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HOMEWORK 3 – Question 3 and 4 in the attached "Homework.py"

Question 1:

(a)

$$Let\ x\ \in\ R^n\ given\ as\ x = (x_1,\ x_2, \dots, x_n)$$

$$\|x\|_\infty = \max_{1\le i\le n}\{|x_i|\} \le \sum_{i=1}^{n} |x_i| = \|x\|_1$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \le \sum_{i=1}^{n} \max_{1\le i\le 0}\{|x_i|\} = n \times \max_{1\le i\le n}\{|x_i|\} = n\|x\|_\infty$$

Again

$$\|x\|_\infty = \max_{1\le i\le n}\{|x_i|\}$$

$$= \left(\left(\max_{1\le i\le n}\{|x_i|\}\right)^2\right)^{\frac{1}{2}}$$

$$\le \left({x_1}^2 + {x_2}^2 + \cdots + \left(\max_{1\le i\le n}\{|x_i|\}\right)^2 + \cdots + {x_n}^2\right)^{\frac{1}{2}}$$

$$= \|x\|_2$$

$$\|x\|_2 = ({x_1}^2 + \cdots + {x_n}^2)^{\frac{1}{2}}$$

$$\le \left[\sum_{i=1}^{n} \left(\max_{1\le i\le n}\{|x_i|\}\right)^2\right]^{\frac{1}{2}}$$

$$= \left[\sum_{i=1}^{n} (\|x\|_\infty)^2\right]^{\frac{1}{2}}$$

$$= (n(\|x\|_\infty)^2)^{\frac{1}{2}}$$

$$= \sqrt{n}\ \|x\|_\infty$$

(b)

$$let\ c_1, c_2\ be\ positive\ costant\ such\ that$$

$$c_1\ \|x\|_a \le \|x\|_b \le c_2\ \|x\|_a \qquad \forall\ x\ \in R^n$$

$$let\ M \in R^{n\times n}.\ Then\ Mx \in R^n \quad \forall\, x \in R^n$$

$$\therefore \quad c_1 \|Mx\|_a \le \|Mx\|_b \le c_2 \|Mx\|_a \quad \forall\, x \in R^n$$

$$Taking\ supremun\ over\ R^n\ of\ above\ inequality, we\ gets$$

$$Sup\,\{c_1 \|Mx\|_a \mid x \in R^n, \|x\|_a = 1\} \le Sup\,\{\|Mx\|_b \mid x \in R^n, \|x\|_b = 1\}$$

$$\le Sup\,\{c_2 \|Mx\|_a \mid \|x\|_a = 1, x \in R^n\}$$

$$\therefore \quad c_1\, Sup\,\{\|Mx\|_a \mid x \in R^n, \|x\|_a = 1\} \le \|M\|_b \le c_2\, Sup\,\{\|Mx\|_a \mid x \in R^n, \|x\|_a = 1\}$$

$$\because c_1, c_2 > 0$$

$$\therefore\ c_1 \|M\|_a \le \|M\|_b \le c_2 \|M\|_a$$

Question 2:

Given,

$$A = \begin{bmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix}$$

(a)

$$Determinant\ for\ a\ matrix \begin{bmatrix} a & b \\ c & d \end{bmatrix} is\ given\ by\ ad - bc$$

$$\therefore Determinant\ for\ A\ or\ \det(A)\ or\ |A|$$

$$|A| = [\,1.1 - (1+\varepsilon)(1-\varepsilon)\,]$$

$$= 1 - [\,1^2 - \varepsilon^2\,]$$

$$= 1 - 1 + \varepsilon^2$$

$$|A| = \varepsilon^2$$

(b)

$$For\ determenant\ |A|\ to\ be\ zero, the\ value\ of\ \varepsilon\ should\ be\ 0$$

$$|A| = 0$$

$$\varepsilon^2 = 0$$

$$\varepsilon = 0$$

(c)

$$LU\ factorization\ of\ A$$

$$A = LU$$

$$For\ lower\ triangular\ matrix\ L, the\ enties\ of\ the\ diagonal\ should\ be\ 1, the\ entries$$

$above\ the\ diagonals\ should\ be\ 0, and\ the\ entries\ below\ the\ diagonals\ can\ be$

$anything$

$\therefore\ the\ 2 \times 2\ L\ can\ be\ represented\ as$

$$L = \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix}$$

$For\ upper\ triangular\ matrix\ U, the\ entries\ of\ the\ diagonals\ can\ be\ anything, the\ entries$

$above\ the\ diagonal\ can\ be\ anything\ and\ entries\ below\ the\ diagonal\ should\ be\ 0.$

$\therefore\ the\ 2 \times 2\ U\ can\ be\ represented\ as$

$$U = \begin{bmatrix} & \\ 0 & \end{bmatrix}$$

$To\ find\ the\ in\ matrix\ U, we\ multiply\ Elementory\ matrix (E)\ with\ the\ given\ matrix\ A$

$$E \times A = U$$

$The\ Elementry\ matrix\ E\ is\ represented\ as$

$$E = \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix} = \begin{bmatrix} & \\ 0 & \end{bmatrix}$$

$Now. let\ us\ assume\ blank\ entity\ in\ E\ as\ x\ and\ perform\ matrix\ multiplication\ of$

$second\ row\ of\ E\ and\ first\ column\ of\ A\ which\ is\ equal\ to\ 0.$

$$x \times 1 + 1 \times (1-\varepsilon) = 0$$

$$x + (1-\varepsilon) = 0$$

$$x = -(1-\varepsilon)$$

$$\therefore\ E = \begin{bmatrix} 1 & 0 \\ -(1-\varepsilon) & 1 \end{bmatrix}$$

$$now, E \times A = U$$

$$\begin{bmatrix} 1 & 0 \\ -(1-\varepsilon) & 1 \end{bmatrix} \begin{bmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot (1-\varepsilon) & 1 \cdot (1-\varepsilon) + 0 \cdot 1 \\ -(1-\varepsilon) + (1-\varepsilon) & -(1-\varepsilon)(1-\varepsilon) + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+\varepsilon \\ 0 & -(1-\varepsilon^2) + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+\varepsilon \\ 0 & -1+\varepsilon^2+1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1+\varepsilon \\ 0 & \varepsilon^2 \end{bmatrix}$$

$To\ find\ the\ value\ of\ matirx\ \ L$

$$E \times A = U \implies A = E^{-1} \times U$$

$$A = L \times U$$

$$\therefore \quad L = E^{-1}$$

$$\therefore L = \begin{bmatrix} 1 & 0 \\ -(1-\varepsilon) & 1 \end{bmatrix}^{-1}$$

$$L = \begin{bmatrix} 1 & 0 \\ 1-\varepsilon & 1 \end{bmatrix}$$

(d)

$$For\ U\ to\ be\ singular, its\ determenat\ |U| = 0$$

$$|U| = 1 \cdot \varepsilon^2 - 0 \cdot (1+\varepsilon)$$

$$|U| = \varepsilon^2$$

$$To\ satisfy\ the\ condition\ |U| = 0$$

$$\varepsilon^2 = 0$$

$$\varepsilon = 0$$

$$The\ value\ of\ \varepsilon = 0\ for\ U\ to\ be\ singular.$$