$C_b^1([a,b]) = \{ f \in C([a,b]) : \exists f^{(1)} \in C_b([a,b]) \}$

With $\|f\|_\infty = \sup_{x \in [a,b]} |f|$ This is a Banach Space

Def Bananch Space: X is a Banach Space with norm $\|\cdot\|$ if the norm introduces a metric $d(x,y) = \|x-y\|$ s.t X is complete w.r.t d

i.e if x_n is a Cauchy Sequence in X, then $\|x_n - x\| \to 0$, $x \in X$

Hilbert Space: Special Case of Banarch where norm is inner product

A Seperable Banarch Space: $(X, \|\cdot\|)$ s.t we have Countable/finite $D \subset \underset{\text{Vector space}}{X}$ s.t $x_j \in D$ is the limit of a Cauchy Sequence in X $\{x_1, \dots, x_n\} \subseteq X$

Operator: A map btwn Sets of fcts

$\frac{\partial}{\partial x} : C_b^1([a,b]) \longrightarrow C_b([a,b])$ $\quad G : \underset{\text{fct Space}}{P} \to \underset{\text{fct Space}}{V}$

Matrix isomorphic to linear operator

Neural Operator: Mapping from some parametric functional dependence into our Solution Space

- A model that takes a fct as input & returns a fct as output.

Learning Procedure: Let $D \subseteq \mathbb{R}^d$ be bounded & open, and $\mathcal{A} = \mathcal{A}(D; \mathbb{R}^{d_a})$, $\mathcal{U}(D; \mathbb{R}^{d_u})$ are Seperable Banarch Spaces Containing fct's taking values in $\mathbb{R}^{d_a}$ & $\mathbb{R}^{d_u}$ respectively

Now let $G^\dagger : \mathcal{A} \to \mathcal{U}$ be a possibly nonlinear map

Now, Suppose we have observations $\{a_j, u_j\}_{j=1}^N$ $a_j \in \mathcal{A}$ & $u_j \in \mathcal{U}$, where $a_j \sim \mu$ for some prob measure μ is a Sequence of a IID r.v's from μ supported on $\mathcal{A}$, and $u_j = G^\dagger(a_j)$.

Goal: Construct an approx of $G^\dagger$ by fitting some $G_\theta : \mathcal{A} \to \mathcal{U} \longleftrightarrow G : \mathcal{A} \times \Theta \to \mathcal{U}$

$G : \mathcal{A} \times \Theta \to \mathcal{U}$
$G_\theta : \mathcal{A} \longrightarrow \mathcal{U}$

Choosing $\theta^\dagger \in \Theta$ s.t $G(\cdot, \theta^\dagger) \approx G^\dagger$

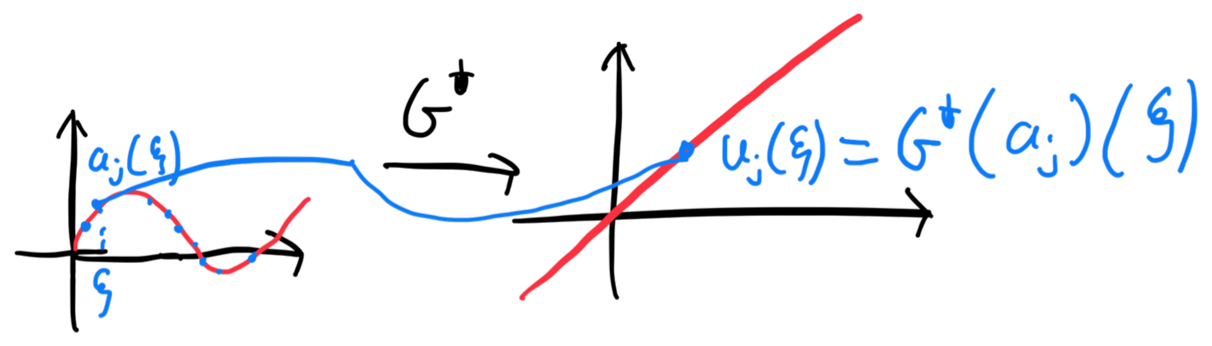
So, let us construct a minimization problem. Define Some cost functional $C : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$

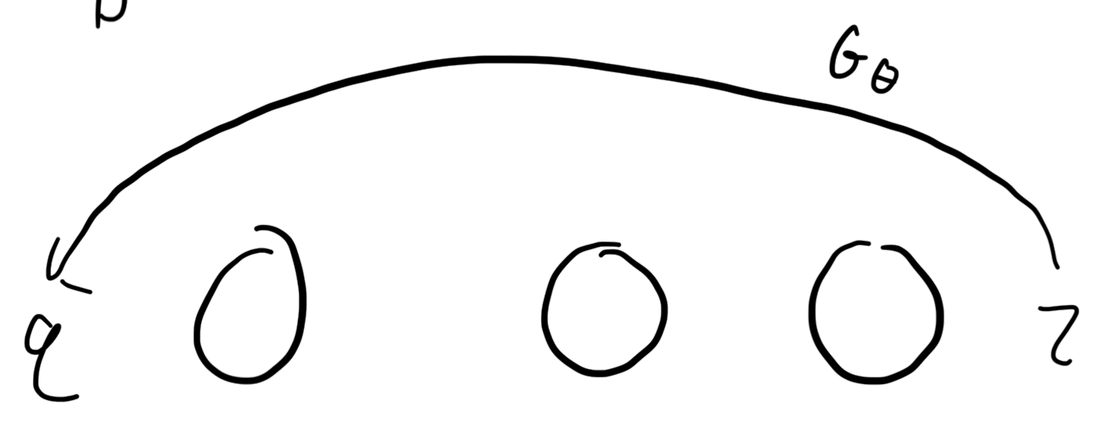
Thus, $\theta^\dagger = \underset{\theta \in \Theta}{\arg\max}\ \mathbb{E}_{a \sim \mu}\left[C(G(a,\theta), G^\dagger(a)) \right]$

Discretization: We model our data as pointwise evaluations of a_j & u_j

Ksi

(x_j, y_j)
$(a_j(\xi), u_j(\xi))$
D





Diffusion Model

Discretization: We model our data as pointwise evaluations of a_j & u_j.

Assume $D_j = \{x_1, \dots, x_n\} \subseteq D$ pointwise discritization of D, and are observations $a_j|_{D_j} \in \mathbb{R}^{n \times d_a}$, $u_j|_{D_j} \in \mathbb{R}^{n \times d_u}$

$$G_{\theta^\dagger}(a(x_j)) \approx G^\dagger(a(x_j)) \quad \forall j$$

We can use this setup to fit $G_{\theta^\dagger}$ s.t

$$G_{\theta^\dagger}(a(x)) \approx G^\dagger(a(x)) \quad \text{for } x \in D \setminus D_j$$