Theory of Koopman MPC

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Consider a discrete-time nonlinear controlled dynamical system

$$x^{+}=f(x,u), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathcal{U} \subset \mathbb{R}^m$. Following the Koopman framework, this system can be lifted into a linear regieme where

$$\begin{aligned} z^{+} &= Az+Bu \\ \hat{x} &= Cz \end{aligned} \tag{2}$$

where $z \in \mathbb{R}^N$ and $\hat{x}$ is the prediction of x^+. $B \in \mathbb{R}^{N\times m}$ and $C \in \mathbb{R}^{n\times N}$. Initial condition is given by

$$z_0=\boldsymbol{\psi}\left(x_0\right):=\left[\begin{array}{c}\psi_1\left(x_0\right) \\ \vdots \\ \psi_N\left(x_0\right)\end{array}\right] \tag{3}$$

where $\psi_i : \mathbb{R}^n \rightarrow \mathbb{R}, i=1,\ldots,N$ are the (typically) nonlinear lifting functions.

Model predictive control on the lifted state solves at each time instance k of the closed-loop operator the optimization problem:

$$\begin{array}{ll} \underset{u_i, z_i}{\operatorname{minimize}} & J\left(\left(u_i\right)_{i=0}^{N_p-1},\left(z_i\right)_{i=0}^{N_p}\right) \\ \text{subject to} & z_{i+1}=A z_i+B u_i, \quad i=0, \ldots, N_p-1 \\ & E_i z_i+F_i u_i \leq b_i, \quad i=0, \ldots, N_p-1 \\ & E_{N_p} z_{N_p} \leq b_{N_p} \\ \text{parameter} & z_0=\boldsymbol{\psi}\left(x_k\right) \end{array} \tag{4}$$

where N_p is the prediction horizon, matrices $E_i \in \mathbb{R}^{n_c\times N}$, $F_i \in \mathbb{R}^{n_c\times m}$ and vector $b_i \in \mathbb{R}^{n_c}$ define state and input polyhedral constraints. The quadratic cost function J is given by

$$\begin{aligned} & J\left(\left(u_i\right)_{i=0}^{N_p-1},\left(z_i\right)_{i=0}^{N_p}\right)=z_{N_p}^{\top} Q_{N_p} z_{N_p}+q_{N_p}^{\top} z_{N_p} \\ & +\sum_{i=0}^{N_p-1} z_i^{\top} Q_i z_i+u_i^{\top} R_i u_i+q_i^{\top} z_i+r_i^{\top} u_i \end{aligned} \tag{5}$$

where $q_i \in \mathbb{R}^N$ places cost on the lifted state (i.e. penalizing nonlinear functions of the original state x).
The dependence on the lifting dimension for solving optimization 4 can be eliminated via dense transform

$$\begin{aligned} & \underset{U \in \mathbb{R}^{m N_p}}{\operatorname{minimize}} \quad U^{\top} H U^{\top}+h^{\top} U+z_0^{\top} G U \\ & \text{subject to } L U+M z_0 \leq c \\ & \text{parameter} \quad z_0=\boldsymbol{\psi}\left(x_k\right) \end{aligned} \tag{6}$$

for some positive-semidefinite matrix $H \in \mathbb{R}^{mN_p\times mN_p}$ and some matrices and vectors $h \in \mathbb{R}^{mN_p}, G \in \mathbb{R}^{N\times mN_p}, L \in \mathbb{R}^{n_c\times N_p\times mN_p}$, $M \in \mathbb{R}^{n_cN_p\times N}$ and $c \in \mathbb{R}^{n_cN_p}$. The optimization is over the vector of predicted control inputs $U=\left[u_0^{\top}, u_1^{\top}, \ldots, u_{N_p-1}^{\top}\right]^{\top}$. Parameters in 6 are:

$$H=\mathbf{R}+\mathbf{B}^{\top} \mathbf{Q B}, \quad h=\mathbf{B}^{\top} \mathbf{q}+\mathbf{r}, \quad G=2 \mathbf{A}^{\top} \mathbf{Q B} \tag{7}$$

$$L=\mathbf{F}+\mathbf{E B}, \quad M=\mathbf{E A}, \quad c=\left[b_0^{\top}, \ldots, b_{N_p}^{\top}\right]^{\top} \tag{8}$$

where

$$\mathbf{A}=\left[\begin{array}{c} I \\ A \\ A^2 \\ \vdots \\ A^{N_p} \end{array}\right] \tag{9}$$

$$\mathbf{B}=\left[\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ B & 0 & \cdots & 0 \\ A B & B & \cdots & 0 \\ \vdots & \ddots & \ddots & \\ A^{N_p-1} B & \cdots & A B & B \end{array}\right] \tag{10}$$

$$\mathbf{F}=\left[\begin{array}{cccc} F_0 & 0 & \cdots & 0 \\ 0 & F_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & F_{N_p-1} \\ 0 & 0 & \cdots & 0 \end{array}\right] \tag{11}$$

$$\mathbf{Q}=\operatorname{diag}\left(Q_0, \ldots, Q_{N_p}\right), \mathbf{R}=\operatorname{diag}\left(R_0, \ldots, R_{N_p-1}\right) \tag{12}$$

$$\mathbf{E}=\operatorname{diag}\left(E_0, \ldots, E_{N_p}\right), \tag{13}$$

$$\mathbf{q}=\left[q_0^{\top}, \ldots, q_{N_p}^{\top}\right]^{\top}, \tag{14}$$

$$\mathbf{r}=\left[r_0^{\top}, \ldots, r_{N_p-1}^{\top}\right]^{\top} \tag{15}$$

with diag$(\cdot, \ldots, \cdot)$ denoting a block-diagonal matrix composed of the arguments.
The closed-loop operation of the lifting-based MPC can be summarized by the following algorithm, where $U^{\star}_{1:m}$ denotes the first m components of U^*:

Algorithm 1 Koopman MPC

1: **for** $k=0,1,\ldots$ **do**
2: Set $z_0 := \boldsymbol{\psi}\left(x_k\right)$
3: Solve 6 to get an optimal solution U^*
4: Set $u_k = U^{\star}_{1:m}$
5:
6: $x_{k+1} = f\left(x_k, u_k\right)$ ▷ Solve using ODE45 to next timestep