Asset Allocation in a
Value-at-Risk Framework

Ronald Huisman, Kees G. Koedijk and Rachel A.J. Pownall

April 1999

In this paper we develop an asset allocation model which allocates assets by maximising expected return subject to the constraint that the expected maximum loss should meet the Value-at-Risk limits set by the risk manager. Similar to the mean-variance approach a performance index like the Sharpe index is constructed. Furthermore it is shown that the model nests the mean-variance approach in case of normally distributed expected returns. We provide an empirical analysis using two assets: US stocks and bonds. The results highlight the influence of non-normal characteristics of the expected return distribution on the optimal asset allocation.

Correspondence:

Rachel A. J. Pownall

Erasmus University Rotterdam, Faculty of Business Administration

Financial Management, 3000 DR Rotterdam

The Netherlands

Tel: +31 10 4081255 Fax: +31 10 4089017

1 Introduction

Modern portfolio theory aims to allocate assets by maximising the expected risk premium per unit of risk. In a mean-variance framework risk is defined in terms of the possible variation of expected portfolio returns. The focus on standard deviation as the appropriate measure for risk implies that investors weigh the probability of negative returns equally against positive returns. However it is highly unlikely that the perception of investors to downside risk faced on investments is the same as the perception to the upward potential. Indeed risk measures such as semi-variance were originally constructed in order to measure the downside risk separately.

Another approach that has been taken is to incorporate downside risk directly into the asset allocation model. The optimal portfolio is then selected by maximising the expected return over candidate portfolios so that some shortfall criterium is met. Leibowitz and Kogelman (1991), and Lucas and Klaassen (1998) for example construct portfolios by maximising expected return subject to a shortfall constraint, defined such, that a minimum return should be gained over a given time horizon for a given confidence level. Roy (1952) and Arzac and Bawa (1977) define the shortfall constraint such that the probability that the value of the portfolio falling below a specified disaster level is limited to a specified disaster probability; these are examples of the safety-first approach to asset allocation. Although the concept of using shortfall constraints is more in line with investors' perception to risk, their applicability is rather limited since the disaster levels, minimum returns, confidence levels or disaster probabilities are hard to specify.

In this paper we extend the literature on asset allocation subject to shortfall constraints. We adress the criticism concerning the definition of disaster levels and probabilities

2

through the use of Value-at-Risk (VaR). Banks and financial institutions have adopted VaR as the measure for market risk[1], whereby VaR is defined as the maximum expected loss on an investment over a specified horizon given some confidence level[2]. It therefore reflects the potential downside risk faced on investments in terms of nominal losses. We therefore show how VaR can be implemented into the asset allocation framework using shortfall constraints. The advantage being that the shortfall constraint is then clearly defined in terms of a widely accepted market risk measure. Furthermore the methodology extends the richness of VaR as a risk management tool. At present risk managers determine the VaR of their portfolios, for a chosen confidence level, whereby VaR is used as an *ex-post* measure to evaluate the current exposure to market risk and the decision as to whether exposure ought to be reduced. The asset allocation framework subject to a VaR shortfall constraint extends the importance of VaR as an *ex-ante* market risk control measure. In our framework risk managers thus define a VaR limit, i.e. set the maximum allowed loss that will only be exceeded by a small probability, and then let the portfolio manager allocate assets in such a way that the maximum expected loss on the portfolio meets the VaR limit.

In developing the model we proceed along the lines of Arzac and Bawa (1977). Their approach is superior to the others mentioned above, since they are able to derive a performance measure equivalent to the Sharpe index, which gives portfolio managers an easy tool with which they can evaluate the efficiency of several (candidate) portfolios. Furthermore market equilibrium can be derived under specific assumptions regarding the distributional characteristics of expected returns on portfolios. For example the asset allocation model leads to the same optimal portfolio as a mean-variance approach in the case of normally distributed expected returns. It therefore generates the comfortable feeling that the framework is closely related to traditional models. It is widely known

however that the expected returns on many financial assets are not normally distributed (they are not symmetric and exhibit excess leptokurtosis); we are also able to incorporate this issue into the asset allocation decision. The model need not make any assumptions regarding the skewness and the structure of the tails of the distribution, thus enabling us to choose a distribution, which can correctly reflect the apparent tail fatness of financial return distribution: an important feature for a framework that focuses on downside risk when finding the optimal allocation of assets.

The plan of the paper is as follows. We introduce the framework in the following section. The third section then provides empirical results of the optimal portfolio allocation for a variety of asset classes. We also shall address the importance of the non-normal characteristics of expected return distributions in such a framework. Conclusions and practical implications are drawn in the final section.

2 Methodology

In this section we present a portfolio construction model subject to a VaR limit set by the risk manager for a specified horizon. In other words we derive an optimal portfolio such that the maximum expected loss would not exceed the VaR for a chosen investment horizon at a given confidence level. Using VaR as the measure for risk in this framework is in accordance with the banking regulations in practice and provides a clear interpretation of investors' behaviour of minimising downside risk. The degree of risk aversion is set according to the VaR limit; hence avoiding the limitations of expected utility theory as to the degree of risk aversion, which an investor is thought to exhibit.

2.1   The asset allocation problem and shortfall constraints

Suppose the risk management department sets the portfolio manager's VaR limit on an amount W(0) to be invested for an investment horizon T. The portfolio manager can invest this amount plus an amount B representing borrowing (B > 0) or lending (B < 0). The manager may invest in n assets whereby γ(i) denotes the fraction invested in the risky asset, i. Hence the γ(i)'s must sum to one. Let P(i,t) be the price of asset i at time t (t = 0 reflects the current decision period). Equation (1) gives the initial value of the portfolio as the following budget constraint:

$$W(0) + B = \sum_{i=1}^{n} \boldsymbol{g}(i)\, P(i,0) \tag{1}$$

According to 0 the manager needs to choose the fractions γ(i) to be invested with wealth W(0) and the amount to be borrowed or lent at time 0: we assume r_f is the interest rate at which the investor can borrow and lend for the period T. The portfolio allocation problem arises on allocating the assets in the portfolio and choosing the amount to borrow or lend such that the maximum expected level of final wealth is achieved.

The introduction of Value-at-Risk into risk management requires the portfolio manager to be highly concerned about the value of the portfolio falling below the VaR constraint. VaR is defined as the worst expected loss over a chosen time horizon within a given confidence interval c (see Jorion 1997). For example a 99% VaR for a 10-day holding period[3], implies that the maximum loss incurred over the next 10 days should only exceed the VaR limit once in every 100 cases. Chosing the desired level of Value-at-Risk as VaR* we therefore formulate the downside risk constraint as follows:

$$\Pr\{W(0) - W(T) \geq VaR^*\} \leq (1 - c) \tag{2}$$

Pr denotes the expected probability conditioned on the information available at time 0. Equation 0 is equivalent to:

$$\Pr\{W(T) \leq W(0) - VaR^*\} \leq (1 - c) \tag{3}$$

VaR is thus the worst expected loss over the investment horizon T, that can be expected with confidence level c. The investor's level of risk aversion is reflected in the VaR level, and the confidence level associated with it. Hence the optimal portfolio which is derived such that equation 0 holds will reflect this.

2.2   Optimal portfolio construction

Portfolio construction subject to a shortfall constraint is not new. Leibowitz and Kogelman (1991) and recently Lucas and Klaassen (1998) construct portfolios by maximising expected return subject to a shortfall constraint defined such that a minimum return should be gained over the given time horizon within a given confidence level. Arzac and Bawa (1977) developed a similar framework according to the safety-first principle introduced by Roy (1952). In that framework investors maximise the expected return of their portfolio subject to a shortfall constraint defined such that the probability that the end-of-investment horizon value of the portfolio falls below a disaster level s is smaller than a disaster probability α. Arzac and Bawa develop the optimal asset allocation for risk-averse investors and are also able to show that the model nests the CAPM for specific distribution characteristics of the expected returns. We therefore believe that the Arzac and Bawa model is the most serious contender to a theory of portfolio choice for risk-averse investors. The limitation however is in specifying the investor's disaster level and the associated probability of disaster.

The introduction of VaR however provides us with a shortfall constraint (denoted by equation 0) that fits perfectly into the Arzac and Bawa framework. We therefore build upon their results to derive an optimal asset allocation model. The investor is interested in maximising wealth at the end of the investment horizon. Let r(p) be the expected total return on a portfolio p in period T; assume that asset i is included with fraction γ(i,p) in portfolio p. The expected wealth from investing in portfolio p at the end of the investment horizon becomes:

$$E_0(W(T, p)) = (W(0) + B)(1 + r(p)) - B(1 + r_f) \qquad (4)$$

Substituting in for B as given in equation (1), we are able to express final wealth in terms of the risk-free rate of return and the expected portfolio risk premium (r(p)-r_f):

$$E_0(W(T, p)) = W(0)(1 + r_f) + (\sum_{i=1}^{n} g(i, p)P(i,0))(r(p) - r_f) \qquad (5)$$

Equation 0 shows that as long as the expected risk premium is positive a risk-averse investor will always invest some fraction of his wealth in the risky assets. In order to determine the optimal portfolio that maximises the expected final wealth subject to the VaR constraint 0 we substitute 0 into 0:

$$\Pr\{r(p) \le r_f - \frac{VaR^* + W(0)r_f}{\sum_{i=1}^{n} g(i, p)P(i,0)}\} \le (1 - c) \qquad (6)$$

Equation 0 simply defines the quantile q(c,p) that corresponds to probability (1-c) that can be read off the cdf of the expected return distribution for portfolio p. This quantile is used to derive the following expression from 0:

$$\sum_{i=1}^{n} g(i, p)P(i,0) = \frac{VaR^* + W(0)r_f}{r_f - q(c, p)} \qquad (7)$$

Substituting 0 back into 0 leads us to the following expression for the expected final wealth in terms of the quantile q(c,p):

$$E_0(W(T, p)) = W(0)(1+r_f) + \frac{(r(p)-r_f)}{(r_f - q(c, p))}(VaR^* + W(0)r_f) \qquad (8)$$

Dividing 0 by initial wealth W(0) we obtain the following expression for the expected return on the initial wealth:

$$E_0(\frac{W(T, p)}{W(0)}) = (1+r_f) + \frac{(r(p)-r_f)}{(W(0)r_f - W(0)q(c, p))}(VaR^* + W(0)r_f) \qquad (9)$$

It can be seen from equation 0 that the final expected return on wealth is maximised for an investor concerned about the downside risk for the portfolio which maximises S(p) in equation (10). We denote this maximising portfolio as p′.

$$p': \max_p S(p) = \frac{r(p)-r_f}{W(0)r_f - W(0)q(c, p)} \qquad (10)$$

Note that although initial wealth is in the denomenator of S(p) it does not affect the choice of the optimal portfolio since it is only a scale constant in the maximisation. The asset allocation process is thus independent of wealth. The advantage however of having initial wealth in the denomenator of 0 is in its interpretation. S(p) equals the ratio of the expected risk premium offered on portfolio p to the risk, reflected by the maximum expected loss on portfolio p that is incurred with probability 1-c relative to the risk-free rate. Since the negative quantile of the return distribution multiplied by the initial wealth is the Value-at-Risk associated with the portfolio for a chosen confidence level, then we are able to derive an expression for the risk faced by the investor as φ. Letting VaR(c,p) denote portfolio p's Value-at-Risk, the denominator of (10) may be written as:

$$\boldsymbol{j}(c, p) = W(0)r_f - VaR(c, p) \tag{11}$$

Such a measure for risk is in fitting with investors' behaviour of focussing on the risk free rate of return as the benchmark return with risk being measured as the potential for losses to be made with respect to the riskfree rate as the point of reference. Indeed the measure for risk can be seen as a possible measure for regret, since it measures the potential opportunity loss of investing in risky assets. Investors will therefore only accept greater returns if they can tolerate the regret occuring from the greater potential wealth-at-risk. The risk-return ratio S(p), which is maximised for the optimal portfolio p′ can therefore be written as:

$$p': \max_{p} S(p) = \frac{r(p) - r_f}{\boldsymbol{j}(c, p)} \tag{12}$$

S(p) is thus a performance measure like the Sharpe index that can be used to evaluate the efficiency of portfolios (see Sharpe 1994 for more details). Indeed under the assumption that expected portfolio returns are normally distributed, and the risk free rate is zero, S(p) collapses to a multiple of the Sharpe index. In this case the VaR is expressed as a multiple of the standard deviation of the expected returns so that the point at which both performance indices are maximised will lead to the same optimal portfolio being chosen. Only a minimal difference in the optimal portfolio weights occurs for positive risk free rates, so that both approaches will lead to almost identical results. Since our performance index S(p) does not rely on any distributional assumptions it has the advantage of being able to incorporate non-normalities into the asset allocation problem, through the use of other distributional assumptions. The existence of non-normalities may lead to the choice of different optimal portfolios, an empirical investigation of which we shall encounter later.

The optimal portfolio that maximises S(p) in 0 is chosen independently from the level of initial wealth. It is also independent from the desired VaR, since the risk measure φ for the various portfolios depends on the estimated portfolio VaR rather than the desired Value-at-Risk. Investors first allocate the risky assets and then, the amount of borrowing or lending will reflect by how much the VaR of the portfolio differs from the VaR limit set; thus two-fund separation holds like in the mean-variance framework. However since the investors' degree of risk aversion is captured by the chosen Value-at-Risk level, the amount of borrowing or lending required to meet the VaR constraint may be determined. The amount of borrowing is therefore found by substituting (1) and (11) into equation (7):

$$B = \frac{W(0)(VaR^* - VaR(c, p'))}{\textbf{\textit{j}}\,'(c, p')} \tag{13}$$

Of course the critical assumption in finding the maximising portfolio and hence the optimal asset allocation is in the choice of distributional assumption for the future distribution of returns. It is to this question which we now turn, with the use of an empirical example for US stocks and bonds.

3   Optimal Asset Allocation for US stocks and Bonds

In this section we provide an empirical example in which a portfolio manager needs to select the optimal allocation of US stocks and Bonds such that a VaR constraint is met. We employ daily data from US stock and bond indices from January 1980 until December 1998, providing us with 2364 observations. We use data obtained from Datastream for the S&P 500 Composite Return Index for the US, the 10- Year Datastream Benchmark US Government Bond Return Index and the 3- Month US

Treasury Bill rate for the risk free rate. The average annual return on the S&P 500 over the sample period was 16.81%, just over twice as high as the average annual return on the 10-year Government Bond Index of 8.35%. The annual standard deviation is also higher on the S&P 500 at 13.42% per annum, compared to the less volatile nature of the Government Bonds with an annual standard deviation of only 6.31%. Both series exhibit significant negative skewness, -0.45 and –0.42 for the S&P 500 Index and Bond Index respectively, and significant kurtosis, with greater excess kurtosis on the S&P 500 Index, 6.88, than on the Bond Index, 3.33.

3.1   Optimal allocation using the empirical distribution

To maximise the performance index S(p) in 0 we estimate the expected return r(p) and the daily Value-at-Risk(c,p) for various combinations of US stocks and bonds using the whole sample period. Figure 1 shows the efficient VaR frontier for a 95% confidence level. The VaR level for the 95% level is directly read off from the empirical return distribution for the various combinations of stocks and bonds.

Insert Figure 1

The VaR efficient frontier is similar to a mean-variance frontier except for the definition of risk: VaR relative to the benchmark return (φ) instead of standard deviation (σ). The lower-left point represents a portfolio containing 100% bonds and the upper-right portfolio depicts a 100% investment into stocks.

Below we assume that the portfolio manager needs to select a portfolio that has the same 95% VaR as has been observed in the past. The optimum position on the efficient VaR frontier, for an investor, who wants to be 95% confident that his wealth will not drop by more than the daily VaR limit, occurs where the return per unit of risk is maximised. This

occurs where equation (9) is maximised, and is independent of the level of wealth. Using the empirical sample we are therefore able to determine the optimal allocation between stocks and bonds, whereby the last available 3-month Treasury Bill rate in the sample period is used for the risk free rate of return. Taking 4.47% as the risk free rate, we find that the optimal allocation between US stocks and bonds occurs when 36% of wealth is held in stocks and 64% in bonds for an investor with a VaR limit at the 95% confidence level. If the desired daily Value-at-Risk, which the risk manager sets is different from the VaR associated with this 95% empirical estimate, then part of the wealth will need to be lent, or additional funds borrowed, at the risk free rate in accordance with equation (13). This involves moving along the Capital Market Line, also shown in Figure 1, until the desired trade off is attained.

The combinations for stocks and bonds for a variety of confidence levels are provided in the first two columns of Table 1.

Insert Table 1

Taking the desired VaR level as the daily VaR at the 95% confidence level from the empirical distribution as our benchmark we can observe how additional lending is required so that this benchmark VaR level is met for higher confidence levels. We can therefore see how sensitive the asset allocation decision is to changes in the confidence level associated with the Value-at-Risk limit. Allocating 36% in Stocks and 64% in Bonds generates a 95% VaR on the portfolio of $6.86 and of course no borrowing or lending is required to meet the VaR with 95% confidence. If however the risk manager desires greater confidence in the probability that the initial wealth will not drop by more than the VaR level, then the VaR associated with the portfolio allocation will be greater than the VaR limit and hence results in too much risk being taken[4]. In order to meet the

benchmark VaR less risk will have to be taken and hence $313.20 of the initial $1000 wealth is lent at the risk free rate. In the last column of the table we show the nominal amounts needed to be lent so that the daily VaR of $6.86 is met for the various confidence levels. The final portfolio allocations are given in the bottom segment of Table 1, and we see that the greater the confidence level, hence the lower the risk tolerance of the investor, the greater the proportion of wealth that needs to be lent at the risk free rate[5].

We have now shown how we can find the optimal allocation from the use of the empirical distribution, whereby we assumed that future returns are distributed in exactly the same manner as in the past. The reliance on a large sample is therefore crucial, so that the quantiles are estimated accurately. The choice of sample period is also extremely crucial to the estimation process. It may therefore be more desirable to assume a parametric distribution for characterising the distribution of future returns. Modifying the methodology in such a way certainly levies some benefits. Firstly, estimation risk is reduced - especially crucial for high confidence levels - hence providing more accurate estimates of quantiles. Secondly, quantile estimation is simplified since the quantiles are a function of the parameters determining the distribution. The characteristic parameters are derived from the historical data and the quantile is obtained by inferring the quantile point from the fitted distribution function. Finally, certain parametric assumptions enable a unique solution to be found for the maximising portfolio, regardless of the quantile used in the maximisation process. Regardless therefore of the investors risk preferences the choice of risky assets will be identical up to a scale constant for all investors. Specific parametric distributional assumptions therefore enables a market equilibrium model to be developed and hence allows its applicability to be tested empirically by means of a testable model of asset valuation.

If we assume that the future distribution of returns can be accurately proxied by the normal distribution, the only risk factor in our downside risk measure is the standard deviation of the distribution. This means that the quantile estimate is merely a multiple of standard deviations, and our risk measure φ in equation (12) is a multiple of the standard deviation alone. This results in the risk-return trade off being identical to that derived under the mean-variance framework. The maximisation will occur at the same point as which the Sharpe ratio is maximised, and the market model will collapse to the CAPM. Of course since we also have the possibility of assuming different distributional assumptions, we need not constrain ourselves to optimising our portfolio according to the first two moments of the distribution only and hence are able to include the possibilities of non-normalities into asset allocation. Indeed any parametric distribution used to represent the future distribution of returns, which can provide a unique solution to our maximising equation, equation (9), regardless of the quantile chosen, will allow for the derivation of a market equilibrium model.

Before entering the discussion on alternative market equilibrium models it is important to first analyse the extent to which non-normalities effect the optimal allocation of assets. We therefore compare the optimal allocation of assets derived using both the normal distribution and a fatter tailed distribution, the student-t, whereby we use the same sample period of data as before.

3.2   Optimal Allocation of US Stocks and Bonds under Parametric distributions

We first shall assume that the distribution of future returns can be captured using the normal distribution. To see how accurate the use of standard deviation alone is as the measure for risk we can plot the efficient VaR frontier for the risk-return trade off at the

95% confidence level for the normal distribution against the empirical VaR frontier derived earlier. It can also be seen in Figure 1 that at the 95% VaR level the assumption of normality reflects the actual risk-return trade off fairly well. On average the assumption of normality for the future distribution of returns at the 95% level means that the risk is only slightly overestimated[6] for a given level of return. The risk is minimised at the optimal allocation of 40% stocks and 60% bonds, and as mentioned earlier this optimum will occur at the same point for various confidence levels. This is presented in table 2[7].

Insert Table 2

In order to compare how well the normal distribution proxies for the empirical distribution, we again use the historical VaR at the 95% level as our benchmark VaR level. With wealth equal to $1000, an amount of $44.43 needs to be lent at the risk free rate to meet the empirical VaR at the 95% level.

We have seen that the assumption of normality renders the investors attitude to risk as unimportant in the optimisation process. Only after the optimising portfolio has been found do the individuals risk preferences come into play. The non-parametric nature of the empirical distribution however, led to the changing optimum allocation of assets for various confidence levels[8], whereby the optimal allocation of assets resulted in a proportionally greater increase in lending to meet the desired VaR level for higher confidence levels. Under the assumption of normality with standard deviation alone as the measure for risk this effect is not captured, and it appears that for a desired confidence level of 99%, too aggressive an investment strategy will result. It would appear that the trade off between risk and return is underestimated for higher confidence levels. We can see this graphically by comparing the efficient VaR frontiers at the 99%

confidence level for both the normal and the empirical distributions. This is presented in Figure 2.

Risk, as measured by the empirical VaR for the portfolio is higher for all combinations of stocks and bonds than captured by the use of standard deviation alone. This underestimation of risk far out in the tails of the distribution will be greater the greater the deviation from normality[9]. The greater probability of extreme negative returns in the empirical distribution implies greater downside risk than is captured by the measure of standard deviation alone. The use therefore of the normal distribution to assess the risk-return trade off will result in an incorrect allocation of assets for investors with low risk tolerance and risk managers wishing to set 99% confidence levels. Since non-normalities will cause errors to be made in the asset proportions held, it would appear more desirable to use a parametric distribution that more accurately reflects the distributional characteristics of financial assets for the whole range of confidence levels associated with the left tail of the distribution.

The student-t with 5 degrees of freedom, a fatter tailed distribution than the normal distribution, has been shown to represent the tail of the distribution of many financial assets more accurately than the normal distribution. In order to determine the extent to which non-normalities effect the optimal allocation of assets we compare both the efficient VaR frontier and the optimal allocation of assets using the assumption that returns are distributed as a student-t with 5 degrees of freedom[10]. We therefore derive efficient VaR frontiers using the empirical distribution, against the normal and the student-t. In figure 1 we see that the student-t with 5 degrees of freedom represents the empirical trade off at the 95% confidence level fairly well. This confirms that the

additional downside risk associated with the presence of fat tails in the distribution only becomes apparent for investors and risk managers wishing to know with greater certainty the probability of exceeding their VaR limits.

As we move to higher confidence levels, as shown in figure 2 for the 99% level, we find that it becomes vital that the additional downside risk from fat tails is incorporated into the risk-return trade off[11]. Comparing the optimal portfolio allocation in Table 2 we find that the proportions held in the various risky assets are again identical to the outcome when we assumed normally distributed returns. The measure for risk is however greater when using the student-t distribution and hence the amount of lending required to meet the same VaR level as before is greater. This is presented in the final two columns of Table 2. Indeed the proportion of wealth held in the various assets is much more in line with the optimal allocation when using the empirical distribution.

The use therefore of a risk measure, which is able to capture higher moments of the distribution appears to capture the true trade-off between risk and return as observed in financial markets. It indeed may provide a better explanation for the size of the risk premium, an interesting area for future research. Furthermore the results show that if a risk manager is concerned about his 99% VaR (as recommended by the regulatory framework of the Basle Committee) the use of the standard deviation alone in determining the correct risk-return trade off with which to optimise the allocation decision is incorrect. The use therefore of a parametric distribution, which is able to capture some of the additional downside risk, such as the student-t, would appear to allocate assets more in accord with the risk-return trade off observed in US financial markets.

17

We find that the use of a student-t distribution with 5 degrees of freedom is better able than the assumption of normality to correctly assess the risk return trade off in financial markets, however the exact nature of the parametric distribution which best represents the distribution of future returns in the negative tail of the distribution is still open to empirical research. Of further vital importance is the correct assessment of the correlation between assets as we move further into the tails of the distribution, and the notion that correlation increases for large negative returns may be able to explain the under assessment of the downside risk at high confidence levels[12].

4   Concluding Remarks

The move towards greater risk management has highlighted the need to be able to control and manage financial risk. Until now Value-at-Risk as a risk management tool has been used in the assessment of risk *ex-post* in financial markets. In this paper we highlight the need for taking an *ex-ante* approach to risk management, such that assets are allocated so that expected return is maximised and that the risk of the portfolio's value falling below a critical level is known. The introduction of VaR allows us to develop such an asset allocation model within a Value-at-Risk framework, so that the focus is on downside risk rather than standard deviation as the crucial measure for risk. The use of Value-at-Risk therefore enables the level of risk aversion to be measured, and hence the desired risk-returns profile of the investor to be attained: a limitation of modern portfolio theory.

In this paper we provide a model for asset allocation which is able to move away from the use of standard deviation alone as the approriate measure for risk in financial markets. We focus on the use of downside risk, as measured by Value-at-Risk, and hence

are able to allocate assets more in accordance with the risk perceptions which investors hold. The model is derived without having to impose distributional assumptions about the future distribution of returns, and hence enables an approach which is also able to encompasses the mean-variance approach, however through distributional assumptions other than normality we are able to incorporate non-normalities into the asset allocation process. The use of the normal distribution, or the student-t, allows for a market equilibrium model to be derived[13], whereas the use of normality enables the model to collapse to the CAPM[14].

Empirical results for US stock and bonds provide evidence of additional downside risk from skewness and kurtosis, and thus the use of the normal distribution results in an incorrect estimation of the risk-return trade off for investors wanting to know the probability of their portfolio falling below the VaR level with high confidence. Since the Basle committee currently recommends bank's capital requirements to be a multiple of the 99% Value-at-Risk, allocating assets such that the 99% VaR level is met is of crucial importance in practice. Desirable is therefore a model of asset allocation, which is well able to assess the risk-return trade-off observed in financial markets, such that the VaR constraint is met. The inclusion therefore of a non-normal distribution in asset allocation models would appear to be much more desirable than the present assumption that assets are normally distributed in the tails. A move away from current portfolio theory, such that assets can be allocated to meet risk management constraints, and the possible inclusion of non-normalities in the asset allocation process as outlined in this paper therefore appears a crucial direction in which to move.

5 References

Arzac, E.R. and V.S. Bawa (1977) 'Portfolio Choice and Equilibrium in Capital Markets with Safety-First Investors'. *Journal of Financial Economics*, 4, 277-288.

Harlow, W. V. (1991) 'Asset Allocation in a Downside-Risk Framework', *Financial Analysts Journal,* 28.

Huisman, R., C.G. Koedijk, and R.A. Pownall (1998) 'VaR-x: Fat Tails in Financial Risk Management'. *Journal of Risk*, 1, 47-61.

Jansen, D.W., C.G. Koedijk, and C.G. de Vries (1998) 'Portfolio Selection with Limited Downside Risk'. *LIFE Working paper.*

Jorion, P. (1997) *Value at Risk: The New Benchmark for Controlling Derivatives Risk*, McGraw-Hill Publishers.

Leibowitz, M.L. and S. Kogelman (1991) 'Asset Allocation under Shortfall Constraints'. *Journal of Portfolio Management*, Winter, 18-23.

Longin, F. and B. Solnik (1995) 'Is the International Correlation of Equity Returns Constant: 1960-1990?'. *Journal of International Money and Finance*, February 1995.

Lucas, A. and P. Klaassen (1998) 'Extreme Returns, Downside Risk, and Optimal Asset Allocation'. *Journal of Portfolio Management*, Fall, 71-79.

Roy, A.D. (1952) 'Safety-First and the Holding of Assets'. *Econometrica*, 20, 431-449.

Sharpe, W. (1994) 'The Sharpe Ratio'. *Journal of Portfolio Management*, 21, 49-58.

Stutzer, M. (1998) 'A Portfolio Performance Index and Its Implications'. *U. of Iowa Working Paper.*

Table 1

Optimal Asset Allocation using Empirical Distribution

The table gives the optimal allocation between Stocks and Bonds using data on the S&P 500 Composite Returns Index and the 10- Year Datastream US Benchmark Government Bond Index over the period January 1980 - December 1998. The risk-return trade off maximises equation (10), whereby the risk free return is the return on the one month Treasury Bill on 31.12.98 of 4.47%. The historical distribution is used to estimate the Value-at-Risk. The VaRs for the portfolio are given, the amount of borrowing or lending required to meet the daily historical VaR at the 95%, and the final portfolio allocation such that the VaR constraint is met.

Asset allocation to maximise risk-return trade-off

Empirical	Stocks	Bonds	VaR	Borrowing	VaR*
95%	36%	64%	6.836	0	6.836
97.5%	51%	49%	10.016	-313.20	6.836
99%	34%	66%	11.400	-395.59	6.836

Optimal Portfolio to meet VaR constraint

Empirical	Stocks	Bonds	Cash
95%	36.00%	64.00%	0.00%
97.5%	35.03%	33.65%	31.32%
99%	20.55%	39.89%	39.56%

Table 2

Optimal Asset Allocation under Normality and Student-t

The table gives the optimal allocation between Stocks and Bonds using data on the S&P 500 Composite Returns Index and the 10- Year Datastream US Benchmark Government Bond Index over the period January 1980 - December 1998. The risk-return trade off maximises equation (10), whereby the risk free return is the return on the one month Treasury Bill on 31.12.98 of 4.47%. The normal distribution and the Student-t with 5 degrees of freedom are used to estimate the Value-at-Risk. The VaRs for the portfolio are given, the amount of borrowing or lending required to meet the daily historical VaR at the 95%, and the final portfolio allocation such that the VaR constraint is met.

Asset allocation to maximise risk-return trade-off

	Optimal Portfolio		**Normality**		**Student-t**	
	Stocks	***Bonds***	**VaR**	**Borrowing**	**VaR**	**Borrowing**
95%	40%	60%	7.160	-44.43	6.771	9.45
97.5%	40%	60%	8.621	-203.82	8.766	-216.76
99%	40%	60%	10.32	-333.17	11.618	-406.80

Optimal Portfolio to meet VaR constraint

	Normality			**Student-t**		
Normal	***Stocks***	***Bonds***	***Cash***	***Stocks***	***Bonds***	***Cash***
95%	38.22%	57.33%	4.44%	40.38%	60.57%	-0.95%
97.5%	31.85%	47.77%	20.38%	31.33%	46.99%	21.68%
99%	26.67%	40.01%	33.32%	23.73%	35.59%	40.68%

Figure 1

Efficient VaR Frontier - 95% Student-t, Normal and Empirical VaR

The figure presents the risk return trade off for portfolios of Stocks and Bonds whereby risk is measured by the VaR of the portfolio at the 95% confidence level. The returns and VaR estimates are obtained using data on the S&P 500 Composite Returns Index and the 10- Year Datastream US Benchmark Government Bond Index for the period January 1980 until December 1998. We present the efficient frontier for the empirical distribution, the parametric normal approach and under the assumption of a Student-t distribution with 5 degrees of freedom.
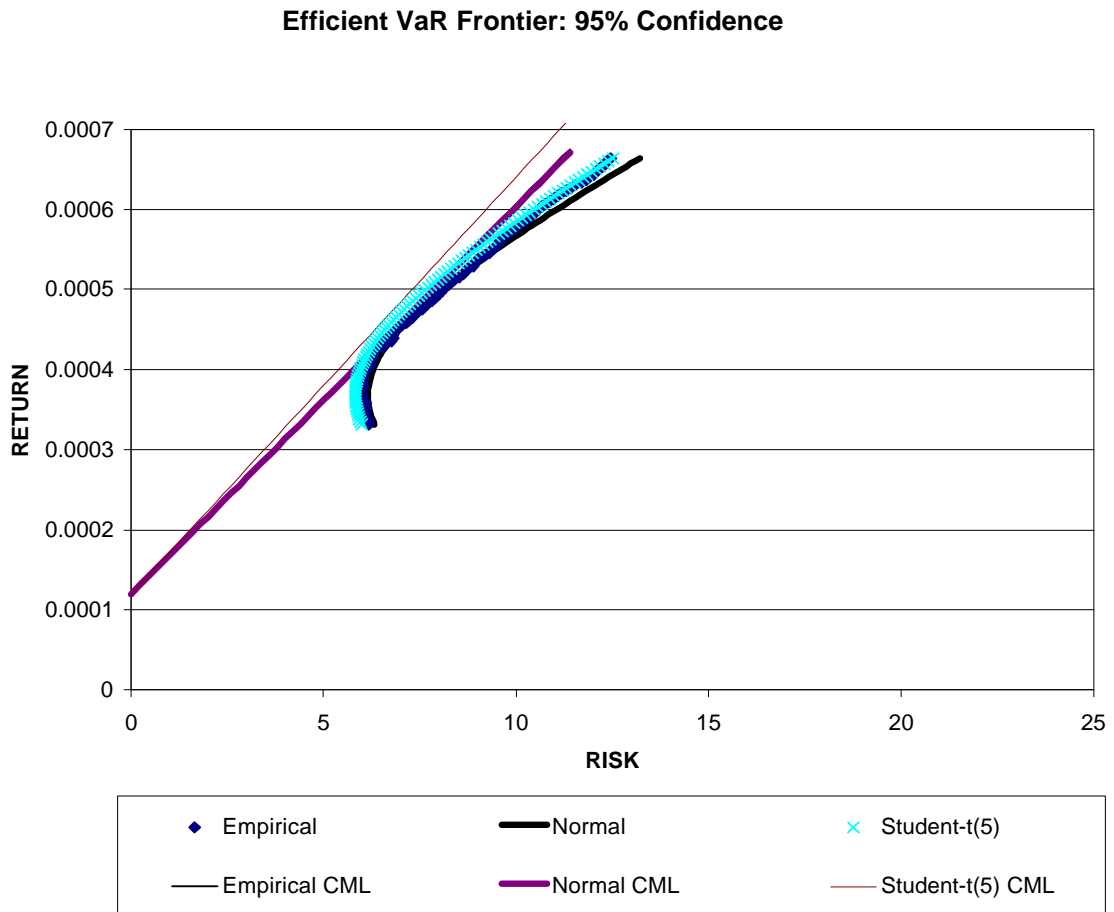
Efficient VaR Frontier: 95% Confidence

Figure 2

Efficient VaR Frontier - 99% Student-t, Normal and Empirical VaR

The figure presents the risk return trade off for portfolios of Stocks and Bonds whereby risk is measured by the VaR of the portfolio at the 99% confidence level. The returns and VaR estimates are obtained using data on the S&P 500 Composite Returns Index and the 10- Year Datastream US Benchmark Government Bond Index for the period January 1980 until December 1998. We present the efficient frontier for the empirical distribution, the parametric normal approach and under the assumption of a Student-t distribution with 5 degrees of freedom.

Efficient VaR Frontier: 99% Confidence
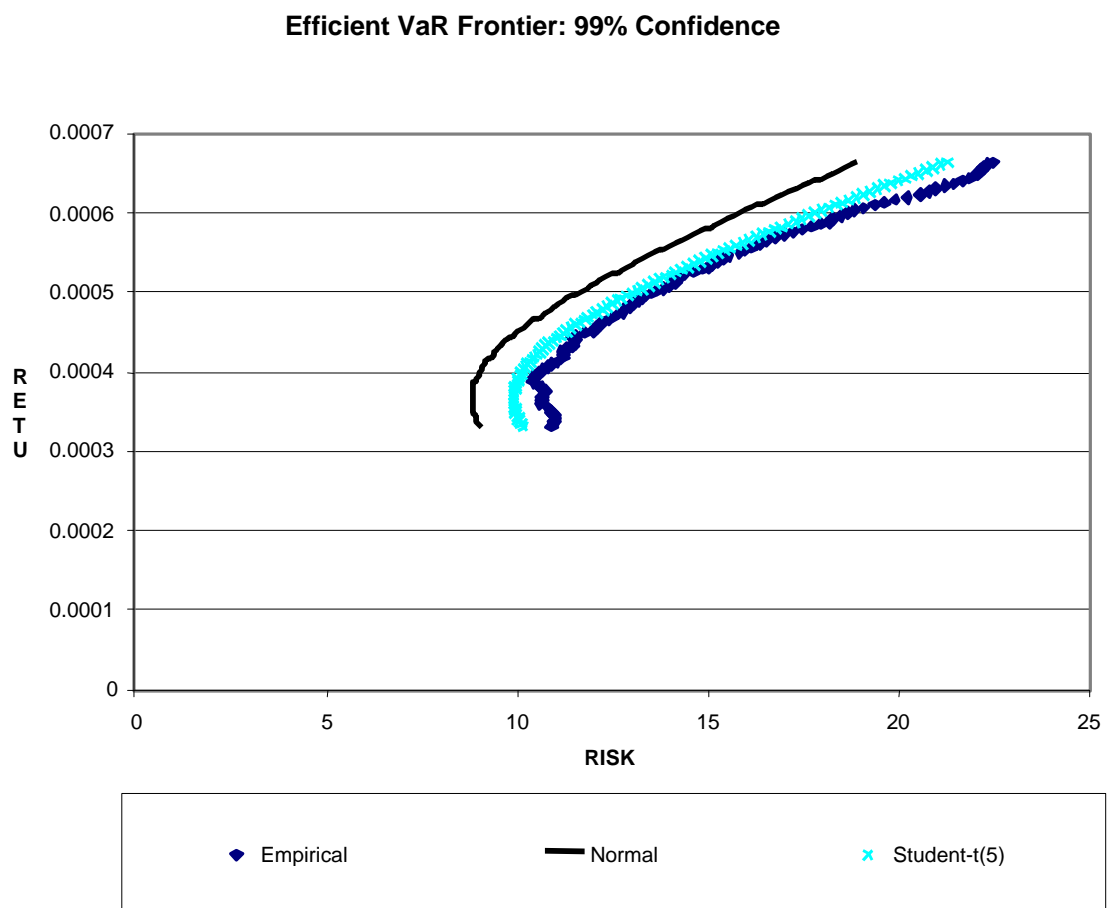
Endnotes

[1] See Jorion (1997) for a comprehensive introduction into Value-at-Risk methodology.

[2] In practice these confidence levels range from 95% through 99%, whereby the Basle Committee recommends 99%.

[3] This is the VaR recommended by the Basle Committee for Banking Regulation used in establishing a bank's capital adequacy requirements.

[4] A higher confidence level by definition will result in a higher Value-at-Risk.

[5] In a similar manner specifying a confidence level below that used for the optimisation the risk manager would want to take on additional risk by borrowing additional funds at the risk free rate, and going short in the 3-month Treasury Bill.

[6] On average in the sample the normal distribution significantly overestimates the risk-return trade off by 2.12%.

[7] Maximising the Sharpe ration also generates an optimal asset allocation of 40% stocks and 60% bonds.

[8] A market equilibrium model could however be derived under other assumptions, such as imposing the assumption that all investors focus on the same confidence level for VaR.

[9] See Huisman, Koedijk and Pownall (1998).

[10] The smaller the number of degrees of freedom used to parameterise the student-t distribution the fatter the tails of the distribution and the greater the severity of the difference between the normal distribution. For consistency we use five degrees of freedom throughout the empirical analysis.

[11] Of course the use of a fatter tailed distribution with more degrees of freedom will be better able to capture the risk-return trade off as seen in the empirical VaR frontier.

[12] See Longin and Solnik (1995) for empirical analysis into the instability of international covariance and correlation matrices over time.

[13] See Arzac and Bawa (1977) for the derivation of a market equilibrium model.

[14] This result enables us to test the VaR approach to asset allocation when other distributional assumptions are assumed.