CHAPTER – 2

QUADRATIC EQUATIONS

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"If a variable occurs in an equation with all positive integer powers and the highest power is two, then it is called a Quadratic Equation (in that variable)."

In other words, a second degree polynomial in x equated to zero will be a quadratic equation. For such an equation to be a quadratic equation, the co-efficient of x^2 should not be zero.

The most general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$ (and a, b, c are real)

Some examples of quadratic equations are

$x^2 - 5x + 6 = 0$ .... (1)
$x^2 - x - 6 = 0$ .... (2)
$2x^2 + 3x - 2 = 0$ .... (3)
$2x^2 + x - 3 = 0$ .... (4)

Like a first degree equation in x has one value of x satisfying the equation, a quadratic equation in x will have TWO values of x that satisfy the equation. The values of x that satisfy the equation are called the ROOTS of the equation. These roots may be real or imaginary.

For the four quadratic equations given above, the roots are as given below:

Equation (1) : $x = 2$ and $x = 3$
Equation (2) : $x = -2$ and $x = 3$
Equation (3) : $x = 1/2$ and $x = -2$
Equation (4) : $x = 1$ and $x = -3/2$

In general, the roots of a quadratic equation can be found out in two ways.
(i) by factorising the expression on the left hand side of the quadratic equation
(ii) by using the standard formula

All the expressions may not be easy to factorise whereas applying the formula is simple and straight forward.

Finding the roots by factorisation

If the quadratic equation $ax^2 + bx + c = 0$ can be written in the form $(x - \alpha)(x - \beta) = 0$, then the roots of the equation are α and β.

To find the roots of a quadratic equation, we should first write it in the form of $(x - \alpha)(x - \beta) = 0$, i.e., the left hand side $ax^2 + bx + c$ of the quadratic equation $ax^2 + bx + c = 0$ should be factorised into two factors.

For this purpose, we should go through the following steps. We will understand these steps with the help of the equation $x^2 - 5x + 6 = 0$ which is the first of the four quadratic equations we looked at as examples above.

- First write down b (the co-efficient of x) as the sum of two quantities whose product is equal to ac.
  In this case –5 has to be written as the sum of two quantities whose product is 6. We can write –5 as (–3) + (–2) so that the product of (–3) and (–2) is equal to 6.
- Now rewrite the equation with the 'bx' term split in the above manner.
  In this case, the given equation can be written as $x^2 - 3x - 2x + 6 = 0$
- Take the first two terms and rewrite them together after taking out the common factor between the two of them. Similarly, the third and fourth terms should be rewritten after taking out the common factor between the two of them. In other words, you should ensure that what is left from the first and the second terms (after removing the common factor) is the same as that left from the third and the fourth term (after removing their common factor).
  In this case, the equation can be rewritten as $x(x - 3) - 2(x - 3) = 0$; Between the first and second terms as well as the third and fourth terms, we are left with $(x - 3)$ is a common factor.
- Rewrite the entire left hand side to get the form $(x - \alpha)(x - \beta)$.
  In this case, if we take out $(x - 3)$ as the common factor, we can rewrite the given equation as $(x - 3)(x - 2) = 0$
- Now, α and β are the roots of the given quadratic equation.
  $\therefore$ For $x^2 - 5x + 6 = 0$, the roots of the equation are 3 and 2.

For the other three quadratic equations given above as examples, let us see how to factorise the expression and get the roots.

For equation (2), i.e., $x^2 - x - 6 = 0$, the co-efficient of x which is –1 can be rewritten as (–3) + (+2) so that their product is –6 which is equal to ac (1 multiplied by –6). Then we can rewrite the equation as $(x - 3)(x + 2) = 0$ giving us the roots as 3 and –2.

For equation (3), i.e., $2x^2 + 3x - 2 = 0$, the co-efficient of x which is 3 can be rewritten as (+4) + (–1) so that their product is –4 which is the value of ac (–2 multiplied by 2). Then we can rewrite the equation as $(2x - 1)(x + 2) = 0$ giving the roots as 1/2 and –2.For equation (4), i.e., $2x^2 + x - 3 = 0$, the co-efficient of x which is 1 can be rewritten as (+3) + (–2) so that their product is –6 which is equal to ac (2 multiplied by –3). Then we can rewrite the given equation as$(x - 1)(2x + 3) = 0$ giving us the roots as 1 and –3/2.

Finding the roots by using the formula

If the quadratic equation is $ax^2 + bx + c = 0$, then we can use the standard formula given below to find out the roots of the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The roots of the four quadratic equations we took as examples above can be taken and their roots found out by using the above formula. The student is advised to check it out for himself that the roots can be obtained by using this formula also.

SUM AND PRODUCT OF ROOTS OF A QUADRATIC EQUATION

For the quadratic equation $ax^2 + bx + c = 0$, the sum of the roots and the product of the roots can be given by the following:

Sum of the roots = $-b/a$
Product of the roots = c/a

These two rules will be very helpful in solving problems on quadratic equation.

NATURE OF THE ROOTS

We mentioned already that the roots of a quadratic equation with real co-efficients can be real or complex. When the roots are real, they can be equal or unequal. All this will depend on the expression $b^2 - 4ac$. Since $b^2 - 4ac$ determines the nature of the roots of the quadratic equation, it is called the "DISCRIMINANT" of the quadratic equation.

If $b^2 - 4ac > 0$, then the roots of the quadratic equation will be real and distinct.

If $b^2 - 4ac = 0$, the roots are real and equal.

If $b^2 - 4ac < 0$, then the roots of the quadratic equation will be complex conjugates.

Thus we can write down the following about the nature of the roots of a quadratic equation when a, b and c are all rational.

when $b^2 - 4ac < 0$	the roots are complex and unequal
when $b^2 - 4ac = 0$	the roots are rational and equal
when $b^2 - 4ac > 0$ and a perfect square	the roots are rational and unequal
when $b^2 - 4ac > 0$ but not a perfect square	the roots are irrational and unequal

Whenever the roots of the quadratic equation are irrational, (a, b, c being rational) they will be of the form $a + \sqrt{b}$ and $a - \sqrt{b}$, i.e. whenever $a + \sqrt{b}$ is one root of a quadratic equation, then $a - \sqrt{b}$ will be the second root of the quadratic equation and vice versa.

SIGNS OF THE ROOTS

We can comment on the signs of the roots, i.e., whether the roots are positive or negative, based on the sign of the sum of the roots and the product of the roots of the quadratic equation. The following table will make clear the relationship between the sum and the product of the roots and the signs of the roots themselves.

Sign of product of the roots	**Sign of sum of the roots**	**Sign of the roots**
+ ve	+ ve	Both the roots are positive
+ ve	– ve	Both the roots are negative
– ve	+ ve	The numerically larger root is positive and the other root is negative.
– ve	– ve	The numerically larger root is negative and the other root is positive.

CONSTRUCTING A QUADRATIC EQUATION

We can build a quadratic equation in the following three cases:

- when the roots of the quadratic equation are given
- when the sum of the roots and the product of the roots of the quadratic equation are given.
- when the relation between the roots of the equation to be framed and the roots of another equation is given.

If the roots of the quadratic equation are given as α and β, the equation can be written as
$(x - \alpha)(x - \beta) = 0$ i.e., $x^2 - x(\alpha + \beta) + \alpha\beta = 0$
If p is the sum of the roots of the quadratic equation and q is the product of the roots of the quadratic equation, then the equation can be written as $x^2 - px + q = 0$.

CONSTRUCTING A NEW QUADRATIC EQUATION BY CHANGING THE ROOTS OF A GIVEN QUADRATIC EQUATION

If we are given a quadratic equation, we can build a new quadratic equation by changing the roots of this equation in the manner specified to us.

For example, let us take a quadratic equation $ax^2 + bx + c = 0$ and let its roots be α and β respectively. Then we can build new quadratic equations as per the following patterns:

(i) A quadratic equation whose roots are the **reciprocals** of the roots of the given equation $ax^2 + bx + c = 0$, i.e., the roots are $1/\alpha$ and $1/\beta$:

This can be obtained by substituting $1/x$ in place of x in the given equation giving us $cx^2 + bx + a = 0$, i.e., we get the equation required by interchanging the co-efficient of x^2 and the constant term.

(ii) A quadratic equation whose roots are **k more** than the roots of the equation $ax^2 + bx + c = 0$, i.e., the roots are $(\alpha + k)$ and $(\beta + k)$
This can be obtained by substituting $(x - k)$ in place of x in the given equation.

(iii) A quadratic equation whose roots are **k less** than the roots of the equation $ax^2 + bx + c = 0$, i.e., the roots are $(\alpha - k)$ and $(\beta - k)$

This can be obtained by substituting (x + k) in place of x in the given equation.

(iv) A quadratic equation whose roots are **k times** the roots of the equation $ax^2 + bx + c = 0$, i.e., the roots are $k\alpha$ and $k\beta$.
This can be obtained by substituting x/k in place of x in the given equation.

(v) A quadratic equation whose roots are **1/k times** the roots of the equation $ax^2 + bx + c = 0$, i.e., the roots are α/k and β/k

This can be obtained by substituting kx in place of x in the given equation.

An equation whose degree is 'n' will have n roots.

MAXIMUM OR MINIMUM VALUE OF A QUADRATIC EXPRESSION

An equation of the type $ax^2 + bx + c = 0$ is called a quadratic equation. An expression of the type $ax^2 + bx + c$ is called a "quadratic expression". The quadratic expression $ax^2 + bx + c$ takes different values as x takes different values.
As x varies from $-\infty$ to $+\infty$, (i.e. when x is real) the quadratic expression $ax^2 + bx + c$

(i) has a minimum value whenever a > 0 (i.e. a is positive). The minimum value of the quadratic expression is $(4ac - b^2) / 4a$ and it occurs at $x = -b/2a$.

(ii) has a maximum value whenever a < 0 (i.e. a is negative). The maximum value of the quadratic expression is $(4ac - b^2) / 4a$ and it occurs at $x = -b/2a$.

Polynomials and Polynomial Equations:

An expression of the type $a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1x + a_0$ is called a polynomial. If we denote it as f(x), f(x) = 0 is a polynomial equation.

Remainder Theorem:
When a polynomial p(x) of degree n is divided by x – a (a linear polynomial), there results a quotient polynomial q(x) (of degree (n – 1)) and a remainder (of degree 0) i.e. a constant.

i.e., p(x) = (x – a) q(x) + R
This relation is true for all values of x. In particular, for x = a, we get, p(a) = R. This result is the Remainder Theorem.

Note:
1. If p(a) = 0, we say that ‘a’ is a zero of the polynomial p(x).
2. If p(x) is a polynomial and ‘a’ is a zero of p(x), then p(x) = (x – a) q(x).
3. If p(x) is divided by ax + b, then the remainder is given by $p\left(\frac{-b}{a}\right)$.
4. If p(x) is divided by ax – b, then the remainder is given by $p\left(\frac{b}{a}\right)$.
5. The degree of remainder is always less than the degree of divisor.

2.01. $f(x) = x^2 + 12x + 20$. Find the remainder when f(x) is divided by x + 14.

Sol. $f(x)\ ) = x^2 + 12x + 20$
Remainder theorem: If f(x) (i.e. a function of x) is divided by x − a, the remainder of the division is f(a).
The remainder of the division of f(x) by x + 14 i.e., x – (−14) is f(−14).
$f(-14) = (-14)^2 + 12(-14) + 20 = 196 - 168 + 20 = 48$

2.02. $g(x) = 4x^2 + 12x + 20$. Find the remainder when g(x) is divided by 2x – 1.

Sol. If f(x) is divided by ax ± b, the remainder of the division is $f\left(\mp\frac{b}{a}\right)$
$g(x) = 4x^2 + 12x + 20$
If g(x) is divided by 2x − 1, the remainder of the division is $g\left(\frac{1}{2}\right)$.
$$g\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 + 12\left(\frac{1}{2}\right) + 20 = 4\frac{1}{4} + 6 + 20 = 27$$

2.03. A linear function of x leaves a remainder of 20 when divided by x − 5. The function is divisible by x + 5. Find the linear function.

Sol. The linear function is divisible by x + 5.
∴ It must be k(x + 5), where k is a constant.
Let us denote the linear function by g(x).
g(x) = k(x + 5)
Remainder theorem: If f(x) is divided by x – a, then the remainder of the division is f(a).
The remainder of the division of g(x) by x – 5 is g(5). This is also equal to 20.
$g(5) = 20 \Rightarrow k(5 + 5) = 20$
$\Rightarrow k = 2$
g(x) = k(x + 5) = 2x + 10.

Factor theorem:

If R = 0, i.e p(a) = 0, then x – a is a factor of p(x) and conversely, if x – a is a factor of p(x), then p(a) = 0. This immediate consequence of the Remainder Theorem is called the Factor Theorem. This can be restated as follows: The number a is a root of p(x) = 0, if and only if (x – a) is a factor of p(x).

2.04. If $(a - 2)x^3 + (a + 1)x^2 + x - 2a$ is divisible by x + 3, find the value of a.

Sol. Let $g(x) - (a - 2)x^3 + (a + 1)x^2 + x - 2a$
Factor theorem: x – a is a factor of f(x) if and only if f(a) = 0.
g(x) is divisible by x + 3 (i.e. x− (−3)) is a factor of g(x).
∴ g(–3) = 0
g(–3) = (a – 2)(–27) + (a + 1)(9) + (–3) – 2a
= –20a + 60
g(–3) = 0
$\Rightarrow -20a + 60 = 0$
$\Rightarrow a = 3$

2.05. If $ax^2 + bx + c$ is divisible by $3x - 4$, the value of $16a + 12b + 9c$ is _______.

Sol. Factor theorem: $x - a$ is a factor of $f(x)$ if and only if $f(a) = 0$.

If $ax \pm b$ is a factor of $f(x)$ $f\left(\mp\frac{b}{a}\right) = 0$

($\because$ This follows from the Factor theorem)

$ax^2 + bx + c$ is divisible by $3x - 4$. $\therefore$ $3x - 4$ is a factor of $ax^2 + bx + c$.

Let $g(x) = ax^2 + bx + c$

As $3x - 4$ is a factor of $g(x)$, $g\left(\frac{4}{3}\right) = 0$

$a\left(\frac{4}{3}\right)^2 + b\left(\frac{4}{3}\right) + c = 0$

$16a + 12b + 9c = 0$

Problems based on factor and remainder theorems:

2.06. $h(x) = 4x^2 + 12x + 20$. Find the remainder when $h(x)$ is divided by $2x + 1$.

Sol. If $f(x)$ is divided by $ax \pm b$, the remainder of the division is $f\left(\mp\frac{b}{a}\right)$.

$h(x) = 4x^2 + 12x + 20$

if $h(x)$ is divided by $2x + 1$, the remainder of the division is $h\left(\frac{-1}{2}\right)$.

$h\left(\frac{-1}{2}\right) = 4\left(\frac{-1}{2}\right)^2 + 12\left(\frac{-1}{2}\right) + 20 = 15$

2.07. $Q(x)$ is a quadratic expression and $Q(0) = 12$. The remainder of $Q(x)$ divided by $x - 1$ is 5. The remainder of $Q(x)$ divided by $x + 2$ is 38. Find $Q(x)$.

Sol. Let $Q(x) = ax^2 + bx + c$. $Q(0) = 12$. $\therefore$ $c = 12$

$Q(x) = ax^2 + bx + 12$

Remainder theorem: If $f(x)$ is divided by $x - a$, the remainder of the division is $f(a)$.

The remainder of $Q(x)$ divided by $x – 1$ is $Q(1)$. This is also equal to 5.

$Q(1) = 5$ Also $Q(-2) = 38$

$a + b + 12 = 5$ and $4a - 2b + 12 = 38$

$\therefore$ $a = 2$ and $b = -9$

$Q(x) = ax^2 + bx + 12 = 2x^2 - 9x + 12$

2.08. $f(x) = ax^3 - 2x^2 – x + b$. If $f(x)$ is divisible by $x^2 - 1$, the value of (a, b) is ____.

Sol. $f(x)$ is divisible by $x^2 - 1$ i.e., $(x + 1)(x - 1)$

$\therefore$ $(x + 1)(x - 1)$ is a factor of $f(x)$.

$x + 1$ and $x – 1$ are both factors of $f(x)$.

Factor theorem: $x – a$ is a factor of $f(x)$ if and only if $f(a) = 0$

$f(–1) = a(–1)^3 – 2(–1)2 – (–1) + b = 0$ …….(1)

and $f(1) = a – 2 – 1 + b = 0$…….(2)

Solving (1) and (2), $a = 1$ and $b = 2$

2.09. A quadratic expression in x leaves remainders of 9 and 4 when divided by $x + 2$ and $x - 3$ respectively. The expression is divisible by $x + 1$. Find the quadratic expression.

Sol. Let the quadratic expression in x be

Let $Q(x) = ax^2 + bx + c$

Remainder theorem: If $f(x)$ is divided by $x – a$, then the remainder of the division is $f(a)$.

The respective remainders when $Q(x)$ is divided by $x + 2$, $x - 3$, $x + 1$are 9 ,4.,0.

$Q(–2) = 4a – 2b + c = 9$……(1)

$Q(3)$ $9a + 3b + c = 4$………(2)

$Q(–1) = a – b + c = 0$ …….(3)

(1) – (2): $–5a – 5b = 5 \Rightarrow a + b = –1$

(1) – (3): $3a – b = 9$

Solving these, $a = 2$. $\therefore$ $b = –3$. $\therefore$ $c = –5$

$Q(x) = 2x^2 – 3x – 5$

2.10. If $3x - 1$ is a common factor of $6x^2 + (6b + 1)x - (a + 1)$ and $3ax^2 - (b - 1)x - 1$,then $(a, b) =$

Sol. Let $g(x) = 6x^2 + (6b + 1)x - (a + 1)$ and

$h(x) = 3ax^2 - (b - 1)x – 1$

Factor theorem: $x - a$ is a factor of $f(x)$ if and only if $f(a) = 0$

If $ax + b$ is a factor of $f(x)$, $f\left(\frac{-b}{a}\right) = 0$ ($\because$ This follows from the Factor theorem)

$3x – 1$ is a common factor of $g(x)$ and $h(x)$.

$\therefore$ $g\left(\frac{-1}{3}\right) = h\left(\frac{1}{3}\right) = 0$

$g\left(\frac{1}{3}\right) = 6\left(\frac{1}{3}\right)^2 + (6b + 1)\left(\frac{1}{3}\right) - (a + 1)$

$= 0$……..(1)

$h\left(\frac{1}{3}\right) = 3a\left(\frac{1}{3}\right)^2 - (b - 1)\frac{1}{3} - 1 = 0$……….(2)

$\Rightarrow a = 2b$

$\Rightarrow \frac{a}{3} - \frac{(b-1)}{3} - 1 = 0$

(1) $\Rightarrow 2 + (6.6 + 1) - 3(a+1) = 0 \Rightarrow a = 26$

(2) $\Rightarrow a – (b – 1) = 3 \Rightarrow a-b =2$

$\therefore$ $(a, b) = (4, 2)$

2.11. Find the remainder of x^{999} divided by $x^2 - 6x +5$.

Sol. Let the quotient and the remainder of x^{999} divided by $x^2 - 6x + 5$ i.e., $(x - 1)(x - 5)$ be $q(x)$ and $r(x)$ respectively.

$x^{999} = (x - 1)(x - 5)\, q(x) + r(x)$

As the divisor $x^2 - 6x + 5$ is a quadratic expression, the remainder $r(x)$ must be a linear expression.

Let $r(x) = ax + b$, where a and b are constants.

$x^{999} = (x - 1)(x - 5)\, q(x) + ax + b$…….(1)

Setting $x = 1$ in (1): $1 = a + b$………(2)

Setting $x = 5$ in (1):, $5^{999} = 5a + b$……. (3)

(3) − (2): $5^{999} – 1 = 4a$

$a = \frac{5^{999} - 1}{4}$

$\therefore$ $b = \frac{-5^{999} + 5}{4}$ ($\because$ From (2) or (3))

Remainder is $ax + b$, $\left(\frac{5^{999} - 1}{4}x + \frac{-5^{999} + 5}{4}\right)$.

2.12. Find the remainder when x^5 is divided by $x^3 - 9x$.

Sol. Let the quotient and the remainder of x^5 divided by $x^3 - 9x$ i.e., $x(x + 3)(x - 3)$ be $q(x)$ and $r(x)$ respectively.

$x^5 = x(x + 3)(x - 3)\, q(x) + r(x)$

As the divisor $x^3 - 9x$ is a cubic expression, the remainder r(x) must be an expression whose degree is at most 2 i.e. it must be a quadratic expression or a linear expression.

Let $r(x) = ax^2 + bx + c$ (if a = 0, r(x) is a linear expression. Otherwise, it is a quadratic expression)

$x^5 = x(x + 3)(x - 3)\, q(x) + ax^2 + bx + c$ - (1)

Setting x = 0 in (1): 0 = c

Setting x = −3 in (1): −243 = 9a – 3b + c

Setting x = 3 in (1): 243 = 9a + 3b + c

−243 = 9a – 3b and 243 = 9a + 3b

−81 = 3a – b and 81 = 3a + b

Solving these, a = 0 and b = 81

$r(x) = ax^2 + bx + c = 0\,x^2 + 81x + 0 = 81x$

Division of a polynomial by a polynomial:

Long division method:

Step 1: First arrange the terms of the dividend and the divisor in the descending order of their degrees.

Step 2: Now the first term of the quotient is obtained by dividing the first term of the dividend by the first term of the divisor.

Step 3: Then multiply all the terms of the divisor by the first term of the quotients and subtract the result from the dividend.

Step 4: Consider the remainder as new dividend and proceed as before.

Step 5: Repeat this process till we obtain a remainder which is either 0 or a polynomial of degree less than that of the divisor.

Example : Divide $2x^3 + 9x^2 + 4x - 15$ by $2x + 5$.

Solution: $2x + 5)\ 2x^3 + 9x^2 + 4x - 15\ (x^2 + 2x - 3$

$2x^3 + 5x^2$

(–) (–)

$4x^2 + 4x$

$4x^2 + 10x$

(–) (–)

$-6x - 15$

$-6x - 15$

(+) (+)

0

$\therefore (2x^3 + 9x^2 + 4x - 15) \div (2x + 5) = x^2 + 2x - 3$

Example :

Find the quotient and the remainder when $x^4 + 4x^3 - 31x^2 - 94x + 120$ is divided by $x^2 + 3x - 4$.

Solution: $x^2 + 3x - 4)\ x^4 + 4x^3 - 31x^2 - 94x + 120\ (x^2 + x - 30$

$x^4 + 3x^3 - 4x^2$

– – +

$x^3 - 27x^2 - 94x + 120$

$x^3 + 3x^2 - 4x$

– – +

$-30x^2 - 90x + 120$

$-30x^2 - 90x + 120$

+ + –

0

$\therefore$ the quotient is $x^2 + x - 30$ and the remainder is ‘0’

Relations between Roots and Coefficients:

An n^{th} order equation has n roots. Corresponding to every root, there is a factor. If α is a root of f(x) = 0, then $x - \alpha$ is a factor of f(x). Sometimes $(x - \alpha)^2$ may also be a factor. In such a case, α is said to be a double root. Similarly equations can have triple roots, quadruple roots and roots of multiplicity m. If m is the greatest value of k, for which $(x - \alpha)^k$ is a factor of f(a), then α is said to be a root of multiplicity m. If all the roots are counted by taking their multiplicity into account, the number of roots is equal to n, the degree of the equation.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not necessarily distinct) are the roots of f(x) = 0, then

$f(x) = a_n(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$

$= a_n[x^n - S_1 x^{n-1} + S_2 x^{n-2} \ldots + (-1)^n S_n]$

where S_1 = the sum of the roots

S_2 = the sum of the products of the roots taken 2 at a time

S_3 = the sum of the product of the roots taken 3 at a time and so on.

S_n = the 'sum' of the product of the roots taken n(or all) at a time. Thus, S_n is a single term.

$S_n = \alpha_1 \alpha_2 \ldots \alpha_n$

Let us write down the polynomial f(x) in two forms:

The standard form

$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} \ldots + a_1 x + a_0$

In terms of the roots of the corresponding equation.

$f(x) = a_n [x_n - S_1 x^{n-1} + S_2 x^{n-2} + \ldots + (-1)^{x-1} S_{n-1} x + (-1)^n S_n]$

These polynomials are identically equal, i.e, equal for all values of x. Therefore the corresponding coefficients are equal. The sum of the roots $S_1 = -a_{n-1} / a_n$

The sum of the products of the roots, taken two at a time, $S_2 = a_{n-2} / a_n$

The sum of the products of the roots, taken three at a time, $S_3 = -a_{n-3} / a_n$ and so on.

The 'sum' of the 'products' of the roots taken m ($m \le n$) at a time $S_m = \Sigma \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_m = (-1)^m \dfrac{a_{n-m}}{a_n}$

$\therefore S_n = \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \dfrac{a_0}{a_n}$

For example, consider the polynomial equation

$(x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6 = 0$

(We can see immediately that the roots are 1, 2, 3)

The sum of roots = (1 + 2 + 3) = –(–6)/1

The sum of the products of the roots, taken two at a time

S_2 = 1(2) + 1(3) + 2(3) = 11 = 11/1

We can drop the word 'sum' and 'products' for the last relation, because there is only one term (only one product). The product = 1(2)(3) = 6 = –(–6)/1.

Roots of Equations and Descartes’ Rule:

If the coefficients are all real and the complex number z_1, is a root of f(x) = 0, then the conjugate of z_1, viz, $\overline{z_1}$ is also a root of f(x) = 0. Thus, for equations with real, coefficients, complex roots occur in pairs.

A consequence of this is that any equation of an odd degree must have at least one real root.

The number of roots is related to very simple properties of the equation as illustrated below.
Let α_1 be a positive root, ie $x - \alpha_1$, is a factor.
Let α_2 be another positive root, ie, $x^2 - (\alpha_1+\alpha_2)x + \alpha_1 \alpha_2$ is a factor.
Let α_3 be another positive root ie $x^3 - (\alpha_1+\alpha_2+\alpha_3)x^2 + (\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1) x - \alpha_1 \alpha_2 \alpha_3$ is a factor
We note that every positive root introduces a sign change in the polynomial. For 1 root, there is 1 sign change (the coefficient of x is positive and $-\alpha_1$ is negative)

The second root results in a second sign change [$x^2 - (\alpha_1 + \alpha_2)x + \alpha_1 \alpha_2$ has 2 sign changes] and so on.
But every sign change need not correspond to a real positive root. (For example $x^2 - 2x + 4$ has two sign changes but the corresponding equation $x^2 - 2x + 4 = 0$ has no real roots.
The number of positive roots is at the most equal to the number of sign changes. It could also be less than that by 2, 4… i.e., if there are k sign changes in f(x), the number of positive roots could be k, k–2, k–4, …

This is called **Descartes' Rule of Signs**. This rule can be extended to negative roots as follows. The number of negative roots of f(x) = 0 is equal to the number of positive roots of g(x) = f(–x) = 0

For example, consider
$f(x) = x^5 - 3x^3 + 6x^2 - 28x + 24$.There are 4 sign changes in f(x)
$\therefore$ The number of positive roots could be 4, 2 or 0,
Consider g(x) = f(–x)
$(-x)^5 - 3(-x)^3 + 6(-x)^2 - 28(-x) + 24$
$= -x^5 + 3x^3 + 6x^2 + 28x + 24$
There is only one sign change in f(– x). $\therefore$ The number of negative roots of f(x) = 0 is 1. (It can't be –1, –3, …)

The following table shows the various possibilities for the roots.

Negative	Positive	Complex
1	4	0
1	2	2
1	0	4

We have considered one specific equation and this specific equation has 5 specific roots. We can use more advanced techniques to find the actual roots. But even without that, using only Descartes Rule, we expect exactly one of the 3 situations shown in the table above.

Examples:

2.13. Find the roots of the equation $2x^2 + 13x + 18 = 0$.

Sol. To find the roots of a quadratic equation, following steps are required. First write the coefficient of x i.e. 13 as the sum (or difference) of two parts such that the product of these two parts is equal to the coefficient of x^2 term and constant term i.e. product of 2 and 18 which is 36.
We see that 13 can be written as the sum of 4 and 9 and the product of these two numbers is 36.
$2x^2 + 13x + 18 = 0 \Rightarrow 2x^2 + 4x + 9x + 18 = 0$

Taking 2x common from the first two terms and taking 9 common from the last two terms, we have:
$\Rightarrow 2x(x + 2) + 9(x + 2) = 0$
$\Rightarrow (x + 2)(2x + 9) = 0$, $x = -2$ or $-\frac{9}{2}$.

2.14. Find the roots of the equation $x^2 + x - 12 = 0$.

Sol. Given equation is $x^2 + x - 12 = 0$
Applying the procedure described above, we have
$\Rightarrow x^2 + 4x - 3x - 12 = 0$
$\Rightarrow x(x + 4) - 3(x + 4) = 0$
$\Rightarrow (x - 3)(x + 4) = 0 \Rightarrow x = 3$ or $x = -4$.

2.15. Find the roots of the equation $11x^2 - 37x + 30 = 0$.

Sol. We have to write –37 as the sum of two parts whose product should be equal to $(11) \times (30)$
$(-22) + (-15) = -37$ and $(-22)(-15) = 11 \times 30$
Therefore, $11x^2 - 37x + 30 = 0$
$\Rightarrow 11x^2 - 22x - 15x + 30 = 0$
$\Rightarrow 11x(x - 2) - 15(x - 2) = 0$
$\Rightarrow (11x - 15)(x - 2) = 0$ $x = \frac{15}{11}$ or 2.

2.16. Discuss the nature of the roots of the equation $8x^2 - 2x - 4 = 0$.

Sol. For the quadratic equation $ax^2 + bx + c = 0$ the nature of the roots is given by the discriminant $b^2 - 4ac$.
Discriminant of $8x^2 - 2x - 4 = 0$ is
$(-2)^2 - 4(8)(-4) = 132$.
Since the discriminant is positive but not a perfect square, the roots of the equation are irrational and unequal.

2.17. Comment on the nature of the roots of $3x^2 - x - 4 = 0$.

Sol. Discriminant of $3x^2 - x - 4 = 0$ is $(-1)^2 - 4(3)(-4) = 1 + 48 = 49$. Since the discriminant is positive and a perfect square, the roots of the equation are rational and unequal.

2.18. If the sum of the roots of the equation $Rx^2 + 5x - 24 = 0$ is 5/11, then find the product of the roots of that equation.

Sol. For a quadratic equation $ax^2 + bx + c = 0$, the sum of the roots is (–b/a) and the product of the roots is (c/a).
Sum of the roots of the equation
$Rx^2 + 5x - 24 = 0$ is $\left(\frac{-5}{R}\right)$ which is given as $\frac{5}{11}$
$\therefore R = -11$
In the given equation, product of the roots $= \frac{-24}{R} = \frac{-24}{-11} = +\frac{24}{11}$.

2.19. Find the value of k, so that the roots of $6x^2 - 12x - k = 0$ are reciprocals of each other.

Sol. If the roots of the equation are reciprocals of each other, then the product of the roots should be equal to 1.
$\Rightarrow \frac{-k}{6} = 1$. Therefore k = –6.

2.20. If $4 + \sqrt{7}$ is one root of a quadratic equation with rational co-efficients, then find the other root of the equation.

Sol. When the coefficients of a quadratic equation are rational and the roots are irrational, they occur only in pairs like $p \pm \sqrt{q}$ i.e., if $p + \sqrt{q}$ is one root, then the other root of the equation will be $p - \sqrt{q}$. So, in this case, the other root of the equation will be $4 - \sqrt{7}$.

2.21. Find the positive value of k if one root of the equation $x^2 - kx + 243 = 0$ is thrice the other root.

Sol. If one root of the equation is α, then the other root will be 3α.
We have $(\alpha)(3\alpha) = 3\alpha^2 = 243$
$\Rightarrow \alpha^2 = 81$
$\Rightarrow \alpha = \pm 9$. Hence $3\alpha = \pm 27$.
Sum of the roots $= -\left(\frac{-k}{1}\right) = k = 4\alpha = \pm 36$.
Since we need the positive value of k, so k = 36.

2.22. Form a quadratic equation whose roots are 4 and 21.

Sol. Sum of the roots = 4 + 21 = 25.
Product of the roots $= 4 \times 21 = 84$.
We know that if p is the sum of the roots and q is the product of the roots of a quadratic equation, the equation will be $x^2 - px + q = 0$. Hence the required equation will be $x^2 - 25x + 84 = 0$.

2.23. Form a quadratic equation with rational coefficients, one of whose roots is $5 + \sqrt{6}$.

Sol. If $5 + \sqrt{6}$ is one root, then the other root is $5 - \sqrt{6}$ (because the coefficients are rational).
Sum of the roots $= 5 + \sqrt{6} + 5 - \sqrt{6} = 10$.
Product of the roots $= (5 + \sqrt{6})(5 - \sqrt{6})$
$= 25 - 6 = 19$.
Thus the required equation is
$x^2 - 10x + 19 = 0$.

2.24. If the price of each book goes up by ₹5, then Atul can buy 20 books less for ₹1200. Find the original price and the number of books Atul could buy at the original price.

Sol. Let the original price of each book be x.
Then the new price of each book will be x + 5.
Number of books that can be bought at the original price $= \frac{1200}{x}$
Number of books that can be bought at the new price $= \frac{1200}{x+5}$
Given that Atul gets 20 books less at new price
i.e. $\frac{1200}{x} - \frac{1200}{x+5} = 20$

$\Rightarrow \frac{60}{x} - \frac{60}{x+5} = 1 \Rightarrow \frac{60(x+5-x)}{x^2+5x} = 1$
$\Rightarrow 300 = x^2 + 5x \Rightarrow x^2 + 5x - 300 = 0$
$\Rightarrow (x + 20)(x - 15) = 0 \Rightarrow x = -20$ or 15
As the price cannot be negative, the original price is ₹15.

2.25. If α and β are the roots of the equation $x^2 - 3x - 180 = 0$ such that $\alpha < \beta$, then find the values of

(i) $\alpha^2 + \beta^2$ (ii) $\frac{1}{\alpha} + \frac{1}{\beta}$ (iii) $\alpha - \beta$

Sol. From the given equation, we get $\alpha + \beta = 3$ and $\alpha\beta = -180$
(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= (3)^2 - 2(-180) = 369$
(ii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{+3}{-180} = \frac{-1}{60}$
(iii) $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$
$\Rightarrow \alpha - \beta = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$
$= \pm\sqrt{(+3)^2 - 4(-180)} = \pm\sqrt{9+720}$
$= \pm\sqrt{729} = \pm 27$; as $\alpha < \beta$, $\alpha - \beta = -27$

2.26. If $\sqrt{x+4} + \sqrt{x+8} = 7$, then find the value of x.

Sol. Given $\sqrt{x+4} + \sqrt{x+8} = 7$
Squaring on both sides, we get
$x + 4 + x + 8 + 2\left(\sqrt{x+4}\sqrt{x+8}\right) = 49$
$\Rightarrow 2x + 12 + 2\sqrt{x^2+12x+32} = 49$
$\Rightarrow 2x - 37 = -2\sqrt{x^2+12x+32}$
Squaring again on both sides, we have
$(2x - 37)^2 = 4(x^2 + 12x + 32)$
$\Rightarrow 4x^2 - 148x + 1369 = 4x^2 + 48x + 128$
$\Rightarrow 1241 = 196x$
$\Rightarrow x = \frac{1241}{196}$

2.27. If $4^{2x+1} + 4^{x+1} = 80$, then find the value of x.

Sol. Given $4^{2x+1} + 4^{x+1} = 80$
$\Rightarrow 4^{2x} \cdot 4 + 4^x \cdot 4 = 80$
$4^{2x} + 4^x = 20$
Substituting $4^x = a$,
we get $a^2 + a = 20$
$\Rightarrow a^2 + a - 20 = 0$
$\Rightarrow (a + 5)(a - 4) = 0$
$\Rightarrow a = -5$ or 4
If $4^x = -5$, there is no possible value for x as no power of 4 gives negative value.
If $4^x = 4$, then x = 1.

2.28. Find the nature of roots of the equation, $f(x) = x^3 + x - 2 = 0$.

Sol. There is only 1 change of sign in f(x).
We know that when f(x) has r changes of sign then f(x) has r, r – 2, r – 4, ….. positive roots.
$\therefore$ f(x) = 0 has one positive root.

Now
$f(-x) \equiv -x^3 - x - 2 = 0$. q = 0
Since there is no
Hence f(x) = 0 has 1 real root and two complex roots.

2.29. How many non real-roots does the equation $x^4 - 2x^2 + 3x - 2 = 0$ have?

Sol. Let $f(x) = x^4 - 2x^2 + 3x - 2$
f(x) has 3 sign changes
$\therefore$ f(x) has 3 or 1 positive roots.
$f(-x) = x^4 - 2x^2 - 3x - 2$
$\therefore$ f(– x) has one sign change
$\therefore$ f(x) has exactly one negative root.
As the sum of the co-efficient of f(x) is zero,
x = 1 is a root f f(x) = 0
$\therefore f(x) = (x - 1)(x^3 + x^2 - x + 2) = (x - 1) f_1(x)$
By trial, $f_1(-2) = 0$
$\therefore f_1(x) = (x + 2)(x^2 - x + 1)$
We can see that $x^2 - x + 1 = 0$ has two non real roots.
$\therefore$ f(x) has one positive, one negative and two non-real roots.

2.30. If p – q, p, p + q are the roots of the equation $x^3 - 18x^2 + 99x - 162 = 0$, then find the values of p and q.

Sol. Given p – q, p, p + q are the roots of the equation.
$\therefore$The sum of the roots is (p – q) + p + (p + q) = 18
$\Rightarrow 3p = 18 \Rightarrow p = 6$
and the product of the roots is (p – q) p (p + q) = 162
$p^2 - q^2 = \frac{162}{6} = 27 \Rightarrow 36 - q^2 = 27$
$\Rightarrow q = \pm 3 \therefore p = 6$ and $q = \pm 3$

2.31. Find the range of the expression $\frac{x-2}{x^2+x+3}$ where x is real.

Sol. Let $f(x) = \frac{x-2}{x^2+x+3} = y$
$\Rightarrow x^2y + xy + 3y = x - 2$
i.e, $x^2y + x(y - 1) + 3y + 2 = 0$
f(x) can have any value y, provided the above equation in x has real roots
$\therefore b^2 - 4ac \geq 0$
$\Rightarrow (y - 1)^2 - 4y(3y + 2) \geq 0$
i.e, $11y^2 + 10y - 1 \leq 0$
$(11y - 1)(y + 1) \leq 0 \Rightarrow -1 \leq y \leq \frac{1}{11}$
$\therefore$ The range of y is $\left[-1, \frac{1}{11}\right]$

2.32. Solve the equation $x^4 - 2x^3 - 19x^2 + 8x + 60 = 0$, given that two of the roots are α and $-\alpha$, where $\alpha > 0$.

Sol. Let the roots of the equation be $\alpha, -\alpha, \beta, \gamma$. Let $\beta > \gamma$
The sum of the roots = $\alpha - \alpha + \beta + \gamma = 2$
$\Rightarrow \beta + \gamma = 2$
The sum of the products of the roots taken three at a time =
$-\alpha^2\beta - \alpha^2\gamma + \alpha\beta\gamma - \alpha\beta\gamma = -8$
$\alpha^2(\beta + \gamma) = 8. \therefore \alpha^2 = 4$
Product of the roots = $-\alpha^2\beta\gamma = 60$. $\beta\gamma = -15$
$\beta + \gamma = 2$ and $\beta\gamma = -15$, $\alpha^2 = 4$
$\beta = 5$ and $\gamma = -3$, $\alpha = \pm 2$
$\Rightarrow$ The roots are –2, 2, – 3 and 5.

2.33. Find the sum of the squares of the roots of the equation $x^3 - 4x^2 + x + 6 = 0$.

Sol. Let α, β, γ be the roots of the given equation.
$\alpha + \beta + \gamma = 4$, $\alpha\beta + \beta\gamma + \alpha\gamma = 1$ (and $\alpha\beta\gamma = -6$).
$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$
$= (4)^2 - 2(1) = 14$

Concept Review Questions

Directions for questions 1 to 25: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. Find the roots of the quadratic equation
$x^2 - 9x + \frac{41}{4} = 0$.
(A) –1, –10 (B) 1, 10
(C) $\frac{9+5\sqrt{2}}{2}, \frac{9-5\sqrt{2}}{2}$ (D) None of these

2. Find the quadratic equation in x whose roots are 5 and 6.
(A) $x^2 + 11x + 30 = 0$ (B) $x^2 - 11x - 30 = 0$
(C) $x^2 - 11x + 30 = 0$ (D) $x^2 + 11x - 30 = 0$

3. The sum of the roots of a quadratic equation is 8 and the constant term of the equation is 15. Find the quadratic equation.
(A) $x^2 - 8x - 15 = 0$
(B) $x^2 - 8x + 15 = 0$
(C) $x^2 + 8x - 15 = 0$
(D) Cannot be determined

4. Find the sum of the roots of $9x^2 - 144x + 92 = 0$.

$\boxed{}$

5. If the sum of the roots of a quadratic equation is 20 and the product of its roots is 36, find the equation.
(A) $x^2 + 20x + 36 = 0$
(B) $x^2 - 20x + 36 = 0$
(C) $x^2 - 20x - 36 = 0$
(D) $x^2 + 20x - 36 = 0$

6. If the sum of the roots of a quadratic equation is 22 and the product of its roots is 72, find the roots of the equation.
(A) 1, 72 (B) 2, 36 (C) 3, 24 (D) 4, 18

7. Find the nature of the roots of $4x^2 + 8x - 11 = 0$.
(A) Real and equal
(B) Complex conjugates
(C) Conjugate surds
(D) Rational and unequal

8. For what value of p does the equation $x^2 + px + 81 = 0$ have equal roots?
(A) 18 (B) 9
(C) ± 9 (D) ± 18

9. Find the number of roots of $(x^n - b)^2 = 0$.
(A) n
(B) n^2
(C) 2n
(D) Infinitely many

10. Find the signs of the roots of $x^2 - 14x + p^2 = 0$, where p is a non-zero real number.
(A) Both positive
(B) Both negative
(C) One positive, one negative
(D) Cannot be determined

11. Find the quadratic equation whose roots are 3 more than the roots of $x^2 - 4x + 10 = 0$.
(A) $x^2 - 10x - 23 = 0$
(B) $x^2 + 10x - 31 = 0$
(C) $x^2 + 10x + 23 = 0$
(D) $x^2 - 10x + 31 = 0$

12. Find the quadratic equation whose roots are the reciprocals of the roots of $2x^2 + 5x + 3 = 0$.
(A) $3x^2 + 5x - 2 = 0$
(B) $3x^2 + 5x + 2 = 0$
(C) $3x^2 - 5x + 2 = 0$
(D) $3x^2 - 5x - 2 = 0$

13. What can be said about the roots of $ax^2 + bx + a = 0$?
(A) Roots are equal in magnitude but opposite in sign
(B) The roots are reciprocals of each other
(C) One root is twice the reciprocal of the other
(D) Both (A) and (B)

14. Find the quadratic equation whose roots are the reciprocals of the roots of $3x^2 + 4x + 2 = 0$.
(A) $2x^2 + 4x + 3 = 0$
(B) $4x^2 + 2x + 3 = 0$
(C) $2x^2 + 3x + 4 = 0$
(D) $4x^2 + 3x + 2 = 0$

15. Find the quadratic equation whose roots are half of the roots of $x^2 + 7x + 11 = 0$.
(A) $4x^2 + 7x + 44 = 0$
(B) $4x^2 + 14x + 11 = 0$
(C) $4x^2 + 7x + 11 = 0$
(D) $x^2 + 14x + 44 = 0$

16. The quadratic expression $px^2 + qx + r$ has its maximum or minimum value at _______.
(A) $\frac{-q}{2p}$ (B) $\frac{q}{2p}$
(C) $\frac{-2q}{p}$ (D) $\frac{2q}{p}$

17. For the quadratic expression $px^2 + 9x + 8$, $\frac{32p-81}{4p}$ is the
I. minimum value when p > 0
II. minimum value when p < 0
III. maximum value when p > 0
IV. maximum value when p < 0
Which of the above statements is/are true?
(A) Only I
(B) Only IV
(C) Both I and IV
(D) Both II and III

18. Find the maximum or minimum value of the expression $x^2 + x + 5$.
(A) Minimum value of $\frac{19}{4}$
(B) Minimum value of $-\frac{19}{2}$
(C) Maximum value of $\frac{19}{4}$
(D) Maximum value of $\frac{19}{2}$

19. If the roots of a quadratic equation are reciprocals of one another and the constant term is 3, what is the coefficient of the second degree term?
(A) 1
(B) 3
(C) –3
(D) Cannot be determined

20. One of the roots of a quadratic equation with rational coefficients is $3 + 2\sqrt{2}$. What is the sum of the roots of the equation?
(A) $-4\sqrt{2}$ (B) $-4\sqrt{2}$
(C) 6 (D) – 6

21. The remainder when $x^3 + 7x^2 + 4x + 5$ is divided by x is $\boxed{}$

22. The remainder of $3x^2 - 5x - 2$ divided by $x - 2$ is

$\boxed{}$

23. If _______ is added to the cubic expression $5x^3 - 2x^2 - 3x - 2$, then the resulting expression will be exactly divisible by $x - 1$.
(A) 4 (B) −2 (C) −4 (D) 2

24. If $ax^2 - bx - c$ is exactly divisible by $x + 1$, then it follows that _______.
(A) b + c = a
(B) a + c = b
(C) a + b = c
(D) None of these

25. If $9x^2 - 13x + c$ is divided by $x + 1$, then the remainder is 12. Find the value of c.

$\boxed{}$

Exercise – 2(a)

Directions for questions 1 to 40: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

1. Solve for x

(i) $x^4 - 41x^2 + 400 = 0$.
(A) ±3, ±4 (B) ±4, ±5
(C) ±3, ±5 (D) ±4, ±6

(ii) $3.2^{2x+1} - 5.2^{x+2} + 16 = 0$ (x is an integer)
(A) 1 (B) 2
(C) 3 (D) 4

(iii) $\sqrt{2x+5} + \sqrt{10x-4} = 7$
(A) 2 (B) 149/8
(C) $\frac{9}{8}$ (D) Either (A) or (B)

(iv) $\sqrt{3x+4} - \sqrt{2x+1} = \sqrt{x-3}$.
(A) 1 (B) –3/2
(C) 4 (D) 7

(v) $\sqrt{x^2 - x - 2} + \sqrt{6x - 8 - x^2} = \sqrt{2x^2 - x - 6}$
(A) –2, 3 (B) 2, 3
(C) 2, –3, 1/2 (D) –2, 3, –1/2

2. If one root of the equation $x^2 - px + 10p = 0$ is twice the other, find the value of p. ($p \neq 0$)

$\boxed{}$

3. If the difference between the roots of the equation $2x^2 - mx + 15 = 0$ is 1/2, find m.
(A) ±10 (B) ±11 (C) 14 (D) ±17

4. Which of the following must be true if the roots of the equation $ax^2 + bx + c = 0$ are in the ratio 2 : 3?
(A) $25b^2 = 6ac$ (B) $6b^2 = 25ac$
(C) $8b^2 = 37ac$ (D) $37b^2 = 8ac$

5. Which of the following must be true if $x^2 + px - q = 0$ has one root as a square root of the other?
(A) $p^2 + q^3 = p(1 - 3q)$
(B) $p^3 + q^2 = q(1 - 3p)$
(C) $p^3 + q^2 = p(1 + 3p)$
(D) $p^3 + q^2 = q(1 + 3p)$

6. If (x + 1) (x + 3) (x + 5) (x + 7) = 5760, find the real values of x.
(A) 5, –13 (B) –5, 13
(C) –5, –13 (D) 5, 13

7. If x is rational and $4(x^2 + 1/x^2) + 16(x + 1/x) - 57 = 0$, then find x.
(A) 2, 1/2 (B) 4, 3/5
(C) 4, 1/4 (D) 3, 1/3

8. There are three constants p, q, r and $r \neq -1$. What is the relation between p and q, such that the roots of the equation $p/(x - p) + q/(x - q) = r$, are equal in magnitude but opposite in sign? (Assume p, q and r are constants and $r \neq -1$.)
(A) pq = 0 (B) pq = – 1
(C) p + q = –1 (D) p + q = 0

9. If α and β are the roots of the equation $2x^2 - 8x + 5 = 0$, find the value of $(\alpha/\beta + \beta/\alpha) + (1/\alpha + 1/\beta) - \alpha\beta$.
(A) –6/5 (B) –2/5 (C) 2/5 (D) 1

10. The roots of $ax^2 + bx + c = 0$ where a, b, c are real numbers, are α and β. The coefficient of x^2 is equal to the constant term. What can be said about $\alpha^2 + \beta^2 + 2$?
(A) It is always positive
(B) It is always negative
(C) It is always non-positive
(D) It is always non-negative

11. The roots of quadratic equation $ax^2 + bx + c = 0$ are α and β. The sum (S) of the roots of another quadratic equation is $(b/c)^2 - 2(a/c)$. Find S in terms of α and β.
(A) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (B) $\frac{1}{\alpha} + \frac{1}{\beta}$
(C) $\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2}$ (D) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

12. One root of the equation $x^2 + mx + n = 0$ is also a root of the equation $x^2 + nx + m = 0$ and $m \neq n$. The value of m + n is

$\boxed{}$

13. Four different numbers are obtained by decreasing a number by 4, decreasing it by 3, decreasing it by 2 and increasing it by 1. The resultant four numbers are such that product of the first and the last is equal to product of the second and the third numbers. Find the original number.
(A) 9 (B) 11 (C) 5 (D) 7

14. A two digit number is such that it is four times the sum of its digits and twice the product of the digits. Find the number.

$\boxed{}$

15. A and B have 25 items between them. A sells all his items at a certain price and B sells all his items at a different price and both of them get the same amounts. Had A sold his items at B's price and B at A's price they would have got ₹576 and ₹676 respectively. How many items did A have?
(A) 12 (B) 13 (C) 15 (D) 17

16. A and B are integers. $A \geq 2$ and $B \leq 6$. If $Ax^2 + Bx + 2 = 0$ has real roots, how many values does (A, B) have?
(A) ∞ (B) 2 (C) 3 (D) 4

17. Find the value of the following expression:
$\sqrt{12(4)\sqrt{12 - 4\sqrt{12 - 4\sqrt{12 \ldots\ldots \infty}}}}$
(A) $2\sqrt{3}$ (B) $4\sqrt{3}$ (C) $4\sqrt{6}$ (D) $6\sqrt{6}$

18. Find the values that $(x^2 - 2x + 4) / (x^2 + 2x + 4)$ can take, if x is a real quantity.
(A) [1/3, 3] (B) (1/3, 3)
(C) [2/3, 4] (D) (2/3, 4)

19. Mohan and Sohan attempted to solve a quadratic equation in x. Mohan made a mistake in writing down the constant term. He obtained the roots as 10 and 18. Sohan made a mistake in writing down the coefficient of x. He obtained the roots as 8 and 24. Find the correct roots.
(A) –16 and –12 (B) 16 and 12
(C) 16 and –12 (D) –16 and 12

20. If both p and q are natural numbers not exceeding 5, the number of equations of the form $px^2 + qx + 1 = 0$ having real roots is

$\boxed{}$

21. The number of real roots of the equation $\frac{P^2}{y+1} + \frac{Q^2}{y} = 1$, where P and Q are real constants with at most one of them being 0 is ______.
(A) 0 (B) 1 (C) 2 (D) 1 or 2

22. If p and q are the roots of $x^2 - 19x + 60 = 0$, compute $\frac{\sqrt{p}}{\sqrt{q}} + \frac{\sqrt{q}}{\sqrt{p}}$.
(A) $\frac{-19}{2\sqrt{15}}$ (B) $\frac{-19\sqrt{15}}{2}$
(C) $\frac{19}{2\sqrt{15}}$ (D) $\frac{19\sqrt{15}}{2}$

23. If α_1, α_2 are the roots of the equation $x^2 - (2k^2 - 3)x + 2k^3 - 3k^2 + k - 5 = 0$ where k = 2, find the value of $\alpha_1^3\alpha_2 + \alpha_1\alpha_2^3$.
(A) 23 (B) 20 (C) –14 (D) –1

24. The maximum value of $\frac{x-1}{x^2 - x + 4}$, where x is real is _______.
(A) $-\frac{1}{5}$ (B) $-\frac{1}{3}$ (C) $\frac{1}{5}$ (D) $\frac{1}{3}$

25. If the magnitude of the difference between the roots (which are not necessarily real) of $x^2 + kx + k = 0$ is less than 6, the range of k is _______.
(A) $\left[2-\sqrt{10}, 2+\sqrt{10}\right]$
(B) $\left(2-\sqrt{10}, 2+\sqrt{10}\right)$
(C) $\left(2-2\sqrt{10}, 2+2\sqrt{10}\right)$
(D) $\left[2-\sqrt{30}, 2+\sqrt{30}\right]$

26. If $y = 2x^2 - 3x + 5$, then the minimum value of y – 2x is _______.
(A) $\frac{15}{8}$ (B) $\frac{31}{8}$ (C) $\frac{-33}{8}$ (D) $-\frac{15}{8}$

27. The roots of $x^2 - ax + 18 = 0$ are positive integers and not coprime to each other. Which of the following can be equal to a^2?
(A) 81 (B) 63 (C) 18 (D) 9

28. If the equation $x^3 + 3x + a = 0$ has no negative root, then which of the following could be the value of a?
(A) 8 (B) 7 (C) – 4 (D) 9

29. If the equation $x^6 + 5x^5 + 11x^4 + 34x^2 + 20x + 24 = 0$ has exactly four non-real roots, then the number of negative roots is

$\boxed{}$

30. A student finds, by trial, two negative and one positive root(s) of the equation $x^5 + 5x^4 + 2802x + 3024 = 103x^3 + 329x^2$. How many non–real roots does the equation have?

$\boxed{}$

31. If one of the roots of the equation $x^3 + 5x^2 - 12x - 36 = 0$ is thrice another root, then the third root is _______.
(A) – 6 (B) 3 (C) –2 (D) – 89/13

32. Find the value of k for which one of the roots of $x^2 - 3x + 2k = 0$ is half that of one of the roots of $x^2 - 10x + 24k = 0$.
(A) 1 (B) 2/3
(C) $\frac{1}{3}$ (D) Either (A) or (B)

33. The roots of the equation $x^3 + \ell x^2 + mx + n = 0$ are the real numbers α, β and γ, where α < β < γ. Also one of these roots is 2. If $\ell - 3 = m$, $m + 8 = n$, then the ratio β : γ is _______.
(A) $\frac{1}{2}$ (B) 1
(C) $\frac{1}{3}$ (D) $\frac{1}{4}$

34. α, β, γ are the roots of the equation $x^3 - 7x - 6 = 0$. Also, 2, α, α – γ are in arithmetic progression. The equation with roots α and γ is _______.
(A) $x^2 + 2x - 3 = 0$
(B) $x^2 - 2x + 3 = 0$
(C) $x^2 - 2x - 3 = 0$
(D) $x^2 + 2x + 3 = 0$

35. The number of rational roots of $x^3 - 6x^2 - 2x + 5 = 0$ is

$\boxed{}$

36. Find the roots of the equation $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$.
(A) – 1, – 4, 1, 6
(B) 4, –3, – 2, – 1
(C) – 1, – 2, 2, 6
(D) – 3, – 1, 2, 4

37. All the roots of $x^4 - 12x^3 + ax^2 + bx + 81 = 0$ are non-negative. The ordered pair(a, b) can be _______.
(A) (9, 36) (B) (27, –108)
(C) (54, –108) (D) (36, 108)

38. The remainder of $x^{72} + x^{60} + x^{48} + x^{36} + x^{24}$ divided by $x^3 - x$ is _______.
(A) $2x^2$ (B) $5x^2$
(C) $3x^2$ (D) x^2

39. If $x + \frac{6}{x} = 2\sqrt{3}$, $x^8 + x^{12}$ = _______.

(A) $36^2(35)$ (B) $36^2(37)$

(C) $-36^2(37)$ (D) $-36^2(35)$

40. F(x) is a polynomial whose degree is atleast 3. When it is divided by x − 5, the remainder is 17. When F(x) is divided by x − 6, the remainder is 19. Find the remainder when F(x) is divided by (x − 5) (x − 6).

(A) 3x + 2 (B) 4x + 3 (C) 4x − 5 (D) 2x + 7

Exercise – 2(b)

Directions for questions 1 to 45: For the Multiple Choice Questions, select the correct alternative from the given choices. For the Non-Multiple Choice Questions, write your answer in the box provided.

Very Easy / Easy

1. Solve for x: $2x^6 + 5x^3 - 7 = 0$. (Assume x is positive).

(A) 1 (B) 2 (C) 7/2 (D) 5/2

2. Solve for x: $\sqrt{x^2+6} + \sqrt{x^2+3} = 5$ (Assume x is positive).

(A) $\pm\frac{\sqrt{46}}{5}$ (B) $\pm 2\sqrt{2}$ (C) $\pm\frac{\sqrt{46}}{10}$ (D) ±2

3. If a quadratic equation $x^2 - px + q = 0$ has two equal roots, then $\frac{p}{2}$ = _______.

(A) $\pm\sqrt{q}$ (B) q (C) q^2 (D) 4q

4. Find the value(s) of k so that the equation $x^2 - 2kx + 9 = 0$ will have equal roots.

(A) ± 1 (B) ± 3 (C) ± 4 (D) ± 9

5. If a positive number is decreased by 2 and then squared the resultant number is 2 less than the original number. Find a possible value of the number.

(A) 1 (B) 4 (C) 3 (D) 7

6. If a number is increased by 4, then result is equal to 5 more than twice the reciprocal of the number. Find the original number.

(A) 1 (B) 2

(C) –1 (D) Either (B) or (C)

7. If I had walked 2 km/hr faster, I would have taken 60 minutes less to walk 12 km. What is my usual speed?

(A) 1 km/hr (B) 2 km/hr

(C) 3 km/hr (D) 4 km/hr

8. The sum of the squares of two positive numbers is 185 and the sum of the larger and thrice the smaller numbers is 35. Find the larger of the two numbers.

(A) 11 (B) 8 (C) 13 (D) 14

9. A positive number exceeds its reciprocal by 168/13. Find the number.

(A) 12 (B) 13 (C) 14 (D) 15

Moderate

10. Find the value of p, if one of the roots of the equation $x^2 - 7x + 2p = 0$ is 3 more than the other root.

(A) 4 (B) 5 (C) 7 (D) 10

11. If the roots of the equation $x^2 + px + 3 = 0$ are two successive odd natural numbers, then find the value of p.

(A) 2 (B) –4 (C) –6 (D) 8

12. A quadratic equation is formed by interchanging a and b of $ax^2 + bx + c = 0$. The equation so formed has its roots as α and β. Find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

(A) $\left(\frac{a}{c}\right)^2 - 2\left(\frac{b}{c}\right)$ (B) $\left(\frac{b}{c}\right)^2 - 2\left(\frac{a}{c}\right)$

(C) $\left(\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$ (D) $\left(\frac{a}{c}\right)^2 - 2\left(\frac{c}{b}\right)$

13. Find the value of k, if the product of the roots of the equation

$x^2 - \left(\sqrt{2} + \sqrt{3k}\right)x + 4y^{2\log_y k} = 0$ is 256.

(A) 8 (B) ±8

(C) 4 (D) ± 4

14. A journey of 1600 km by an express train takes 24 hours less than that by an ordinary train. If the ordinary train is 15 km/hr slower than the express train, find the speed of the ordinary train. (in km/hr)

15. Find the larger of two positive numbers whose sum is 24 and twice the square of the smaller number is 4 more than the square of the larger number.

16. The roots of the equation $2x^2 + 5x + 2 = 0$ are k less than those of the equation $px^2 + qx + r = 0$. Which of the following could be true?

(A) k(pk + 4p + q) + 4p + 2q + r = 0

(B) k(pk – 4p + q) + 4p – 2q + r = 0

(C) k(pk + 4p + q) + 4p – 2q + r = 0

(D) k(pk – 4p + q) + 4p + 2q + r = 0

17. Two boys and two girls went for a movie. Each boy had spent ₹x while each girl spent ₹50 less than each boy. If the product of the total amount spent by the boys and the total amount spent by the girls (both in rupees) is 416, then find the amount spent by each girl. (in ₹)

18. In a joint family, there are 3 couples. The ages (all in years) of the husbands are 4x, 3x and 2x. The age of each women is 4 years less than her husband's age. The product of the ages of the first couple is more than the sum of the products of the ages of the other two couples by 832. Find x.

19. One root of the quadratic equation $ax^2 + bx + c = 0$ is $\frac{2}{\sqrt{3}+\sqrt{5}}$. If c/a is rational, then which of the following is true?
(A) $2b = 2a\left(\sqrt{5}-\sqrt{3}\right)+c\left(\sqrt{5}+\sqrt{3}\right)$
(B) $2b = a\left(\sqrt{5}-\sqrt{3}\right)+2c\left(\sqrt{5}+\sqrt{3}\right)$
(C) $-2b = 2a\left(\sqrt{5}-\sqrt{3}\right)+c\left(\sqrt{5}+\sqrt{3}\right)$
(D) $-2b = a\left(\sqrt{5}-\sqrt{3}\right)+2c\left(\sqrt{5}+\sqrt{3}\right)$

20. If the roots of the equation $cx^2 + bx + a = 0$ are α and β, then find the equation whose roots are α^3 and β^3.
(A) $a^3x^2 - (3abc - b^3)x + c^3 = 0$
(B) $a^3x^2 + (3abc + b^3)x + c^3 = 0$
(C) $c^3x^2 - (3abc - b^3)x + a^3 = 0$
(D) $a^3x^2 + (3abc - b^3)x + c^3 = 0$

21. If $x + 2y = 5$, then find the maximum (or) minimum possible value of $x^2 + y^2$.
(A) Maximum value, 8
(B) Minimum value, 8
(C) Minimum value, 5
(D) Maximum value, 5

22. K is a positive integer satisfying $K^2 \le 36$. How many equations of the form $x^2 + Kx + 4 = 0$ exist such that the roots are real and unequal?

$\boxed{}$

23. If one root of the equation $x^2 - 2Rx + 6 = 0$ is twice one root of the equation $x^2 - 5x + 6 = 0$, then R can be ________.
(A) 7/2 (B) 11/4
(C) Either (A) or (B) (D) –7/2

24. For the equation $x^2 - 2px + 2p = 0$ where p is real, find the nature of the roots.
(A) Real and unequal
(B) Real and equal
(C) Imaginary
(D) Cannot be determined

25. A and B had a total of 50 chocolates. The sum of the squares of the numbers of chocolates with them is 1300. A sells each chocolate at ₹5 and B sells each chocolate at ₹6. Find the total amount they would get, if each had as many chocolates as the other had. How much more/ less would they realize totally in selling the chocolates?
(A) Amount = 270/-, ₹10 more
(B) Amount = 270/-, ₹10 less
(C) Amount = 280/-, ₹10 less
(D) Either (A) or (C)

26. If the sum of the squares of the roots of $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ are equal and $b^2 \ne 2ac$, then which of the following could be true?
(A) $a - c = 0$ (B) $c - a = 1$
(C) $a + c = 1$ (D) $a - c = 1$

27. Solve: $[(x+3)/(x-1)]^4 + 2[(x+3)/(x-1)]^2 - 8 = 0$
(A) 2, 6 (B) 5, 7
(C) 3, 6 (D) None of these

28. If one root of the equation $x^2 - 8x + 15 = 0$ is the same as one root of the equation $x^2 - 10x + k = 0$, then find the values of k.
(A) 21, 12 (B) 15, 25
(C) 21, 25 (D) 12, 25

29. A teacher wrote a quadratic equation of the form $x^2 + bx + c = 0$ on the board and asked his students to find the roots. One student copied the coefficient of x incorrectly and found the roots as 3 and 18. Another student copied the constant term incorrectly and found the roots as 10 and 5. Find the correct equation.
(A) $x^2 + 15x + 54 = 0$
(B) $x^2 - 15x + 54 = 0$
(C) $x^2 + 15x - 54 = 0$
(D) $x^2 - 15x - 54 = 0$

30. If the price of a chocolate goes up by ₹5, 18 less chocolates can be bought for a total amount of ₹1800. Find the number of chocolates bought.
(A) 80 (B) 90 (C) 100 (D) 96

31. The denominator of a fraction is one more than the square of the numerator. When the numerator is increased by 2 and the denominator is increased by 9, the new fraction formed is equal to $\frac{1}{5}$. If the numerator is positive, find the original fraction.
(A) $\frac{1}{2}$ (B) $\frac{2}{5}$ (C) $\frac{3}{10}$ (D) $\frac{5}{26}$

32. If the equation $3x^4 - 13x^3 + 7x^2 + 17x + a - 10 = 0$ has exactly three positive roots, then a can be ________.
(A) 11 (B) 4 (C) 13 (D) 12

33. The number of non–real roots of the equation $x^{15} + 1 = 0$ is

$\boxed{}$

34. Find the value of r for which the following pair of equations yields a unique solution for p, which is positive.
$p^2 - q^2 = 0$
$(p + r)^2 + q^2 = 1$
(A) –2 (B) 2 (C) $-\sqrt{2}$ (D) $\sqrt{2}$

35. If $x^2 - 2x - 15$ is a factor of $x^4 + px^2 + q$, then (p, q) = ________.
(A) (– 34, 225) (B) (34, – 225)
(C) (– 34, – 225) (D) (34, 225)

36. Three of the roots of the equation $x^4 + \ell x^3 + mx^2 + nx + 24 = 0$ are 3, 1 and –2. Which of the following could be the value of $\ell + m - n$?
(A) 0 (B) 1 (C) 2 (D) 3

37. a, b, c are in arithmetic progression and α, β, γ are the roots of $x^3 + ax^2 + bx + c = 0$. Which of the following is true if $\gamma = -1$?
(A) $3\alpha\beta - \alpha - \beta = 1$
(B) $\alpha\beta + \alpha + \beta = 1$
(C) $2\alpha\beta - \alpha - \beta = -1$
(D) $\alpha\beta - \alpha - \beta = 1$

38. If $12x^2 + 25x + k$ is divisible by $3x + 4$, then which of the following is a factor of $12x^2 + 25x + k$?

(A) $4x + 1$ (B) $4x + 3$

(C) $4x + 2$ (D) $4(x+1)$

39. If $2x + \frac{1}{x} = -\sqrt{6}$, then the value of $x^{12} - x^6 =$ _______.

(A) $\frac{9}{64}$ (B) $\frac{7}{64}$

(C) $\frac{11}{64}$ (D) $\frac{5}{64}$

40. The remainder of $x^{1021} + x^2 + x + 5$ divided by $x + 1$ is $\boxed{}$

Difficult / Very Difficult

41. How many roots of the equation $x^3 - 3x^2 + 4 = 0$ are also roots of the equation $x^2 + x - 6 = 0$?

$\boxed{}$

42. If two of the roots of the equation $x^3 + 3x^2 - 10x - 24 = 0$ are such that one is twice the other, then the third root is $\boxed{}$.

43. Two of the roots of the equation $x^3 - 7x^2 + 36 = 0$ are such that one is numerically thrice the other. These roots have opposite signs. Find the difference of the greatest two roots.

$\boxed{}$

44. If one of the roots of $x^2 - 12x + k = 0$ lies between 0 and 1, then the range of k is _______.

(A) $(-\sqrt{13}, 0) \cup (0, \sqrt{13})$

(B) $(-\sqrt{17}, 0) \cup (0, \sqrt{17})$

(C) $(-\sqrt{11}, 0) \cup (0, \sqrt{11})$

(D) None of these.

45. If $x^{10} + x^{11} + \ldots + x^{20}$ is divided by $x^3 + x$, then what is the remainder?

(A) x (B) $-x$

(C) x^2 (D) $-x^2$

Data Sufficiency

Directions for questions 46 to 55: Each question is followed by two statements, I and II. Answer each question using the following instructions:

Choice (A) if the question can be answered by using one of the statements alone, but cannot be answered by using the other statement alone.

Choice (B) if the question can be answered by using either statement alone.

Choice (C) if the question can be answered by using both statements together, but cannot be answered by using either statement alone.

Choice (D) if the question cannot be answered even by using both the statements together."

46. What is the value of k given $ax^2 + kx + 72 = 0$ where a and k are positive integers?

I. The roots of the equation are real and equal.

II. $a = 8$

47. Does $x^2 + y^2 + 4x - 6y + b = 0$ have a unique solution?

I. $b = 13$

II. $b = 0$

48. What is the minimum value of the quadratic expression $3x^2 + bx + c$?

I. The two positive roots of the equation $3x^2 + bx + c = 0$ are reciprocal to each other.

II. The sum of the roots of $3x^2 + bx + c = 0$ is 2.

49. If a, b, c are real numbers, does the quadratic equation $ax^2 + bx + c = 0$ have only real roots?

I. One of the roots is complex

II. One of the roots is real.

50. In the quadratic equation $5x^2 + bx + c = 0$, is $b > 0$?

I. The roots of the equation are positive and reciprocals of each other.

II. The roots of the equation are equal

51. If the two quadratic equations $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ have a root in common, what is the value of k?

I. k is an integer

II. k + 5 is a natural number.

52. If a is an integer how many roots of the equation $ax^2 - (a + 1)x + 1 = 0$ are integers?

I. $|a| \neq 1$

II. $-5 < a < 0$

53. The roots of the equation $x^2 - 5x + a = 0$ are p and q and a is an integer. Is $\left(\frac{p^3 + q^3}{pq}\right)$ an integer?

I. $1 < a < 5$

II. $0 < a < 6$

54. $ax^2 + 3x + 2 = 0$ has roots α and β. Is $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ greater than zero?

I. $a < 0$

II. $2 > a > 0$

55. What is the sum of the roots of $ax^2 + bx + c = 0$?

I. Product of the roots of $cx^2 - ax + b = 0$ is 3.

II. Sum of the roots of $cx^2 + ax + b = 0$ is 4.

Key

Concept Review Questions

1. D	6. D	11. D	16. A	21. 5
2. C	7. C	12. B	17. C	22. 0
3. D	8. D	13. B	18. A	23. D
4. 16	9. C	14. A	19. B	24. C
5. B	10. A	15. B	20. C	25. –10

Exercise – 2(a)

1. (i) B	6. A	15. A	24. C	33. A
(ii) A	7. A	16. A	25. C	34. C
(iii) A	8. D	17. C	26. A	35. 1
(iv) C	9. D	18. A	27. A	36. D
(v) B	10. D	19. B	28. C	37. C
2. 45	11. D	20. 12	29. 2	38. B
3. B	12. –1	21. D	30. 0	39. D
4. B	13. C	22. C	31. B	40. D
5. B	14. 36	23. A	32. A	

Exercise – 2(b)

1. A	9. B	17. 2	25. D	33. 14	41. 2	49. B
2. A	10. B	18. 16	26. A	34. C	42. 3	50. A
3. A	11. B	19. C	27. D	35. A	43. 3	51. B
4. B	12. A	20. C	28. C	36. D	44. D	52. A
5. C	13. A	21. C	29. B	37. D	45. B	53. A
6. D	14. 25	22. 2	30. B	38. B	46. C	54. A
7. D	15. 14	23. C	31. D	39. A	47. B	55. C
8. A	16. B	24. D	32. B	40. 4	48. C	