

This is the seventh homework assignment. Students should tick in [TUWEL](#) problems they have solved and upload their detailed solutions by **20:00 on Monday November 27, 2023**.

1. **z-test (without R)**

In the context of a two-sided z -test let $\bar{x} = \frac{4}{3}$, $\sigma = 1$, $n = 9$ and $H_0 : \mu = \frac{2}{3}$. The α -quantiles q_α of $\mathcal{N}(0, 1)$ are as follows

α	0.005	0.010	0.015	0.020	0.025	0.030	0.035	0.040
q_α	-2.576	-2.326	-2.170	-2.054	-1.960	-1.881	-1.812	-1.751

How do you decide for a significance level of 1%, (b) 3%, (c) 6% and (d) 10% by only using the information above? Justify your answers.

2. **Interpretation of test results**

In the context of a statistical test at significance level α , the test statistic lies in the rejection region. Comment on the following statements.

- (a) The null hypothesis is rejected at the α -level
- (b) The null hypothesis is rejected at the $\alpha/2$ -level.
- (c) The null hypothesis is rejected at the 2α -level.
- (d) The p -value was at least α .
- (e) The null hypothesis was significant.
- (f) The null hypothesis is not true.
- (g) The null hypothesis is probably not true.

3. **Test power in the z -test**

Let X_1, \dots, X_{25} be i.i.d. random variables with $X_1 \sim \mathcal{N}(2, 4)$, and $H_0 : \mu = 1$.

- (a) Compute the test power of the left-sided z -test. Express it through the cumulative distribution function of the standard normal distribution, depending on the significance level α .
- (b) Comment on the impact of α on the test power.

4. Type I error and Type II error

Let X_1, \dots, X_{16} be i.i.d. random variables with $X_1 \sim \mathcal{N}(0, 4)$. Assume that for a realization it holds $\bar{x} = 4$. In the context of a right-sided z -test, let $H_0 : \mu = 2$ and the rejection area $R = [3, +\infty)$. Which of the following statements are correct?

- (a) We will commit a Type I error
- (b) We will commit a Type II error
- (c) We will not commit a Type II error.
- (d) If we increase the significance level of the test, then we obtain a higher test power.

Hint: The expectation of X_1 is fixed at zero. Is the null hypothesis true?

Remark: Note that this is a theoretical consideration of the z -test. Here, the distribution of X_1 is fixed (as the expectation is zero), while usually we assume the expectation (resp. *the population*) to be unknown. In this theoretical consideration we know whether the null hypothesis is true and we are thus able to make statements about the errors. From a practical point of view, the setup is not useful, because the reason to perform a statistical test is the fact that the population of interest is unknown.

5. Law of large numbers

Visualize the law of the large numbers assuming exponential distributed random variables. Let X_1, X_2, \dots be i.i.d. random variables, with $X_1 \sim \exp(\lambda)$.

For $m = 1, 2, \dots$ the mean of the first m random variables is given as

$$\bar{X}_m = \frac{1}{m} \sum_{i=1}^m X_i, \quad m \in \mathbb{N}.$$

Further, for $n \in \mathbb{N}$ a sequence of means is given as $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n$. Assume $\lambda = 0.5$ and $n = 200$ and plot 20 realized sequences of means in one graph. Mark the expectation of X_1 . Comment on the obtained result.