Ay 20 #2 - The Solar System: planets, moons, and minor planets

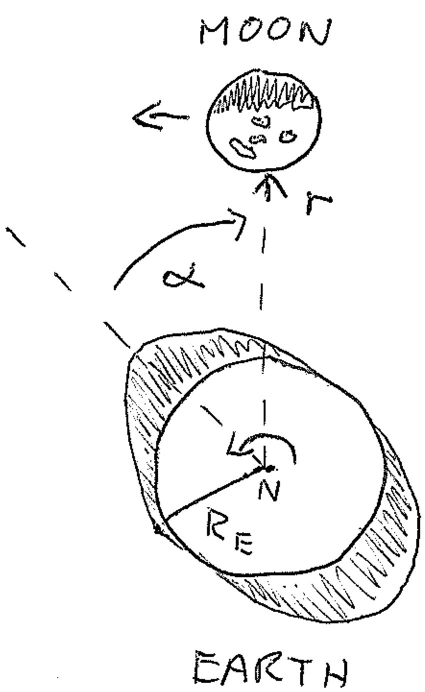
* Gravity (Newtonian)

$$\underline{F} = \frac{G m_1 m_2}{|\underline{r}|^3} \underline{r}, \quad U = -\frac{G m_1 m_2}{|\underline{r}|}.$$

From this, and the standard laws of classical mechanics, most astrophysical dynamics can be solved. And more!

Consider two examples ...

Tidal locking and spin-down



The "tidal" force gradient across the Earth excites a deformation in the Earth, which leads the lunar orbit due to the Earth's rotation. The asymmetry causes the Earth to spin down.

A rough calculation:

We need to estimate the torque

$$\tau = \frac{dL}{dt} = I_E \frac{d\Omega}{dt}. \quad \left(I_E \approx \frac{2}{5} M_E R_E^2\right)$$

First, $\tau \approx \underset{?}{2} R_E F_T \sin\alpha$

F_T "force due to tides" not "tidal force".

$$F_T \approx \frac{dF_G}{dr} \cdot \underset{?}{2} R_E \cos\alpha, \quad \text{where}$$

$$\frac{dF_G}{dr} = -\frac{2GMm_T}{r^3} = -\frac{2GM}{r^3}\pi R_E^2 h\rho$$

h is the mean tidal height. Finally,

$$2\pi \frac{d\nu}{dt} = -\frac{2R_E \sin\alpha}{\frac{2}{5}M_E R_E^2} \cdot 2R_E\cos\alpha \cdot \frac{2GM}{r^3}\pi R_E^2 h\rho.$$

$$\frac{d\nu}{dt} = -\frac{5h\rho GMR_E^2 \sin 2\alpha}{M_E r^3}$$

Constants: $h \sim 10^2\,\mathrm{cm}$, $\rho \sim 1\,\mathrm{g\,cm^{-3}}$,

$G = 6.7 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$, $M = 7 \times 10^{25}\,\mathrm{g}$,

$R_E = 6.4 \times 10^{8}\,\mathrm{cm}$, $\sin 2\alpha \sim 1/13$,

$M_E = 6 \times 10^{27}\,\mathrm{g}$, $r \sim 4 \times 10^{10}\,\mathrm{cm}$.

Then,
$$\frac{d\nu}{dt} \simeq -2 \times 10^{-22}\ \mathrm{Hz\ s^{-1}}$$

Now, because
$$\frac{dP}{dt} = -P^2 \frac{d\nu}{dt} \simeq 1.5 \times 10^{-12}\ \mathrm{s\ s^{-1}}$$

or 1s every $\sim$ 20,000 years.

Note the (several) first-order approximations!

Gravitational radiation

The LIGO detectors are sensitive to "strain amplitudes", or distortions in the space-like part of the metric, of $\sim 10^{-21}$ at 10^2 Hz.

Consider units where $G = c = 1$.

The gravitational wave (GW) strain is easily approximated by

$$h \sim \frac{M}{D}.$$

The luminosity is then $L \sim h^2$.

In "real" terms,

$$h \sim \frac{G}{c^2}\frac{M}{D}, \quad L \sim \frac{c^3}{16\pi G} h^2 \cdot 4\pi D^2$$

Where do the pre-factors come from?

Binary neutron star: $M = 2 \times 2.8 M_{\odot}$.
solar mass

Gives $h_{LIGO} = 10^{-21}$,

$$D = 4 \times 10^{26} \text{ cm} = 130 \text{ Mpc}.$$

$L = 2 \times 10^{49}$ erg s^{-1}, or 2×10^{28} g converted to energy per s (3 Earths!).

* Gravitational orbits.

In polar coordinates, the equations of motion is

$$r\left(\frac{d\theta}{dt}\right)^2 - \frac{d^2r}{dt^2} = \frac{GM_1}{r^2}. \quad (\theta \text{ is "mean anomaly"})$$

At any time, the total energy of the system is

$$E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r}, \quad M = M_1 + M_2,$$
$$\mu = \frac{M_1 M_2}{M} \quad (\text{reduced mass})$$

Either relation, together with the assumption of constant angular momentum $L = \mu r^2 \frac{d\theta}{dt}$, can be solved to derive Kepler's laws.
(exercise for student - can use standard, Lagrangian or Hamiltonian methods)

Solution:

$$r = \frac{h^2}{GM}\,\frac{1}{1 + e\cos\theta}, \quad h = r^2\frac{d\theta}{dt} = \sqrt{GM_1 p} \quad (\text{semi latus rectum})$$

is specific angular momentum.

Circular orbits $\quad e = 0$ - super special.

$$r = \frac{h^2}{GM} = \frac{r^2 v_{orb}^2}{GM} \Rightarrow v_{orb} = \sqrt{\frac{GM}{r}}.$$

Elliptical orbits $\quad 0 < e < 1$

If a is the semi-major axis,

$$2a = \frac{p}{1-e} + \frac{p}{1+e} \quad \text{(apoapsis + periapsis)}$$

$$\Rightarrow p = \frac{h^2}{GM} = a(1-e^2).$$

With this substitution, $r = \dfrac{a(1-e^2)}{1+e\cos\theta}$,

$$P^2 = \frac{4\pi^2}{GM} a^3, \quad v_{orb} = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}.$$

Parabolic orbits $\quad e = 1$ - super special

Consider energy @ periapsis ($\theta = 0$).

First, $r_p = \dfrac{h^2}{2GM} \Rightarrow r_p = \dfrac{2GM}{v_p^2}$.

Then, $$E_p = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r_p} = 0 \; !$$

Must be 0 everywhere! Then,

$$v = \sqrt{\frac{2GM}{r}} \quad (\text{escape velocity}).$$

<u>Hyperbolic orbits</u> $e > 1$

Excess velocity @ $r = \infty$ $\quad v_\infty = \sqrt{\frac{GM}{a}}$.

Properties @ periapse (velocity, distance) can then be derived from orbit equation and conservation of energy (exercise for student).