CHAPTER – 2

NUMBERS – II

In the previous chapter we discussed various concepts on numbers. In this chapter we will discuss models on remainders as relevant to the MBA entrance exams.

THE LAST DIGIT OF ANY POWER

The last digits of the powers of any number follow a cyclic pattern - i.e., they repeat after certain number of steps. If we find out after how many steps the last digit of the powers of a number repeat, then we can find out the last digit of any power of any number.

Let us look at the powers of 2.
Last digit of 2^1 is 2
Last digit of 2^2 is 4
Last digit of 2^3 is 8
Last digit of 2^4 is 6
Last digit of 2^5 is 2

Since last digit of 2^5 is the same as the last digit of 2^1, then onwards the last digit will start repeating, i.e. digits of 2^5, 2^6, 2^7, 2^8 will be the same as those of 2^1, 2^2, 2^3, 2^4. Then the last digit of 2^9 is again the same as the last digit of 2^1 and so on. So, we have been able to identify that for powers of 2 the last digits repeat after every 4 steps. In other words whenever the power is a multiple of 4, the last digit of that number will be the same as the last digit of 2^4.

Suppose we want to find out the last digit of 2^{67}, we should look at a multiple of 4 which is less than or equal to the power 67. Since 64 is a multiple of 4, the last digit of 2^{64} will be the same as the last digit of 2^4.

Then the last digits of 2^{65}, 2^{66}, 2^{67} will be the same as the last digits of 2^1, 2^2, 2^3 respectively. Hence the last digit of 2^{67} is the same as the last digit of 2^3 i.e. 8.

Similarly, we can find out the last digit of 3^{74} by writing down the pattern of the powers of 3.

Last digit of 3^1 is 3
Last digit of 3^2 is 9
Last digit of 3^3 is 7
Last digit of 3^4 is 1
Last digit of 3^5 is 3

The last digit repeats after 4 steps (like in the case of powers of 2)

To find the last digit of 3^{74}, we look for a multiple of 4 which is less than or equal to 74. Since 72 is multiple of 4, the last digit of 3^{72} will be the same as that of 3^4. Hence the last digit of 3^{74} will be the same as the last digit of 3^2, i.e. 9.

LAST DIGIT OF A SUM OR PRODUCT

The problem consists of finding the last digit of the sum of two numbers each of which is a power of some integer. For example, you may be asked to find out the last digit of the sum $2^{67} + 3^{74}$

In general, when we want to find out the last digit of the sum of two numbers, we can just take the last digit of the two numbers and add them up. That will be the last digit of the sum. The last digit of 243 + 456 will be the same as the sum of the last digits of the two numbers, i.e., the sum of 3 and 6, which is 9. Similarly, in the case of $2^{67} + 3^{74}$ also, the last digit will be equal to the sum of the last digits of the two terms 2^{67} and 3^{74}.

We have already looked at finding out the last digit of powers like 2^{67} and 3^{74}. Hence the last digit of $2^{67} + 3^{74}$ is 8 + 9 i.e. 7.

Similarly, the last digit of a product will be equal to the last digit of the product of the last digits of the two given numbers.

For example, the last digit of the product $2^{67} \times 3^{74}$ will be equal to the last digit of the product of the last digit of 2^{67} and the last digit of 3^{74}, i.e. the last digit of 8 x 9, i.e., 2. Hence the last digit of $2^{67} \times 3^{74}$ is 2.

Examples:

2.01. Find the last digit of $2^{412} \times 4^{428}$

Sol: Writing down the powers of 2 and 4 to check the pattern of the last digits, we have
Last digit of 2^1 – 2
Last digit of 2^2 – 4
Last digit of 2^3 – 8
Last digit of 2^4 – 6
Last digit of 2^5 – 2
Last digit of 4^1 – 4
Last digit of 4^2 – 6
Last digit of 4^3 – 4
Last digit of 4^4 – 6

We find that the last digit of powers of 2 repeat after 4 steps, the last digit of any power of 4 is 4 for an odd power and 6 for an even power. The last digit of 2^{412} will be the same as 2^4 as 412 is a multiple of 4. So the last digit of 2^{412} is 6
Last digit of 4^{428} is 6. Since the power of 4 is even.

Hence the last digit of $2^{412} \times 4^{428}$ will be equal to the last digit of 6 x 6 i.e., 6

FINDING THE REMAINDER IN DIVISIONS INVOLVING POWERS OF NUMBERS

There is one particular model of problem that appeared about 3 times in CAT papers. It is explained below with the help of an example.

2.02. Find the remainder of the division $5^{64}/6$.

Sol: Let us find the pattern that remainders follow when successive powers of 5 are divided by 6.
Remainder of $5^1/6$ is 5.
Remainder of $5^2/6$ is 1.
Remainder of $5^3/6$ is 5.
Remainder of $5^4/6$ is 1.
We find that the remainders are repeated after every two powers. So, remainder of 5^{64} when divided by 6 is the same as 5^2 when divided by 6, since 64 is a multiple of 2. Hence the remainder is 1.

PATTERN METHOD

Similar to the last digit of the powers of a number repeating in a certain pattern, the remainders of powers of a number also follow a certain pattern. If we identify the pattern in which the remainders repeat, we can find out the remainder of any division given (This type of a problem was given 3 to 4 times in CAT paper from 1990 onwards).

To solve the example given above, let us find the pattern that remainders follow when various powers of 2 are divided by 7.

Remainder when 2^1 is divided by 7 is 2
Remainder when 2^2 is divided by 7 is 4
Remainder when 2^3 is divided by 7 is 1
Remainder when 2^4 is divided by 7 is 2

We find that the remainder repeats in the fourth step, i.e., after 3 steps. So,

- the remainder of 2^4 when divided by 7 is the same as that when 2^1 is divided by 7, i.e., 2
- the remainder of 2^5 when divided by 7 is the same as that when 2^2 is divided by 7, i.e., 4
- the remainder of 2^6 when divided by 7 is the same as that when 2^3 is divided by 7, i.e., 1
- the remainder of 2^7 when divided by 7 is the same as that when 2^1 is divided by 7, i.e., 2
  and so on.

If we take 2^{54}, since 54 is divisible by 3, 2^{54} itself completes a cycle of 3 steps and hence the remainder when 2^{54} is divided by will be the same as that when 2^3 is divided by 7. Hence the remainder is 1.

REMAINDER THEOREM METHOD

We can apply Remainder Theorem to find the remainder in problems like the one discussed above. Let us first look at Remainder Theorem and understand it.

Remainder Theorem states that when f(x), a polynomial function in x is divided by x – a, the remainder is f(a).

A polynomial function in x is a function where x will appear only in the form of x^n and not in any other form, where n is a positive integer.

Let us take an example to understand Remainder Theorem.

When the function $x^2 + 2x – 3$ is divided by x – 1, the remainder will be f(1). This is because, as per Remainder Theorem, when the divisor is (x – a), the remainder is f(a). Here the divisor is x – 1 and hence the remainder is f(1). To get f(1), we should substitute x = 1 in the given equation. As we get f(1) = 0, the remainder in this case is 0. {Note that when f(x) is divided by x – a, if the remainder is 0, then x – a will be a factor of f(x). So, in this case, (x – 1) is a factor of $x^2 + 2x – 3$}.

When the function $x^2 + 2x + 3$ is divided by x + 1, the reminder will be f (–1) which is $(–1)^2 + 2 (–1) + 3$, i.e., 2.

Now let us take the example of finding the remainder when 2^{54} is divided by 7 (which was solved by the Pattern Method above) and solve it by Remainder Theorem Method.

In the division $2^{54}/7$, the dividend is 2^{54} and the divisor is 7. Since the numerator is in terms of powers of 2, express the denominator also in terms of powers of 2. In this case, 7 can be written as 8 – 1 which is $2^3 – 1$. So, now the denominator is in terms of 2^3, the numerator, i.e., the dividend should be rewritten in terms of 2^3 which will be $(2^3)^{18}$. Now, the given problem reduces to finding out the remainder when $(2^3)^{18}$ is divided by $2^3 – 1$. Here, if we consider 2^3 as x, it is equivalent to finding out the remainder when x^{18} is divided by (x – 1) which, as per Remainder Theorem, is f(1), i.e., the remainder is obtained by substituting 1 in place of x. So, the remainder will be $(1)^{18}$, i.e., 1.

2.03. Find the remainder of the division $2^{34}/5$.

Sol: In the division, since the numerator is in terms of power of 2, the denominator should also be expressed in terms of power of 2 i.e., as $(2^2 + 1)$. Now, as the denominator is in terms of 2^2, the numerator should also be rewritten in terms of 2^2 as $(2^2)^{17}$. The problem reduces to finding the remainder when $(2^2)^{17}$ is divided by $2^2 – (–1)$. This remainder, as per the Remainder Theorem is $(–1)^{17} = –1$; and –1 + 5 = 4 (the divisor is added to get a positive remainder).

2.04. Find the remainder of the division $2^{56}/31$.

Sol: In this division, since the numerator is in terms of powers of 2, the denominator 31 should also be expressed in terms of 2, as $2^5 – 1$. Now as the denominator is in terms of 2^5, the numerator 2^{56} should also be rewritten in terms of 2^5 as $(2^5)^{11} \times 2^1$. The problem now reduces to finding the remainder when $2(2^5)^{11}$ is divided by $2^5 – 1$. This remainder as per the Remainder Theorem is $2(1)^{11} = 2$

2.05. Find the remainder of the division $2^{58}/24$.

Sol: Pattern method
The remainders of powers of 2 when divided by 24 are as follows.
The remainder when 2^1 is divided by 24 is 2
The remainder when 2^2 is divided by 24 is 4
The remainder when 2^3 is divided by 24 is 8
The remainder when 2^4 is divided by 24 is 16
The remainder when 2^5 is divided by 24 is 8
The remainder when 2^6 is divided by 24 is 16
The remainder repeats in such a way (excluding the remainder when 2^1 and 2^2 are divided by 24) that the remainder is 16 when an even power of 2 is divided by 24 and 8. When an odd power of 2 is divided by 24. When 2^{58} is divided by 24, the remainder is 16.

2.06. Find the remainder of the division $3^{98}/10$.

Sol: Pattern method
The remainder when 3 is divided by 10 is 3.
The remainder when 3^2 is divided by 10 is 9.
The remainder when 3^3 is divided by 10 is 7.
The remainder when 3^4 is divided by 10 is 1.
The remainder when 3^5 is divided by 10 is 3.
Since the remainder is repeating after 4 steps, the remainder of $3^{98}/10$ is the same as remainder of $3^2/10$ (since 98 = 4 x 24 + 2)

Remainder Theorem Method

In the division $3^{98}/10$, the numerator is in terms of powers of 3, so the denominator can be written as $3^2 + 1$. Since the denominator is written in terms of 3^2, the numerator is expressed as $(3^2)^{49}$. So, the remainder of $(3^2)^{49}$ divided by $(3^2 + 1)$, as per the Remainder Theorem, is $(–1)^{49} = –1$
Hence remainder is –1 +10 = 9

2.07. Find the remainder of the division 7^{93} divided by 10.

Sol: Pattern method
The remainders of powers of 7 when divided by 10 are as follows.
Remainder when 7^1 is divided by 10 is 7.
Remainder when 7^2 is divided by 10 is 9.
Remainder when 7^3 is divided by 10 is 3.
Remainder when 7^4 is divided by 10 is 1.
Remainder when 7^5 is divided by 10 is 7.
Since the remainder is repeating after 4 steps, the remainder of $7^{93}/10$ is the same as that of $7^1/10$= 7 (Since $93 = 4 \times 23 + 1$).

As is evident from the above examples, the remainder theorem is more suited to cases where the denominator (i.e., the divisor) can be written in the form of one more or one less than some power of the base in the numerator. For example, in case of $2^{54}/7$, since the base in the numerator is 2, the denominator 7 has to be written as one more or one less than some power of 2. In this case it can be written as $2^3 – 1$. In cases where it is not possible to write it in this manner, then applying the Pattern Method is the easiest method.

Remainder of a number when divided by $10^n \pm 1$

This is best illustrated with examples:

2.08. Find the remainder when 123,123, … (up to 300 digits) is divided by 999.

Sol: To find the remainder when some number (say N) is divided by 9 (or $10^1 – 1$), we add up all the digits of N to get (say S_1) and divide S_1 by 9. Similarly to find the remainder when N is divided by 99(or $10^2 – 1$), we start at the right end of N, group the digits two at a time and add up all the groups to get, say S_2. Then we find the remainder of $S_2/99$.
In general to find the remainder when N is divided by D_n = 99 . . . . 9(n nines) or $(10^n - 1)$, we start at the right end of N, group the digits n at a time and add up all the groups to get say S_n.

$$\text{Rem } \frac{N}{D_n} = \text{Rem } \frac{S_n}{D_n}$$

Similarly we can start with the remainder rule for 11 and work out the corresponding rules for 101, 1001, 10001 etc. All this is an application of Remainder theorem.

Here, N = 123, 123, ..., 123 (a total of 300 digits or 100 groups) = 123 (1000^{99}) + 123 (1000^{98}) + . . . + 123 (1000^1) + 123

Now, let N = f(1000); When N or f(1000) is divided by 999 or (1000 – 1), the remainder is f(1) i.e., 123(100) by remainder theorem. [i.e. S_3 = 123 (100)]

$$\therefore \text{Rem}\frac{N}{999} = \text{Rem } \frac{12300}{999} = \text{Rem } \frac{12+300}{999} = 312$$

2.09. Let N = 345345345 …. upto 300 digits. What is the remainder when N is divided by 999? Also find the remainder when N is divided by 1001.

Sol: N = 345, 345, ……… 345 (upto 300 digits or 100 groups of 3 digits) = 345 $[10^{3(99)} + 10^{3(98)} + 10^{3(97)} + \ldots + 10^3 + 1]$

$$\text{Rem}\frac{N}{999} = \text{Rem } \left(\frac{N}{(10^3-1)}\right) \text{Rem}\frac{(345)(100)}{999}$$

($\because$ By remainder theorem)

$$= \text{Rem}\frac{34{,}500}{999} N = \text{Rem}\frac{34+500}{999} = 534$$

To get $\text{Rem}\frac{N}{1001}$, we need U and Th, where U is the sum of all the alternate groups starting with the rightmost (the group containing the units digit) and Th is the sum of all the alternate groups starting with the second rightmost (the group consisting of the thousands digit)
U = 345(50) = 17250 and Th = 345(50) = 17250

$$\therefore \text{Rem}\frac{N}{1001} = \text{Rem}\frac{U-Th}{1001} = 0$$

Last two digits of any power

The terms of any Geometric progression (GP) leave a cyclic pattern of remainders when divided by any divisor. The sequence of powers of the base ‘a’ is a GP with common ratio equal to ‘a’.

If we take the divisor as 100, the remainder is simply the last two digits. We’ll find it convenient to consider the following 4 cases separately.
(1) The base ends in 0
(2) The base ends in 5
(3) The base ends in 1, 3, 7 or 9
(4) The base ends in 2, 4, 6 or 8

The first two cases are very simple.
(1) If a ends in 0, the square and all higher powers end in at least 2 zeroes.
(2) If a ends in 5, the powers either all end in 25 or end alternately in 25 and 75, depending on whether the tens digit of a is even or odd.
(3) If the base ends in 1, 3, 7 or 9, there is a cycle of at the most 20 distinct remainders. The twentieth power ends in 01. (The cycle length could also be some factor of 20 ie.1, 2, 4, 5 or 10)
(4) If the base ends in 2, 4, 6 or 8, there is a cycle of at the most 20 distinct remainders. The twentieth power ends in 76. The cycle length could also be some factor of 20.

(4.1) Moreover, if a = 4k, the second set and all the subsequent sets of 20 remainders are exactly the same as the first set.

(4.2) But if a = 4k + 2, it is not possible to get 4k + 2 as the last two digits in any higher power. All such powers are multiples of 4. Consequently, of the forty ‘two-digit’ numbers (02, 04, 06, 08, 12, 14, 16, 18 ----- 92, 94, 96, 98) only twenty, viz 04, 08, 12, 16, 24,…., 92,96 can occur as the last two digits in the higher powers. If 02, 06, 14 etc do occur, they can occur only as the first power. We find that the last two digits of a^{21} are obtained by adding 50 to 4k + 2 (For example, 2^{21} ends in 52, 6^{21} ends in 56, 14^{21} ends in 64 etc). Therefore, while the second set of 20 remainders is almost the same as the first set (differing only in the first remainder), all subsequent sets are exactly the same as the second set.

The examples below will illustrate these points.

Consider point (3) above
The last two digits of successive powers of 13 are 13, 69, 97, 61 …. 01; 13, 69, 97, 61….01 etc

Consider point (4.1) above
The last two digits of successive powers of 4 are 04, 16, 64, 56, ….76; 04,16,….76; etc.

Consider point (4.2) above
The last two digits of successive powers of 2 are 02, 04, 08, 16, ….76; 52,04,08,16, ….76(instead of the 02, we get 52)

These 6 points, 1, 2, 3, 4, 4.1, 4.2 (whichever is applicable) should be used in all problems on the last two digits.

2.10. Find the remainder when $N = 817^{673}$ is divided by 100. Alternatively, find the last two digits of N.

Sol: We are interested only in the last two digits of N. We need to consider only the last two digits of 817, i.e 17. Successive powers of 17 (or any other number) show a cyclic pattern, when divided by 100(or any other divisor). We can list these remainders until we discover the point, where the repetition starts.

17	57	97	37	77
89	69	49	29	09
13	73	33	93	53
21	41	61	81	01

$17^1 = 17$

To get the next number, we take only the last two digits of 17^2, i.e. 89. To get the next number, we take only the last two digits of 17(89). We need not perform the complete multiplication. We need only the units and tens digits. The units digit is 3 and there are 3 parts to the tens digit – the carry over of 6, the units digit of 9(1) and 8(7) i.e. 6 + 9 + 6. Again we need only the units digit of this which is 1.

$\therefore$ The last two digits of 17^3 are 13. Similarly, we can work out the other numbers.

It is convenient to break the column after every 4 steps (the units digit is found to be the same in each row. This serves as a check to our calculations). After we get 01, the next 20 powers show the same pattern.

In the given example, as 673 = 20 (33) + 13.

$\therefore$The 13th number in the list, i.e. 37, is our answer.

In general, we find that if we are interested in the last 2 digits, we need to go up to at most 20 steps. In some cases the period may be some factor of 20 (1, 2, 4, 5 or 10).

Consider the powers of 01. The pattern is 01; 01 etc.
The period is 1.

Consider powers of 49, 51 or 99. The patterns are
49, 01; 49, 01; etc
51, 01; 51, 01; etc
99, 01; 99, 01; etc, i.e. the period is 2.

Consider powers of 07, 43, 57 or 93. The patterns are
07, 49, 43, 01; etc
43, 49, 07, 01; etc
57, 49, 93, 01; etc
93, 49, 57, 01; etc, i.e. the period is 4.

Consider powers of 21,
The pattern is 21, 41, 61, 81, 01; etc. The period is 5.

Consider powers of 29, 71 or 79. The patterns are
29, 41, 89, 81, 49, 21, 09, 61, 69, 01; etc
71, 41, 11, 81, 51, 21, 91, 61, 31, 01; etc
79, 41, 39, 81, 99, 21, 59, 61, 19, 01; etc
The period is 10.

2.11. What are the last two digits of 37^{12345}?

Sol: $N = 37^{12345} = 37^{12340}\, 37^5 = 37^{20(617)}.37^5$
37^{20} ends in 01,
while $37^5 \equiv (37)^4\, 37 \equiv (1369)^2\,(37) \equiv (69)^2\,(37) \equiv (4761)\,(37)$
$\equiv (61)\,(37) = 2257.$ $\therefore$ N ends in 57
Note: $a \equiv b$ means $a - b$ is divisible by the considered divisor

2.12. Find the remainder when 164^{359} is divided by 100

Sol: $N = 164^{359}$. We need the last two digits.
These digits for successive powers form a pattern of cycle length 20. As 359 = 340 + 19 and 164 = 100 + 64, we can think of $64^{19} = 2^{114}$
Now $2^{14} = 16384$, It ends in 84.
$\therefore$ N also ends in 84

2.13. Find the last two digits of 282^{822}

Sol: $N = 282^{822}$. We can think of $82^2 = 6724$
$\therefore$ N ends in 24

Some important theorems

Binomial Theorem: For any natural number n, $(a + b)^n = {}^nC_0a^n + {}^nC_1\, a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots\ldots + {}^nC_{n-1}\, ab^{n-1} + {}^nC_nb^n$.
where ${}^nC_r$ is the number of ways of choosing r objects out of n distinct objects and is given by

$${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1(2)(3)\ldots\ldots(r)} = \frac{n!}{r!(n-r)!}$$

It can be observed that $(a + b)^n = a^n$ + (A multiple of b) = (A multiple of a) + b^n

2.14. Show that $(a + b)^7 - a^7 - b^7$ is a multiple of 7 for all positive integral values of a and b.

Sol: $(a + b)^7$
$= {}^7C_0a^7 + {}^7C_1\, a^6\, b + {}^7C_2\, a^5\, b^2 + {}^7C_3\, a^4\, b^3 + \ldots + {}^7C_7\, b^7$
$\therefore (a + b)^7 - a^7 - b^7$
$= {}^7C_1\, a^6\, b + {}^7C_2\, a^5\, b^2 + {}^7C_3\, a^4\, b^3 + \ldots\ldots + {}^7C_7\, b^7$ ---- (1)
If p is any prime number,
$${}^pC_r = \frac{p(p-1)\ldots\ldots(p-r+1)}{1(2)\ldots\ldots(r)} = p(\text{an integer}) \text{ for all } r < p.$$
$\therefore$ The RHS of (1) (and hence the LHS of (1)) is a multiple of 7.

Note: When n is prime, $(a+b)^n = a^n + b^n +$ (a multiple of n)

2.15. Find the remainder when 2^{1000} is divided by 33

Sol: $2^{1000} = (2^5)^{200} = (33-1)^{200} = (33^{200} + {}^{200}C_1\,(33)^{199}(-1) + {}^{200}C_2\,(33)^{198}(-1)^2 + \ldots + {}^{200}C_{199}\,(33)(-1)^{199} + (-1)^{200}$ M $(33) + (-1)^{200} =$ (A multiple of 33) + 1

Fermat's little theorem: If p is prime and HCF (a, p) = 1, then $a^{p-1} - 1$ is a multiple of p.

For example, take p = 5, a = 3. From the theorem, $3^4 - 1$ or 80 is a multiple of 5.

If we take successive powers of 3, we get all the possible remainders.
$3^1 = 3$, $3^2 = 4$, $3^3 = 2$, $3^4 = 1$ (also $3^5 = 3$, $3^6 = 3^2$, $3^7 = 3^3$ etc). At a certain stage, we get a remainder of 1 and after that, the pattern repeats. In this example, the pattern is 3, 4, 2, 1; 3,4,2,1; etc. The pattern length is 4. In general, it would be (p – 1) or some factor of (p – 1).

2.16. What is the remainder when 5^{119} is divided by 59?

Sol: $N = 5^{119}$ We need $\text{Rem}\dfrac{N}{59}$

By Fermat's Little Theorem, $5^{58} = 59k + 1$ (where k is a natural number)

$5^{59} = 59\,(5k) + 5$ or $5^{59} \equiv 5$

$\therefore\ 5^{118} \equiv 25$ and $5^{119} \equiv 125 \equiv 7$

2.17. Find the remainder when 26^{57} is divided by 29

Sol: $\text{Rem}\dfrac{26^{57}}{29} = \text{Rem}\dfrac{(26)(26)^{56}}{29}$

$= \left\{\text{Rem}\dfrac{26}{29}\right\}\left\{\text{Rem}\dfrac{26^{56}}{29}\right\}$

$= \{26\}\{1\} = 26.$

$\therefore$ The remainder is 26.

Wilson's Theorem: If p is prime, (p – 1)! +1 is a multiple of p.

For example,
(2 – 1)! + 1= 2(1), (3-1)! + 1 = 3(1), (5-1)! + 1 = 5(5), (7 – 1)! + 1 = 721 = 7(103) and so on.

2.18. What is the remainder when 28! is divided by 29?

Sol: By Wilson's theorem, $\text{Rem}\,\dfrac{28!+1}{29} = 0$

$\Rightarrow \text{Rem}\,\dfrac{28!}{29} = -1$ or $-1 + 29 = 28$

RULES PERTAINING TO $a^n + b^n$ or $a^n - b^n$

Sometimes, there will be problems involving **numbers that can be written in the form** $a^n + b^n$ or $a^n - b^n$ which can be simplified using simple rules. Let us first look at the rules pertaining to both $a^n + b^n$ and $a^n - b^n$, a, b and n being positive integers.

The following rules should be remembered for numbers in the form of $\mathbf{a^n - b^n}$.

1. It is always (i.e. when n is even as well as odd) divisible by a – b.
2. When n is even it is also divisible by a + b.
3. When n is odd it is divisible by a + b, if a + b is a factor of $2.b^n$.

The following rules should be remembered for numbers in the form of $\mathbf{a^n + b^n}$.

1. When n is odd it is divisible by a + b.
2. When n is odd, it is divisible by a – b, when a – b is a factor of $2.b^n$.
3. When n is even, it is divisible by a + b, if a + b is a factor of $2b^n$.

2.19. Which of the following statements is true about $15^n + 1$?
(a) It is divisible by 16, when n is even.
(b) It is always divisible by 16.
(c) It is never divisible by 16.
(d) It is never divisible by 14.

Sol: $15^n + 1$ is in the form of $a^n + b^n$.
The divisor 16 is 15 + 1, which is in the form of a + b.
Hence 16 divides $15^n + 1$ only when n is odd. Hence first three options are wrong.
14 = 15 – 1 and hence is in the form a – b.
$a^n + b^n$ is divisible by a – b when a – b is a factor of a + b. Here, a – b = 14 and a + b = 16. 14 is not a factor of 16. Hence $15^n + 1$ is not divisible by 14.
The fourth option is correct.

2.20. Which of the statements is true about $31^n - 1$?
(a) It is always divisible by 32.
(b) It is divisible by 32 when n is odd.
(c) It is never divisible by 30.
(d) It is always divisible by 30.

Sol: $31^n - 1$ is in the form of $a^n - b^n$.
(a – b) is a factor of $a^n - b^n$ for even as well as odd values of n.
Hence $31^n - 1$ is always divisible by (31 – 1) which is 30. Option (d) is correct.
Note : 32 = 31 + 1; i.e. it is in the form of a + b.
$a^n - b^n$ is divisible by a + b when
(i) n is even or
(ii) n is odd and $2b^n$ is a multiple of (a + b). As neither condition is satisfied, options (a) and (b) are not correct.

Note:
$a^N - b^N = (a - b)(a^{N-1} + a^{N-2}b + a^{N-3}b^2 + \ldots + a^2b^{N-3} + ab^{N-2} + b^{N-1})$ for all positive integer values of N.
$a^N + b^N = (a+b)(a^{N-1} - a^{N-2}b + a^{N-3}b^2 - a^{N-4}b^3 + \ldots - a^3b^{N-4} + a^2b^{N-3} - ab^{N-2} + b^{N-1})$ for all odd positive integer values of N.

Concept Review Questions

Directions for questions 1 to 20: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. $11^{103} + 14^{103}$ is divisible by _____.
(A) 45 (B) 35 (C) 25 (D) 75

2. The greatest number which always divides $38^{2n} – 11^{2n}$ among the following is _____.
(A) 26 (B) 441 (C) 676 (D) 1323

3. Find the units digit of $3^{200} \times 4^{500}$.

4. For what values of n is the following statement true? If n is a natural number, 3^{3n-1} is divisible by 26.
(A) Even values of n
(B) Odd values of n
(C) All values of n

5. For what values of n is the following statement true? If n is a natural number, 2^{5n+1} is divisible by 33.
(A) Even values of n
(B) Odd values of n
(C) All values of n

6. Find the units digit of $(13687)^{3265}$.

7. Find the remainder when 1643276569 is divided by 25.

8. Find the remainder when 367543216 is divided by 9.

9. Find the remainder when 18^{18} is divided by 19.
(A) 1 (B) 18 (C) 11 (D) 4

10. Find the last digit in the product of any 10 consecutive odd natural numbers.

11. Find the tens digit in the product of the first 14 natural numbers.

12. Find the largest 4-digit number which when divided by 19 leaves a remainder of 6.
(A) 9984 (B) 9978 (C) 9999 (D) 9981

13. Find the least 4-digit number which leaves a remainder of 10 when divided by 36.

14. Find the last remainder when 192 is successively divided by 7, 2 and 4.

15. Is the five-digit number pqr86 a perfect square?
(A) Yes (B) No (C) Can't say

16. Is the six-digit number 2a4b75 a perfect square?
(A) Yes (B) No (C) Can't say

17. Is the four-digit number PQ76 a perfect square?
(A) Yes (B) No (C) Can't say

18. If the three-digit number PQ1 is a perfect square, is Q odd?
(A) Yes (B) No (C) Can't say

19. If the three-digit number P6Q is a perfect square, is P even?
(A) Yes (B) No (C) Can't say

20. If the three-digit number A9B is a perfect square, A + B = _____.
(A) 7
(B) 6
(C) Can't be determined

Exercise – 2(a)

Directions for questions 1 to 25: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. Find the last digit of $2^{48} \times 7^{40} \times 4^{48}$.

2. Find the units digit in $57867^{192567} - 1452^{876}$.

3. If (x – 1) and (x + 1) are factors of the expression $5x^3 - 2x^2 - ax - b$, find the value of a.
(A) 3 (B) 5 (C) –2 (D) 7

4. Find the remainder when 2^{360} is divided by 7.

5. Find the remainder when 3^{50} is divided by 11.

6. If $y = 21^3 + 22^3 + 23^3 + 24^3$, find the remainder when y is divided by 90.

7. What is the remainder when 495149514951.... upto 600 digits is divided by 101?
(A) 95 (B) 6 (C) 98 (D) 3

8. Find the remainder when 676767.....upto 900 digits is divided by 999.

9. What is the remainder when 347^{347} is divided by 100?
(A) 23 (B) 43 (C) 63 (D) 83

10. What is the remainder when 78^{1234} is divided by 100?

11. What is the remainder when 326^{972} is divided by 100?
(A) 74 (B) 24 (C) 26 (D) 76

12. What is the sum of the coefficients in the expansion of $(1 + x + 2x^2)^{100}$?
(A) 2^{200} (B) 2^{100} (C) 2^{400} (D) 2^{101}

13. What is the remainder when 2^{275} is divided by 137?
(A) 2 (B) 16
(C) 4 (D) 8

14. What is the remainder when 32^{180} is divided by 149?

15. What is the remainder when 24^{1202} is divided by 1446?
(A) 1290 (B) 576
(C) 879 (D) None of these

16. What is the remainder when 95! is divided by 97?

17. What is the remainder when 50! is divided by 47^2?
(A) 282 (B) 41 (C) 1886 (D) 1927

18. N =1! + 2! + 3! + ……. 100!. Find the remainder when N is divided by 168.

19. N = 127127 …… (a total of 202 digits). Find the remainder when N is divided by 143.
(A) 1 (B) 142 (C) 127 (D) 16

20. $N = (1111)^{2222}$. Find the remainder when N is divided by 19.
(A) 1 (B) 18 (C) 7 (D) 17

21. If $x = \dfrac{60^{99} - 58^{99}}{60^{98} + 58^{98}}$, which of the following holds true?
(A) $0 < x \le 0.5$ (B) $0.5 < x \le 1$
(C) $1 < x \le 2$ (D) $x > 2$

22. N is a natural number. The remainder of $2^{14N + 7} + 3^{10N + 5} - 7$ divided by 371 is ______.
(A) 360 (B) 356
(C) 364 (D) Dependent on N

23. $X = 10^{57} - 450$
Consider the following statements:
I. X is divisible by 11.
II. X leaves the remainder 4 when divided by 7.

Which of the following statements is/are true?
(A) Only I (B) Only II
(C) Both I and II (D) Neither I nor II

24. P is a positive integer not more than 100. If the difference of 7^P and P^3 is divisible by 10, how many values of P are there?

25. Find the remainder when $289(256^{15})$ is divided by 17^4.

Exercise – 2(b)

Directions for questions 1 to 35: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

Very Easy / Easy

1. Find the units digit of $7^{33} \times 14^{31} \times 6^{30}$.

2. Find the units digit of $3^{44} + 131 \times 56 + 34 \times 46$.

3. 70^{5340} has its rightmost non-zero digit equal to ______.
(A) 7 (B) 9 (C) 3 (D) 1

4. What is the last digit of $424^{782} + 179^{137}$?
(A) 7 (B) 6 (C) 5 (D) 4

5. What are the last two digits of 1576^{689}?
(A) 74 (B) 26 (C) 24 (D) 76

6. When $(99999)^{1000}$ is divided by 100,000, what will be the remainder?

Moderate

7. What is the units digit of the remainder when $(19^{23} + 17^{23})$ is divided by 36?

8. Find the remainder when 2^{76} is divided by 15.

9. Find the remainder when 2^{66} is divided by 65.
(A) 1 (B) 17
(C) 33 (D) 64

10. The remainder when $2582^{801} – 2579^{401}$ is divided by 3 is

11. What is the remainder when 7^{83} is divided by 20?
(A) 13 (B) 7
(C) 3 (D) None of these

12. Find the remainder when 56785678.....upto 1000 digits is divided by 99.
(A) 19 (B) 45 (C) 38 (D) 80

13. What is the remainder when 468468468....upto 333 digits is divided by 1001?

14. What is the remainder when 793^{1008} is divided by 25?
(A) 24 (B) 1 (C) 23 (D) 22

15. What is the remainder when 114^{210} is divided by 25?
(A) 23 (B) 24 (C) 1 (D) 22

16. What are the last two digits of 784^{489}?

17. What is the remainder when 1532^{786} is divided by 25?
(A) 4 (B) 24 (C) 19 (D) 14

18. What is the remainder when $71^{72} + 73^{72}$ is divided by 72?

19. What is the remainder when $91^{150} + 95^{150}$ is divided by 31?

20. What is the sum of the coefficients in the expansion of $(2 + 3x)^{75}$?
(A) 2^{75} (B) 5^{76} (C) 5^{75} (D) 2^{76}

21. What is the remainder when 6^{722} is divided by 73?
(A) 3 (B) 36
(C) 70 (D) 37

22. What is the remainder when 81^{225} is divided by 179?

23. What is the remainder when 101! is divided by 103?
(A) 101 (B) 1 (C) 102 (D) 2

24. Find the remainder when 7^{349} is divided by 100.
(A) 49 (B) 7 (C) 43 (D) 1

25. $N = 12^1 + 12^2 + 12^3 + 12^4 + \ldots\ldots. 12^{100.}$
Find the remainder when N is divided by 7.

26. Find the remainder when $1750 \times 1752 \times 1754$ is divided by 13.

27. Find the remainder when $3333333333 + 3^{144}$ is divided by 16.
(A) 2 (B) 10 (C) 14 (D) 6

28. Find the remainder when $31^{3300} – 3332$ is divided by 32.
(A) 21 (B) 25 (C) 29 (D) 17

Difficult / Very Difficult

29. Find the sum of all the possible distinct remainders which can be obtained when the square of a natural number is divided by 9.

30. What is the remainder when 10^{1283} is divided by 514?
(A) 285 (B) 229
(C) 232 (D) None of these

31. Find the remainder when 25! is divided by 529.
(A) 46 (B) 483
(C) 24 (D) None of these

32. Find the remainder when $1^4 + 2^4 + 3^4 + \ldots\ldots\ldots 100^4$ is divided by 7.

$\boxed{}$

33. N = 1111 ….11 (a total of 363 digits). Find the remainder when N is divided by 13.
(A) 1 (B) 12 (C) 7 (D) 6

34. X and Y are positive integers. X leaves a remainder of 1 when divided by 40. Y leaves a remainder of 2 when divided by 40.

Consider the following statements:
I. $3^X - 3X$ is divisible by 10.
II. $7^Y + 7(Y + 1)$ is divisible by 10.

Which of the two statements is/are definitely true?
(A) Only I (B) Only II
(C) Both I and II (D) Neither I nor II

35. N is the number formed by writing all the positive integers from 120 to 165 side-by-side. Find the remainder when N is divided by 9.
(A) 3 (B) 6 (C) 0 (D) 4

Key

Concept Review Questions

1. C	5. D	9. A	13. 1018	17. C
2. D	6. 7	10. 5	14. 1	18. B
3. 6	7. 19	11. 0	15. B	19. B
4. C	8. 1	12. D	16. B	20. A

Exercise – 2(a)

1. 6	6. 0	11. D	16. 1	21. D
2. 7	7. C	12. A	17. D	22. C
3. B	8. 666	13. D	18. 33	23. C
4. 1	9. C	14. 73	19. C	24. 10
5. 1	10. 44	15. B	20. D	25. 19941

Exercise – 2(b)

1. 8	8. 1	15. C	22. 158	29. 12
2. 1	9. D	16. 64	23. B	30. D
3. D	10. 0	17. B	24. B	31. B
4. C	11. C	18. 2	25. 3	32. 3
5. D	12. C	19. 2	26. 11	33. C
6. 1	13. 468	20. C	27. D	34. C
7. 0	14. B	21. B	28. C	35. A