

# Notes on resonator dynamics

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## Cavity with two ports

These notes are focused on solving the problem of a cavity connected to the environment through two ports, as shown in Figure 1. The interaction strength with this environment is  $\kappa$ , while  $\hat{b}_{in/out}(\omega)$  and  $\hat{c}_{in/out}(\omega)$  are the input/output field operators of the two signals interacting with the cavity through the ports. The cavity modes are represented by the operators  $\hat{a}$  and  $\hat{a}^\dagger$ . Here, no intrinsic losses are considered for the cavity.

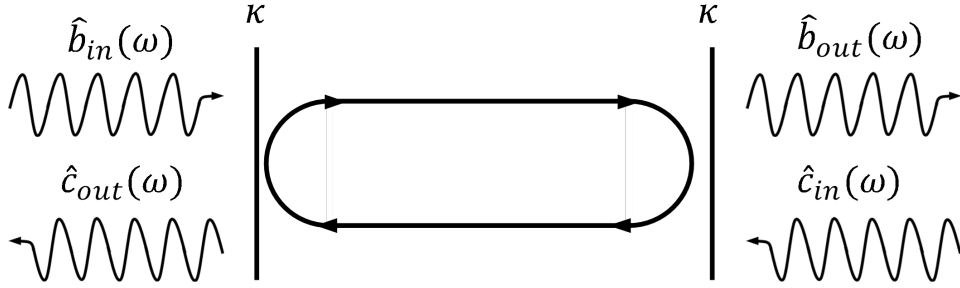


Figure 1. Scheme of the cavity with two ports.

In the continuum, the Hamiltonian for the interaction is:

$$\hat{H}_{int} = \frac{\hbar}{\sqrt{2\pi}} \left[ \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[ \hat{a}(t) \hat{b}^\dagger(t, \omega') + \hat{a}^\dagger(t) \hat{b}(t, \omega') \right] + \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[ \hat{a}(t) \hat{c}^\dagger(t, \omega') + \hat{a}^\dagger(t) \hat{c}(t, \omega') \right] \right] \quad (1)$$

While the Hamiltonian of the cavity is just:

$$\hat{H}_c = \hbar \omega_r \hat{a}^\dagger(t) \hat{a}(t) \quad (2)$$

And the Hamiltonian of the input/output fields is:

$$\hat{H}_{ext} = \hbar \int_{-\infty}^{\infty} d\omega' \omega' \hat{b}^\dagger(t, \omega') \hat{b}(t, \omega') \quad (3)$$

To simplify the notation, from now on  $\hat{a} = \hat{a}(t)$ ,  $\hat{b} = \hat{b}(t, \omega')$  and  $\hat{c} = \hat{c}(t, \omega')$ .

Now, the master equation rules the dynamics of the cavity interacting with the input/output signals:

$$\frac{d\hat{a}}{dt} = -\frac{i}{\hbar} [\hat{a}, \hat{H}_c] - \frac{i}{\hbar} [\hat{a}, \hat{H}_{ext}] - \frac{i}{\hbar} [\hat{a}, \hat{H}_{int}] \quad (4)$$

The first part is easy to solve:

$$[\hat{a}, \hat{H}_c] = \hbar \omega_r [\hat{a}, \hat{a}^\dagger \hat{a}] = \hbar \omega_r \cancel{[\hat{a}, \hat{a}^\dagger]}^1 + \hbar \omega_r \cancel{[\hat{a}, \hat{a}]}^0 = \hbar \omega_r \quad (5)$$

The second term is zero since  $\hat{a}$  commutes with  $\hat{b}$ , while the third term goes as:

$$[\hat{a}, \hat{H}_{int}] = \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[ \hat{a}, \hat{a} \hat{b}^\dagger(\omega') + \hat{a}^\dagger \hat{b}(\omega') \right] + \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[ \hat{a}, \hat{a} \hat{c}^\dagger(\omega') + \hat{a}^\dagger \hat{c}(\omega') \right] \quad (6)$$

With:

$$\Rightarrow [\hat{a}, \hat{a} \hat{b}^\dagger(\omega') + \hat{a}^\dagger \hat{b}(\omega')] = [\hat{a}, \hat{a} \hat{b}^\dagger(\omega')] + [\hat{a}, \hat{a}^\dagger \hat{b}(\omega')] = \cancel{[\hat{a}, \hat{a}]}^0 \hat{b}^\dagger + \hat{a} [\hat{a}, \hat{b}^\dagger(\omega')] + \cancel{[\hat{a}, \hat{a}^\dagger]}^0 \hat{b} + \hat{a}^\dagger [\hat{a}, \hat{b}(\omega')] \quad (7)$$

So,

$$[\hat{a}, \hat{a}\hat{c}^\dagger(\omega') + \hat{a}^\dagger\hat{b}(\omega')] = \hat{b}(\omega') \quad (8)$$

$$[\hat{a}, \hat{a}\hat{c}^\dagger(\omega') + \hat{a}^\dagger\hat{c}(\omega')] = \hat{c}(\omega') \quad (9)$$

The master equation for the temporal evolution of the cavity modes  $\hat{a}$  reads as:

$$\frac{d\hat{a}}{dt} = -i\omega_r - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} [\hat{b}(\omega') + \hat{c}(\omega')] \quad (10)$$

Mind that there are only complex terms in the equation, as a result of not considering the cavity losses in the master equation.

Similarly, there is a master equation for the input/output field:

$$\frac{d\hat{b}(\omega')}{dt} = -\frac{i}{\hbar} [\hat{b}(\omega'), \hat{H}_c] - \frac{i}{\hbar} [\hat{b}(\omega'), \hat{H}_{ext}] - \frac{i}{\hbar} [\hat{b}(\omega'), \hat{H}_{int}] \quad (11)$$

The first term is zero since  $\hat{a}$  commutes with  $\hat{b}$ :

$$[\hat{b}(\omega'), \hat{H}_c] = \hbar\omega_r [\hat{b}(\omega'), \hat{a}^\dagger\hat{a}] = \hbar\omega_r \left[ \cancel{\hat{b}(\omega'), \hat{a}^\dagger}^0 \hat{a} + \hbar\omega_r \hat{a}^\dagger \cancel{[\hat{b}(\omega'), \hat{a}]}^0 \right] = 0 \quad (12)$$

The second term is:

$$\begin{aligned} [\hat{b}(\omega'), \hat{H}_{ext}] &= \\ &= \hbar \int_{-\infty}^{\infty} d\omega'' \omega'' [\hat{b}(\omega'), \hat{b}^\dagger(\omega'')\hat{b}(\omega'')] = \\ &= \hbar \int_{-\infty}^{\infty} d\omega'' \omega'' \left\{ [\hat{b}(\omega'), \hat{b}^\dagger(\omega'')] \hat{b}(\omega'') + \hat{b}^\dagger(\omega'') [\hat{b}(\omega'), \hat{b}(\omega'')] \right\} \end{aligned} \quad (13)$$

In the integral, the first commutator  $[\hat{b}(\omega'), \hat{b}^\dagger(\omega'')]$  is one only when  $\omega'' = \omega'$  and it is zero otherwise. Hence, the integral disappears for this term. The second commutator  $[\hat{b}(\omega'), \hat{b}(\omega'')]$  is always zero. So:

$$[\hat{b}(\omega'), \hat{H}_{ext}] = \hbar\omega' \hat{b}(\omega') \quad (14)$$

The third term goes as:

$$\begin{aligned} [\hat{b}(\omega'), \hat{H}_{int}] &= \\ &= \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} [\hat{b}(\omega'), \hat{a}\hat{b}^\dagger(\omega') + \hat{a}^\dagger\hat{b}(\omega')] + \\ &+ \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} [\hat{b}(\omega'), \hat{a}\hat{c}^\dagger(\omega') + \hat{a}^\dagger\hat{c}(\omega')] \end{aligned} \quad (15)$$

Where:

$$\begin{aligned} &\Rightarrow [\hat{b}(\omega'), \hat{a}\hat{b}^\dagger(\omega') + \hat{a}^\dagger\hat{b}(\omega')] = \\ &= \cancel{[\hat{b}(\omega'), \hat{a}]}^0 \hat{b}^\dagger(\omega') + \hat{a} \cancel{[\hat{b}(\omega'), \hat{b}^\dagger(\omega'')]}^1 + \cancel{[\hat{b}(\omega'), \hat{a}^\dagger]}^0 \hat{b}(\omega'') + \hat{a}^\dagger \cancel{[\hat{b}(\omega'), \hat{b}(\omega'')]}^0 = \hat{a} \end{aligned} \quad (16)$$

Commutator  $[\hat{b}(\omega'), \hat{b}^\dagger(\omega'')]$  is one only when  $\omega' = \omega''$  and it is zero otherwise so the integral in equation (15) disappears.

Likewise, the second integral in (15) is zero since the  $\hat{b}(\omega')$  does always commutes with  $\hat{c}(\omega'')$  and then the commutator is always zero. Now, under the Markovian approximation is possible to treat the  $\kappa$  as independent of the frequency so  $\kappa(\omega') = \kappa$ .

Finally, these three differential equations model the dynamics of the system (no intrinsic losses are considered for the cavity). The first equation corresponds to the cavity, while the other two correspond to the input/output fields:

$$\begin{aligned}
 \frac{d\hat{a}}{dt} &= -i\omega_r - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} d\omega' \sqrt{\kappa} \left[ \hat{b}(\omega') + \hat{c}(\omega') \right] \\
 \frac{d\hat{b}(\omega')}{dt} &= -i\omega' \hat{b}(\omega') - \frac{i}{\sqrt{2\pi}} \sqrt{\kappa} \hat{a}(t) \\
 \frac{d\hat{c}(\omega')}{dt} &= -i\omega' \hat{c}(\omega') - \frac{i}{\sqrt{2\pi}} \sqrt{\kappa} \hat{a}(t)
 \end{aligned} \tag{17}$$

Solving these equations requires transforming them into rotating frame: