Notes on qubit dynamics

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Introduction

We want to study the temporal evolution of a two-level system (an idealized qubit) driven by an oscillating external input. Rabi oscillations are the oscillations in the probability of measuring an eigenstate of the qubit ($|0\rangle$ or $|1\rangle$) that appear when the qubit is driven by an external input. When we make several projective measurements on the qubit state as a function of the time length or the amplitude of the pulse, we can reproduce the statistics of the system and hence, these Rabi oscillations.

Rabi oscillations

First, we define the Hamiltonian of the system, which consists in two terms, the two-level system and the driving:

$$\hat{H} = \frac{\hbar\omega_q}{2}\hat{\sigma}^z + A\left(e^{-i(\omega_d t+\varphi)}\hat{\sigma}^+ + e^{i(\omega_d t+\varphi)}\hat{\sigma}^-\right) \tag{1}$$

With $A = \frac{\hbar\Omega}{2}$. We choose A in this way since we want to express the final result in terms of a frequency called Rabi frequency (Ω). We have introduced an arbitrary phase (φ) in the driving term. Also, we must remember that $\hat{\sigma}^+ = |1\rangle\langle 0|$ and $\hat{\sigma}^- = |0\rangle\langle 1|$.

To study the temporal evolution we take the Schrödinger equation:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \tag{2}$$

with $|\psi(t)\rangle = a(t)|0\rangle + b(t)|1\rangle$ being the wave function of the two-level system and $a(t)$, $b(t)$ the probability amplitudes depending on time. Hence, we have the following system of differential equations:

$$i\begin{pmatrix}\dot{a}\\ \dot{b}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-\omega_q & \Omega e^{i(\omega_d t+\varphi)}\\ \Omega e^{-i(\omega_d t+\varphi)} & \omega_q\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} \tag{3}$$

Now, we go into the rotating frame of the driving to get rid of the exponentials. To do that, we change $a(t) = \tilde{a}(t)e^{i\frac{\omega_d}{2}t}$ and $b(t) = \tilde{b}(t)e^{-i\frac{\omega_d}{2}t}$. Hence:

$$i\dot{\tilde{a}}e^{i\frac{\omega_d}{2}t} - \frac{\omega_d}{2}\tilde{a}e^{i\frac{\omega_d}{2}t} = -\frac{\omega_q}{2}\tilde{a}e^{i\frac{\omega_d}{2}t} + \frac{\Omega}{2}\tilde{b}e^{i\frac{\omega_d}{2}t}e^{i\varphi} \tag{4}$$

$$i\dot{\tilde{b}}e^{-i\frac{\omega_d}{2}t} + \frac{\omega_d}{2}\tilde{b}e^{-i\frac{\omega_d}{2}t} = \frac{\omega_q}{2}\tilde{b}e^{-i\frac{\omega_d}{2}t} + \frac{\Omega}{2}\tilde{a}e^{-i\frac{\omega_d}{2}t}e^{-i\varphi} \tag{5}$$

So, defining the detuning as $\Delta = \omega_q - \omega_d$, the system of differential equations is:

$$\begin{pmatrix}\dot{\tilde{a}}\\ \dot{\tilde{b}}\end{pmatrix} = \frac{i}{2}\begin{pmatrix}\Delta & -\Omega e^{i\varphi}\\ -\Omega e^{-i\varphi} & -\Delta\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} \tag{6}$$

This system is really easy to solve. First, we need to find the eigenvalues of the characteristic matrix:

$$\begin{vmatrix}\frac{i}{2}\Delta-\lambda & -\frac{i}{2}\Omega e^{i\varphi}\\ -\frac{i}{2}\Omega e^{-i\varphi} & -\frac{i}{2}\Delta-\lambda\end{vmatrix} = (\frac{i}{2}\Delta-\lambda)(-\frac{i}{2}\Delta-\lambda) + \frac{\Omega^2}{4} = \lambda^2 + \frac{\Delta^2}{4} + \frac{\Omega^2}{4} = 0 \tag{7}$$

The solution to this second order equation is:

$$\lambda = \pm\frac{i}{2}\sqrt{\Delta^2+\Omega_R^2} \tag{8}$$

From now on, we will define $\Omega_R = \sqrt{\Delta^2+\Omega^2}$ as the generalized Rabi frequency, being Ω the Rabi frequency in resonance. This means that, in resonance, the amplitude of the pulse will define the frequency of the Rabi Oscillations.

Out of resonance, the frequency of the oscillations depends on the amplitude of the pulse and the detuning from the qubit frequency.

The general solution for this system of differential equations is then:

$$\tilde{a}(t) = A\cos\frac{\Omega_R t}{2} + B\sin\frac{\Omega_R t}{2} \tag{9}$$

$$\tilde{b}(t) = C\cos\frac{\Omega_R t}{2} + D\sin\frac{\Omega_R t}{2} \tag{10}$$

From the initial condition $|\psi(0)\rangle = a_0|0\rangle + b_0|1\rangle$ at t = 0 we know that $A = a_0$ and $C = b_0$. From the derivative we can calculate B and D:

$$-a_0\frac{\Omega_R}{2}\sin\frac{\Omega_R t}{2} + B\frac{\Omega_R}{2}\cos\frac{\Omega_R t}{2} = \frac{i\Delta}{2}a_0\cos\frac{\Omega_R t}{2} + \frac{i\Delta}{2}B\sin\frac{\Omega_R t}{2} - \frac{i\Omega}{2}e^{i\varphi}b_0\cos\frac{\Omega_R t}{2} - \frac{i\Omega}{2}e^{i\varphi}D\sin\frac{\Omega_R t}{2} \tag{11}$$

With t = 0:

$$B\Omega_R = i\Delta a_0 - i\Omega e^{i\varphi}b_0 \tag{12}$$

So:

$$B = \frac{i\Delta}{\Omega_R}a_0 - \frac{i\Omega}{\Omega_R}e^{i\varphi}b_0 \tag{13}$$

Proceeding in the same way, for the other equation we have:

$$-b_0\frac{\Omega_R}{2}\sin\frac{\Omega_R t}{2} + D\frac{\Omega_R}{2}\cos\frac{\Omega_R t}{2} = -\frac{i\Delta}{2}b_0\cos\frac{\Omega_R t}{2} + \frac{i\Delta}{2}D\sin\frac{\Omega_R t}{2} - \frac{i\Omega}{2}e^{-i\varphi}a_0\cos\frac{\Omega_R t}{2} - \frac{i\Omega}{2}e^{i\varphi}D\sin\frac{\Omega_R t}{2} \tag{14}$$

With t = 0:

$$D\Omega_R = -i\Delta b_0 - i\Omega e^{-i\varphi}a_0 \tag{15}$$

So:

$$D = -\frac{i\Delta}{\Omega_R}b_0 - \frac{i\Omega}{\Omega_R}e^{i\varphi}a_0 \tag{16}$$

The general solution for the system of differential equations is:

$$a(t) = a_0\cos\left(\frac{\Omega_R t}{2}\right) + \left(\frac{i\Delta}{\Omega_R}a_0 - \frac{i\Omega}{\Omega_R}e^{i\varphi}b_0\right)\sin\left(\frac{\Omega_R t}{2}\right) \tag{17}$$

$$b(t) = b_0\cos\left(\frac{\Omega_R t}{2}\right) - \left(\frac{i\Delta}{\Omega_R}b_0 + \frac{i\Omega}{\Omega_R}e^{i\varphi}a_0\right)\sin\left(\frac{\Omega_R t}{2}\right) \tag{18}$$

Now, we compute the probability of measuring the qubit on the excited state by doing:

$$\mathcal{P}(|1\rangle) = \langle 1|\psi(t)\rangle\langle\psi(t)|1\rangle = b(t)b^*(t) \tag{19}$$

The general expression for the probability of measuring the qubit on the excited state for an arbitrary initial state and a drive with an arbitrary phase is:

$$\begin{aligned}\mathcal{P}(|1\rangle) = |b_0|^2\cos^2\left(\tfrac{\Omega_R t}{2}\right) + \tfrac{1}{\Omega_R^2}\left(\Delta^2|b_0|^2 + \Omega^2|a_0|^2 + 2\Omega\Delta\,\mathrm{Re}\{(e^i\varphi a_0 b_0^*\}\right)\sin^2\left(\tfrac{\Omega_R t}{2}\right) - \\ -2\tfrac{\Omega}{\Omega_R}\,\mathrm{Im}\{(e^i\varphi a_0 b_0^*)\}\cos\left(\tfrac{\Omega_R t}{2}\right)\sin\left(\tfrac{\Omega_R t}{2}\right)\end{aligned} \tag{20}$$

We can particularize this expression for a initial condition $|\psi(t=0)\rangle = |0\rangle$ (so $a_0 = 1$ and $b_0 = 0$):

$$\mathcal{P}(|1\rangle) = \frac{\Omega^2}{\Omega_R^2}\sin^2\left(\frac{\Omega_R t}{2}\right) \tag{21}$$

We can notice that the frequency at which the probability of measuring the excited state ($\mathcal{P}(|1\rangle)$) oscillates is twice the frequency of the oscillation for the probability amplitudes ($a(t)$ and $b(t)$), since the trigonometric functions in $\mathcal{P}(|1\rangle)$ are second order. Because of that, we have chosen the generalized Rabi frequency as $\Omega_R = \sqrt{\Delta^2 + \Omega^2}$ and not as $\Omega_R = \frac{1}{2}\sqrt{\Delta^2 + \Omega^2}$. We have done this since in the experiment we measure the probability, not the probability amplitudes, so we want the probability expressed in terms of the generalized Rabi frequency.

Figure 1a shows the Rabi oscillations in the probability of measuring the excited state as a function of the duration and frequency of the driving, the so-called Rabi chevrons. The frequency axis has been re-scaled to show the detuning with respect to the qubit frequency ($\Delta = \omega_q - \omega_d$). The amplitude of the driving is fixed at $\Omega = 30$ MHz, while the initial state is $|\psi(t=0)\rangle = |0\rangle$. The frequency of the oscillations (the generalized Rabi frequency Ω_R) depends on the detuning. The slower oscillations occur at resonance, when $\Omega_R = \Omega$. Figure 1b shows these oscillations only as a function of the length of the driving for three different values of the detuning. In both figures the time axis has been re-scaled to Ωt (remember that Ω is the Rabi frequency at zero-detuning). In this way, it is clear that in resonance $\mathcal{P}(|1\rangle) = 1$ when $\Omega t = (2n+1)\pi$. Hence, the pulses with a time length $\Omega t = \pi$ are called π-*pulse*, and they change the state of the qubit from $|0\rangle$ to $|1\rangle$. As the detuning increases, the frequency of these oscillations (Ω_R) increases, while the probability of finding the qubit in the excited state decreases.

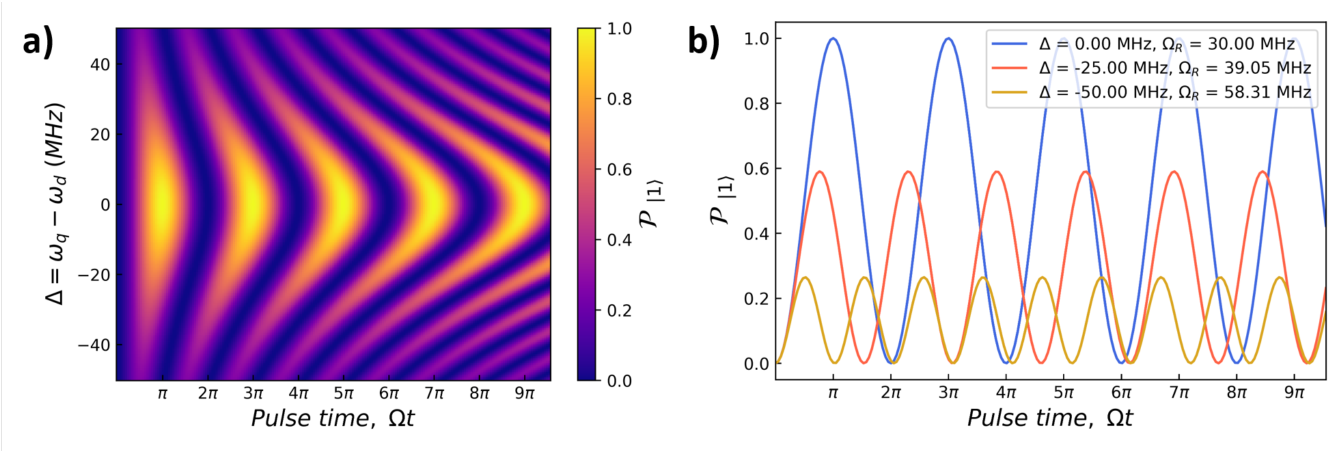

Figure 1. Rabi oscillations as a function of drive length and frequency. a) Probability of measuring the excited state as a function of the time length and the frequency of the driving input. The horizontal axis has been re-scaled to Ωt to show clearly when the qubit state is at a half turn ($(2n+1)\pi$) or a complete turn ($2n\pi$) in the Bloch sphere (see Figure 2 for more details). The vertical axis has been re-scaled to show the detuning with respect to the qubit frequency ($\Delta = \omega_q - \omega_d$). **b)** Probability of measuring the qubit in the excited state as a function of the time length of the driving for three different driving frequencies (these frequencies are referenced to the frequency of the qubit). Blue curve corresponds to resonant condition while red and yellow curves correspond to off-resonant condition. For both figures, the amplitude of the pulse is fixed at $\Omega = 30$ MHz.

Figure 2 shows the trajectories followed by the state of the qubit in the Bloch sphere when a drive is applied. It depicts two different situations: on resonance (upper row) and out of resonance (lower row). The amplitude of the pulse is $\Omega = 80$ MHz, while the initial state is $|\psi(0)\rangle = |0\rangle$. When the drive is in resonance with the qubit ($\Delta = 0$) a π-*pulse* takes the qubit to the state $|\psi(t)\rangle = |1\rangle$ (shown in 2a) and the probability of measuring this state will be 1. If the pulse is twice longer, the qubit will be back to the state $|\psi(t)\rangle = |0\rangle$ (shown in 2b). The number of turns does not change the result, the stat of the qubit will be going back and forth from $|\psi(t)\rangle = |0\rangle$ to $|\psi(t)\rangle = |1\rangle$ (see 2c).

On the other hand, if the drive is out of resonance ($\Delta \neq 0$), the angle between the qubit trajectory and the *z-axis* is different from zero (actually, this angle increases with the detuning), then the qubit will never reach the state $|\psi(t)\rangle = |1\rangle$ even if we apply a π-*pulse* (see 2d). Actually, the fidelity of this off-$resonance$ π-*pulse* will be lower than one since the probability of measuring the state $|1\rangle$ is not 1. Naturally, the fidelity of the π-*pulse* gets worst when the detuning between the driving and the qubit increases. When we apply a 2π-*pulse* we recover the initial state (shown in 2e). Moreover, when we apply a long pulse (for example the 4π-*pulse* in 2f) the trajectory of the state of the qubit follows two circles deflected from the *z-axis*. These circles intersect at $|0\rangle$

In Figure 1 we saw the Rabi oscillations when the time of the drive changes for a fixed amplitude. What happens if we sweep the amplitude of the drive maintaining the time length fixed? Figure 3a shows this situation. Again, the horizontal axis has been re-scaled to Ωt, but in this case the amplitude of the pulse is changing while the time length is constant. The initial state is again $|\psi(0)\rangle = |0\rangle$, while the time length of the pulse is $t = 200$ ns.

We observe that the Rabi oscillations pattern obtained by sweeping the amplitude of the drive is quite different to the Rabi chevrons shown in Figure 1a. In resonance, the profile of the oscillations does not change. One can see this by comparing Figure 3b and Figure 1b for the resonant condition.

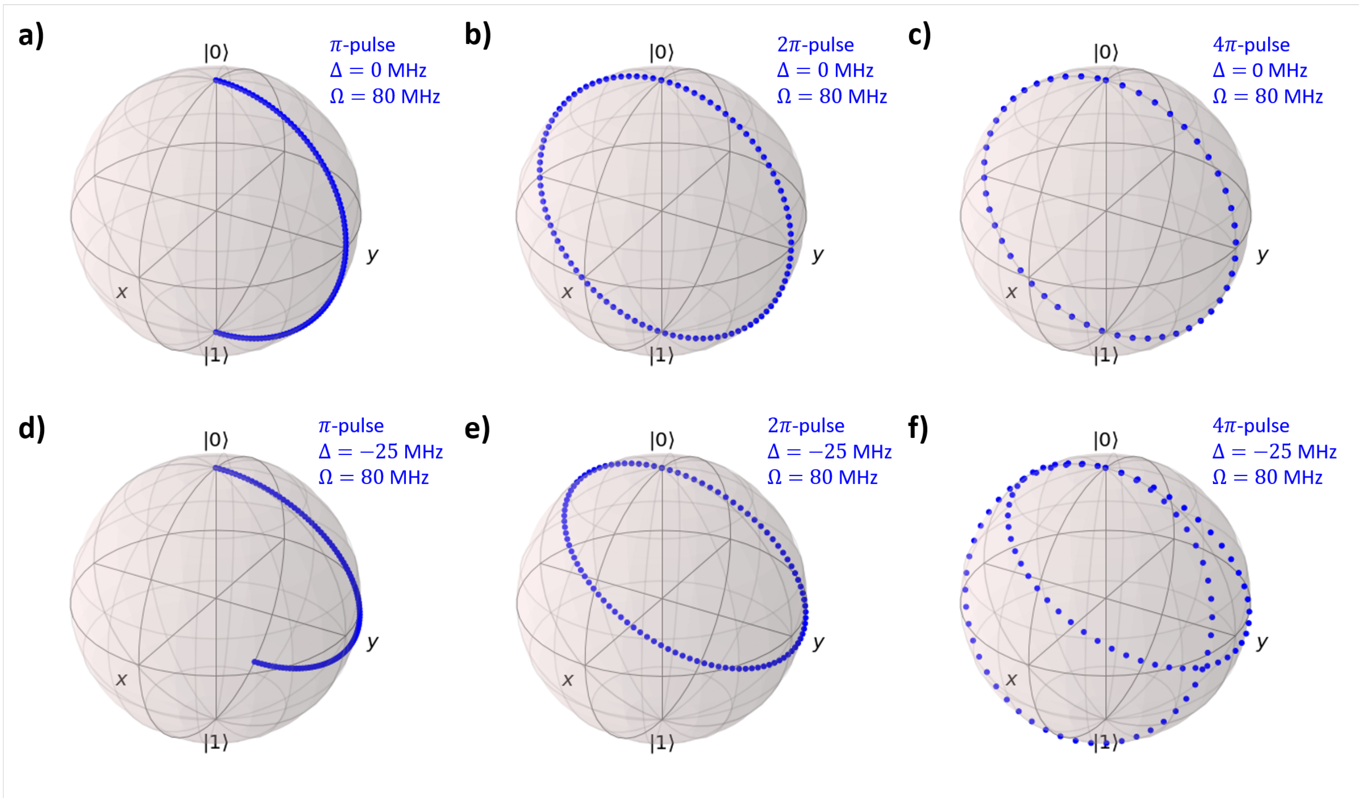

Figure 2. Visualization of the Rabi Oscillations in the Bloch sphere. This panel shows the trajectories of the qubit state for different pulses in resonant condition (upper row) and out of resonance (lower row). The amplitude of the drive is always $\Omega = 80$ MHz. **a)** *on-resonance π-pulse*, **b)** *on-resonance 2π-pulse*, **c)** *on-resonance 4π-pulse*, **d)** *off-resonance π-pulse*, **e)** *off-resonance 2π-pulse*, and **f)** *off-resonance 4π-pulse*. Off-resonance pulses have a detuning value of -25 MHz. The time length of the pulse has been calibrated as $\Omega_R t = n\pi$.

Otherwise, when the detuning is different from zero the probability of measuring the state $|1\rangle$ increases with the number of oscillations. This behaviour is consequence of the dependence of Ω_R with Ω, which now is changing. The deflection of the trajectory of the qubit state from the *z-axis* in the Bloch sphere depends on the ratio between the pulse amplitude Ω and the generalized Rabi frequency Ω_R. When the amplitude of the pulse is high, this ratio approaches to 1, so the probability of measuring the state $|1\rangle$ approaches also to 1 when $\Omega_R t = (2n+1)\pi$. One can understand this by analyzing equation 21. Also, Figure 4 shows different trajectories of the qubit state in the Bloch sphere for a fixed value of the detuning and different pulse amplitudes. For a fixed value of the detuning, a higher amplitude of the pulse means a lower deflection angle from the *z-axis*, meaning that the probability of measuring the $|1\rangle$ is higher (the oscillation gets closer to $|1\rangle$ as the ratio between Δ and Ω_R decreases).

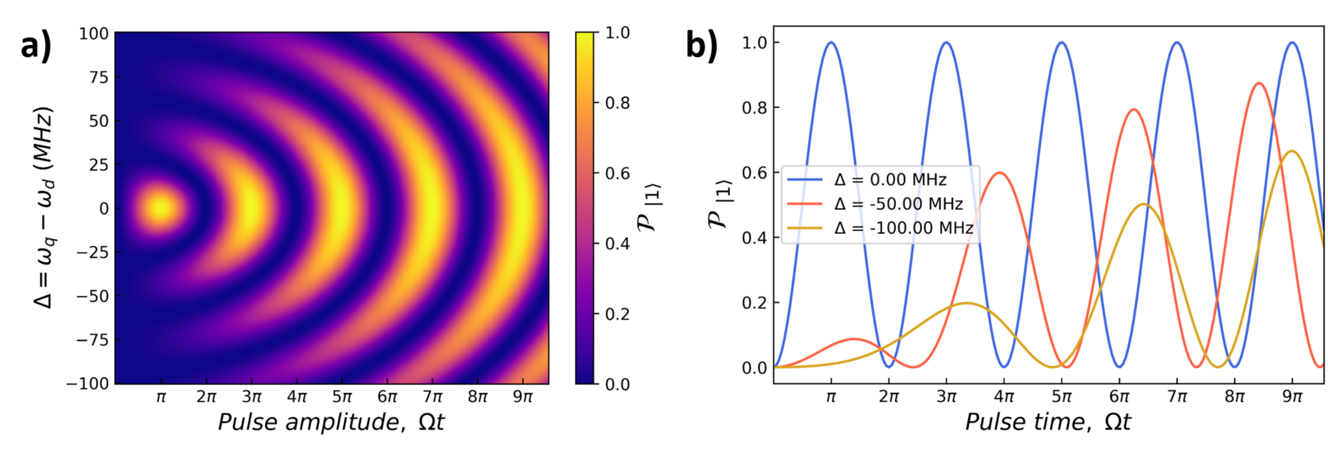

Figure 3. Rabi oscillations as a function of drive length and frequency. a) Probability of measuring the excited state as a function of the amplitude and the frequency of the driving input. The horizontal axis has been re-scaled to Ωt to show clearly when the qubit state is at a half turn ($(2n+1)\pi$) or a complete turn ($2n\pi$) in the Bloch sphere. The vertical axis has been re-scaled to show the detuning with respect to the qubit frequency ($\Delta = \omega_q - \omega_d$).**b)** Probability of measuring the qubit in the excited state as a function of the amplitude of the driving for three different driving frequencies (these frequencies are referenced to the frequency of the qubit).

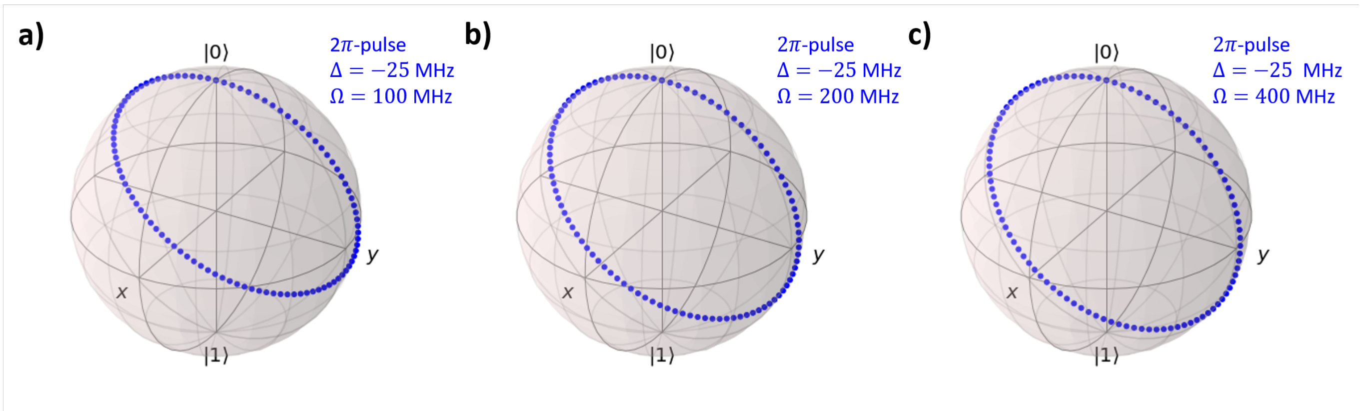

Figure 4. Visualization of the Rabi Oscillations in the Bloch sphere for different drive amplitudes. This panel shows three different trajectories of the qubit state for a fixed detuning value and different pulse amplitudes: **a)** $\Omega = 100$ MHz, **b)** $\Omega = 200$ MHz, and **c)** $\Omega = 400$ MHz. The time of the pulse is calibrated to always have $\Omega_R t = 2\pi$. At the half of the drive length, the trajectory approaches to $|1\rangle$ as the amplitude of the drive is higher. In other words, the deflection angle with respect to the z-$axis$ is lower.

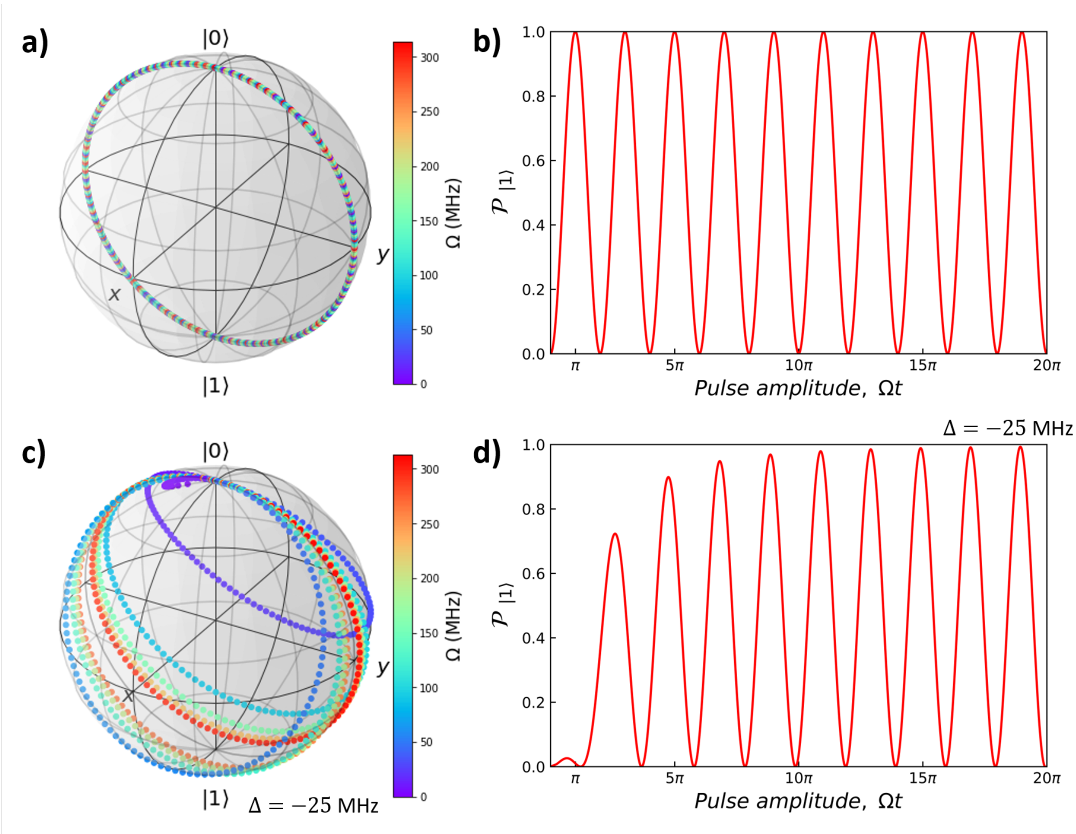

Figure 5. Trajectories of the qubit state in the Bloch sphere during the pulse amplitude sweeping. a) Evolution of the qubit state in resonant condition as a function of the driving pulse amplitude. Colorbar indicates the driving amplitude of the pulse. Each point in the bloch sphere corresponds to a different amplitude, indicated by its color. **b)** Evolution of the probability of measuring the qubit at state $|1\rangle$ in resonant condition. The horizontal axis has been re-scaled to Ωt. **c)** Evolution of the qubit state in off-resonant condition as a function of the driving pulse amplitude. Colorbar indicates the driving amplitude of the pulse. **d** Evolution of the probability of measuring the qubit at state $|1\rangle$ in off-resonant condition. The horizontal axis has been re-scaled to Ωt.

Figures 5a and 5c show the evolution of the qubit state in the Bloch sphere when a drive is applied in resonant condition (5a) and off-resonant condition (5c). Figures 5b and 5d show the probability of measuring the qubit in the state $|1\rangle$ as a function of the drive amplitude (the axis has been re-scaled to Ωt. From these calculations we understand that, out of resonance, the trajectory moves towards the z-$axis$ as the amplitude of the pulse increases (5c). Hence, the probability of measuring the state $|1\rangle$ approaches to 1 as the amplitude increases. In resonance, the qubit state follows always the same trajectory in the Bloch sphere (5a) and the probability of measuring the state $|1\rangle$ is always one when $\Omega_R t = (2n+1)\pi$ (5b).

In other words, increasing the amplitude of the pulse lowers the effect of the detuning between the qubit and the pulse frequencies on the Rabi oscillations. This effect is shown in Figure 6 with a comparison of the Rabi chevrons for two different pulse amplitudes: $\Omega = 50$ MHz (6a) and $\Omega = 100$ MHz (6b). It is clear that the chevrons flatten as the amplitude of the driving pulse increases, a consequence of the behaviour we just described.

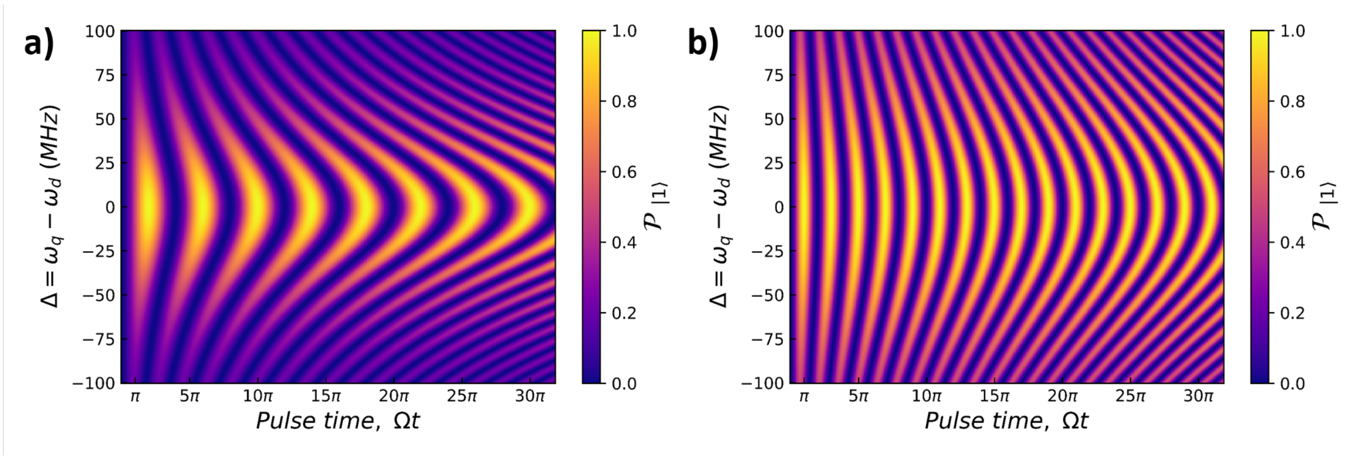

Figure 6. Comparison of the Rabi chevrons for two different pulse amplitudes. Probability of measuring the qubit at state $|1\rangle$ as a function of pulse time and pulse frequency for the pulse amplitudes: **a)** $\Omega = 50$ MHz and **b)** $\Omega = 100$ MHz.

Introducing decay and dephase

In this section we will introduce the decay and dephase rates of the qubit in the dynamics. To do this, we use the Lindblad master equation, which is an extension of the Schrödinger equation to deal with mixed states. First, we need to remember that the wavefunction $|\psi\rangle$ only can describe pure states of the system. To deal with mixed states, which are a statistical ensemble of pure states, we need to introduce the density matrix:

$$\hat{\rho} = \sum_{ij} p_{ij} |\psi_i\rangle\langle\psi_j| \tag{22}$$

In our case of study (a two level system driven by an oscillatory signal), we have that the density matrix is quite simple:

$$\hat{\rho} = |\psi\rangle\langle\psi| = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \tag{23}$$

We can easily map the coefficients in $\psi(t)\rangle = a(t)|0\rangle + b(t)|1\rangle$:

$$\hat{\rho} = \begin{pmatrix} |a(t)|^2 & a(t)b^*(t) \\ a^*(t)b(t) & |b(t)|^2 \end{pmatrix} \tag{24}$$

Now we can introduce the Lindblad master equation:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \sum_j \hat{L}_j \hat{\rho} \hat{L}_j^\dagger - \frac{1}{2}\{\hat{L}_j^\dagger \hat{L}_j, \hat{\rho}\} \tag{25}$$

In this equation, the second term $\sum_j \hat{L}_j \hat{\rho} \hat{L}_j^\dagger - \frac{1}{2}\{\hat{L}_j^\dagger \hat{L}_j, \hat{\rho}\}$ is the Lindbladian superoperator, representing the losses of the system. Here, $\hat{L}_r = \sqrt{\gamma_r}\hat{\sigma}^-$ and $\hat{L}_r^\dagger = \sqrt{\gamma_r}\hat{\sigma}^+$, with γ_r being the decay rate of the qubit. To take into account the dephasing rate we need to introducide a second Lindbladian term, $\hat{L}_\phi = \sqrt{\gamma_\phi/2}\hat{\sigma}_z$ and $\hat{L}_\phi^\dagger = \sqrt{\gamma_\phi/2}\hat{\sigma}_z$ (the h.c. of $\hat{\sigma}_z$ is itself).

Then, we need to solve the following master equation:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \gamma_r\left[\hat{\sigma}^- \hat{\rho} \hat{\sigma}^+ - \frac{1}{2}\{\hat{\sigma}^+\hat{\sigma}^-, \hat{\rho}\}\right] + \frac{\gamma_\phi}{2}\left[\hat{\sigma}_z \hat{\rho} \hat{\sigma}_z - \frac{1}{2}\{\hat{\sigma}_z\hat{\sigma}_z, \hat{\rho}\}\right] \tag{26}$$

After the calculations (see Appendix A1) we have that:

$$-\frac{i}{\hbar}[\hat{H}, \hat{\rho}] = -i\omega_q \begin{pmatrix} 0 & -\rho_{01} \\ \rho_{10} & 0 \end{pmatrix} - i\frac{\Omega}{2}e^{-i(\omega_d t + \varphi)} \begin{pmatrix} -\rho_{01} & 0 \\ \rho_{00} - \rho_{11} & \rho_{11} \end{pmatrix} + i\frac{\Omega}{2}e^{i(\omega_d t + \varphi)} \begin{pmatrix} \rho_{10} & \rho_{11} - \rho_{00} \\ 0 & -\rho_{10} \end{pmatrix} \tag{27}$$

The decay term is:

$$\gamma_r\left[\hat{\sigma}^- \hat{\rho} \hat{\sigma}^+ - \frac{1}{2}\{\hat{\sigma}^+\hat{\sigma}^-, \hat{\rho}\}\right] = \frac{\gamma_r}{2}\begin{pmatrix} 2\rho_{11} & -\rho_{01} \\ -\rho_{10} & -2\rho_{11} \end{pmatrix} \tag{28}$$

And the dephasing term is:

$$\frac{\gamma_\phi}{2}\left[\hat{\sigma}_z \hat{\rho} \hat{\sigma}_z - \frac{1}{2}\{\hat{\sigma}_z\hat{\sigma}_z, \hat{\rho}\}\right] = \gamma_\phi \begin{pmatrix} 0 & \rho_{01} \\ \rho_{10} & 0 \end{pmatrix} \tag{29}$$

Now, we have a system of four coupled differential equations, one for each component of the density matrix. We can eventually simplify the problem considering these two conditions: (1) $\rho_{00} + \rho_{11} = 1$, and (2) $\rho_{01} = \rho_{10}^*$, so we only write the equations for $\dot{\rho}_{00}$ and $\dot{\rho}_{01}$. These two differential equations are:

$$\dot{\rho}_{00} = i\frac{\Omega}{2}e^{-i(\omega_d t+\varphi)}\rho_{01} - i\frac{\Omega}{2}e^{i(\omega_d t+\varphi)}\rho_{10} + \gamma_r\rho_{11} \tag{30}$$

$$\dot{\rho}_{01} = i\omega_q\rho_{01} + i\frac{\Omega}{2}e^{i(\omega_d t+\varphi)}\rho_{00} - i\frac{\Omega}{2}e^{i(\omega_d t+\varphi)}\rho_{11} - \left(\frac{\gamma_r}{2} + \gamma_\phi\right)\rho_{01} \tag{31}$$

Now we transform the equations in to the rotating frame of the driving signal: (1) $\tilde{\rho}_{00} = \rho_{00}$, (2) $\tilde{\rho}_{01}e^{i\omega_d t} = \rho_{01}$, (3) $\tilde{\rho}_{10}e^{-i\omega_d t} = \rho_{10}$ and (4) $\tilde{\rho}_{11} = \rho_{11}$ (see appendix A1 for complete calculation).

$$\dot{\tilde{\rho}}_{00} = i\frac{\Omega}{2}e^{-i\varphi}\tilde{\rho}_{01} - i\frac{\Omega}{2}e^{i\varphi}\tilde{\rho}_{10} + \gamma_r\tilde{\rho}_{11} \tag{32}$$

$$\dot{\tilde{\rho}}_{01} = i\Delta\tilde{\rho}_{01} + i\frac{\Omega}{2}e^{i\varphi}\tilde{\rho}_{00} - i\frac{\Omega}{2}e^{i\varphi}\tilde{\rho}_{11} - \left(\frac{\gamma_r}{2} + \gamma_\phi\right)\tilde{\rho}_{01} \tag{33}$$

The conditions (1) $\rho_{00} + \rho_{11} = 1$, and (2) $\rho_{01} = \rho_{10}^*$ are then applied, as well as some complex calculus rules to get the final equations (again, see the Appendix A1). The component $\tilde{\rho}_{00}$ is real, but the off-diagonal terms of the density matrix are complex numbers, so we have three differential equations to solve. One differential equation is for the $\tilde{\rho}_{00}$ component and the other two are for the real and complex parts of $\tilde{\rho}_{01}$ respectively. For simplicity, we use the notation $x = \tilde{\rho}_{00}$, $y = \mathrm{Re}(\tilde{\rho}_{01})$, and $z = \mathrm{Im}(\tilde{\rho}_{01})$:

$$\dot{x} = -\gamma_r x - \Omega\,\mathrm{Im}(e^{-i\varphi})y - \Omega\,\mathrm{Re}(e^{-i\varphi}) + \gamma_r \tag{34}$$

$$\dot{y} = -\Omega\,\mathrm{Im}(e^{i\varphi})x - \left(\frac{\gamma_r}{2} + \gamma_\phi\right)y - \Delta z + \frac{\Omega}{2}\,\mathrm{Im}(e^{i\varphi}) \tag{35}$$

$$\dot{z} = \Omega\,\mathrm{Re}(e^{i\varphi})x + \Delta y - \left(\frac{\gamma_r}{2} + \gamma_\phi\right)z - \frac{\Omega}{2}\,\mathrm{Re}(e^{i\varphi}) \tag{36}$$

This differential equation system is not solvable analytically (or at least is extremely complicated) so we use a python code to solve it. The code is provided in Appendix A2.