Ay 20 #9 – Opacity mechanisms & convection

We have previously discussed the generation of energy in the Sun, and the transport of radiation in its atmosphere. The goal of this lecture is to understand how energy is transported from the core to the atmosphere.

Radiation and convection are the two most important effects, as conduction is unimportant.

* Radiation effects.

Recall the definition of radiation pressure for isotropic intensity:

$$P = \frac{4\pi}{3c} I .$$

In this regime (LTE), I is described only by the local Temperature:

$$I = \int_0^\infty B_\nu \, d\nu = \frac{\sigma T^4}{\pi} \Rightarrow P = \frac{4\sigma}{3c} T^4 .$$

Planck law

Further, in a plane-parallel (one-dimensional) transport geometry (e.g., radially outward), recall that

$$\frac{dP}{d\tau_\nu} = -\frac{1}{\bar{\kappa}\rho}\frac{dP}{dr} = \overset{\substack{\text{radiative} \\ \text{flux} \\ \nwarrow}}{\frac{F}{c}} = \frac{L_r}{4\pi r^2 c},$$

where L_r is the luminosity @ radius r generated by all mass interior to r:

$$\frac{dL_r}{dr} = 4\pi r^2 \rho\,\varepsilon \swarrow$$

energy generated per unit mass per unit time.

So, combining the above,

$$\frac{dP}{dr} = \frac{16\sigma}{3c}T^3\frac{dT}{dr} = -\frac{\bar{\kappa}\rho L_r}{4\pi r^2 c}$$

$$\Rightarrow \quad \frac{dT}{dr} = -\frac{3\bar{\kappa}\rho L_r}{64\pi\sigma r^2 T^3}$$

This describes the temperature gradient in a (static, 1D, LTE) star where energy transport is dominated by radiation.

What are the dependencies?

And what exactly is $\bar{\kappa}$?

Let's repeat the above, but maintain a ν-dependence in the opacity, etc.

At a frequency ν, setting $I_\nu = B_\nu$,

$$\frac{dP_\nu}{dr} = \frac{4\pi}{3c} \frac{dB_\nu}{dT} \frac{dT}{dr} = -\frac{\kappa_\nu \rho F_\nu}{c}$$

Rearranging and integrating over frequency,

$$\int_0^\infty F_\nu \, d\nu = \frac{L_r}{4\pi r^2} = -\frac{4\pi}{3\rho} \frac{dT}{dr} \int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} \, d\nu$$

We define the Rosseland mean opacity as

$$\frac{1}{\bar{\kappa}_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} \, d\nu}{\int_0^\infty \frac{dB_\nu}{dT} \, d\nu} = (4\sigma T^3)^{-1} \int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} \, d\nu.$$

This results in the previous expression for $\frac{dT}{dr}$ with $\bar{\kappa} \to \bar{\kappa}_R$.

* Opacity mechanisms.

We've already discussed Thompson scattering in the context of the Eddington luminosity:

$$\kappa_T = \frac{n_e \sigma_T}{\rho}, \quad \sigma_T \sim 6.6 \times 10^{-25}\ cm^2.$$

(Thompson cross section)

Other mechanisms include

<u>Free</u> - <u>free</u> <u>opacity</u>

Nice illustration

of Kirchhoff's Law $\left(\frac{j_\nu}{\kappa_\nu} \propto B_\nu(T) \right)$.

The interaction of ν & e^- results in the e^- being transitioned to a higher energy with respect to the plasma. For <u>spontaneous emission</u>,

$$\rho j_\nu \propto n^2 T^{-1/2} e^{-h\nu/kT}$$ Boltzmann factor.

Therefore, given $B_\nu \propto \nu^3 \left(e^{h\nu/kT} - 1\right)^{-1}$,

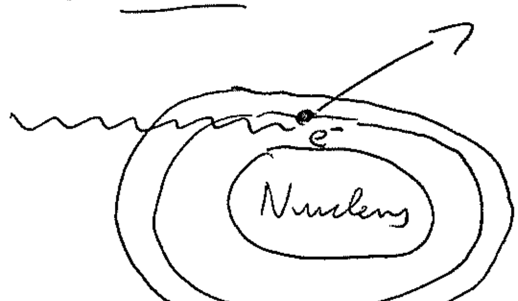
$$\kappa_\nu \propto \rho T^{-1/2} \nu^{-3} \left(1 - e^{-h\nu/kT}\right)$$

<u>Bound</u> - <u>free</u> <u>opacity</u> <u>(photoionization)</u>

For hydrogen,

$$\sigma_{bf} \propto \frac{1}{n^5} \lambda^3$$

quantum state



Useful quantity: for ground state ($n=1$), the ionization potential of hydrogen is 13.6 eV. Scales as n^{-2}.

The distribution of atoms in different states n is defined by the Saha equation:

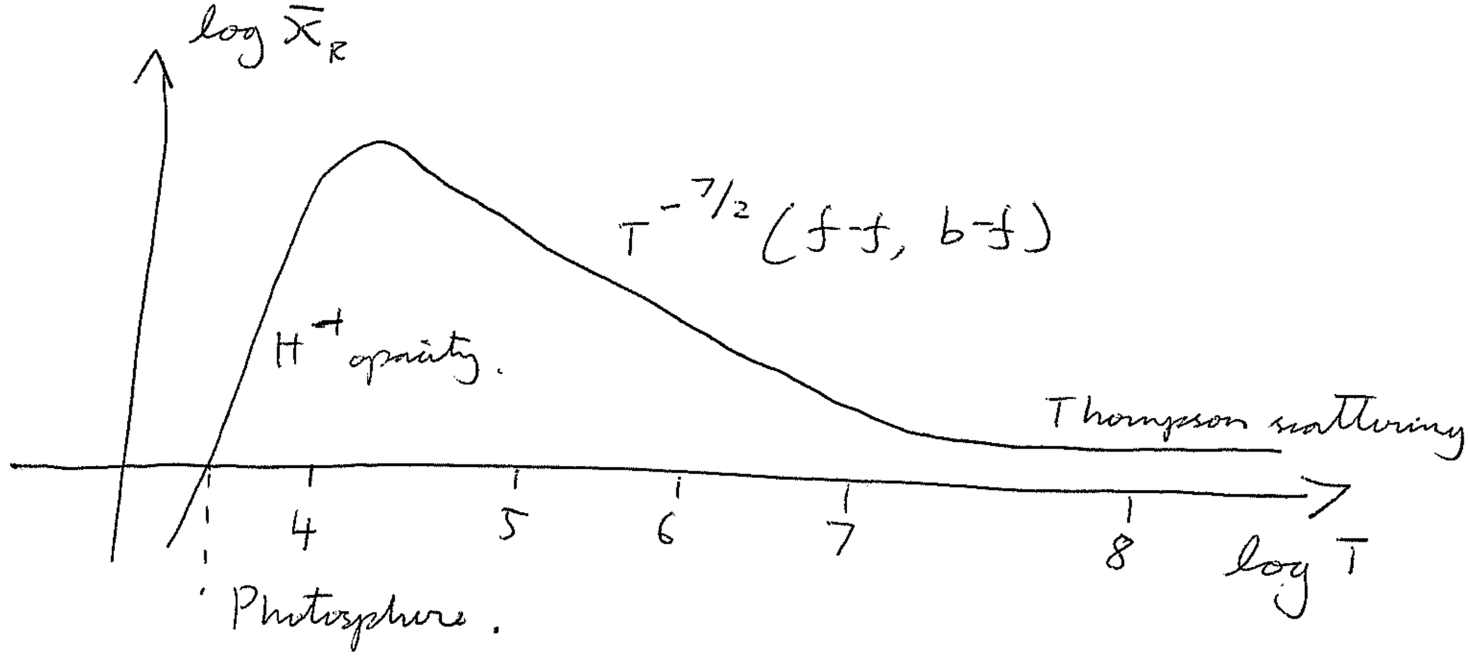
$$\frac{N_{n+1}}{N_n} \propto \frac{1}{n_e} \frac{Z_{n+1}}{Z_n} T^{3/2} e^{-\chi_n/kT},$$

Z is the partition function (# of configurations of electrons given n), and χ_n is the ionization potential.

In general, the free-free & bound-free opacities scale as $\rho T^{-3.5}$ (Kramer's Law - see problem set!).

Finally, at low temperatures, hydrogen recombines, and free-free interactions cease. Bound-free interactions, which rely on photoionizing photons, also stop. The opacity is then dominated by free-free interactions with $H^- = H + e^-$!

In summary,



But radiation isn't the whole picture ...

* Convection in stars.

Consider the adiabatic expansion case. The pressure and density of a gas blob are related as

$$P = K\rho^{\gamma} \qquad \left(\gamma = \tfrac{5}{3} \text{ for monatomic gas}\right).$$

If, as a gas blob rises, the change in density $d\rho$ is greater than the change in density of the surrounding medium ($d\rho'$), while the pressure change remains the same ($dP = dP'$), the blob will be bouyant and convection will occur.

Adiabatic expansion implies (for small $d\rho$)

$$\frac{P}{\rho^\gamma} = \frac{P-dP}{(\rho-d\rho)^\gamma} = \frac{P-dP}{\rho^\gamma - \gamma\rho^{\gamma-1}d\rho} \Rightarrow \frac{dP}{P} = \gamma\frac{d\rho}{\rho}.$$

Then, we have for the surrounding medium, with $dP = dP'$ and $d\rho > d\rho'$,

$$\frac{dP'}{P'} > \gamma\frac{d\rho'}{\rho'}, \text{ or } \frac{d\ln P}{d\ln\rho} > \gamma.$$

The ideal gas law implies

$$\frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma-1} \quad \text{(exercise for student)}$$

With a low pressure gradient, or high temperature gradient, convection will occur.

These conditions are satisfied in the outer layer of the Sun, where recombination implies a low pressure gradient (energy deposition with no T-increase)

Let's derive the temperature gradient in the presence of adiabatic convection.

Adiabatic equation of state:

$$P = K\rho^{\gamma} \Rightarrow \frac{dP}{dr} = \gamma K \rho^{\gamma-1} \frac{d\rho}{dr}$$

$$= \gamma \frac{P}{\rho} \frac{d\rho}{dr}.$$

Ideal gas law:

$$P = nkT = \frac{\rho k T}{\mu m_H} \Rightarrow \frac{dP}{dr} = \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$$

$$\left(\text{if } \frac{d\mu}{dr} = 0\right).$$

Combining, $\gamma \frac{P}{\rho}\frac{d\rho}{dr} = \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$

$$\frac{dT}{dr} = \frac{d\rho}{dr}\frac{T}{\rho}(\gamma-1) = \frac{T}{\gamma P}\frac{dP}{dr}(\gamma-1)$$

Re-instating the ideal gas law and the hydrostatic equilibrium condition,

$$\frac{dT}{dr} = -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{G M_r}{r^2}.$$

Aside - adiabatic sound speed.

$$v_s = \left(\rho^{-1} \underbrace{V \frac{\partial P}{\partial V}} \right)^{1/2}$$

bulk modulus of gas, assumed to be adiabatic.

From adiabatic gas law,

$$v_s = \sqrt{\frac{\gamma P}{\rho}} \qquad (\gamma = 5/3 \text{ in monatomic gas}).$$

$$= \sqrt{\frac{\gamma k T}{m_H}} \quad \text{in ideal H-dominated gas.}$$

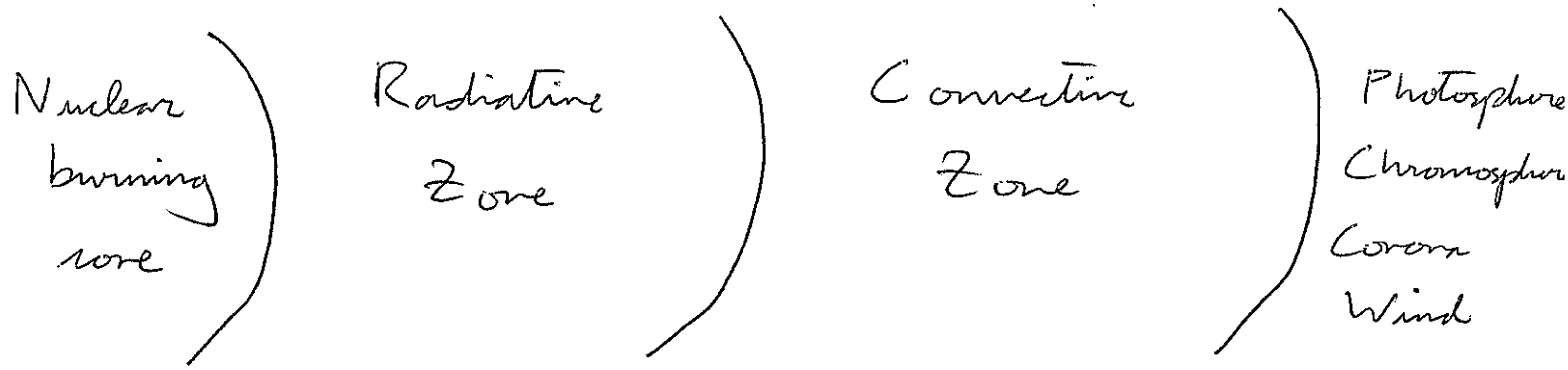
For example, in the dense star-forming regions, $T \approx 100\,K$, and $v_s \approx 1\ km\ s^{-1}$.

In the Solar photosphere, $v_s \approx 10\ km\ s^{-1}$.

Sound speeds are low in most astrophysical scenarios! What does this imply for the fast winds & outflows ($\lesssim$ several $\times 100\ km\ s^{-1}$) impacting the ISM?

Ay 20 #10 – Equations of Stellar Structure

Zones of the Sun:



$$\longrightarrow r \; : \; \frac{dP}{dr}, \; \frac{dM_r}{dr}, \; \frac{dL_r}{dr}, \; \frac{dT}{dr}$$

We are interested in tracing P (pressure), M_r & L_r (mass & radiation flux), and T (temperature) with radius. These depend on the physics determining pressure (ρ, T, composition), opacity (ρ, T, composition), and the generation of energy (ρ, T, composition).

You've seen all the equations:

Hydrostatic equilibrium: $$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2}.$$

Mass profile: $$\frac{dM_r}{dr} = 4\pi r^2 \rho \quad \text{(problem set 3)}$$

Radiation flux through spherical shell:

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon$$

ε ← energy generation / time / density.

Temperature gradient in 1D (plane parallel)

LTE star:

$$\frac{dT}{dr} = -\frac{3\bar{\kappa}_R \rho L_r}{64\sigma\pi T^3 r^2}$$

(radiation dominated)

Temperature gradient with adiabatic convection:

$$\frac{dT}{dr} = -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{G M_r}{r^2}.$$

To gain some familiarity with these equations ...

* What is a crude estimate of the pressure at the center of the Sun?
* How would you then estimate the temperature?
* Finally, what about density?