Assignment-4

Name : Bommakanti Venkata Naga Sai Aditya

Entry No : 2019CS50471

1) $B(x)$: x is a barber

$S(x, y)$: x shaves y

(a) Every barber shaves all persons who don't shave themselves.

$$F_1 = \forall x \forall y \, (\neg S(x,x) \rightarrow (B(y) \rightarrow S(y,x)))$$

$$F_1 = \forall x \forall y \, (S(x,x) \vee \neg B(y) \vee S(y,x))$$

(b) No barber shaves any person who shaves himself

$$F_2 = \forall x \forall y \, (S(x,x) \rightarrow (B(y) \rightarrow \neg S(y,x)))$$

$$F_2 = \forall x \forall y \, (\neg S(x,x) \vee \neg B(y) \vee \neg S(y,x))$$

Negation of (c)

(c) There exist a barber

$$F_3 = \exists x \, B(x)$$

If we show that $\square$ can be derived from the resolution of F_1, F_2, F_3 then it implies that (c) is a consequence of (a), (b).

F_1, F_2 are already in skolem.

Converting F_3 into skolem, we get ~~B(a~~ $F_3 = B(a)$

F_1, F_2, F_3 are defined over a signature with single constant symbol 'a'. So, ground term is 'a'

$$\frac{\dfrac{\{S(a,a), \neg B(a), S(a,a)\}, \quad \{\neg S(a,a), \neg B(a), \neg S(a,a)\}}{\{\neg B(a)\}}, \quad \{B(a)\}}{\square}$$

We derived $\square$ from ground resolution of F_1, F_2, F_3.

So, $a \wedge b \wedge \neg c$ is not satisfiable.

So, $a, b \models c$.

2) Properties of relation $\sim$:

- If $A \sim B$ then for every atomic formula F, $A \models F$ iff $B \models F$
- If $A \sim B$, then for each variable x,

  (i) for each $a \in U_A$, $\exists\, b \in U_B$ such that $A_{[x \to a]} \sim B_{[x \to b]}$

  (ii) for each $b \in U_B$, $\exists\, a \in U_A$ such that $A_{[x \to a]} \sim B_{[x \to b]}$

Claim: If $A \sim B$, then $A \models F$ iff $B \models F$ for any formula F

Proof: By induction on structure of formula

Base Case: F is atomic formula. $A \models F$ iff $B \models F$ from the property of relation $\sim$.

Induction step:

Case 1: F is of the form $\neg G$

$A \models F$
iff $A \models \neg G$
iff $A \not\models G$
iff $B \not\models G$ (Induction Hypothesis)
iff $B \models \neg G$ iff $B \models F$.

So, $A \models F$ iff $B \models F$

Case 2: F is of the form $F_1 \wedge F_2$

$A \models F$
iff $A \models F_1 \wedge F_2$
iff $A \models F_1$ and $A \models F_2$
iff $B \models F_1$ and $B \models F_2$ (Induction Hypothesis)
iff $B \models F_1 \wedge F_2$
iff $B \models F$

So, $A \models F$ iff $B \models F$

Case 3: F is of the form $\exists x\, G$

Claim-1: $A_{[x \to a]} \models G$ iff $B_{[x \to b]} \models G$ for some $a \in U_A, b \in U_B$

Proof: From the properties of $\sim$

$A_{[x \to a]} \sim B_{[x \to b]}$ for some $a \in U_A$, $b \in U_B$.

So, $A_{[x \to a]} \models G$ iff $B_{[x \to b]} \models G$ for some $a \in U_A, b \in U_B$

$A \models F$

iff $A \models \exists x G$

iff $A_{[x \to a]} \models G$ for some $a \in U_A$

iff $B_{[x \to b]} \models G$ for some $b \in U_B$ (from Claim-1)

iff $B \models \exists x G$

iff $B \models F$

So, $A \models F$ iff $B \models F$

From all 3 Cases, we can conclude that if $A \sim B$ then $A \models F$ iff $B \models F$

3)

(a) Any person is happy if all their children are rich.

Contrapositive: If a person is not happy, then some of their children is not rick

$F_1 = \forall x \exists y \, (\neg H(x) \rightarrow (C(x,y) \wedge \neg R(y)))$

$F_1 = \forall x \exists y \, (H(x) \vee (C(x,y) \wedge \neg R(y)))$

$F_1 = \forall x \exists y \, ((H(x) \vee C(x,y)) \wedge (H(x) \vee \neg R(y)))$

$F_1 = \forall x \, ((H(x) \vee C(x, f(x))) \wedge (H(x) \vee \neg R(f(x))))$

(b) All graduates are rick

$F_2 = \forall x \, (G(x) \rightarrow R(x))$

$F_2 = \forall x \, (\neg G(x) \vee R(x))$

(c) Someone is graduate if they are a child of graduate

$F_3 = \forall x \forall y \, ((C(x,y) \wedge G(x)) \rightarrow G(y))$

$F_3 = \forall x \forall y \, (\neg C(x,y) \vee \neg G(x) \vee G(y))$

Negation of (d)

(d) Some graduate is not happy

$F_4 = \exists x \; G(x) \wedge \neg H(x)$

$F_4 = G(a) \wedge \neg H(a)$

F_1, F_2, F_3, F_4 are in skolem form with a constant symbol 'a' and a function 'f'. So, ground terms are $\{a, f(a), f(f(a)), \ldots\}$

First Order Resolution: (Each variable is subscripted with line no. because there should not be common variables b/w clauses we are resolving

1. $H(x_1) \vee C(x_1, f(x_1))$ Premise
2. $H(x_2) \vee \neg R(f(x_2))$ Premise
3. $\neg G(x_3) \vee R(x_3)$ Premise
4. $\neg C(x_4, y_4) \vee \neg G(x_4) \vee G(y_4)$ Premise
5. $G(a)$ Premise
6. $\neg H(a)$ Premise

7. $H(x_7) \vee \neg G(x_7) \vee G(f(x_7))$ Res. 1,4 $[x_1|x_4][f(x_1)|y_4][x_7|x_1]$

8. $H(a) \vee G(f(a))$ Res 5,7 $[a|x_7]$

9. $G(f(a))$ Res. 6,8

10. $H(x_{10}) \vee \neg G(f(x_{10}))$ Res 2,3 $[f(x_2)|x_3][x_{10}|x_2]$

11. $\neg G(f(a))$ Res 6,10 $[a|x_{10}]$

12. $\square$ Res 9, 11

So, statements a, b, c entail d.

4) $A(x_1, x_2, \ldots, x_n)$ be a formula with no quantifiers and no function symbol

Let $F = \forall x_1 \forall x_2 \ldots \forall x_n A(x_1, \ldots x_n)$

($\Rightarrow$) F is satisfiable

Let M be a model which satisfies F, with universe U.

Let $P_1, P_2, \ldots P_m$ be the interpretations of predicates in M.

Now, we define a model M' with universe $U' = \{a\}$ for some $a \in U$. Let $P_1', P_2', \ldots P_m'$ be interpretations of predicates in M'. We define them as:

$(a, a, \ldots a) \in P_i'$ iff $(a, a, \ldots a) \in P_i$ $\forall x_1 \ldots \forall x_n A(x_1, \ldots x_n)$

Claim: M' is a model for F i.e. $M \models F \Rightarrow M' \models \forall x_1 \ldots \forall x_n A(x_1, \ldots x_n)$

As the universe contains only one element a, we have to verify if $A(a, a, \ldots a)$ is true or not. to prove $M' \models \forall x_1 A(x_1, \ldots x_n)$

~~We know that~~ $\cancel{M \not\models \forall x_1 \forall x_2 \ldots \forall x_n A(x_1, \ldots x_n)}$ ~~and~~ ~~so~~ ~~So,~~ $\cancel{A(a,a,\ldots a)}$ We know that A contains only boolean operators.

Proof by Induction: (on structure of A)

Base Case: A is atomic formula. This is true by construction. $A = P_i$. As $M \models F$, $(a, a, \ldots a) \in P_i$ So, $A(a, a, \ldots a)$ is true.

Induction Step: 1) A is of the form $F_1 \vee F_2$

$M \models \forall x_1 \ldots \forall x_n (F_1 \vee F_2)$ -

$\Rightarrow$ $M \models \forall x_1 \ldots \forall x_n F_1$ or $M \models \forall x_1 \ldots \forall x_n F_2$

$\Rightarrow$ $F_1(a, a, \ldots a)$ is T or $F_2(a, a, \ldots a)$ is T (Induction Hypothesis)

$\Rightarrow$ $A(a, a \ldots a)$ is T

2) A is of the form $F_1 \wedge F_2$

$M \models \forall x_1 \ldots \forall x_n (F_1 \wedge F_2)$

$\Rightarrow M \models \forall x_1 \ldots \forall x_n F_1$ and $M \models \forall x_1 \ldots \forall x_n F_2$

$\Rightarrow$ $F_1(a, a, \ldots a)$ is T and $F_2(a, a, \ldots a)$ is T (Induction Hypothesis)

$\Rightarrow$ $A(a, a, \ldots a)$ is T

3) A is of the form $\neg F$

$M \models \forall x_1 \ldots \forall x_n \neg F$

$\Rightarrow M \not\models \forall x_1 \ldots \forall x_n F$

$\Rightarrow F(a, a \ldots a)$ is F

$\Rightarrow A(a, a, \ldots a)$ is T

So, $M' \models \forall x_1 \ldots \forall x_n A$.

($\Leftarrow$) F is satisfiable in an interpretation with one element

Clearly, F has a model

So, F is satisfiable.

5)

a) Consider the formula $F_n = \exists x_1 \ldots \exists x_n \bigwedge_{\substack{i,j \\ i<j}} (R(x_i, x_j) \wedge \neg R(x_j, x_i))$

For $n=2$, $F_2 = \exists x_1 \exists x_2 \; R(x_1, x_2) \wedge \neg R(x_2, x_1)$

Similarly for $n=3$, $F_3 = \exists x_1 \exists x_2 \ldots \exists x_3 \; R(x_1, x_2) \wedge \neg R(x_2, x_1)$
$\wedge R(x_1, x_3) \wedge \neg R(x_3, x_1)$
$\wedge R(x_2, x_3) \wedge \neg R(x_3, x_2)$

Claim 1: F_n is satisfiable

We define a model M with universe $U = \{1, 2, \ldots n\}$

$R = \{(i, j) \mid i < j, \; i,j \in U\}$

Clearly, M satisfies F_n because let $x_i = i$

for each i,j $i<j$ $R(x_i, x_j) = R(i, j) = T$

$\neg R(x_j, x_i) = \neg R(j, i) = \neg F = T$

So, $R(x_i, x_j) \wedge \neg R(x_j, x_i)$ is T for all i,j $i<j$

So, $\bigwedge_{\substack{i,j \\ i<j}} (R(x_i, x_j) \wedge \neg R(x_j, x_i))$ is T

So, $M \models \exists x_1 \ldots \exists x_n \bigwedge_{\substack{i,j \\ i<j}} (R(x_i, x_j) \wedge \neg R(x_j, x_i))$

Claim 2: Every model A of F_n has n elements

Proof by contradiction

let A has less than n elements

For any $(x_1, x_2 \ldots x_n)$ we have some i,j $(i<j)$ such that

$x_i = x_j$ (By pigeon hole principle)

So, for this i,j $R(x_i, x_i) \wedge \neg R(x_i, x_i)$ is False(F)

So, F_n is 'F' for every $(x_1, \ldots x_n)$

So, $A \not\models F_n$. This is a contradiction

So, every model A of F_n has n elements.

(b) Signature σ contains unary predicate symbols $P_1, \ldots, P_k$

Let F be any satisfiable σ-formula.

So, there exists a ~~model~~ assignment M which satisfies the given formula.

If M has ~~less than~~ atmost 2^k elements then we are done.

Lets assume that there are more than 2^k elements in the universe. 'U'.

For every element $e \in U$, let $S_e = \{P_1(e), P_2(e), \ldots P_k(e)\}$

Each predicate can be T/F. So, there will be at most 2^k possibilities of $\{P_1(x), \ldots, P_k(x)\}$

But we have more than 2^k elements.

So, there exists two elements $e, e' \in U$ such that $\{P_1(e), P_2(e), \ldots P_k(e)\} = \{P_1(e'), P_2(e'), \ldots P_k(e')\}$ (By pigeon hole principle)

So, e and e' have same valuation for each predicate and from

So, we remove e' from universe and from assignment interpretations of predicates. Clearly this new assignment also satisfies the formula because e' is providing same interpretation as e.

We repeat the process till there are less than 2^k elements in the universe.

So, we have a model where universe has at most 2^k elements.

So, any σ-Satisfiable formula has a model where the universe has atmost 2^k elements.