Number Theory

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1 Introduction/Review

1.1 Introduction

Number Theory is the theory of the ring $\mathbb{Z}$ and other related rings. A ring (in this course) is a set R with two binary operations $+$ and $*$ such that:

- $(R, +)$ is an abelian group
- $*$ is associative, commutative and has an identity element 1
- $x(y+z) = xy + xz \quad \forall x, y, z \in \mathbb{R}$

Examples of rings:

- $\mathbb{Z}$ is a ring
- Every field is a ring, (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}$)
- $\mathbb{Z}/n \qquad \mathbb{Z}$ modulo $n = \{0, \ldots, n-1\}$
- $\mathbb{F}[\mathrm{X}] = \{$ polynomials $f(x)$ with coefficients in $\mathbb{F}$

1.2 Review

1.2.1 Congruences

Let n be a positive integer. Given $x, y \in \mathbb{Z}$, we say x is congruentt to y modulo n if $x - y$ is a multiple of n.

$$x \equiv y(n) \qquad \text{or} \qquad x \equiv y \bmod n$$

E.g $\quad 2 \equiv 12 \ (10)$
$\qquad\quad \equiv -8 \ (10)$

We write $\mathbb{Z}/n$ for the ring of congruency classes modulo n, i.e. the elements are integer, with two of them regarded as the same if they are congruent modulo n.

Since every integer is congruent to a unique integer in the set $\{0, \ldots, n-1\}$, we have $\mathbb{Z}/n = \{0, \ldots, n-1\}$.

An element x of $\mathbb{Z}/n$ is called "invertible" or a "unit" if $\exists y \in \mathbb{Z}/n$ such that $xy \equiv 1(n)$.

Theorem 1.1. *x is invertible modulo n iff x and n are coprime*

Recall Two numbers are coprime if their highest common factor is 1.

Here's how we find the inverse of x in $\mathbb{Z}/n$. Since X and n are coprime we can find $h, k \in \mathbb{Z}$ such that $hx + kn = 1 \implies hx = 1 \ (n)$. So h is the inverse of x modulo n.

E.g We'll find the inberse of 7 modulo 25 using Euclid's algorithm

$$\begin{aligned} 25 &= 3 \times 7 + 4 \\ 7 &= 1 \times 4 + 3 \\ 4 &= 1 \times 3 + 1 \end{aligned} \qquad \begin{aligned} 1 &= 4 - 1(3) \\ 1 &= 4 - 1(7 - 1(4)) = 2(4) - 1(7) \\ 1 &= 2(25 - 3(7)) - 1(7) = 2(25) - 7(7) \end{aligned}$$

$$\begin{aligned} 2(25) - 7(7) &= 1 \\ -7(7) &= 1 \ (25) \end{aligned}$$

$$\begin{aligned} (7^{-1}) &= -7 = 18 \ (25) \\ 7 \times 18 &= 126 = 1 \ (25) \end{aligned}$$

We'll write $(\mathbb{Z}/n)^{\times}$ for the invertible elements in $\mathbb{Z}/n$

E.g
$(\mathbb{Z}/3)^{\times} = \{ \cancel{0}, 1, 2 \}$
$(\mathbb{Z}/6)^{\times} = \{ \cancel{0}, 1, \cancel{2}, \cancel{3}, \cancel{4}, 5 \}$

Theorem 1.2. $(\mathbb{Z}/n)^{\times}$ *is a group with the operation of multiplicity.*

1.2.2 Solving Linear Congruences

Suppose we want to solve $ax \equiv b \ (n)$ (given a, b and n).

Case 1: If a is coprime to n then we can find a^{-1} modulo n by Euclid's algorithm, $x \equiv a^{-1}b \ (n)$

Case 2: If a is a factor of n, then there are two possibilities:
- **2a)** if a is also a factor of b then $ax \equiv b \quad (n)$ is equivalent to $x = \frac{b}{a} \quad (\frac{n}{a})$
- **2b)** if a is not a factor of b then there are no solutions

E.g. Solve $5x = 11 \ (13)$
This is case 1 because 5 and 13 are coprime

$$\begin{aligned} 13 &= 2\times 5+3 \\ 5 &= 1\times 3+2 \\ 3 &= 1\times 2+1 \end{aligned} \qquad \begin{aligned} 1 &= (3)-1(2) \\ 1 &= (3)-1(5-1(3)) = 2(3)-(5) \\ 1 &= 2(13-2(5))-(5) = 2(13)-5(5) \end{aligned}$$

$$\begin{aligned} 1 &\equiv -5(5) \quad (13) \\ 5^{-1} &\equiv -5 \equiv 8 \quad (13) \end{aligned}$$

$$\begin{aligned} 5x \equiv& 11 \quad (13) \\ x \equiv 8\times 11 \equiv& 88 \quad (13) \\ \\ x \equiv& 10 \quad (13) \end{aligned}$$

E.g. Solve $7x \equiv 84 \quad (490)$
7 is a factor of 490 so case 2)
7 is a factor of 84 so case 2a)

$7x \equiv 84 \quad (490)$
$x \equiv 12 \quad (70)$

E.g. Solve $7x \equiv 85 \quad (490)$
This is case 2b (7 is a factor of 490 but not of 85) $\therefore$ No solutions

$7x \equiv 85 \quad (490)$
$\implies 7x = 85 + 490y$ for some $y \in \mathbb{Z}$
$\implies 0 \equiv 1 \quad (7)$

E.g. Solve $6x \equiv 3 \quad (21)$
This is neither case 1 nor case 2 but we can rewrite as:

$3(2x) \equiv 3 \quad (21)$

By case 2 we can solve for $2x \equiv 1 \quad (7)$
but now 2 is invertible modulo 7 so now solve by case 1

$\therefore x \equiv 4 \quad (7)$

1.3 Chinese Remainder Theorem

Suppost we know the congruency class of x modulo 10. Then we can work out its congruency class mod 2 and mod 5.
E.g. if $x \equiv 7 \ (10)$, then $x \equiv 1 \ (2)$ and $x \equiv 2 \ (5)$

Then the Chinese Remainder Theorem allows us to do the opposite, i.e. if we know x modulo 2 and modulo 5, then we can work out the value of x modulo 10.

Suppose n & m are coprime positive integers, let $a \in (\mathbb{Z}/n)$ and $b \in (\mathbb{Z}/m)$ then there is a unique

$$x \in (\mathbb{Z}/nm) \text{ such that } \begin{aligned} x &\equiv a \quad (n) \\ x &\equiv b \quad (m) \end{aligned}$$

Proof of existence part:
Since n & m are coprime, we can find $h, k \in \mathbb{Z}$ such that $hn + km = 1$.

Let $x = hnb + kma$

Check that this a solution to both congruences:

$$\begin{aligned} &x \equiv kma \quad (n) \\ &x \equiv (1 - hn)a \quad (n) \\ &x \equiv (1)a \quad (n) \\ &x \equiv a \quad (n) \end{aligned}$$

Similarly, this holds for $x \equiv b \quad (m)$.

E.g. Solve the simultaneous congruence:

$$\begin{aligned} &x \equiv 3 \quad (8) \\ &x \equiv 4 \quad (5) \end{aligned}$$

By the Chinese Remainder Theorem, there is unique solution modulo 40. To find the solution we let $x = hnb + kma$.
First find h, k by Euclid's algorithm.

$$\begin{aligned} 8 &= 1 \times 5 + 3 & 1 &= (3) - 1(2) \\ 5 &= 1 \times 3 + 2 & 1 &= (3) - 1(5 - 1(3)) = 2(3) - (5) \\ 3 &= 1 \times 2 + 1 & 1 &= 2(8 - 2(5)) - (5) = 2(8) - 5(5) \end{aligned}$$

$$\begin{aligned} \therefore x &= (2 * 8 * 4) - (3 * 5 * 3) \\ x &= 64 - 45 \\ \implies x &\equiv 19 \quad (40) \end{aligned}$$

Remark: We can use the Chinese Remainder Theorem to solvoe a congruence modulo nm, by first solving mod n and then mod m and then combining the results.
E.g. Solve $x^2 \equiv 2$ (119). Note 119 = 7 * 17.
By CRT this is equivalent to:

$$\begin{aligned} x^2 &\equiv 2 \quad (7) & &\implies x \equiv \pm 3 \quad (7) \\ x^2 &\equiv 2 \quad (17) & &\implies x \equiv \pm 6 \quad (17) \end{aligned}$$

Now we combine the solutions:

$$\begin{aligned} 17 &= 2 * 7 + 3 & 1 &= (7) - 2(3) \\ 7 &= 2 * 3 + 1 & 1 &= (7) - 2(17 - 2(7)) \\ & & 1 &= 5(7) - 2(17) \end{aligned}$$

Since

$$x \equiv \pm 3 \quad (7) \qquad \text{We get } x \equiv 5*7*(\pm 6) - 2*17*(\pm 3)$$
$$x \equiv \pm 6 \quad (17) \qquad x \equiv \pm 11 \text{ or } \pm 45 \quad (119)$$

1.4 Prime numbers

Defintion 1.3. *An integer $p \geq 2$ is a prime number if the only factors of p are $\pm 1, \pm p$*

We'll write $\mathbb{F}_p$ for $\mathbb{Z}/p$. This is because:

Theorem 1.4. *If p is prime, then $\mathbb{F}_p$ is a field*

Proof. Need to check that the non-zero elements of $\mathbb{F}_p$ all have inverses.

Let $x \in \mathbb{F}_p$ with $x \not\equiv 0 \;\; (p)$ i.e. x is not a multiple of p

$\therefore$ hcf$(x, p) = 1$

$\therefore x$ & p coprime □

1.5 Fermat's Little Theorem

Theorem 1.5. *Let p be a prime number. If x is not a multiple of p then $x^{p-1} \equiv 1 \;\; (p)$*

Proof. $x \in \mathbb{F}_p^x = \{1, 2, \ldots, p-1\}$ a group with $p-1$ elements.
Let n be the order of x in this group.
(order of x is smallest $n > 0$ such that $x^n \equiv 1 \;\; (p)$)
By corollary to Lagrange's Theorem, $p-1$ is a multiple of n

$x^n \equiv 1 \;\; (p)$
$x^{p-1} \equiv 1 \;\; (p)$ □

Theorem 1.6. *Lagrange's Theorem: If H is a subgroup of a finite group G, then $|H|$ is a factor of $|G|$.*

Corollary 1.7. *Order of an element is a factor of $|G|$*

We can use Fermat's Little Theorem to do calculations.
E.g. Calculate 10^{100} modulo 19

By Fermat's Little Theorem: $10^{18} \equiv 1 \;\; (19)$

$$\begin{aligned} 10^{100} &\equiv (10^{18})^5 * 10^{10} \quad (19) \\ &\equiv 100^5 \quad (19) \\ &\equiv 5^5 \quad (19) \\ &\equiv 25 * 125 \equiv 6 * 11 \equiv 9 \quad (19) \end{aligned}$$

Also using Fermat's Little Theorem we can solve congruence of the form $x^a \equiv b \;\; (p)$ as long as p prime and a inverible modulo $p-1$

1.5.1 General method to solve $x^a \equiv b \ \ (p)$

Let

$$c = a^{-1} \ \ (p-1)$$
$$ac = 1 + (p-1)r$$

Raise both sides of the congruence to power c:

$$\therefore x^{ac} \equiv b^c \ \ (p)$$
$$x^{1+(p-1)r} \equiv b^c \ \ (p)$$
$$x \equiv b^c$$

So the solution is $x \equiv b^c \ \ (p)$
E.g. Solve $x^5 \equiv 2 \ \ (19)$

19 is prime and 5 is coprime to 18.

Find $c = 5^{-1}$ mod 18

$$18 = 3 * 5 + 3$$
$$5 = 2 * 3 - 1$$

$$1 = 2 * 3 - 5$$
$$1 = 2(18 - 3 * 5) - 5$$
$$1 = 2 * 18 - 7 * 5$$

$$\therefore 5^{-1} \equiv -7 \ \ (18)$$
$$\equiv 11 \ \ (18)$$

$$\therefore x \equiv 2^{11} \ \ (19)$$
$$\equiv 2048 \ \ (19)$$
$$\equiv 15 \ \ (19)$$

1.6 Fundamental Theorem of Arithmetic

If n is a positive integer then there is a unique factorisation, $n = p_1 p_2 \dots p_r$ with p_i prime. "Unique" means up to reordering the primes $p_1, \dots, p_r$. Showing that a factorisation exists is easy. For the uniqueness part we use:

1.6.1 Euclid's Lemma

Lemma 1.8. *Suppose p prime, and $p|ab$. Then $p|a$ or $p|b$.*

To prove Euclid's lemma we use Bezout's lemma.

Proof. Assume $p|ab$ but $p \nmid a$. Then hcf$(a, p) = 1$

By Bezout's lemma, $\exists h, k$ such that:

$1 = ha + kp$

$b = hab + kpb$ Both hab and kpb are multiples of p.

$\therefore p|b$ ∎

1.6.2 Checking whether a number is prime

If n is composite then the smallest factor of n is (apart from 1) is a prime number $p \leq \sqrt{n}$, i.e. to show that n is prime, we just need to show that none of the primes up to $\sqrt{n}$ are factors of n.

E.g. Is 199 prime?

$\sqrt{199} < 15$ since $15^2 = 225$

The primes up to 15 are $\cancel{2}, \cancel{3}, \cancel{5}, \cancel{7}, \cancel{1}1, \cancel{1}3$

$199 \equiv 3 \quad (7)$

$199 \equiv 4 \quad (13)$

$\therefore$ 199 is prime

Theorem 1.9. *There are infinitely many primes*

Proof. Suppose $p_1, \ldots, p_n$ are all the primes.
Let $N = p_1 \ldots p_n + 1$
$\therefore N$ has no prime factors $\lightning$ $\square$

Similarly there are infinitely many primes $p \equiv 2 \ \ (3)$

Proof. Assume there are only finitely many primes, call them $p_1, p_2, \ldots, p_r$. All other primes are either 3 or are congruent to 1 mod 3.
Let $N = 3p \ldots p_{r-1}$. Since $3 \nmid N$ and $p_i \nmid N$ then all the prime factor of N are congruent to 1 mod 3.
$\therefore N \equiv 1 \ \ (3) \implies$ because clearly $N \equiv 2 \ \ (3)$ $\square$

2 Elementary Number Theory

2.1 Euler Totient Function

Recall $(\mathbb{Z}/n)^\times$ is the group of invertible elements in $\mathbb{Z}/n$.
E.g. $(\mathbb{Z}/6)^\times = \{1, 6\}$
$(\mathbb{Z}/8)^\times = \{1, 3, 5, 7\}$
These are groups with the multiplication operation, * . The multiplication table for $(\mathbb{Z}/8)^\times$ is given below.

*	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Defintion 2.1. *The Euler Totient function is* $\phi(n) = |(\mathbb{Z}/n)^\times|$

E.g. $\phi(6) = 2$
$\phi(8) = 4$

If p prime then $(\mathbb{Z}/p)^\times = \{1, \dots, p-1\}$ so $\phi(p) = p - 1$

Theorem 2.2. *Euler's Theorem- Let* $x \in (\mathbb{Z}/n)^\times$ *then* $x^{\phi(n)} \equiv 1 \pmod{n}$

In the case $n = p$ is prime, this is just Fermat's Little Theorem.

Proof. Let d be the order of x, i.e. $x^d \equiv 1 \pmod{n}$. By a corollary to Lagrange's Theorem, d is a factor of $\phi(n) \implies x^{\phi(n)} \equiv 1 \pmod{n}$ □

We can use Euler's theorem to solve congruences and calculate powers mod n. To use the theorem, we need a quick way of calculating $\phi(n)$.

Lemma 2.3. *Let* $n = p^a$ *where* p *is prime* $a > 0$. *Then* $\phi(n) = (p-1)p^{a-1}$

E.g. $\phi(8) = \phi(2^3) = (2-1)2^{3-1} = 4$

Proof. An integer is coprime to p^a as long as it's not a multiple of p.
$\therefore$ The elements of $\mathbb{Z}/p^a$ which are not invertible are the multiples of p. $0, p, 2p, \dots, p^a - p$.
There are $p^a - 1$ of these:

$\therefore |(\mathbb{Z}/p^a)^\times| = p^a - p^{a-1} = (p-1)p^{a-1}$ □

Theorem 2.4. *Let* n *and* m *be coprime. Then there is an isomorphism:*

$$(\mathbb{Z}/nm)^\times \cong (\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$$

We'll use the theorem before we prove it.
Remark: If G and H are groups, $G \times H = \{(x, y) : x \in G, y \in H\}$, then $G \times H$ is a group with the operation , $(x, y)(x', y') = (xx', yy')$ and $G \times H$ is the "direct product" of G and H

Corollary 2.5. *If n and m are coprime then $\phi(nm) = \phi(n)\phi(m)$*

Proof.

$$\begin{aligned}\phi(nm) = |(\mathbb{Z}/n)^\times| &= |(\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times| \\ &= |(\mathbb{Z}/n)^\times| * |(\mathbb{Z}/m)^\times| \\ &= \phi(n)\phi(m)\end{aligned}$$

□

Corollary 2.6. *(Corollary of the corollary): Suppose $n = p_1^{a_1} \dots p_r^{a_r}$ with $p_1, \dots p_r$ distinct primes and $a_i > 0$. Then*

$$\phi(n) = (p_1 - 1)p^{a_1 - 1} \, * \cdots * \, (p_r - 1)p_r^{a_r - 1}$$

Proof. Since $p_1^{a_1}, \dots, p_r^{a_r}$ are coprime,

$$\begin{aligned}\phi(n) &= \phi(p_1^{a_1}) \dots \phi(p_r^{a_r}) && \text{by the corollary} \\ &= (p_1 - 1)p_1^{a_1 - 1} \dots (p_r - 1)p_r^{a_r - 1} && \text{by the lemma}\end{aligned}$$

□

E.g. Calculate $\phi(200)$

$$\begin{aligned}\phi(200) &= \phi(2^3 * 5^2) \\ &= (2-1)2^{3-1} * (5-1)5^{2-1} \\ &= 4 * 4 * 5 \\ &= 80\end{aligned}$$

Theorem 2.7. *Suppose n and m are coprime, then $(\mathbb{Z}/nm)^\times \cong (\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$*
The isomorphism is the map $x \mapsto (x \mod n, x \mod m)$

E.g. $n = 4$, $m = 5$

$$\begin{aligned}(\mathbb{Z}/4)^\times &= \{1, 3\} \\ (\mathbb{Z}/5)^\times &= \{1, 2, 3, 4\} \\ \therefore (\mathbb{Z}/4)^\times * (\mathbb{Z}/5)^\times &= \{(1,1), (1,2), (1,3), (1,4), \\ &\quad\; (3,1), (3,2), (3,3), (3,4)\} \\ (\mathbb{Z}/20)^\times &= \{1, 3, 7, 9, 11, 13, 17, 19\}\end{aligned}$$

The isomorphism is:

$$\begin{aligned}1 &\mapsto (1,1) & 11 &\mapsto (3,1) \\ 3 &\mapsto (3,3) & 13 &\mapsto (1,3) \\ 7 &\mapsto (3,2) & 17 &\mapsto (1,2) \\ 9 &\mapsto (1,4) & 19 &\mapsto (3,4)\end{aligned}$$

Proof. Let $\Phi : \mathbb{Z}/nm \mapsto \mathbb{Z}/n * \mathbb{Z}/m$

$\Phi(x) = (x \mod n, x \mod m)$

This is a bijection by the Chinese Remainder Theorem.

We'll next show that x is invertible mod $nm \iff x$ is invertible mod n and mod m

$(\implies)$ Suppose x is invertible mod nm

$$\begin{aligned} \text{Let } xy &\equiv 1 \pmod{nm} \\ \therefore xy &\equiv 1 \pmod{n} \\ xy &\equiv 1 \pmod{m} \end{aligned}$$

$\therefore x$ invertible mod n and m

$(\impliedby)$ Suppose x invertible mod n and m

$$\begin{aligned} xa &\equiv 1 \pmod{n} \\ xb &\equiv 1 \pmod{n} \end{aligned}$$

By the Chinese Remainder Theorem, $\exists y$ such that $y \equiv a \pmod{n}$
$y \equiv b \pmod{m}$

$\therefore xy \equiv xa \equiv 1 \pmod{n}$ $\square$