

Stationarity: Properties and Examples

✚ For a Weakly Stationary Process, $-1 \leq \rho(t) \leq 1$ 

This is, of course, perfectly analogous to the property that $-1 \leq \rho \leq 1$ from elementary statistics. If you have had a linear algebra course, this may feel familiar (that is, if you showed that $|x^T y| \leq \|x\|_2 \|y\|_2$). We know variances are non-negative, so set up a linear combination

$$V[a X_1 + b X_2] \geq 0$$

In particular, in the spirit of autocorrelation, set up for lag spacing τ

$$V[a X(t) + b X(t + \tau)] \geq 0$$

Your probability teacher probably told you (time and time again) that

$$V[X + Y] = V[X] + V[Y] + 2 \operatorname{cov}(X, Y)$$

As well as

$$V[aX] = a^2 V[X]$$

So, immediately,

$$V[a X(t) + b X(t + \tau)] = a^2 V[X(t)] + b^2 V[X(t + \tau)] + 2ab \operatorname{cov}(X(t), X(t + \tau)) \geq 0$$

We are assuming weak stationarity, so replace variance operators with a notation which suggests constants

$$a^2 \sigma^2 + b^2 \sigma^2 + 2ab \operatorname{cov}(X(t), X(t + \tau)) \geq 0$$

Two special cases: (1) Let $a = b = 1$

$$2 \sigma^2 \geq -2 \operatorname{cov}(X(t), X(t + \tau)), \quad \sigma^2 \geq -\operatorname{cov}(X(t), X(t + \tau))$$

$$1 \geq -\frac{\operatorname{cov}(X(t), X(t + \tau))}{\sigma^2} = -\frac{\gamma(\tau)}{\gamma(0)} = -\rho(\tau)$$

This gives us

$$\rho(\tau) \geq -1$$

(2) Let $a = 1, b = -1$

It's your turn- take a moment to show

$$\rho(\tau) \leq 1$$

We have already seen a few simple models: noise, random walks, and moving averages. Can we now show that some of our simple models are, in fact, weakly stationary?

Examples

+ White Noise

Is it obvious to you that Gaussian white noise is weakly stationary? Consider a discrete family of independent, identically distributed normal random variables

$$X_t \stackrel{iid}{\sim} N(\mu, \sigma)$$

The mean function $\mu(t)$ is obviously constant, so look at

$$\gamma(t_1, t_2) = \begin{cases} 0 & t_1 \neq t_2 \\ \sigma^2 & t_1 = t_2 \end{cases}$$

And

$$\rho(t_1, t_2) = \begin{cases} 0 & t_1 \neq t_2 \\ 1 & t_1 = t_2 \end{cases}$$

~~We are evidently weakly stationary, and could even show strict stationarity if we wanted to.~~

+ Random Walks

Simple random walks are obviously **not** stationary. Think of a walk with N steps built off of IID Z_t where $E[Z_t] = \mu$, $V[Z_t] = \sigma^2$. We would create

$$\begin{aligned} X_1 &= Z_1 \\ X_2 &= X_1 + Z_2 \\ X_3 &= X_2 + Z_3 = X_1 + X_2 + X_3 \\ &\vdots \\ X_t &= X_{t-1} + Z_t = \sum_{i=1}^t Z_i \end{aligned}$$

For the mean, using the idea that “the mean of the sum is the sum of the means”:

$$E[X_t] = E\left[\sum_{i=1}^t Z_i\right] = \sum_{i=1}^t E[Z_i] = t \cdot \mu$$

For the variance, using the idea that “the variance of the sum is the sum of the variances when the random variables are independent”:

$$V[X_t] = V\left[\sum_{i=1}^t Z_i\right] = \sum_{i=1}^t V[Z_i] = t \cdot \sigma^2$$

(Independent random variables have variances which add. All random variables have means which add).

Even if $\mu = 0$ the variances will still increase along the time series.

✚ Moving Average Processes, $MA(q)$

A moving average process will create a new set of random variables from an old set, just like the random walk does, but now we build them as, for IID Z_t with $E[Z_t] = 0$ and $V[Z_t] = \sigma_Z^2$

$$MA(q) \text{ process: } X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}$$

The parameter q tells us how far back to look along the white noise sequence for our average. Since the Z_t are independent, we immediately have (using the usual linear operator results)

$$E[X_t] = \beta_0 E[Z_t] + \beta_1 E[Z_{t-1}] + \cdots + \beta_q E[Z_{t-q}] = 0$$

$$V[X_t] = \beta_0^2 V[Z_t] + \beta_1^2 V[Z_{t-1}] + \cdots + \beta_q^2 V[Z_{t-q}] = \sigma_Z^2 \sum_{i=0}^q \beta_i^2$$

The autocovariance isn't all that hard to find either. Consider random variables k steps apart and set up their covariance.

$$\begin{aligned} cov[X_t, X_{t+k}] &= cov[\beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}, \\ &\quad \beta_0 Z_{t+k} + \beta_1 Z_{t+k-1} + \cdots + \beta_q Z_{t+k-q}] \end{aligned}$$

This is a little tricky, but please stay focused. There are two numbers to keep track of, the lag spacing k and the support of the MA process, q .

Now

$$cov[X_t, X_{t+k}] = E[X_t \cdot X_{t+k}] - E[X_t]E[X_{t+k}] = E[X_t \cdot X_{t+k}]$$

Since $E[X_t] = 0$ we really just need

$$E[X_t \cdot X_{t+k}] = E[(\beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}) \cdot (\beta_0 Z_{t+k} + \beta_1 Z_{t+k-1} + \cdots + \beta_q Z_{t+k-q})]$$

We can rely on matrix results concerning linear combinations of random variables or just work directly. The patient among us will write out

$$\begin{aligned} E[X_t \cdot X_{t+k}] &= \beta_0 \beta_0 E[Z_t Z_{t+k}] + \beta_0 \beta_1 E[Z_t Z_{t+k-1}] + \cdots + \beta_0 \beta_q E[Z_t Z_{t+k-q}] \\ &\quad + \beta_1 \beta_0 E[Z_{t-1} Z_{t+k}] + \beta_1 \beta_1 E[Z_{t-1} Z_{t+k-1}] + \cdots + \beta_1 \beta_q E[Z_{t-1} Z_{t+k-q}] \\ &\quad + \cdots + \\ &\quad \beta_q \beta_0 E[Z_{t-q} Z_{t+k}] + \beta_q \beta_1 E[Z_{t-q} Z_{t+k-1}] + \cdots + \beta_q \beta_q E[Z_{t-q} Z_{t+k-q}] \end{aligned}$$

The key to simplifying this is to notice that, since the Z_t are independent, we can say that the expectation of the product is the product of the expectations and so we have

$$E[Z_i \cdot Z_j] = E[Z_i]E[Z_j] = \begin{cases} 0 & i \neq j \\ \sigma_Z^2 & i = j \end{cases}$$

When the lag spacing k is greater than the order of the process q then the subscripts can never be the same (there is no overlap on the underlying Z_t 's) and we have $cov[X_t, X_{t+k}] = 0$. When the lag spacing is small enough to have contributions, that is if $q - k \geq 0$, you can visualize the sum like this (we just need to keep track of the β 's):

$$\begin{array}{cccccccccccc} Z_{t-q} & Z_{t-q+1} & \cdots & Z_{t-(q-k)} & \cdots & Z_{t-1} & Z_t & \cdots & & & & \\ \beta_q & \beta_{q-1} & & \beta_{q-k} & & \beta_1 & \beta_0 & & 0 & 0 & & \\ & & & & & & & & & & & \\ & & & Z_{t+k-q} & \cdots & Z_{t+k-k+1} & Z_{t+k-k} & \cdots & Z_{t+k-1} & Z_{t+k} & & \\ 0 & 0 & 0 & \beta_q & & \beta_{k+1} & \beta_k & & \beta_1 & \beta_0 & & \end{array}$$

This should make clear that, when $k \leq q$

$$E[X_t, X_{t+k}] = \sigma_Z^2 \cdot \sum_{i=0}^{q-k} \beta_i \beta_{i+k} \quad (\text{no } t \text{ dependence})$$

Summing up, then, we have found that

$$\gamma(t_1, t_2) = \gamma(k) = \begin{cases} 0 & k > q \\ \sigma_Z^2 \cdot \sum_{i=0}^{q-k} \beta_i \beta_{i+k} & k \leq q \end{cases}$$

We know that the mean function is constant, in fact $\mu(t) = 0$ and the autocovariance function has no t dependence, so we conclude that the $MA(q)$ process is (weakly) stationary.

Let's finish this lecture by finding the autocorrelation function. In general

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$$

Obviously, then $\rho(0) = 1$. It is easy to see that

$$\gamma(0) = \sigma_Z^2 \cdot \sum_{i=0}^q \beta_i \beta_i = \sigma_Z^2 \cdot \sum_{i=0}^q \beta_i^2$$

Finally

$$\rho(k) = \frac{\sum_{i=0}^{q-k} \beta_i \beta_{i+k}}{\sum_{i=0}^q \beta_i^2}$$

In the next lecture we will simulate an $MA(q)$ process and validate these results numerically.