Assignment 5 (*Marks: 7, Due date: Nov. 15*) **COL703, Aug-Nov 2022**

You are required to submit answers of only those questions that carry marks.

Some of the question below (not to be submitted, of course) are from topics that have not been covered yet, but we will cover them in the next few lectures.

Please note that there will be zero tolerance for dishonest means like copying solutions from others, and even letting others copy your solution, in any assignment/quiz/exam. If you are found indulging in such an activity, your answer-paper/code will not be evaluated and your participation/submission will not be counted. Second-time offenders will be summarily awarded an F grade. The onus will be on the supposed offender to prove his or her innocence.

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1. Unify the following pairs of atomic formulas, if possible.
   (a) $P(a, x, f(g(y))),\ P(y, f(z), f(z))$
   (b) $P(x, g(f(a)), f(x)),\ P(f(a), y, y)$
   (c) $P(x, g(f(a)), f(x)),\ P(f(y), z, y)$
   (d) $P(a, x, f(g(y))),\ P(z, h(z, u), f(u))$
2. A substitution $\theta = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ is *idempotent* if $\theta = \theta\theta$. Let V be the set of variables occurring in the terms $\{t_1, \ldots, t_n\}$. Prove that θ is idempotent if and only if $V \cap \{x_1, \ldots, x_n\} = \emptyset$. Show that the most general unifier produced by the unification algorithm is idempotent.
3. Show that the composition of substitutions is not commutative.
4. Let S be a finite set of expressions and θ is a unifier of S. Prove that θ is an idempotent *mgu* if and only if for every unifier σ of S, $\sigma = \theta\sigma$.
5. Implement the predicate resolution technique (including the unification algorithm) so that you can automatically obtain the derivation of an empty clause for all the unsatisfiable examples discussed in the class/slides/notes.
6. Prove or disprove that the following formulas are modal tautologies.
   (a) $\Box\phi \wedge \Box(\phi \to \psi) \to \Box\psi$
   (b) $\phi \to \Diamond\phi$
   (c) $\Box\phi \to \Diamond\phi$
   (d) $\Diamond(\phi \vee \psi) \leftrightarrow (\Diamond\phi \vee \Diamond\psi)$
   (e) $\Diamond(\phi \wedge \psi) \leftrightarrow (\Diamond\phi \wedge \Diamond\psi)$
7. What classes of frames are characterized by the following formulas?
   (a) $\Diamond\alpha \to \Box\alpha$
   (b) $\Diamond\alpha \to \Diamond\Diamond\alpha$
   (c) $\alpha \to \Box\alpha$
8. Draw the ROBDDs for $(x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \wedge (x_3 \leftrightarrow y_3)$ for orderings $x_1 < x_2 < x_3 < y_1 < y_2 < y_3$ and $x_1 < y_1 < x_2 < y_2 < x_3 < y_3$. What variable ordering would you recommend for constructing a small ROBDD for $(x_1 \leftrightarrow y_1) \wedge \ldots \wedge (x_k \leftrightarrow y_k)$?

9. Consider the following boolean formulas. Compute their unique reduced OBDDs with respect to the ordering $x < y < z$.

   (a) $f(x, y) = x.y$

   (b) $f(x, y) = x + y$

   (c) $f(x, y) = x \oplus y$

   (d) $f(x, y, z) = (x \oplus y).(\overline{x} + z)$

10. Let $f(x, y, z) = y + \overline{z}.x + z.\overline{y} + y.x$ be a boolean formula. Compute f's Shannon expansion with respect to

    (a) x

    (b) y

    (c) z

11. Show that the boolean formulas f and g are semantically equivalent if, and only if, the boolean formula $(\overline{f} + g).(f + \overline{g})$ computes 1 for all possible assignments of 0s and 1s to their variables.

12. A Quantified Boolean Formula (QBF) is a formula of the form

$$\psi \equiv Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \ \phi(y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_n) \qquad \text{where}$$

$y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_n$ are propositional variables, and $Q_1, Q_2, \ldots, Q_n$ are existential or universal quantifiers and $\phi(y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_n)$ is a CNF propositional formula on the variables $y_1, y_2, \ldots, y_m, x_1, x_2, \ldots, x_n$. Note that ψ does not strictly follow the syntax of first-order logic formulae. Specifically, variables cannot be combined with propositional connectives in first-order logic; instead they must appear as terms in the arguments of predicates before propositional connectives can be used.

The QBF formula ψ is said to be satisfiable iff the first-order formula

$$\psi' \equiv Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \ \phi(P(y_1), P(y_2), \ldots, P(y_m), P(x_1), P(x_2), \ldots, P(x_n))$$

is satisfied by the structure $\mathcal{A}$, where $U_{\mathcal{A}} = \{\mathit{false}, \mathit{true}\}$, $P_{\mathcal{A}}(\mathit{false}) = \mathit{false}$, and $P_{\mathcal{A}}(\mathit{true}) = \mathit{true}$. Note that $\mathcal{A}$ must also specify the assignment to the free variables $y_1, y_2, \ldots, y_m$. A satisfiability checker for the QBF formula ψ outputs the assignment of free variables for which $\mathcal{A} \models \psi'$. Since $U_{\mathcal{A}}$ is finite, it is easy to see that satisfiability checking of QBF formulas is decidable.

In this question, we would like to use a QBF satisfiability solver (i.e., a QBF-SAT solver) to help us win a board game, whenever possible. The game is described as below.

We have a 3×3 board, divided into 9 squares as shown in the figure below. In the figure, the squares on the board are numbered (0,0) through (2,2). The game is to be played between you and an opponent. You are required to insert As in the squares, and your opponent must insert Bs, one at a time, and in alternation.

(0,0)	(0,1)	(0,2)
(1,0)	(1,1)	(1,2)
(2,0)	(2,1)	(2,2)

Suppose you start the game by first inserting an A in one of the 9 squares. Your opponent then inserts a B in one of the remaining 8 squares. You are now allowed to insert an A only in those remaining squares that cannot be reached by a diagonal from the square in which a B has been placed by your opponent. If you cannot find such an empty square, you lose the game. Otherwise, you can insert an A in an empty square not reachable along a diagonal from the square containing B. Your opponent then inserts a B in one of the remaining 6 squares, and you are again required to insert an A in one of the remaining 5 squares that cannot be reached along a diagonal from any of the squares containing B. If you can find such a square and insert the third A in that square, you win the game. Otherwise, you lose the game. Thus, if you start the game, your goal at each step is to insert an A in one of the unfilled squares that is not reachable along a diagonal from any of the squares already containing a B. You win the game in this case if you can insert three As in this way.

If your opponent starts the game, however, by first filling in a B in one of the 9 squares, then your goal at each step is to insert an A in one of the unfilled squares such that your opponent eventually cant insert three Bs, none of which are reachable along a diagonal from previously inserted As. In this case, you win the game, if you can prevent the opponent from inserting three Bs.

We wish to solve the problem of determining your (winning) sequence of moves in the above board game using a QBF-SAT solver. For this purpose, we will use 27 propositions named $a_{i,j}, b_{i,j}$, and $m_{i,j}$, where $i, j \in \{0, 1, 2\}$. We wish to associate the following meanings with these propositions:

- $a_{i,j}$ is *true* iff an A has already been inserted in the i^{th} row and j^{th} column of the board in an earlier step.
- $b_{i,j}$ is *true* iff your opponent has already inserted a B in the i^{th} row and j^{th} column of the board in an earlier move.
- if $m_{i,j}$ is *true*, then in the current move, you can place an A in the i^{th} row and j^{th} column of the board to eventually win the game.

(a) **[4 marks]** Give a QBF formula ϕ_1 using the above propositional variables, with $m_{i,j}$'s as free variables, such that if you start the game, and wish to determine a move that will eventually lead you to a win (for all possible moves of the opponent), you can use the following procedure.

  i. Simplify ϕ_1 by assigning *true* and *false* values for $a_{i,j}$'s and $b_{i,j}$'s depending on previous moves made by you and the opponent.

  ii. Feed the simplified formula to a QBF-SAT solver.

  iii. If the QBF-SAT solver returns an environment specifying values of the free variables for which the formula can be satisfied, you place an A in the $(i, j)^{th}$ square if $m_{i,j}$ is set to true in the satisfying assignment. Note that there could be several choices of (i, j) from the satisfying assignment, and you can pick one at random.

  iv. If the QBF-SAT solver says that the simplified formula is unsatisfiable, then you give up and declare yourself to be a loser.

(b) **[3 marks]** Using only the above propositional variables, is it possible to give a propositional logic formula ϕ_2 that is to be used exactly as in the previous subquestion to determine a move that will eventually lead you to a win, but only if your opponent starts the game. Give justification if your answer is in the negative. Otherwise, describe how would obtain the required formula ϕ_2, without necessarily giving the formula.

(c) There are quite a few interesting QBF-SAT solvers available in the public domain (see `http://www.qbflib.org`). It might be instructive to try this problem using one of them.