B.V.N.S. Aditya
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1) $X \subseteq \Phi$, $\alpha \in \Phi$

To Prove: $X \models \alpha$ iff $X \vdash \alpha$

($\Rightarrow$) If $X \models \alpha$ then $X \vdash \alpha$

From compactness theorem, $\exists\, Y \subseteq_{fin} X$ such that $Y \models \alpha$

Let $Y = \{\beta_1, \beta_2, \ldots, \beta_n\}$

So, $(\beta_1 \to (\beta_2 \to (\ldots (\beta_n \to \alpha) \ldots)))$ is valid

From completeness theorem we have

$\vdash (\beta_1 \to (\beta_2 \to \ldots (\beta_n \to \alpha) \ldots)$

Now, we apply deduction theorem 'n' times, we get

$\{\beta_1, \beta_2, \ldots \beta_n\} \vdash \alpha$

$\Rightarrow Y \vdash \alpha$

$\Rightarrow X \models \alpha$

($\Leftarrow$) If $X \vdash \alpha$, then $X \models \alpha$.

Let $Y$ be the set of formulas of $X$ used in deriving $\alpha$.

So, $Y \vdash \alpha$, $Y \subseteq_{fin} X$ ( $Y$ always exists because derivation length of $\alpha$ from $X$ is finite)

~~From soundness,~~ $Y \models \alpha$, so $X \models \alpha$

~~So, if $X \vdash \alpha$~~ then $X \models \alpha$

~~Hence, $X \models \alpha$ iff $X \vdash \alpha$~~

Let $Y = \{\beta_1, \beta_2, \ldots \beta_n\}$. So $\{\beta_1, \beta_2, \ldots \beta_n\} \vdash \alpha$

Using deduction theorem, $\vdash (\beta_1 \to (\beta_2 \to \ldots (\beta_n \to \alpha) \ldots)$

From soundness, we have $\models (\beta_1 \to (\beta_2 \to \ldots (\beta_n \to \alpha) \ldots)$

Which implies $\{\beta_1, \beta_2, \ldots \beta_n\} \models \alpha$

So, $Y \models \alpha$, So $X \models \alpha$

So, if $X \vdash \alpha$ then $X \models \alpha$

Hence, $X \models \alpha$ iff $X \vdash \alpha$.

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2. Let $X$ be a set of formulas. $X$ is FSS (Finitely Satisfiable Set) if every $Y \subseteq_{fin} X$ is satisfiable.

(a) let $X$ be an arbitary FSS. Let $\alpha_0, \alpha_1, \dots,$ ~~$\alpha$~~ be an enumeration of $\phi$

We define an infinite sequence of sets $X_0, X_1, X_2 \dots$ as below.

$X_0 = X$

for $i \geq 0$, $X_{i+1} = \begin{cases} X_i \cup \{\alpha_i\} & \text{if } X_i \cup \{\alpha_i\} \text{ is FSS} \\ X_i & \text{otherwise} \end{cases}$

From construction, $X_0 \subseteq X_1 \subseteq X_2 \dots$ and each $X_i$ is FSS.

Let us define $Y = \bigcup_{i \geq 0} X_i$

Claim: $Y$ is maximal FSS

1) To prove: $Y$ is FSS.
Let's assume that $Y$ is not FSS. So, $\exists Z \subseteq_{fin} Y$ such that $Z$ is not satisfiable. let $Z = \{\beta_1, \beta_2, \dots \beta_n\}$ which can be written as $\{\alpha_{i_1}, \alpha_{i_2}, \dots \alpha_{i_n}\}$. Indices here correspond to our enumeration in $\phi$. So, $Z \subseteq_{fin} X_{K+1}$ where $K = \max(i_1, i_2, \dots i_n)$. So, ~~$z$~~ as $Z$ is not satisfiable, $X_{K+1}$ is not a FSS. This is a contradiction. So, $Y$ is FSS.

2) To prove: $Y$ is maximal
Let's assume that $Y$ is not maximal. So, $\exists \alpha_i \in \phi$ such that $Y \cup \{\alpha_i\}$ is FSS. $\alpha_i$ is considered while forming $X_{i+1}$. As ~~$X_i$~~ $\alpha_i$ is not added, $X_i \cup \{\alpha_i\}$ is not FSS. So, $\exists Z \subseteq X_i$ such that $Z \cup \{\alpha_i\}$ is not satisfiable. We know that $X_i \subseteq Y$, so $Z \subseteq Y$. So, $Y \cup \{\alpha_i\}$ can't be FSS because $Z \cup \{\alpha_i\}$ is not satisfiable. This is a contradiction. So, $Y$ is maximal.

Hence, $Y$ is maximal FSS.
So, every FSS can be extended to a maximal FSS.

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(b) $X$ is a maximal FSS.

1) We prove that ~~$\{\alpha, \neg\alpha\}$~~ both $\alpha$, $\neg\alpha$ don't belong to $X$.
lets assume that both $\alpha$, $\neg\alpha$ belong to $X$.
$\{\alpha, \neg\alpha\} \subseteq X$. ~~So~~, $\{\alpha, \neg\alpha\}$ is not satisfiable because
$\neg(\alpha \wedge \neg\alpha) = \alpha \vee \neg\alpha$ is valid. So, $X$ is not FSS. This is a contradiction. So, ~~to~~ $\{\alpha, \neg\alpha\} \not\subseteq X$

Now, we show that atleast one of $\alpha$, $\neg\alpha$ is in $X$.
lets assume that both $\alpha$, $\neg\alpha$ don't belong to $X$. So,
there exists sets $B_1 \subseteq_{fin} X$, $B_2 \subseteq_{fin} X$ such that
$B_1 \cup \{\alpha\}$ is not satisfiable, $B_2 \cup \{\neg\alpha\}$ is not satisfiable

let $B_1 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ let $\hat{\beta} = \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_n$
$B_2 = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ let $\hat{\gamma} = \gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \gamma_m$

So, $\neg(\hat{\beta} \wedge \alpha)$ is ~~not~~ valid $\Rightarrow$ $\neg\hat{\beta} \vee \neg\alpha$ is valid —① 
$\neg(\hat{\gamma} \wedge \neg\alpha)$ is valid $\Rightarrow$ $\neg\hat{\gamma} \vee \alpha$ is valid —②

From ①, ② $\neg\hat{\beta} \vee \neg\alpha \vee \neg\hat{\gamma} \vee \alpha$ is valid
$\Rightarrow$ $\neg\hat{\beta} \vee \neg\hat{\gamma}$ is valid ($\because \neg\alpha \vee \alpha$ is $T$)
$\Rightarrow$ $\neg(\hat{\beta} \wedge \hat{\gamma})$ is valid
$\Rightarrow$ $B_1 \cup B_2$ is not satisfiable

$B_1 \cup B_2 \subseteq_{fin} X$.
So, ~~$B_1 \cup B_2$~~ $X$ is not FSS.
This is a contradiction.
So, atleast one of $\alpha$, $\neg\alpha$ is in $X$.

So, $\alpha \in X$ iff $\neg\alpha \notin X$.

(c) $X$ is a maximal FSS

($\Rightarrow$) If $(\alpha \vee \beta) \in X$, then $\alpha \in X$ or $\beta \in X$

Contrapositive: If $\alpha \notin X$ and $\beta \notin X$ then $\alpha \vee \beta \notin X$

$\alpha \notin X \Rightarrow \neg\alpha \in X$ (from part b)

$\beta \notin X \Rightarrow \neg\beta \in X$

Assume that $\alpha \vee \beta \in X$

Consider the set $Y = \{\neg\alpha, \neg\beta, \alpha \vee \beta\} \subseteq X$

$Y$ is not satisfiable because $\neg(\neg\alpha \wedge \neg\beta \wedge (\alpha \vee \beta))$

$= (\alpha \vee \beta \vee \neg\alpha) \wedge (\alpha \vee \beta \vee \neg\beta)$

is valid

So, this is a contradiction

$\alpha \vee \beta \notin X$

($\Leftarrow$) If $\alpha \in X$ or $\beta \in X$ then $\alpha \vee \beta \in X$

Contrapositive: If $\alpha \vee \beta \notin X$, then $\alpha \notin X$ and $\beta \notin X$

$\alpha \vee \beta \notin X \Rightarrow \neg(\alpha \vee \beta) \in X$

$\Rightarrow \neg\alpha \wedge \neg\beta \in X$

Consider the set $\{\alpha, \neg\alpha \wedge \neg\beta\} \subseteq X$

1) If $\alpha \in X$, then consider the set $\{\alpha, \neg\alpha \wedge \neg\beta\} \subseteq X$

This set is not satisfiable because $\neg(\alpha \wedge \neg\alpha \wedge \neg\beta)$ is valid

So, $\alpha \notin X$

2) If $\beta \in X$, then consider the set $\{\beta, \neg\alpha \wedge \neg\beta\} \subseteq X$

This set is not satisfiable because $\neg(\beta \wedge \neg\alpha \wedge \neg\beta)$ is valid

So, $\beta \notin X$

So, $\alpha \notin X$ and $\beta \notin X$

So, if $X$ is a maximal FSS, $(\alpha \vee \beta) \in X$ iff ($\alpha \in X$ or $\beta \in X$)

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(d) To prove: Every maximal FSS $x$ generates a $\vartheta_x$ such that for every ~~for~~ formula $\alpha$, $\vartheta_x \vDash \alpha$ iff $\alpha \in x$
Here, $\vartheta_x = \{p \in \mathcal{P} \mid p \in x\}$
Proof by induction on structure of $\alpha$

Base Case: $\alpha$ is atomic proposition. From the definition of $\vartheta_x$, $\vartheta_x \vDash \alpha$ iff $\alpha \in X$

Induction Step:

1) $\alpha$ is of form $\neg\beta$:

$\vartheta_x \vDash \neg\beta$ iff $\vartheta_x \nvDash \beta$ (Definition of valuation ~~$\vartheta_x$~~)

$\vartheta_x \vDash \beta$ iff $\beta \notin X$ (Induction Hypothesis)

$\beta \notin X$ iff $\neg\beta \in X$ (From 2(b))

So, $\vartheta_x \vDash \neg\beta$ iff $\neg\beta \in X$

2) $\alpha$ is of form $\beta \vee \gamma$:

$\vartheta_x \vDash \beta \vee \gamma$ iff $(\vartheta_x \vDash \beta$ or $\vartheta_x \vDash \gamma)$ (Definition of valuation)

$\vartheta_x \vDash \beta$ or $\vartheta_x \vDash \gamma$ iff $\beta \in X$ or $\gamma \in X$ (Induction Hypothesis)

$\beta \in X$ or $\gamma \in X$ iff $\beta \vee \gamma \in X$ (Proved in 2(c))

So, $\vartheta_x \vDash \beta \vee \gamma$ iff $\beta \vee \gamma \in X$

So, every maximal FSS $X$, ~~$\exists\, \vartheta_x$~~ generates a valuation $\vartheta_x$ such that for every formula $\alpha$, $\vartheta_x \vDash \alpha$ iff $\alpha \in X$

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(e) Given that $X$ is FSS.
From Part (a), every FSS $X$ can be extended to a maximal FSS $Y$
From Part (d), every maximal FSS $Y$ generates a valuation such that $v_Y \models \alpha$ iff $\alpha \in Y$
As $X \subseteq Y$, we can have $v_Y \models \alpha$ iff $\alpha \in Y$
So, we define $v_X = \{p \in \mathcal{P} \mid p \in Y\} = v_Y$
So, for every FSS $X$, there exists $v_X$ such that $v_X \models X$.
So, any FSS $X$ is simultaneously satisfiable

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(f) To Prove: For all $X$ and forall $\alpha$, $X \models \alpha$ iff there exists $Y \subseteq_{fin} X$ such that $Y \models \alpha$.

($\Rightarrow$) If $X \models \alpha$ then there exists $Y \subseteq_{fin} X$ such that $Y \models \alpha$

We know that if $X \models \alpha$, then $X \cup \{\neg\alpha\}$ is not satisfiable.

~~So~~, Let $X' = X \cup \{\neg\alpha\}$

As $X'$ is not satisfiable, $\exists$ a subset $Y'$ of $X'$ ($Y' \subseteq_{fin} X'$) which is not satisfiable

Let $Y = Y' \setminus \{\neg\alpha\}$

$Y \subseteq_{fin} X$. So, $Y$ should be satisfiable

~~So~~ As $Y \cup \{\neg\alpha\}$ is not satisfiable, $Y \models \alpha$.

So, $\exists\ Y \subseteq_{fin} X$ such that $Y \models \alpha$.

($\Leftarrow$) If $\exists\ Y \subseteq_{fin} X$ such that $Y \models \alpha$, then $X \models \alpha$

This is trivially true because

If $v \models X$, then $v \models Y$, then $v \models \alpha$

So, $v \models X \Rightarrow v \models \alpha$

So, $X \models \alpha$

Hence, For all $X$, for all $\alpha$, $X \models \alpha$ iff $Y \subseteq_{fin} X$ such that $Y \models \alpha$.