COL703 Assignment - 1

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Contents

1 Problem 1

1.a Part - b

1. $\neg p$	Premise
2. $p \lor q$	Premise
3. p	Premise
4. $\perp$	$\neg e$ 1, 3
5. q	$\perp e$ 4
6. q	Premise
7. q	$\lor e$ 5, 6

So, $\neg p, p \lor q \vdash q$ is valid.

1.b Part - d

1. $p \land \neg p$	Premise
2. p	$\land e_1$ 1
3. $\neg p$	$\land e_2$ 1
4. $\perp$	$\neg e$ 2, 3
5. $\neg(r \rightarrow q) \land (r \rightarrow q)$	$\perp e$ 4

So, $p \land \neg p \vdash \neg(r \rightarrow q) \land (r \rightarrow q)$ is valid.

2 Problem 3

Rule for $\leftrightarrow$ Introduction ($\leftrightarrow$i):

$$\frac{\phi \to \psi, \psi \to \phi}{\phi \leftrightarrow \psi}$$

Rule-1 for $\leftrightarrow$ Elimination ($\leftrightarrow e_1$):

$$\frac{\phi \leftrightarrow \psi, \phi}{\psi}$$

Rule-2 for $\leftrightarrow$ Elimination ($\leftrightarrow e_2$):

$$\frac{\phi \leftrightarrow \psi, \psi}{\phi}$$

Rule-3 for $\leftrightarrow$ Elimination ($\leftrightarrow e_3$):

$$\frac{\phi \leftrightarrow \psi, \neg\phi}{\neg\psi}$$

Rule-4 for $\leftrightarrow$ Elimination ($\leftrightarrow e_4$):

$$\frac{\phi \leftrightarrow \psi, \neg\psi}{\neg\phi}$$

Proof for Introduction:

1. $\phi \to \psi$	Premise
2. $\psi \to \phi$	Premise
3. $(\phi \to \psi) \wedge (\psi \to \phi)$	$\wedge$i 1, 2
4. $\phi \leftrightarrow \psi$	Interpretation of 1

Proof for Elimination - 1:

1. $\phi \leftrightarrow \psi$	Premise
2. $(\phi \to \psi) \wedge (\psi \to \phi)$	Interpretation of 1
3. ϕ	Premise
4. $\phi \to \psi$	$\wedge e_1$ 2
5. ψ	MP 4, 3

Proof for Elimination - 2:

1. $\phi \leftrightarrow \psi$	Premise
2. $(\phi \to \psi) \wedge (\psi \to \phi)$	Interpretation of 1
3. ψ	Premise
4. $\psi \to \phi$	$\wedge e_2$ 2
5. ϕ	MP 4, 3

Proof for Elimination - 3:

1. $\phi \leftrightarrow \psi$	Premise
2. $(\phi \to \psi) \wedge (\psi \to \phi)$	Interpretation of 1
3. $\neg\, \phi$	Premise
4. $\psi \to \phi$	$\wedge e_2$ 2
5. $\neg\, \psi$	MT 4, 3

Proof for Elimination - 4:

1. $\phi \leftrightarrow \psi$	Premise
2. $(\phi \to \psi) \wedge (\psi \to \phi)$	Interpretation of 1
3. $\neg\, \psi$	Premise
4. $\phi \to \psi$	$\wedge e_2$ 2
5. $\neg\, \phi$	MT 4, 3

So, we have proofs for Introduction and Elimination using basic proof rules. So, the rules of introduction, elimination of the connective $\leftrightarrow$ are derived rules.

3 Problem 4

3.a Part c

1. $(p \to r) \land (q \to r)$	Premise
2. $(p \to r)$	$\land e_1$ 1
3. $p \land q$	Assumption
4. p	$\land e$ 3
5. r	$\to e$ 2, 4
6. $(p \land q) \to r$	$\to i$ 3 - 5

So, $(p \to r) \land (q \to r) \vdash (p \land q) \to r$ is valid.

3.b Part d

1. $p \to q \land r$	Premise
2. p	Assumption
3. $q \land r$	MP 1, 2
4. q	$\land e_1$ 3
5. $p \to q$	$\to i$ 2 - 4
6. p	Assumption
7. $q \land r$	MP 1, 6
8. r	$\land e_2$ 7
9. $p \to r$	$\to i$ 6 - 8
10. $(p \to q) \land (p \to r)$	$\land i$ 5, 9

So, $p \to q \land r \vdash (p \to q) \land (p \to r)$ is valid.

4 Problem 7

Truth Table of p $\leftrightarrow$ q:

p	q	p $\leftrightarrow$ q
T	T	T
T	F	F
F	T	F
F	F	T

Claim - 1: Any formula formed using {$\leftrightarrow$, $\neg$} having more than 1 atom has even number of T's in its truth table
Proof:
Induction on number of connectives
Base Case: Let number of connectives = 1. Formula we can form using one connective and having more than 1 atom is p $\leftrightarrow$ q. From the truth table of p $\leftrightarrow$ q, we can see that it has 2 T's. So, our claim is true.

Induction Hypothesis: Let's assume that our claim is true for all formulas with number of connectives less than k.

Induction Step: Let's consider a formula F with k connectives.
F can be of form A $\leftrightarrow$ B or $\neg$A where A, B are formulas with less than k connectives
Case - 1: F = A $\leftrightarrow$ B. From Induction Hypothesis, truth table of A, B have even number of T's. Number of entries in truth table is also even (because number of atoms > 1). Let no of entries in truth tables of A, B be 2*k1, 2*k2 respectively. Let number of T's in A = 2*m. So, number of F's in A = 2*k1 - 2*m. Similarly, number of T's in B = 2*n. So, no of F's in B = 2*k2 - 2*n. From the truth table of $\leftrightarrow$, we can say that F will be true only when both A, B are true or both A, B are false. So, F will be true min(2*m, 2*n) + min(2*k1 - 2*m, 2*k2 - 2*n) times which is even. Hence, our claim is true in this case.
Case - 2: F = $\neg$A. From Induction Hypothesis, A has even number of T's in its truth table. Let no of entries in the truth table of 2*k1 and no of T's be 2*m. So, no of F's in truth table of A are 2*k1 - 2*m. So, no of T's in the truth table of F are 2*k1 - 2*m which is even. So, our claim is true in this case.
Hence, our claim is proved.

Claim - 2: A formula of type A $\rightarrow$ B can't be represented by using {$\leftrightarrow$, $\neg$} and has only one atom.
Proof:
Proof by Contradiction.
As we can use only one atom, we can use only $\neg$ connective. Let the propositional atoms be {p, q}. Let's assume that a formula F formed is equivalent to p $\rightarrow$ q. Now, F has only one propositional atom, let it be p. Let v, w be defined as v(p) = T, v(q) = T, w(p) = T, w(q) = F. From here, v(p $\rightarrow$ q) $\neq$ w(p $\rightarrow$ q) but v(p) = v(q). This is a contradiction. Hence, we can't represent a formula of form p $\rightarrow$ q using only one atom

The formula p $\rightarrow$ q have odd number of T's in their truth tables. And from claim-2, we need atleast 2 atoms to represent it. So, we can't represent this using {$\leftrightarrow$, $\neg$} because of Claim - 1 (As any formula formed have even number of T's). So, {$\leftrightarrow$, $\neg$} is not adequate.

5 Problem 8

5.a Part b

Given sequent: $\neg r \to (p \vee q), r \wedge \neg q \vdash r \to q$

p	q	r	¬q	¬r	p ∨ q	¬r → (p ∨ q)	r ∧ ¬q	r → q
T	T	T	F	F	T	T	F	T
T	T	F	F	T	T	T	F	T
T	F	T	T	F	T	T	T	F
F	T	T	F	F	T	T	F	T
T	F	F	T	T	T	T	F	T
F	T	F	F	T	T	T	F	T
F	F	T	T	F	F	T	T	F
F	F	F	T	T	F	F	F	T

From rows 3, 7, we can observe that for valuations p = T/F, q = F, r = T, the truth values of formulas on left is T and the truth value for formula on right is F. So, the given sequent is not valid.

5.b Part d

Given sequent: $p \to (q \to r) \vdash p \to (r \to q)$

p	q	r	q → r	r → q	p → (q → r)	p → (r → q)
T	T	T	T	T	T	T
T	T	F	F	T	F	T
T	F	T	T	F	T	F
F	T	T	T	T	T	T
T	F	F	T	T	T	T
F	T	F	F	T	T	T
F	F	T	T	F	T	T
F	F	F	T	T	T	T

From row 3, we can observe that for valuation p = T, q = F, r = T, the truth values of formulas on left is T and the truth value for formula on right is F. So, the given sequent is not valid.