MATH7701 Number Theory

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Abstract

What did the number theorist say as he drowned?

Log, log, log, log....

For an up to date version of this pdf, check my GitHub :)

https://github.com/vrvinny/number-theory

Contents

1 Introduction/Review

1.1 Introduction

Number Theory is the theory of the ring $\mathbb{Z}$ and other related rings. A ring (in this course) is a set R with two binary operations $+$ and $*$ such that:

- $(R, +)$ is an abelian group
- $*$ is associative, commutative and has an identity element 1
- $x(y+z) = xy + xz \quad \forall x, y, z \in \mathbb{R}$

Examples of rings:

- $\mathbb{Z}$ is a ring
- Every field is a ring, (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}$)
- $\mathbb{Z}/n \qquad \mathbb{Z}$ modulo $n = \{0, \ldots, n-1\}$
- $\mathbb{F}$[X] = { polynomials $f(x)$ with coefficients in $\mathbb{F}$

1.2 Review

1.2.1 Congruences

Let n be a positive integer. Given $x, y \in \mathbb{Z}$, we say x is congruentt to y modulo n if $x - y$ is a multiple of n.

$$x \equiv y(n) \qquad \text{or} \qquad x \equiv y \bmod n$$

E.g $\quad 2 \equiv 12 \ (10)$
$\equiv -8 \ (10)$

We write $\mathbb{Z}/n$ for the ring of congruency classes modulo n, i.e. the elements are integer, with two of them regarded as the same if they are congruent modulo n.

Since every integer is congruent to a unique integer in the set $\{0, \ldots, n-1\}$, we have $\mathbb{Z}/n = \{0, \ldots, n-1\}$.
An element x of $\mathbb{Z}/n$ is called "invertible" or a "unit" if $\exists y \in \mathbb{Z}/n$ such that $xy \equiv 1(n)$.

Theorem 1.1. *x is invertible modulo n iff x and n are coprime*

Recall Two numbers are coprime if their highest common factor is 1.
Here's how we find the inverse of x in $\mathbb{Z}/n$. Since X and n are coprime we can find $h, k \in \mathbb{Z}$ such that $hx + kn = 1 \implies hx = 1 \ (n)$. So h is the inverse of x modulo n.

E.g We'll find the inberse of 7 modulo 25 using Euclid's algorithm

$$
\begin{aligned}
25 &= 3 \times 7 + 4 & 1 &= 4 - 1(3) \\
7 &= 1 \times 4 + 3 & 1 &= 4 - 1(7 - 1(4)) = 2(4) - 1(7) \\
4 &= 1 \times 3 + 1 & 1 &= 2(25 - 3(7)) - 1(7) = 2(25) - 7(7)
\end{aligned}
$$

$$
\begin{aligned}
2(25) - 7(7) &= 1 \\
-7(7) &= 1 \ (25)
\end{aligned}
$$

$$
\begin{aligned}
(7^{-1}) &= -7 = 18 \ (25) \\
7 \times 18 = 126 &= 1 \ (25)
\end{aligned}
$$

We'll write $(\mathbb{Z}/n)^{\times}$ for the invertible elements in $\mathbb{Z}/n$

E.g
$(\mathbb{Z}/3)^{\times} = \{\ \cancel{0}, 1, 2\ \}$
$(\mathbb{Z}/6)^{\times} = \{\ \cancel{0}, 1, \cancel{2}, \cancel{3}, \cancel{4}, 5\ \}$

Theorem 1.2. $(\mathbb{Z}/n)^{\times}$ *is a group with the operation of multiplicity.*

1.2.2 Solving Linear Congruences

Suppose we want to solve $ax \equiv b\ (n)$ (given a, b and n).

Case 1: If a is coprime to n then we can find a^{-1} modulo n by Euclid's algorithm, $x \equiv a^{-1}b\ (n)$

Case 2: If a is a factor of n, then there are two possibilities:
2a) if a is also a factor of b then $ax \equiv b \quad (n)$ is equivalent to $x = \frac{b}{a} \quad (\frac{n}{a})$
2b) if a is not a factor of b then there are no solutions

E.g. Solve $5x = 11 \ (13)$
This is case 1 because 5 and 13 are coprime

$$\begin{aligned} 13 &= 2 \times 5 + 3 \qquad & 1 &= (3) - 1(2) \\ 5 &= 1 \times 3 + 2 \qquad & 1 &= (3) - 1(5 - 1(3)) = 2(3) - (5) \\ 3 &= 1 \times 2 + 1 \qquad & 1 &= 2(13 - 2(5)) - (5) = 2(13) - 5(5) \end{aligned}$$

$$\begin{aligned} 1 &\equiv -5(5) \quad (13) \\ 5^{-1} &\equiv -5 \equiv 8 \quad (13) \end{aligned}$$

$$\begin{aligned} 5x &\equiv 11 \quad (13) \\ x \equiv 8 \times 11 &\equiv 88 \quad (13) \end{aligned}$$

$$x \equiv 10 \quad (13)$$

E.g. Solve $7x \equiv 84 \quad (490)$
7 is a factor of 490 so case 2)
7 is a factor of 84 so case 2a)

$7x \equiv 84 \quad (490)$
$x \equiv 12 \quad (70)$

E.g. Solve $7x \equiv 85 \quad (490)$
This is case 2b (7 is a factor of 490 but not of 85) $\therefore$ No solutions

$$\begin{aligned} & 7x \equiv 85 \quad (490) \\ \implies & 7x = 85 + 490y \text{ for some } y \in \mathbb{Z} \\ \implies & 0 \equiv 1 \quad (7) \end{aligned}$$

E.g. Solve $6x \equiv 3 \quad (21)$
This is neither case 1 nor case 2 but we can rewrite as:

$3(2x) \equiv 3 \quad (21)$

By case 2 we can solve for $2x \equiv 1 \quad (7)$
but now 2 is invertible modulo 7 so now solve by case 1

$\therefore x \equiv 4 \quad (7)$

1.3 Chinese Remainder Theorem

Suppost we know the congruency class of x modulo 10. Then we can work out its congruency class mod 2 and mod 5.
E.g. if $x \equiv 7 \ (10)$, then $x \equiv 1 \ (2)$ and $x \equiv 2 \ (5)$

Then the Chinese Remainder Theorem allows us to do the opposite, i.e. if we know x modulo 2 and modulo 5, then we can work out the value of x modulo 10.

Suppose n & m are coprime positive integers, let $a \in (\mathbb{Z}/n)$ and $b \in (\mathbb{Z}/m)$ then there is a unique

$$x \in (\mathbb{Z}/nm) \text{ such that } \begin{aligned} x &\equiv a \quad (n) \\ x &\equiv b \quad (m) \end{aligned}$$

Proof of existence part:
Since n & m are coprime, we can find $h, k \in \mathbb{Z}$ such that $hn + km = 1$.

Let $x = hnb + kma$

Check that this a solution to both congruences:

$$\begin{aligned} x &\equiv kma \quad (n) \\ x &\equiv (1 - hn)a \quad (n) \\ x &\equiv (1)a \quad (n) \\ x &\equiv a \quad (n) \end{aligned}$$

Similarly, this holds for $x \equiv b \quad (m)$.

E.g. Solve the simultaneous congruence:

$$\begin{aligned} x &\equiv 3 \quad (8) \\ x &\equiv 4 \quad (5) \end{aligned}$$

By the Chinese Remainder Theorem, there is unique solution modulo 40. To find the solution we let $x = hnb + kma$.
First find h, k by Euclid's algorithm.

$$\begin{aligned} 8 &= 1 \times 5 + 3 & 1 &= (3) - 1(2) \\ 5 &= 1 \times 3 + 2 & 1 &= (3) - 1(5 - 1(3)) = 2(3) - (5) \\ 3 &= 1 \times 2 + 1 & 1 &= 2(8 - 2(5)) - (5) = 2(8) - 5(5) \end{aligned}$$

$$\begin{aligned} \therefore x &= (2 * 8 * 4) - (3 * 5 * 3) \\ x &= 64 - 45 \\ \implies x &\equiv 19 \quad (40) \end{aligned}$$

Remark: We can use the Chinese Remainder Theorem to solvoe a congruence modulo nm, by first solving mod n and then mod m and then combining the results.
E.g. Solve $x^2 \equiv 2$ (119). Note 119 = 7 * 17.
By CRT this is equivalent to:

$$\begin{aligned} x^2 &\equiv 2 \quad (7) & &\implies x \equiv \pm 3 \quad (7) \\ x^2 &\equiv 2 \quad (17) & &\implies x \equiv \pm 6 \quad (17) \end{aligned}$$

Now we combine the solutions:

$$\begin{aligned} 17 &= 2 * 7 + 3 & 1 &= (7) - 2(3) \\ 7 &= 2 * 3 + 1 & 1 &= (7) - 2(17 - 2(7)) \\ & & 1 &= 5(7) - 2(17) \end{aligned}$$

Since

$$x \equiv \pm 3 \quad (7) \qquad \text{We get } x \equiv 5 * 7 * (\pm 6) - 2 * 17 * (\pm 3)$$
$$x \equiv \pm 6 \quad (17) \qquad x \equiv \pm 11 \text{ or } \pm 45 \quad (119)$$

1.4 Prime numbers

Defintion 1.3. *An integer $p \geq 2$ is a prime number if the only factors of p are $\pm 1, \pm p$*

We'll write $\mathbb{F}_p$ for $\mathbb{Z}/p$. This is because:

Theorem 1.4. *If p is prime, then $\mathbb{F}_p$ is a field*

Proof. Need to check that the non-zero elements of $\mathbb{F}_p$ all have inverses.

Let $x \in \mathbb{F}_p$ with $x \not\equiv 0 \ \ (p)$ i.e. x is not a multiple of p

$\therefore$ hcf$(x, p) = 1$

$\therefore x$ & p coprime ☐

1.5 Fermat's Little Theorem

Theorem 1.5. *Let p be a prime number. If x is not a multiple of p then $x^{p-1} \equiv 1 \ \ (p)$*

Proof. $x \in \mathbb{F}_p^x = \{1, 2, \ldots, p-1\}$ a group with $p-1$ elements.
Let n be the order of x in this group.
(order of x is smallest $n > 0$ such that $x^n \equiv 1 \ \ (p)$)
By corollary to Lagrange's Theorem, $p-1$ is a multiple of n

$x^n \equiv 1 \ \ (p)$
$x^{p-1} \equiv 1 \ \ (p)$ ☐

Theorem 1.6. *Lagrange's Theorem: If H is a subgroup of a finite group G, then $|H|$ is a factor of $|G|$.*

Corollary 1.7. *Order of an element is a factor of $|G|$*

We can use Fermat's Little Theorem to do calculations.
E.g. Calculate 10^{100} modulo 19

By Fermat's Little Theorem: $10^{18} \equiv 1 \ \ (19)$

$$\begin{aligned}
10^{100} &\equiv (10^{18})^5 * 10^{10} \quad (19) \\
&\equiv 100^5 \quad (19) \\
&\equiv 5^5 \quad (19) \\
&\equiv 25 * 125 \equiv 6 * 11 \equiv 9 \quad (19)
\end{aligned}$$

Also using Fermat's Little Theorem we can solve congruence of the form $x^a \equiv b \ \ (p)$ as long as p prime and a inverible modulo $p-1$

1.5.1 General method to solve $x^a \equiv b \pmod{p}$

Let

$$\begin{aligned} c &= a^{-1} \quad (p-1) \\ ac &= 1 + (p-1)r \end{aligned}$$

Raise both sides of the congruence to power c:

$$\begin{aligned} \therefore x^{ac} &\equiv b^c \quad (p) \\ x^{1+(p-1)r} &\equiv b^c \quad (p) \\ x &\equiv b^c \end{aligned}$$

So the solution is $x \equiv b^c \quad (p)$
E.g. Solve $x^5 \equiv 2 \quad (19)$
19 is prime and 5 is coprime to 18.
Find $c = 5^{-1}$ mod 18

$$\begin{aligned} 18 &= 3 * 5 + 3 \\ 5 &= 2 * 3 - 1 \end{aligned} \qquad \begin{aligned} 1 &= 2 * 3 - 5 \\ 1 &= 2(18 - 3 * 5) - 5 \\ 1 &= 2 * 18 - 7 * 5 \end{aligned}$$

$$\begin{aligned} \therefore 5^{-1} &\equiv -7 \quad (18) \\ &\equiv 11 \quad (18) \end{aligned}$$

$$\begin{aligned} \therefore x &\equiv 2^{11} \quad (19) \\ &\equiv 2048 \quad (19) \\ &\equiv 15 \quad (19) \end{aligned}$$

1.6 Fundamental Theorem of Arithmetic

If n is a positive integer then there is a unique factorisation, $n = p_1 p_2 \dots p_r$ with p_i prime. "Unique" means up to reordering the primes $p_1, \dots, p_r$. Showing that a factorisation exists is easy. For the uniqueness part we use:

1.6.1 Euclid's Lemma

Lemma 1.8. *Suppose p prime, and $p|ab$. Then $p|a$ or $p|b$.*

To prove Euclid's lemma we use Bezout's lemma.

Proof. Assume $p|ab$ but $p \nmid a$. Then $\text{hcf}(a, p) = 1$
By Bezout's lemma, $\exists h, k$ such that:
$1 = ha + kp$
$b = hab + kpb$ Both hab and kpb are multiples of p.
$\therefore p|b$ $\square$

1.6.2 Checking whether a number is prime

If n is composite then the smallest factor of n is (apart from 1) is a prime number $p \leq \sqrt{n}$, i.e. to show that n is prime, we just need to show that none of the primes up to $\sqrt{n}$ are factors of n.

E.g. Is 199 prime?

$\sqrt{199} < 15$ since $15^2 = 225$

The primes up to 15 are $\cancel{2}, \cancel{3}, \cancel{5}, \cancel{7}, \cancel{1}1, \cancel{1}3$

$199 \equiv 3 \quad (7)$

$199 \equiv 4 \quad (13)$

$\therefore$ 199 is prime

Theorem 1.9. *There are infinitely many primes*

Proof. Suppose $p_1, \ldots, p_n$ are all the primes.

Let $N = p_1 \ldots p_n + 1$

$\therefore N$ has no prime factors ϟ $\square$

Similarly there are infinitely many primes $p \equiv 2 \pmod 3$

Proof. Assume there are only finitely many primes, call them $p_1, p_2, \ldots, p_r$. All other primes are either 3 or are congruent to 1 mod 3.

Let $N = 3p \ldots p_{r-1}$. Since $3 \not| N$ and $p_i \not| N$ then all the prime factor of N are congruent to 1 mod 3.

$\therefore N \equiv 1 \pmod 3 \implies$ because clearly $N \equiv 2 \pmod 3$ $\square$

2 Elementary Number Theory

2.1 Euler Totient Function

Recall $(\mathbb{Z}/n)^\times$ is the group of invertible elements in $\mathbb{Z}/n$.
E.g. $(\mathbb{Z}/6)^\times = \{1, 6\}$
$(\mathbb{Z}/8)^\times = \{1, 3, 5, 7\}$
These are groups with the multiplication operation, * . The multiplication table for $(\mathbb{Z}/8)^\times$ is given below.

*	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Defintion 2.1. *The Euler Totient function is* $\phi(n) = |(\mathbb{Z}/n)^\times|$

E.g. $\phi(6) = 2$
$\phi(8) = 4$

If p prime then $(\mathbb{Z}/p)^\times = \{1, \ldots, p-1\}$ so $\phi(p) = p - 1$

Theorem 2.2. *Euler's Theorem- Let* $x \in (\mathbb{Z}/n)^\times$ *then* $x^{\phi(n)} \equiv 1 \pmod{n}$

In the case $n = p$ is prime, this is just Fermat's Little Theorem.

Proof. Let d be the order of x, i.e. $x^d \equiv 1 \ (n)$. By a corollary to Lagrange's Theorem, d is a factor of $\phi(n) \implies x^{\phi(n)} \equiv 1 \ (n)$ □

We can use Euler's theorem to solve congruences and calculate powers mod n. To use the theorem, we need a quick way of calculating $\phi(n)$.

Lemma 2.3. *Let* $n = p^a$ *where* p *is prime* $a > 0$*. Then* $\phi(n) = (p-1)p^{a-1}$

E.g. $\phi(8) = \phi(2^3) = (2-1)2^{3-1} = 4$

Proof. An integer is coprime to p^a as long as it's not a multiple of p.
$\therefore$ The elements of $\mathbb{Z}/p^a$ which are not invertible are the multiples of p. $0, p, 2p, \ldots, p^a - p$.
There are $p^a - 1$ of these:

$\therefore |(\mathbb{Z}/p^a)^\times| = p^a - p^{a-1} = (p-1)p^{a-1}$ □

Theorem 2.4. *Let* n *and* m *be coprime. Then there is an isomorphism:*

$$(\mathbb{Z}/nm)^\times \cong (\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$$

We'll use the theorem before we prove it.
Remark: If G and H are groups, $G \times H = \{(x, y) : x \in G, y \in H\}$, then $G \times H$ is a group with the operation , $(x, y)(x', y') = (xx', yy')$ and $G \times H$ is the "direct product" of G and H

Corollary 2.5. *If n and m are coprime then $\phi(nm) = \phi(n)\phi(m)$*

Proof.

$$\begin{aligned}\phi(nm) = |(\mathbb{Z}/n)^\times| &= |(\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times| \\ &= |(\mathbb{Z}/n)^\times| * |(\mathbb{Z}/m)^\times| \\ &= \phi(n)\phi(m)\end{aligned}$$

□

Corollary 2.6. *(Corollary of the corollary): Suppose $n = p_1^{a_1} \dots p_r^{a_r}$ with $p_1, \dots p_r$ distinct primes and $a_i > 0$. Then*

$$\phi(n) = (p_1 - 1)p^{a_1 - 1} \; * \cdots * \; (p_r - 1)p_r^{a_r - 1}$$

Proof. Since $p_1^{a_1}, \dots, p_r^{a_r}$ are coprime,

$$\begin{aligned}\phi(n) &= \phi(p_1^{a_1}) \dots \phi(p_r^{a_r}) && \text{by the corollary} \\ &= (p_1 - 1)p_1^{a_1 - 1} \dots (p_r - 1)p_r^{a_r - 1} && \text{by the lemma}\end{aligned}$$

□

E.g. Calculate $\phi(200)$

$$\begin{aligned}\phi(200) &= \phi(2^3 * 5^2) \\ &= (2-1)2^{3-1} * (5-1)5^{2-1} \\ &= 4 * 4 * 5 \\ &= 80\end{aligned}$$

Theorem 2.7. *Suppose n and m are coprime, then $(\mathbb{Z}/nm)^\times \cong (\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$*
The isomorphism is the map $x \mapsto (x \mod n, x \mod m)$

E.g. $n = 4$, $m = 5$

$$\begin{aligned}(\mathbb{Z}/4)^\times &= \{1, 3\} \\ (\mathbb{Z}/5)^\times &= \{1, 2, 3, 4\} \\ \therefore (\mathbb{Z}/4)^\times * (\mathbb{Z}/5)^\times &= \{(1,1), (1,2), (1,3), (1,4), \\ &\quad (3,1), (3,2), (3,3), (3,4)\} \\ (\mathbb{Z}/20)^\times &= \{1, 3, 7, 9, 11, 13, 17, 19\}\end{aligned}$$

The isomorphism is:

$$\begin{aligned}1 &\mapsto (1,1) & 11 &\mapsto (3,1) \\ 3 &\mapsto (3,3) & 13 &\mapsto (1,3) \\ 7 &\mapsto (3,2) & 17 &\mapsto (1,2) \\ 9 &\mapsto (1,4) & 19 &\mapsto (3,4)\end{aligned}$$

Proof. Let $\Phi : \mathbb{Z}/nm \mapsto \mathbb{Z}/n * \mathbb{Z}/m$

$\Phi(x) = (x \mod n, x \mod m)$

This is a bijection by the Chinese Remainder Theorem.

We'll next show that x is invertible mod $nm \iff x$ is invertible mod n and mod m

($\implies$) Suppose x is invertible mod nm

$$\begin{aligned} \text{Let } xy &\equiv 1 \pmod{nm} \\ \therefore xy &\equiv 1 \pmod{n} \\ xy &\equiv 1 \pmod{m} \end{aligned}$$

$\therefore x$ invertible mod n and m

($\impliedby$) Suppose x invertible mod n and m

$$\begin{aligned} xa &\equiv 1 \pmod{n} \\ xb &\equiv 1 \pmod{n} \end{aligned}$$

By the Chinese Remainder Theorem, $\exists y$ such that $y \equiv a \pmod{n}$
$y \equiv b \pmod{m}$

$$\left.\begin{aligned} \therefore xy \equiv xa &\equiv 1 \pmod{n} \\ \equiv xb &\equiv 1 \pmod{m} \end{aligned}\right\} \implies xy \equiv 1 \pmod{nm} \text{ by the Chinese Remainder Theorem}$$

We've shown that Φ gives a bijection between $(\mathbb{Z}/nm)^\times$ and $(\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$. We'll next check that $\Phi(xy) = \Phi(x)\Phi(y)$.

$$\begin{aligned} \Phi(xy) &= (xy \mod n, xy \mod m) \\ &= (x \mod n, x \mod m) * (y \mod n, y \mod m) \\ &= \Phi(x)\Phi(y) \end{aligned}$$

$\square$

2.2 Euler's Theorem

If $x \in (\mathbb{Z}/n)^\times$ then $x^{\phi(n)} \equiv 1 \pmod{n}$ and $\phi(p_1^{a_1} \dots p_r^{a_r}) = (p_1 - 1)p^{a_1 - 1} \dots (p_r - 1)p_r^{a_r - 1}$

E.g. Calculate $7^{135246872002} \mod 10000$

$$\begin{aligned} &7 \text{ coprime to } 10000 \text{ so } 7^{\phi(10000)} \equiv 1 \pmod{10000} \\ &10000 = 2^4 * 5^4 \\ \therefore &\phi(10000) = (2-1)2^3 * (5-1) * 5^3 = 8 * 500 \\ &7^{4000} \equiv 1 \pmod{10000} \implies 7^n \text{ depends only on } n \text{ mod } 4000 \\ &135246872002 \equiv 2 (4000) \\ \therefore &7^{135246872002} \equiv 7^2 \equiv 49 \pmod{10000} \end{aligned}$$

We can also use Euler's THeorem to solve congruence with powers

2.2.1 Solving equations of the form $x^a \equiv b \pmod{n}$

Supose we want to solve $x^a \equiv b \pmod{n}$ where b is coprime to n and a is coprime to $\phi(n)$.
Clearly any solution x must be coprime to n by Euler's Theorem $x^{\phi(n)} \equiv 1 \pmod{n}$.
$\therefore$ The congruency class of $x^y \mod n$ depends only $y \mod \phi(n)$

Let

$$c = a^{-1} \mod \phi(n)$$

Raise both sides of the congruence to power c:

$$x^{ac} \equiv x^1 \equiv b^c \pmod{n}$$

$\therefore$ The solution is $x \equiv b^c \pmod{n}$

E.g. $x^7 \equiv 3 \pmod{50}$
3 is coprime to 50,

$$\begin{aligned} 50 &= 2 * 5^2 \\ \implies \phi(50) &= 1 * 4 * 5 = 20 \end{aligned}$$

7 is coprime to $\phi(50)$. To solve, we need to find

$$\begin{aligned} c &\equiv 7^{-1} \mod \phi(50) \\ &\equiv 3 \pmod{20} \end{aligned}$$

$$x \equiv 3^3 \equiv 27 \pmod{50}$$

E.g. $x^{27} \equiv 5 \pmod{123}$
5 is coprime to 123,

$$\begin{aligned} 123 &= 3 * 41 \\ \implies \phi(123) &= 2 * 40 = 80 \end{aligned}$$

27 is coprime to 80
To solve, we find $27^{-1} \mod 80$

$$\begin{aligned} 80 &= 3 * 27 - 1 \\ \implies 1 &= 3 * 27 - 80 \end{aligned}$$

$$27^{-1} = 3$$

$$\begin{aligned} x &= 5^3 \\ x &= 125 \equiv 2 \pmod{123} \end{aligned}$$

2.3 Primitive roots

Recall, let G be a finite group. G is called a cyclic group if $\exists x \in G$ such that, every element in G has the form x^n for some $n \in \mathbb{Z}$, i.e. $G = \{1, x, x^2, \ldots, x^{n-1}\}$ where n is the order of x, equivalentl the order of x is $|G|$. The element x is called a generator of G.

Theorem 2.8. *(Gauss' Theorem), For ever prime number p, the group $\mathbb{F}_p^\times$ is cyclic*

Defintion 2.9. *A generator of $\mathbb{F}_p^\times$ is called a primitive root. Equivalently, this is an element of order $p-1$*

E.g. $p = 7, x = 3$ We'll see that 3 is a primitive root modulo 7

$$\text{Powers of 3 in } F_7^\times : \quad \begin{array}{rll} 3^0 = 1 & 3^3 \equiv 6 \pmod 7 & 3^6 \equiv 1 \pmod 7 \\ 3^1 = 3 & 3^4 \equiv 4 \pmod 7 & \\ 3^2 \equiv 2 \pmod 7 & 3^5 \equiv 1 \pmod 7 & \end{array}$$

so 3 is a primitive root modulo 7. There is a quicker way to check whether x is a primitive root.

Proposition 2.10. *Let $x \in \mathbb{F}_p^\times$, then x is a primitive root modulo p if and only if for every prime factor q of $p-1$:*

$$x^{\frac{p-1}{q}} \not\equiv 1 \pmod p$$

Proof. Assume the second statement is false, so $\exists$ prime factor q of $p-1$ such that:

$$x^{\frac{p-1}{q}} \equiv 1 \pmod p \qquad \therefore \text{order of } x \leq \frac{p-1}{q} < p-1$$

$$\therefore x \text{ is not a primitive root}$$

Conversely, assume x is not a primitive root, so x doe not have order $p-1$. But the order of x is a factor of $p-1$.
Suppose the order of x is $\frac{p-1}{d}$, $d > 1$.
Let q be a prime factor of $d \implies q|p-1$
$\frac{p-1}{q}$ is a multiple of $\frac{p-1}{d}$ but $x^{\frac{p-1}{q}} \equiv 1 \pmod p \implies x^{\frac{p-1}{q}} \equiv 1 \pmod p$

□

E.g. $p = 29$
By the proposition x is a primitive root mod 29 $\iff x^{28/2} \not\equiv 1 \pmod{29}$ and $x^{28/7} \not\equiv 1 \pmod{29}$

$\iff x^{14} \not\equiv 1 \pmod{29}$ and $x^4 \not\equiv 1 \pmod{29}$

$$\text{Try } x = 2 : \quad \begin{array}{l} 2^4 \equiv 16 \not\equiv 1 \pmod{29} \\ 2^{14} \equiv 128^2 \equiv 12^2 \equiv 144 \equiv -1 \pmod{29} \end{array}$$

$\therefore$ 2 is a primitive root mod 29

Another trick to speed up the calculation:
$\mathbb{F}_p$ is a field $\therefore$ every polynomimal of d has no more than d in $\mathbb{F}$ (proved in 2201).
$\therefore$ if $x^2 \equiv 1 \pmod{p}$ then $x \equiv \pm 1 \pmod{p}$

This means that checking whether $x^{14} \equiv 1 \ (29)$ is equivalent to checking whether $x^7 \equiv \pm 1 \ (29)$.

E.g 3 is also a primitive root modulo 29

$$3^2 \equiv 9 \not\equiv \pm 1 \ \ (29)$$
$$3^4 \equiv 1 \ \ (29)$$

$$\begin{aligned} 3^7 &\equiv 27^2 * 3 \ \ (29) \\ &\equiv (-2)^2 * 3 \equiv 12 \ \ (29) \\ &\equiv \pm 1 \ \ (29) \end{aligned}$$

$$\therefore 3^{14} \not\equiv 1 \ \ (29)$$

2.4 Roots of unity and Cyclotomic Polynomials

A complex number ζ is called an n^{th} root of unity if $\zeta^n = 1$. The n^{th} roots of unity are $e^{2\pi i \frac{a}{n}}$ for $a = \{0, 1, \ldots, n-1\}$
We call ζ a primitive n^{th} root of unity if n smaller power than ζ^n is equal to 1, i.e. ζ has order n in $\mathbb{C}^\times$ if ζ is not a primitive n^{th} root of unity $\zeta = e^{2\pi i \frac{b}{d}}$ where $b = \{0, \ldots, d-1\}$ for $d < n$
$\therefore \frac{a}{n} = \frac{b}{d}$

The cancellation happens when a is not coprime to n. This shows that the primitive n^{th} of unity are $e^{2\pi i \frac{a}{n}}$, $a \in (\mathbb{Z}/n)^\times$.

Corollary 2.11. *There are exactly $\phi(n)$ primitive n^{th} roots of unity*

We'll actually prove a more precise version of Gauss' Theorem.

Theorem 2.12. *For every factor d of p^{-1} there are $\phi(d)$ elements in $\mathbb{F}_p^\times$ of order d.*

Defintion 2.13. *The n^{th} cyclotomic polynomial is:*

$$\Phi_n(x) = \Pi_{\substack{primtive \\ n^{th} roots \\ of\ unity\ \zeta}} (X - \zeta)$$

i.e $\zeta^n = 1$ and no smaller power of ζ is 1, $\zeta = e^{2\pi i \frac{a}{n}}, a \in (\mathbb{Z}/n)^\times$

This has degree $\phi(n)$.

E.g. n=4
Primitive 4^{th} roots of unity are $i, -i$:

$$\begin{aligned}\Phi_4(x) &= (x-i)(x-(-i)) \\ &= x^2+1\end{aligned}$$

Lemma 2.14. *For every $n > 0$:*

$$x^n - 1 = \Pi_{\substack{factors \\ d\ of\ n}} \Phi_d(x)$$

E.g. Calculate $\Phi_6(x)$

By the lemma

$$\begin{aligned} x^6 - 1 &= \Phi_1\Phi_2\Phi_3\Phi_6 \\ x^3 - 1 &= \Phi_1\Phi_3 \end{aligned} \qquad x^6 - 1 = (x^3-1)\Phi_2\Phi_6$$

$$\therefore \Phi_6 = \frac{x^6-1}{(x^3-1(x+1)} = \frac{x^3+1}{x+1} = x^2 - x + 1$$

Let p be a prime number. A primitive root mod p is an $x \in \mathbb{F}_p^\times$, such that x generates $\mathbb{F}_p^\times$. Equivalently order $= p-1$

2.4.1 How to calculate $\Phi_n(x)$

Lemma 2.15. $x^n - 1 = \Pi_{d|n}\Phi_d(x)$

E.g. $n = 4$

$$\begin{aligned} x^4 - 1 &= \Phi_1\Phi_2\Phi_4 \\ &= (x-1)(x+1)(x^2+1) \end{aligned}$$

$$\begin{aligned} \Phi_1 &= x - 1 \\ \Phi_2 &= (x-(-1)) = x+1 \\ \Phi_4 &= (x-i)(x-(-i)) = x^2+1 \end{aligned}$$

Proof.

$$x^n - 1 = \Pi_{\substack{\zeta \text{ is an} \\ n^{th} \text{ root of} \\ \text{unity}}} (x - \zeta)$$

but every n^{th} root of unity is a primitive d^{th} root of unity for some $d|n$.

$$x^n = \Pi_{d|n}(\Pi_{\substack{\text{primitive} \\ d^{th}\text{roots} \\ \text{of unity}}}(x-\zeta)) = \Pi_{d|n}\Phi_x(x)$$

$\square$

E.g. Calculate $\Phi_5(x)$

$$\begin{aligned} x^5 - 1 &= \Phi_1(x)\Phi_5(x) \\ &= (x-1)\Phi_5(x) \end{aligned}$$

$$\Phi_5(x) = \frac{x^5-1}{x-1} = 1 + x + x^2 + x^3 + x^4$$

More generally if p prime then $x^p - 1 = (x-1)\Phi_p(x) \implies \Phi_p(x) = 1 + x + \cdots + x^{p-1}$
E.g. Calculate $\Phi_8(x)$

$$x^8 - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x)\Phi_8(x)$$

$$x^4 - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x) \implies \Phi_8(x) = \frac{x^8-1}{x^4-1} = x^4 + 1$$

Corollary 2.16. *$\Phi_n(x)$ has coefficients in $\mathbb{Z}$*

Proof. $\Phi_n(x) = \dfrac{x^n - 1}{\Pi_{\substack{d|n \\ d \neq n}} \Phi_d(x)}$

We'll prove the corollary by induction on n, clearly true when $n = 1$. Assume Φ_d has integer coefficients $\forall d < n$.
It is proved in Algebra 3 (MATH2201) that, if $f, g \in \mathbb{Z}[X]$ and g monic then $f = qg + r$ where $deg(r) < deg(g)$ and $g, r \in \mathbb{Z}[x]$.
Using this, we get that the denominator $\Pi_{\substack{d|n \\ d \neq n}} \Phi_d(x)$ is a monic polynomial with coefficients in $\mathbb{Z} \implies \Phi_n \in \mathbb{Z}[X]$. □

2.4.2 Gauss' Theorem

Theorem 2.17. *Let n be a factor of $p-1$, where p is prime. Then there are exactly $\phi(n)$ elements of order n in $\mathbb{F}_p^\times$. These are the roots of Φ in $\mathbb{F}_p^\times$. In particular there are $\phi(p-1)$ primitive roots.*

Proof. Let $f(x) = x^{p-1} - 1$
By Fermat's Little theorem, $f(x) = 0 \ \ (p)$ for $x = 1, \ldots, p-1$ for $(x \neq 0)$

$$\begin{aligned} \therefore f(x) &= (x-1)(x-2)\ldots(x-(p-1)) \\ &= \Pi_{n|p-1}\Phi_n(x) \end{aligned}$$

This implies that:

- Each Φ_n (for $n|p-1$) factorises completely into linear factors with no repeated roots $\therefore \Phi_n$ has $\phi(n)$ roots in $\mathbb{F}_p$
- Every element of $\mathbb{F}_p^\times$ is a root of exactly one of the polynomials Φ_n with $n|p-1$

It remains to show that the roots of $\Phi_n(x)$ in $\mathbb{F}_p$ has order of exactly n.
Suppose $\Phi_n(x) \equiv 0 \pmod{p}$

By the lemma $\Phi_n(x)$ is a factor $x^n - 1$

$$\therefore x^n - 1 \equiv 0 \pmod{p}$$
$$\therefore x^n \equiv 1 \pmod{p}$$

Suppose $x^m \equiv 1 \pmod{p}$ for some $m|n$, $m < n$

$$\implies x^m - 1 \equiv 0 \pmod{p}$$

By the lemma $\Pi_{d|m}\Phi_d(x) \equiv 0 \pmod{p}$

$$\implies \Phi_d(x) \equiv 0 \pmod{p} \text{ for some } d \quad \lightning$$

We already know that x is only a root of 1 of the cyclotomic polynomials, therefore x has order n. $\square$

2.5 Quadratic reciprocity (Quadratic equations modulo prime numbers)

Recall we can solve $x^a \equiv b \pmod{p}$ as long as a is coprime to $p-1$. This won't work if $a = 2$ because a will not be invertible mod $p-1$. An easier question to ask is, which quadratic equations have solutions modulo p?
E.g. Does $x^2 \equiv 37 \pmod{149}$ have solutions?

Notation: We always let p be an odd prime (i.e. $p \neq 2$)

An element $a \in \mathbb{F}_p^\times$ is a quadratic residue if $x^2 \equiv a \pmod{p}$ has solutions.
An element $a \in \mathbb{F}_p^\times$ is a quadratic non-residue if there are no solutions.

The quadratic residue symbol is defined for $a \in \mathbb{F}_p^\times$ by

$$(\frac{a}{p}) = \begin{cases} 1 & \text{a quadratic residue} \\ -1 & \text{a quadratic non-residue} \end{cases}$$

Lemma 2.18. *Let g be a primitive root modulo p (p odd prime). Then g^r is a quadratic residue iff r even.*

Proof.

$(\impliedby)$Assume r even

Clearly g^r is a square in $\mathbb{F}_p^\times$

So g^r is a quadratic residue

$(\implies)$Assume $g^r \equiv x^2 \pmod{p}$

$x \equiv g^s \pmod{p}$ $\quad (s \in \mathbb{Z})$ since g primitive roots

$\therefore g^r \equiv g^{2s} \pmod{p}$

$g^{r-2s} \equiv 1 \pmod{p}$

g has order $p-1$,so $r - 2s$ is a multiple of $p-1$

p odd $\implies p-1$ is even $\implies r$ is even

$\square$

E.g. $p = 7$

x	$x^2 \mod 7$
± 1	1
± 2	4
± 3	2

$\implies$

a	$(\frac{a}{7})$
1	1
2	1
3	-1
4	1
5	-1
6	-1

So 1,2,4 are quadratic residues; 3,4,6 are quadratic non-residues

Corollary 2.19. *There are exactly $\frac{p-1}{2}$ quadratic residues and $\frac{p-1}{2}$ quadratic non-residues mod p*

Defintion 2.20. *Euler's criterion: Let p be an odd prime and $a \in \mathbb{F}_p^\times \implies (\frac{a}{p}) \equiv a^{\frac{p-1}{2}} \pmod{p}$*
Also $(\frac{ab}{p}) = (\frac{a}{p})(\frac{b}{p})$

Proof. $(a^{\frac{p-1}{2}})^2 \equiv 1 \pmod{p}$ by Fermat's Little theorem.

$$\therefore a^{\frac{p-1}{2}} \equiv \pm 1 \pmod{p}$$

Let $a = g^r$ where g is a primitive root $\implies a^{\frac{p-1}{2}} \equiv g^{(p-1)\frac{r}{2}}$

$$\begin{aligned} a \text{ is a quadratic residue} &\iff r \text{ is even} \\ &\iff (p-1)\frac{r}{2} \text{ is a multiple of } p-1 \\ &\iff g^{(p-1)\frac{r}{2}} \equiv 1 \pmod{p} \\ &\iff a^{\frac{p-1}{2}} \equiv 1 \pmod{p} \end{aligned}$$

□

To calculate $(\frac{a}{p})$, we'll use three theorems:

2.5.1 Quadratic Reciprocity Law

Let p, q be distinct odd prime numbers. Then $(\frac{p}{q}) = (-1)^{\frac{(p-1)(q-1)}{4}}$

$$\text{i.e. } (\frac{p}{q}) = \begin{cases} (\frac{q}{p}) & \text{if } p \equiv 1 \pmod 4 \text{ or } q \equiv 1 \pmod 4 \\ -(\frac{q}{p}) & \text{if } p \equiv q \equiv -1 \pmod 4 \end{cases}$$

2.5.2 First Nebensatz

If p is an odd prime, then $(\frac{-1}{p}) = (-1)^{\frac{p-1}{2}}$

$$\text{i.e. } (\frac{-1}{p}) = \begin{cases} 1 & p \equiv 1 \pmod 4 \\ -1 & p \equiv -1 \pmod 4 \end{cases}$$

2.5.3 Second Nebensatz

Let p be an odd prime, then $(\frac{2}{p}) = (-1)^{\frac{p^2-1}{8}}$

$$\text{i.e. } (\frac{2}{p}) = \begin{cases} 1 & p \equiv \pm 1 \pmod 8 \\ -1 & p \equiv \pm 3 \pmod 8 \end{cases}$$

We'll prove the theorems later.

E.g. Does the congruence $x^2 \equiv 37 \ \ (199)$ have solutions?

$$\begin{aligned}
\text{199 is an odd prime } (\frac{37}{199}) &= +(\frac{199}{37}) && \text{by quadratic reciprocity} \\
&\equiv (\frac{14}{37}) && \text{because } 199 \equiv 14 \ \ (37) \\
&\equiv (\frac{2}{37})(\frac{7}{37}) && \text{by the corollary} \\
&\equiv (-1)(\frac{7}{37}) && \text{by the } 2^{nd} \text{ Nebensatz} \\
&\equiv (-1)(+1)(\frac{37}{7}) && \text{by the quadratic reciprocity law} \\
&\equiv -(\frac{2}{7}) && \text{because } 37 \equiv 2 \ \ (7) \\
&\equiv -(+1) && \text{by the } 2^{nd} \text{ Nebensatz} \\
&\equiv -1 && \therefore x^2 \equiv 37 \ \ (199) \text{ has no solutions}
\end{aligned}$$

E.g. $x^2 \equiv 47 \ \ (53)$ have solutions?

$$(\frac{47}{53}) = +(\frac{53}{47}) = (\frac{6}{47}) = (\frac{2}{47})(\frac{3}{47}) = (+1)(-1)(\frac{47}{3}) = -(-\frac{1}{3}) = -(-1) = +1$$

This shows that 47 is a quadratic residue mod 53, so $x^2 \equiv 47 \ \ (53)$ does have solutions. $(x = 10)$

We can speed up the test for primitive roos using quadratic reciprocity,

$$x \text{ is a primitive root mod } p \iff \forall q | p-1, q \text{ prime } x^{\frac{p-1}{q}} \not\equiv 1 \ \ (p)$$

This means we need to calculate $x^{\frac{p-1}{q}} \ \ (p)$ for primes $q|p-1$, the biggest power of x to calculate is $x^{\frac{p-1}{2}}$. But we can calculate this, because it is $(\frac{x}{p})$ by Euler's criterion.

E.g. Is 35 a primitive root modulo 83?

The primes q dividing 82 are $2, 41$, need to check $35^2, 35^{41}$
$35^2 \not\equiv 1 \ \ (83)$ because $35 \not\equiv \pm 1 \ \ (83)$, a quadratic equation cannot have more than 2 roots.
$35^{41} \equiv (\frac{35}{83}) \ \ (83) = (\frac{5}{83})(\frac{7}{83}) = (\frac{83}{5})(-1)(\frac{83}{7}) = (\frac{3}{5})(-1)(\frac{-1}{7}) = (\frac{5}{3})(-1)(-1) = (\frac{2}{3})$
$= -1 \not\equiv 1 \ \ (83)$
So 35 is a primitive root modulo 83.

Proof. First Nebensatz:
By Euler's criterion, $(\frac{-1}{p}) \equiv (-1)^{\frac{p-1}{2}} \;\; (p)$.
Both sides are ± 1, and $+1 \not\equiv -1 \;\; (p)$ because $p \geq 3 \implies$ they are equal. $\square$

E.g. Find the first primitive root modulo 41
$40 = 2^3 * 5$

$$x \in \mathbb{F}_{41}^{\times} \text{ is a primitive root } \iff \begin{cases} x^{\frac{40}{2}} \not\equiv 1 \;\; (41) \\ x^{\frac{40}{5}} \not\equiv 1 \;\; (41) \end{cases}$$

$$\text{We can then simplify the conditions to: } \begin{cases} \frac{x}{41} = -1 \\ x^4 \not\equiv \pm 1 \;\; (41) \end{cases}$$

Try $x = 2 :$ $(\dfrac{2}{41}) = 1$ $\implies$ not a primitive root

Try $x = 3 :$ $(\dfrac{3}{41}) = (\dfrac{41}{3}) = (\dfrac{2}{3}) = -1$ and $3^4 = 81 \equiv -1 \;\; (41)$ $\implies$ not a primitive root

Try $x = 4 :$ $\implies$ not a primitive root

Try $x = 5$ $(\dfrac{5}{41}) = (\dfrac{41}{5}) = (\dfrac{1}{5}) = 1$ $\implies$ not a primitive root

Try $x = 6 :$ $(\dfrac{6}{41}) = (\dfrac{2}{41})(\dfrac{3}{41}) = 1 * -1 = -1$

$2^4 * 3^4 = -2^4 \equiv 16 \;\; (41) \not\equiv \pm 1$ $\implies$ so 6 is a primitive root

E.g. For which primes p does the congruence $x^2 \equiv -3 \;\; (p)$ have solutions?

Notice $x = 1$ is a solution mod 2,
$x = 2$ is a solution mod 3.
For primes $p \neq 2, 3$ it depends on $(\frac{-3}{p})$

We'll calculate $(\dfrac{-3}{p})$

$$\begin{aligned} (\frac{-3}{p}) &= (\frac{-1}{p})(\frac{3}{p}) \\ &= (-1)^{\frac{p-1}{2}}(\frac{3}{p}) \\ &= (-1)^{\frac{(3-1)(p-1)}{4}}(\frac{p}{3}) \\ &= (\frac{p}{3}) \end{aligned}$$

List the squares mod 3, $1^2 = 1 \;\; (3), 2^2 = 1 \;\; (3)$

$$\therefore (\frac{p}{3}) = \begin{cases} 1 & \text{if } p \equiv 1 \;\; (3) \\ -1 & \text{if } p \equiv 2 \;\; (3) \end{cases}$$

We've shown that $x^2 \equiv -3 \;\; (p)$ has solutions iff $p \neq 2$ or $p \equiv 1 \mod 3$.

Corollary 2.21. *There are infinitely many primes $p \equiv 1 \;\; (3)$*

Proof. Assume there are only finitely many, and call them $p_1, p_2, \dots, p_r$
Let $N = n^2 + 3$ where $n = 2p_1 \dots p_r$
Take a prime factor q of N

$N \equiv 0 \pmod{q}$
$n^2 + 3 \equiv 0 \pmod{q}$
$n^2 \equiv -3 \pmod{q}$
We've just shown that this implies $q = 2$ or 3 or $q \equiv 1 \pmod{3}$ but $q \neq 2, 3, q \not\equiv 1 \pmod{3}$ ↯ □

Before we prove the 2^{nd} Nebensatz, we need to know about a new ring.

Let $\zeta = e^{\frac{2\pi i}{8}}$, a primitive 8^{th} root of unity.
We'll use the ring $\mathbb{Z}[\zeta] = \{f(\zeta) \colon f \in \mathbb{Z}\} = \{a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_n\zeta^n : a_i \in \mathbb{Z}\}$
This is clearly a ring (closed under +,*).

2.6 Uniqueness Lemma

Every $A \in \mathbb{Z}[\zeta]$ can be written uniquely as $A = W + x\zeta + y\zeta^2 + z\zeta^3$ with $w, x, y, z \in \mathbb{Z}$.
We'll use congruence modulo p in the ring $\mathbb{Z}[\zeta]$ to prove the 2^{nd} Nebensatz.

Defintion 2.22. *Let $A, B \in \mathbb{Z}[\zeta]$*
We'll say $A \equiv B \pmod{p\mathbb{Z}[\zeta]}$ if $A - B = pC$ for some $c \in \mathbb{Z}[\zeta]$

$$\begin{aligned} \text{Suppose } A &= a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 \\ B &= b_0 + b_1\zeta + b_2\zeta^2 + b_3\zeta^3 \\ C &= c_0 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 \end{aligned}$$

The equation $A - B = pC$ is equivalent (by uniqueness lemma) to:

$$\begin{aligned} a_0 - b_0 &= pC_0, \\ a_1 - b_1 &= pC_1, \\ a_2 - b_2 &= pC_2, \\ a_3 - b_3 &= pC_3, \end{aligned}$$

This implies that the congruence $A \equiv B \pmod{p\mathbb{Z}[\zeta]}$ is equivalent to $a_i \equiv b_i \pmod{p}$ for $i = 0, 1, 2, 3$

Corollary 2.23. $1 \not\equiv -1 \pmod{p\mathbb{Z}[\zeta]}$ *if p is an odd prime.*

This means that to calculate $(\frac{2}{p})$ it is enough to calculate its congruency class mod $(p\mathbb{Z}[\zeta])$
The uniquess lemma is implied by a more general result:

2.6.1 General Uniqueness Lemma

Let $m \in \mathbb{Z}[X]$ be monic and irreducible over $\mathbb{Q}$ of degree d. If $\alpha \in \mathbb{C}$ is a root of m, then every element of $\mathbb{Z}[\alpha]$ can be written uniquely as $a_0 + a_1 + \cdots + a_{d-1}\alpha^{d-1}$ with $a_i \in \mathbb{Z}$.
The uniqueness lemma for $\mathbb{Z}[\zeta]$ follows because ζ is a root of $m(x) = \Phi_8(x) = x^4 + 1$. It is proved in (7202 Groups & Rings) that $x^4 + 1$ is irreducible over $\mathbb{Q}$.

Proof. (General Uniqueness Lemma)

Let $A \in \mathbb{Z}[\alpha]$ and $m(\alpha) = 0$

Existence: $A = f(\alpha)$ for some $f \in \mathbb{Z}[X]$

divide f by m with remainder, $f = q * m + r$ $\qquad deg(r) < deg(m) < d$

$\therefore \quad f(\alpha) = q(\alpha)m(\alpha) + r(\alpha)$

$\therefore \quad A = r(\alpha)$

Uniqueness: Suppose $A = f(\alpha) = g(\alpha)$ $(f \neq g)$ where f & g both have degree $< d$

$\therefore \quad h(\alpha) = 0$ where $h = f - g$ $(\neq 0)$

m is irreducible over $\mathbb{Q}$ and has a bigger degree than h

$\therefore \quad m \nmid h$ in $\mathbb{Q}[x]$, so m and h are coprime in $\mathbb{Q}[x]$

$\exists a, b \in \mathbb{Q}[x]$ such that :

$1 = am + bh = a(\alpha)m(\alpha) + b(\alpha)h(\alpha) = 0 \qquad m(\alpha) = 0 \quad h(\alpha) = 0$

$\implies \quad 1 = 0$

$\implies \quad f = g$

$\square$

Lemma 2.24. *In any ring R with any prime p*

$$(x + y)^p \equiv x^p + y^p \ \ (pR) \text{ for any } x, y \in R$$

Proof. Sufficient to show that each binomial coefficient:

$$c = \frac{p!}{i!(p-i)!} \qquad i = 1, 2, \ldots, p-1 \text{ is a multiple of p}$$

$$i!(p-i)! \not\equiv 0 \ \ (p) \implies \in \mathbb{F}_p^{\times}$$

$\square$

Proof. 2^{nd} Nebensatz

Let p be an odd prime and let $G = \zeta + \zeta^{-1} = \sqrt{2}$. We'll calculate G^p *mod* $(p\mathbb{Z}[\zeta])$ in two ways.

First Calculation:

$$\begin{aligned} G^P &= (\zeta + \zeta^{-1})^p \\ &= \zeta^p + \zeta^{-p} \mod (p\mathbb{Z}[\zeta]) \text{ by the lemma} \end{aligned}$$

Since $\zeta^8 = 1$ this only depends p modulo 8 if $p \equiv \pm 1(8)$ then,

$$G^p = \zeta + \zeta^{-1} \equiv G \ \ (p\mathbb{Z}[\zeta])$$

If $p \equiv \pm 3 \ \ (8)$ then,

$$G^p \equiv \zeta^3 + \zeta^{-3} \equiv -G \ \ (p\mathbb{Z}[\zeta])$$

So in summary,

$$G^p \equiv (-1)^{\frac{p^2-1}{8}} G \ \ (p\mathbb{Z}[\zeta])$$

Second Calculation:

Since $G^2 = 2$,

$$\begin{aligned} G^p &= G * 2^{\frac{p^2-1}{2}} \\ &= G * (\frac{2}{p}) \ \ (p\mathbb{Z}[\zeta]) \text{ by Euler's criterion} \end{aligned}$$

Comparing the results of these two calculations we get:

$$(\frac{2}{p})G = (-1)^{\frac{p^2-1}{8}} G \ \ (p\mathbb{Z}[\zeta])$$

Note $G^2 * \frac{p+1}{2} \equiv 1 \ \ (p\mathbb{Z}[\zeta])$, i.e. G is invertible modulo $p\mathbb{Z}[\zeta]$ with inverse $G * \frac{p+1}{2}$

$$\implies (\frac{2}{p}) \equiv (-1)^{\frac{p^2-1}{8}} \ \ (p\mathbb{Z}[\zeta])$$

Since $1 \equiv -1 \ \ (p\mathbb{Z}[\zeta])$,

$$(\frac{2}{p}) = (-1)^{\frac{p^2-1}{8}}$$

$\square$

The proof of the 2^{nd} Nebensatz worked because $\sqrt{2} \in \mathbb{Z}[\zeta]$
To prove the quadratic reciprocity law, we'll show that $\sqrt{\pm p)}$ is in another cyclotomic ring

Let $\zeta_p = e^{\frac{2\pi i}{p}}$, a primtive p^{th} root of unity. We'll work in the ring modulo $q\mathbb{Z}[\zeta]$.

Defintion 2.25. *The p^{th} Gauss sum (where p is an odd prime):*

$$G(p) = \sum_{a=1}^{p-1} (\frac{a}{p})\zeta_p^a \in \mathbb{Z}[\zeta_p]$$

Lemma 2.26. $G(p)^2 = (-1)^{\frac{p-1}{2}}$

Proof.

$$\begin{aligned}
G(p)^2 &= \left(\sum_{a=1}^{p-1} (\frac{a}{p})\zeta_p^a\right)\left(\sum_{b=1}^{p-1} (\frac{b}{p})\zeta_p^b\right) \\
&= \sum_{a,b\in\mathbb{F}_p^\times} (\frac{a}{p})(\frac{b}{p})\zeta_p^a\zeta_p^b \\
&= \sum_{a,b\in\mathbb{F}_p^\times} (\frac{ab}{p})\zeta_p^{a+b}
\end{aligned}$$

Let $c \equiv a^{-1}b \pmod{p}$, as b runs through $\mathbb{F}_p^\times$, so does c

$$\begin{aligned}
&= \sum_{a,c\in\mathbb{F}_p^\times} (\frac{a^2}{p})\zeta_p^{a+ac} \\
&= \sum_{c\in\mathbb{F}_p^\times} (\frac{c}{p})\left(\sum_{a=1}^{p-1} (\zeta_p^{1+c})^a\right)
\end{aligned}$$

Note the second summation is a geometric progression. Recall that,

$$\sum_{i=1}^{p-1} r^i = \begin{cases} \dfrac{r^p - 1}{r - 1} & r \neq 1 \\ p - 1 & r = 1 \end{cases}$$

Summing the geometric progression:

$$\sum_{a=1}^{p-1} \left(\zeta_p^{1+c}\right)^a = \begin{cases} \dfrac{\left(\zeta_p^{1+c}\right)^p - \zeta_p^{1+c}}{\zeta_p^{1+c} - 1} & \text{if } c \not\equiv 1 \pmod{p} \\ p - 1 & \text{if } c \not\equiv -1 \pmod{p} \end{cases}$$

$$= \begin{cases} -1 & c \not\equiv -1 \pmod{p} \\ p - 1 & c \equiv -1 \pmod{p} \end{cases}$$

$$\therefore G(p)^2 = \sum_{c \in \mathbb{F}_p^\times} (\frac{c}{p})(-1) + p(\frac{-1}{p})$$

$\sum_{c \in \mathbb{F}_p^\times} (\frac{c}{p})(-1) = 0$ since there are $\frac{p-1}{2}$ quadratic residues and quadratic non-residues.

$$\begin{aligned} &= p(\frac{-1}{p}) \\ &= (-1)^{\frac{p-1}{2}} p && \text{by the } 1^{st} \text{ Nebensatz} \end{aligned}$$

□

2.6.2 Uniqueness Lemma for $\mathbb{Z}[\zeta_p]$

Every element $A \in \mathbb{Z}[\zeta_p]$ can be written uniquely as:

$$A = a_0 + a_1\zeta + \cdots + a_{p-2}\zeta^{p-2} \qquad \text{with } a_i \in \mathbb{Z}$$

This is because ζ_p is a root of $m(x) = \Phi_p(x) = 1 + x + \cdots + x^{p-1}$. It's proved in 7202 that Φ_p is irreducible over $\mathbb{Q}$.

Proof. Quadratic Reciprocity law

We'll calculate $G(p)^q \pmod{q\mathbb{Z}[\zeta_p]}$ in two ways.

First Calculation:

$$\begin{aligned} G(p)^q &= \left(\sum_{a=1}^{p-1} (\frac{a}{p})\zeta_p^a\right)^q \\ &= \sum_{a=1}^{p-1} \left((\frac{a}{p})\zeta_p^a\right)^q \pmod{p\mathbb{Z}[\zeta]} \end{aligned}$$

Since q is odd, $(\frac{a}{p})^q = (\frac{a}{p})$

$$G(p)^q \equiv \sum_{a \in \mathbb{F}_p^\times} (\frac{a}{p})\zeta_p^{aq}$$

Let $b \equiv aq \pmod{p}$, and as a runs through $\mathbb{F}_p^\times$ so does b

$$\begin{aligned} G(p)^q &\equiv \sum_{b \in \mathbb{F}_p^\times} (\frac{bq^{-1}}{p})\zeta_p^b \\ &= (\frac{q^{-1}}{p}) \sum_{b \in \mathbb{F}_p^\times} (\frac{b}{p})\zeta_p^b \end{aligned}$$

Note that $G(p) = \sum_{b \in \mathbb{F}_p^\times} (\frac{b}{p})\zeta_p^b$ which implies,

$$\begin{aligned} G(p)^q &\equiv (\frac{q^{-1}}{p})G(p) \quad (q\mathbb{Z}[\zeta_p]) \\ &\equiv (\frac{q}{p})G(p) \quad (q\mathbb{Z}[\zeta_p]) \end{aligned}$$

Second Calculation:

Since $G(p)^2 = (-1)^{\frac{p-1}{2}} p$,

$$\begin{aligned} G(p)^q &= G(p)\left((-1)^{\frac{p-1}{2}} p\right)^{\frac{q-1}{2}} \\ &= G(p)(-1)^{\frac{(p-1)(q-1)}{4}} p^{\frac{q-1}{2}} \\ \therefore G(p)^q &\equiv G(p)(-1)^{\frac{(p-1)(q-1)}{4}} (\frac{p}{q}) \quad (q\mathbb{Z}[\zeta_p]) && \text{by Euler's criterion} \end{aligned}$$

Comparing the two results we get:

$$(\frac{q}{p})G(p) \equiv (-1)^{\frac{(p-1)(q-1)}{4}} (\frac{p}{q})G(p) \quad (q\mathbb{Z}[\zeta_p])$$

We need to check that $G(p)$ is invertible modulo $q\mathbb{Z}[\zeta_p]$,

$G(p)^2 = \pm p$, which is invertible modulo q

$G(p)$ has inverse $G(p) * (\pm p)^{-1} \quad (q\mathbb{Z}[\zeta_p])$

$\therefore (\frac{p}{q}) \equiv (-1)^{\frac{(p-1)(q-1)}{4}} (\frac{p}{q}) \quad (q\mathbb{Z}[\zeta_p])$

Since $1 \equiv -1 \quad (q\mathbb{Z}[\zeta_p])$, it follows that $(\frac{q}{p}) = (-1)^{\frac{(p-1)(q-1)}{4}} (\frac{p}{q})$ $\square$

3 P-adic Number theory

This means methods for congruences modulo p^n, p prime and n large.

If we want to solve $f(x) = 0$, $x \in \mathbb{R}$ we can use the Newton-Raphson method:

- Begin with an "approximate solution" a_0
- Define a sequence recursively $a_{n+1} = a_n - \dfrac{f(a_n)}{f'(a_n)}$

Very often a_n converge to a limit a and $f(a) = 0$.
We can use the same method in number theory for solving congruences. Suppose $f(x)$ is a polynomial with coefficients in $\mathbb{Z}$ and we want to solve $f(x) \equiv 0 \ \ (p^N)$ (p prime, n large)
We can try this:

- Find a solution a_0 to $f(a_0) \equiv 0 \ \ (p^r)$ where r is small
- Define a recursive sequence $a_{n+1} = a_n - \dfrac{f(a_n)}{f'(a_n)}$

If n is large enough, then often $f(a_n) \equiv 0 \ \ (p^N)$

E.g. Let $f(x) = x^2 + 2$, $p = 3$
Suppose we want to solve $x^2 + 2 \equiv 0 \ \ (3^N)$

Let $a_0 = 1 : f(a_0) = 1^2 + 2 = 3 \equiv 0 \ \ (3)$

Define the sequence a_n by $a_{n+1} = a_n = \dfrac{f(a_n)}{f'(a_n)} = a_n - \dfrac{a_n^2 + 2}{2a_n} = \dfrac{a_n}{2} - \dfrac{1}{a_n}$

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= \frac{1}{2} - 1 = \frac{-1}{2} \\
a_2 &= \frac{-1}{4} + 2 = \frac{7}{4}
\end{aligned}$$

It turns out that $\frac{-1}{2}$ is a solution mod 9 $\implies -1 * 2^{-1} \mod 9$
$\frac{7}{4}$ is a solution mod 81 $\implies 7 * 4^{-1} \mod 9$

$2^{-1} \equiv 5 \ \ (9) \implies a_1 \equiv 4 \ \ (9)$

$4^{-1} \equiv -20 \ \ (81) \implies a_2 = \frac{7}{4} \equiv -140 \equiv 22 \ \ (81)$

a_3 would be a solution mod 3^8.

In this example, we're reducing rational numbers mod p^n not just integers. If $\frac{a}{b}$ is arational numbe,r then we can reduce this modulo p^n as long as b is invertible mod p^n, i.e. when b is not a multiple of p. We'll write:

$$\mathbb{Z}_{(p)} = \{\frac{a}{b} : a, b \in \mathbb{Z}, p \not| b\}$$

$\mathbb{Z}_{(p)}$ is closed under $+, *$, so $\mathbb{Z}_{(p)}$ is a ring contained in $\mathbb{Q}$ containing $\mathbb{Z}$. This is called the "local ring of p" and is the set of rational number which can be reduce modulo p^n $(\forall n)$

Defintion 3.1. *If p is a prime number and $n \in \mathbb{Z}$, then the valuation of n, at p is:*

$$V_p(n) = \begin{cases} max\{a : p^a | n\} & n \neq 0 \\ \infty & n = 0 \end{cases}$$

A simple statement that can be made is, $V_p(nm) = V_p(n) + V_p(m)$. We can also extend V_p to a function on $\mathbb{Q}$, $V_p(\frac{n}{m}) = V_p(n) - V_p(m)$.
With this notation:

$$Z_{(p)} = \{x \in \mathbb{Q} : V_p(x) \geq 0\}$$

$$x \equiv y \quad (p^a) \iff V_p(x - y) \geq a$$

E.g

$$V_2(\frac{7}{12}) = -2 \qquad V_2(\frac{7}{12}) = -1 \qquad V_5(\frac{7}{12}) = 0 \qquad V_7(\frac{-7}{12}) = +1$$

3.1 Hensel's Lemma

Let p be a prime number. Let $f \in \mathbb{Z}_{(p)}[x]$ and $a_0 \in \mathbb{Z}_{(p)}$ such that $f(a_0) \equiv 0 \quad (p^{2c+1})$ where $c = V_P(f'(a_0))$.
Then if we define $a_{n+1} = a_n - \dfrac{f(a_n}{f'(a_n)}$ then $a_n \in \mathbb{Z}_{(p)}$ and $f(a_n) \equiv 0 \quad (p^{2c+2^n})$

Proof. We'll prove the following by induction on n

1. $a_n \in \mathbb{Z}_{(p)}$ and $a_n \equiv a_0 \quad (p^{c+1})$
2. $V_p(f'(a_n)) = c$
3. $f(a_n) \equiv 0 \quad (p^{2c+2^n})$

If $n = 0$ then the statements 1,2,3 are all true for a by assumption. Now assume 1,2,3 for a_n, we'll prove them for a_{n+1}

Let $a_{n+1} = a_n - \delta$ where $\delta = \dfrac{f(a_n)}{f'(a_n)}$

1:

$$\begin{aligned} V_p(\delta) = V_p(f(a_n)) - V_p(f'(a_n)) \\ &= c && \text{by } \mathbf{2:} \\ \geq 2c + 2^n && && \text{by } \mathbf{3:} \\ V_p(\delta) \geq 2c + 2^n - c \\ V_p(\delta) \geq c + 2^n && && \mathbf{(*)} \end{aligned}$$

By **(*)**

$$V_p(\delta) \geq 0 \implies \delta \in \mathbb{Z}_{(p)}$$

$$\therefore a_{n+1} = a_n - \delta \in \mathbb{Z}_{(p)}$$

By **(*)**

$$V_p \geq c+1 \implies \delta \equiv 0 \pmod{p^{c+1}}$$

$$\begin{aligned} a_{n+1} &\equiv a_n \pmod{p^{c+1}} \\ &\equiv a_0 \pmod{p^{c+1}} && \text{by } \mathbf{1} \end{aligned}$$

2: We've shown that $a_{n+1} \equiv a_0 \pmod{p^{c+1}}$

$$\begin{aligned} \therefore \quad f'(a_{n+1}) &\equiv f'(a_0) \pmod{p^{c+1}} \\ &\not\equiv 0 && \text{because } V_p(f'(a_0)) = c \\ & && \text{also } f'(a_{n+1}) \equiv f'(a_0) \pmod{p^c} \\ & && \qquad\qquad\quad \equiv 0 \pmod{p^c} \\ & && \text{because } V_p(f'(a_0)) = c \pmod{p^c} \end{aligned}$$

$$\therefore V_p(f'(a_{n+1})) = c$$

3: Must show that $f(a_{n+1}) \equiv 0 \pmod{p^{2c+2^{n+1}}}$

$$a_{n+1} = a_n - \delta$$

$$\begin{aligned} a_{n+1}^r &= (a_n - \delta)^r \\ &= a_n^r - r a_n^r \delta + \text{multiples of } \delta^2 \end{aligned}$$

By **(*)**:

$$V_p(\delta) \geq c + 2^n$$

$$\therefore V_p(\delta^2) \geq 2c + 2^{n+1}$$

$$\therefore \delta^2 \equiv 0 \pmod{p^{2c+2^{n+1}}}$$

This implies $a_{n+1}^r \equiv a_n^r - r a_n^{r-1}\delta \pmod{p^{2c+2^{n+1}}}$
Suppose $f(x) = \sum c_r * x$. Substiuting a_{n+1}, we get:

$$\begin{aligned} f(a_{n+1}) &= \sum c_r(a_n^r - r a_n^{r-1}\delta) \pmod{p^{2c+2^{n+1}}} \\ &= \sum c_r a_n^r - \left(\sum r c_r a_n^{r-1}\right)\delta \pmod{p^{2c+2^{n+1}}} \\ &= f(a_n) - f'(a_n) * \frac{f(a_n)}{f'(a_n)} \equiv 0 \pmod{p^{2c+2^{n+1}}} \end{aligned}$$

$\square$

E.g. $f(x) = x^3 + x + 1$, $p = 3$
Find a root of f mod 81

Note that $f'(x) = 3x^2 + 1$ and $f(1) = 3 \equiv 0$

Try $a_0 = 1$

$$\begin{aligned} c &= V_3(f'(a_0)) \\ &= V_3(4) \\ &= 0 \end{aligned}$$

$3^{2c+1} = 3$ and a_0 is a root of f modulo 3
$\therefore a_0 = 1$ satisfies the conditions of Hensel's lemma.

$$\begin{aligned} a_1 &= 1 - \frac{a_0}{f'(a_0)} \\ &= 1 - \frac{3}{4} \end{aligned}$$

It is sufficient to work out a_1 mod 9

$$4^{-1} \equiv 1 \pmod{3} \qquad \frac{3}{4} \equiv 3 * 1 \pmod{9} \qquad a_1 \equiv -2 \pmod{9}$$

Check

$$\begin{aligned} f(a_1) &\equiv (-2)^3 + (-2) + 1 \\ f(2) &= -9 \equiv 0 \pmod{9} \end{aligned}$$

$$\begin{aligned} a_2 &= -2 - \frac{f(-2)}{f'(-2)} \\ &= -2 - \frac{-9}{13} \end{aligned}$$

This should be a root of f modulo 81.

$$13^{-1} \equiv -2 \pmod{9}$$

$$\implies \frac{9}{13} \equiv -18$$

$$a_2 \equiv -2 - 18 \equiv -20$$

$$\begin{aligned} \text{Check } f(a_2) = (-20)^3 - 20 + 1 &\equiv -8000 - 19 \\ &= -8019 \\ &= -81 * 99 \\ &= 0 \pmod{81} \end{aligned}$$

3.2 Quadratic Congruences

We'll see how to find out whether $x^2 \equiv b \pmod{n}$ has solutions.

Suppose $n = p_1^{a_1} \dots p_r^{a_r}$ (p_i distinct primes). There are solutions modulo $n \iff \forall i$, there are solutions modulo $p_i^{a_i}$ by the Chinese Remainder Theorem.

Proposition 3.2. *Suppose p is an odd prime not dividing b. If $x^2 \equiv b \pmod{p}$ has solutions then $x^2 \equiv b \pmod{p^r}$ has solutions for all r*

Proof. Suppose there is a solution a_0 modulo p, i.e. $a_0^2 \equiv b \pmod{p}$

Let $f(x) = x^2 - b$. We'll check that a_0 satisfies the conditions of Hensel's lemma.

$$\begin{aligned} c &= V_p(f'(a_0)) \\ &= V_p(2a_0) && \text{and since } p \neq 2 \\ \implies c &= V_p(a_0) \end{aligned}$$

Also since $p \not| b$, we know $p \not| a_0$:

$$\begin{aligned} &\therefore\ c = 0 \\ &\therefore\ f(a_0) \equiv \ \ (p^{2c+1}) \implies a_0 \text{ satisfies the conditions of Hensel's lemma} \\ &\therefore\ \text{We have roots of } f \text{ modulo all powers of } p \end{aligned}$$

□

Remark
Suppose we want a root of f modulo p^{13}
Choose n so that $2c + 2^n \geq 13$
$f(a_n) \equiv 0 \pmod{p^{2c+2^n}} \implies f(a_n) \equiv 0 \pmod{p^{13}}$

The proposition would be false if we allowed $p = 2$
E.g. Let $b = 3$

$x^2 \equiv 3 \pmod{2}$ has a solution
$x^2 \equiv 3 \pmod{4}$ has no solutions

x	x^2 mod 4
0	0
1	1
2	0
3	1

if $b = 5$

$x^2 \equiv 5 \pmod{2}$ has a solution
$x^2 \equiv 5 \pmod{4}$ has solutions
$x^2 \equiv 5 \pmod{8}$ has no solutions

x	x^2 mod 8
0	0
± 1	1
± 2	4
± 3	1
± 4	0

Proposition 3.3. *Suppose b is odd. If $x^2 =\equiv b \pmod 8$ has solutions then $x^2 \equiv b \pmod{2^r}$ has solutions for all r*

Proof. Suppose $a_0 \equiv b \pmod 8$, this implies a_0 is odd.
Let $f(x) = x^2 - b$

$\therefore c = V_2(f'(a_0)) = V_2(2a_0) = 1$ because a_0 is odd

$\therefore 2^{2c+1} = 8$

$\therefore a_0$ is a root of f modulo p^{2c+1}

By Hensel's lemma, there are solutions modulo all powers of 2.

□

E.g. For which n does the congruence $x^2 \equiv 5 \pmod 5$ have solutions?

First consider the case $n \equiv p^r \quad (p \text{ prime})$
If $p \neq 2, 5$ then by the first propositiong, there are solutions $p^n \iff (\frac{5}{p}) = 1$

$(\frac{5}{p}) = +(\frac{p}{5})$ depends on p mod 5

x	s
1	1
2	-1
3	-1
4	1

(different x)
The congruence $x^2 \equiv 5 \pmod p$ has solutions
$\iff p \equiv 1, 4 \pmod 5$ (in the cases $p \neq 2, 5$)

For $p = 2$, $\quad x^2 \equiv 5 \pmod 2$ has a solution, $x = 1$
$\qquad\qquad x^2 \equiv 5 \pmod 4$ has a solution, $x = 1$
But the only odd square mod 8 is 1. So $x^2 \equiv 5 \pmod 8$ has no solutions.
$\therefore$ no solutions mod 2^n if $n \geq 3$

For $p = 5 \quad x^2 = 5 \pmod 5$ has solutions, here's how we check. Assume:

$$x^2 \equiv 5 \pmod{25}$$
$$\therefore x^2 \equiv 0 \pmod 5$$
$$\text{So } 5|x^2$$
$$\text{So } 5|x$$
$$\therefore x^2 \equiv 0 \pmod{25} \ ↯$$

So there are solution modulo n if $n = 2^a * 5^b * \Pi p_i^{c_i}$ where $a \leq 2, b \leq 1, p_i \equiv 1 \pmod 5, c_i \in \mathbb{N}$

E.g. For which n does $x^2 \equiv -7 \pmod n$ have solutions?

Assume p is a prime $\neq 2, 7$

$$\left(\frac{-7}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{7}{p}\right)$$
$$= (-1)^{\frac{p-1}{2}}(-1)^{\frac{(7-1)(p-1)}{4}}\left(\frac{p}{7}\right)$$
$$= (-1)^{\frac{p-1}{2}+\frac{3(p-1)}{2}}\left(\frac{p}{7}\right)$$
$$= (+1)\left(\frac{p}{7}\right) \text{ depends on } p \bmod 7$$

x	$\left(\frac{x}{7}\right)$
1	1
2	1
3	-1
4	1
5	-1
6	-1

$3^2 = 9 \equiv 2 \mod 7$

$x^2 \equiv -7 \ (p^r)$ has solutions

$\implies p \equiv 1, 2, 4 \mod 7$

For $p = 2$: $-7 \equiv 1 \ (8)$ so -7 is a square modulo 8 by the proposition.
$x^2 \equiv -7 \ (2^r)$ has solutions for all r.

For $p = 7$: $x^2 \equiv -7 \ (7)$ has a solution $x = 0$ but $x^2 \equiv -7 \ (7^2)$ has no solutions. Suppose

$$x^2 \equiv -7 \ (7^2)$$
$$\therefore x^2 \equiv 0 \ (7)$$
$$\therefore 7|x^2$$
$$\implies 7|x$$
$$\implies x^2 \equiv 0 \ (49) \ \lightning$$

So $x^2 \equiv -7 \ (n)$ has solutions $\iff n = 7^a * \Pi p_i^{b_i}$ where $a \leq 1$, $p_i \equiv 1, 2, 4 \ (7)$, $b_i \in \mathbb{N}$

3.3 P-adic congruence

Suppose we have a series $\sum_{n=1}^{\infty} x_n$ for $x_n \in \mathbb{Z}_{(p)}$. We'll say that the series converges **p-adically** if for every a, there are only finitely many terms x_n with $x_n \not\equiv \ (p^a)$. We can add up the series in $\mathbb{Z}/p^a$ because only finitely many terms are non zero.

Lemma 3.4. *$\sum x_n$ converges p-adically $\iff V_p(x_n) \to \infty$*

Proof. If $V_p(x_n) \to \infty$ then for n significantly large, $V_p(x_n) \geq a$, i.e., $x_n \equiv 0 \ (p^a)$

$\square$

E.g. p=3

$$(1+3x)^{\frac{1}{2}} = 1 + \frac{1}{2}(3x) + \frac{(\frac{1}{2})(\frac{-1}{2})(3x)^2}{2!} + \frac{(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})(3x)^3}{3!}$$

if $x \in \mathbb{Z}_{(3)}$ then this series converge 3-adically.

$$
\begin{aligned}
(1+3x)^{\frac{1}{2}} &\equiv 1 \quad (3)\\
&\equiv 1+\frac{3x}{2} \quad (9)\\
&\equiv 1+\frac{3x}{2}+\frac{9}{8}x^2 \quad (27)\\
&\equiv 1+\frac{3x}{2}+\frac{9}{8}x^2+\frac{27}{16}x^3 \quad (27)
\end{aligned}
$$

We can write these polynomials with integer coefficients.

$$
\begin{aligned}
(1+3x)^{\frac{1}{2}} &\equiv 1+15x+9x^2 \quad (27)\\
&\equiv 1+42x+9x^2+27x^3 \quad (81)
\end{aligned}
$$

Important point; these polynomials play the same role in number theory $(1+3x)^{\frac{1}{2}}$ does in analysis $\sqrt{(1+3x)}$

E.g.

$$
\begin{aligned}
(1+15x+9x^2)^2 &= 1+(30)x+(18+15^2)x^2+(2*9*15)x^3+(81)x^4\\
&\equiv 1+3x \quad (27)
\end{aligned}
$$

E.g. Find a square root of 7 in $\mathbb{Z}/81$

$$
\begin{aligned}
7^{\frac{1}{2}} &= (1+3*2)^{\frac{1}{2}}\\
&\equiv 1+42*2+9*2^2+27*2^3 \quad (81)\\
&\equiv 1+84+36-27 \quad (81)\\
&\equiv 13 \quad (81)
\end{aligned}
$$

Check $13^2 = 169 \equiv 7 \quad (81)$

This works because of a result called the power series trick.

Notation We'll write $\mathbb{Z}_{(p)}\Big[[x]\Big]$ for the set of power series in x with coefficient in $\mathbb{Z}_{(p)}$. $\mathbb{Z}_{(p)}\Big[[x]\Big]$ is a ring with addition and multiplication of power series as operations. We can often compose two power series $f, g \in \mathbb{Z}_{(p)}\Big[[x]\Big]$ to get a new power series $f \circ g$. $(f \circ g)(x) = f(g(x))$.
We can define $f \circ g$ as long as either f is a polynomial or g has zero constant term. Suppose

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n x^n\\
g(x) &= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

We'll see that $f \circ g$ is a power series

$$
\begin{aligned}
f(g(x)) &= \sum_{n=0}^{\infty} a_n (\sum_{m=1}^{\infty} b_m x^m)^n \\
&= \sum_{n=0}^{\infty} a_n \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} b_{m_1} \dots b_{m_n} x^{m_1+\cdots+m_n}
\end{aligned}
$$

so $f(g(x)) = \sum c_d x^d$ where

$$c_d = \underbrace{\sum_{m_1,\dots,m_n=1}^{\infty} a_n b_{m_1} \dots b_{m_n}}_{\text{finite sum in } \mathbb{Z}_{(p)}}$$

Note $f \circ g$ is not defined otherwise.
E.g

$$
\begin{aligned}
f(x) &= 1 + x + x^2 + \dots \\
g(x) &= 1 + x
\end{aligned}
$$

$$\implies f(g(x)) = 1 + (1+x) + (1+x)^2$$

This has constant term $1 + 1 + 1 + 1 + \dots$, so $f \circ g$ is not defined.

3.4 Power Series Trick

Suppose f, g, h are power series with coefficients in $\mathbb{Z}_{(p)}$. Assume either f is a polynomial or g has no constant term. Also assume:

- For small real numbers x, $f(x), g(x), h(x)$ converge and $f(g(x)) = h(x)$
- For all $x \in \mathbb{Z}_{(p)}$, $f(x), g(x)$ and $h(x)$ converge p-adically

Then for all $x \in \mathbb{Z}_{(p)}$, $f(g(x)) \equiv h(x) \quad (p^n)$

In the example $f(x) = x^2, g(x) = (1+3x)^{\frac{1}{2}}, h(x) = 1 + 3x$.
For small real x, $f(g(x)) = h(x)$, so as long as we know that $g(x)$ converges 3-adically $(\forall x \in \mathbb{Z}_{(3)})$ the power series trick implies $g(x)^2 \equiv 1 + 3x \ \ (3^n)$

How do we check for p-adic convergence?

Lemma 3.5. *$\sum x_n$ converge p-adically if and only if $V_p(x_n) \to \infty$*

We need a way of calculating valuations of $n^t h$ term of a square.

Proposition 3.6. $V_p(n!) = \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \cdots \leq \frac{n}{p-1}$

We'll prove this later, first use the properties to show that $(1+3x)^{\frac{1}{2}}$ converge 3-adically for all $x \in \mathbb{Z}_{(3)}$

$$\begin{aligned}(1+3x)^{\frac{1}{2}} &= 1 + \frac{1}{2}(3x) + \frac{(\frac{1}{2})(\frac{-1}{2})(3x)}{2!} + \dots \\ n^{th} \text{ term } &= \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-n+1)}{n!}(3x)^n\end{aligned}$$

$$\begin{aligned}V_3(n^{th} \text{ term}) &= V_3\Bigg((\frac{1}{2})(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)\Bigg) - V_3(n!) + V_3((3x)^n) \\ &\geq 0 - \frac{n}{3-1} + n \\ &\geq \frac{n}{2} \to \infty \text{ as } n \to \infty \\ &\implies \text{series converges 3-adically}\end{aligned}$$

E.g. Assume p is an odd prime

Let $exp(px) = 1 + px + \frac{(px)^2}{2!} + \frac{(px)^3}{3!} + \dots$.
We'll see that this converges for all $x \in \mathbb{Z}_{(p)}$

$$n^{th} \text{ term } = \frac{(px)^n}{n!}$$

$$\begin{aligned}V_p(n^{th} \text{ term } &= V_p((px)^n) - V_p(n!) \\ &= \underset{1}{(n * V_p(p))} + \underset{\geq 0}{(n * V_p(x)))} - \underset{\leq \frac{n}{p-1}}{V_p(n!)} \\ &\geq n - \frac{n}{p-1} \\ &\geq (\frac{p-2}{p-1}) \to \infty \text{ for } p \neq 2\end{aligned}$$

E.g. $\log(1+px)$ converges p-adically for all $x \in \mathbb{Z}_{(p)}$

$$\begin{aligned}V_p\Bigg(\pm\frac{(px)^n}{n}\Bigg) &= \underset{nV_p(px)}{V_p\Big((px)^n\Big)} - \underset{< V_p(n!)}{V_p(n)} \\ &\geq n - \frac{n}{p-1} \\ &\geq \Bigg(\frac{p-2}{p-1}\Bigg)n \to \infty \text{ if } p \neq 2\end{aligned}$$

Remark A quick way to remember the series for $\log(1+px)$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots \qquad \text{geometric series}$$
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$
$$\log(1+px) = x - \frac{(px)^2}{2} + \frac{(px^3}{3} - \frac{(px)^4}{4} + \dots$$

Proof. Calculating $V_p(n!)$

$n! = 1 * 2 * \cdots * n$
$V_p(n) = \sum_{i=1}^{n} V_p(i) \quad (*)$

The number of i between 1 & n which are multiples of p is $\lfloor \frac{n}{p} \rfloor$.
There are $\frac{n}{p^2}$ values of i which are multiples of p^2, etc.

$\lfloor \frac{n}{p} \rfloor$ - $\lfloor \frac{n}{p^2} \rfloor$ values of i are mutliples of p, but not of p^2, i.e. $V_p(i) = 1$

So $\lfloor \frac{n}{p} \rfloor$ - $\lfloor \frac{n}{p^2} \rfloor$ terms in the sum $(*)$ are equal to 1.

Similarly $\lfloor \frac{n}{p^2} \rfloor$- $\lfloor \frac{n}{p^3} \rfloor$ terms in the sum $(*)$ are equal to 2.
In general there are exactly $\lfloor \frac{n}{p^a} \rfloor$- $\lfloor \frac{n}{p^{a+1}} \rfloor$ terms in $(*)$ which are equal to a

$$\therefore V_p(n!) = 1 * \text{no of terms equal to } 1 + 2 * \text{number of terms equal to } 2 \ + \dots$$

$$\begin{aligned} Vp(n!) =& 1 * (\lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor) \\ & + 2 * (\lfloor \frac{n}{p^2} \rfloor - \lfloor \frac{n}{p^3} \rfloor) \\ & + 3 * (\lfloor \frac{n}{p^3} \rfloor - \lfloor \frac{n}{p^4} \rfloor) \\ & + \dots \\ =& \lfloor \frac{n}{p} \rfloor + (2-1) \lfloor \frac{n}{p^2} \rfloor + (3-2) \lfloor \frac{n}{p^3} \rfloor + \dots \\ =& \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \dots \end{aligned}$$

Using this we can prove the upper bound.

$$\begin{aligned} V_p(n!) &\le \frac{n}{p} + \frac{n}{p^2} + \dots \\ &\le \frac{n}{p} \underbrace{(1 + \frac{1}{p} + \frac{1}{p^2} + \dots)}_{\text{geometric series } \frac{1}{1-\frac{1}{p}}} \\ &\le \frac{n}{p-1} \end{aligned}$$

□

3.4.1 P-adic log & exp

Let p be an odd prime. Use the notation

$$p\mathbb{Z}/p^n = \{px \colon x \in p\mathbb{Z}/p^n\}$$

E.g.

$$3\mathbb{Z}/27 = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$$

$p\mathbb{Z}/p^n$ is closed under $+$, so it is a subgroup of $(\mathbb{Z}/p^n)^\times$

$$1 + p\mathbb{Z}/p^n = \{1 + px \colon x \in p\mathbb{Z}/p^n\}$$

E.g.

$$1 + 3\mathbb{Z}/27 = \{1, 4, 7, 10, 13, 16, 19, 22, 25\}$$

$1 + p\mathbb{Z}/p^n$ is closed under $*$, so $1 + p\mathbb{Z}/p^n$ is a subgroup $(\mathbb{Z}/n)^\times$. Both subgroups have p^{n-1} elements, but one is additive and the other is multiplicative. But actually there are isomorphic. This isomorphism is $\exp \,\&\, \log$.

Theorem 3.7. *Let p be an odd prime. Then there is an isomorphism:*

$$p\mathbb{Z}/p^n \leftrightarrows^{log}_{exp} 1 + p\mathbb{Z}/p^n$$

$$\begin{aligned} px &\longmapsto \exp(px) \\ 1 + px &\longmapsto \log(1 + px) \end{aligned}$$

E.g. $\mathbb{Z}/27$ $(p = 3)$ We'll find the isomorphisms in this case.

$$\begin{aligned} \exp(3x) &\equiv 1 + 3x + \frac{3^2x^2}{2!} + \frac{3^3x^3}{3!} \quad (27) \\ &\equiv 1 + 3x + 18x^2 + 18x^3 \quad (27) \end{aligned}$$

$$\begin{aligned} \log(1 + 3x) &\equiv 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} \quad (27) \\ &\equiv 3x + 9x^2 + 9x^3 \quad (27) \end{aligned}$$

Check:

$$\begin{aligned}
\log(\exp(3x)) &\equiv \log(1+3(x+6x^2+6x^3)) \\
&\equiv 3(x+6x^2+6x^3)+9(x+6x^2+6x^3)^2+9(x+6x^2+6x^3) \\
&\equiv 3x+18x^2+18x^3+9x^2+9x^3 \\
&\equiv 3x
\end{aligned}$$

Similarly $\exp(\log(1+3x)) \equiv 1+3x \pmod{27}$
We can use the theorem to solve congruences.

E.g. Solve $7^x \equiv 13 \pmod{27}$

7 and 13 are in $1+3\mathbb{Z}/27$, so we can take their logarithms.

$$x\log(7) \equiv \log(13)$$

Using the formula for $\log(1+3x)$, we get:

$$\begin{aligned}
\log(7) &= \log(1+6) \\
&\equiv 6-\frac{6^2}{2}+\frac{6^3}{3}-\frac{6^4}{4} \\
&\equiv 6-18+72 \\
&\equiv 6 \pmod{27}
\end{aligned}$$

$$\begin{aligned}
\log(13) &\equiv \log(1+12) \\
&\equiv 12-\frac{12^2}{2}+\frac{12^3}{3} \pmod{27} \\
&\equiv 12-72+3^2*4^3 \pmod{27} \\
&\equiv 12-72+9 \pmod{27} \\
&\equiv 3 \pmod{27}
\end{aligned}$$

So $7^x \equiv 13 \pmod{27}$ reduces to:

$$\begin{aligned}
&7^x \equiv 13 \pmod{27} \\
\implies &6x \equiv 3 \pmod{27} \\
\implies &2x \equiv 1 \pmod{9} \\
\implies &x \equiv 5 \pmod{9}
\end{aligned}$$

Proof.

We'll use the power series trick. We've shown that $\log(1+px), \exp(px)$ converge p-adically for $x \in \mathbb{Z}_{(p)}$ and they converge for small real numbers and for small real x

$$\log\Big(\exp(px)\Big) = px$$
$$\exp\Big(\log(1+px)\Big) = 1 + px$$

By the power series trick:

$$\exp\Big(\log(1+px)\Big) \equiv 1 + px \quad (p^n)$$
$$\log\Big(\exp(px)\Big) \equiv px \quad (p^n)$$

$\therefore \log$ and $\exp$ are inverse functions, so they are bijective.

Remains to show that $\exp(px+py) \equiv \exp(px) * \exp(py) \quad (p^n)$

For any $a \in \mathbb{N}$: $\exp(pax) = \exp(px))^a$ for small real x

By the power series trick with:

$$f(x) = x^a$$
$$g(x) = \exp(px)$$
$$h(x) = \exp(pax)$$

$$\exp(pax) \equiv (\exp(px))^a \quad (p^a)$$

Take $x = 1$

$$\exp(pa) \equiv exp(p)^a \quad (p^a)$$
$$\therefore \exp(pa+pb) \equiv \exp(p)^{a+b} \quad (p^a)$$
$$\equiv \exp(p)^a * \exp(p)^b$$
$$\equiv \exp(pa) * \exp(pb) \quad (p^n)$$

We've proved this when a & b are positive integers, but every element of $\mathbb{Z}/p^n$ can be written as a positive integer. $\square$

3.5 Teichmüller Lifts

Let p be an odd prime. We saw that $(\mathbb{Z}/p^n)^\times$ has a big subgroup $1 + p\mathbb{Z}/p^n$ and we can easily do calculations in the subgroup. Teichmüller lifts is another subgroup.

$(\mathbb{Z}/p^n)^\times$ = Teichmüller lifts $*\,(1 + p\mathbb{Z}/p^n)$

Let $x \in \mathbb{Z}_{(p)}$ and assume $x \not\equiv 0 \pmod{p}$:

$$x, x^p, x^{p^2}, x^{p^3}, \ldots$$

All these terms are constant mod p:

$$\begin{aligned} x^{p-1} &\equiv 1 \pmod{p} \\ x^p &\equiv x \pmod{p} \end{aligned}$$

The sequence is constant mod p^2, but all terms after the 2^{nd} are constant mod p^2.

E.g. $p = 3, x = 2$

We'll look at the sequence mod 9:

$$\begin{aligned} 2^3 &\equiv 8 \pmod{9} \\ 2^9 \equiv 8^3 &\equiv 8 \pmod{9} \\ 2^{27} &\equiv 8 \pmod{9} \qquad \textit{etc} \end{aligned}$$

The sequence is eventually constant modulo p^n

Defintion 3.8. *The Teichüller lift of x modulo p^n is:*

$$T(x) \equiv x^{p^{n-1}} \mod p^n$$

To calculate Teichmüller lifts, we use:

Lemma 3.9. *Suppose $x \equiv y \pmod{p^n}$ then $x^p \equiv y^p \pmod{p^{n+1}}$*

Proof. Let $x \equiv y + p^n \implies$

$$x^p \equiv (y + p^n)^p$$

$$x^p \equiv y^p + py^{p-1}p^n + \text{ multiples of } p^{2n}$$

$$x^p \equiv y^p \pmod{p^{n+1}}$$

□

E.g. Calculate $T(12) \mod 125$

Definition is $12^{25} \mod 125$. Using the lemma:

$$12 \equiv 2 \quad (5)$$

$$\begin{aligned} 12^5 &\equiv 2^5 \quad (5^2) \\ &\equiv 32 \equiv 7 \quad (25) \\ 12^5 &\equiv 7 \quad (25) \end{aligned}$$

$$12^{25} \equiv 7^5 \quad (5^3)$$

$$\begin{aligned} T(12) &\equiv (2+5)^5 \quad (125) \\ &\equiv 2^5 + \quad 5(2^4)*5 + \quad 10*2^3*5^2 + \text{ multiples of } 125 \\ &\equiv 2^5 + \quad 5^2 * 2^4 \quad (125) \\ &\equiv 2^5 + \quad 25*16 \quad (125) \qquad\qquad \text{note } 16 \equiv 1 \quad (5) \\ &\equiv 2^5 + \quad 25*1 \quad (125) \\ &\equiv 32 + 25 \quad (125) \\ &\equiv 57 \quad (125) \end{aligned}$$

x	$T(x) \mod 125$
1	1
2	57
3	$T(-1)*T(2) = -1*57 \equiv 68 \quad (125)$
4	$(-1)^{25} \equiv -1$

Theorem 3.10.

1. *If* $r > n-1$ *then* $x^{p^r} \equiv T(x) \quad (p^n)$
2. $T(x)^{p-1} \equiv 1 \quad (p^n)$
3. $T(x)$ *depends only on* $x \mod p$ *and* $T(x) \equiv x \quad (p)$
4. $T\colon \mathbb{F}_p^\times \mapsto (\mathbb{Z}/p^n)^\times$ *is an injective homomorphism*

Proof.

By Euler's theorem, $\phi(p^n) = (p-1)p^{n-1}$

$$\begin{aligned}
&\implies \underbrace{x^{(p-1)p^{n-1}}}_{T(x)^{p-1}} \equiv 1 \pmod{p^n} \\
&\implies T(x)^{p-1} \equiv 1 \pmod{p^n}
\end{aligned}$$

This proves ***2***.

$\therefore T(x)^p \equiv T(x) \ \ (p^n)$
Doing this several times we get $T(x) \equiv T(x)^p \equiv T(x)^{p^2} \equiv \dots \ \ (p^n)$
This proves ***1***.

Suppose:

$$\begin{aligned}
x &\equiv y \ \ (p) \\
x^p &\equiv y^p \ \ (p^2) && \text{by the lemma} \\
x^{p^2} &\equiv y^{p^2} \ \ (p^3) && \text{by the lemma} \\
&\ \vdots \\
T(x) &\equiv T(y) \ \ (p^n) && \text{by Fermat's Little Theorem} \\
x &\equiv x^p \equiv x^{p^2} \equiv \cdots \equiv T(x) \ \ (p)
\end{aligned}$$

This proves ***3***.

$$\begin{aligned}
T(xy) \equiv (xy)^{p^{n-1}} &\equiv x^{p^{n-1}} y^{p^{n-1}} \\
&\equiv T(x)T(y) \ \ (p)
\end{aligned}$$

So T is a homomorphism, suppose:

$$\begin{aligned}
T(x) &\equiv T(y) \ \ (p^n) \\
\therefore T(x) &\equiv T(y) \ \ (p) \\
x &\equiv y \ \ (p) && \text{by } \textbf{\textit{3}}.
\end{aligned}$$

$$\therefore T\colon \mathbb{F}_p^\times \mapsto (\mathbb{Z}/p^n)^\times \text{is injective}$$

□

Corollary 3.11. *Let p be an odd prime, every element in $(\mathbb{Z}/p^n)^\times$ can be written uniquely in the form:*

$$\begin{aligned}
T(x) * \exp(py) \ \textit{with} \ x &\in \mathbb{F}_p^\times \\
py &\in p\mathbb{Z}/p^n
\end{aligned}$$

E.g $22 \in (\mathbb{Z}/125)^\times$

$$\begin{aligned} 22 &= T(x)\exp(5y) \quad (125) \\ &\equiv 2 \quad (5) \\ &\equiv x \quad (5) \end{aligned}$$

$$\implies x \equiv 2 \quad (5)$$

$$\begin{aligned} 22 &= T(2)\exp(5y) \quad (125) \\ 22 * T(2^{-1}) &\equiv \exp(5y) \quad (125) \end{aligned} \qquad \text{from the table } T(3) = 68$$

$$\begin{aligned} \exp(5y) &\equiv 22 * 68 \quad (125) \\ &\equiv 121 \quad (125) \\ &\equiv -4 \quad (125) \end{aligned}$$

$$\begin{aligned} \therefore 5y &\equiv \log(-4) \quad (125) \\ &\equiv \log(1-5) \quad (125) \\ &\equiv -5 - \frac{25}{2} - \frac{125}{3} + \dots \\ &\equiv -5 - 75 \\ &\equiv 45 \quad (125) \end{aligned}$$

$$\therefore 22 = T(2)\exp(45) \quad (125)$$

E.g Calculate $22^{37} \mod 125$

$$\begin{aligned} 22^{37} &\equiv \big(T(2)\exp(45)\big)^{37} \\ &\equiv T(2^{37}) * \underbrace{\exp(45*37)}_{\equiv 40} \quad (125) \\ &\equiv 2 \quad (5) \end{aligned}$$

$$\begin{aligned} \therefore 22^{37} &\equiv \underbrace{T(2)}_{\equiv 57} * \underbrace{\exp(40)}_{\equiv 841} \\ &\equiv 57 * 91 \quad (125) \\ &\equiv 62 \quad (125) \end{aligned}$$

E.g. Calculate $T(23) \mod 7^3$

$$\begin{aligned}23 &\equiv 2 \ \ (7)\\ 23^7 &\equiv 2^7 \equiv 128 \equiv 30 \ \ (7^2)\\ 23^{7^2} &\equiv 30^7 \ \ (7^3)\end{aligned}$$

Using the binomial theorem:

$$\begin{aligned}23^{7^2} &\equiv (2+4\cdot 7)^7 \ \ (7^3)\\ &\equiv 2^7 + 7\cdot 2^6\cdot 4 + \text{multiples of } 7^3 \ \ (7^3)\end{aligned}$$

Since $2^6 \cdot 4 \equiv 4 \mod 7$, it follows that $(7^2 \cdot 2^6 \cdot 4) \equiv (49 * 4) \equiv 196 \mod 7^3$. This shows that $T(23) \equiv 128 + 196 \equiv 324 \mod 7^3$

The following corollary was stated but not proved:

Corollary 3.12. *Let p be an odd prime number. Every element of $(\mathbb{Z}/p^n)^\times$ can be written uniquely in the form $T(x)\cdot \exp(py)$ where $x \in \mathbb{F}_p^\times$ and $py \in p\mathbb{Z}/p^n$. Thi is an isomorphism of groups:*

$$(\mathbb{Z}/p^n)^\times \cong \mathbb{F}_p^\times * p\mathbb{Z}/p^n$$

Proof. Take any $a \in (\mathbb{Z}/p^n)^\times$ and let $x \equiv a \mod p$. We have $a \equiv x \equiv T(x) \mod p$. Therefore $aT(x)^{-1} \equiv 1 \mod p$. This implies that $\log(\frac{a}{T(x)})$ converges p-adically. Let $py = \log(aT(x)^{-1})$. Then obviously $a = T(x)\exp(py)$.

For uniqueness, suppose $T(x)\cdot \exp(py) \equiv T(x')\cdot \exp(py') \mod p^n$. Since the image of exp is congruent to $1 \mod p$, we have $T(x) \equiv T(x') \mod p$.

This implies $x \equiv x' \mod p$. Therefore $T(x) \equiv T(x') \mod p^n$. From this we get $\exp(py) \equiv \exp(py') \mod p^n$. Taking logs we get $py \equiv py' \mod p^n$

□

This corollary can be used to solve the following types of equations:

E.g $x^{21} \equiv 71 \mod 81$
Note 21 is not coprime to $\phi(81) = 54$, so previous methods cannot be used to solve this equation. Also $(1+70)^{\frac{1}{21}}$ does not converge 3-adically. Start with $71 \equiv 2 \mod 3$:

$$\implies 71 \equiv T(2)\exp(3y) \mod 81$$

$T(2) \equiv -1 \mod 81$,

$$\begin{aligned}&\implies \exp(3y) \equiv -71\\ &\implies 3y \equiv \log(1+9) = 9 - \frac{81}{2} + \cdots \equiv 9 \mod 81\\ &\implies 71 = T(2)\exp(9) \mod 81\end{aligned}$$

Suppose we also decompose $x = T(u)\exp(3v)$. Then

$$x^{21} = T(u^{21})\exp(3 * 21v) = T(2)\exp(9)$$

Since such a representation is unique, this gives us two simultaneous equations:

$$u^{21} \equiv 2 \mod 3 \implies u \equiv -1 \equiv 2 \mod 3$$

$$\begin{aligned} 63v \equiv 9 \mod 81 \implies & 7v \equiv 1 \mod 9 \\ & v \equiv 4 \mod 9 \\ & 3v \equiv 12 \mod 27 \end{aligned}$$

$$x \equiv T(2)\exp(12) \mod 27$$

$$\begin{aligned} \exp(12) &\equiv 1 + 12 + \frac{12^2}{2} + \frac{12^3}{6} + \frac{12^4}{4!} + \dots \\ &\equiv 1 + 12 + 72 + 288 \\ &\equiv 1 + 12 + 18 + 18 \\ &\equiv 22 \mod 27 \end{aligned}$$

$$T(2) \equiv -1 \mod 27$$

So $x \equiv 5 \mod 27 \implies x \equiv 5, 32, 59 \mod 81$

3.6 Fractional Powers

If p is an odd prime n and $a \equiv 1 \mod p$ and $b \in \mathbb{Z}_{(p)}$ then a^b modulo p^n is:

$$a^b \equiv \exp(b\log(a)) \mod p^n$$

The usual rules hold for powers:

- $(ab)^c \equiv a^c b^c \mod p^n$
- $a^{b+c} \equiv a^b a^c \mod p^n$
- $a^{bc} \equiv (a^b)^c \mod p^n$

E.g $4^{\frac{1}{2}} \mod 27$ First find $\log(4) \mod 27$

$$\begin{aligned}
\log(4) &\equiv (1+3) \mod 27 \\
&\equiv 3 - \frac{9}{2} + \frac{27}{3} \mod 27 \\
&\equiv 3 + 9 + 9 \mod 27 \\
&\equiv 6 \mod 27
\end{aligned}$$

So

$$\begin{aligned}
4^b &\equiv \exp(-6b) && \mod 27 \\
&1 - 6b + \frac{36b^2}{2} - \frac{6^3b^3}{6} && \mod 27 \\
&1 - 6b + 18b^2 - 36b^3 && \mod 27 \\
&1 - 6b - 9b^2 - 9b^3 && \mod 27
\end{aligned}$$

$$\begin{aligned}
4^{\frac{1}{2}} &\equiv 1 - 3 - 9(\frac{1}{4} + \frac{1}{8}) \\
&\equiv 1 - 3\frac{27}{8} \\
&\equiv 1 - 3 \\
&\equiv -2 \mod 27
\end{aligned}$$

3.7 P-adic integers

This section is slighty more highbrow way of looking at the results of the previous lectures. We've defined several congruency classes such as $\exp(px) \mod p^n$, $T(a) \mod p^n$, $\log(1+px) \mod p^n$. It's a little bit more convenien to be able to write down just $\exp(px)$, $T(a)$, etc ... without needing to write modulo p^n everywhere. The problem is that there is no integer (or even an element of the local ring) which is congruent $T(a) \mod p^n$ for all n. Instead, we work in a bigger ring, the ring $\mathbb{Z}_p$ of p-adic integers. In this ring, the expression $T(a)$, $\exp(px)$ etc all make sense.

Let p be any prime number. By a p-adic integer, we shall mean a p-adically convergent series

$$\sum_{i=1}^{\infty} a_i \qquad\qquad a_i \in \mathbb{Z}_{(p)}, V_p(a_i) \longrightarrow \infty$$

Recall that any series represents and element of $\mathbb{Z}/p^n$ for every n. We call two p-adic integers equal if they are congruent modulo p^n for every n. THe set of all p-adic integers is whether $\mathbb{Z}_p$ (without the brackets around the p). Note that we can add, subtract and multiply p-adically convergent series, so in fact $\mathbb{Z}_p$ is a ring.

The advantage of this kind of notation is that we can write (for example) $\log(1+px)$ to mean a p-adic integer, without having to reduce modulo p^n. This allows us to state many of the recent theorems more simply. If $a \in \mathbb{Z}$ or $\in \mathbb{Z}_{(p)}$, then we can regard a as the series $a = a + 0 + 0 + 0 + \ldots$ and so a is a p-adic integer as well.
Therefore $\mathbb{Z} \subset \mathbb{Z}_{(p)} \subset \mathbb{Z}$

However, it turns out that there are many more p-adic integers than there are elements in the local ring $\mathbb{Z}_{(p)}$. For example consider the following 5-adic integer:

$$\begin{aligned} a &= (1+5)^{\frac{1}{2}} \\ &= 1 + \frac{1}{2} * 5 + \frac{\frac{1}{2} * \frac{-1}{2}}{2} + 5^2 + \ldots \end{aligned}$$

In fact a is a square root of 6. We've shown earlier that $a^2 \equiv 6 \mod 5^n$ for all n and therefore $a^2 \equiv 6 \in \mathbb{Z}_5$. However, the local ring $\mathbb{Z}_{(5)}$ has no square roots of 6 since its elements are rational numbers. This shows that a is in $\mathbb{Z}_5$ but not $\mathbb{Z}_{(5)}$.

Proposition 3.13. *Every p-adic integer can be written uniquely in the form:*

$$\sum_{i=0}^{\infty} a_i p_i$$

with coefficients $a_i \in \{0, 1, \ldots, p-1\}$

Proof.
Let x be a p-adic integer, so x is defined modulo p^n for all n. There is a unique choice of a_0 such that $a_0 \equiv x \mod p$.
This means that $x - a_0$ is a multiple of p. There is a unique choice of a_1 such that $a_1 \equiv \frac{x-a_0}{p} \mod p$.
This implies $pa_1 \equiv x - a_0 \mod p^2$, so $x \equiv a_0 + a_1 p \mod p^2$. This implies $x - a_0 - a_1 p$ is a multiple of p^2 and there is a unique a_2 such that $p^2 a_2 \equiv x - a_0 - a_1 p \mod p^3$, etc. □

We've already seen what it means for a series to converge p-adically. We'll now make a coresponding definition for sequences.

Defintion 3.14. *Let a_n be a sequence for elements of $\mathbb{Z}_{(p)}$. We'll say that this sequence converges p-adically if the corresponding series:*

$$a_0 + (a_1 - a_0) + (a_2 - a_1) + \ldots$$

If this is the case, then we define the limit of the sequence to be this series, regarded as an element of $\mathbb{Z}_p$. Note that the partial sums of the series above are exactly the terms of the sequence a_n. In fact, we have already seen manyt examples of p-adic limits.
Suppose $a_0 \in \mathbb{Z}_{(p)}$ satisfies the conditions of Hensel's lemma for a polynomial $f(x)$, i.e. $f(a_0) \equiv 0 \mod p^{2c+1}$, where $c = V_p(f'(a_0))$. Consider the series:

$$a = a_0 + (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots$$

We'll show that this series converges p-adically. Recall that $a_{n+1} - a_n = \frac{f(a_n)}{f'(a_n)}$

When proving Hensel's lemma, we showed that $f(a_n) \equiv 0 \mod p^{2c+2^n}$, $V_p(f'(a_n)) = c$

Therefore $V_p(a_{n+1} - a_n) \geq 2c + 2^n - c = c + 2^n \to \infty$

Hence a is a p-adic integer, and is congruent to $a_n \mod p^{c+2^n}$. We can re-interpret Hensel's lemma as saying the following:

Proposition 3.15. *Let a_0 and f satisfy the conditions of Hensel's lemma and let $a \in \mathbb{Z}_p$ be the p-adic integer defined above. Then $f(a) = 0$*

Proof.
We just need to prove that $f(a) \equiv 0 \mod$ any power of p. But we have $f(a) \equiv f(a_n) \equiv 0 \mod p^{c+2^n}$ $\square$

Next consider Teichmüller lifts. For an odd prime p and an element $a \in \mathbb{Z}_{(p)}$ such that $p \not| a$ let:

$$T(a) = a + (a^p - a) + (a^{p^2} - a^p) + (a^{p^3} - a^{p^2}) + (a^{p^4} - a^{p^3}) + \dots$$

We've shown that $a^{p^n} - a^{p^{n-1}} \equiv 0 \mod p^n$, and therefore the valuation of the $n-th$ term is at least n. This shows that the series converges p-adically, so $T(a) \in \mathbb{Z}_p$. Now the properties of Teichmüller lifts can be restated as follows:

Proposition 3.16. *The p-adic integer $T(a)$ depends onlt on the congruence class of a modulo p, and the map $T\colon \mathbb{F}_p^\times \mapsto \mathbb{Z}_p^\times$ is an injective group homomorphism.*

Proof.
Since $T(x) \equiv x \mod p$, it follows that T is injective. For every n, we have
$T(xy) \equiv T(x)T(y) \mod p^n$. Therefore $T(xy) = T(x)T(y) \in \mathbb{Z}_p$ $\square$

4 Quadratic rings

An integer d is called square-free if d is not a multiple of a square (apart from 1^2). Let d be a square-free integer with $d \not\equiv 1$. Define a complex number α by:

$$\alpha = \begin{cases} \sqrt{d} & \text{when } d \not\equiv 1 \pmod 4 \\ \frac{1+\sqrt{d}}{2} & \text{when } d \equiv 1 \pmod 4 \end{cases}$$

Consider the set $\{x + y\alpha : \quad x, y \in \mathbb{Z}\}$. This is called a "Quadratic Ring".

Lemma 4.1. *Every quadratic ring is a ring, i.e. closed under* $+, \times$.

Proof. Clearly closed under $+$

$$(x + y\alpha)(r + s\alpha) = xr + (xs + yr)\alpha + ys\alpha^2$$

Sufficient to show that α^2 is in the quadratic ring.
Case 1:

$$\alpha = \sqrt{d} \implies \alpha^2 = d \text{ which is in the quadratic ring}$$

Case 2:

$$\begin{aligned} \alpha &= \frac{1+\sqrt{d}}{2} & & d \equiv 1 \pmod 4 \\ (\alpha - \frac{1}{2})^2 &= \frac{d}{4} \\ \alpha^2 - \alpha + \frac{1}{4} &= \frac{d}{4} \\ \alpha &\equiv \alpha + \frac{d-1}{4} & d - 1 \equiv 0 \pmod 4 \implies & \frac{d-1}{4} \in \mathbb{Z} \end{aligned}$$

$\therefore \alpha^2$ is in the quadratic ring. ∎

We call $\mathbb{Z}[\alpha] = \{x + y\alpha : x, y, \in \mathbb{Z}\}$:

- A real quadratic ring if $d > 0$
- A complex quadratic ring if $d < 0$

E.g $d = -1$
$-1 \equiv 1 \pmod 4$
$\therefore \alpha = \sqrt{-1} = i$

$\mathbb{Z}[i] = \{x + iy : x, y \in \mathbb{Z}\}$ is the ring of Gaussian integers.

E.g $d = -3$
$-3 \equiv 1 \pmod 4$ so $\alpha = \frac{1+\sqrt{3}}{2}$
This is the ring of Eisenstein integers. It is the same as $\mathbb{Z}[\zeta_3] = \zeta = e^{\frac{2\pi i}{3}}$.

Defintion 4.2. *Let $\mathbb{Z}[\alpha]$ be a quadratic ring. The elements all have the form $A = x + y\sqrt{d}$ where x, y are rational. The conjugate of such an element $\bar{A} = x - y\sqrt{d}$*

4.0.1 Properties of conjugates

1 $\bar{\alpha} =$

1. $\bar{\alpha} = \begin{cases} -\alpha & d \not\equiv 1 \pmod 4 \\ 1-\alpha & d \equiv 1 \pmod 4 \end{cases}$
2. $\overline{A+B} = \bar{A} + \bar{B}$
   $\overline{AB} = \overline{A}\,\overline{B}$
3. $\bar{A} \in \mathbb{Z}[\alpha]$ if $A \in \mathbb{Z}[\alpha]$
4. $\bar{\bar{A}} = A$

Proof.
1. If $d \not\equiv 1 \pmod 4$ then $\alpha = \sqrt{d}$
$\implies \bar{\alpha} = -\sqrt{d} = -\alpha$

If $d \equiv 1 \pmod 4$ then $\alpha = \frac{1+\sqrt{d}}{2}$
$\implies \bar{\alpha} = \frac{1-\sqrt{d}}{2} = 1 - \alpha$

2. Suppose:
$A = x + y\sqrt{d}$
$B = r + s\sqrt{d}$

Clearly $\overline{A+B} = \bar{A} + \bar{B}$

$$\begin{aligned} \overline{A+B} &= \overline{(x+y\sqrt{d}))(r+s\sqrt{d})} \\ &= \overline{(xr+dys)+(xs+yr)\sqrt{d}} \\ &= \overline{(xr+dys)-(xs+yr)\sqrt{d}} \end{aligned}$$

$$\begin{aligned} \bar{A}\cdot\bar{B} &= (x-y\sqrt{d})(r-s\sqrt{d}) \\ &= (xr+dys)+(-xs-yr)\sqrt{d} \end{aligned}$$

3. Let $A = x + y\alpha \qquad x, y \in \mathbb{Z}$

by **2.** $\bar{A} = x + y\alpha$
by **1.** $\bar{\alpha} \in \mathbb{Z}[\alpha]$
$\therefore \bar{A} \in \mathbb{Z}[\alpha]$

4. Trivial $\square$

Defintion 4.3. *For an element $A \in \mathbb{Z}$ we define $N(A) = A\bar{A}$ - The norm of A*

Remark: If $\mathbb{Z}[\alpha]$ is a complex quadratic ring then $\bar{A}$ is the complex conjugate of A. This means $N(A) = |A|^2$

E.g. $d = -1 \implies \mathbb{Z}[\alpha] = \mathbb{Z}[i]$
The elements have the form $x + iy, \ \ x, y \in \mathbb{Z}$

$$\begin{aligned} N(x+iy) &= (x+iy)(x-iy) \\ &= x^2 + y^2 \end{aligned}$$

E.g. $d = -3 \implies -3 \equiv 1 \ \ (4)$, so $\alpha = \frac{1+\sqrt{-3}}{2}$

$$\begin{aligned} N(x+y\alpha) &= (x+y\alpha)(x+y\bar{\alpha}) \\ &= x^2 + xy(\underbrace{\alpha + 1 - \alpha}_{=1}) + y^2(\underbrace{\alpha(1-\alpha)}_{=1}) \end{aligned} \qquad \bar{\alpha} = 1 - \alpha$$

Note that:

$$\alpha = \frac{1+\sqrt{d}}{2}$$

$$\implies (\alpha - \frac{1}{2})^2 = \frac{d}{4}$$

$$\implies \alpha^2 - \alpha + \frac{1}{4} = \frac{d}{4}$$

$$\implies \alpha(1-\alpha) = \frac{1-d}{4}$$

In this case:

$$N(x+y\alpha) = x^2 + xy + y^2$$

4.0.2 Formula for norms

The general formula for norms is given by:

$$N(x+y\alpha) = \begin{cases} x^2 - dy^2 & \text{if } d \not\equiv 1 \ \ (4) \\ x^2 + xy + \frac{1-d}{4}y^2 & \text{if } d \equiv 1 \ \ (4) \end{cases}$$

Case 1: $d \not\equiv 1 \pmod 4$
$\implies \quad \bar{\alpha} \equiv -\alpha$

$$N(x+y\alpha) = (x+y\alpha)(x-y\alpha) = x^2 - dy^2 \qquad\qquad \alpha^2 = d$$

Case 2: $\bar{\alpha} = 1 - \alpha$
$\implies \quad \alpha = \frac{1+\sqrt{d}}{2}$

$$\begin{aligned} N(x+y\alpha) &= (x+y\alpha)(x+y(1-\alpha)) \\ &= x^2 + xy + y^2(\underbrace{\alpha - \alpha^2}_{\frac{1-d}{4}}) \end{aligned}$$

4.0.3 Properties of norms

1. $N(A) \in \mathbb{Z}$
2. $N(AB) = N(A)N(B)$
3. If $N(A) = 0$ then $A = 0$

Proof.

1. Follows from formulas for norms
2. $N(AB) = AB\overline{AB}$
3. If $N(A) = 0$ then $A\bar{A} = 0$
   $\therefore$ either $A = 0$ or $\bar{A} = 0$
   $\therefore \bar{A} = 0$ then $A = \bar{\bar{A}} = \bar{0} = 0$

$\square$

Recall - A unit in a ring R is an element with an inverse in R.

E.g. $1 + \sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}] \implies (1+\sqrt{2})(\sqrt{2}-1) = 1$

Corollary 4.4. *An element $A \in \mathbb{Z}[\alpha]$ is a unit if and only if $N(A) = \pm 1$*

Proof. If $N(A) = \pm 1$ then $A\bar{A} = \pm 1$
$\implies A^{-1} = \pm \bar{A} \in \mathbb{Z}[\alpha]$. So A is a unit. $\square$

If A is a unit with inverse B, $AB = 1 \implies N(A)N(B) = N(AB) = N(1) = 1$
Using this proposition, it's easy to find all the units in any complex quadratic ring.
E.g The units in $\mathbb{Z}[i]$ are $\pm 1, \pm i$

Proof. Since $N(x+iy) = x^2 + y^2$, the units correspond to the solutions to $x^2 + y^2 = 1$.
These solutions are $x = \pm 1$, $y = 0$ and $x = 0$, $y = \pm 1$ $\square$

E.g. The units in the Eisenstein integers $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ are $\pm 1, \pm\alpha, \pm(\alpha - 1)$. Equivalently these are $\pm 1, \pm\zeta_3, \pm\zeta_3^2$

Proof. Find all solutions to $x^2 + xy + y^2 = 1$. We can complete the square to get:

$$(x + \frac{1}{2}y)^2 + \frac{3}{4}y^2 = 1$$

$y^2 < \frac{4}{3}$ and since y is an integer $|y| < 1$
Similarly $|x| < 1$.

If $y = 0$ then $x \pm 1$
If $y = \pm 1$ then $x^2 + xy + 1 = 1 \implies x = 0, -y$ are solutions.

So the 6 solutions are $(1, 0), (-1, 0), (0, 1), (-1, 1), (0, -1), (1, -1)$

□

Corollary 4.5. *If $d < 0$ and $d \neq -1, -3$ then the units in $\mathbb{Z}[\alpha]$ are $\{$1,-1$\}$*

Proof. Assume first that $d \not\equiv 1 \mod 4$

$\implies N(x + y\alpha) = x^2 - dy^2 \qquad x, y \in \mathbb{Z}$ and $-d > 1$ so $y = 0 \implies x = \pm 1$

So $(1, 0), (-1, 0)$ give us the two units $1, -1$

Assume now $d \equiv 1 \mod 4 \implies -d \geq -7$ and need to find solutions to the equation:

$$x^2 + xy + \frac{1-d}{4}y^2 = 1$$

$$\implies (x + \frac{1}{2}y)^2 - \frac{d}{4}y^2 = 1$$

Since $\frac{d}{4} > 1, y^2 < 1 \implies y = 0 \implies x = \pm 1$

□

4.1 Norm-Euclidean quadratic rings

Defintion 4.6. *A quadratic ring $\mathbb{Z}[\alpha]$ is norm-Euclidean if $\forall A, B \in \mathbb{Z}[\alpha]$ with $B \neq 0$ $\exists Q, R, \in \mathbb{Z}[\alpha]$ such that:*

- $A = QB + R$
- $|N(R)| < N(B)$

Finitely many of the quadratic rings are norm-Euclidean.

Theorem 4.7. *If $\mathbb{Z}[\alpha]$ is norm-Euclidean then every non-zero element of $\mathbb{Z}[alpha]$ can be factorised as $UQ_1 \dots Q_r$*

Q_i are irreducible elements of $\mathbb{Z}[\alpha]$, U is a unit.
This factorisation is unique in the sense that if $U_1Q_1 \dots Q_r = U_2R_1 \dots R_s$ then $r = s$ and (after reordering Q_r, R_i is a unit for each U

E.g Let $d = -7$ so $\alpha = \frac{1+\sqrt{-7}}{2}$. This is norm-Euclidean.
Suppose $z = x + y\sqrt{-7} \quad x, y \in \mathbb{Q}$. We'll show that there is an element $Q \in \mathbb{Z}[\alpha]$ such that $|N(Z - Q)| < 1$.

Choose $b \in \mathbb{Z}$ such that $|y - \frac{b}{2}| < \frac{1}{4}$

Note that $z - b\alpha = (x - \frac{b}{2}) + (y - \frac{b}{2})\sqrt{-7}$

Then choose $a \in \mathbb{Z}$ so that $|x - \frac{b}{2} - a| \leq \frac{1}{2}$. Also note that the maximum distance to the closest integer is $\frac{1}{2}$.

We let $Q = a + b\alpha$ and we have:

$$Z - Q = (x - \frac{b}{2} - a) + (y - \frac{b}{2})\sqrt{-7}$$

$$N(Z - Q) \leq (\frac{1}{2})^2 + 7(\frac{1}{4})^2 = \frac{11}{16} < 1$$

Now to show the ring is norm-Euclidean. Choose $A, B \in \mathbb{Z}[\alpha]$ with $B \neq 0$. By what we've shown there is an element $Q \in \mathbb{Z}[\alpha]$ such that $|N(\frac{A}{B} - Q)| \leq 1$

Let $R = A - QB$ then $A = QB + R$

$$\begin{aligned} |N(R)| &= |N(A - QB)| \\ &= |N(\frac{A}{B} - Q)N(B)| \\ &< |N(B)| \end{aligned}$$

E.g Let $d = 3$. In this case $\alpha = \sqrt{3}$. We'll show that the quadratic ring $\mathbb{Z}[\sqrt{3}]$ is norm-Euclidean.

Let $z = x + y\sqrt{3}$ with $x, y \in \mathbb{Q}$. Need to show there is an element $Q = r + s\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$ such that $|N(Z - Q)| < 1$. Choose $r, s \in \mathbb{Z}$ such that:

$$|x - r| \leq \frac{1}{2} \qquad\qquad |y - s| \leq \frac{1}{2}$$

$$\begin{aligned} &\implies N(Z - A) = (x - r)^2 - 3(y - s)^2 \\ &\implies -\frac{3}{4} \leq N(Z - Q) \leq \frac{1}{4} \\ &\implies |N(Z - Q)| < 1 \end{aligned}$$

Now to show that $\mathbb{Z}[\sqrt{3}]$ is norm-Euclidean. Suppose $A, B \in \mathbb{Z}[\sqrt{3}]$ with $B \neq 0$, there is already a $Q \in \mathbb{Z}[\sqrt{3}]$ such that $|N(\frac{A}{B} - Q)| < 1$.

Let $R = A - QB$. This implies $A = QB + R$

$$\begin{aligned} \implies |N(R)| &= |N(A - QB)| \\ &= |N(\frac{A}{B} - Q)N(B)| \\ &< |N(B)| \end{aligned}$$

Hence $\mathbb{Z}[\sqrt{3}]$ is norm-Euclidean.

Theorem 4.8. *The disappointing theorem - The quadratic rings with* $d = -1, -2, -3, -7, 1, 2, 3, 5, 13$ *are norm-Euclidean.*

Defintion 4.9. *Suppose $A, B \in \mathbb{Z}[\alpha]$. A highest common factor of A and B is an element $C \in \mathbb{Z}[\alpha]$ with the following properties:*

- *C is a factor of both A and B i.e. $\frac{A}{C}$ and $\frac{B}{C}$ are both in $\mathbb{Z}[\alpha]$*
- *If D is a factor of both A and B then D is a factor of C (and hence$|N(D)| \leq |N(C)|$)*

If C is a highest common factor of A and B, then so is UC fo every unit U, but these are all the highest common factors. Hence highest common factors, if they exit are unique up to multiplication by a unit.

Lemma 4.10. *Bezout's Lemma - Let $\mathbb{Z}[\alpha]$ be norm-Euclidean ring and let $A, B \in \mathbb{Z}[\alpha]$ not both 0. Then there is a highest common factor C of A, B and there exist $H, K \in \mathbb{Z}[\alpha]$, such that $HA + KB = C$*

Proof. The proof goes similarly to in the ring $\mathbb{Z}$. We prove by induction on $\min(|N(A)|, |N(B)|)$. The induction step consists of writing $A = QB + R$ with $N(R) < |N(B)|$ and using the lemma. To prove the start of the induction, we assume $B = 0$. But then it's easy to check that A is a highest common factor. $\square$

Defintion 4.11. *An element $P \in \mathbb{Z}[\alpha]$ is called irreducible if:*

- *P is not a unit*
- *If $P = AB$ with $A, B \in \mathbb{Z}[\alpha]$ then either A or B is a unit*

Defintion 4.12. *We'll say that a quadratic ring $\mathbb{Z}[\alpha]$ has unique factorisation if the following is true:*

- *For every non-zero element $A \in \mathbb{Z}[\alpha]$ there is a factorisation $A = UP_1 \dots P_r$ with U a unit and each P_i irreducible*
- *If we have another factorisation $A = U'Q_1 \dots Q_s$, then $r = s$ and we can reorder $Q_1, \dots, Q_s$ so that each P_i/Q_i is a unit.*

Lemma 4.13. *Let $\mathbb{Z}[\alpha]$ be norm-Euclidean. Let p be irreducible and suppose $P|AB$ in $\mathbb{Z}[\alpha]$. Then $P|A$ or $P|B$*

Proof. Suppose P does not divide A. Then the highest common factor of P and A is not P, so it must be 1. Therefore we can find $H, K \in \mathbb{Z}[\alpha]$ such that $HP + KA = 1$. This implies $B = HPB + KPB$ which is a multiple of P. $\square$

Theorem 4.14. *If $\mathbb{Z}[\alpha]$ is norm-Euclidean, then $\mathbb{Z}[\alpha]$ has unique factorisation.*

Proof.
The proof is exactly as for $\mathbb{Z}$ (using the previous lemma for the uniqueness part), except that we prove by induction on $N(A)$. $\square$

In fact, there are many examples when $\mathbb{Z}[\alpha]$ has unique factorisation, even though it is not norm-Euclidean. It's known that a complex quadratic ring has unique factorisation for exactly the following values of d and no more:

$$d = -1, -2, -3, -7, -11, -19, -43, -67, -163$$

In contrast, it is much more common for a real quadratic ring to have unique factorisation. In fact, the following is believed (but not proved):

Conjecture: There are infinitely many positive square-free integers d such that $\mathbb{Z}[\alpha]$ has unique factorisation. On the other hand, there are many quadratic rings which do not have unique factorisation.

E.g In the ring $\mathbb{Z}[\sqrt{-5}]$ we have non-unique factorisation. For example
$6 = 2 * 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.

The elements $2, 3, 1 \pm \sqrt{-5}$ are all irreducible. To see this, note that they have norms $2, 9,$ $6\&6$. Hence any proper factors would have norm 2 and 3. However the ring $\mathbb{Z}[\sqrt{-5}]$ has no elements of norm 2 and 3 since $x^2 + 5y^2$ is never equal to 2 or 3 fo r integers x, y.

4.2 The Decomposition Theorem

Assume that $\mathbb{Z}[\alpha]$ is a quadratic ring with unique factorisation into irreducible elements, e.g. $\mathbb{Z}[\alpha]$ could be norm-Euclidean.

Lemma 4.15. *If Q is an irreducible element in $\mathbb{Z}[\alpha]$ then there exists a unique prime number p such that $Q|p$*

Proof.

$$Q|N(Q) = \pm p_1 p_2 \dots p_r \quad (p_i \text{ prime})$$

By uniqueness of factorisation $Q|p_i$ for some i if $Q|p$ and $Q|q$ where p, q are distinct primes, $hcf(p, q) = 1 = hp + kq$ $(h, k \in \mathbb{Z}) \implies Q|1$ ↯ $\square$

The lemma means that to find all the irreducible elements, we just need to factorise all the primes in $\mathbb{Z}[\alpha]$. Suppose $Q|p$, where Q is irreducible in $\mathbb{Z}[\alpha]$, p prime:

$$\implies N(Q)|N(p) = p^2$$
$$\implies N(Q) = \pm p \text{ or } \pm p^2$$

If $N(Q) = \pm p^2$ then $Q = unit * p$ so p is irreducible.

- If $P = Q_1 Q_2$ where $\frac{Q_1}{Q_2}$ is not a unit, then we say P is **split** in $\mathbb{Z}[\alpha]$
- If $P = UQ^2$ (U a unit, Q irreducible) then we say P is **ramified** in $\mathbb{Z}[\alpha]$
- If P is irreducible in $\mathbb{Z}[\alpha]$ then we say that P is **inert** in $\mathbb{Z}[\alpha]$

E.g $d = -1 \quad \alpha = \sqrt{-1} = i \quad N(x + iy) = x^2 + y^2$

A prime number p factorises in $\mathbb{Z}[i] \implies$ there is an element with norm $\pm p$

- $2 = 1^2 + 1^2 = N(1 + i) = (1 + i)(1 - i) = -i(1 + i)^2 \implies$ 2 is ramified
- 3 is inert
- $5 = 2^2 + 1^2 = (2 + i)(2 - i)$ 5 is split
- 7 is inert
- 11 is inert
- $13 = 3^2 + 2^2 = (3 + 2i)(3 - 2i)$ 13 is split

Check if a number is ramified by dividing one by the other and checking if the result is in the ring.

E.g $d = -3 \quad \alpha = \frac{1+\sqrt{3}}{2} \quad N(x + y\alpha) = x^2 + xy + y^2$

- 2 inert
- $3 = -\sqrt{-3}^2 = -(1 - 2\alpha)^2$ 3 ramified
- 5 inert
- $7 = N(2 + \alpha) = (2 + \alpha)(3 - \alpha)$
- 11 inert

Assume $\mathbb{Z}[\alpha]$ is a quadratic ring with unique factorisation. Let p be an odd prime number.

$$p \text{ is ramified} \iff p|d \qquad\qquad 2 \text{ splits} \iff d \equiv 1 \pmod 8$$
$$p \text{ is split} \iff \left(\frac{d}{p}\right) = 1 \qquad\qquad 2 \text{ inert} \iff d \equiv 5 \pmod 8$$
$$p \text{ is inert} \iff \left(\frac{d}{p}\right) = -1 \qquad\qquad \text{in other cases 2 is ramified}$$

Idea of proof:

Assume $d \not\equiv 1 \pmod 4$, $N(x+y\alpha) = x^2 - dy^2$

If p factorises then $\exists x, y \in \mathbb{Z}$, $x^2 - dy^2 = \pm p$

$$\begin{aligned} &\implies x^2 \equiv dy^2 \ \ (p) \\ &\implies \left(\frac{x}{y}\right)^2 \equiv d \ \ (p) \end{aligned}$$

If d is a quadratic residue then $x^2 \equiv d \ \ (p) \qquad p|(x+\sqrt{d})(x-\sqrt{d})$.

If p were inert then $p|x+\sqrt{d}$ or $x-\sqrt{d}$ $\lightning$ (number not in ring)

$\implies$ factorises

4.3 Solving $|N(A) = n|$

Assume that $\mathbb{Z}[\alpha]$ has unique factorisation, does the equation $|N(A)| = n$ have solutions?

E.g $d = -1 \qquad \mathbb{Z}[\alpha] = \mathbb{Z}[i] \qquad N(x+iy) = x^2 + y^2$

$2 = 1^2 + 1^2$	$8 = 2^2 + 2^2$
$3\times$	$9 = 3^2 + 0^2$
$4 = 2^2 + 0^2$	$10 = 3^2 + 1^2$
$5 = 2^2 + 1^2$	$11\times$
$6\times$	$12\times$
$7\times$	$13 = 3^2 + 2^2$

The answer is a corollary to the Decomposition Theorem.

Corollary 4.16. *Assume $\mathbb{Z}[\alpha]$ has unique factorisation and let n be a positive integer. Then the following are equivalent:*

1. $\exists A \in \mathbb{Z}[\alpha] : |N(A)| = n$
2. $\forall$ *inert primes* $p|n$, $V_p(n)$ *is even*

Proof.

1 $\implies$ 2 Assume $|N(A)| = n$

$$\begin{aligned} A = Q_1^{a_1} \dots Q_r^{a_r} &\text{ for } Q_i \text{ irreducible in } \mathbb{Z}[\alpha] \\ Q_i|P_i &\ \ (p_i \text{ prime}) \end{aligned}$$

$$|N(Q_i)| = \begin{cases} P_i & \text{if } P_i \text{ splits or is ramified} \\ P_i^2 & P_i \text{ inert} \end{cases}$$

$$n = |N(A)| = \left(\prod_{\substack{P_i \text{ split} \\ \text{or ramified}}} P_i^{a_i} \right) * \left(\prod_{P_i \text{ inert}} \right)$$

So powers of inert primes are even.

2 $\implies$ **1** Let

$$n = \left(\prod_{\substack{P_i \text{ split} \\ \text{or ramified}}} P_i^{a_i} \right) * \left(\prod_{P_i \text{ inert}} P_i^{2a_i} \right)$$

Choose an element Q_i with norm $\pm P_i$ if P_i is split or ramified:

$$n = N\left(\prod_{\substack{P_i \text{ ramified} \\ \text{or split}}} Q_i^{a_i} \quad \times \prod_{P_i \text{ inert}} P_i \right)$$

□

E.g Solve $x^2 + y^2 = 585$ i.e. $N(x + iy) = 585$
Note $585 = 3^2 * 5 * 13$

- 3 is inert because $\left(\frac{-1}{3}\right) = -1$
- 5 is split because $\left(\frac{-1}{5}\right) = +1$
- 3 is split because $\left(\frac{-1}{13}\right) = +1$

The only inert prime factor of 585 is 3 and its power is even so $x^2 + y^2 = 585$ will have solutions.

$$5 = 2^2 + 1^2 = N(2 + i) = (2 + i)(2 - i)$$
$$13 = 3^2 + 2^2 = N(3 + 2i) = (3 + 2i)(3 - 2i)$$

$$\begin{aligned} 585 &= 3^2 * 5 * 13 \\ &= N(3 * (2 + i)(3 + 2i)) \\ &= N(3(6 + 7i - 2)) \\ &= N(12 + 21i) \\ &= 12^2 + 21^2 \end{aligned}$$

The other elements of norm 585 are unit multiples of it:

- $3(2+i)(3-2i) = 24-3i$
- $3(2-i)(3+2i) = 24+3i$
- $3(2-i)(3-2i) = 12-21i$

E.g. $x^2+xy+y^2 = 84$
Note that $84 = 2^2 * 3 * 7$

- 2 is inert in $\mathbb{Z}[\alpha]$
- $3 = -(\sqrt{3})^2 = -(1-2\alpha)^2 \implies$ ramified
- $7 \; \left(\frac{-3}{7}\right) = \left(\frac{4}{7}\right) = 1 \implies$ 7 splits

  $7 = N(2+2\alpha) = (2+\alpha)(3-\alpha)$

So the elements with norm 84 are:

- $2(1-2\alpha)(2+\alpha)*$ unit
- $2(1-2\alpha)(3-\alpha)*$ unit

There are 6 units in this ring, therefore there are 12 solutions to $x^2+xy+y^2 = 84$

One possible solution is:

$$\begin{aligned} 2(1-2\alpha)(2+\alpha) &= 2(2-3\alpha = 3\alpha^2) \\ &= 2(2-3\alpha-2\alpha+2) \\ &= 8-10\alpha \end{aligned}$$

so $N(8-10\alpha) = 84$

Super cool trick

Using $\alpha = \frac{1+\sqrt{d}}{2}$

$$\begin{aligned} \left(\alpha - \frac{1}{2}\right)^2 &= \frac{d}{4} \\ \alpha^2 - \alpha + \frac{1}{4} &= \frac{d}{4} \\ \alpha^2 &= \frac{d-1}{4} + \alpha \end{aligned}$$

4.4 Continued Fractions

A finite continued fraction is $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cfrac{1}{\ddots a_n}}}}$ where $a_0 \in \mathbb{Z}$

and $a_1, \ldots, a_n \in \mathbb{Z} > 0$.

We'll use the notation $[a_0, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cfrac{1}{\ddots a_n}}}}$

E.g. $[1, 2, 3, 2] = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2}}} = 1 + \cfrac{1}{2 + \cfrac{1}{7/2}} = 1 + \cfrac{1}{2 + \cfrac{2}{7}} = 1 + \frac{7}{16} = \frac{23}{16}$

More generally if $\alpha \in \mathbb{R}$, $\alpha > 0$

$$[a_0, \ldots, a_n, \alpha] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \cfrac{1}{\ddots a_n + \cfrac{1}{\alpha}}}}}$$

and as a consequence $[a_0, \ldots, a_n] = [a_0, \ldots, a_r, [a_{r+1}, \ldots, a_r]]$

E.g $[1, 2, 3, 2] = [1, 2, [3, 2]] = [1, 2, \frac{7}{2}] = [1, 2 + \frac{2}{7}] = [1, \frac{16}{7}] = 1 + \frac{7}{16} = \frac{23}{16}$

Clearly every finite continued fraction is in $\mathbb{Q}$. Conversely if $fracnm \in \mathbb{Q}$, then we can write $\frac{n}{m}$ as a finite continued fraction.

E.g. By Euclid's algorithm:

$$89 = 2 * 39 + 11 \qquad 89/39 = 2 + \frac{11}{39}$$

$$39 = 3 * 11 + 6 \qquad 39/11 = 3 + \frac{6}{11}$$

$$11 = 1 * 6 + 5 \qquad 11/6 = 1 + \frac{5}{6}$$

$$6 = 1 * 5 + 1 \qquad 6/5 = 1 + \frac{1}{5}$$

$$5 = 1 * 5 + 0 \qquad 5/1 = 5$$

Therefore $\frac{89}{39} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5}}}} = [2, 3, 1, 1, 5]$

Now suppose we have a sequence $a_n \in \mathbb{Z}$ for all $n, a_1, a_2, \dots > 0$. For any n we have a finite continued fraction $[a_0, \dots, a_n] = \dfrac{h_n}{k_n} \in \mathbb{Q}$

We define $[a_0, a_1, \dots] = \lim\limits_{n\to\infty}[a_0, \dots, a_n] = \lim\limits_{n\to\infty}\dfrac{h_n}{k_n}$

Defintion 4.17. $[a_0, a_1, \dots]$ *is called an infinite continued fraction*

Theorem 4.18. *For any sequence of integers $a_n > 0$ for $n > 0$, the limit $[a_0, \dots]$ exists. If $\alpha = [a_0, a_1, \dots]$ then $\left|\alpha - \dfrac{h_n}{k_n}\right| < \dfrac{1}{k_n^2}$ (will be proved later)*

Sometimes we can calculate infinite continued fractions.

E.g. $\alpha = [1, 2, 1, 2, 1, 2, 1, 2, \dots]$ Which real number is α?

$$\begin{aligned}
\alpha &= [1, 2, \alpha] \\
&= \left[1, 2 + \frac{1}{\alpha}\right] \\
&= \left[1, \frac{2\alpha + 1}{\alpha}\right] \\
&= 1 + \frac{\alpha}{2\alpha + 1} \\
&= \frac{3\alpha + 1}{2\alpha + 1}
\end{aligned}$$

$$2\alpha^2 + \alpha = 3\alpha + 1$$
$$2\alpha^2 - 2\alpha - 1 = 0$$

$$\alpha = \frac{1 \pm \sqrt{3}}{2}$$

Since $\alpha = 1 + \dfrac{2}{1 + \dots} > 1$, $\alpha = \dfrac{1 + \sqrt{3}}{2}$.

Every infinite continued fraction converges to a real number. Conversely if α is an irrational real number, then we can write α as an infinite continued fraction.

Method: We define a sequence $\alpha_n \in \mathbb{R}$, $a_n \in \mathbb{Z}$ such that $\alpha_0 = \alpha$ and $a_n = \lfloor \alpha_n \rfloor$

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n} > 1 \qquad\qquad a_n > 0$$

From this definition:

$$\begin{aligned}
\alpha &= \alpha_0 \\
&= a_0 + \frac{a_0}{\alpha_1} \\
&= a_0 + \cfrac{1}{a_1 + \cfrac{1}{\alpha_2}} \\
&= [a_0, a_1, a_2, \alpha_3] \text{ etc}
\end{aligned}$$

Using this we can show that $\alpha = [a_0, a_1, \dots]$

E.g. Write $\sqrt{2}$ as an infinite continued fraction

$$\begin{aligned}
\alpha_0 &= \sqrt{2} & a_0 &= \lfloor \sqrt{2} \rfloor = 1 \\
\alpha_1 &= \frac{1}{\alpha_0 - a_0} & a_1 &= \lfloor \sqrt{2} + 1 \rfloor = 2 \\
&= \frac{1}{\sqrt{2} - 1} \\
&= \frac{\sqrt{2} + 1}{(\sqrt{2} - 1)(\sqrt{2} + 1)} \\
&= \frac{\sqrt{2} + 1}{2 - 1} \\
&= \sqrt{2} + 1
\end{aligned}$$

$$\alpha_2 = \frac{1}{\alpha_1 - a_1} \qquad a_2 = \lfloor \alpha_2 \rfloor = \lfloor \alpha_1 \rfloor = 2$$
$$= \frac{1}{(\sqrt{2}+1) - 2}$$
$$= \frac{1}{\sqrt{2} - 1}$$
$$= \alpha_1$$

$$\alpha_3 = \frac{1}{\alpha_2 - \alpha_2} \qquad a_3 = \lfloor a_1 \rfloor = 2$$
$$= \frac{1}{\alpha_1 - a_1}$$
$$= \alpha_2$$

So $\alpha_2 = \alpha_3 = \alpha_4 = \cdots = \dfrac{1}{\sqrt{2}-1} = \sqrt{2}+1$ and $a_3, a_4, a_5 = 2$

Therefore $\sqrt{2} = [a_0, a_1, a_2, \ldots] = [1, 2, 2, 2, \ldots]$.

Using this method we can write $\sqrt{d}$ for any $+ve\ d$ as an infinite continued fraction.

Recall that an element in $\mathbb{Z}[\sqrt{2}]$ is a unit if its norm is ± 1, i.e. elements $x + y\sqrt{2}$ where $x^2 - 2y^2 = \pm 1$ i.e. $\left| \left(\dfrac{x}{y}\right)^2 - 2 \right| = 1$ so $\dfrac{x}{y}$ is close to $\sqrt{2}$.

Let $\dfrac{h_n}{k_n} = \underbrace{[1, 2, 2, \ldots, 2]}_{n \text{ terms}}$

This is close to $\sqrt{2}$

$$[1] = 1/1 = 1 \qquad 1^2 - 2 * 1^2 = -1$$
$$[1,2] = 1 + 1/2 = 3/2 \qquad 3^2 - 2 * 2^2 = +1$$
$$[1,2,2] = 7/5 \qquad 7^2 - 2 * 5^2 = -1$$
$$[1,2,2,2] = 17/12 \qquad 17^2 - 2 * 12^2 = +1$$

In this case when $\dfrac{h}{k} = [1, 2, , \ldots, 2]$, we always have $h^2 - 2 * k^2 = \pm 1$, so $h + k\sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$

4.5 Pell's equation and units in real quadratic rings

Let $d > 1$ be a square free integer. Pell's equation is $x^2 - dy^2 = 1$. We'll see how to find the solutions (x, y) in integers.

Let $A = x + y\sqrt{d}$. Pell's equation $\leftrightarrow N(A) = 1$. Therefore A is a unit in $\mathbb{Z}[\sqrt{d}]$ with norm 1.

There are obvious solutions $x = \pm 1$, $y = 0$. We'll call these the trivial solutions, these correspond to the units $A = \pm 1$

Theorem 4.19. *For any d, there are non-trivial solutions*

Defintion 4.20. *The smallest solution (x, y) with $x, y > 0$ is called the fundamental solution*

E.g. $d = 2$
$x^2 - 2y^2 = 1$ and so $(x, y) = (3, 2)$ is the fundamental solution

A^n is also a unit with norm 1. This gives an infinite sequence of solutions to Pell's equations.
$A^2 = (3 + 2\sqrt{2})^2 = 9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2}$
$A^3 = (3 + 2\sqrt{2})(17 + 12\sqrt{2}) = 99 + 70\sqrt{2}$

So $(17, 12)$ and $(99, 70)$ are also the solutions to $x^2 - 2y^2 = 1$

Proposition 4.21. *If (x, y) is the fundamental solution, then any other solution in positive integers will be (x_n, y_n) where $(x_n + y_n\sqrt{d}) = (x + y\sqrt{d})^n$*

Proof. Let $A = x + y\sqrt{d}$
A is the smallest unit with norm 1 such that $A > 1$
Let B be any unit in $\mathbb{Z}[\sqrt{d}]$ which is bigger than 1, and has norm 1.
Want to show $B = A^n$ for some n
$1 < A < A^2 < A^3 < \cdots \to \infty$
There exists n such that $A^n \leq B < A^{n+1} \implies 1 \leq A^{-n}B < B$
$A^{-n}B$ is a unit with norm 1. By choice of A, $A^{-n}B = 1$ and $B = A^n$ $\square$

Once we have the fundamental solution, we've solved the equation. Sometimes the fundamental solution is big so it's difficult to find, for example:
$d = 151 \implies x^2 - 151y^2 = 1$
Fundamental solution $x = 1728148040$, $y = 140634693$
So we need a fast way of finding the fundamental solution.

If $\alpha \in \mathbb{R}$ and α is irrational then it has an infinite continued fraction expansion.

$$\alpha = [a_0, a_1, \dots] = \lim_{n\to\infty} a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots \cfrac{1}{\ddots a_n}}}$$

The limit above converges to $\sqrt{d}$ and $[a_0, \ldots, a_n]$.

Let $\dfrac{h_n}{k_n} = [a_0, \ldots, a_n]$.

Defintion 4.22. *The rational numbers* $\dfrac{h_n}{k_n}$ *are called* ***convergents*** *of* α

Theorem 4.23. *If* $\dfrac{h}{k} \in \mathbb{Q}$ *with* $\left|\alpha - \dfrac{h}{k}\right| < \dfrac{1}{2k^2}$ *then* $\dfrac{h}{k}$ *must be one of the convergents of* α

Corollary 4.24. *If* (x, y) *is a solution to Pell's equation* $x^2 - dy^2 = 1$ *for* $x > y > 0$*, then* $\dfrac{x}{y}$ *is a convergent in the continued fraction of* $\sqrt{d}$

Proof. (Corollary)

$$\begin{aligned} x^2 - dy^2 &= 1 \\ (x + y\sqrt{d})(x - y\sqrt{d}) &= 1 \end{aligned}$$

$$\begin{aligned} |x - y\sqrt{d}| &= \frac{1}{x + y\sqrt{d}} < \frac{1}{2y} \\ |\frac{x}{y} - \sqrt{d}| &< \frac{1}{2y^2} \end{aligned}$$

$\dfrac{x}{y}$ is a convergent. □

E.g. $d = 7 \implies x^2 - 7y^2 = 1$
To find the fundamental solution we find $\sqrt{7}$ as a continued fraction.

$$\alpha_0 = \sqrt{7} \qquad a_n = \lfloor \alpha_n \rfloor$$

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n}$$

$$\alpha_1 = \frac{1}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{(\sqrt{7}-2)(\sqrt{7}+2)} = \frac{\sqrt{7}+2}{3} \qquad a_1 = 1$$

$$\alpha_2 = \frac{1}{\frac{\sqrt{7}+2}{3}-1} = \frac{3}{\sqrt{7}-1} = \frac{3(\sqrt{7}+1)}{(\sqrt{7}-1)(\sqrt{7}+1)} = \frac{\sqrt{7}+1}{2} \qquad a_2 = 1$$

$$\alpha_3 = \frac{1}{\frac{\sqrt{7}+1}{2}-1} = \frac{2}{\sqrt{7}-1} = \frac{2(\sqrt{7}+1)}{6} = \frac{\sqrt{2}+1}{3} \qquad a_3 = 1$$

$$\alpha_4 = \frac{1}{\frac{\sqrt{7}+1}{3}-1} = \frac{3}{\sqrt{7}-2} = \frac{3(\sqrt{7}+2)}{3} = \sqrt{7}+2 \qquad a_4 = 4$$

$$\alpha_5 = \frac{1}{\sqrt{7}+2-4} = \frac{1}{\sqrt{7}-2} = \alpha_1 \qquad a_5 = 1$$

So $\sqrt{7} = [2, 1, 1, 1, 4, 1, 1, 1, 4, \dots]$

$$\begin{aligned} [2] &= 2/1 & 2^2 - 7 * 1^2 &= -3 \\ [2,1] &= 3/1 & 3^2 - 7 * 1^2 &= 2 \\ [2,1,1] &= 5/2 & 5^2 - 7 * 2^2 &= -3 \\ [2,1,1,1] &= 8/3 & 8^2 - 7 * 3^2 &= 1 \end{aligned}$$

Therefore $(8, 3)$ is fundamental solution.

Proposition 4.25. *If (x, y) is any solution in integers to $x^2 - dy^2 = 1$, with $x > y > 0$ then $\frac{x}{y}$ is a convergent.*

E.g. If $d = 13$, find the fundamental solution to $x^2 - 13y^2 = 1$

We define sequences α_n, a_n by $\alpha_0 = \sqrt{d}$, $a_n = \lfloor \alpha_n \rfloor$, $\alpha_{n+1} = \dfrac{1}{\alpha_n - a_n} = \dfrac{1}{\alpha_n - \lfloor \alpha_n \rfloor}$

$$\alpha_0 = \sqrt{13} \qquad\qquad a_0 = 3$$

$$\alpha_1 = \frac{1}{\sqrt{13}-3} \qquad\qquad a_1 = 4$$

$$= \frac{\sqrt{13}+3}{(\sqrt{13}+3)(\sqrt{13}-3)}$$

$$= \frac{\sqrt{13}+3}{4}$$

$$\alpha_2 = \frac{1}{\frac{\sqrt{13}+3}{4}-1} \qquad\qquad a_2 = 1$$

$$= \frac{4}{\sqrt{13}-1}$$

$$= \frac{4(\sqrt{13}+1)}{(\sqrt{13}-1)(\sqrt{13}+1)}$$

$$= \frac{4(\sqrt{13}+1)}{12}$$

$$= \frac{\sqrt{13}+1}{3}$$

$$\alpha_3 = \frac{1}{\frac{\sqrt{13}+1}{3}-1} \qquad\qquad a_3 = 1$$

$$= \frac{3}{\sqrt{13}-2}$$

$$= \frac{3(\sqrt{13}+2)}{9}$$

$$= \frac{\sqrt{13}+2}{3}$$

$$\alpha_4 = \cfrac{1}{\cfrac{\sqrt{13}+2}{3} - 1} \qquad a_4 = 1$$

$$= \frac{3}{\sqrt{13}-1}$$

$$= \frac{3(\sqrt{13}+1)}{12}$$

$$= \frac{\sqrt{13}+1}{4}$$

$$\alpha_5 = \cfrac{1}{\cfrac{\sqrt{13}+1}{4} - 1} \qquad a_5 = 6$$

$$= \frac{4}{\sqrt{13}-3}$$

$$= \frac{4(\sqrt{13}+3)}{4}$$

$$= \sqrt{13}+3$$

$$\alpha_6 = \frac{1}{\sqrt{13}-3} = \alpha_1$$

$$\alpha_7 = \alpha_2 \quad etc$$

So $\sqrt{13} = [3, 1, 1, 1, 1, 6, 1, 1, 1, 6, \ldots] = [3, \overline{1, 1, 1, 1, 6}]$

$$\begin{array}{rr}
[3] = 3/1 & 3^2 - 13 * 1^2 = -4 \\
[3,1] = 4/1 & 4^2 - 13 * 1^2 = +3 \\
[3,1,1] = 7/2 & 7^2 - 13 * 2^2 = -3 \\
[3,1,1,1] = 11/3 & 11^2 - 13 * 3^2 = +4 \\
[3,1,1,1,1] = 18/5 & 18^2 - 13 * 5^2 = -1
\end{array}$$

$$
\begin{aligned}
&N(18+5\sqrt{13}) = -1 \\
\implies &N((18+5\sqrt{13})^2) = 1
\end{aligned}
$$

$$
\begin{aligned}
(18+5\sqrt{13})^2 &= 324 + 180\sqrt{13} + 325 \\
&= 649 + 180\sqrt{13}
\end{aligned}
$$

This means that $649^2 - 13 * 180^2 = 1$. If we find a unit of norm -1, before any unit of norm $+1$, then its square will be the fundamental solution.

In general if $\sqrt{d} = [a_0, \overline{a_1, \ldots, a_n, 2a_0}]$ and if $[a_0, \ldots, a_n] = \dfrac{h_n}{k_n}$ then $h_n^2 - dk_n^2 = (-1)^{n+1}$

4.5.1 Convergence of continued fractions

Let $[a_0, a_1, \ldots]$ be a continuous fraction. Then $x_n = [a_0, \ldots, a_n]$ is the n^{th} convergent. Want a formula for numerator and denominator of x_n

$$
x_0 = \frac{a_0}{1} \qquad\qquad x_1 = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1}
$$

Define sequences of integers h_n, k_n by:

$$
\begin{aligned}
h_0 &= a_0 & k_0 &= 1 \\
h_1 &= a_1 a_0 + 1 & k_1 &= a_1 \\
h_n &= a_n h_{n-1} + h_{n-2} & k_n &= a_n k_{n-1} + k_{n-2}
\end{aligned}
$$

Lemma 4.26. $x_n = \dfrac{h_n}{k_n}$

Proof. By induction on n. True for $n = 0, 1$.

$$
\begin{aligned}
x_n &= [a_0, \ldots, a_n] \\
&= \left[a_0, \ldots, a_{n-1} + \frac{1}{a_n}\right] \\
&= \frac{h_{n-1}}{k_{n-1}}
\end{aligned}
$$

By the inductive hypothesis:

$$
\begin{aligned}
h'_{n-1} &= (a_{n-1} + \frac{1}{a_n})h_{n-2} + h_{n-3} \\
k'_{n-1} &= (a_{n-1} + \frac{1}{a_n})k_{n-2} + k_{n-3}
\end{aligned}
$$

This means that:

$$x_n = \frac{(a_{n-1} + \dfrac{1}{a_n})h_{n-2} + h_{n-3}}{(a_{n-1} + \dfrac{1}{a_n})k_{n-2} + k_{n-3}}$$

$$= \frac{a_n a_{n-1} h_{n-2} + h_{n-2} + a_n h_{n-3}}{a_n a_{n-1} k_{n-2} + k_{n-2} + a_n k_{n-3}}$$

$$= \frac{a_n \overbrace{(a_{n-1}h_{n-2} + h_{n-3})}^{h_{n-1}} + h_{n-2}}{a_n \underbrace{(a_{n-1}k_{n-2} + k_{n-3})}_{k_{n-1}} + k_{n-2}}$$

$$x_n = \frac{\overbrace{a_n h_{n-1} + h_{n-2}}^{h_n}}{\underbrace{a_n k_{n-1} + k_{n-2}}_{k_n}}$$

Therefore $x_n = \dfrac{h_n}{k_n}$

Since $k_0 = 1 > 0, \quad k_1 = a_1 > 0, \quad k_n = a_n k_{n-1} + k_{n-2} > k_{n-1}$, the denominators are an increasing sequence of positive integers.

$\square$

Lemma 4.27. *h_n and k_n are coprime and $h_{n+1}k_n - h_n k_{n+1} = (-1)^n$*

Proof. By induction on n. Check in cases $n = 0, 1$. Assume true for $n - 1 > 1$ and prove for n.

$$\begin{aligned} h_{n+1}k_n - h_n k_{n+1} &= (\cancel{a_{n+1}h_n} + h_{n-1})k_n - h_n(\cancel{a_{n+1}k_n} + k_{n-1}) \\ &= -(h_n k_{n-1} - h_{n-1}k_n \\ &= -(-1)^{n-1} \\ &= (-1)^n \end{aligned}$$

$\square$

Theorem 4.28. *The continued fraction $[a_0, \dots]$ converges to a real number α and* $\left|\alpha - \dfrac{h_n}{k_n}\right| < \dfrac{1}{k_n^2}$

Alternating Series Test Suppose y_n is decreasing and $y_n \to 0$. Then $\sum_{n=1}^{\infty}(-1)^n y_n$ converges if $S = \sum_{n=1}^{\infty}(-1)^n y^n$ then S is between $\sum_{n=1}^{N}(-1)^n y_n$ and $\sum_{n=1}^{N+1}(-1)^n y_n$

Proof. Let $x_n = \dfrac{h_n}{k_n}$

$$x_{n+1} - x_n = \frac{h_{n+1}}{k_{n+1}} - \frac{h_n}{k_n} = \frac{h_{n+1}k_n - h_n k_{n+1}}{k_n k_{n+1}} = \frac{(-1)^n}{k_n k_{n+1}}$$

$$\begin{aligned} x_n &= x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \\ &= x_0 + \frac{1}{k_0 k_1} - \frac{1}{k_1 k_2} + \frac{1}{k_2 k_3} - \cdots + \frac{-1}{k_{n-1}k_n} \end{aligned}$$

Therefore x_n converges to some $\alpha \in \mathbb{R}$ by the alternating series test. Also α in between x_n and x_{n+1}

$$|x_n - \alpha| < |x_n - x_{n+1}| \implies \frac{1}{k_n k_{n+1}} < \frac{1}{k_n^2}$$

□

Using the theorem we'll prove:

Theorem 4.29. *For any square-free $d > 1$, Pell's equation has non trivial solutions in integers. Equivalently, every real quadratic ring has non trivial units.*

Proof. $\sqrt{d}$ has a continued fraction expansion. For any convergent $\dfrac{h}{k}$ we have

$$\begin{aligned} \left|\frac{h}{k} - \sqrt{d}\right| &< \frac{1}{k^2} \\ |h - k\sqrt{d}| &< \frac{1}{k} \\ |h^2 - k^2 d| &= |h + k\sqrt{d}| * |h - k\sqrt{d}| \\ &< \left|\frac{h}{k} + \sqrt{d}\right| < 2\sqrt{d} + 1 \end{aligned}$$

This shows that for the convergents $\dfrac{h}{k}$ to $\sqrt{d}$, $h^2 - dk^2$ takes only finitely many values.
There exists n which can be written as $h^2 - dk^2$ in infinitely many ways. The values of h and k mod n have only finitely many possibilities but we have infinitely many pairs (h, k) such that $h^2 - dk^2 = n$
Choose two solutions (h, k),(h', k') where $h = h' \pmod{n}$ and $k = k' \pmod{n}$.

Let $A = \dfrac{h + k\sqrt{d}}{h' + k'\sqrt{d}}$. Claim A is a unit in $\mathbb{Z}[\sqrt{d}]$.

Clearly $N(A) = \dfrac{N(h + k\sqrt{d})}{N(h' + k'\sqrt{d})} = \dfrac{n}{n} = 1$. Remains to show that $A \in \mathbb{Z}[\sqrt{d}]$.

$$A = \frac{h + k\sqrt{d}}{h' + k'\sqrt{d}} = \frac{(h + k\sqrt{d})(h' - k'\sqrt{d})}{h'^2 - dk'^2} = \frac{(hh' - dkk') + (kh' - hk')\sqrt{d}}{n}$$

Recall $h = h' \ \ (n)$ and $k = k' \ \ (n)$.

Therefore

$$\begin{aligned} hh' - dkk' = h^2 - dk^2 = n \equiv 0 \ \ (n) \\ kh' - hk' \equiv kh - hk \equiv 0 \ \ (n) \end{aligned}$$

So $A \in \mathbb{Z}[\sqrt{d}]$ and A is a unit with norm 1 in $\mathbb{Z}[\sqrt{d}]$. $\square$

Theorem 4.30. *Let $\alpha \in \mathbb{R}$ be irrational. If $\frac{a}{b} \in \mathbb{Q}$ with $\left|\frac{a}{b} - \alpha\right| < \frac{1}{2b^2}$ then $\frac{a}{b}$ is a convergent of α*

In order to solve this, we will state and prove the following lemma:

Lemma 4.31. *Let α be an irrational real number $\frac{h_n}{k_n}$ and the n^{th} convergent of α. If $\frac{a}{b}$ is any rational number with $b > 0$ and $b < k_{n+1}$ and $\frac{a}{b}$ is not a convergent then* $|a - b\alpha| > |h_n - k_n\alpha|$

Proof. Consider these simultaneous equations:

$$\begin{aligned} h_n x + h_{n+1} y = a \\ k_n x + k_{n+1} y = b \end{aligned}$$

The matrix $\begin{pmatrix} h_n & h_{n+1} \\ k_n & k_{n+1} \end{pmatrix}$ has determinant ± 1 which means the solutions of x, y are integers $x, y \neq 0$ because $\frac{a}{b} \neq \frac{h_n}{k_n}, \frac{h_{n+1}}{k_{n+1}}$. Plug $x = 0$ or $y = 0$ for a contradiction.

Also x, y have opposite signs because $b < k_{n+1}$ and α is between $\frac{h_n}{k_n}$ and $\frac{h_{n+1}}{k_{n+1}}$.

Therefore $\frac{h_n}{k_n} - \alpha$ and $\frac{h_{n+1}}{k_{n+1}} - \alpha$ have opposite signs.

Therefore $h_n - k_n\alpha$ and $h_{n+1} - k_{n+1}$ have opposite signs.

Therefore $x(h_n - k_n\alpha)$ and $y(h_{n+1} - k_{n+1}\alpha)$ have the same sign.

$$\begin{aligned} |a - b\alpha| &= |(h_n x + h_{n+1} y) - (k_n x + k_{n+1} y)\alpha| \\ &= |x(h_n - k_n\alpha) + y(h_{n+1} - k_{n+1}\alpha)| \\ &= |x| * |h_n - k_n\alpha| + |y| * |h_{n+1} - k_{n+1}\alpha| \\ &> |h_n - k_n\alpha| \end{aligned}$$

$\square$

Proof. Assume $\frac{a}{b}$ is not a convergent to α, choose an a such that $k_n \le b < k_{n+1}$. By the lemma $|h_n - k_n\alpha| < \underbrace{|a - b\alpha|}_{<\frac{1}{2b}} = |b| * \underbrace{\left|\frac{a}{b} - \alpha\right|}_{<\frac{1}{2b^2}}$

This means that $\left|\frac{h_n}{k_n} - \alpha\right| < \frac{1}{2bk_n}$

$$\frac{a}{b} \neq \frac{h_n}{k_n} \implies \left|\frac{a}{b} - \frac{h_n}{k_n}\right| \geq \frac{1}{bk_n}$$

$$\therefore \frac{1}{bk_n} \leq \left|\frac{a}{b} - \frac{h_n}{k_n}\right| = \left|\left(\frac{a}{b} - \alpha\right) + \left(a - \frac{h_n}{k_n}\right)\right| \underbrace{<}_{\frac{1}{bk_n}<\frac{1}{bk_n}\lightning} \frac{1}{2b^2} + \underbrace{\frac{1}{2bk_n}}_{\frac{1}{2bk_n}+\frac{1}{2bk_n}=\frac{1}{bk_n}}$$

So $\frac{a}{b}$ must be a convergent. $\square$