



Yule-Walker Equations in matrix form

PRACTICAL TIME SERIES ANALYSIS

THISTLETON AND SADIGOV

Objectives

- ▶ Rewrite Yule – Walker equations in matrix form for $AR(p)$ processes

AR(p) process

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots \phi_p X_{t-p} + Z_t$$

where

$$Z_t \sim \text{Normal}(0, \sigma_Z^2)$$

Note that, if we take expectation from both sides of the model

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots \phi_p X_{t-p} + Z_t$$

We get

$$\mu = \phi_0 + \phi_1 \mu + \phi_2 \mu + \cdots \phi_p \mu$$

Subtract side by side

$$X_t - \mu = \phi_1 (X_{t-1} - \mu) + \phi_2 (X_{t-2} - \mu) + \cdots \phi_p (X_{t-p} - \mu) + Z_t$$

If $\tilde{X}_t = X_t - \mu$, then $E[\tilde{X}_t] = 0$, and

$$\tilde{X}_t = \phi_1 \tilde{X}_{t-1} + \phi_2 \tilde{X}_{t-2} + \cdots \phi_p \tilde{X}_{t-p} + Z_t$$

AR(p) process with $\mu = 0$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots \phi_p X_{t-p} + Z_t$$

where

$$Z_t \sim \text{Normal}(0, \sigma_Z^2)$$

Yule –Walker equations

Autocorrelation function obeys

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p)$$

for $k \geq 1$, $\rho(0) = 1$ and $\rho(k) = \rho(-k)$ for $k < 0$.

Lets write them for $k = 1, 2, \dots, p$.

Yule Walker equations

$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2) + \cdots + \phi_p \rho(k-p)$$

For $k = 1, 2, \dots, p$,

$$\rho(1) = \phi_1 \rho(0) + \phi_2 \rho(-1) + \phi_3 \rho(-2) + \cdots + \phi_p \rho(1-p)$$

$$\rho(2) = \phi_1 \rho(1) + \phi_2 \rho(0) + \phi_3 \rho(-1) + \cdots + \phi_p \rho(2-p)$$

$$\rho(3) = \phi_1 \rho(2) + \phi_2 \rho(1) + \phi_3 \rho(0) + \cdots + \phi_p \rho(3-p)$$

\vdots

$$\rho(p-1) = \phi_1 \rho(p-2) + \phi_2 \rho(p-3) + \phi_3 \rho(p-4) + \cdots + \phi_p \rho(1)$$

$$\rho(p) = \phi_1 \rho(p-1) + \phi_2 \rho(p-2) + \phi_3 \rho(p-3) + \cdots + \phi_p \rho(0)$$

Recall $\rho(k) = \rho(-k)$.

i.e., $\rho(-1) = \rho(1)$, $\rho(-2) = \rho(2)$, ..., $\rho(2-p) = \rho(p-2)$, $\rho(1-p) = \rho(p-1)$

$$\rho(-k) = \rho(k)$$

$$\rho(1) = \phi_1\rho(0) + \phi_2\rho(1) + \phi_3\rho(2) + \cdots + \phi_p\rho(p-1)$$

$$\rho(2) = \phi_1\rho(1) + \phi_2\rho(0) + \phi_3\rho(1) + \cdots + \phi_p\rho(p-2)$$

$$\rho(3) = \phi_1\rho(2) + \phi_2\rho(1) + \phi_3\rho(0) + \cdots + \phi_p\rho(p-3)$$

$$\vdots$$

$$\rho(p-1) = \phi_1\rho(p-2) + \phi_2\rho(p-3) + \phi_3\rho(p-4) + \cdots + \phi_p\rho(1)$$

$$\rho(p) = \phi_1\rho(p-1) + \phi_2\rho(p-2) + \phi_3\rho(p-3) + \cdots + \phi_p\rho(0)$$

$$\rho(0) = 1$$

$$\rho(1) = \phi_1 + \phi_2\rho(1) + \phi_3\rho(2) + \cdots + \phi_p\rho(p-1)$$

$$\rho(2) = \phi_1\rho(1) + \phi_2 + \phi_3\rho(1) + \cdots + \phi_p\rho(p-2)$$

$$\rho(3) = \phi_1\rho(2) + \phi_2\rho(1) + \phi_3 + \cdots + \phi_p\rho(p-3)$$

$$\vdots$$

$$\rho(p-1) = \phi_1\rho(p-2) + \phi_2\rho(p-3) + \phi_3\rho(p-4) + \cdots + \phi_p\rho(1)$$

$$\rho(p) = \phi_1\rho(p-1) + \phi_2\rho(p-2) + \phi_3\rho(p-3) + \cdots + \phi_p$$

Matrix form: Yule- Walker equations

$$\begin{bmatrix} \rho(1) \\ \rho(2) \\ \rho(3) \\ \vdots \\ \rho(p-1) \\ \rho(p) \end{bmatrix} = \begin{bmatrix} 1 & \rho(1) & \rho(2) & \dots & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \dots & \rho(p-2) \\ \rho(2) & \rho(1) & 1 & \dots & \rho(p-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho(p-2) & \rho(p-3) & \rho(p-4) & \dots & \rho(1) \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \dots & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix}$$

b

R

ϕ

$$b = R\phi$$

$$R^{-1} b = \phi$$

Sample ACF $r_k \approx \rho(k)$

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{p-1} \\ r_p \end{bmatrix} = \begin{bmatrix} 1 & r_1 & r_2 & \cdots & r_{p-1} \\ r_1 & 1 & r_1 & \cdots & r_{p-2} \\ r_2 & r_1 & 1 & \ddots & r_{p-3} \\ & \vdots & & \ddots & \vdots \\ r_{p-2} & r_{p-3} & r_{p-4} & \cdots & r_1 \\ r_{p-1} & r_{p-2} & r_{p-3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{p-1} \\ \phi_p \end{bmatrix}$$

\hat{b}	\hat{R}	$\hat{\phi}$
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$$\hat{b} = \hat{R}\hat{\phi}$$

$$\hat{R}^{-1} \hat{b} = \hat{\phi}$$

Matrices R and \hat{R}

- ▶ These matrices are symmetric matrices
- ▶ They are positive semidefinite matrices
- ▶ All eigenvalues are nonnegative
- ▶ Inverses of these matrices exist

$$\begin{bmatrix} 1 & r_1 & r_2 & \cdots & r_{p-1} \\ r_1 & 1 & r_1 & \cdots & r_{p-2} \\ r_2 & r_1 & 1 & \cdots & r_{p-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{p-2} & r_{p-3} & r_{p-4} & \cdots & r_1 \\ r_{p-1} & r_{p-2} & r_{p-3} & \cdots & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & \rho(1) & \rho(2) & \cdots & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \cdots & \rho(p-2) \\ \rho(2) & \rho(1) & 1 & \cdots & \rho(p-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho(p-2) & \rho(p-3) & \rho(p-4) & \cdots & \rho(1) \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \cdots & 1 \end{bmatrix}$$

- ▶ $\hat{b} = \hat{R}\hat{\phi}$ has a unique solution

Example – AR(2)

We (will) estimate coefficients of the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$



by first finding r_1, r_2 using `acf()` routine, then solving the system of equations

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 & r_1 \\ r_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix}$$

Example – AR(3)

We (will) estimate coefficients of the model


$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t$$

by first finding r_1, r_2, r_3 , using `acf()` routine, then solving the system of equations

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \end{bmatrix}$$

What We've Learned

- ▶ Matrix form of Yule – Walker equations
- ▶ How to estimate the coefficients of an AR process using Yule-Walker equations



Estimating model parameters – AR(2) Simulation

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Objectives

- ▶ Estimate variance of a white noise in a simulated AR(2) processes
- ▶ Estimate coefficients of a simulated AR(2) process using Yule-Walker equations in matrix form

AR(2) process (with mean zero)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

where

$$Z_t \sim \text{Normal}(0, \sigma_Z^2)$$

We simulate this process for

$$\phi_1 = \frac{1}{3}, \phi_2 = \frac{1}{2}, \sigma_Z = 4$$

Yule –Walker equations

We estimate coefficients of the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

by first finding r_1, r_2 using `acf()` routine, then solving the system of equations

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 & r_1 \\ r_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix}$$

σ_Z Estimation

Since

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$$

We have

$$\text{Var}(X_t) = \phi_1^2 \text{Var}(X_{t-1}) + \phi_2^2 \text{Var}(X_{t-2}) + 2\phi_1\phi_2 \text{Cov}(X_{t-1}, X_{t-2}) + \sigma_Z^2$$

Thus

$$\sigma_Z^2 = \gamma(0) \left[1 - \phi_1^2 - \phi_2^2 - \frac{2\phi_1\phi_2\gamma(1)}{\gamma(0)} \right] = \gamma(0)[1 - \phi_1^2 - \phi_2^2 - 2\phi_1\phi_2\rho_1]$$

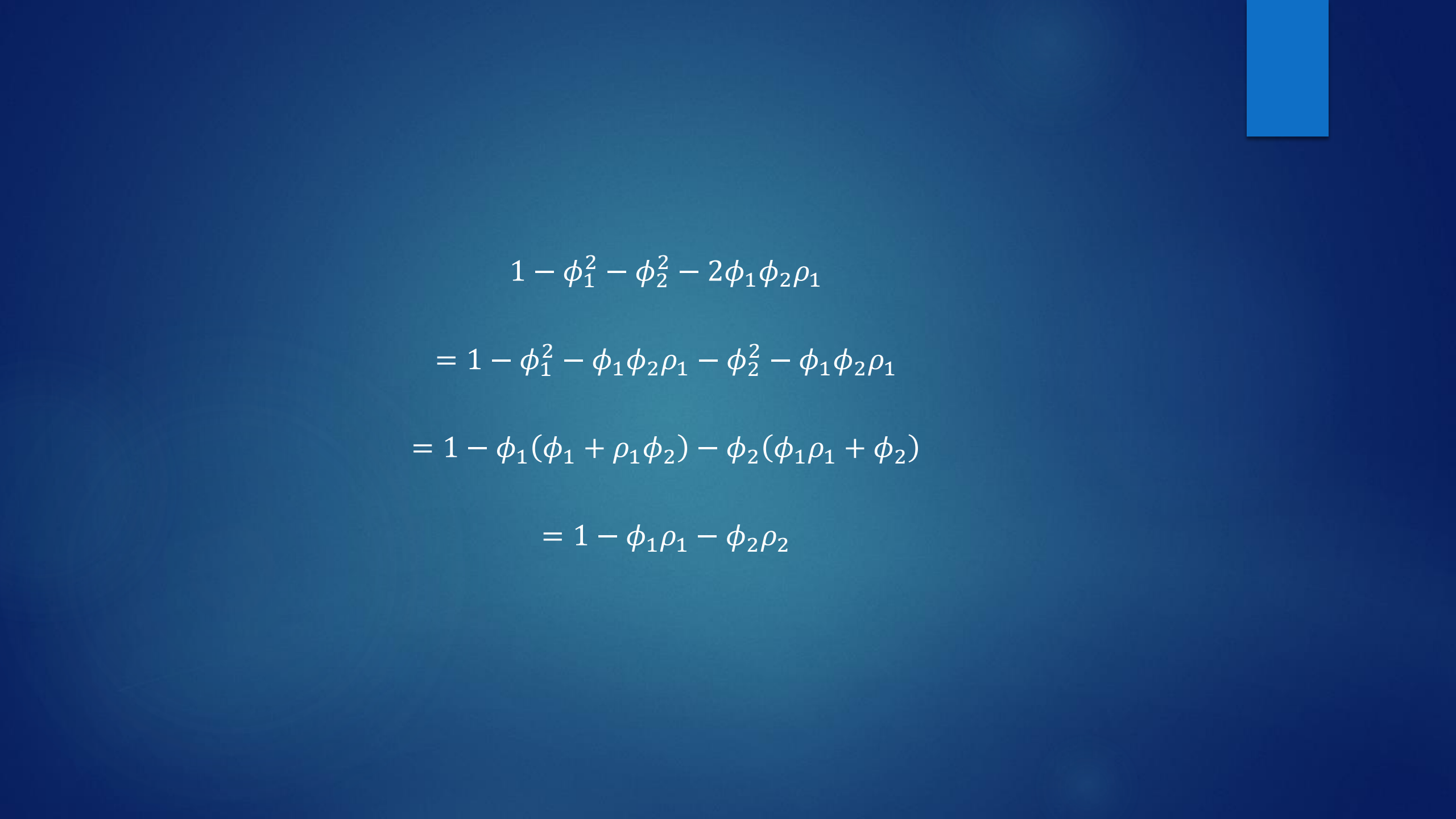
Since

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

we have

$$\rho_1 = \phi_1 + \rho_1 \phi_2$$

$$\rho_2 = \phi_1 \rho_1 + \phi_2$$


$$1 - \phi_1^2 - \phi_2^2 - 2\phi_1\phi_2\rho_1$$

$$= 1 - \phi_1^2 - \phi_1\phi_2\rho_1 - \phi_2^2 - \phi_1\phi_2\rho_1$$

$$= 1 - \phi_1(\phi_1 + \rho_1\phi_2) - \phi_2(\phi_1\rho_1 + \phi_2)$$

$$= 1 - \phi_1\rho_1 - \phi_2\rho_2$$

σ_Z Estimation cont.

Thus,

$$\sigma_Z^2 = \gamma(0)[1 - \phi_1\rho_1 - \phi_2\rho_2]$$

Yule –Walker estimator

$$\hat{\sigma}_Z^2 = c_0 [1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2]$$

Simulation

- ▶ Number of data points, $n = 10000$

Routines that we use:

- ▶ `arima.sim()` # simulating
- ▶ `plot()` # plotting the series
- ▶ `acf()` # autocorrelation function
- ▶ `matrix(,m,n)` # matrix with dimensions m by n
- ▶ `solve(R,b)` # finds the solution to $Rx=b$

Code details

- ▶ `sigma=4`
- ▶ `phi[1:2]=c(1/3,1/2)`
- ▶ `n=10000`
- ▶ `set.seed(2017)`
- ▶ `ar.process=arima.sim(n, model=list(ar=c(1/3,1/2)), sd=4)`
- ▶ `ar.process[1:5]`

4.087685, 5.598492, 3.019295, 2.442354, 5.398302



Code details cont.

► `r[1:2]=acf(ar.process, plot=F)$acf[2:3]`

$$\begin{aligned}r[1] &= 0.6814103 \\ r[2] &= 0.7255825\end{aligned}$$

`R=matrix(1,2,2)` # matrix of dimension 2 by 2, with entries all 1's.

$$R = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

► `R[1,2]=r[1]` # only diagonal entries are edited

► `R[2,1]=r[1]` # only diagonal entries are edited

$$R = \begin{bmatrix} 1 & r_1 \\ r_1 & 1 \end{bmatrix}$$

Code details cont.

- ▶ `b=matrix(r,2,1)` # b- column vector entires from r

$$b = \begin{bmatrix} r[1] \\ r[2] \end{bmatrix} = \begin{bmatrix} 0.6814103 \\ 0.7255825 \end{bmatrix}$$

Code details cont.

► `solve(R,b)`

$$\begin{bmatrix} 0.3490720 \\ 0.4877212 \end{bmatrix}$$

► `phi.hat=matrix(c(solve(R,b)[1,1], solve(R,b)[2,1]),2,1)`

$$\begin{aligned} \hat{\phi}_1 &= 0.3490720 \\ \hat{\phi}_2 &= 0.4877212 \end{aligned}$$

Code details cont.

- ▶ `c0= acf(ar.process, type='covariance', plot=F)$acf[1]`
- ▶ `var.hat= c0*(1-sum(phi.hat*r))`
- ▶ `par(mfrow=c(3,1))`
- ▶ `plot(ar.process, main='Simulated AR(2)')`
- ▶ `acf(ar.process, main='ACF')`
- ▶ `pacf(ar.process, main='PACF')`

Results

- ▶ $\phi_1 \approx \hat{\phi}_1 = 0.3490720$
- ▶ $\phi_2 \approx \hat{\phi}_2 = 0.4877212$
- ▶ $\sigma_Z^2 \approx \widehat{\sigma_Z^2} = 16.37169$

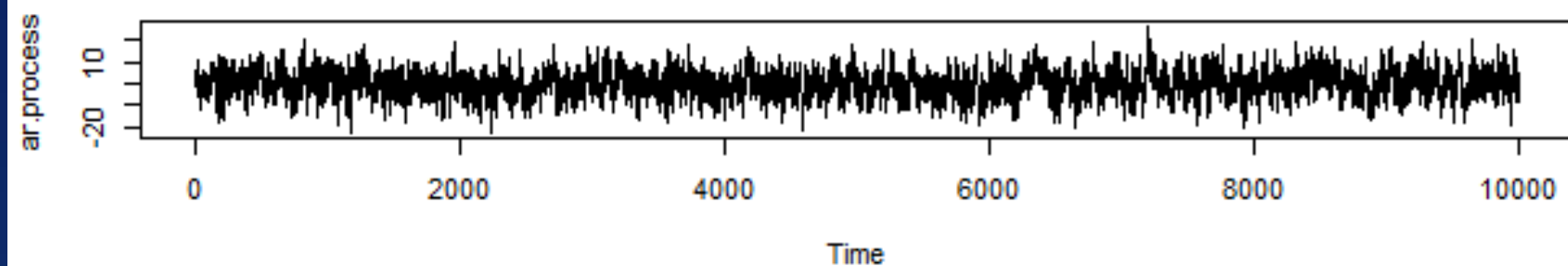
Actual Model

$$X_t = 0.3 X_{t-1} + 0.5 X_{t-2} + Z_t, \quad Z_t \sim N(0, 16)$$

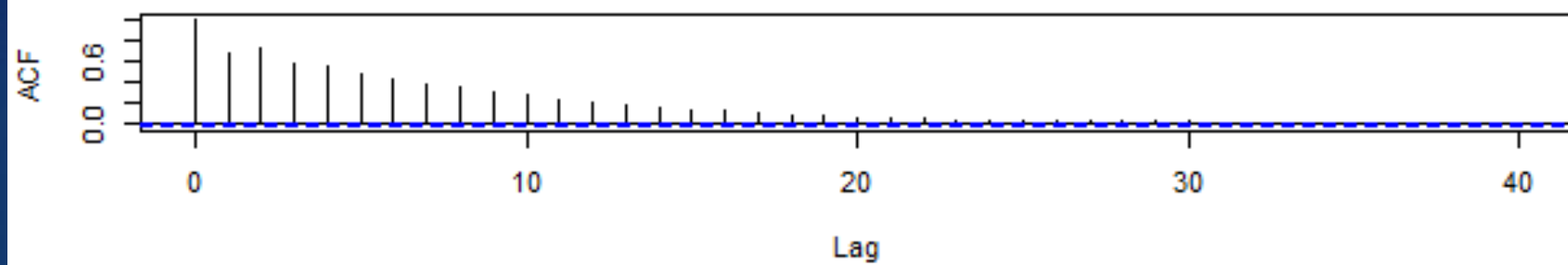
Fitted model

$$X_t = 0.3490720 X_{t-1} + 0.4877212 X_{t-2} + Z_t, \quad Z_t \sim N(0, 16.37169)$$

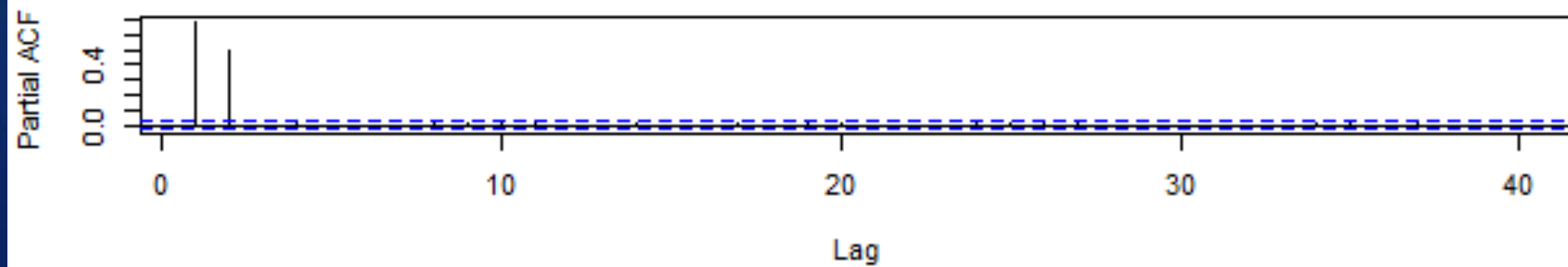
Simulated AR(2)



ACF




PACF



What We've Learned

- ▶ Estimating model parameters of a simulated AR(2) process using Yule-Walker equations in a matrix form



Estimating model parameters – AR(3) Simulation

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Objectives

- ▶ Estimate model parameters of a simulated AR(3) process using Yule-Walker equations in matrix form

AR(2) process (with mean zero)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t$$

where

$$Z_t \sim \text{Normal}(0, \sigma_Z^2)$$

We simulate this process for

$$\phi_1 = \frac{1}{3}, \phi_2 = \frac{1}{2}, \phi_3 = \frac{7}{100}, \sigma_Z = 4$$

Yule –Walker equations

We (will) estimate coefficients of the model

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t$$

by first finding r_1, r_2 using `acf()` routine, then solving the system of equations

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & r_2 \\ r_1 & 1 & r_1 \\ r_2 & r_1 & 1 \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \end{bmatrix}$$

σ_Z Estimation

Yule – Walker estimator for σ_Z^2

$$\hat{\sigma}_Z^2 = c_0(1 - \sum_{i=1}^p \phi_i r_i)$$

Results (set.seed(2017))

- ▶ $n = 100000$
- ▶ $\phi_1 \approx 0.3381245$
- ▶ $\phi_2 \approx 0.4984999$
- ▶ $\phi_3 \approx 0.06849712$
- ▶ $\sigma_Z^2 \approx 15.979$

Actual Model

$$X_t = 0.3X_{t-1} + 0.5X_{t-2} + 0.07X_{t-3} + Z_t, \quad Z_t \sim N(0,16)$$

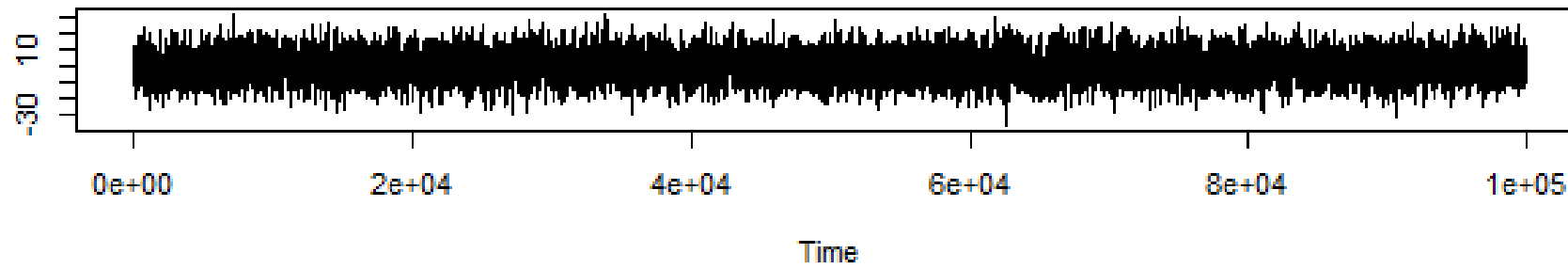
Fitted model

$$X_t = 0.3381245 X_{t-1} + 0.4984999 X_{t-2} + 0.06849712 X_{t-3} + Z_t,$$

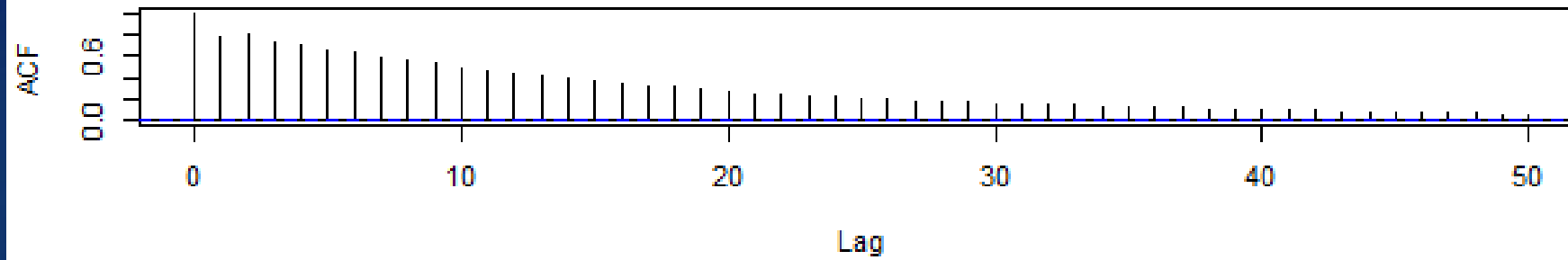
$$Z_t \sim N(0,15.979)$$

Simulated AR(3)

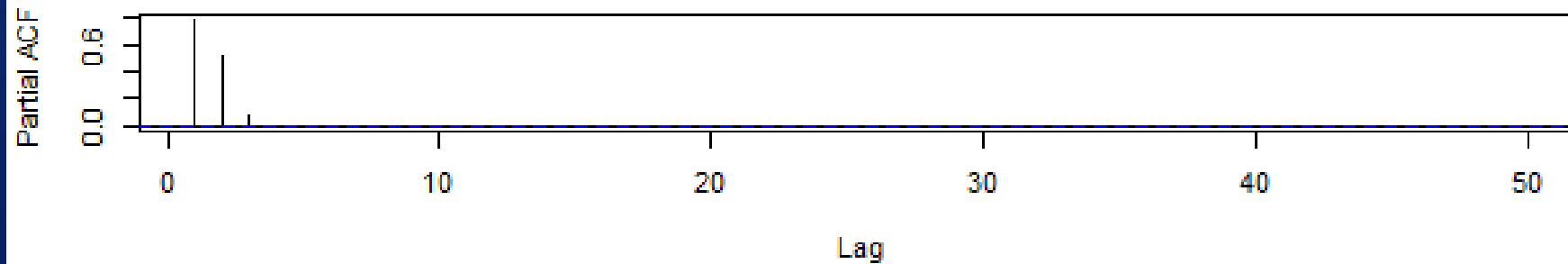
ar3.process



ACF



PACF



What We've Learned

- ▶ Estimating model parameters of a simulated AR(3) process using Yule-Walker equations in a matrix form



Parameter estimation: Recruitment

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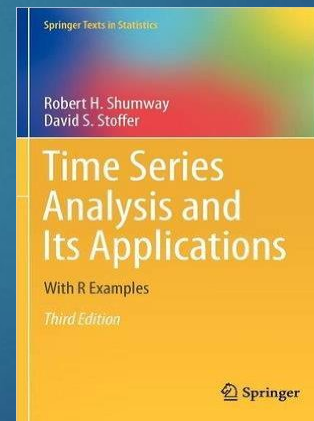
Objectives

- ▶ To fit an AR(p) model to recruitment (number of new fish) for a period of 453 months ranging over the years 1950-1987.
- ▶ Use Yule-Walker equations in matrix form to estimate parameters of the fitted model

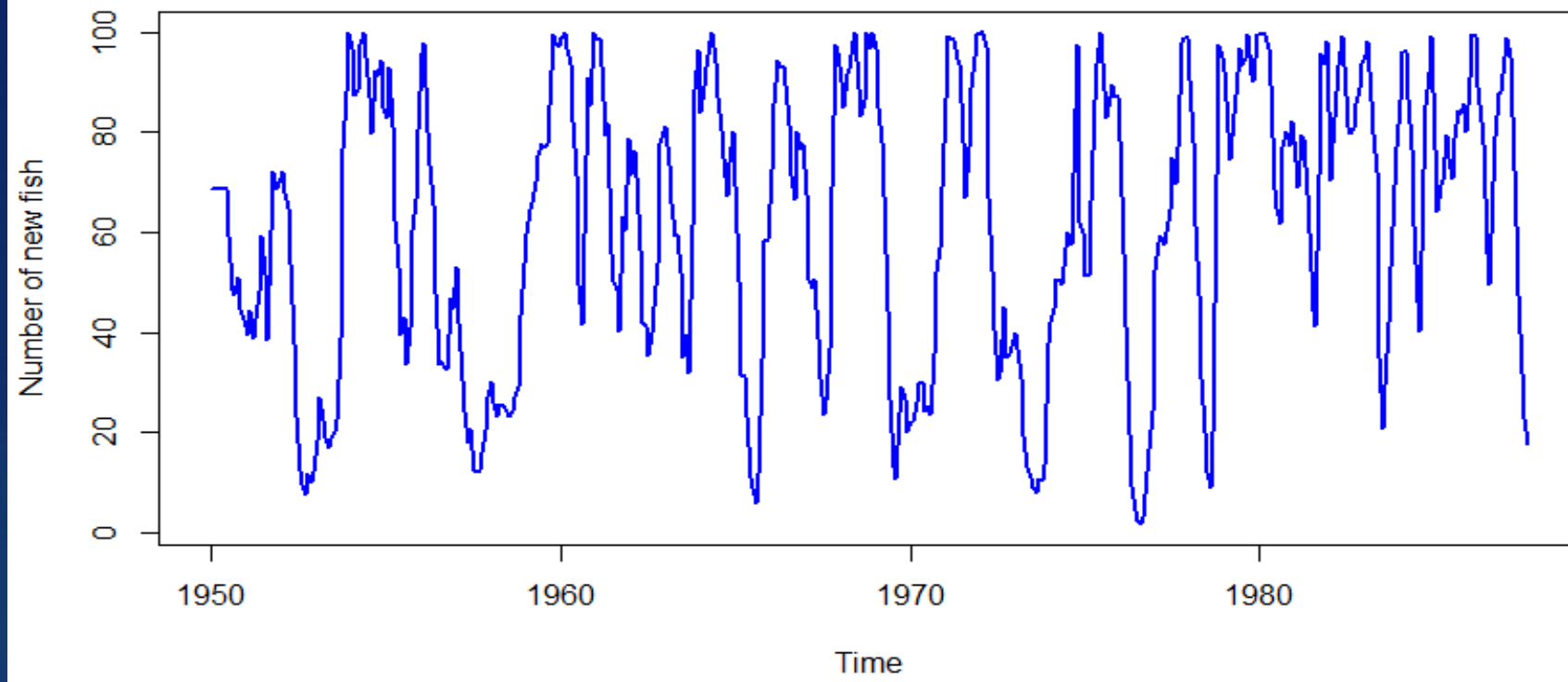
rec {astsa}

- ▶ Recruitment (number of new fish) for a period of 453 months ranging over the years 1950-1987.
- ▶ Monthly time series
- ▶ Source: “astsa” package

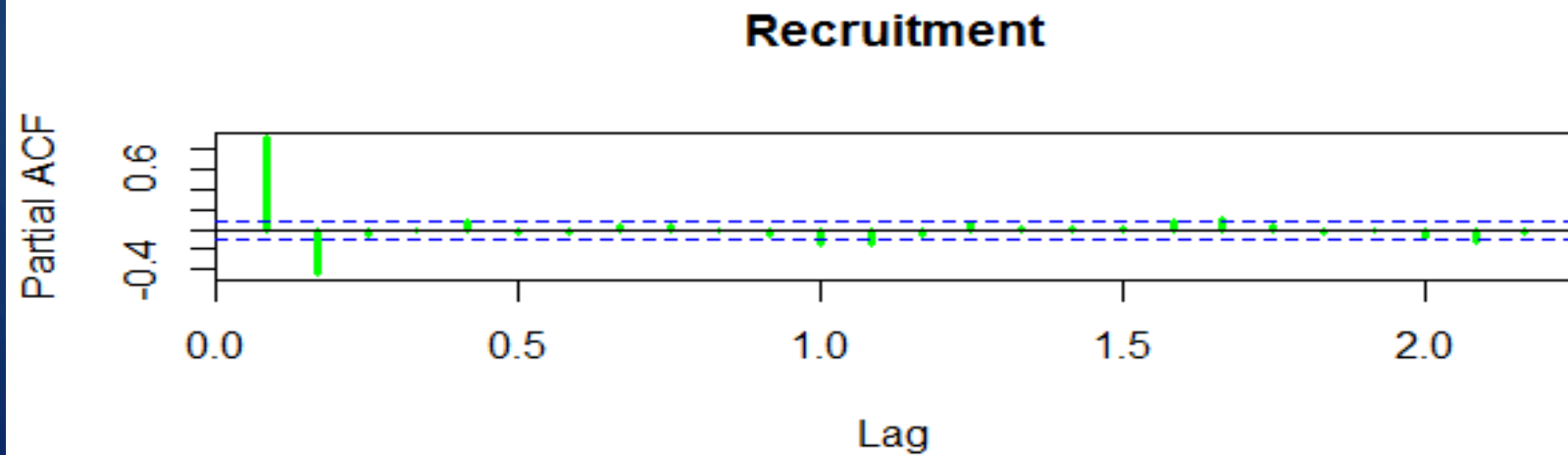
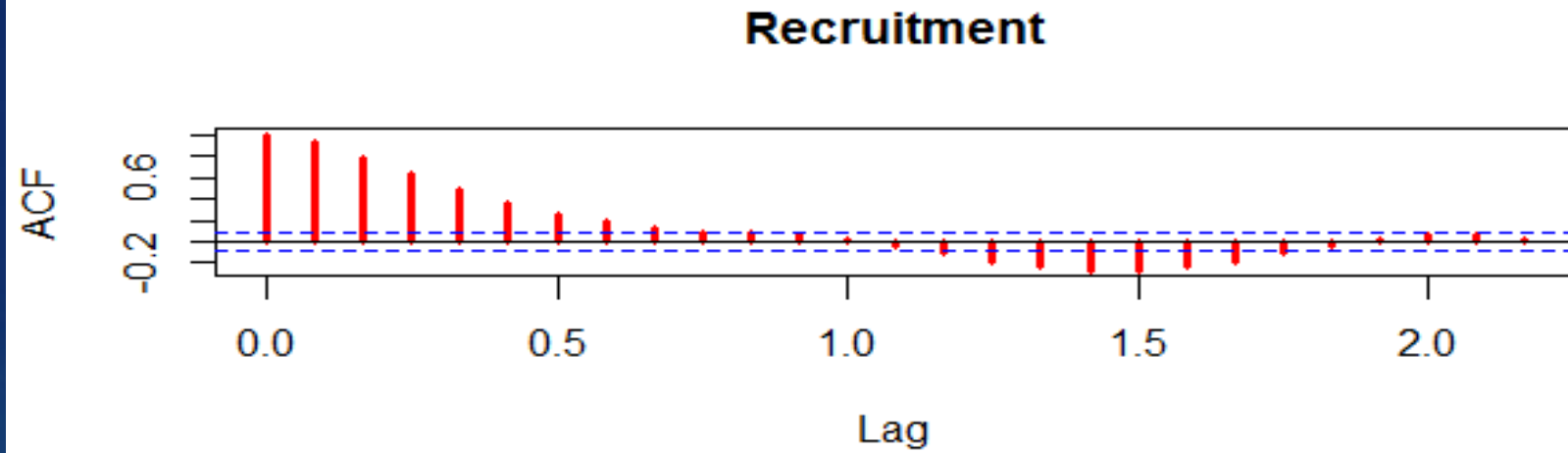
Shumway, R.H. and Stoffer, D.S. (2000)
Time Series Analysis and its Applications
With R examples
Third Edition
Springer



Recruitment time series



ACF and PACF



The parsimony principle

- ▶ Choose 'simplest explanation that fits the evidence'
- ▶ Simplest of competing theories is to be preferred
- ▶ PACF \Rightarrow AR(2)
- ▶ Yule-Walker equations in matrix form

Code

- ▶ `ar.process=rec-mean(rec)`

$$X_t - \mu$$

- ▶ `p=2`
- ▶ Yule-Walker equations: $\hat{R}\hat{\phi} = \hat{b}$
- ▶ Sample autocorrelation coefficients, vector r
for(i in 1:p+1){
 `r[i-1]<-acf(ar.process, plot=F)$acf[i]`
}

Matrix \hat{R}

$$\begin{bmatrix} 1 & r_1 & r_2 & \cdots & r_{p-1} \\ r_1 & 1 & r_1 & \cdots & r_{p-2} \\ r_2 & r_1 & 1 & \cdots & r_{p-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{p-2} & r_{p-3} & r_{p-4} & \cdots & r_1 \\ r_{p-1} & r_{p-2} & r_{p-3} & \cdots & 1 \end{bmatrix}$$

Realize

$$\hat{R}(i, j) = \hat{R}_{ij} = r_{|i-j|}$$



► `R=matrix(1,p,p)` # matrix of dimension p by p, with entries all 1's.

► `for(i in 1:p){`
 `for(j in 1:p){`
 `if(i!=j)`
 `R[i,j]=r[abs(i-j)]`
 `}`
 `}`

- ▶ # b-column vector on the right

```
b=matrix(,p,1)# b- column vector with no entries
for(i in 1:p){
  b[i,1]=r[i]
}
```

- ▶ # solve(R,b) solves $Rx=b$, and gives $x=R^{-1}b$ vector

```
phi.hat=NULL
for(i in 1:p){
  phi.hat[i]=solve(R,b)[i,1]
}
```


Model

$$X_t - \bar{x} = \hat{\phi}_1(X_{t-1} - \bar{x}) + \hat{\phi}_2(X_{t-2} - \bar{x}) + \cdots + \hat{\phi}_p(X_{t-p} - \bar{x}) + Z_t$$

Thus

$$X_t = \hat{\phi}_0 + \hat{\phi}_1 X_{t-1} + \hat{\phi}_2 X_{t-2} + \cdots + \hat{\phi}_p X_{t-p} + Z_t$$

where

$$\hat{\phi}_0 = \bar{x} \left(1 - \sum_{i=1}^p \hat{\phi}_i \right)$$

$$p = 2$$

Fitted model is

$$X_t = 7.033036 + 1.331587 X_{t-1} - 0.4445447 X_{t-2} + Z_t$$

$$Z_t \sim \text{Normal} (0, 94.17131)$$

What We've Learned

- ▶ Fitting an AR($p=2$) model to Recruitment (number of new fish) from 'astsa' package using Yule-Walker equations in matrix form

Parameter estimation: Johnson&Johnson (AR attempt)

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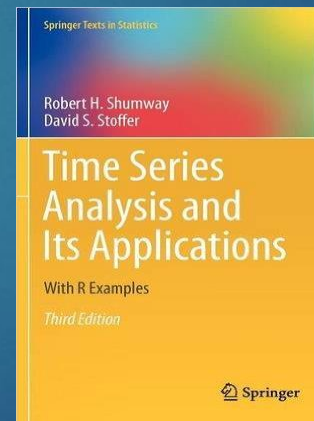
Objectives

- ▶ To fit an AR(p) model to Quarterly earnings (dollars) per Johnson & Johnson share 1960–80.
- ▶ Use Yule-Walker equations in matrix form to estimate parameters of the fitted model

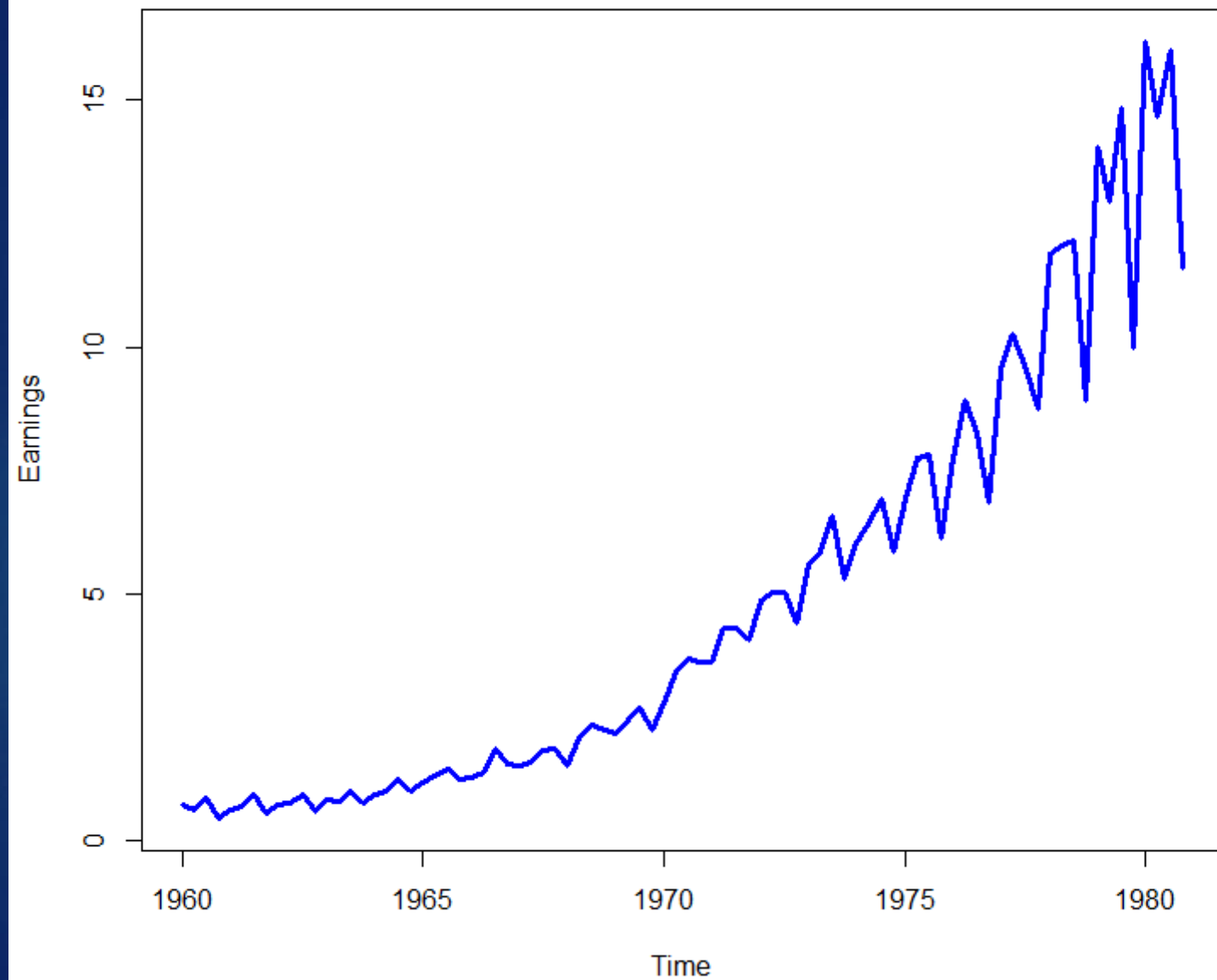
JohnsonJohnson {datasets}

- ▶ Quarterly earnings (dollars) per Johnson & Johnson share 1960–80.
- ▶ Quarterly time series
- ▶ Source: “astsa” package

Shumway, R.H. and Stoffer, D.S. (2000)
Time Series Analysis and its Applications
With R examples
Third Edition
Springer



Quarterly Earnings per Johnson&Johnson share (Dollars)



Transformation

Log-return a time series $\{X_t\}$

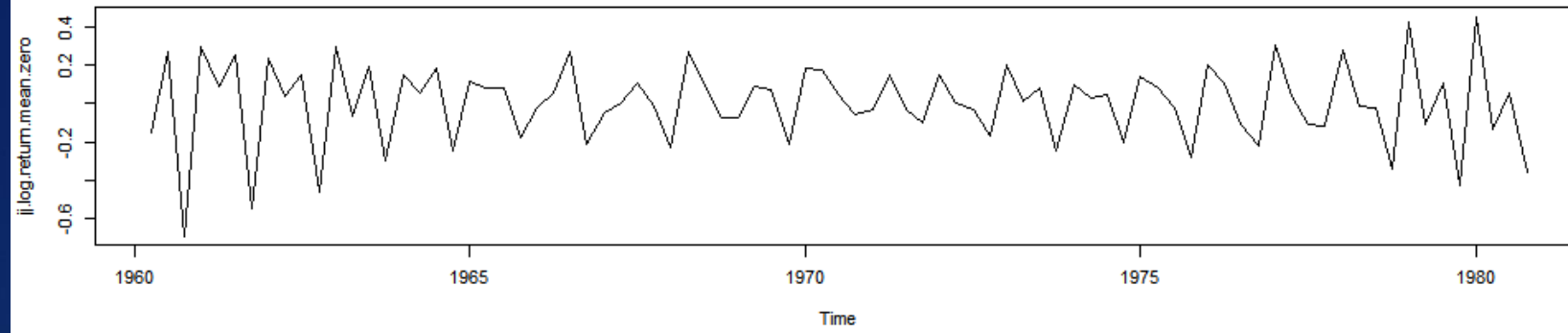
is defined as

$$r_t = \log\left(\frac{X_t}{X_{t-1}}\right) = \log(X_t) - \log(X_{t-1})$$

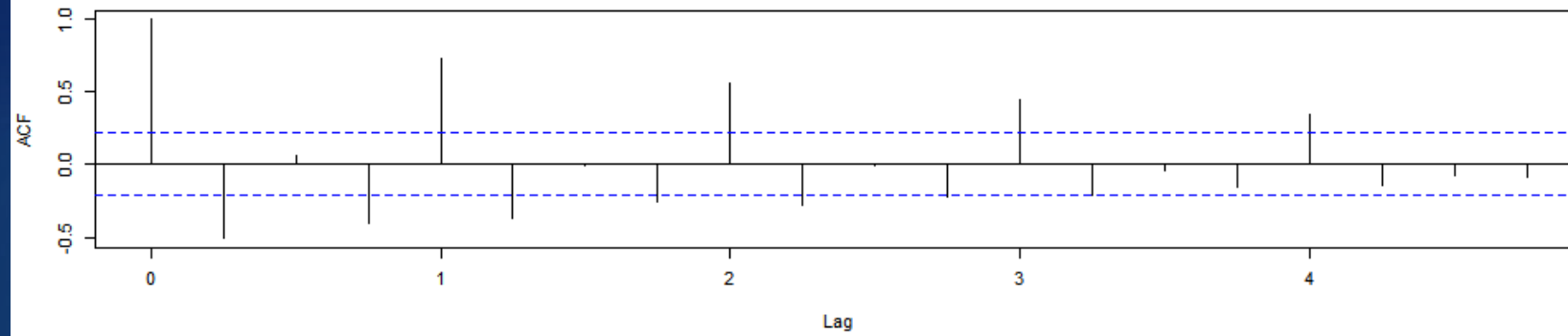
In R,

diff(log())

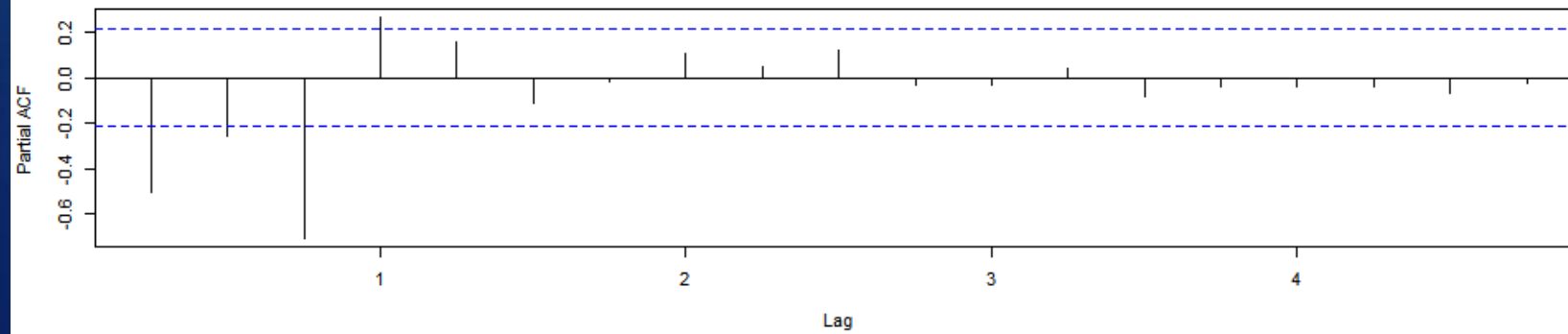
Log-return (mean zero) of Johnson&Johnsn earnings per share



ACF



PACF



The parsimony principle

- ▶ Choose 'simplest explanation that fits the evidence'
- ▶ Simplest of competing theories is to be preferred
- ▶ PACF \Rightarrow AR(4)
- ▶ Yule-Walker equations in matrix form

$$p = 4$$

Fitted model is

$$r_t = 0.079781 - 0.6293492 r_{t-1} - 0.5171526 r_{t-2} - 0.4883374 r_{t-3} + 0.2651266 r_{t-4} + Z_t$$

$$Z_t \sim \text{Normal}(0, 0.01419242)$$

where

$$r_t = \log\left(\frac{X_t}{X_{t-1}}\right)$$

What We've Learned

- ▶ Fitting an $AR(p=4)$ model to log-return of Johnson & Johnson quarterly earnings from 'astsa' package using Yule-Walker equations in matrix form