Notes on resonator dynamics

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Cavity with two ports

These notes are focused on solving the problem of a cavity connected to the environment through two ports, as shown in Figure 1. The intreaction strength with this environment is κ, while $\hat{b}_{in/out}(\omega)$ and $\hat{c}_{in/out}(\omega)$ are the input/output field operators of the two signals interacting with the cavity through the ports. The cavity modes are represented by the operators $\hat{a}$ and $\hat{a}^\dagger$. Here, no intrinsic losses are considered for the cavity.

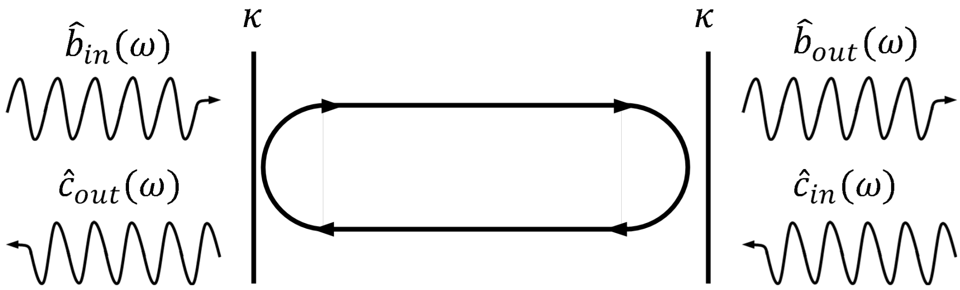

Figure 1. Scheme of the cavity with two ports.

In the continuum, the Hamiltonian for the interaction is:

$$\hat{H}_{int} = \frac{\hbar}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{a}(t)\hat{b}^\dagger(t,\omega') + \hat{a}^\dagger(t)\hat{b}(t,\omega') \right] + \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{a}(t)\hat{c}^\dagger(t,\omega') + \hat{a}^\dagger(t)\hat{c}(t,\omega') \right] \right] \tag{1}$$

While the Hamiltonian of the cavity is just:

$$\hat{H}_c = \hbar\omega_r \hat{a}^\dagger(t)\hat{a}(t) \tag{2}$$

And the Hamiltonian of the input/output fields is:

$$\hat{H}_{ext} = \hbar \int_{\infty}^{\infty} d\omega' \omega' \hat{b}^\dagger(t,\omega')\hat{b}(t,\omega') \tag{3}$$

To simplify the notation, from now on $\hat{a} = \hat{a}(t)$, $\hat{b} = \hat{b}(t,\omega')$ and $\hat{c} = \hat{c}(t,\omega')$.
Now, the master equation rules the dynamics of the cavity interacting with the input/output signals:

$$\frac{d\hat{a}}{dt} = -\frac{i}{\hbar}\left[\hat{a}, \hat{H}_c\right] - \frac{i}{\hbar}\left[\hat{a}, \hat{H}_{ext}\right] - \frac{i}{\hbar}\left[\hat{a}, \hat{H}_{int}\right] \tag{4}$$

The first part is easy to solve:

$$\left[\hat{a}, \hat{H}_c\right] = \hbar\omega_r \left[\hat{a}, \hat{a}^\dagger\hat{a}\right] = \hbar\omega_r \cancelto{1}{\left[\hat{a}, \hat{a}^\dagger\right]} + \hbar\omega_r \cancelto{0}{\left[\hat{a}, \hat{a}\right]} = \hbar\omega_r \tag{5}$$

The second term is zero since $\hat{a}$ commutes with $\hat{b}$, while the third term goes as:

$$\left[\hat{a}, \hat{H}_{int}\right] = \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{a}, \hat{a}\hat{b}^\dagger(\omega') + \hat{a}^\dagger\hat{b}(\omega')\right] + \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{a}, \hat{a}\hat{c}^\dagger(\omega') + \hat{a}^\dagger\hat{c}(\omega')\right] \tag{6}$$

With:

$$\Rightarrow \left[\hat{a}, \hat{a}\hat{b}^\dagger(\omega') + \hat{a}^\dagger\hat{b}(\omega')\right] = \left[\hat{a}, \hat{a}\hat{b}^\dagger(\omega')\right] + \left[\hat{a}, \hat{a}^\dagger\hat{b}(\omega')\right] = \cancelto{0}{\left[\hat{a}, \hat{a}\right]}\hat{b}^\dagger + \hat{a}\cancelto{0}{\left[\hat{a}, \hat{b}^\dagger(\omega')\right]} + \cancelto{1}{\left[\hat{a}, \hat{a}^\dagger\right]}\hat{b}(\omega') + \hat{a}^\dagger\cancelto{0}{\left[\hat{a}, \hat{b}(\omega')\right]} \tag{7}$$

So,

$$\left[\hat{a}, \hat{a}\hat{c}^{\dagger}(\omega') + \hat{a}^{\dagger}\hat{b}(\omega')\right] = \hat{b}(\omega') \tag{8}$$

$$\left[\hat{a}, \hat{a}\hat{c}^{\dagger}(\omega') + \hat{a}^{\dagger}\hat{c}(\omega')\right] = \hat{c}(\omega') \tag{9}$$

The master equation for the temporal evolution of the cavity modes $\hat{a}$ reads as:

$$\frac{d\hat{a}}{dt} = -i\omega_r - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{b}(\omega') + \hat{c}(\omega')\right] \tag{10}$$

Mind that there are only complex terms in the equation, as a result of not considering the cavity losses in the master equation.

Similarly, there is a master equation for the input/output field:

$$\frac{d\hat{b}(\omega')}{dt} = -\frac{i}{\hbar}\left[\hat{b}(\omega\acute{}), \hat{H}_c\right] - \frac{i}{\hbar}\left[\hat{b}(\omega\acute{}), \hat{H}_{ext}\right] - \frac{i}{\hbar}\left[\hat{b}(\omega'), \hat{H}_{int}\right] \tag{11}$$

The first term is zero since $\hat{a}$ commutes with $\hat{b}$:

$$\left[\hat{b}(\omega'), \hat{H}_c\right] = \hbar\omega_r \left[\hat{b}(\omega'), \hat{a}^{\dagger}\hat{a}\right] = \hbar\omega_r \cancelto{0}{\left[\hat{b}(\omega'), \hat{a}^{\dagger}\right]} \hat{a} + \hbar\omega_r \hat{a}^{\dagger} \cancelto{0}{\left[\hat{b}(\omega'), \hat{a}\right]} = 0 \tag{12}$$

The second term is:

$$\begin{aligned} &\left[\hat{b}(\omega'), \hat{H}_{ext}\right] = \\ &= \hbar \int_{-\infty}^{\infty} d\omega''\omega'' \left[\hat{b}(\omega'), \hat{b}^{\dagger}(\omega'')\hat{b}(\omega'')\right] = \\ &= \hbar \int_{-\infty}^{\infty} d\omega''\omega'' \left\{ \left[\hat{b}(\omega'), \hat{b}^{\dagger}(\omega'')\right] \hat{b}(\omega'') + \hat{b}^{\dagger}(\omega'') \left[\hat{b}(\omega'), \hat{b}(\omega'')\right] \right\} \end{aligned} \tag{13}$$

In the integral, the first commutator $\left[\hat{b}(\omega'), \hat{b}^{\dagger}(\omega'')\right]$ is one only when $\omega'' = \omega'$ and it is zero otherwise. Hence, the integral dissapears for this term. The second commutator $\left[\hat{b}(\omega'), \hat{b}(\omega'')\right]$ is always zero. So:

$$\left[\hat{b}(\omega'), \hat{H}_{ext}\right] = \hbar\omega'\hat{b}(\omega') \tag{14}$$

The third term goes as:

$$\begin{aligned} &\left[\hat{b}(\omega'), \hat{H}_{int}\right] = \\ &= \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{b}(\omega'), \hat{a}\hat{b}^{\dagger}(\omega') + \hat{a}^{\dagger}\hat{b}(\omega')\right] + \\ &+ \frac{\hbar}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \sqrt{\kappa(\omega')} \left[\hat{b}(\omega'), \hat{a}\hat{c}^{\dagger}(\omega') + \hat{a}^{\dagger}\hat{c}(\omega')\right] \end{aligned} \tag{15}$$

Where:

$$\begin{aligned} &\Rightarrow \left[\hat{b}(\omega'), \hat{a}\hat{b}^{\dagger}(\omega') + \hat{a}^{\dagger}\hat{b}(\omega')\right] = \\ &= \cancelto{0}{\left[\hat{b}(\omega'), \hat{a}\right]} \hat{b}^{\dagger}(\omega'') + \hat{a} \cancelto{1}{\left[\hat{b}(\omega'), \hat{b}^{\dagger}(\omega'')\right]} + \cancelto{0}{\left[\hat{b}(\omega'), \hat{a}^{\dagger}\right]} \hat{b}(\omega'') + \hat{a}^{\dagger} \cancelto{0}{\left[\hat{b}(\omega'), \hat{b}(\omega'')\right]} = \hat{a} \end{aligned} \tag{16}$$

Commutator $\left[\hat{b}(\omega'), \hat{b}^{\dagger}(\omega'')\right]$ is one only when $\omega' = \omega''$ and it is zero otherwise so the integral in equation (15) disappears.

Likewise, the second integral in (15) is zero since the $\hat{b}(\omega')$ does always commutes with $\hat{c}(\omega'')$ and then the commutator is always zero.

Finally, these three differential equations model the dynamics of the system (no intrinsic losses are considered for the cavity). The first equation corresponds to the cavity, while the other two correspond to the input/output fields:

$$\begin{aligned}\frac{d\hat{a}}{dt} &= -i\omega_r\hat{a} - \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} d\omega' \sqrt{\kappa(\omega')}\left[\hat{b}(\omega') + \hat{c}(\omega')\right] \\ \frac{d\hat{b}(\omega')}{dt} &= -i\omega'\hat{b}(\omega') - \frac{i}{\sqrt{2\pi}}\sqrt{\kappa(\omega')}\hat{a}(t) \\ \frac{d\hat{c}(\omega')}{dt} &= -i\omega'\hat{c}(\omega') - \frac{i}{\sqrt{2\pi}}\sqrt{\kappa(\omega')}\hat{a}(t)\end{aligned} \tag{17}$$

The first equation in the system represents the cavity, while the other two equations represent the in/out electromagnetic modes from left to right ($\hat{b}(\omega')$) and conversely from right to left ($\hat{c}(\omega')$). Solving these equations requires transforming them into a rotating frame where:

$$\begin{aligned}\hat{b}(\omega') &= \hat{\tilde{b}}(\omega')e^{i\omega' t} \\ \hat{c}(\omega') &= \hat{\tilde{c}}(\omega')e^{i\omega' t}\end{aligned} \tag{18}$$

Then,

$$\frac{d}{dt}\left[\hat{\tilde{b}}(\omega')e^{i\omega' t}\right] = \frac{d\hat{\tilde{b}}(\omega')}{dt}e^{i\omega' t} + i\omega'\hat{\tilde{b}}(\omega')e^{i\omega' t} \tag{19}$$

so:

$$\frac{d}{dt}\left[\hat{\tilde{b}}(\omega')e^{i\omega' t}\right] = -\frac{i}{\sqrt{2\pi}}\sqrt{\kappa(\omega')}\hat{\tilde{a}}(t)e^{i\omega' t} \tag{20}$$

The integral as a function of time from the past t_0 to a time t is (with $\hat{\tilde{b}}_0(\omega') = \hat{\tilde{b}}(t_0, \omega')$):

$$\hat{\tilde{b}}(\omega')e^{i\omega' t} - \hat{\tilde{b}}_0(\omega')e^{i\omega' t_0} = -\frac{i}{\sqrt{2\pi}}\int_{t_0}^{t} dt' \sqrt{\kappa(\omega')}\hat{\tilde{a}}(t')e^{i\omega' t'} \tag{21}$$

So:

$$\begin{aligned}\hat{\tilde{b}}(\omega') &= \hat{\tilde{b}}_0(\omega')e^{-i\omega'(t-t_0)} - \frac{i}{\sqrt{2\pi}}\int_{t_0}^{t} dt' \sqrt{\kappa(\omega')}\hat{\tilde{a}}(t')e^{-i\omega'(t-t')} \\ \hat{\tilde{c}}(\omega') &= \hat{\tilde{c}}_0(\omega')e^{-i\omega'(t-t_0)} - \frac{i}{\sqrt{2\pi}}\int_{t_0}^{t} dt' \sqrt{\kappa(\omega')}\hat{\tilde{a}}(t')e^{-i\omega'(t-t')}\end{aligned} \tag{22}$$

These expressions can be introduced in the first equation in (17) for $\frac{d\hat{a}}{dt}$, so:

$$\begin{aligned}&\frac{d\hat{\tilde{a}}}{dt} = -i\omega_r\hat{\tilde{a}}(t) - \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} d\omega' \sqrt{\kappa(\omega')}\hat{\tilde{b}}_0(\omega')e^{i\omega'(t-t_0)} - \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{-\infty} d\omega' \sqrt{\kappa(\omega')}\hat{\tilde{c}}_0(\omega')e^{i\omega'(t-t_0)} \\ &-\frac{1}{\pi}\int_{-\infty}^{-\infty} d\omega' \kappa(\omega')\int_{t_0}^{t} dt' \hat{\tilde{a}}(t')e^{-i\omega'(t-t')}\end{aligned} \tag{23}$$

Now, under the Markovian approximation it is possible to treat the κ as independent of the frequency so $\kappa(\omega') = \kappa$. Markovian approximation ensures that the system couples uniformly to a broad band of in/out field frequency modes, causing the fields to act as "memoryless" signals. This approximation is typically very good in systems with relatively weak system-bath interactions $\kappa(\omega') << \omega'$ such that the system-field interaction is narrowband.Since the Markovian approximation is applied, the second and third terms in (23) can be called:

$$\begin{aligned}\hat{\tilde{b}}_{in}(t) &= \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega' \hat{\tilde{b}}_0(\omega')e^{-i\omega'(t-t_0)} \\ \hat{\tilde{c}}_{in}(t) &= \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega' \hat{\tilde{c}}_0(\omega')e^{-i\omega'(t-t_0)}\end{aligned} \tag{24}$$

Hence, equation (23) becomes:

$$\frac{d\hat{\tilde{a}}}{dt} = -i\omega_r\hat{\tilde{a}}(t) - \sqrt{\kappa}\left[\hat{\tilde{b}}_{in}(t) + \hat{\tilde{c}}_{in}(t)\right] - \frac{\kappa}{\pi}\int_{-\infty}^{-\infty} d\omega' \int_{t_0}^{t} dt' \hat{\tilde{a}}(t')e^{-i\omega'(t-t')} \tag{25}$$

The following two properties help solving the last term with two integrals:

$$\begin{aligned} &1. \int_{-\infty}^{\infty} d\omega' e^{-i\omega'(t-t')} = 2\pi\delta(t-t') \\ &2. \int_{t_0}^{t} dt' \hat{\hat{a}}(t')\delta(t-t') = \frac{\hat{\hat{a}}(t)}{2} \end{aligned} \tag{26}$$

So:

$$\frac{\kappa}{\pi}\int_{-\infty}^{-\infty} d\omega' \int_{t_0}^{t} dt' \hat{\hat{a}}(t') e^{-i\omega'(t-t')} = 2\kappa \int_{t_0}^{t} dt' \hat{\hat{a}}(t')\delta(t-t') = \kappa\hat{\hat{a}}(t) \tag{27}$$

So differential equation (25) for $\hat{\hat{a}}(t)$ is rewritten as:

$$\frac{d\hat{\hat{a}}}{dt} = -i\omega_r\hat{\hat{a}}(t) - \kappa\hat{\hat{a}}(t) - \sqrt{\kappa}\left[\hat{\hat{b}}_{in}(t) + \hat{\hat{c}}_{in}(t)\right] \tag{28}$$

This equation is called the quantum Langevin equation for the input fields.

Now, the input-output relation must be derived. Integrating over the entire spectrum, the first equation (22) reads as:

$$\int_{-\infty}^{\infty} d\omega' \hat{\hat{b}}(\omega') = \int_{-\infty}^{\infty} d\omega' \hat{\hat{b}}_0(\omega') e^{-i\omega'(t-t_0)} - i\sqrt{\frac{\kappa}{2\pi}}\int_{-\infty}^{\infty} d\omega' \int_{t_0}^{t} dt' \hat{\hat{a}}(t') e^{-i\omega'(t-t')} \tag{29}$$

With equation (24) and the properties in (26) the equations (22) are rewritten as:

$$\begin{aligned} \int_{-\infty}^{\infty} d\omega' \hat{\hat{b}}(\omega') &= -i\sqrt{2\pi}\hat{\hat{b}}_{in}(t) - i\sqrt{\frac{\pi\kappa}{2}}\hat{\hat{a}}(t) \\ \int_{-\infty}^{\infty} d\omega' \hat{\hat{c}}(\omega') &= -i\sqrt{2\pi}\hat{\hat{c}}_{in}(t) - i\sqrt{\frac{\pi\kappa}{2}}\hat{\hat{a}}(t) \end{aligned} \tag{30}$$

From equation (19) the same relations can be extracted for $\hat{\hat{b}}_{out}(\omega')$ and $\hat{\hat{c}}_{out}(\omega')$ if the integration is performed from the present to a time t_1 in the future:

$$\begin{aligned} \hat{\hat{b}}(\omega') &= \hat{\hat{b}}_0(\omega') e^{-i\omega'(t-t_0)} + i\sqrt{\frac{\kappa}{2\pi}}\int_{t}^{t_1} dt' \sqrt{\kappa(\omega')}\hat{\hat{a}}(t') e^{-i\omega'(t-t')} \\ \hat{\hat{c}}(\omega') &= \hat{\hat{c}}_1(\omega') e^{-i\omega'(t-t_0)} + i\sqrt{\frac{\kappa}{2\pi}}\int_{t}^{t_1} dt' \sqrt{\kappa(\omega')}\hat{\hat{a}}(t') e^{-i\omega'(t-t')} \end{aligned} \tag{31}$$

As in (29), integrating over the entire spectrum equations (31) read as:

$$\int_{-\infty}^{\infty} d\omega' \hat{\hat{b}}(\omega') = \int_{-\infty}^{\infty} d\omega' \hat{\hat{b}}_1(\omega') e^{-i\omega'(t-t_0)} + i\sqrt{\frac{\kappa}{2\pi}}\int_{-\infty}^{\infty} d\omega' \int_{t_1}^{t} dt' \hat{\hat{a}}(t') e^{-i\omega'(t-t')} \tag{32}$$

So, the relations for $\hat{\hat{b}}_{out}(\omega')$ and $\hat{\hat{c}}_{out}(\omega')$ result in:

$$\begin{aligned} \int_{-\infty}^{\infty} d\omega' \hat{\hat{b}}(\omega') &= -i\sqrt{2\pi}\hat{\hat{b}}_{out}(t) + i\sqrt{\frac{\pi\kappa}{2}}\hat{\hat{a}}(t) \\ \int_{-\infty}^{\infty} d\omega' \hat{\hat{c}}(\omega') &= -i\sqrt{2\pi}\hat{\hat{c}}_{out}(t) + i\sqrt{\frac{\pi\kappa}{2}}\hat{\hat{a}}(t) \end{aligned} \tag{33}$$

Putting together equations (30) and (33) the input-output relations are:

$$\begin{aligned} \hat{\hat{b}}_{out}(t) - \hat{\hat{b}}_{in}(t) &= \sqrt{\kappa}\hat{\hat{a}}(t) \\ \hat{\hat{c}}_{out}(t) - \hat{\hat{c}}_{in}(t) &= \sqrt{\kappa}\hat{\hat{a}}(t) \\ \hat{\hat{b}}_{out}(t) - \hat{\hat{b}}_{in}(t) &= \hat{\hat{c}}_{out}(t) - \hat{\hat{b}}_{in}(t) \end{aligned} \tag{34}$$