Notes on resonator dynamics

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Cavity with two ports

These notes are focused on solving the problem of a cavity connected to the environment through two ports, as shown in Figure 1. The intreaction strength with this environment is κ, while $\hat{b}_{in/out}(\omega)$ and $\hat{c}_{in/out}(\omega)$ are the input/output field operators of the two signals interacting with the cavity through the ports. The cavity modes are represented by the operators $\hat{a}$ and $\hat{a}^\dagger$. Here, no intrinsic losses are considered for the cavity.

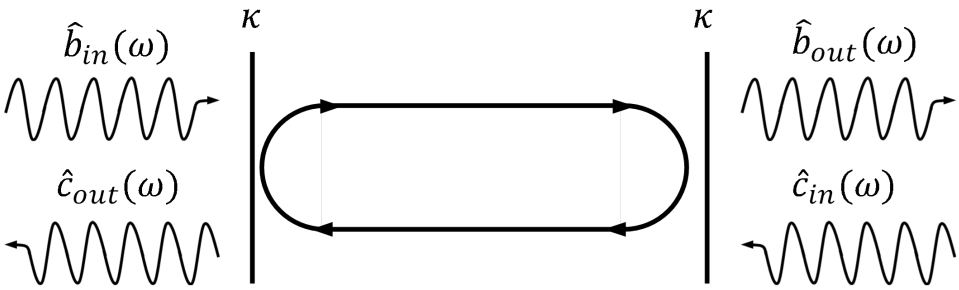

Figure 1. Scheme of the cavity with two ports.

In the continuum, the Hamiltonian for the interaction is:

$$\hat{H}_{int} = \frac{\hbar}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{a}(t)\hat{b}^\dagger(t,\omega') + \hat{a}^\dagger(t)\hat{b}(t,\omega') \right] + \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{a}(t)\hat{c}^\dagger(t,\omega') + \hat{a}^\dagger(t)\hat{c}(t,\omega') \right] \right] \tag{1}$$

While the Hamiltonian of the cavity is just:

$$\hat{H}_c = \hbar\omega_r \hat{a}^\dagger(t)\hat{a}(t) \tag{2}$$

And the Hamiltonian of the input/output fields is:

$$\hat{H}_{ext} = \hbar \int_{\infty}^{\infty} d\omega' \omega' \hat{b}^\dagger(t,\omega')\hat{b}(t,\omega') \tag{3}$$

To simplify the notation, from now on $\hat{a} = \hat{a}(t)$, $\hat{b} = \hat{b}(t,\omega')$ and $\hat{c} = \hat{c}(t,\omega')$.
Now, the master equation rules the dynamics of the cavity interacting with the input/output signals:

$$\frac{d\hat{a}}{dt} = -\frac{i}{\hbar}\left[\hat{a}, \hat{H}_c\right] - \frac{i}{\hbar}\left[\hat{a}, \hat{H}_{ext}\right] - \frac{i}{\hbar}\left[\hat{a}, \hat{H}_{int}\right] \tag{4}$$

The first part is easy to solve:

$$\left[\hat{a}, \hat{H}_c\right] = \hbar\omega_r \left[\hat{a}, \hat{a}^\dagger\hat{a}\right] = \hbar\omega_r \cancelto{1}{\left[\hat{a}, \hat{a}^\dagger\right]} + \hbar\omega_r \cancelto{0}{\left[\hat{a}, \hat{a}\right]} = \hbar\omega_r \tag{5}$$

The second term is zero since $\hat{a}$ commutes with $\hat{b}$, while the third term goes as:

$$\left[\hat{a}, \hat{H}_{int}\right] = \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{a}, \hat{a}\hat{b}^\dagger(\omega') + \hat{a}^\dagger\hat{b}(\omega')\right] + \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{a}, \hat{a}\hat{c}^\dagger(\omega') + \hat{a}^\dagger\hat{c}(\omega')\right] \tag{6}$$

With:

$$\Rightarrow \left[\hat{a}, \hat{a}\hat{b}^\dagger(\omega') + \hat{a}^\dagger\hat{b}(\omega')\right] = \left[\hat{a}, \hat{a}\hat{b}^\dagger(\omega')\right] + \left[\hat{a}, \hat{a}^\dagger\hat{b}(\omega')\right] = \cancelto{0}{[\hat{a}, \hat{a}]}\hat{b}^\dagger + \hat{a}\cancelto{0}{\left[\hat{a}, \hat{b}^\dagger(\omega')\right]} + \cancelto{1}{[\hat{a}, \hat{a}^\dagger]}\hat{b}(\omega') + \hat{a}^\dagger \cancelto{0}{\left[\hat{a}, \hat{b}(\omega')\right]} \tag{7}$$

So,

$$\left[\hat{a}, \hat{a}\hat{c}^{\dagger}(\omega') + \hat{a}^{\dagger}\hat{b}(\omega')\right] = \hat{b}(\omega') \tag{8}$$

$$\left[\hat{a}, \hat{a}\hat{c}^{\dagger}(\omega') + \hat{a}^{\dagger}\hat{c}(\omega')\right] = \hat{c}(\omega') \tag{9}$$

The master equation for the temporal evolution of the cavity modes $\hat{a}$ reads as:

$$\frac{d\hat{a}}{dt} = -i\omega_r - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{b}(\omega') + \hat{c}(\omega')\right] \tag{10}$$

Mind that there are only complex terms in the equation, as a result of not considering the cavity losses in the master equation.

Similarly, there is a master equation for the input/output field:

$$\frac{d\hat{b}(\omega')}{dt} = -\frac{i}{\hbar}\left[\hat{b}(\omega'), \hat{H}_c\right] - \frac{i}{\hbar}\left[\hat{b}(\omega'), \hat{H}_{ext}\right] - \frac{i}{\hbar}\left[\hat{b}(\omega'), \hat{H}_{int}\right] \tag{11}$$

The first term is zero since $\hat{a}$ commutes with $\hat{b}$:

$$\left[\hat{b}(\omega'), \hat{H}_c\right] = \hbar\omega_r \left[\hat{b}(\omega'), \hat{a}^{\dagger}\hat{a}\right] = \hbar\omega_r \cancelto{0}{\left[\hat{b}(\omega'), \hat{a}^{\dagger}\right]} \hat{a} + \hbar\omega_r \hat{a}^{\dagger} \cancelto{0}{\left[\hat{b}(\omega'), \hat{a}\right]} = 0 \tag{12}$$

The second term is:

$$\begin{aligned}
\left[\hat{b}(\omega'), \hat{H}_{ext}\right] &= \\
&= \hbar \int_{-\infty}^{\infty} d\omega''\omega'' \left[\hat{b}(\omega'), \hat{b}^{\dagger}(\omega'')\hat{b}(\omega'')\right] = \\
&= \hbar \int_{-\infty}^{\infty} d\omega''\omega'' \left\{ \left[\hat{b}(\omega'), \hat{b}^{\dagger}(\omega'')\right] \hat{b}(\omega'') + \hat{b}^{\dagger}(\omega'') \left[\hat{b}(\omega'), \hat{b}(\omega'')\right] \right\}
\end{aligned} \tag{13}$$

In the integral, the first commutator $\left[\hat{b}(\omega'), \hat{b}^{\dagger}(\omega'')\right]$ is one only when $\omega'' = \omega'$ and it is zero otherwise. Hence, the integral dissapears for this term. The second commutator $\left[\hat{b}(\omega'), \hat{b}(\omega'')\right]$ is always zero. So:

$$\left[\hat{b}(\omega'), \hat{H}_{ext}\right] = \hbar\omega'\hat{b}(\omega') \tag{14}$$

The third term goes as:

$$\begin{aligned}
\left[\hat{b}(\omega'), \hat{H}_{int}\right] &= \\
&= \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{b}(\omega'), \hat{a}\hat{b}^{\dagger}(\omega') + \hat{a}^{\dagger}\hat{b}(\omega')\right] + \\
&+ \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{b}(\omega'), \hat{a}\hat{c}^{\dagger}(\omega') + \hat{a}^{\dagger}\hat{c}(\omega')\right]
\end{aligned} \tag{15}$$

Where