system desired output with the current output. The difference is then subtracted from the input signal to make the output same as the desired output.

The expectations from a stable system is that it should reject the permanent noise in the input/system and should address corrective actions for the disturbances which affect the output of the system. A feedback loop can achieve this when the system noise and the disturbances have a difference in bandwidth/frequency. For example, if we consider the graph in Figure 2 showing three different types of errors which are very common, (set point error, Disturbance , and noise), and their bandwidths, as shown in Figure 3, has a clear frequency gap and a frequency filter can easily reject the noise if used in a feedback loop.

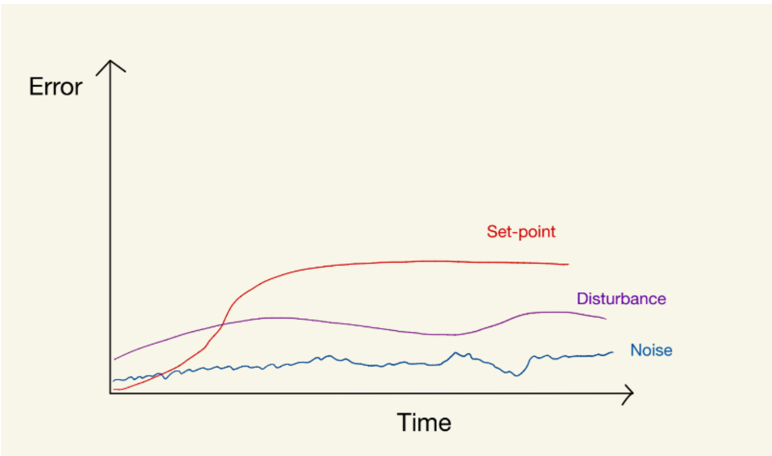


Figure 2: Relation between Set point Error, Disturbance and Noise

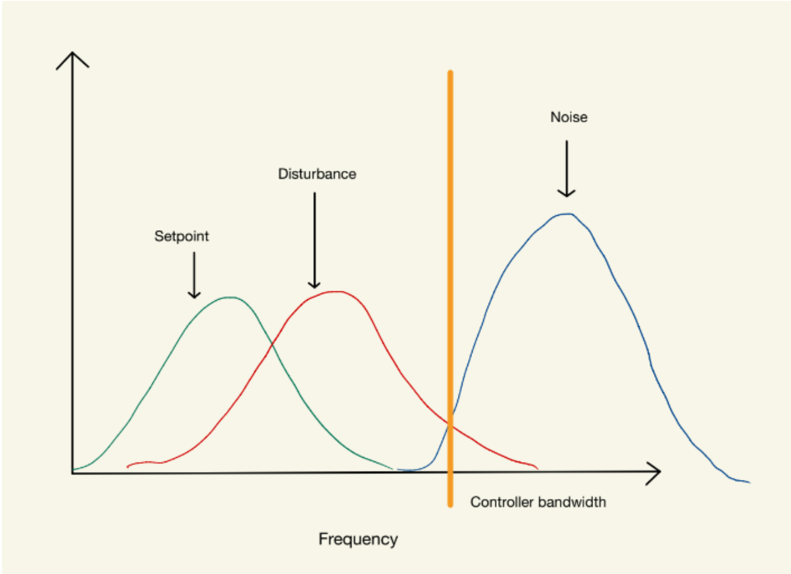


Figure 3: Bandwidth for noise with substantial frequency difference

But if the frequency of noise is similar to other disturbances, then feedback loop cannot clearly filter out the noise from the other disturbances. By reducing the controller bandwidth this can still be achieved, but it leads to a large reaction time for the controller i.e. the reaction time of the controller is affected which is undesirable. As seen in graph of Figure 4 the filtering out of noise is difficult because of the overlap.

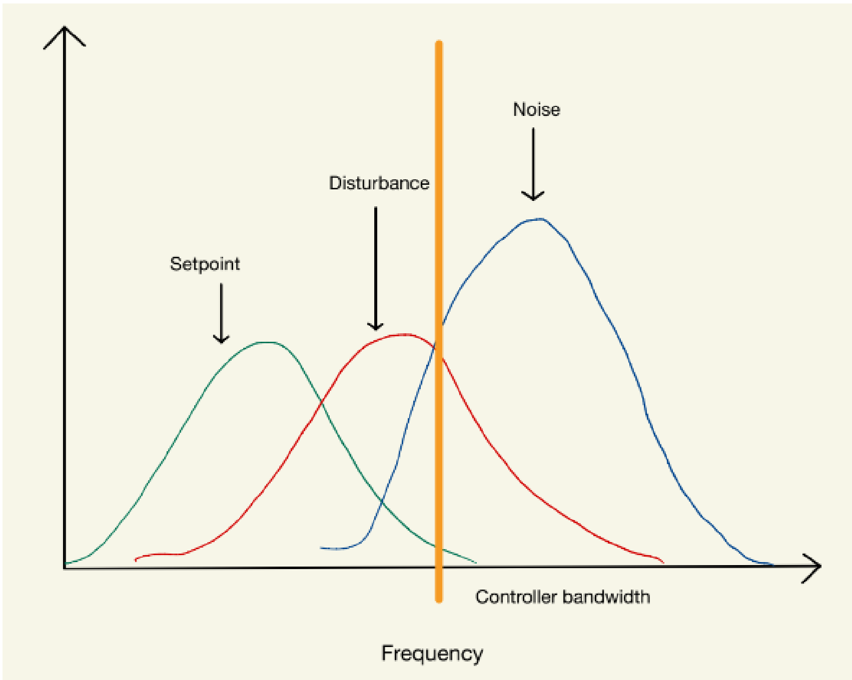


Figure 4: Bandwidth for noise with small frequency difference

To overcome this limitation of feedback control, we use feedforward control technique. This method tends to map the error in the system model itself so that the error can be eliminated in the input. This is done by compensating for the time varying nature of the uncertain parameters. The possible disturbances for the system are modelled mathematically and then these terms are introduced in the robot model. This leads to the error being rejected at the controller level before it is ever observed in the output. The problem with feedforward control is that making an exact mathematical model of the system is very difficult given the multiple parameters which might affect the system at any given point of time. Below model is a representation of feedback and feedforward in the same system.

Include control diagram here:

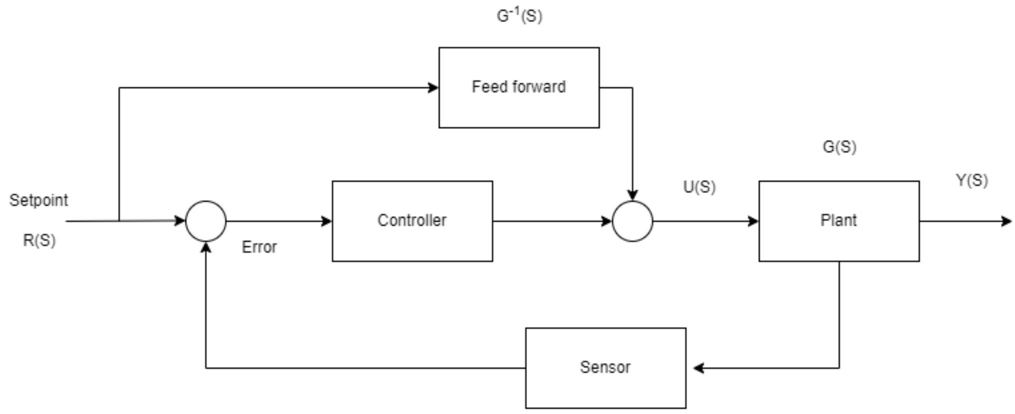


Figure 5: Block Diagram of Feed forward System

Now that we know about the control terms and how they help to stabilize the system, let us revisit the concept of stability from the boundedness perspective. Stability of a system is usually measured by checking if the output of the system is bounded within a certain range. This is called boundedness of a system, i.e. if the output of the system is bounded for a range of system inputs. Here we will introduce the term called UUB (Uniform Ultimate Boundedness). UUB basically refers to the bounds of the output of a system when the system is out of its transient state and reached its stable state.

To understand this better, let us look into the mathematical derivation and intuition:

Suppose we have a dynamical system

$$\dot{x} = f(x, y)$$

- We are assuming that mathematical model is valid in some region and the region is subset of R^n.

$$x \in D$$

- Whenever we apply Lyapunov Analysis that equilibrium point $x = 0$ lies in the region D. If the equilibrium point is unknown or changing with time, then is it possible to apply Lyapunov analysis to find bound of equation.
- Diagram of initial condition:

We try to find bound on $x(t)$. The band is for all $t >= t_0$
When plotting the response:

- The initial trajectory creates an overshoot.
- During the transient period the bound of solution goes on decreasing.
- We have 2 different responses in Linear Systems:
  - Transient Response
  - Steady State Response
- Once the transient response dies the steady state bound is significantly decreased for most of systems, same goes for $x(t)$.
- For a dynamical system where the equilibrium point is not known, the trajectory $x(t)$ is still bounded by a finite value.
- Lyapunov Analysis can be used to show boundedness of the solution of the state equation even when there is not equilibrium point at the origin.

Now let us consider a Scalar system given as:

$$\dot{x} = -x + \delta \sin(t); x(t_0) = a; a > \delta > b$$

when $x = \delta \sin(t)$; then $\dot{x} = 0$

- This equilibrium point is changing as time changes and lies between $[-\delta, \delta]$ i.e. $x^* \in [-\delta, \delta]$
- The general solution if the equation is given by :

$$\vec{X}(t) = e^{-A(t_1-t_0)}\vec{X}(t_0) + \int_{t_0}^{t_1} e^{A(t_1-\tau)}\vec{B}u(\tau)d\tau$$

- $\therefore$ Solution of given scalar system is:

$$X(t) = e^{-(t-t_0)}a + \delta \int_{t_0}^{t} e^{-(t-\tau)}\sin(\tau)d\tau$$

  where $A = -1, B = \delta, u = \sin(t)$
- $e^{-(t-t_0)}$ is always decreasing with time.
- a is some positive quantity.
- $\sin(t)$ is bounded from -1 to 1.
- The solution satisfies the bounds :

$$|x(t)| \leq e^{-(t-t_0)} + e^{(t-t_0)}a \int_{t_0}^{t} e^{t-\tau}d\tau + \delta[1 - e^{-(t-t_0)}]$$

  when $t = t_0, |x(t)| \leq a; \forall t \geq t_0$ since $a > \delta > 0$
- Above equation shows that the solution is bounded for all t>to uniformly in t_o, that is with a bound independent to t_0.
- As we know that the bound reduces significantly after transient period has elapsed. The above bound is the worst case bound. We can get the bound for the steady state period in further analysis.
- While this bound is valid for all $t \geq t0$, it becomes a consecutive estimation of the solution as time progresses, because it does not take into consideration the exponential decay term.
- We assume b such that $\delta < b < a$; it can be easily seen that

$$|x(t)| \leq b; \forall t \geq t_0 + ln(\frac{a-\delta}{b-\delta})$$

- The bound is independent of t_0 gives a better estimation of the solution after a transient period has passed.
- In this case, the solution is said to be uniformly ultimately bounded and b is said to be the ultimate bound.
- This means that we cannot reduce b below δ. Hence here we proved the concept of UUB and that it is the ultimate boundedness of any system.

As in the previous topics we saw multiple mentions of Lyapunov stability, we will discuss this in detail with proofs to have a better intuition.

Lyapunov Stability Theorem:
Let x=0 be an equilibrium point for $\dot{(x)} = f(x)$ and $D \in R_n$ be a domain containing x = 0
Let $V : D \to R$ be a continuously differentiable function such that V(0) = 0 and V(x) >0 in D-0. $\dot{V}(x) \leq 0$ in D. The derivative of V(x) along the trajectory of $\dot{x} = f(x)$, denoted by $\dot{V}$, is given by:

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = \frac{\partial V}{\partial x} f(x) \tag{1}$$

then, x=0 is stable. Moreover, if $\dot{V}(x) < 0$ in D-0, then x = 0 is asymptotically stable.

To derive the above result we need to define a few terms before proceeding.

Let us consider a nonlinear time-variant system $\dot{x} = f(x)$, where $f : R^n \to R^n$.
a point $x_e \in R^n$ is an equilibrium point of the system $f(x_e) = 0$
x_e is an equilibrium point $\Leftrightarrow$x(t) = x$_e$ is a trajectory
suppose x-e is an equilibrium point.

- system is globally asymptotically stable (G.A.S) if for every trajectory $x(t)$, we have $x(t) \to x_e$ as $t \to \infty$ (implies x_e is the unique equilibrium point)
- system is locally asymptotically stable (L.A.S) near or at x_e if there is an $R > 0$ such that $||x(0) - x_e|| \leq R \Rightarrow x(t) \to x$e as $t \to \infty$
- often we change coordinates so that $x_e = 0$ (i.e., we use$x$ $^{=x-x_0}$)
- a linear system $\dot{(x)} = Ax$ is G.A.S. (with x-e = 0) $\Leftrightarrow \mathbb{R}\lambda_i(A) < 0, i = 1, ..., n$
- a linear system $\dot{(x)} = Ax$ is G.A.S. (with x-e = 0) $\Leftrightarrow \mathbb{R}\lambda_i(A) < 0, i = 1, ..., n$ (so for linear systems, L.A.S. $\Leftrightarrow$ G.A.S.)
- there are many other variants on stability (e.g. stability, uniform stability, exponential stability,...)
- when f is nonlinear establishing any kind of stability is usually very difficult.

Positive Definite Functions:
a function$V : R^n \to R$is positive definite (PD) if

- $V(z) \geq 0 \forall z$
- $V(z) = 0$ iff $z = 0$
- all sublevel sets of V are bounded

last condition equivalent to$V(z) \to \infty$ as $z \to \infty$
example : $V(z) = z^T Pz$, with $P = P^T$, is Pd iff P>0

Lyapunov Theory:
Lyapunov theory is used to make conclusions about trajectories of a system $\dot{x} = f(x)$ (e.g., G.A.S.) without finding the trajectories (i.e., solving the differential equation)
A typical Lyapunov theorem has the form:

- if there exists a function $V : R^n \to R$ that sarisfies some coiditions on $V and \dot{V}$
- then, trajectoreis of system satisfy some property

if such a function V exists we call it a Lapunov function (that proves property holds for the trajectories)
Lyapunov function V can be thought of as generalized energy function for system.

Lyapunov boundedness theorem:
Suppose there is a function that satisfies

- all sublevel sets of V are bounded
- $\dot{V}(x) \leq 0$ for all z.

then, all trajectories are bounded, i.e., for each trajectory x there is an R such that $||x(t)|| \leq R$ for all $t \geq 0$
In this case, V is called a Lyapunov function (for the system) that proves the trajectories are bounded. To prove it, we note that for any trajectory x

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \leq V(x(0))$$

so the whole trajectory lies in $x|V(z) \leq V(x(0))$, which is bounded
also shows that every sublevel set $x|V(z) \leq a$ is invariant

Lyapunov global asymptotic stability theorem:
Suppose there is a function V such that

- V is a positive definite
- $\dot{V}(z) < 0$ for all $z \neq 0, \dot{V}(0) = 0$

then, every trajectory of $\dot{x} = f(x)$ converges to zero as $t \to \infty$ (i.e. the system is globally asymptotically stable)
interpretaion:

- V is positive definite generalized every function
- energy is always dissipated, except at 0

Proof
suppose trajectory x(t) does not converge to zero $V(x(t))$ is decreasing and non negative, so it converges to, say ϵ as $t \to$, Since x(T) does not converge to 0, we must have $\epsilon > 0$, so for all t, $e \leq V(x(t)) \leq V(x(0))$.

$C = z|\epsilon \leq V(z) \leq V(x(0))$ is closed and bounded, hence compact. so V (assumed continuous) attains its supremum on C, i.e., $sup_{z \in C}\dot{V} = -a < 0$. Since $\dot{V}(x(t)) \leq -a$ for all t, we have

$$V(x(t)) = V(x(0)) + \int_T^0 \dot{V}(x(t))\, dt \leq V(x(0)) - aT \tag{2}$$

which for T>V(x(0))/a implies V(x(0))<0, a contradiction.
So every trajectory x(t) converges to 0, i.e, $\dot{x} = f(x)$ is G.A.S.

A Lyapunov exponential stability theorem
suppose there is a function V and constant $\alpha > 0$ such that

- V is positive definite
- $\dot{V}(z) \leq= (z)$ for all z

then there is an M such hat every trajectory of $\dot{x} = f(x)$ satisfies $||x(t)|| \leq Me^{-\alpha t/2}||x(0)||$
(this is called global exponential stability (G.E.S))
ides:$\dot{V} \leq -\alpha V$gives guaranteed minimum dissipation rate, proportional to energy.

Finding Lyapunov functions

- There are many different types of Lyapunov theorems
- The key in all cases is to find a Lyapunov function and verify that it has the required properties
- there are several approaches to finding Lyapunov functions and verifying the properties.

One common approach:

- Decide form of Lyapunov function (e.g., quadratic), parametrized by some parameters (called a Lyapunov function candidate)
- Try to find values of parameters so that the required hypotheses hold

In this report to check the stability of the non-linear system we will use the Lyapunov stability criteria.

There are multiple control techniques by which a system can be stabilized. We will look into three such methods : Optimal Control, Robust Control and Adaptive control.

Optimal control seeks to optimize a performance index over a span of time. It assumes that the model is perfect and optimizes the provided functional. If the model is imperfect, then the optimal controller is not necessarily optimal, also it is optimal only for a specific cost function and not for all the cases. Common examples are LQR and LQG controllers.

Robust control assumes that the model provided is not perfect and there are certain parameters which needs to be taken in a range (i.e. the exact values of the parameters are not known). In short, robust control seeks to optimize the stability and quality of the controller given uncertainty in the plant model, feedback sensors, and actuators.

In adaptive control, the control systems keep on redesigning itself in response to the continuously varying uncertain parameters. This is one of the advance techniques and is difficult to design and implement. We will use this control technique for stabilizing our system in this report. In below few paragraphs we will develop the motivation for using Adaptive control and its relevance in today's industries.

Now that all the basic terms are introduced, we will discuss the problem statement for this paper, the developments done in this field, and why it is still an unsolved problem. It has been established that traditional gradient-based update laws can compensate for time invariant uncertain parameters yielding asymptotic convergence. Moreover, even for slowly varying uncertain parameters, such adaptive update laws result in UUB results (i.e. bounded in steady state) using a Lyapunov based analysis, under the assumption of bounded parameters and their time derivatives. Some adaptive control method talks about controlling the system with an external signal fed to the controller using an external probe which tracks the uncertain parameter, but no method developed yet focuses on stabilizing the system internally with the control law only where parameters are uncertain and time varying.

Some recent results discuss on improvement in the parameter estimation and tracking for systems with unknown time-varying parameters. Fast adaptation law [4] is one of the approaches, where the improvement in tracking and estimation is done using a matrix of time-varying learning rates under finite excitation condition. Another method rejects the effect of parametric variation by using a set-theoretic control architecture while also restricting the system error within the prescribed performance bound. While the approaches can potentially yield improved transient response, they yield UUB error systems.

For linear systems, there exists control techniques which have shown asymptotically vanishing errors for time- varying uncertain parameters. Results from papers [6] and [7] shows asymptotically vanishing time-varying parameter variations for asymptotic tracking of linear systems. For nonlinear systems involving periodic time varying uncertain parameters with known periodicity, repetitive/iterative learning based approaches such as [8] yield asymptotic tracking. However, it is challenging to extend these results to nonlinear systems with sustained and aperiodic uncertain parameters.

Results using robust adaptive control approaches such as in [9] yield asymptotic adaptive tracking for systems with time-varying uncertain parameters using an adaptive sliding mode-like design, and [10] use a continuous robust design. However as we analyzed earlier in feedback control the importance of noise frequency, these papers exploit high-gain or high-frequency feedback without any additional adaptive feedforward term that is specifically designed to target the uncertainty through adaptation.

Before moving forward, we will explain in brief about the Regressor matrix for a robot and its significance. The manipulator regressor, often denoted by $Y(q,\dot{q},\dot{q})$, is a key quantity in derivation as well as implementation of the many established adaptive motion and force control algorithms. This is because its availability enables one to express the dynamics of a robot arm as $Y\theta = \tau$ with $\theta \in R^r$ representing the manipulator parameters, thus a Lyapunov approach may lead to a *Linear Law* for updating the parameters.

In principle, the regressor is to formulate the manipulator dynamics as $H(q)\dot{q}+C(q,\dot{q})\dot{q}+G(q) = \tau$. This can be accomplished by using the Newton-Euler or Lagrange formulations [28]. Having done this, the second step of the approach defines a parameter vector θ and then works on every entry on the left-hand side of above equation to extract vecotr θ, leading above equation to the regressor formulation $Y\theta = \tau$. So we see computationally that this is an indirect approach that formulating the above equation plus a parameter extraction procedure. As the entries of θ are a general, spread over all the entries of $H(q), C(q,\dot{q})$, and G(q), the second step is also computationally complicated. We will see the use of regressor matrix when we develop the dynamic model in section 2 of the paper.

Results as in [11] yield asymptotic tracking using a method called congelation of variables. In this method, each unknown time-varying parameter is treated as a nominal constant unknown parameter perturbed by a time-varying perturbation, and the control input consists of an adaptive feedforward term to compensate for the nominal constant parameters, while a robust highgain term is designed to compensate the time-varying perturbation. While the congelation of variables based approach can compensate for fast-varying parameters, it requires the regression matrix to vanish with the state, which might be restrictive for a wide variety of applications.

More recent results as in [15] utilizes the dynamic regressor extension and mixing technique to yield finite-time parameter convergence for systems with unknown piecewise linearly time-varying parameters. Note that these results concern only adaptive parameter estimation, without developing an adaptive feedforward control term for closed-loop implementation.

In the field of fault-tolerant control design, system faults are typically modeled as unknown piecewise constant time-varying parameters such as in [16], for which, classical adaptive control techniques are used. In this report, we consider the more challenging problem of continuously time-varying parameters, which necessitates an alternative adaptive update law.

Let us consider the following example of a scalar dynamical system, and we will try to illustrate the technical challenges associated with developing an adaptive feedforward term for systems with time-varying parametric uncertainty,

$$\dot{x}(t) = a(t)x(t) + b(t)\cos(x(t)) + u(t) \tag{3}$$

with the controller $u(t) = -kx(t) - \hat{a}(t)x(t) - \hat{b}(t)\cos(x(t))$, where k is a positive constant gain; $a(t)$ and $b(t)$ are unknown time-varying parameters; $\hat{a}(t)$ and $\hat{b}(t)$ are the parameter estimates of $a(t)$ and $b(t)$, respectively; and the parameter estimation errors $\tilde{a}(t)$ and $\tilde{b}(t)$ are defined as $\tilde{a}(t) \triangleq a(t) - \hat{a}(t)$ and $\tilde{b}(t) \triangleq b(t) - \hat{b}(t)$, respectively. The traditional stability analysis approach for such problems is to consider the candidate Lyapunov function $V(x(t), \tilde{a}(t), \tilde{b}(t)) = \frac{1}{2}x^2(t) + \frac{1}{2\gamma_a}\tilde{a}^2(t) + \frac{1}{2\gamma_b}\tilde{b}^2(t)$, where γ_a and γ_b are positive constant gains. The given definitions and controller yield the following time-derivative of the candidate Lyapunov function: $\dot{V}(t) = -kx^2(t) + \tilde{a}(t)x^2(t) + \tilde{b}(t)x(t)\cos(x(t)) + \frac{\tilde{a}(t)}{\gamma_a}(\dot{a}(t) - \dot{\hat{a}}(t)) + \frac{\tilde{b}(t)}{\gamma_b}(\dot{b}(t) - \dot{\hat{b}}(t))$. For the constant parameter case, i.e., $\dot{a}(t) = \dot{b}(t) = 0$, the well-known adaptive update laws $\dot{\hat{a}}(t) = \gamma_a x^2(t)$ and $\dot{\hat{b}}(t) = \gamma_b x(t)\cos(x(t))$, respectively, will cancel $\tilde{a}(t)x^2(t)$ and $\tilde{b}(t)x(t)\cos(x(t))$ in $\dot{V}(t)$, leading to Lyapunov stability and asymptotic tracking. However, when the parameters are time-varying, it is unclear how to address $\dot{a}(t)$ and $\dot{b}(t)$ via a feedforward adaptive update law, such that $\dot{V}(t)$ becomes at least negative semidefinite. Alternatively to obtain a negative semidefinite derivative of the Lyapunov-like function (which is a contribution of this article), the typical approach to design adaptive controllers for the time-varying parameter case is to consider a robust modification of the update laws and assume some constant upper bounds on $|a(t)|, |b(t)|, |\dot{a}(t)|$, and $|\dot{b}(t)|$ to obtain a UUB result. For instance, consider a standard gradient update law with sigma-modification [3], $\dot{\hat{a}}(t) = \gamma_a x^2(t) - \gamma_a \sigma \hat{a}(t)$ and $\dot{\hat{b}}(t) = \gamma_b x(t)\cos(x(t)) - \gamma_b \sigma \hat{b}(t)$, which yields $\dot{V}(t) = -kx^2(t) - \sigma\tilde{a}^2(t) - \sigma\tilde{b}^2(t) + \tilde{a}(t)\left(\frac{\dot{a}(t)}{\gamma_a} + \sigma a(t)\right) + \tilde{b}(t)\left(\frac{\dot{b}(t)}{\gamma_b} + \sigma b(t)\right)$, implying a UUB result when the parameters $a(t)$ and $b(t)$, and their time-derivatives $\dot{a}(t)$ and $\dot{b}(t)$ are bounded. More modern approaches ([4]) provide additional modifications to yield UUB results with improved transient performance.

It would be desirable to have a sliding-mode like term based on $\tilde{a}(t)$ and $\tilde{b}(t)$ (i.e., $\mathrm{sgn}(\tilde{a})$ and $\mathrm{sgn}(\tilde{b})$ in the adaptation law) if only $\tilde{a}(t)$ and $\tilde{b}(t)$ were known. Another approach could be to use a pure robust controller, e.g., $u(t) = -kx(t) - \bar{a}x(t) - \bar{b}\,\mathrm{sgn}(x(t))$, where $\bar{a}$ and $\bar{b}$ are known constant upper-bounds on the norms of parameters $|a(t)|$ and $|b(t)|$, respectively. If the bounds $\bar{a}$ and $\bar{b}$ are unknown, an adaptation law could be designed to yield their adaptive estimates, i.e., $\hat{\bar{a}}$ and $\hat{\bar{b}}$. Either of these approaches would yield an asymptotic tracking result (cf., [9]), but as stated earlier, these approaches require a discontinuous pure sliding-mode term in the control input, and do not include an adaptive feedforward term to compensate for the uncertainty. The congelation of variable-based approach in [11] may help avoid some of the aforementioned challenges; however, it is not applicable for uncertain terms like $b(t)\cos(x(t))$, which do not vanish with the state.

The major challenge in achieving asymptotic tracking is that the time-derivative of the parameter acts like an unknown exogenous disturbance in the parameter estimation dynamics, which is difficult to cancel with an adaptive update law in a Lyapunov-based stability analysis. We address this technical challenge through new insights into the closed-loop error system development and stability analysis, coupled with a new adaptive update law design. Specifically, because of the challenges associated with including the uncertain parameter estimation error in the Lyapunov function, we omit such terms, and include a P-function based on [17], while also formulating the closed-loop error system so that they appear in the Lyapunov-based derivative in a manner that facilitates an adaptive update law. We address the unique challenge associated with incorporating the time-varying parameter estimation error in the analysis by formulating the update law so that it contains a signum function of the tracking error term multiplied by a desired regressor. The update law also involves a projection algorithm to ensure that the parameter estimates stay within a known bounded set. However, the projection algorithm introduces a potentially destabilizing term in the time-derivative of the candidate Lyapunov function, leading to an additional technical obstacle to obtain asymptotic tracking. This challenge is resolved by using an additional term in the control input, which compensates for terms that result from using a projection operator. The developed Lyapunov-based stability analysis yields semiglobal asymptotic tracking and boundedness of the closed-loop signals. Additionally, the time-varying uncertain function approximation error is shown to converge to zero. The dynamics of a two-link manipulator are used in a simulation to demonstrate the asymptotic tracking and function approximation error convergence result, and the tracking performance is compared with a robust e-modification update law [18] based controller.

From the above statements and results we can say that their exists a lot of research in the field of adaptive control for systems with varying parameters. But none of them offers a solution for the Adaptive control of nonlinear dynamical systems with time-varying uncertain parameters and thus it is an open and practically relevant problem. Through this report we explain how this problem can be approached, and we will do the simulation of our results for a 2 link manipulator.

3 DYNAMIC MODEL

3.1 Euler Lagrange Equations

In order to determine the Euler-Lagrange equations in a specific situation, one has to form the Lagrangian of the system, which is the difference between the kinetic energy and potential energy. These are set of differential equations that describe the time evolution of mechanical systems subjected to holonomic constraints when the constraint forces satisfy the principle of virtual work.
The equations can e derived using two methods one based on the method of virtual work and second using

Hamilton's principle of least action.

$$\mathcal{L} = \mathcal{K} - \mathcal{P} \tag{4}$$

where $\mathcal{K}$ is the kinetic energy and $\mathcal{P}$ is the potential energy.
$\mathcal{L}$ is called the Lagrangian of the system.
Note that
$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial \mathcal{L}}{\dot{y}}$ and $\frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial \mathcal{P}}{\partial y}$
And the below equation is called the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = f \tag{5}$$

To develop the Euler Lagrange equations we first write down the kinetic and potentials energies of the system in terms of set of so-called generalized co ordinates $(q_1, ...q_n)$, where n is the number of degrees of freedom of the system, and then computes the equation of motion of the n-DOF system according to

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k; k = 1, ..., n \tag{6}$$

where τ_k is the (generalized) force associated with q_k. The kinetic energy of a rigid object is the sum of two terms, the transnational kinetic energy obtained by the concentrating the entire mass of the object at the centre of the mass, and the rotational kinetic energy of the body about the centre of mass.
The kinetic energy of the rigid body is given as

$$K = \frac{1}{2}mv^T v + \frac{1}{2}\omega^T \mathcal{I}\omega \tag{7}$$

where m is the total mass of the object ,v and ω are the linear and angular velocity vectors, respectively, and $\mathcal{I}$ is a symmetric matrix called a s a inertia tensor.
ω is found from a Skew-symmetric matrix

$$S(\omega) = \dot{R}R^T \tag{8}$$

where R is the orientation transformation from the body-attached frame and the inertial frame.
The kinetic energy is a quadratic function of the vector $\dot{q}$ of the form

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{i,j} d_{i,j}(q)\dot{q}_i\dot{q}_j \tag{9}$$

where $d_{i,j}$ are the entries of the $n \times n$ inertia matrix D(q), which is a symmetric and positive definite for each $q \in \mathbb{R}^n$, and second the potential energy P=P(q) is independent of $\dot{q}$.

The Euler-Lagrange equations for such a system can be derived as follows.

$$\mathcal{L} = \mathcal{K} - \mathcal{P} = \frac{1}{2}\sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j - P(q) \tag{10}$$

The partial derivative of the Lagrangian with respect t the k^{th} joint velocity is given by

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_j d_{kj}\dot{q}_j \tag{11}$$

and therefore

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_j d_{kj}\ddot{q}_j + \sum_j \frac{d}{dt}d_{kj}\dot{q}_j = \sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j \tag{12}$$

Similarly he partial dervative of the lagrangian with respect to the k^{th} joint position is given by

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}j - \frac{\partial P}{\partial q_k} \tag{13}$$

Thus , for each k1,...,n, the Euler-Lagrange equations can be written as

$$\sum_j d_{kj}\ddot{q}_j + \sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k \tag{14}$$

by interchanging the order of the summation and taking the advantage of symmetry, one can show that

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{\partial d_{ki}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j \tag{15}$$

Hence

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j = \sum_{i,j}\frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} + \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j = \sum_{i,j} c_{ijk}\dot{q}_i\dot{q}_j \tag{16}$$

where we define

$$c_{ijk} := \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} + \frac{\partial d_{ij}}{\partial q_k}\right\} \tag{17}$$

The terms c_{ijk} are known as Christoffel symbols. Finally if we define

$$g_k = \frac{\partial P}{\partial q_k} \tag{18}$$

then we can write the Euler Lagrange equation as

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_k + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k, k = 1, \ldots, n \tag{19}$$

In the above equation there a re three types of terms. The first type involves the second derivative of the generalized co ordinates. The second type involves the quadratic term in the first derivatives of q, where the co efficient may depend on q. These latter terms are further classified into those involving a product of type $\dot{q}_i^2$ and those involving a product of the type $\dot{q}_i\dot{q}_j$ where $i \neq j$. Terms of type $\dot{q}_i^2$ are called **centrifugal**, while terms of the type $\dot{q}_i\dot{q}_j$ are called **Coriolis** terms. The third type arises from differentiating the potential energy. It is a common to write the equation in matrix form as

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \tag{20}$$

The subsequent development is based on the general uncertain nonlinear Euler-Lagrange (EL) dynamics given by [20, Sec. 2.2] where we see the development of the dynamic model as shown above and adding a component of Exogenous disturbance τ_d and a force of dissipation F.
We use changed notations for subsequent analysis to provide consistency throughout the solution where
$D(q) \to M(q)$ is the inertia matrix
$C(q, \dot{q}) \to V_m(q, \dot{q})$ is denoting the Coriolis and centrifugal forces
$g(q) \to G(q)$ is the gravity vector.
All the coefficients are functions of time as the system is time varying in nature.

$$M(q(t), t)\ddot{q}(t) + V_m(q(t), \dot{q}(t), t)\dot{q}(t) + G(q(t), t) + F(\dot{q}(t), t) + \tau_d(t) = \tau(t) \tag{21}$$

where $t \in [t_0, \infty)$ denotes time,
$t_0 \in \mathbb{R}_{\geq 0}$ denotes the initial time,
$q : [t_0, \infty) \to \mathbb{R}^n$ denotes a vector of generalized positions,
$M : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^{n\times n}$ denotes a generalized inertia matrix,
$V_m : \mathbb{R}^n \times \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^{n\times n}$ denotes the Coriolis and centrifugal forces matrix,
$G : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n$ denotes a generalized vector of potential forces,
$F : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n$ denotes a generalized vector of dissipation,
$\tau_d : [t_0, \infty) \to \mathbb{R}^n$ represents an exogenous disturbance acting on the system,
and $\tau : [t_0, \infty) \to \mathbb{R}^n$ represents a generalized control input vector [20, Ch. 2].
The subsequent development is based on the assumption that only $q(t)$ and $\dot{q}(t)$ are measurable. The following assumptions about the EL system are made in the subsequent development [20, Sec. 2.3].

Assumption 1: The inertia matrix satisfies $m_1\|\xi\|^2 \leq \xi^T M\ (q(t), t)\xi \leq \bar{m}(q)\|\xi\|^2 \forall \xi \in \mathbb{R}^n$, where $m_1 \in \mathbb{R}_{>0}$ is a known bounding constant, $\bar{m} : \mathbb{R}^n \to \mathbb{R}_{>0}$ is a known bounding function, and $\|\cdot\|$ denotes the Euclidean norm for a vector argument or the spectral norm for a matrix argument.

Assumption 2: The functions $M(q(t), t)$, $V_m(q(t), \dot{q}(t), t)$, $G(q(t), t)$, and $F(\dot{q}(t), t)$ are second order differentiable such that their second time derivatives are bounded if $q^{(i)} \in \mathcal{L}_\infty \forall i = 0, 1, 2, 3$, where $\mathcal{L}_\infty$ denotes the space of essentially bounded Lebesgue-measurable functions.
A measurable function is a function between the underlying sets of two measurable spaces that preserves the structure of the spaces: the preimage of any measurable set is measurable.

A function $f : \mathbb{R} \to \mathbb{R}$ is called a Lebesgue Measurable (or measurable with respect to the measurable space $(\mathbb{R}, \mathcal{L})$) if for each $a \in \mathbb{R}$

$$\{x \in \mathbb{R} : f(x) \leq a\} = f^{-1}((-\infty, a]) \in \mathcal{L} \tag{22}$$

Assumption 3: The equations of motion are defined in terms of certain parameters such as link masses, moments of inertia,etc., that must be determined for each particular system, in order, to simulate the equations or to tune controller. The complexity of the dynamics equations makes the determination of these parameters a difficult task.

The equations however are linear in these inertia parameters in the following sense. There exist an $n \times m$ function, $Y_p(q(t), \dot{q}(t), \ddot{q}(t), t)$ and an $m - dimensional$ vector $\theta_p(t)$ such that the Euler-lagrange equations can be linearly parameterized
The function $Y_p(q(t), \dot{q}(t), \ddot{q}(t), t)$ is called the regressor and $\theta_p(t) \in \mathbb{R}^m$ is the parameter vector. the dimension of the parameter space, that is the number of parameters needed to write the dynamics in this way is not unique. The dynamics in (21) can be linearly parameterized as

$$Y_p(q(t), \dot{q}(t), \ddot{q}(t), t)\theta_p(t) = M(q(t), t)\ddot{q}(t) + F(\dot{q}(t), t) + G(q(t), t) + V_m(q(t), \dot{q}(t), t)\dot{q}(t) \tag{23}$$

where $Y_p : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^{n \times m}$ is a known regression matrix and $\theta_p : [t_0, \infty) \to \mathbb{R}^m$ is a vector of time-varying unknown parameters.

In the above case, a linear parameterization is considered for simplicity. For systems that do not satisfy the linear-in-the-parameters assumption, the parameterization can yet be linearized according to [21, eq. (7)], where the linearization error can be upper bounded using [21, Lemma 1]. Subsequently, the adaptive design approach of this article is then applicable. The disturbance parameter vector $\tau_d(t)$ can be appended to the $\theta_p(t)$ vector, yielding an augmented parameter vector $\theta : [t_0, \infty) \to \mathbb{R}^{n+m}$ as

$$\theta(t) \triangleq \begin{bmatrix} \theta_p(t) \\ \tau_d(t) \end{bmatrix} \tag{24}$$

and the augmented regressor $Y : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^{n \times (n+m)}$ can be designed as

$$Y(q(t), \dot{q}(t), \ddot{q}(t), t) \triangleq [Y_p(q(t), \dot{q}(t), \ddot{q}(t), t)I_n] \tag{25}$$

Substituting the parameterization in (23)-(25) into (21) yields

$$M(q(t), t)\ddot{q}(t) + F(\dot{q}(t), t) + V_m(q(t), \dot{q}(t), t)\dot{q}(t) + G(q(t), t) + \tau_d(t) = Y(q(t), \dot{q}(t), \ddot{q}(t), t)\theta(t) \tag{26}$$

where $Y(q(t), \dot{q}(t), \ddot{q}(t), t)\theta(t) = \tau(t)$.

Assumption 4: The time-varying augmented parameter $\theta(t)$ and its time derivatives, i.e., $\dot{\theta}(t)$ and $\ddot{\theta}(t)$ are bounded by known constants, i.e., $\|\theta(t)\| \leq \zeta_0, \|\dot{\theta}(t)\| \leq \zeta_1$ and $\|\ddot{\theta}(t)\| \leq \zeta_2$, where $\zeta_0, \zeta_1, \zeta_2 \in \mathbb{R}_{>0}$ are known bounding constants.

For practical applications, it is often not difficult to develop sufficiently large bounds on uncertain parameters or their rate of change. For example, variation in a friction coefficient due to wear is difficult to model, but it is not difficult to obtain an upper bound on the friction coefficient. Similarly, it is possible to develop an upper bound on the inertia and drag coefficient parameters of an aircraft. We refer the reader to the result in [22, Sec. 4] for an example of an aerospace system with bounded time-varying parameters. For systems with unknown bounds, robust adaptive control methods such as [9, Sec. IV] may provide insight for a solution, but such an extension is beyond the scope of the contributions of this article.

4 CONTROL DESIGN

4.1 Control Objective

The objective is to design a controller such that the state tracks a smooth bounded reference trajectory, despite the time-varying nature of the uncertain parameters. The objective is quantified by defining the tracking error $e_1 : [t_0, \infty) \to \mathbb{R}^n$ as [3]

$$e_1 \triangleq q - q_d \tag{27}$$

where $q_d : [t_0, \infty) \to \mathbb{R}^n$ is a desired trajectory. Time-dependency is suppressed for the sake of brevity, except where explicit time-dependency adds clarity. To facilitate the subsequent analysis, filtered tracking errors e_2 and $r : [t_0, \infty) \to \mathbb{R}^n$ are defined as

$$e_2 \triangleq \dot{e}_1 + \alpha_1 e_1 \tag{28}$$

$$r \triangleq \dot{e}_2 + \alpha_2 e_2 \tag{29}$$

respectively, where $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$ are constant control gains.

Assumption 5: The desired trajectory $q_d(t)$ is bounded and smooth, such that $\|q_d(t)\| \leq \delta_0, \|\dot{q}_d(t)\| \leq \delta_1$, and $\|\ddot{q}_d(t)\| \leq \delta_2$, where δ_0, δ_1, and $\delta_2 \in \mathbb{R}_{>0}$ are known bounding constants.

Substituting (27)-(29) into (26) yields the open-loop error system

$$M(q,t)r = \tau + S(t) - Y_d\theta(t) \tag{30}$$

where

$S(t) \triangleq V_m(q_d, \dot{q}_d, t)\dot{q}_d - V_m(q, \dot{q}, t)\dot{q} + G(q_d, t) - G(q,t) + F(\dot{q}_d, t) - F(\dot{q}, t) + (M(q_d, t) - M(q,t))\ddot{q}_d + M(q,t)(\alpha_1(e_2 - \alpha_1 e_1) + \alpha_2 e_2)$ and $Y_d \triangleq Y(q_d, \dot{q}_d, \ddot{q}_d, t)$ denotes the desired regression matrix.

4.2 Control and Update Law Development

From the subsequent stability analysis, the continuous control input is designed as

$$\tau \triangleq Y_d\hat{\theta} - ke_2 + \mu \tag{31}$$

where $k \in \mathbb{R}_{>0}$ is a constant control gain, $\mu : [t_0, \infty) \to \mathbb{R}^n$ is a subsequently defined auxiliary control term, and $\hat{\theta} : [t_0, \infty) \to \mathbb{R}^{n+m}$ denotes the parameter estimate of $\theta(t)$. Substituting the control input in (31) into the open-loop error system in (30) yields the following closed-loop error system:

$$M(q,t)r = -Y_d\tilde{\theta}(t) + \mu - ke_2 + S(t) \tag{32}$$

where $\tilde{\theta} : [t_0, \infty) \to \mathbb{R}^{n+m}$ denotes the parameter estimation error, i.e., $\tilde{\theta}(t) \triangleq \theta(t) - \hat{\theta}(t)$. Taking the time-derivative of (32) yields

$$M(q,t)\dot{r} = -\dot{M}(q,t)r - \dot{Y}_d\tilde{\theta}(t) - Y_d\dot{\theta}(t) + Y_d\dot{\hat{\theta}} - k\dot{e}_2 + \dot{\mu} + \dot{S}(t) \tag{33}$$

The control variables $\dot{\hat{\theta}}(t)$ and $\dot{\mu}(t)$ now appear in the higher-order dynamics in (33), and these control variables are designed with the use of a continuous projection algorithm [23, Appendix E]. The projecton algorithm is used to prevent the problem of singularity The projection algorithm constrains $\hat{\theta}(t)$ to lie inside a bounded convex set $\mathcal{B} = \{\sigma \in \mathbb{R}^{(n+m)} \mid \|\sigma\| \leq \zeta_0\}$ by switching the adaptation law to its component tangential to the boundary of the set $\mathcal{B}$ when $\hat{\theta}(t)$ reaches the boundary. A continuously differentiable convex function $f : \mathbb{R}^{(n+m)} \to \mathbb{R}$ is used to describe the boundaries of the bounded convex set $\mathcal{B}$ such that $f(\sigma) < 0 \forall \|\sigma\| < \zeta_0$ and $f(\sigma) = 0 \forall \|\sigma\| = \zeta_0$. Based on the subsequent analysis, the continuous adaptation law is designed as

$$\dot{\hat{\theta}} \triangleq \operatorname{proj}(\Lambda_0) = \begin{cases} \Lambda_0, |\hat{\theta}\| < \zeta_0 \vee (\nabla f(\hat{\theta}))^T \Lambda_0 \leq 0 \\ \Lambda_1, |\hat{\theta}\| \geq \zeta_0 \wedge (\nabla f(\hat{\theta}))^T \Lambda_0 > 0 \end{cases} \tag{34}$$

where $\|\hat{\theta}(0)\| < \zeta_0$, and $\vee$ and $\wedge$ denote the logical "or," "and" operators, respectively; ∇ represents the gradient operator, i.e., $\nabla f(\hat{\theta}) = \left[\frac{\partial f}{\partial \phi_1} \cdots \frac{\partial f}{\partial \phi_{n+m}}\right]^T_{\phi = \hat{\theta}}$; and $\Lambda_0, \Lambda_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n+m}$ are designed as

$$\Lambda_0 \triangleq -\Gamma Y_d^T \left(Y_d \Gamma Y_d^T\right)^{-1} [k\alpha_2 e_2 + \beta \operatorname{sgn}(e_2)] \tag{35}$$

$$\Lambda_1 \triangleq \left(I_{m+n} - \frac{(\nabla f(\hat{\theta}))(\nabla f(\hat{\theta}))^T}{\|\nabla f(\hat{\theta})\|^2}\right)\Lambda_0 \tag{36}$$

respectively. The proof of the matrix $\left(Y_d \Gamma Y_d^T\right)^{-1}$ being invertible is shown in Appendix Lemma 1.

In (35), $\beta \in \mathbb{R}_{>0}$ is a constant control gain, and $\Gamma \in \mathbb{R}^{(n+m)\times(n+m)}$ is a constant, positive-definite, control gain matrix with a block diagonal structure, i.e., $\Gamma \triangleq \begin{bmatrix} \Gamma_1 & 0_{m\times n} \\ 0_{n\times m} & \Gamma_2 \end{bmatrix}$, with $\Gamma_1 \in \mathbb{R}^{m\times m}$, $\Gamma_2 \in \mathbb{R}^{n\times n}$ and $I_{m+n} \in \mathbb{R}^{(n+m)\times(n+m)}$ is an identity matrix. The continuous auxiliary term $\mu(t)$ used in the control input in (31) is designed as a generalized solution to

$$\dot{\mu} \triangleq Y_d(\Lambda_0 - \operatorname{proj}(\Lambda_0)) \tag{37}$$

where $\mu(t_0) = 0$. Substituting (34) and (37) into (33), the closed-loop error system can be obtained as

$$M(q,t)\dot{r} = -\dot{M}(q,t)r - \dot{Y}_d\tilde{\theta}(t) - Y_d\dot{\theta}(t) - \beta \operatorname{sgn}(e_2) - kr + \dot{S}(t) \tag{38}$$

for both cases, i.e., when $\|\hat{\theta}\| < \zeta_0 \vee (\nabla f(\hat{\theta}))^T \Lambda_0 \leq 0$ or $\|\hat{\theta}\| \geq \zeta_0 \wedge (\nabla f(\hat{\theta}))^T \Lambda_0 > 0$. Let

$$z \triangleq \begin{bmatrix} e_1^T & e_2^T & r^T \end{bmatrix}^T \in \mathbb{R}^{3n} \tag{39}$$

denote a composite error vector. To facilitate the subsequent analysis, (38) can be rewritten as

$$M(q,t)\dot{r} = -\frac{1}{2}\dot{M}(q,t)r + \tilde{N}(z,t) + N_B(\tilde{\theta},t) - \beta\,\mathrm{sgn}\,(e_2) - kr - e_2 \tag{40}$$

where $\tilde{N} : \mathbb{R}^{3n} \times [t_0, \infty) \to \mathbb{R}^n$ and $N_B : \mathbb{R}^{n+m} \times [t_0, \infty) \to \mathbb{R}^n$ are defined as $\tilde{N}(z,t) \triangleq -\frac{1}{2}\dot{M}(q,t)r + \dot{S}(t) + e_2$ and $N_B(\tilde{\theta},t) \triangleq -\dot{Y}_d\tilde{\theta} - Y_d\dot{\theta}(t)$, respectively.

Before moving forward we introduce the Mean Value Theorem (MVT) which states that if a function f is continuous on the closed interval [a,b] and differentiable on the open interval (a,b), then there exists a point c in the interval (a,b) such that f'(c) is equal to the function's average rate of change over [a,b]. The MVT can be used to develop the following upper bound on the term $\tilde{N}(z,t)$:

$$\|\tilde{N}(z,t)\| \leq \rho(\|z\|)\|z\| \tag{41}$$

where $\rho : \mathbb{R}^{3n} \to \mathbb{R}$ is a positive, globally invertible, and non decreasing function. By Assumptions 4 and 5 , Corollary 1 in the Appendix, and the bounding effect of projection algorithm on $\hat{\theta}(t)$, the term $N_B(\tilde{\theta},t)$ and its time-derivative $\dot{N}_B(\tilde{\theta},z,t)$ can be upper bounded using known constants $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}_{>0}$ as

$$\left\|N_B(\tilde{\theta},t)\right\| \leq \gamma_1, \quad \left\|\dot{N}_B(\tilde{\theta},z,t)\right\| \leq \gamma_2 + \gamma_3 \left\|e_2\right\| \tag{42}$$

respectively.

5 STABILITY ANALYSIS

In this section first we will introduce some terms which we will use in the stability analysis of our control law. First of them being the **Lasalla's Theorem**:

In continuation of the Lyapunov theorem stated in introduction, Lasalla's theorem allows us to conclude G.A.S. of a system with $\dot{V} \leq 0$, along with an observability type condition
we consider $\dot{x} = f(x)$
suppose there is a function $V : R^n \to R$ such that

- V is positive definite
- $\dot{V}(z) \leq 0$
- the only solution of $\dot{w} = f(w), \dot{V}(w) = 0$ is w9t) $= 0$ for all t

then, the system $\dot{x} = f(x)$ is G.A.S.

- last condition means no nonzero trajectory can hide in the "zero dissipation" set.
- unlike most other Lyapunov theorems, which extend to time-varying systems, Lasalle's theorem requires time-invariance.

Now we proceed with the analysis of our model. Let $y : [t_0, \infty) \to \mathbb{R}^{3n+1}$ be defined as

$$y \triangleq \begin{bmatrix} z^T & \sqrt{P} \end{bmatrix}^T \tag{43}$$

where $P : [t_0, \infty) \to \mathbb{R}$ is a generalized solution to the differential equation

$$\dot{\mathrm{P}} \triangleq -\mathrm{L} \tag{44}$$

In(24)

$$P\left(t_0\right) \triangleq \beta \left\|e_2\left(t_0\right)\right\|_1 - e_2\left(t_0\right)^T N_B\left(\tilde{\theta}\left(t_0\right), t_0\right) \tag{45}$$

and

$$L \triangleq r^T\left(N_B(\tilde{\theta},t) - \beta\,\mathrm{sgn}\left(e_2\right)\right) - \gamma_3\left\|e_2\right\|^2 \tag{46}$$

In (45), $\|\cdot\|_1$ denotes the 1 -norm. Provided that the gain condition

$$\beta > \gamma_1 + \frac{\gamma_2}{\alpha_2} \tag{47}$$

is satisfied, $P(t) \geq 0$, (this result is based on reference paper [18]) where the bounds γ_1, γ_2, and γ_3 are introduced in (42), and the control gain α_2 is introduced in (29). Therefore, it is valid to use $P(t)$ in the candidate Lyapunov function in the subsequent stability analysis. Furthermore, we introduce the auxiliary constant $\lambda_3 \triangleq \min\left\{\alpha_1 - \frac{1}{2}, \alpha_2 - \gamma_3 - \frac{1}{2}, \frac{k}{2}\right\}$, where the control gains α_1 and k are introduced in (28) and (35), respectively. The gains α_1, α_2, and k are selected based on the sufficient gain condition

$$\lambda_3 > \frac{\rho^2\left(\sqrt{\frac{\lambda_2(q(t_0))}{\lambda_1}}\left\|y\left(t_0\right)\right\|\right)}{2k} \tag{48}$$

with $\lambda_1 \triangleq \frac{1}{2}\min\{1, m_1\}$ and $\lambda_2(q) \triangleq \frac{1}{2}\max\{2, \bar{m}(q)\}$, where m_1 and $\bar{m}(q)$ are introduced in Assumption 1. From (28), (29), (40), (44), and (46), the differential equations describing the closed-loop system are

$$\dot{e}_1 = e_2 - \alpha_1 e_1 \tag{49}$$

$$\dot{e}_2 = r - \alpha_2 e_2 \tag{50}$$

$$\dot{r} = M^{-1}(q,t)\left(-\frac{1}{2}\dot{M}(q,t)r + \widetilde{N}(z,t) + N_B(\tilde{\theta},t)\right. \tag{51}$$

$$\left.-\beta \operatorname{sgn}\left(e_2\right) - kr - e_2\right) \dot{P} = -r^T\left(N_B(t) - \beta \operatorname{sgn}\left(e_2\right)\right) + \gamma_3\left\|e_2\right\|^2 \tag{52}$$

Theorem 1: Given the EL dynamic system in (21) along with Assumptions 1-5, for any arbitrary initial condition of the states $e_1\left(t_0\right), e_2\left(t_0\right)$, and $r\left(t_0\right)$, selecting $P\left(t_0\right), \alpha_1, \alpha_2, \beta$, and k according to (45), (47), and (48) ensures that $e_1, e_2, r, P \in \mathcal{L}_\infty$, and $\|e_1(t)\| \to 0$ as $t \to \infty$.

Proof: Let $\mathcal{D} \subset \mathbb{R}^{3n+1}$ be the open and connected set defined as

$$\mathcal{D} \triangleq \left\{\sigma \in \mathbb{R}^{3n+1} \mid \|\sigma\| < \rho^{-1}\left(\sqrt{2\lambda_3 k}\right)\right\} \tag{53}$$

and $V_L : \mathcal{D} \times [t_0, \infty) \to \mathbb{R}_{\geq 0}$ be a positive-definite candidate Lyapunov function defined as

$$V_L(y,t) \triangleq \frac{1}{2}r^T M(q,t) r + \frac{1}{2}e_2^T e_2 + \frac{1}{2}e_1^T e_1 + P \tag{54}$$

The candidate Lyapunov function in (54) satisfies

$$\lambda_1\|y\|^2 \leq V_L \leq \lambda_2(q)\|y\|^2 \tag{55}$$

where λ_1 and $\lambda_2(q)$ are defined after (48). Let $\psi \triangleq \left[\begin{array}{llll} e_1^T & e_2^T & r^T & P \end{array}\right]^T$ and $\dot{\psi} \in K[g](\psi,t)$ denote the Filippov differential inclusion corresponding to (49)-(52), where the operator $K[\cdot]$ is defined in [24, (2b)]. Filippov [l] has developed a solution concept for dif- ferential equations with a discontinuous right-hand side.

This concept leads to existence and uniqueness results and is described as follows.Note that $g : \mathbb{R}^{3n+1} \times [t_0, \infty) \to \mathbb{R}^{3n+1}$ is Lebesgue measurable and locally essentially bounded, since it is continuous except in the set with measure zero, $\left\{(\psi,t) \in \mathbb{R}^{3n+1} \times [0,\infty) \mid e_2 = 0\right\}$. Therefore, the existence of an absolutely continuous solution $t \mapsto \psi(t)$ to $\dot{\psi} \in K[g](\psi,t)$ is guaranteed by [25, Proposition 3] which states that the existing solution might have a finite escape time. We rule out this possibility by proving the boundedness of Filippov trajectories under the aforementioned sufficient conditions using Lyapunov-based stability theory. Therefore, dom $\psi = [t_0, \infty)$, i.e., the solution is complete. The solution may not be unique; however, the results are applicable to all the trajectories, since we consider a generalized Filippov solution in the analysis.

Let $\dot{\tilde{V}}_L(y,t) \triangleq \bigcap_{\xi \in \partial V_L(y,t)} \xi^T[K[g](\psi,t);1]$ as defined in [26, (13)], where $\partial V_L(y,t)$ denotes Clarke's generalized gradient [26, (7)]. Since $(y,t) \mapsto V_L(y,t)$ is continuously differentiable, Clarke's gradient is the same as the standard gradient, i.e., $\partial V_L = \{\nabla V_L\}$. Using [26, Th. 2.2], $t \mapsto \dot{V}_L(y(t),t)$ exists almost everywhere (Since $\psi = \left[\begin{array}{ll} z^T & P \end{array}\right]^T$ and $y = \left[\begin{array}{ll} z^T & \sqrt{P} \end{array}\right]^T$, $y(t)$ can be evaluated along a Filippov trajectory $\psi(t)$ by a transformation which involves taking the squareroot of $P(t)$, which is applicable since $P(t) \geq 0, \forall t \in [t_0, \infty)$) and $\dot{V}_L(y,t) \in \dot{\tilde{V}}_L(y,t)$ for almost all time (a.a.t.). Evaluating $\dot{\tilde{V}}_L(y,t)$ and (29)-(32) yields

$$\dot{\tilde{V}}_L \overset{\text{a.a.t.}}{\leq} r^T\left(-\frac{1}{2}\dot{M}(q,t)r + \widetilde{N}(z,t) + N_B(\tilde{\theta},t)\right).$$

$$-\beta K[\operatorname{sgn}]\left(e_2\right) - kr - e_2) + e_2^T\left(r - \alpha_2 e_2\right) + e_1^T\left(e_2 - \alpha_1 e_1\right) - r^T\left(N_B(t) - \beta K[\operatorname{sgn}]\left(e_2\right)\right) + \gamma_3\left\|e_2\right\|^2 + \frac{1}{2}r^T\dot{M}(q,t)r \tag{56}$$

Using (41) and applying Young's inequality on $e_1^T e_2$ in (56), $\dot{V}_L$ can be upper bounded as

$$\dot{V}_L \overset{\text{a.a.t.}}{\leq} \rho(\|z\|)\|z\|\|r\| - k\|r\|^2 - \left(\alpha_2 - \gamma_3 - \frac{1}{2}\right)\|e_2\|^2 \quad - \left(\alpha_1 - \frac{1}{2}\right)\|e_1\|^2$$

To explain the above upper bound, the set of times
$T \triangleq \left\{t \in [t_0, \infty) : r(t)^T \beta\, \mathrm{SGN}\,(e_2(t)) - \ r(t)^T \beta\, \mathrm{SGN}\,(e_2(t)) \neq \{0\}\right\} \subset \mathbb{R}_{\geq 0}$
is equal to the set of times $\{t : e_2(t) = \ 0 \wedge r(t) \neq 0\}$. Using $r = \dot{e}_2 + \alpha_2 e_2$, this set can also be represented by $\{t : e_2(t) = 0 \wedge \dot{e}_2(t) \neq 0\}$. Since e_2 is continuously differentiable because the right-hand side of (30) is continuous, [27, Lemma 2] can be used to show that the set of time instances $\{t : e_2(t) = 0 \wedge \dot{e}_2(t) \neq 0\}$ is isolated, and hence, measure zero; hence, T is measure zero. Therefore, $\dot{\tilde{V}}_L = \left\{\dot{V}_L\right\}$ a.e. in time, and an upper bound on $\dot{V}_L$ can be obtained a.e. in time, using the right-hand side of (36).

Now using Young's Inequality on $\rho(\|z\|)\|z\|\|r\|$ yields $\rho(\|z\|)\|z\|\|r\| \leq \frac{\rho^2(\|z\|)\|z\|^2}{2k} + \frac{1}{2}k\|r\|^2$. Therefore,

$$\dot{\tilde{V}}_L \overset{\text{a.a.t.}}{\leq} \frac{\rho^2(\|z\|)\|z\|^2}{2k} - \frac{k}{2}\|r\|^2 - \left(\alpha_2 - \gamma_3 - \frac{1}{2}\right)\|e_2\|^2 - \left(\alpha_1 - \frac{1}{2}\right)\|e_1\|^2 \leq -\left(\lambda_3 - \frac{\rho^2(\|z\|)}{2k}\right)\|z\|^2.$$

(57)

The expression in (57) can be rewritten as

$$\dot{V}_L \overset{\text{a.a.t.}}{\leq} -W(y) = -c\|z\|^2, \forall y \in \mathcal{D} \tag{58}$$

with some constant $c \in \mathbb{R}_{>0}$, where $W : \mathbb{R}^{3n+1} \to \mathbb{R}$ is a continuous positive semidefinite function.

Whenever $y \in \mathcal{D}, \|y(t)\| < \rho^{-1}\left(\sqrt{2\lambda_3 k}\right)$ by definition of $\mathcal{D}$, which is sufficient to infer $\|z(t)\| < \rho^{-1}\left(\sqrt{2\lambda_3 k}\right)$ using (43). Therefore, if $y(t) \in \mathcal{D}$, $\lambda_3 > \frac{\rho^2(\|z\|)}{2k}$, which implies from (57) that there exists $c \in \mathbb{R}_{>0}$ which satisfies (57), and larger values of λ_3 expand the size of $\mathcal{D}$. Since V_L is nonincreasing, which implies $\|y(t)\| \leq \sqrt{\frac{V_L(t)}{\lambda_1}} \leq \sqrt{\frac{V_L(t_0)}{\lambda_1}}$, it is sufficient to show that $\sqrt{\frac{V_L(t_0)}{\lambda_1}} < \rho^{-1}\left(\sqrt{2\lambda_3 k}\right)$, to obtain $y(t) \in \mathcal{D}$. Since $V_L(t_0) \leq \lambda_2(q(t_0))\|y(t_0)\|^2$, the result $\sqrt{\frac{V_L(t_0)}{\lambda_1}} < \rho^{-1}\left(\sqrt{2\lambda_3 k}\right)$ can be sufficiently obtained from $\sqrt{\frac{\lambda_2(q(t_0))}{\lambda_1}}\|y(t_0)\| < \rho^{-1}\left(\sqrt{2\lambda_3 k}\right)$. Therefore, $\|y(t_0)\| < \sqrt{\frac{\lambda_1}{\lambda_2(q(t_0))}}\rho^{-1}\left(\sqrt{2\lambda_3 k}\right)$, which implies that $\mathcal{S} \triangleq \left\{\sigma \in \mathcal{D} \mid \|\sigma\| < \sqrt{\frac{\lambda_1}{\lambda_2(q(t_0))}}\rho^{-1}\left(\sqrt{2\lambda_3 k}\right)\right\}$ is the region where $y(t_0)$ should lie to guarantee that $y(t) \in \mathcal{D}$ for all $t \in [t_0, \infty)$.

proof of above line: Though the sets $\mathcal{S}$ and $\mathcal{D}$ are defined to include $y(t)$ instead of $\psi(t)$, one can easily construct the bounded sets $\mathcal{S}_\psi$ and $\mathcal{D}_\psi$ such that $y(t_0) \in \mathcal{S}$ and $y(t) \in \mathcal{D}$ imply $\psi(t_0) \in \mathcal{S}_\psi$ and $\psi(t) \in \mathcal{D}_\psi$, respectively, to conclude the uniform boundedness of all Filippov trajectories $\psi(t)$ initializing in the set $\mathcal{S}_\psi$. to be satisfied according to the initial condition, and the region of attraction can be made arbitrarily large to include any initial condition by increasing the gains α_1, α_2, and k accordingly; therefore, the result is semiglobal.

Now using (53), (55), and (58), since g is Lebesgue measurable and essentially locally bounded and uniformly in time, the extension of the LaSalle-Yoshizawa corollary in [27, Corollary 1] can be invoked to show that $e_1, e_2, r, P \in \mathcal{L}_\infty$, and $\|z(t)\| \to 0$ as $t \to \infty$. Therefore, using the definition of z in (19), $\|e_1(t)\| \to 0$ as $t \to \infty$. The gain condition in (28) needs

The parameter estimate $\hat{\theta} \in \mathcal{L}_\infty$ due to the projection operation, which implies $\tilde{\theta}(t) = \theta(t) - \hat{\theta}(t)$ is bounded, because the parameter $\theta \in \mathcal{L}_\infty$ by Assumption 4. Since $e_1, e_2, r \in \mathcal{L}_\infty$, and because $q_d, \dot{q}_d, \ddot{q}_d \in \mathcal{L}_\infty$ by Assumption 5, using (27)-(29) implies that $q, \dot{q}, \ddot{q} \in \mathcal{L}_\infty$. Furthermore, the regression matrix $Y_d \in \mathcal{L}_\infty$ by Assumption 5 , because Y is locally bounded due to Properties 2 and 3 . Therefore, by Corollary 1 in the Appendix, $\dot{\hat{\theta}} \in \mathcal{L}_\infty$.

The expression in (32) indicates that $\underset{\tilde{U}}{\mu\mathcal{L}_\infty}$, because among the remaining terms in (32), $M(q)r$ and $Y_d\tilde{\theta}$ comprise bounded terms because M is locally bounded, and $S \in \mathcal{L}_\infty$ because its definition comprises terms that are locally bounded functions of the bounded errors and states due to Assumption 2. From the expression in (31), since $\hat{\theta}, Y_d, \mu \in \mathcal{L}_\infty, \tau \in \mathcal{L}_\infty$. Moreover, differentiating the right-hand side in (31) yields terms that are bounded, which implies $\dot{\tau} \in \mathcal{L}_\infty$; therefore, τ is continuous. Hence, all the closed-loop signals are bounded.

6 SIMULATION EXAMPLE

To demonstrate the efficacy of the developed method, a simulation example of a horizontal two-link manipulator system is provided, and the results are compared with an e-modification (e-mod)-based controller [18].

A e-mod based controller is derived from the method of σ-modification, in which an additional term of the form $-\sigma\theta$ is introduced in the adaptive law for adjusting the parameter vector θ. In the e-mod based controller the constant σ is replaced by a term proportional to $|e_1|$ where e_1,is the output error. This modification-which is referred as the $e_1 - modification$ is shown to improve the performance of the system in all respects while retaining the advantage disturbances, without requiring additional information regarding the plant or the disturbance.

The dynamics of the manipulator system can be represented in the form of (21), with
$M(q,t) = \begin{bmatrix} p_1(t) + 2p_3(t)c_2 & p_2(t) + p_3(t)c_2 \\ p_2(t) + p_3(t)c_2 & p_2(t) \end{bmatrix}, V_m(q,\dot{q},t) = \begin{bmatrix} -p_3(t)s_2\dot{q}_2 & -p_3(t)s_2\left(\dot{q}_1 + \dot{q}_2\right) \\ p_3(t)s_2\dot{q}_1 & 0 \end{bmatrix}$,
$F(\dot{q},t) = \begin{bmatrix} F_{d1}(t)\dot{q}_1 \\ F_{d2}(t)\dot{q}_2 \end{bmatrix}$, and $\tau_d(t) = [\tau_{d1}(t)\tau_{d2}(t)]^T$,
where $c_2 \triangleq \cos(q_2), s_2 \triangleq \sin(q_2)$, and $p_1, p_2, p_3, F_{d1}, F_{d2}, \tau_{d1}, \tau_{d2} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, and the gravity term $G(q,t)$ is ignored for a horizontal manipulator. The augmented time-varying parameter vector for the manipulator system is given by $\theta(t) = [p_1(t)p_2(t)p_3(t)F_{d1}(t)F_{d2}(t)\tau_{d1}(t)\tau_{d2}(t)]^T$. The control objective is to track a given reference trajectory $q_d(t) = [\cos(0.5t)2\cos(t)]^T$. The time-varying parameters used in the simulation are $p_1(t) = 3.473 + 0.5\sin(3t)$, $p_2(t) = 0.196 + 0.2\exp(-\sin(t))$, $p_3(t) = 0.242 + 0.1\cos(10t)$, $F_{d1}(t) = 5.3 + 2\exp(-0.1t)$, $F_{d2}(t) = 1.1 + \cos(5t)$, and the disturbance terms $\tau_{d1}(t) = 0.5\cos(0.5t)$ and $\tau_{d2}(t) = \sin(t)$. The initial conditions used in the simulation are $q(0) = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$, $\dot{q}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $\hat{\theta}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$

The control gains for each method are obtained using a **Monte-Carlo method**; Monte Carlo methods are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. The underlying concept is to use randomness to solve problems that might be deterministic in principle.
An appropriate range is qualitatively determined for each gain, and 10000 iterations are subsequently run with a uniform random gain sampling within those ranges in an attempt to minimize

$$J = \int_0^{10} \left(a\left\|e_1(t)\right\|^2 + b\|\tau(t)\|^2\right) dt \tag{59}$$

with $a = 1$ and $b = 0.01$. The gains that minimized (39) for the developed method are $K = 18.1502, \alpha_1 = 0.8982, \alpha_2 = 1.0552, \beta = 36.2946$, and $\Gamma = I_2$. For the projection algorithm, $\zeta_0 = 5000$ and the corresponding function $f(\hat{\theta}) = \|\hat{\theta}\|^2 - \zeta_0^2$. For the e-mod update law, i.e., $\dot{\hat{\theta}} = \Gamma_e Y_d^T r - \sigma\left\|e_1\right\|\hat{\theta}$ and the corresponding controller $\tau = Y_d\hat{\theta} - k_e r$, the gains are $\Gamma_e = 12.5, k_e = 9.7877$, and $\sigma = 9.7319$.

6.1 Simulink Model

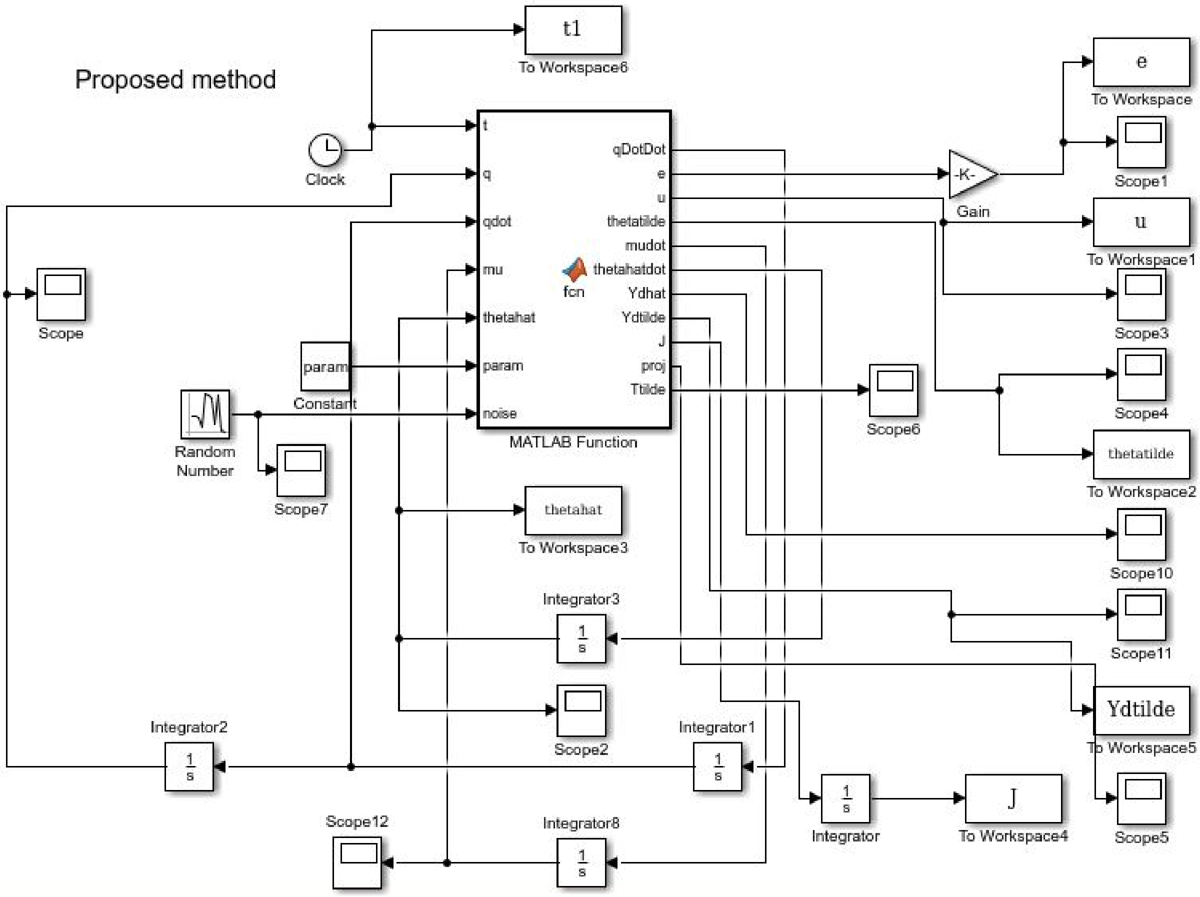


Figure 6: Simulink Model for proposed method

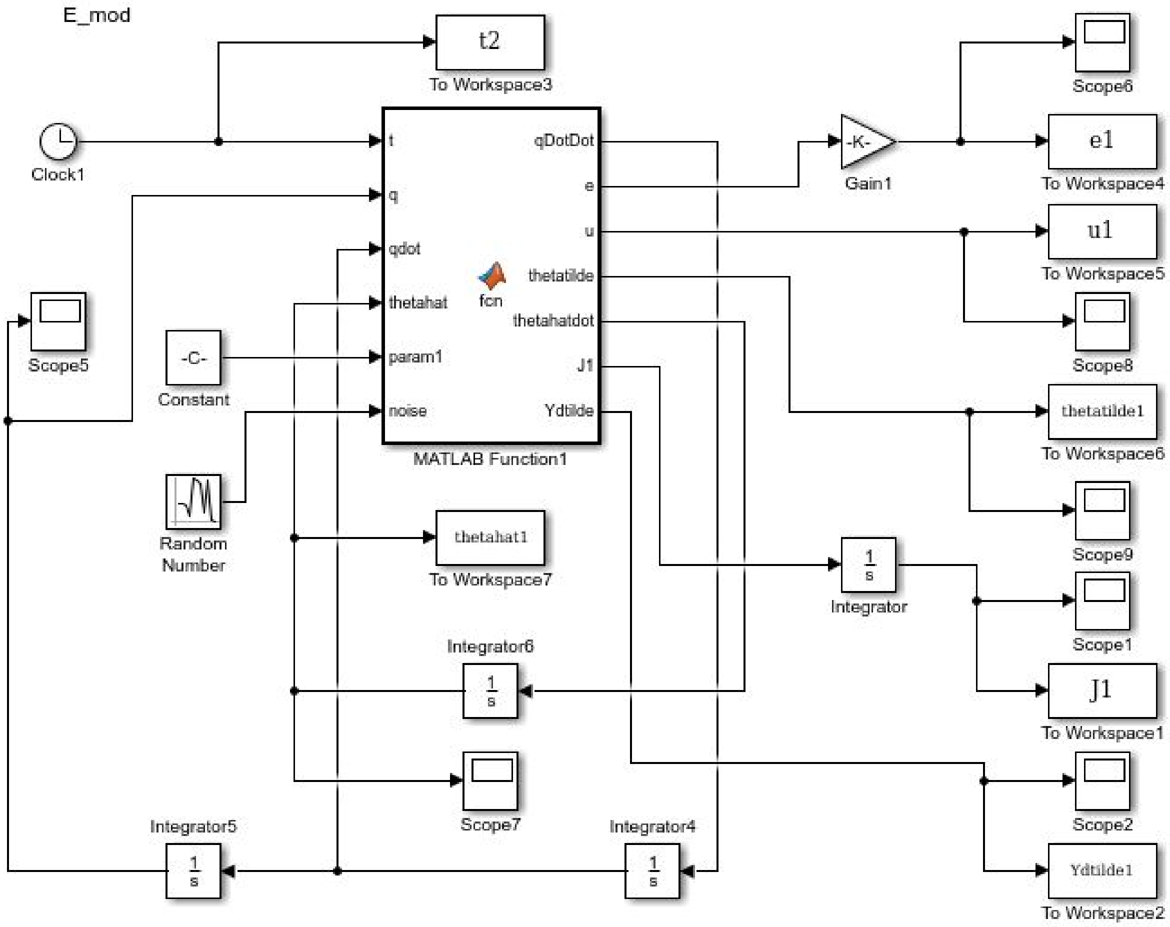


Figure 7: Simulink Model for e-mod method

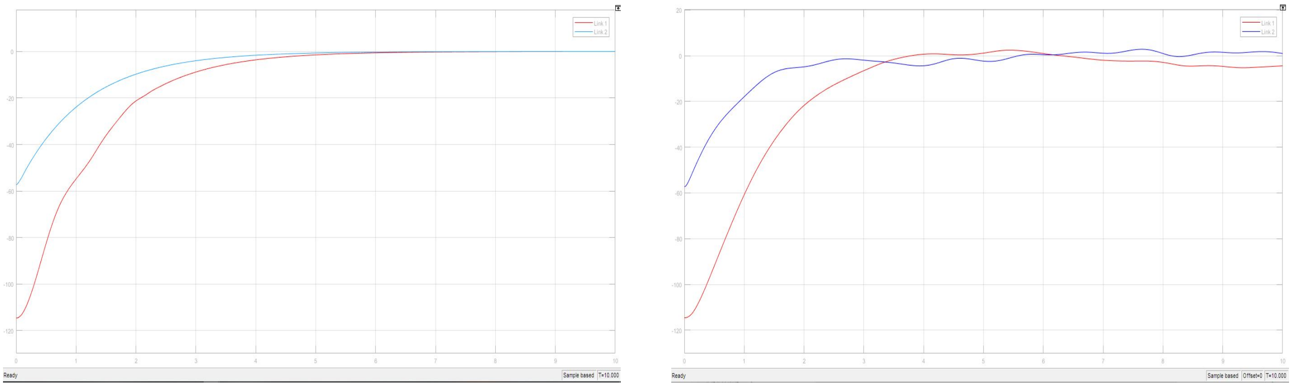
TABLE I
CONTROLLER PERFORMANCE COMPARISON

Method	$\|e_{\mathrm{rms}}\|$	$\|e_{\mathrm{rms,ss}}\|$	$e_{\max,\mathrm{ss}}$	$\|\tilde{Y}_{\mathrm{rms}}\|$	$\|\tau_{\mathrm{rms}}\|$
(31)	29.1340	0.5515	1.4666	1.3654	3.2726
e-mod	52.4838	3.5624	5.2500	6.1292	7.4722

Table I provides a quantitative comparison of the controllers, where e_{rms} is the root-mean-square (rms) of e_1 (in deg) taken over the time interval $[0,10], e_{\mathrm{rms,ss}}$ is the rms of e_1 over the time interval $[5,10]$ (i.e., after reaching the steady state), $e_{\max,\mathrm{ss}}$ is the maximum absolute value of the components of e_1 over the time interval $[5,10], \tilde{Y}_{\mathrm{rms}}$ denotes the rms function estimation error (in Nm) over the interval $[0,10]$, and τ_{rms} denotes the rms simulated torque (in Nm) over the time interval $[0,10]$. The developed method provides a significantly improved tracking and function estimation performance with less rms control effort, upon comparison with e-mod.

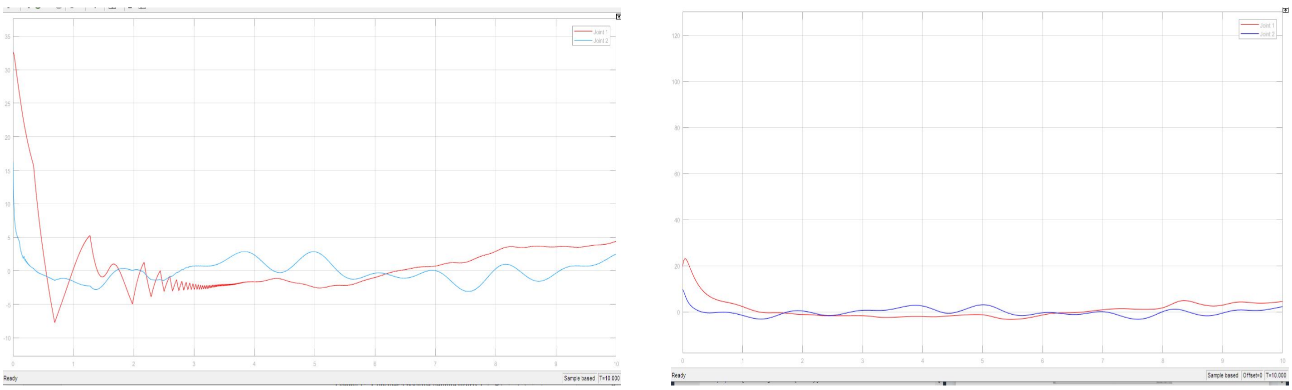
Figure 8-11 demonstrates the asymptotic convergence of the tracking error and the function estimation error $\left(Y\theta - Y_d\hat{\theta}\right)^{12}$ to zero with the developed method in the simulation, as opposed to the UUB tracking with the e-mod scheme.

In the above equation of function estimation error tracking, from an applied perspective, if the upper bound used for projection algorithm, i.e., ζ_0 is selected to be sufficiently high such that the parameter estimates never reach the boundary of the set $\mathcal{B}$, then $\operatorname{proj}\left(\Lambda_0(t)\right) = \Lambda_0(t), \forall t \in [t_0,\infty)$, implying $\mu(t) = 0, \forall t \in [t_0,\infty)$. From (26), $Y\theta = \tau$, and $\tau - Y_d\hat{\theta} = \mu - ke_2$ using (31), therefore if $\mu(t) = 0 \forall t \in [t_0,\infty)$, then the function approximation error $Y\theta - Y_d\hat{\theta} = \mu - ke_2 = -ke_2 \to 0$ as $t \to \infty$. In case the parameter estimates reach the boundary of $\mathcal{B}, Y\theta - Y_d\hat{\theta}$ may not converge to zero, yet it is guaranteed to be bounded using the stability analysis since μ is bounded. Fig. 11 demonstrates the tracking error performance in the presence of additive white Gaussian (AWG) noise with standard deviations of 2deg and 2deg/s in the q and $\dot{q}$ measurements, respectively. The rms steady state tracking error norms in the presence of measurement noise with the developed method and e-mod are 2.9427 and 4.5891, respectively.



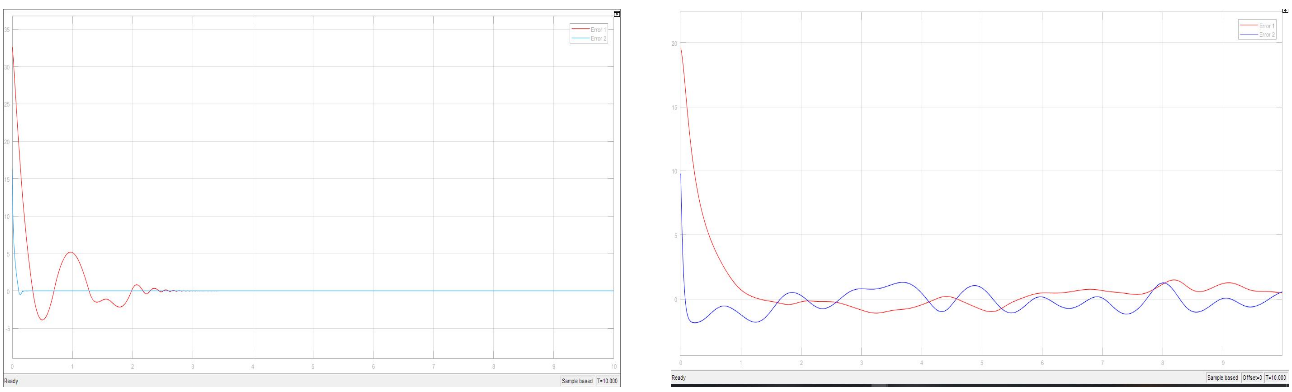
(a) Tracking Error Proposed Method (b) Tracking Error e-mod

Figure 8: Tracking Error Comparison (deg vs time)



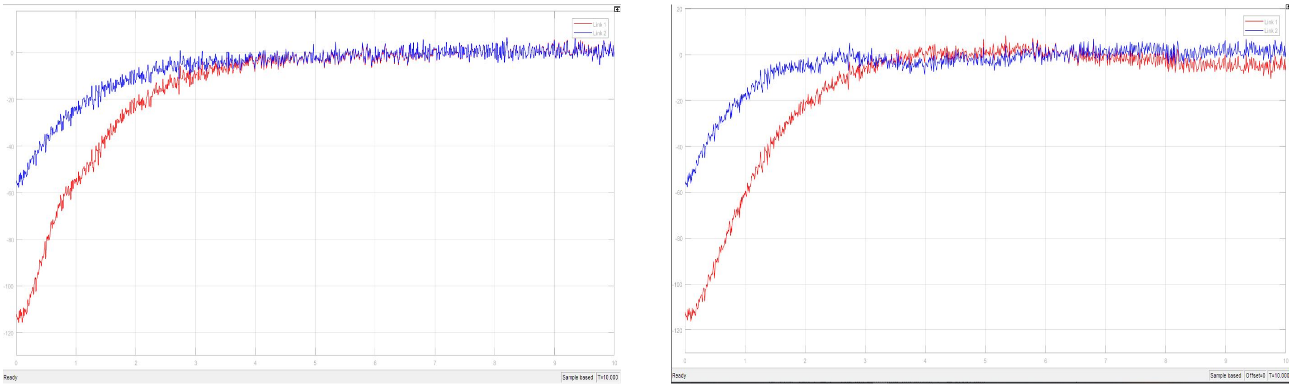
(a) Torque Proposed Method (b) Torque e-mod

Figure 9: Torque Comparison (Nm vs time)



(a) Function Estimation Error Proposed Method (b) Function estimation Error Emod

Figure 10: Function Estimation Error Comparison ($Y\theta$ - $Y_d\hat{\theta}(Nm)$)



(a) AWG Tracking Error (Proposed Method) (b) AWG Tracking Error (e mod

Figure 11: Plots of tracking error (deg) versus time (s) in presence of AWG measurement noise with the proposed method and e-mod.

6.2 Code

MATLAB FUNCTION FOR PROPOSED METHOD

```
% param=[18.1502;36.2946;0.8982;1.0552]; % Run the following command in Matlab terminal before
     running the simulation.
% For Proposed_method control law.

function [qDotDot,e,u,thetatilde,mudot,thetahatdot,Ydhat,Ydtilde,J,proj,Ttilde] = fcn(t,q,
    qdot,mu,thetahat,param,noise)
qd=[cos(0.5*t);2*cos(t)];
qdDot=[-0.5*sin(0.5*t);-2*sin(t)];
qdDotDot=[-0.25*cos(0.5*t);-2*cos(t)];
qdDotDotDot=[0.125*sin(0.5*t);2*sin(t)];
qdDotDotDotDot=[0.0625*cos(0.5*t);2*cos(t)];
qd0=[1;2];
qdDot0=[0;0];

% Adding AWG noise to the q vector. Uncomment below 2 lines to check for AWG performance.
% q=q+noise(1:2);
% qdot=qdot+noise(3:4);

q1=q(1);
q2=q(2);
q1dot=qdot(1);
q2dot=qdot(2);
c2=cos(q2);
s2=sin(q2);
q10=-1;
q20=1;
q1dot0=0;
q2dot0=0;
c20=cos(q20);
s20=sin(q20);
c2d=cos(qd(2));
s2d=sin(qd(2));
c2d0=cos(qd0(2));
s2d0=sin(qd0(2));
e=q-qd;
q0=[q10;q20];
qdot0=[q1dot0;q2dot0];
e10=q0-qd0;
e1dot0=qdot0-qdDot0;
edot=qdot-qdDot;
e1=e;
e1dot=edot;
% alpha=1;
k=param(1);
beta=param(2);
alpha=param(3);
alpha2=param(4);
e2=e1dot+alpha*e1;
e20=e1dot0+alpha*e10;
```

```
Yd=[qdDotDot(1) qdDotDot(2) 2*c2d*qdDotDot(1)+c2d*qdDotDot(2)-s2d*qdDot(1)*qdDot(2)-s2d*(qdDot
    (1)+qdDot(2))*qdDot(2) qdDot(1) 0;
    0 qdDotDot(1)+qdDotDot(2) s2d*qdDot(1)^2+c2d*qdDotDot(2) 0 qdDot(2)];
Yd=[Yd eye(length(e))];

Gamma=1;
% k=0.01;
% alpha2=0.1;


thetabar=5000;
f=norm(thetahat)^2-thetabar^2;
delf=2*thetahat;
prjscl=delf*delf'/norm(delf)^2;
Lambda=Yd'*inv(Yd*Yd')*(k*alpha2*e2+beta*sign(e2));
if((f<0)||(delf'*Lambda<=0))
    thetahatdot=Lambda;
    mudot=[0;0];
    proj=0;
else
    thetahatdot=(eye(length(thetahat))-prjscl)*Lambda;
    mudot=-Yd*prjscl*Lambda;
    proj=1;
end
mu=mu-k*e2;



u=-Yd*thetahat+mu;


p1 = 3.473+0.5*sin(3*t);
p2 = 0.196+0.2*exp(-sin(t));
p3 = 0.242+0.1*cos(10*t);

fd1 = 5.3+2*exp(-0.1*t);
fd2 = 1.1+cos(5*t);
theta=[p1;p2;p3;fd1;fd2];
disturbance=[0.5*cos(0.5*t);sin(t)];
theta=[theta;disturbance];
thetatilde=theta+thetahat;
M = [p1+2*p3*cos(q(2)), p2+p3*cos(q(2)); p2+p3*cos(q(2)), p2];
Vm = [-p3*sin(q(2))*qdot(2), -p3*sin(q(2))*(qdot(1)+qdot(2)); p3*sin(q(2))*qdot(1), 0];
Fd = [fd1, 0; 0, fd2];

qDotDot = M\(u - Vm*qdot - Fd*qdot-disturbance);
Ydhat=Yd*thetahat;
Ydtilde=u+Yd*thetahat;
J=0.01*norm(u)^2+norm(e)^2;
qDot=qdot;
Y=[qDotDot(1) qDotDot(2) 2*c2*qDotDot(1)+c2*qDotDot(2)-s2*qDot(1)*qDot(2)-s2*(qDot(1)+qDot(2))
    *qDot(2) qDot(1) 0 1 0;
    0 qDotDot(1)+qDotDot(2) s2*qDot(1)^2+c2*qDotDot(2) 0 qDot(2) 0 1];
Ttilde=u-Y*theta;
```

MATLAB FUNCTION FOR E-MOD METHOD

```
% param1=[12.5;9.7877;9.7319]; % Run the following command in Matlab terminal before running
    the simulation.
% For E_mod control law.



function [qDotDot,e,u,thetatilde,thetahatdot,J1,Ydtilde] = fcn(t,q,qdot,thetahat,param1,noise
    )

% Declaring the desired q values and its differentials.
qd=[cos(0.5*t);2*cos(t)];
qdDot=[-0.5*sin(0.5*t);-2*sin(t)];
qdDotDot=[-0.25*cos(0.5*t);-2*cos(t)];
qdDotDotDot=[0.125*sin(0.5*t);2*sin(t)];
qdDotDotDotDot=[0.0625*cos(0.5*t);2*cos(t)];
qd0=[1;2];
qdDot0=[0;0];


```

```
% Adding AWG noise to the q vector. Uncomment below 2 lines to check for AWG performance.
% q=q+noise(1:2);
% qdot=qdot+noise(3:4);


q1=q(1);
q2=q(2);
q1dot=qdot(1);
q2dot=qdot(2);
c2=cos(q2);
s2=sin(q2);
q10=-1;
q20=1;
q1dot0=0;
q2dot0=3;
c20=cos(q20);
s20=sin(q20);
c2d=cos(qd(2));
s2d=sin(qd(2));
c2d0=cos(qd0(2));
s2d0=sin(qd0(2));

% Calculating error in e
e=q-qd;
q0=[q10;q20];
qdot0=[q1dot0;q2dot0];

% Calculating tracking error
e10=q0-qd0;
e1dot0=qdot0-qdDot0;
edot=qdot-qdDot;
e1=e;
e1dot=edot;

% Declaring constant control gain alpha
alpha=1;

% Calculating filtered tracking error
e2=e1dot+alpha*e1;
e20=e1dot0+alpha*e10;


% Calculating the regression matrix
Yd=[qdDotDot(1) qdDotDot(2) 2*c2d*qdDotDot(1)+c2d*qdDotDot(2)-s2d*qdDot(1)*qdDot(2)-s2d*(qdDot
    (1)+qdDot(2))*qdDot(2) qdDot(1) 0;
    0 qdDotDot(1)+qdDotDot(2) s2d*qdDot(1)^2+c2d*qdDotDot(2) 0 qdDot(2)];
Yd=[Yd eye(length(e))];


Gamma=param1(1);
k=param1(2);
sigma=param1(3);


thetahatdot=Gamma*Yd'*e2-sigma*norm(e1)*thetahat;
u=-Yd*thetahat-k*e2;


p1 = 3.473+0.5*sin(3*t);
p2 = 0.196+0.2*exp(-sin(t));
p3 = 0.242+0.1*cos(10*t);

fd1 = 5.3+2*exp(-0.1*t);
fd2 = 1.1+cos(5*t);
theta=[p1;p2;p3;fd1;fd2];
disturbance=[0.5*cos(0.5*t);sin(t)];
theta=[theta;disturbance];
thetatilde=theta-thetahat;
M = [p1+2*p3*cos(q(2)), p2+p3*cos(q(2)); p2+p3*cos(q(2)), p2];
Vm = [-p3*sin(q(2))*qdot(2), -p3*sin(q(2))*(qdot(1)+qdot(2)); p3*sin(q(2))*qdot(1), 0];
Fd = [fd1, 0; 0, fd2];

qDotDot = M\(u - Vm*qdot - Fd*qdot-disturbance);
J1=0.02*norm(u)^2+norm(e)^2;
Ydtilde=u+Yd*thetahat;
```

7 CONCLUSION

In this report, we discussed the concepts Adaptive Control. We designed an Adaptive control method to achieve asymptotic tracking for linearly parameterizable non linear systems with time varying uncertain parameters. To achieve asymptotic behaviour, we introduced an adaptive feedforward term along with specialized feedback terms to compensate for the time varying uncertainty. Asymptotic tracking error convergence was guaranteed via a Lyapunov-based stability analysis for an Euler Lagrange system. Additionally, the time-varying uncertain function approximation error was shown to converge to zero. A simulation example of a two-link manipulator was provided to demonstrate the asymptotic tracking result, and a comparison with the e-mod scheme shows a better tracking performance with the proposed method.

8 APPENDIX

Lemma 1: Consider a positive-definite matrix $\Gamma \in \mathbb{R}^{(n+m)\times(n+m)}$ such that Γ has the block diagonal structure as $\Gamma \triangleq \begin{bmatrix} \Gamma_1 & 0_{m\times n} \\ 0_{n\times m} & \Gamma_2 \end{bmatrix}$, where $\Gamma_1 \in \mathbb{R}^{m\times m}$ and $\Gamma_2 \in \mathbb{R}^{n\times n}$. The matrix $Y\Gamma Y^T$ is positive definite and therefore invertible. Furthermore, the inverse of this matrix satisfies the property $\left\| \left(Y\Gamma Y^T\right)^{-1} \right\|_2 \leq \frac{1}{\lambda_{\min}\{\Gamma_2\}}$, where $\|\cdot\|_2$ denotes the spectral norm and $\lambda_{\min}\{\cdot\}$ denotes the minimum eigenvalue of $\{\cdot\}$.

Proof: Substituting the definitions for Y and Γ in $Y\Gamma Y^T$ yields

$$\begin{aligned} Y\Gamma Y^T &= \begin{bmatrix} Y_p & I_n \end{bmatrix} \begin{bmatrix} \Gamma_1 & 0_{m\times n} \\ 0_{n\times m} & \Gamma_2 \end{bmatrix} \begin{bmatrix} Y_p^T \\ I_n \end{bmatrix} \\ &= Y_p \Gamma_1 Y_p^T + \Gamma_2. \end{aligned}$$

Since Γ is selected to be a positive-definite matrix, the block matrices Γ_1 and Γ_2 are both positive-definite, so $Y_p\Gamma_1 Y_p^T$ is positive semidefinite while the second term Γ_2 is positive-definite, hence the sum of these two terms, i.e., $Y\Gamma Y^T$ is positive-definite, and therefore, invertible. Furthermore, the spectral norm satisfies the property, $\|A\|_2 = \sqrt{\lambda_{\max}\{A^T A\}}$ for some $A \in \mathbb{R}^{p\times q}$, where $\lambda_{\max}\{\cdot\}$ denotes the maximum eigenvalue of $\{\cdot\}$. Utilizing this property with $\left\| \left(Y\Gamma Y^T\right)^{-1} \right\|_2$ yields

$$\left\| \left(Y\Gamma Y^T\right)^{-1} \right\|_2 = \sqrt{\lambda_{\max}\left\{ \left(\left(Y\Gamma Y^T\right)^{-1}\right)^T \left(Y\Gamma Y^T\right)^{-1} \right\}} = \lambda_{\max}\left\{ \left(Y\Gamma Y^T\right)^{-1} \right\} = \frac{1}{\lambda_{\min}\{Y\Gamma Y^T\}} \leq \frac{1}{\lambda_{\min}\{\Gamma_2\}} \tag{60}$$

Corollary 1: The norm of the time-derivative of the parameter estimate $\|\dot{\hat{\theta}}\|$ can be upper bounded by as $\|\dot{\hat{\theta}}\| \leq \gamma_4 + \gamma_5 \|e_2\|$, where $\gamma_4, \gamma_5 \in \mathbb{R}_{>0}$ are known bounding constants. Proof: Based on (34)

$$\|\dot{\hat{\theta}}\| = \|\mathrm{proj}(\Lambda_0)\| \leq \|\Lambda_0\| = \left\| \Gamma Y_d^T \left(Y_d \Gamma Y_d^T\right)^{-1} \left(\beta\,\mathrm{sgn}(e_2) + k\alpha_2 e_2\right) \right\| \leq \left\| \Gamma Y_d^T \left(Y_d \Gamma Y_d^T\right)^{-1} \right\| \left(\beta + k\alpha_2 \|e_2\|\right) \tag{61}$$

Applying Holder's inequality to the right-hand side of (61) yields

$$\|\dot{\hat{\theta}}\| \leq \|\Gamma\|_2 \|Y_d\|_2 \left\| \left(Y_d \Gamma Y_d^T\right)^{-1} \right\|_2 \left(\beta + k\alpha_2 \|e_2\|\right) \tag{62}$$

Using the result from Lemma 1 yields

$$\|\dot{\hat{\theta}}\| \leq \frac{\|\Gamma\|_2 \|Y_d\|_2}{\lambda_{\min}\{\Gamma_2\}} \left(\beta + k\alpha_2 \|e_2\|\right)$$

Based on Assumption 5, the spectral norm of the desired regressor may be upper bounded by a constant $\bar{Y}_d \in \mathbb{R}_{>0}$, i.e., $\|Y_d\|_2 \leq \bar{Y}_d$, because Y_d is a continuously differentiable function. Therefore, selecting $\gamma_4 = \frac{\beta\|\Gamma\|_2 \bar{Y}_d}{\lambda_{\min}\{\Gamma_2\}}$ and $\gamma_5 = \frac{k\alpha_2\|\Gamma\|_2 \bar{Y}_d}{\lambda_{\min}\{\Gamma_2\}}$ yields

$$\begin{aligned} \|\dot{\hat{\theta}}\| &\leq \frac{\|\Gamma\|_2 \bar{Y}_d}{\lambda_{\min}\{\Gamma_2\}} \left(\beta + k\alpha_2 \|e_2\|\right) \\ &= \gamma_4 + \gamma_5 \|e_2\| \end{aligned}$$

8.1 Terminology

Definite and Semi definite Matrix
The preceding example can be generalized as follows: if A is an n × n diagonal matrix

$$A = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & d_n \end{bmatrix}$$

then A is:

1. positive definite if and only if $d_i > 0$ for i = 1,2,...,n,
2. negative definite if and only if $d_i < 0$ for i = 1,2,...,n,
3. positive semidefinite if and only if $d_i \geq 0$ for i = 1,2,...,n,
4. negative semidefinite if and only if $d_i \leq 0$ for i = 1,2,...,n,
5. indefinite if and only if $d_i > 0$ for some indices i, $1 \leq i \leq n$, and negative for other indices.

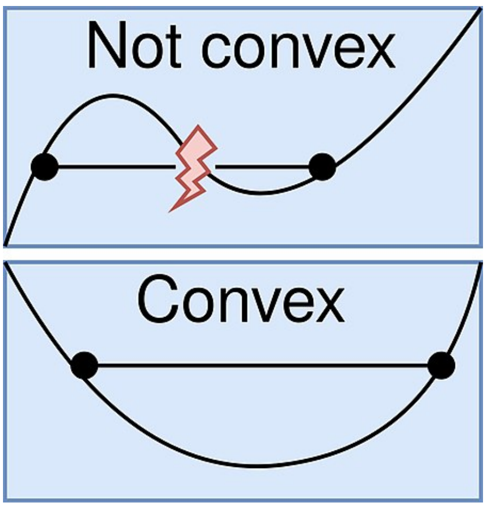
Positive Semi definite:
Let A be a symmetric matrix, and $Q(x) = x^T Ax$ the corresponding quadratic form.
Definitions: Q and A are called positive semidefinite if $Q(x) \leq 0$ for all x. They are called positive definite if $Q(x) > 0$ for all $x = 0$. So positive semidefinite means that there are no minuses in the signature, while positive definite means that there are n pluses, where n is the dimension of the space.

Sign Function:
In mathematics, the sign function or signum function (from signum, Latin for "sign") is an odd mathematical function that extracts the sign of a real number. In mathematical expressions the sign function is often represented as sgn. To avoid confusion with the sine function, this function is usually called the signum function. The signum function of a real number x is a piecewise function which is defined as follows:

$$sgnx := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Convex Function

- In mathematics, a real-valued function is called convex if the line segment between any two points on the graph of the function lies above the graph between the two points. Equivalently, a function is convex if its epigraph (the set of points on or above the graph of the function) is a convex set.
- A twice-differentiable function of a single variable is convex if and only if its second derivative is non-negative on its entire domain.



Continuously Differentiable

A function f is said to be continuously differentiable if its derivative f' exists and is itself a continuous function.
Although the derivative of a differentiable function never has a jump discontinuity, it is possible for the derivative to have an essential discontinuity. For example, the function

$$f(x) = \begin{cases} \mathrm{x}^2 sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is differentiable at 0, but

$$f'(x) = 2xsin(1/x) - cos(1/x)$$

has no limit as $x \to 0$

Lebesgue Measurable Function

Lebesgue measurable functions play an important role in Lebesgue integration. Lebesgue measure is a natural extension of the concept of area, length, or volume, depending on dimension.

Let f be a function defined on a measurable domain E taking values in the extended real number line. We say f is a Lebesgue measurable function if for every real number c the set $x \in E|f(x) > c$ is measurable.

We can also say that an extended real-valued function f defined on $E \in M$ is Lebesgue if it satisfies the following (equivalent) statements for each number c in $\mathbb{R}$

- $x \in E|f(x) > c \in M$
- $x \in E|f(x) \geq c \in M$
- $x \in E|f(x) < c \in M$
- $x \in E|f(x) \leq c \in M$

Lipschitz continuity:

In mathematical analysis, Lipschitz continuity, is a strong form of uniform continuity for functions. Intuitively, a Lipschitz continuous function is limited in how fast it can change: there exists a real number such that, for every pair of points on the graph of this function, the absolute value of the slope of the line connecting them is not greater than this real number; the smallest such bound is called the Lipschitz constant of the function (or modulus of uniform continuity). For instance, every function that has bounded first derivatives is Lipschitz continuous.
Example:

- The function $f(x) = \sqrt{x^2 + 5}$ defined for all real numbers is Lipschitz continuous with the Lipschitz constant K = 1, because it is everywhere differentiable and the absolute value of the derivative is bounded above by 1.
- The sine function is Lipschitz continuous because its derivative, the cosine function, is bounded above by 1 in absolute value.

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