

MATH7701 Number Theory

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Abstract

What did the number theorist say as he drowned?

Log, log, log, log....

For an up to date version of this pdf, check my GitHub :)

<https://github.com/vrvinny/number-theory>

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1 Introduction/Review

1.1 Introduction

Number Theory is the theory of the ring \mathbb{Z} and other related rings. A ring (in this course) is a set R with two binary operations $+$ and $*$ such that:

- $(R, +)$ is an abelian group
- $*$ is associative, commutative and has an identity element 1
- $x(y + z) = xy + xz \quad \forall x, y, z \in R$

Examples of rings:

- \mathbb{Z} is a ring
- Every field is a ring, (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}$)
- \mathbb{Z}/n \mathbb{Z} modulo $n = \{0, \dots, n-1\}$
- $\mathbb{F}[X] = \{ \text{polynomials } f(x) \text{ with coefficients in } \mathbb{F} \}$

1.2 Review

1.2.1 Congruences

Let n be a positive integer. Given $x, y \in \mathbb{Z}$, we say x is congruent to y modulo n if $x - y$ is a multiple of n .

$$x \equiv y(n) \quad \text{or} \quad x \equiv y \pmod{n}$$

E.g $2 \equiv 12 \pmod{10}$
 $\equiv -8 \pmod{10}$

We write \mathbb{Z}/n for the ring of congruency classes modulo n , i.e. the elements are integer, with two of them regarded as the same if they are congruent modulo n .

Since every integer is congruent to a unique integer in the set $\{0, \dots, n-1\}$, we have $\mathbb{Z}/n = \{0, \dots, n-1\}$.

An element x of \mathbb{Z}/n is called "invertible" or a "unit" if $\exists y \in \mathbb{Z}/n$ such that $xy \equiv 1(n)$.

Theorem 1.1. x is invertible modulo n iff x and n are coprime

Recall Two numbers are coprime if their highest common factor is 1.

Here's how we find the inverse of x in \mathbb{Z}/n . Since X and n are coprime we can find $h, k \in \mathbb{Z}$ such that $hx + kn = 1 \implies hx = 1 \pmod{n}$. So h is the inverse of x modulo n .

E.g We'll find the inberse of 7 modulo 25 using Euclid's algorithm

$$25 = 3 \times 7 + 4$$

$$7 = 1 \times 4 + 3$$

$$4 = 1 \times 3 + 1$$

$$1 = 4 - 1(3)$$

$$1 = 4 - 1(7 - 1(4)) = 2(4) - 1(7)$$

$$1 = 2(25 - 3(7)) - 1(7) = 2(25) - 7(7)$$

$$2(25) - 7(7) = 1$$

$$-7(7) = 1 \pmod{25}$$

$$(7^{-1}) = -7 = 18 \pmod{25}$$

$$7 \times 18 = 126 = 1 \pmod{25}$$

We'll write $(\mathbb{Z}/n)^\times$ for the invertible elements in \mathbb{Z}/n

E.g

$$(\mathbb{Z}/3)^\times = \{ \emptyset, 1, 2 \}$$

$$(\mathbb{Z}/6)^\times = \{ \emptyset, 1, \cancel{2}, \cancel{3}, \cancel{4}, 5 \}$$

Theorem 1.2. $(\mathbb{Z}/n)^\times$ is a group with the operation of multiplicity.

1.2.2 Solving Linear Congruences

Suppose we want to solve $ax \equiv b \pmod{n}$ (given a, b and n).

Case 1: If a is coprime to n then we can find a^{-1} modulo n by Euclid's algorithm,
 $x \equiv a^{-1}b \pmod{n}$

Case 2: If a is a factor of n , then there are two possibilities:

2a) if a is also a factor of b then $ax \equiv b \pmod{n}$ is equivalent to $x = \frac{b}{a} \pmod{\frac{n}{a}}$

2b) if a is not a factor of b then there are no solutions

E.g. Solve $5x = 11 \pmod{13}$

This is case 1 because 5 and 13 are coprime

$$13 = 2 \times 5 + 3$$

$$5 = 1 \times 3 + 2$$

$$3 = 1 \times 2 + 1$$

$$1 = (3) - 1(2)$$

$$1 = (3) - 1(5 - 1(3)) = 2(3) - (5)$$

$$1 = 2(13 - 2(5)) - (5) = 2(13) - 5(5)$$

$$1 \equiv -5(5) \pmod{13}$$

$$5^{-1} \equiv -5 \equiv 8 \pmod{13}$$

$$5x \equiv 11 \pmod{13}$$

$$x \equiv 8 \times 11 \equiv 88 \pmod{13}$$

$$x \equiv 10 \pmod{13}$$

E.g. Solve $7x \equiv 84 \pmod{490}$

7 is a factor of 490 so case 2)

7 is a factor of 84 so case 2a)

$$7x \equiv 84 \pmod{490}$$

$$x \equiv 12 \pmod{70}$$

E.g. Solve $7x \equiv 85 \pmod{490}$

This is case 2b (7 is a factor of 490 but not of 85) \therefore No solutions

$$7x \equiv 85 \pmod{490}$$

$$\implies 7x = 85 + 490y \text{ for some } y \in \mathbb{Z}$$

$$\implies 0 \equiv 1 \pmod{7}$$

E.g. Solve $6x \equiv 3 \pmod{21}$

This is neither case 1 nor case 2 but we can rewrite as:

$$3(2x) \equiv 3 \pmod{21}$$

By case 2 we can solve for $2x \equiv 1 \pmod{7}$

but now 2 is invertible modulo 7 so now solve by case 1

$$\therefore x \equiv 4 \pmod{7}$$

1.3 Chinese Remainder Theorem

Suppose we know the congruency class of x modulo 10. Then we can work out its congruency class mod 2 and mod 5.

E.g. if $x \equiv 7 \pmod{10}$, then $x \equiv 1 \pmod{2}$ and $x \equiv 2 \pmod{5}$

Then the Chinese Remainder Theorem allows us to do the opposite, i.e. if we know x modulo 2 and modulo 5, then we can work out the value of x modulo 10.

Suppose n & m are coprime positive integers, let $a \in (\mathbb{Z}/n)$ and $b \in (\mathbb{Z}/m)$ then there is a unique

$$x \in (\mathbb{Z}/nm) \text{ such that } \begin{aligned} x &\equiv a \pmod{n} \\ x &\equiv b \pmod{m} \end{aligned}$$

Proof of existence part:

Since n & m are coprime, we can find $h, k \in \mathbb{Z}$ such that $hn + km = 1$.

Let $x = hnb + kma$

Check that this a solution to both congruences:

$$\begin{aligned} x &\equiv kma \pmod{n} \\ x &\equiv (1 - hn)a \pmod{n} \\ x &\equiv (1)a \pmod{n} \\ x &\equiv a \pmod{n} \end{aligned}$$

Similarly, this holds for $x \equiv b \pmod{m}$.

E.g. Solve the simultaneous congruence:

$$\begin{aligned} x &\equiv 3 \pmod{8} \\ x &\equiv 4 \pmod{5} \end{aligned}$$

By the Chinese Remainder Theorem, there is unique solution modulo 40. To find the solution we let $x = hnb + kma$.

First find h, k by Euclid's algorithm.

$$\begin{aligned} 8 &= 1 \times 5 + 3 & 1 &= (3) - 1(2) \\ 5 &= 1 \times 3 + 2 & 1 &= (3) - 1(5 - 1(3)) = 2(3) - (5) \\ 3 &= 1 \times 2 + 1 & 1 &= 2(8 - 2(5)) - (5) = 2(8) - 5(5) \end{aligned}$$

$$\begin{aligned} \therefore x &= (2 * 8 * 4) - (3 * 5 * 3) \\ x &= 64 - 45 \end{aligned}$$

$$\implies x \equiv 19 \pmod{40}$$

Remark: We can use the Chinese Remainder Theorem to solve a congruence modulo nm , by first solving mod n and then mod m and then combining the results.

E.g. Solve $x^2 \equiv 2 \pmod{119}$. Note $119 = 7 * 17$.

By CRT this is equivalent to:

$$\begin{aligned} x^2 &\equiv 2 \pmod{7} & \implies x &\equiv \pm 3 \pmod{7} \\ x^2 &\equiv 2 \pmod{17} & \implies x &\equiv \pm 6 \pmod{17} \end{aligned}$$

Now we combine the solutions:

$$\begin{aligned} 17 &= 2 * 7 + 3 & 1 &= (7) - 2(3) \\ 7 &= 2 * 3 + 1 & 1 &= (7) - 2(17 - 2(7)) \\ & & 1 &= 5(7) - 2(17) \end{aligned}$$

Since

$$\begin{array}{ll} x \equiv \pm 3 \pmod{7} & \text{We get } x \equiv 5 * 7 * (\pm 6) - 2 * 17 * (\pm 3) \\ x \equiv \pm 6 \pmod{17} & x \equiv \pm 11 \text{ or } \pm 45 \pmod{119} \end{array}$$

1.4 Prime numbers

Definition 1.3. An integer $p \geq 2$ is a prime number if the only factors of p are $\pm 1, \pm p$

We'll write \mathbb{F}_p for \mathbb{Z}/p . This is because:

Theorem 1.4. If p is prime, then \mathbb{F}_p is a field

Proof. Need to check that the non-zero elements of \mathbb{F}_p all have inverses.

Let $x \in \mathbb{F}_p$ with $x \not\equiv 0 \pmod{p}$ i.e. x is not a multiple of p

$$\therefore \text{hcf}(x, p) = 1$$

$\therefore x$ & p coprime □

1.5 Fermat's Little Theorem

Theorem 1.5. Let p be a prime number. If x is not a multiple of p then $x^{p-1} \equiv 1 \pmod{p}$

Proof. $x \in \mathbb{F}_p^\times = \{1, 2, \dots, p-1\}$ a group with $p-1$ elements.

Let n be the order of x in this group.

(order of x is smallest $n > 0$ such that $x^n \equiv 1 \pmod{p}$)

By corollary to Lagrange's Theorem, $p-1$ is a multiple of n

$$\begin{aligned} x^n &\equiv 1 \pmod{p} \\ x^{p-1} &\equiv 1 \pmod{p} \end{aligned} \quad \square$$

Theorem 1.6. Lagrange's Theorem: If H is a subgroup of a finite group G , then $|H|$ is a factor of $|G|$.

Corollary 1.7. Order of an element is a factor of $|G|$

We can use Fermat's Little Theorem to do calculations.

E.g. Calculate 10^{100} modulo 19

By Fermat's Little Theorem: $10^{18} \equiv 1 \pmod{19}$

$$\begin{aligned} 10^{100} &\equiv (10^{18})^5 * 10^{10} \pmod{19} \\ &\equiv 100^5 \pmod{19} \\ &\equiv 5^5 \pmod{19} \\ &\equiv 25 * 125 \equiv 6 * 11 \equiv 9 \pmod{19} \end{aligned}$$

Also using Fermat's Little Theorem we can solve congruence of the form $x^a \equiv b \pmod{p}$ as long as p prime and a invertible modulo $p-1$

1.5.1 General method to solve $x^a \equiv b \pmod{p}$

Let

$$\begin{aligned}c &= a^{-1} \pmod{p-1} \\ac &= 1 + (p-1)r\end{aligned}$$

Raise both sides of the congruence to power c :

$$\begin{aligned}\therefore x^{ac} &\equiv b^c \pmod{p} \\x^{1+(p-1)r} &\equiv b^c \pmod{p} \\x &\equiv b^c\end{aligned}$$

So the solution is $x \equiv b^c \pmod{p}$

E.g. Solve $x^5 \equiv 2 \pmod{19}$

19 is prime and 5 is coprime to 18.

Find $c = 5^{-1} \pmod{18}$

$$\begin{array}{ll}18 = 3 * 5 + 3 & 1 = 2 * 3 - 5 \\5 = 2 * 3 - 1 & 1 = 2(18 - 3 * 5) - 5 \\& 1 = 2 * 18 - 7 * 5\end{array}$$

$$\begin{aligned}\therefore 5^{-1} &\equiv -7 \pmod{18} \\&\equiv 11 \pmod{18}\end{aligned}$$

$$\begin{aligned}\therefore x &\equiv 2^{11} \pmod{19} \\&\equiv 2048 \pmod{19} \\&\equiv 15 \pmod{19}\end{aligned}$$

1.6 Fundamental Theorem of Arithmetic

If n is a positive integer then there is a unique factorisation, $n = p_1 p_2 \dots p_r$ with p_i prime. "Unique" means up to reordering the primes p_1, \dots, p_r . Showing that a factorisation exists is easy. For the uniqueness part we use:

1.6.1 Euclid's Lemma

Lemma 1.8. Suppose p prime, and $p|ab$. Then $p|a$ or $p|b$.

To prove Euclid's lemma we use Bezout's lemma.

Proof. Assume $p|ab$ but $p \nmid a$. Then $\text{hcf}(a, p) = 1$

By Bezout's lemma, $\exists h, k$ such that:

$$1 = ha + kp$$

$$b = hab + kpb$$

Both hab and kpb are multiples of p .

$\therefore p|b$

□

If n is composite then the smallest factor of n is (apart from 1) is a prime number $p \leq \sqrt{n}$, i.e. to show that n is prime, we just need to show that none of the primes up to \sqrt{n} are factors of n .

$$\sqrt{199} < 15 \text{ since } 15^2 = 225$$

The primes up to 15 are ~~2~~, ~~3~~, ~~5~~, ~~7~~, ~~11~~, ~~13~~

$$199 \equiv 3 \pmod{7} \quad (7)$$

$$199 \equiv 4 \pmod{13} \quad (13)$$

$\therefore 199$ is prime

Proof. Suppose p_1, \dots, p_n are all the primes.

$\therefore N$ has no prime factors \nmid

☐

Proof. Assume there are only finitely many primes, call them p_1, p_2, \dots, p_r . All other primes are either 3 or are congruent to 1 mod 3.

$$\therefore N \equiv 1 \pmod{3} \implies \text{because clearly } N \equiv 2 \pmod{3}$$
☐

2 Elementary Number Theory

2.1 Euler Totient Function

Recall $(\mathbb{Z}/n)^\times$ is the group of invertible elements in \mathbb{Z}/n .

E.g. $(\mathbb{Z}/6)^\times = \{1, 5\}$

$(\mathbb{Z}/8)^\times = \{1, 3, 5, 7\}$

These are groups with the multiplication operation, $*$. The multiplication table for $(\mathbb{Z}/8)^\times$ is given below.

$*$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Definition 2.1. The Euler Totient function is $\phi(n) = |(\mathbb{Z}/n)^\times|$

E.g. $\phi(6) = 2$

$\phi(8) = 4$

If p prime then $(\mathbb{Z}/p)^\times = \{1, \dots, p-1\}$ so $\phi(p) = p-1$

Theorem 2.2. Euler's Theorem- Let $x \in (\mathbb{Z}/n)^\times$ then $x^{\phi(n)} \equiv 1 \pmod{n}$

In the case $n = p$ is prime, this is just Fermat's Little Theorem.

Proof. Let d be the order of x , i.e. $x^d \equiv 1 \pmod{n}$. By a corollary to Lagrange's Theorem, d is a factor of $\phi(n) \implies x^{\phi(n)} \equiv 1 \pmod{n}$ \square

We can use Euler's theorem to solve congruences and calculate powers mod n . To use the theorem, we need a quick way of calculating $\phi(n)$.

Lemma 2.3. Let $n = p^a$ where p is prime $a > 0$. Then $\phi(n) = (p-1)p^{a-1}$

E.g. $\phi(8) = \phi(2^3) = (2-1)2^{3-1} = 4$

Proof. An integer is coprime to p^a as long as it's not a multiple of p .

\therefore The elements of \mathbb{Z}/p^a which are not invertible are the multiples of p . $0, p, 2p, \dots, p^a - p$.

There are $p^a - 1$ of these:

$$\therefore |(\mathbb{Z}/p^a)^\times| = p^a - p^{a-1} = (p-1)p^{a-1} \quad \square$$

Theorem 2.4. Let n and m be coprime. Then there is an isomorphism:

$$(\mathbb{Z}/nm)^\times \cong (\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$$

We'll use the theorem before we prove it.

Remark: If G and H are groups, $G \times H = \{(x, y) : x \in G, y \in H\}$, then $G \times H$ is a group with the operation $(x, y)(x', y') = (xx', yy')$ and $G \times H$ is the "direct product" of G and H

Corollary 2.5. *If n and m are coprime then $\phi(nm) = \phi(n)\phi(m)$*

Proof.

$$\begin{aligned}\phi(nm) &= |(\mathbb{Z}/nm)^\times| = |(\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times| \\ &= |(\mathbb{Z}/n)^\times| * |(\mathbb{Z}/m)^\times| \\ &= \phi(n)\phi(m)\end{aligned}$$

□

Corollary 2.6. *(Corollary of the corollary): Suppose $n = p_1^{a_1} \dots p_r^{a_r}$ with p_1, \dots, p_r distinct primes and $a_i > 0$. Then*

$$\phi(n) = (p_1 - 1)p_1^{a_1-1} * \dots * (p_r - 1)p_r^{a_r-1}$$

Proof. Since $p_1^{a_1}, \dots, p_r^{a_r}$ are coprime,

$$\begin{aligned}\phi(n) &= \phi(p_1^{a_1}) \dots \phi(p_r^{a_r}) && \text{by the corollary} \\ &= (p_1 - 1)p_1^{a_1-1} \dots (p_r - 1)p_r^{a_r-1} && \text{by the lemma}\end{aligned}$$

□

E.g. Calculate $\phi(200)$

$$\begin{aligned}\phi(200) &= \phi(2^3 * 5^2) \\ &= (2 - 1)2^{3-1} * (5 - 1)5^{2-1} \\ &= 4 * 4 * 5 \\ &= 80\end{aligned}$$

Theorem 2.7. *Suppose n and m are coprime, then $(\mathbb{Z}/nm)^\times \cong (\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$. The isomorphism is the map $x \mapsto (x \bmod n, x \bmod m)$*

E.g. $n = 4, m = 5$

$$\begin{aligned}(\mathbb{Z}/4)^\times &= \{1, 3\} \\ (\mathbb{Z}/5)^\times &= \{1, 2, 3, 4\} \\ \therefore (\mathbb{Z}/4)^\times * (\mathbb{Z}/5)^\times &= \{(1, 1), (1, 2), (1, 3), (1, 4), \\ &\quad (3, 1), (3, 2), (3, 3), (3, 4)\} \\ (\mathbb{Z}/20)^\times &= \{1, 3, 7, 9, 11, 13, 17, 19\}\end{aligned}$$

The isomorphism is:

$$\begin{array}{ll} 1 \mapsto (1, 1) & 11 \mapsto (3, 1) \\ 3 \mapsto (3, 3) & 13 \mapsto (1, 3) \\ 7 \mapsto (3, 2) & 17 \mapsto (1, 2) \\ 9 \mapsto (1, 4) & 19 \mapsto (3, 4) \end{array}$$

Proof. Let $\Phi : \mathbb{Z}/nm \mapsto \mathbb{Z}/n * \mathbb{Z}/m$

$$\Phi(x) = (x \bmod n, x \bmod m)$$

This is a bijection by the Chinese Remainder Theorem.

We'll next show that x is invertible mod $nm \iff x$ is invertible mod n and mod m

(\implies) Suppose x is invertible mod nm

$$\text{Let } xy \equiv 1 \pmod{nm}$$

$$\therefore xy \equiv 1 \pmod{n}$$

$$xy \equiv 1 \pmod{m}$$

$$\therefore x \text{ invertible mod } n \text{ and } m$$

(\impliedby) Suppose x invertible mod n and m

$$xa \equiv 1 \pmod{n}$$

$$xb \equiv 1 \pmod{m}$$

By the Chinese Remainder Theorem, $\exists y$ such that $y \equiv a \pmod{n}$

$$y \equiv b \pmod{m}$$

$$\left. \begin{array}{l} \therefore xy \equiv xa \equiv 1 \pmod{n} \\ \equiv xb \equiv 1 \pmod{m} \end{array} \right\} \implies xy \equiv 1 \pmod{nm} \text{ by the Chinese Remainder Theorem}$$

We've shown that Φ gives a bijection between $(\mathbb{Z}/nm)^\times$ and $(\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$. We'll next check that $\Phi(xy) = \Phi(x)\Phi(y)$.

$$\begin{aligned} \Phi(xy) &= (xy \bmod n, xy \bmod m) \\ &= (x \bmod n, x \bmod m) * (y \bmod n, y \bmod m) \\ &= \Phi(x)\Phi(y) \end{aligned}$$

□

2.2 Euler's Theorem

If $x \in (\mathbb{Z}/n)^\times$ then $x^{\phi(n)} \equiv 1 \pmod{n}$ and $\phi(p_1^{a_1} \dots p_r^{a_r}) = (p_1 - 1)p_1^{a_1-1} \dots (p_r - 1)p_r^{a_r-1}$

E.g. Calculate $7^{135246872002} \bmod 10000$

$$7 \text{ coprime to } 10000 \text{ so } 7^{\phi(10000)} \equiv 1 \pmod{10000}$$

$$10000 = 2^4 * 5^4$$

$$\therefore \phi(10000) = (2-1)2^3 * (5-1) * 5^3 = 8 * 500$$

$$7^{4000} \equiv 1 \pmod{10000} \implies 7^n \text{ depends only on } n \bmod 4000$$

$$135246872002 \equiv 2 \pmod{4000}$$

$$\therefore 7^{135246872002} \equiv 7^2 \equiv 49 \pmod{10000}$$

We can also use Euler's THEorem to solve congruence with powers

2.2.1 Solving equations of the form $x^a \equiv b \pmod{n}$

Suppose we want to solve $x^a \equiv b \pmod{n}$ where b is coprime to n and a is coprime to $\phi(n)$.

Clearly any solution x must be coprime to n by Euler's Theorem $x^{\phi(n)} \equiv 1 \pmod{n}$.

\therefore The congruency class of $x^y \pmod{n}$ depends only $y \pmod{\phi(n)}$

Let

$$c = a^{-1} \pmod{\phi(n)}$$

Raise both sides of the congruence to power c :

$$x^{ac} \equiv x^1 \equiv b^c \pmod{n}$$

\therefore The solution is $x \equiv b^c \pmod{n}$

E.g. $x^7 \equiv 3 \pmod{50}$

3 is coprime to 50,

$$\begin{aligned} 50 &= 2 * 5^2 \\ \implies \phi(50) &= 1 * 4 * 5 = 20 \end{aligned}$$

7 is coprime to $\phi(50)$. To solve, we need to find

$$\begin{aligned} c &\equiv 7^{-1} \pmod{\phi(50)} \\ &\equiv 3 \pmod{20} \end{aligned}$$

$$x \equiv 3^3 \equiv 27 \pmod{50}$$

E.g. $x^{27} \equiv 5 \pmod{123}$

5 is coprime to 123,

$$\begin{aligned} 123 &= 3 * 41 \\ \implies \phi(123) &= 2 * 40 = 80 \end{aligned}$$

27 is coprime to 80

To solve, we find $27^{-1} \pmod{80}$

$$\begin{aligned} 80 &= 3 * 27 - 1 \\ \implies 1 &= 3 * 27 - 80 \end{aligned}$$

$$27^{-1} = 3$$

$$\begin{aligned} x &= 5^3 \\ x &= 125 \equiv 2 \pmod{123} \end{aligned}$$

2.3 Primitive roots

Recall, let G be a finite group. G is called a cyclic group if $\exists x \in G$ such that, every element in G has the form x^n for some $n \in \mathbb{Z}$, i.e. $G = \{1, x, x^2, \dots, x^{n-1}\}$ where n is the order of x , equivalentl the order of x is $|G|$. The element x is called a generator of G .

Theorem 2.8. (Gauss' Theorem), For ever prime number p , the group \mathbb{F}_p^\times is cyclic

Definition 2.9. A generator of \mathbb{F}_p^\times is called a primitive root. Equivalently, this is an element of order $p - 1$

E.g. $p = 7, x = 3$ We'll see that 3 is a primitive root modulo 7

$$\begin{array}{llll} \text{Powers of 3 in } \mathbb{F}_7^\times : & 3^0 = 1 & 3^3 \equiv 6 \pmod{7} & 3^6 \equiv 1 \pmod{7} \\ & 3^1 = 3 & 3^4 \equiv 4 \pmod{7} & \\ & 3^2 \equiv 2 \pmod{7} & 3^5 \equiv 1 \pmod{7} & \end{array}$$

so 3 is a primitive root modulo 7. There is a quicker way to check whether x is a primitive root.

Proposition 2.10. Let $x \in \mathbb{F}_p^\times$, then x is a primitive root modulo p if and only if for every prime factor q of $p - 1$:

$$x^{\frac{p-1}{q}} \not\equiv 1 \pmod{p}$$

Proof. Assume the second statement is false, so \exists prime factor q of $p - 1$ such that:

$$\begin{array}{ll} x^{\frac{p-1}{q}} \equiv 1 \pmod{p} & \therefore \text{order of } x \leq \frac{p-1}{q} < p-1 \\ & \therefore x \text{ is not a primitive root} \end{array}$$

Conversely, assume x is not a primitive root, so x doe not have order $p - 1$. But the order of x is a factor of $p - 1$.

Suppose the order of x is $\frac{p-1}{d}$, $d > 1$.

Let q be a prime factor of $d \implies q|p-1$

$$\frac{p-1}{q} \text{ is a multiple of } \frac{p-1}{d} \text{ but } x^{\frac{p-1}{q}} \equiv 1 \pmod{p} \implies x^{\frac{p-1}{d}} \equiv 1 \pmod{p}$$

□

E.g. $p = 29$

By the proposition x is a primitive root mod 29 $\iff x^{28/2} \not\equiv 1 \pmod{29}$ and $x^{28/7} \not\equiv 1 \pmod{29}$

$$\iff x^{14} \not\equiv 1 \pmod{29} \text{ and } x^4 \not\equiv 1 \pmod{29}$$

$$\begin{array}{ll} \text{Try } x = 2 : & 2^4 \equiv 16 \not\equiv 1 \pmod{29} \\ & 2^{14} \equiv 128^2 \equiv 12^2 \equiv 144 \equiv -1 \pmod{29} \end{array}$$

$\therefore 2$ is a primitive root mod 29

Another trick to speed up the calculation:

\mathbb{F}_p is a field \therefore every polynomial of d has no more than d in \mathbb{F} (proved in 2201).

\therefore if $x^2 \equiv 1 \pmod{p}$ then $x \equiv \pm 1 \pmod{p}$

This means that checking whether $x^{14} \equiv 1 \pmod{29}$ is equivalent to checking whether $x^7 \equiv \pm 1 \pmod{29}$.

E.g 3 is also a primitive root modulo 29

$$3^2 \equiv 9 \not\equiv \pm 1 \pmod{29}$$

$$3^4 \equiv 1 \pmod{29}$$

$$3^7 \equiv 27^2 * 3 \pmod{29}$$

$$\equiv (-2)^2 * 3 \equiv 12 \pmod{29}$$

$$\equiv \pm 1 \pmod{29}$$

$$\therefore 3^{14} \not\equiv 1 \pmod{29}$$

2.4 Roots of unity and Cyclotomic Polynomials

A complex number ζ is called an n^{th} root of unity if $\zeta^n = 1$. The n^{th} roots of unity are $e^{2\pi i \frac{a}{n}}$ for $a = \{0, 1, \dots, n-1\}$

We call ζ a primitive n^{th} root of unity if n smaller power than ζ^n is equal to 1, i.e. ζ has order n in \mathbb{C}^\times if ζ is not a primitive n^{th} root of unity $\zeta = e^{2\pi i \frac{b}{d}}$ where $b = \{0, \dots, d-1\}$ for $d < n$

$$\therefore \frac{a}{n} = \frac{b}{d}$$

The cancellation happens when a is not coprime to n . This shows that the primitive n^{th} of unity are $e^{2\pi i \frac{a}{n}}$, $a \in (\mathbb{Z}/n)^\times$.

Corollary 2.11. *There are exactly $\phi(n)$ primitive n^{th} roots of unity*

We'll actually prove a more precise version of Gauss' Theorem.

Theorem 2.12. *For every factor d of $p-1$ there are $\phi(d)$ elements in \mathbb{F}_p^\times of order d .*

Definition 2.13. *The n^{th} cyclotomic polynomial is:*

$$\Phi_n(x) = \prod_{\substack{\text{primitive} \\ n^{th} \text{ roots} \\ \text{of unity } \zeta}} (X - \zeta)$$

i.e $\zeta^n = 1$ and no smaller power of ζ is 1, $\zeta = e^{2\pi i \frac{a}{n}}$, $a \in (\mathbb{Z}/n)^\times$

This has degree $\phi(n)$.

E.g. $n=4$

Primitive 4^{th} roots of unity are $i, -i$:

$$\begin{aligned}\Phi_4(x) &= (x - i)(x - (-i)) \\ &= x^2 + 1\end{aligned}$$

Lemma 2.14. For every $n > 0$:

$$x^n - 1 = \prod_{\substack{d \text{ factors} \\ d \text{ of } n}} \Phi_d(x)$$

E.g. Calculate $\Phi_6(x)$

$$\begin{aligned}\text{By the lemma} \quad x^6 - 1 &= \Phi_1 \Phi_2 \Phi_3 \Phi_6 & x^6 - 1 &= (x^3 - 1) \Phi_2 \Phi_6 \\ x^3 - 1 &= \Phi_1 \Phi_3\end{aligned}$$

$$\therefore \Phi_6 = \frac{x^6 - 1}{(x^3 - 1)(x + 1)} = \frac{x^3 + 1}{x + 1} = x^2 - x + 1$$

Let p be a prime number. A primitive root mod p is an $x \in \mathbb{F}_p^\times$, such that x generates \mathbb{F}_p^\times .
Equivalently order = $p - 1$

2.4.1 How to calculate $\Phi_n(x)$

Lemma 2.15. $x^n - 1 = \prod_{d|n} \Phi_d(x)$

E.g. $n = 4$

$$\begin{aligned}x^4 - 1 &= \Phi_1 \Phi_2 \Phi_4 & \Phi_1 &= x - 1 \\ & & \Phi_2 &= (x - (-1)) = x + 1 \\ & & \Phi_4 &= (x - i)(x - (-i)) = x^2 + 1 \\ &= (x - 1)(x + 1)(x^2 + 1)\end{aligned}$$

Proof.

$$x^n - 1 = \prod_{\substack{\zeta \text{ is an} \\ n^{th} \text{ root of} \\ \text{unity}}} (x - \zeta)$$

but every n^{th} root of unity is a primitive d^{th} root of unity for some $d|n$.

$$x^n = \prod_{d|n} (\prod_{\substack{\text{primitive} \\ d^{th} \text{ roots} \\ \text{of unity}}} (x - \zeta)) = \prod_{d|n} \Phi_d(x)$$

□

E.g. Calculate $\Phi_5(x)$

$$\begin{aligned} x^5 - 1 &= \Phi_1(x)\Phi_5(x) \\ &= (x - 1)\Phi_5(x) \end{aligned}$$

$$\Phi_5(x) = \frac{x^5 - 1}{x - 1} = 1 + x + x^2 + x^3 + x^4$$

More generally if p prime then $x^p - 1 = (x - 1)\Phi_p(x) \implies \Phi_p(x) = 1 + x + \dots + x^{p-1}$

E.g. Calculate $\Phi_8(x)$

$$x^8 - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x)\Phi_8(x)$$

$$x^4 - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x) \implies \Phi_8(x) = \frac{x^8 - 1}{x^4 - 1} = x^4 + 1$$

Corollary 2.16. $\Phi_n(x)$ has coefficients in \mathbb{Z}

$$\text{Proof. } \Phi_n(x) = \frac{x^n - 1}{\prod_{\substack{d|n \\ d \neq n}} \Phi_d(x)}$$

We'll prove the corollary by induction on n , clearly true when $n = 1$. Assume Φ_d has integer coefficients $\forall d < n$.

It is proved in Algebra 3 (MATH2201) that, if $f, g \in \mathbb{Z}[X]$ and g monic then $f = qg + r$ where $\deg(r) < \deg(g)$ and $g, r \in \mathbb{Z}[x]$.

Using this, we get that the denominator $\prod_{\substack{d|n \\ d \neq n}} \Phi_d(x)$ is a monic polynomial with coefficients in $\mathbb{Z} \implies \Phi_n \in \mathbb{Z}[X]$. □

2.4.2 Gauss' Theorem

Theorem 2.17. Let n be a factor of $p - 1$, where p is prime. Then there are exactly $\phi(n)$ elements of order n in \mathbb{F}_p^\times . These are the roots of Φ in \mathbb{F}_p^\times . In particular there are $\phi(p - 1)$ primitive roots.

Proof. Let $f(x) = x^{p-1} - 1$

By Fermat's Little theorem, $f(x) = 0 \pmod{p}$ for $x = 1, \dots, p - 1$ for $(x \neq 0)$

$$\begin{aligned} \therefore f(x) &= (x - 1)(x - 2) \dots (x - (p - 1)) \\ &= \prod_{n|p-1} \Phi_n(x) \end{aligned}$$

This implies that:

- Each Φ_n (for $n|p - 1$) factorises completely into linear factors with no repeated roots $\therefore \Phi_n$ has $\phi(n)$ roots in \mathbb{F}_p
- Every element of \mathbb{F}_p^\times is a root of exactly one of the polynomials Φ_n with $n|p - 1$

It remains to show that the roots of $\Phi_n(x)$ in \mathbb{F}_p has order of exactly n .
 Suppose $\Phi_n(x) \equiv 0 \pmod{p}$

By the lemma $\Phi_n(x)$ is a factor $x^n - 1$
 $\therefore x^n - 1 \equiv 0 \pmod{p}$
 $\therefore x^n \equiv 1 \pmod{p}$

Suppose $x^m \equiv 1 \pmod{p}$ for some $m|n, m < n$
 $\implies x^m - 1 \equiv 0 \pmod{p}$

By the lemma $\Pi_{d|m} \Phi_d(x) \equiv 0 \pmod{p}$
 $\implies \Phi_d(x) \equiv 0 \pmod{p}$ for some $d \nmid n$

We already know that x is only a root of 1 of the cyclotomic polynomials, therefore x has order n . \square

2.5 Quadratic reciprocity (Quadratic equations modulo prime numbers)

Recall we can solve $x^a \equiv b \pmod{p}$ as long as a is coprime to $p - 1$. This won't work if $a = 2$ because a will not be invertible mod $p - 1$. An easier question to ask is, which quadratic equations have solutions modulo p ?

E.g. Does $x^2 \equiv 37 \pmod{149}$ have solutions?

Notation: We always let p be an odd prime (i.e. $p \neq 2$)

An element $a \in \mathbb{F}_p^\times$ is a quadratic residue if $x^2 \equiv a \pmod{p}$ has solutions.

An element $a \in \mathbb{F}_p^\times$ is a quadratic non-residue if there are no solutions.

The quadratic residue symbol is defined for $a \in \mathbb{F}_p^\times$ by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{a quadratic residue} \\ -1 & \text{a quadratic non-residue} \end{cases}$$

Lemma 2.18. *Let g be a primitive root modulo p (p odd prime). Then g^r is a quadratic residue iff r even.*

Proof.

(\Leftarrow) Assume r even

Clearly g^r is a square in \mathbb{F}_p^\times

So g^r is a quadratic residue

(\Rightarrow) Assume $g^r \equiv x^2 \pmod{p}$

$x \equiv g^s \pmod{p}$ ($s \in \mathbb{Z}$) since g primitive roots

$\therefore g^r \equiv g^{2s} \pmod{p}$

$g^{r-2s} \equiv 1 \pmod{p}$

g has order $p - 1$, so $r - 2s$ is a multiple of $p - 1$

p odd $\implies p - 1$ is even $\implies r$ is even

\square

E.g. $p = 7$

x	$x^2 \pmod{7}$		a	$\left(\frac{a}{7}\right)$
± 1	1	\implies	1	1
± 2	4		2	1
± 3	2		3	-1
			4	1
			5	-1
			6	-1

So 1,2,4 are quadratic residues; 3,4,6 are quadratic non-residues

Corollary 2.19. *There are exactly $\frac{p-1}{2}$ quadratic residues and $\frac{p-1}{2}$ quadratic non-residues mod p*

Definition 2.20. *Euler's criterion: Let p be an odd prime and $a \in \mathbb{F}_p^\times \implies \left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$*
Also $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

Proof. $(a^{\frac{p-1}{2}})^2 \equiv 1 \pmod{p}$ by Fermat's Little theorem.

$$\therefore a^{\frac{p-1}{2}} \equiv \pm 1 \pmod{p}$$

Let $a = g^r$ where g is a primitive root $\implies a^{\frac{p-1}{2}} \equiv g^{(p-1)\frac{r}{2}}$

$$\begin{aligned} a \text{ is a quadratic residue} &\iff r \text{ is even} \\ &\iff (p-1)\frac{r}{2} \text{ is a multiple of } p-1 \\ &\iff g^{(p-1)\frac{r}{2}} \equiv 1 \pmod{p} \\ &\iff a^{\frac{p-1}{2}} \equiv 1 \pmod{p} \end{aligned}$$

□

To calculate $\left(\frac{a}{p}\right)$, we'll use three theorems:

2.5.1 Quadratic Reciprocity Law

Let p, q be distinct odd prime numbers. Then $\left(\frac{p}{q}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}$

$$\text{i.e. } \left(\frac{p}{q}\right) = \begin{cases} \left(\frac{q}{p}\right) & \text{if } p \equiv 1 \pmod{4} \text{ or } q \equiv 1 \pmod{4} \\ -\left(\frac{q}{p}\right) & \text{if } p \equiv q \equiv -1 \pmod{4} \end{cases}$$

2.5.2 First Nebensatz

If p is an odd prime, then $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$

$$\text{i.e. } \left(\frac{-1}{p}\right) = \begin{cases} 1 & p \equiv 1 \pmod{4} \\ -1 & p \equiv -1 \pmod{4} \end{cases}$$

2.5.3 Second Nebensatz

Let p be an odd prime, then $(\frac{2}{p}) = (-1)^{\frac{p^2-1}{8}}$

$$\text{i.e. } (\frac{2}{p}) = \begin{cases} 1 & p \equiv \pm 1 \pmod{8} \\ -1 & p \equiv \pm 3 \pmod{8} \end{cases}$$

We'll prove the theorems later.

E.g. Does the congruence $x^2 \equiv 37 \pmod{199}$ have solutions?

$$\begin{aligned} 199 \text{ is an odd prime } (\frac{37}{199}) &= +(\frac{199}{37}) && \text{by quadratic reciprocity} \\ &\equiv (\frac{14}{37}) && \text{because } 199 \equiv 14 \pmod{37} \\ &\equiv (\frac{2}{37})(\frac{7}{37}) && \text{by the corollary} \\ &\equiv (-1)(\frac{7}{37}) && \text{by the 2}^{nd} \text{ Nebensatz} \\ &\equiv (-1)(+1)(\frac{37}{7}) && \text{by the quadratic reciprocity law} \\ &\equiv -(\frac{2}{7}) && \text{because } 37 \equiv 2 \pmod{7} \\ &\equiv -(+1) && \text{by the 2}^{nd} \text{ Nebensatz} \\ &\equiv -1 && \therefore x^2 \equiv 37 \pmod{199} \text{ has no solutions} \end{aligned}$$

E.g. $x^2 \equiv 47 \pmod{53}$ have solutions?

$$(\frac{47}{53}) = +(\frac{53}{47}) = (\frac{6}{47}) = (\frac{2}{47})(\frac{3}{47}) = (+1)(-1)(\frac{47}{3}) = -(-\frac{1}{3}) = -(-1) = +1$$

This shows that 47 is a quadratic residue mod 53, so $x^2 \equiv 47 \pmod{53}$ does have solutions. ($x = 10$)

We can speed up the test for primitive roots using quadratic reciprocity,

$$x \text{ is a primitive root mod } p \iff \forall q|p-1, q \text{ prime } x^{\frac{p-1}{q}} \not\equiv 1 \pmod{p}$$

This means we need to calculate $x^{\frac{p-1}{q}} \pmod{p}$ for primes $q|p-1$, the biggest power of x to calculate is $x^{\frac{p-1}{2}}$. But we can calculate this, because it is $(\frac{x}{p})$ by Euler's criterion.

E.g. Is 35 a primitive root modulo 83?

The primes q dividing 82 are 2, 41, need to check $35^2, 35^{41}$
 $35^2 \not\equiv 1 \pmod{83}$ because $35 \not\equiv \pm 1 \pmod{83}$, a quadratic equation cannot have more than 2 roots.
 $35^{41} \equiv (\frac{35}{83}) \pmod{83} = (\frac{5}{83})(\frac{7}{83}) = (\frac{83}{5})(-1)(\frac{83}{7}) = (\frac{3}{5})(-1)(\frac{-1}{7}) = (\frac{5}{3})(-1)(-1) = (\frac{2}{3})$
 $= -1 \not\equiv 1 \pmod{83}$

So 35 is a primitive root modulo 83.

Proof. First Nebensatz:

By Euler's criterion, $\left(\frac{-1}{p}\right) \equiv (-1)^{\frac{p-1}{2}} \pmod{p}$.

Both sides are ± 1 , and $+1 \not\equiv -1 \pmod{p}$ because $p \geq 3 \implies$ they are equal. \square

E.g. Find the first primitive root modulo 41

$$40 = 2^3 * 5$$

$$x \in \mathbb{F}_{41}^\times \text{ is a primitive root} \iff \begin{cases} x^{\frac{40}{2}} \not\equiv 1 \pmod{41} \\ x^{\frac{40}{5}} \not\equiv 1 \pmod{41} \end{cases}$$

$$\text{We can then simplify the conditions to: } \begin{cases} \frac{x}{41} = -1 \\ x^4 \not\equiv \pm 1 \pmod{41} \end{cases}$$

$$\text{Try } x = 2 : \left(\frac{2}{41}\right) = 1 \implies \text{not a primitive root}$$

$$\text{Try } x = 3 : \left(\frac{3}{41}\right) = \left(\frac{41}{3}\right) = \left(\frac{2}{3}\right) = -1 \quad \text{and } 3^4 = 81 \equiv -1 \pmod{41} \implies \text{not a primitive root}$$

$$\text{Try } x = 4 : \implies \text{not a primitive root}$$

$$\text{Try } x = 5 : \left(\frac{5}{41}\right) = \left(\frac{41}{5}\right) = \left(\frac{1}{5}\right) = 1 \implies \text{not a primitive root}$$

$$\begin{aligned} \text{Try } x = 6 : \left(\frac{6}{41}\right) &= \left(\frac{2}{41}\right)\left(\frac{3}{41}\right) = 1 * -1 = -1 \\ 2^4 * 3^4 &= -2^4 \equiv 16 \pmod{41} \not\equiv \pm 1 \implies \text{so 6 is a primitive root} \end{aligned}$$

E.g. For which primes p does the congruence $x^2 \equiv -3 \pmod{p}$ have solutions?

Notice $x = 1$ is a solution mod 2,

$x = 2$ is a solution mod 3.

For primes $p \neq 2, 3$ it depends on $\left(\frac{-3}{p}\right)$

$$\begin{aligned} \text{We'll calculate } \left(\frac{-3}{p}\right) &= \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) \\ &= (-1)^{\frac{p-1}{2}} \left(\frac{3}{p}\right) \\ &= (-1)^{\frac{(3-1)(p-1)}{4}} \left(\frac{p}{3}\right) \\ &= \left(\frac{p}{3}\right) \end{aligned}$$

List the squares mod 3, $1^2 = 1 \pmod{3}, 2^2 = 1 \pmod{3}$

$$\therefore \left(\frac{p}{3}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{3} \\ -1 & \text{if } p \equiv 2 \pmod{3} \end{cases}$$

We've shown that $x^2 \equiv -3 \pmod{p}$ has solutions iff $p \neq 2$ or $p \equiv 1 \pmod{3}$.

Corollary 2.21. *There are infinitely many primes $p \equiv 1 \pmod{3}$*

Proof. Assume there are only finitely many, and call them p_1, p_2, \dots, p_r
Let $N = n^2 + 3$ where $n = 2p_1 \dots p_r$
Take a prime factor q of N

$$N \equiv 0 \pmod{q}$$

$$n^2 + 3 \equiv 0 \pmod{q}$$

$$n^2 \equiv -3 \pmod{q}$$

We've just shown that this implies $q = 2$ or 3 or $q \equiv 1 \pmod{3}$ but $q \neq 2, 3, q \not\equiv 1 \pmod{3}$ \nmid □

Before we prove the 2^{nd} Nebensatz, we need to know about a new ring.

Let $\zeta = e^{\frac{2\pi i}{8}}$, a primitive 8^{th} root of unity.

We'll use the ring $\mathbb{Z}[\zeta] = \{f(\zeta) : f \in \mathbb{Z}\} = \{a_0 + a_1\zeta + a_2\zeta^2 + \dots + a_n\zeta^n : a_i \in \mathbb{Z}\}$

This is clearly a ring (closed under $+, *$).

2.6 Uniqueness Lemma

Every $A \in \mathbb{Z}[\zeta]$ can be written uniquely as $A = W + x\zeta + y\zeta^2 + z\zeta^3$ with $w, x, y, z \in \mathbb{Z}$.

We'll use congruence modulo p in the ring $\mathbb{Z}[\zeta]$ to prove the 2^{nd} Nebensatz.

Definition 2.22. Let $A, B \in \mathbb{Z}[\zeta]$

We'll say $A \equiv B \pmod{p\mathbb{Z}[\zeta]}$ if $A - B = pC$ for some $C \in \mathbb{Z}[\zeta]$

$$\text{Suppose } A = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3$$

$$B = b_0 + b_1\zeta + b_2\zeta^2 + b_3\zeta^3$$

$$C = c_0 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3$$

The equation $A - B = pC$ is equivalent (by uniqueness lemma) to:

$$a_0 - b_0 = pC_0,$$

$$a_1 - b_1 = pC_1,$$

$$a_2 - b_2 = pC_2,$$

$$a_3 - b_3 = pC_3,$$

This implies that the congruence $A \equiv B \pmod{p\mathbb{Z}[\zeta]}$ is equivalent to $a_i \equiv b_i \pmod{p}$ for $i = 0, 1, 2, 3$

Corollary 2.23. $1 \not\equiv -1 \pmod{p\mathbb{Z}[\zeta]}$ if p is an odd prime.

This means that to calculate $\left(\frac{2}{p}\right)$ it is enough to calculate its congruency class mod $(p\mathbb{Z}[\zeta])$

The uniqueness lemma is implied by a more general result:

2.6.1 General Uniqueness Lemma

Let $m \in \mathbb{Z}[X]$ be monic and irreducible over \mathbb{Q} of degree d . If $\alpha \in \mathbb{C}$ is a root of m , then every element of $\mathbb{Z}[\alpha]$ can be written uniquely as $a_0 + a_1\alpha + \dots + a_{d-1}\alpha^{d-1}$ with $a_i \in \mathbb{Z}$.

The uniqueness lemma for $\mathbb{Z}[\zeta]$ follows because ζ is a root of $m(x) = \Phi_8(x) = x^4 + 1$. It is proved in (7202 Groups & Rings) that $x^4 + 1$ is irreducible over \mathbb{Q} .

Proof. (General Uniqueness Lemma)

Let $A \in \mathbb{Z}[\alpha]$ and $m(\alpha) = 0$

Existence: $A = f(\alpha)$ for some $f \in \mathbb{Z}[X]$

divide f by m with remainder, $f = q * m + r$ $\deg(r) < \deg(m) < d$

$$\therefore f(\alpha) = q(\alpha)m(\alpha) + r(\alpha)$$

$$\therefore A = r(\alpha)$$

Uniqueness: Suppose $A = f(\alpha) = g(\alpha)$ ($f \neq g$) where f & g both have degree $< d$

$$\therefore h(\alpha) = 0 \text{ where } h = f - g \text{ } (\neq 0)$$

m is irreducible over \mathbb{Q} and has a bigger degree than h

$$\therefore m \nmid h \text{ in } \mathbb{Q}[x], \text{ so } m \text{ and } h \text{ are coprime in } \mathbb{Q}[x]$$

$\exists a, b \in \mathbb{Q}[x]$ such that :

$$1 = am + bh = a(\alpha)m(\alpha) + b(\alpha)h(\alpha) = 0$$

$$m(\alpha) = 0 \quad h(\alpha) = 0$$

$$\implies 1 = 0$$

$$\implies f = g$$

□

Lemma 2.24. *In any ring R with any prime p*

$$(x + y)^p \equiv x^p + y^p \text{ } (pR) \text{ for any } x, y \in R$$

Proof. Sufficient to show that each binomial coefficient:

$$c = \frac{p!}{i!(p-i)!}$$

$i = 1, 2, \dots, p-1$ is a multiple of p

$$i!(p-i)! \not\equiv 0 \text{ } (p) \implies \in \mathbb{F}_p^\times$$

□

Proof. 2nd Nebensatz

Let p be an odd prime and let $G = \zeta + \zeta^{-1} = \sqrt{2}$. We'll calculate $G^p \bmod (p\mathbb{Z}[\zeta])$ in two ways.

First Calculation:

$$\begin{aligned} G^p &= (\zeta + \zeta^{-1})^p \\ &= \zeta^p + \zeta^{-p} \bmod (p\mathbb{Z}[\zeta]) \text{ by the lemma} \end{aligned}$$

Since $\zeta^8 = 1$ this only depends p modulo 8 if $p \equiv \pm 1(8)$ then,

$$G^p = \zeta + \zeta^{-1} \equiv G \bmod (p\mathbb{Z}[\zeta])$$

If $p \equiv \pm 3(8)$ then,

$$G^p \equiv \zeta^3 + \zeta^{-3} \equiv -G \bmod (p\mathbb{Z}[\zeta])$$

So in summary,

$$G^p \equiv (-1)^{\frac{p^2-1}{8}} G \bmod (p\mathbb{Z}[\zeta])$$

Second Calculation:

Since $G^2 = 2$,

$$\begin{aligned} G^p &= G * 2^{\frac{p^2-1}{2}} \\ &= G * \left(\frac{2}{p}\right) \bmod (p\mathbb{Z}[\zeta]) \text{ by Euler's criterion} \end{aligned}$$

Comparing the results of these two calculations we get:

$$\left(\frac{2}{p}\right)G = (-1)^{\frac{p^2-1}{8}} G \bmod (p\mathbb{Z}[\zeta])$$

Note $G^2 * \frac{p+1}{2} \equiv 1 \bmod (p\mathbb{Z}[\zeta])$, i.e. G is invertible modulo $p\mathbb{Z}[\zeta]$ with inverse $G * \frac{p+1}{2}$

$$\implies \left(\frac{2}{p}\right) \equiv (-1)^{\frac{p^2-1}{8}} \bmod (p\mathbb{Z}[\zeta])$$

Since $1 \equiv -1 \bmod (p\mathbb{Z}[\zeta])$,

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$$

□

The proof of the 2nd Nebensatz worked because $\sqrt{2} \in \mathbb{Z}[\zeta]$
To prove the quadratic reciprocity law, we'll show that $\sqrt{\pm p}$ is in another cyclotomic ring

Let $\zeta_p = e^{\frac{2\pi i}{p}}$, a primitive p^{th} root of unity. We'll work in the ring modulo $q\mathbb{Z}[\zeta]$.

Definition 2.25. The p^{th} Gauss sum (where p is an odd prime):

$$G(p) = \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta_p^a \in \mathbb{Z}[\zeta_p]$$

Lemma 2.26. $G(p)^2 = (-1)^{\frac{p-1}{2}}$

Proof.

$$\begin{aligned} G(p)^2 &= \left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta_p^a \right) \left(\sum_{b=1}^{p-1} \left(\frac{b}{p}\right) \zeta_p^b \right) \\ &= \sum_{a,b \in \mathbb{F}_p^\times} \left(\frac{a}{p}\right) \left(\frac{b}{p}\right) \zeta_p^a \zeta_p^b \\ &= \sum_{a,b \in \mathbb{F}_p^\times} \left(\frac{ab}{p}\right) \zeta_p^{a+b} \end{aligned}$$

Let $c \equiv a^{-1}b \pmod{p}$, as b runs through \mathbb{F}_p^\times , so does c

$$\begin{aligned} &= \sum_{a,c \in \mathbb{F}_p^\times} \left(\frac{a^2}{p}\right) \zeta_p^{a+ac} \\ &= \sum_{c \in \mathbb{F}_p^\times} \left(\frac{c}{p}\right) \left(\sum_{a=1}^{p-1} (\zeta_p^{1+c})^a \right) \end{aligned}$$

Note the second summation is a geometric progression. Recall that,

$$\sum_{i=1}^{p-1} r^i = \begin{cases} \frac{r^p - 1}{r - 1} & r \neq 1 \\ p - 1 & r = 1 \end{cases}$$

Summing the geometric progression:

$$\begin{aligned} \sum_{a=1}^{p-1} (\zeta_p^{1+c})^a &= \begin{cases} \frac{(\zeta_p^{1+c})^p - \zeta_p^{1+c}}{\zeta_p^{1+c} - 1} & \text{if } c \not\equiv 1 \pmod{p} \\ p - 1 & \text{if } c \equiv 1 \pmod{p} \end{cases} \\ &= \begin{cases} -1 & c \not\equiv -1 \pmod{p} \\ p - 1 & c \equiv -1 \pmod{p} \end{cases} \end{aligned}$$

$$\therefore G(p)^2 = \sum_{c \in \mathbb{F}_p^\times} \left(\frac{c}{p}\right)(-1) + p\left(\frac{-1}{p}\right)$$

$\sum_{c \in \mathbb{F}_p^\times} \left(\frac{c}{p}\right)(-1) = 0$ since there are $\frac{p-1}{2}$ quadratic residues and quadratic non-residues.

$$\begin{aligned} &= p\left(\frac{-1}{p}\right) \\ &= (-1)^{\frac{p-1}{2}} p \end{aligned} \quad \text{by the 1}^{st} \text{ Nebensatz}$$

□

2.6.2 Uniqueness Lemma for $\mathbb{Z}[\zeta_p]$

Every element $A \in \mathbb{Z}[\zeta_p]$ can be written uniquely as:

$$A = a_0 + a_1\zeta + \cdots + a_{p-2}\zeta^{p-2} \quad \text{with } a_i \in \mathbb{Z}$$

This is because ζ_p is a root of $m(x) = \Phi_p(x) = 1 + x + \cdots + x^{p-1}$. It's proved in 7202 that Φ_p is irreducible over \mathbb{Q} .

Proof. Quadratic Reciprocity law

We'll calculate $G(p)^q$ ($q\mathbb{Z}[\zeta_p]$) in two ways.

First Calculation:

$$\begin{aligned} G(p)^q &= \left(\sum_{a=1}^{p-1} \left(\frac{a}{p}\right) \zeta_p^a \right)^q \\ &= \sum_{a=1}^{p-1} \left(\left(\frac{a}{p}\right) \zeta_p^a \right)^q \quad (p\mathbb{Z}[\zeta]) \end{aligned}$$

Since q is odd, $\left(\frac{a}{p}\right)^q = \left(\frac{a}{p}\right)$

$$G(p)^q \equiv \sum_{a \in \mathbb{F}_p^\times} \left(\frac{a}{p}\right) \zeta_p^{aq}$$

Let $b \equiv aq \pmod{p}$, and as a runs through \mathbb{F}_p^\times so does b

$$\begin{aligned} G(p)^q &\equiv \sum_{b \in \mathbb{F}_p^\times} \left(\frac{bq^{-1}}{p}\right) \zeta_p^b \\ &= \left(\frac{q^{-1}}{p}\right) \sum_{b \in \mathbb{F}_p^\times} \left(\frac{b}{p}\right) \zeta_p^b \end{aligned}$$

Note that $G(p) = \sum_{b \in \mathbb{F}_p^\times} \left(\frac{b}{p}\right) \zeta_p^b$ which implies,

$$\begin{aligned} G(p)^q &\equiv \left(\frac{q^{-1}}{p}\right) G(p) \pmod{q\mathbb{Z}[\zeta_p]} \\ &\equiv \left(\frac{q}{p}\right) G(p) \pmod{q\mathbb{Z}[\zeta_p]} \end{aligned}$$

Second Calculation:

Since $G(p)^2 = (-1)^{\frac{p-1}{2}} p$,

$$\begin{aligned} G(p)^q &= G(p) \left((-1)^{\frac{p-1}{2}} p \right)^{\frac{q-1}{2}} \\ &= G(p) (-1)^{\frac{(p-1)(q-1)}{4}} p^{\frac{q-1}{2}} \\ \therefore G(p)^q &\equiv G(p) (-1)^{\frac{(p-1)(q-1)}{4}} \left(\frac{p}{q}\right) \pmod{q\mathbb{Z}[\zeta_p]} \quad \text{by Euler's criterion} \end{aligned}$$

Comparing the two results we get:

$$\left(\frac{q}{p}\right) G(p) \equiv (-1)^{\frac{(p-1)(q-1)}{4}} \left(\frac{p}{q}\right) G(p) \pmod{q\mathbb{Z}[\zeta_p]}$$

We need to check that $G(p)$ is invertible modulo $q\mathbb{Z}[\zeta_p]$,

$G(p)^2 = \pm p$, which is invertible modulo q

$G(p)$ has inverse $G(p) * (\pm p)^{-1} \pmod{q\mathbb{Z}[\zeta_p]}$

$$\therefore \left(\frac{p}{q}\right) \equiv (-1)^{\frac{(p-1)(q-1)}{4}} \left(\frac{p}{q}\right) \pmod{q\mathbb{Z}[\zeta_p]}$$

Since $1 \equiv -1 \pmod{q\mathbb{Z}[\zeta_p]}$, it follows that $\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}} \left(\frac{p}{q}\right)$ □

3 P-adic Number theory

This means methods for congruences modulo p^n , p prime and n large.

If we want to solve $f(x) = 0$, $x \in \mathbb{R}$ we can use the Newton-Raphson method:

- Begin with an "approximate solution" a_0
- Define a sequence recursively $a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$

Very often a_n converge to a limit a and $f(a) = 0$.

We can use the same method in number theory for solving congruences. Suppose $f(x)$ is a polynomial with coefficients in \mathbb{Z} and we want to solve $f(x) \equiv 0 \pmod{p^N}$ (p prime, n large)

We can try this:

- Find a solution a_0 to $f(a_0) \equiv 0 \pmod{p^r}$ where r is small
- Define a recursive sequence $a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$

If n is large enough, then often $f(a_n) \equiv 0 \pmod{p^N}$

E.g. Let $f(x) = x^2 + 2$, $p = 3$

Suppose we want to solve $x^2 + 2 \equiv 0 \pmod{3^N}$

Let $a_0 = 1$: $f(a_0) = 1^2 + 2 = 3 \equiv 0 \pmod{3}$

Define the sequence a_n by $a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} = a_n - \frac{a_n^2 + 2}{2a_n} = \frac{a_n}{2} - \frac{1}{a_n}$

$$\begin{aligned} a_0 &= 1 \\ a_1 &= \frac{1}{2} - 1 = \frac{-1}{2} \\ a_2 &= \frac{-1}{4} + 2 = \frac{7}{4} \end{aligned}$$

It turns out that $\frac{-1}{2}$ is a solution mod 9 $\implies -1 * 2^{-1} \pmod{9}$
 $\frac{7}{4}$ is a solution mod 81 $\implies 7 * 4^{-1} \pmod{81}$

$$2^{-1} \equiv 5 \pmod{9} \implies a_1 \equiv 4 \pmod{9}$$

$$4^{-1} \equiv -20 \pmod{81} \implies a_2 = \frac{7}{4} \equiv -140 \equiv 22 \pmod{81}$$

a_3 would be a solution mod 3^8 .

In this example, we're reducing rational numbers mod p^n not just integers. If $\frac{a}{b}$ is a rational number then we can reduce this modulo p^n as long as b is invertible mod p^n , i.e. when b is not a multiple of p . We'll write:

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, p \nmid b \right\}$$

$\mathbb{Z}_{(p)}$ is closed under $+$, $*$, so $\mathbb{Z}_{(p)}$ is a ring contained in \mathbb{Q} containing \mathbb{Z} . This is called the "local ring of p " and is the set of rational number which can be reduce modulo p^n ($\forall n$)

Definition 3.1. If p is a prime number and $n \in \mathbb{Z}$, then the valuation of n , at p is:

$$V_p(n) = \begin{cases} \max\{a : p^a | n\} & n \neq 0 \\ \infty & n = 0 \end{cases}$$

A simple statement that can be made is, $V_p(nm) = V_p(n) + V_p(m)$. We can also extend V_p to a function on \mathbb{Q} , $V_p(\frac{n}{m}) = V_p(n) - V_p(m)$.

With this notation:

$$Z_{(p)} = \{x \in \mathbb{Q} : V_p(x) \geq 0\}$$

$$x \equiv y \pmod{p^a} \iff V_p(x - y) \geq a$$

E.g

$$V_2(\frac{7}{12}) = -2 \quad V_2(\frac{7}{12}) = -1 \quad V_5(\frac{7}{12}) = 0 \quad V_7(\frac{-7}{12}) = +1$$

3.1 Hensel's Lemma

Let p be a prime number. Let $f \in \mathbb{Z}_{(p)}[x]$ and $a_0 \in \mathbb{Z}_{(p)}$ such that $f(a_0) \equiv 0 \pmod{p^{2c+1}}$ where $c = V_p(f'(a_0))$.

Then if we define $a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$ then $a_n \in \mathbb{Z}_{(p)}$ and $f(a_n) \equiv 0 \pmod{p^{2c+2^n}}$

Proof. We'll prove the following by induction on n

1. $a_n \in \mathbb{Z}_{(p)}$ and $a_n \equiv a_0 \pmod{p^{c+1}}$
2. $V_p(f'(a_n)) = c$
3. $f(a_n) \equiv 0 \pmod{p^{2c+2^n}}$

If $n = 0$ then the statements 1,2,3 are all true for a by assumption. Now assume 1,2,3 for a_n , we'll prove them for a_{n+1}

Let $a_{n+1} = a_n - \delta$ where $\delta = \frac{f(a_n)}{f'(a_n)}$

1:

$$\begin{aligned} V_p(\delta) &= V_p(f(a_n)) - V_p(f'(a_n)) \\ &= c && \text{by 2:} \\ &\geq 2c + 2^n && \text{by 3:} \\ V_p(\delta) &\geq 2c + 2^n - c \\ V_p(\delta) &\geq c + 2^n && (*) \end{aligned}$$

By (*)

$$V_p(\delta) \geq 0 \implies \delta \in \mathbb{Z}_{(p)}$$

$$\therefore a_{n+1} = a_n - \delta \in \mathbb{Z}_{(p)}$$

By (*)

$$V_p \geq c + 1 \implies \delta \equiv 0 \pmod{p^{c+1}}$$

$$a_{n+1} \equiv a_n \pmod{p^{c+1}}$$

$$\equiv a_0 \pmod{p^{c+1}}$$

by 1

2: We've shown that $a_{n+1} \equiv a_0 \pmod{p^{c+1}}$

$$\therefore f'(a_{n+1}) \equiv f'(a_0) \pmod{p^{c+1}}$$

$$\not\equiv 0$$

because $V_p(f'(a_0)) = c$

also $f'(a_{n+1}) \equiv f'(a_0) \pmod{p^c}$

$$\equiv 0 \pmod{p^c}$$

because $V_p(f'(a_0)) = c \pmod{p^c}$

$$\therefore V_p(f'(a_{n+1})) = c$$

3: Must show that $f(a_{n+1}) \equiv 0 \pmod{p^{2c+2^{n+1}}}$

$$a_{n+1} = a_n - \delta$$

$$a_{n+1}^r = (a_n - \delta)^r$$

$$= a_n^r - r a_n^{r-1} \delta + \text{multiples of } \delta^2$$

By (*):

$$V_p(\delta) \geq c + 2^n$$

$$\therefore V_p(\delta^2) \geq 2c + 2^{n+1}$$

$$\therefore \delta^2 \equiv 0 \pmod{p^{2c+2^{n+1}}}$$

This implies $a_{n+1}^r \equiv a_n^r - r a_n^{r-1} \delta \pmod{p^{2c+2^{n+1}}}$

Suppose $f(x) = \sum c_r * x$. Substituting a_{n+1} , we get:

$$\begin{aligned} f(a_{n+1}) &= \sum c_r (a_n^r - r a_n^{r-1} \delta) \pmod{p^{2c+2^{n+1}}} \\ &= \sum c_r a_n^r - \left(\sum r c_r a_n^{r-1} \right) \delta \pmod{p^{2c+2^{n+1}}} \\ &= f(a_n) - f'(a_n) * \frac{f(a_n)}{f'(a_n)} \equiv 0 \pmod{p^{2c+2^{n+1}}} \end{aligned}$$

□

E.g. $f(x) = x^3 + x + 1$, $p = 3$

Find a root of $f \bmod 81$

Note that $f'(x) = 3x^2 + 1$ and $f(1) = 3 \equiv 0$

Try $a_0 = 1$

$$\begin{aligned}c &= V_3(f'(a_0)) \\&= V_3(4) \\&= 0\end{aligned}$$

$3^{2c+1} = 3$ and a_0 is a root of f modulo 3

$\therefore a_0 = 1$ satisfies the conditions of Hensel's lemma.

$$\begin{aligned}a_1 &= 1 - \frac{a_0}{f'(a_0)} \\&= 1 - \frac{3}{4}\end{aligned}$$

It is sufficient to work out $a_1 \bmod 9$

$$4^{-1} \equiv 1 \pmod{3} \qquad \frac{3}{4} \equiv 3 * 1 \pmod{9} \qquad a_1 \equiv -2 \pmod{9}$$

Check

$$\begin{aligned}f(a_1) &\equiv (-2)^3 + (-2) + 1 \\f(2) &= -9 \equiv 0 \pmod{9}\end{aligned}$$

$$\begin{aligned}a_2 &= -2 - \frac{f(-2)}{f'(-2)} \\&= -2 - \frac{-9}{13}\end{aligned}$$

This should be a root of f modulo 81.

$$13^{-1} \equiv -2 \pmod{9}$$

$$\implies \frac{9}{13} \equiv -18$$

$$a_2 \equiv -2 - 18 \equiv -20$$

$$\begin{aligned}\text{Check } f(a_2) &= (-20)^3 - 20 + 1 \equiv -8000 - 19 \\&= -8019 \\&= -81 * 99 \\&= 0 \pmod{81}\end{aligned}$$

3.2 Quadratic Congruences

We'll see how to find out whether $x^2 \equiv b \pmod{n}$ has solutions.

Suppose $n = p_1^{a_1} \dots p_r^{a_r}$ (p_i distinct primes). There are solutions modulo $n \iff \forall i$, there are solutions modulo $p_i^{a_i}$ by the Chinese Remainder Theorem.

Proposition 3.2. *Suppose p is an odd prime not dividing b . If $x^2 \equiv b \pmod{p}$ has solutions then $x^2 \equiv b \pmod{p^r}$ has solutions for all r*

Proof. Suppose there is a solution a_0 modulo p , i.e. $a_0^2 \equiv b \pmod{p}$

Let $f(x) = x^2 - b$. We'll check that a_0 satisfies the conditions of Hensel's lemma.

$$\begin{aligned} c &= V_p(f'(a_0)) \\ &= V_p(2a_0) \quad \text{and since } p \neq 2 \\ \implies c &= V_p(a_0) \end{aligned}$$

Also since $p \nmid b$, we know $p \nmid a_0$:

$$\begin{aligned} \therefore c &= 0 \\ \therefore f(a_0) &\equiv 0 \pmod{p^{2c+1}} \implies a_0 \text{ satisfies the conditions of Hensel's lemma} \\ \therefore &\text{ We have roots of } f \text{ modulo all powers of } p \end{aligned}$$

□

Remark

Suppose we want a root of f modulo p^{13}

Choose n so that $2c + 2^n \geq 13$

$$f(a_n) \equiv 0 \pmod{p^{2c+2^n}} \implies f(a_n) \equiv 0 \pmod{p^{13}}$$

The proposition would be false if we allowed $p = 2$

E.g. Let $b = 3$

x	$x^2 \pmod{4}$
0	0
1	1
2	0
3	1

$$x^2 \equiv 3 \pmod{2} \text{ has a solution}$$

$$x^2 \equiv 3 \pmod{4} \text{ has no solutions}$$

if $b = 5$

x	$x^2 \pmod{8}$
0	0
± 1	1
± 2	4
± 3	1
± 4	0

$$x^2 \equiv 5 \pmod{2} \text{ has a solution}$$

$$x^2 \equiv 5 \pmod{4} \text{ has solutions}$$

$$x^2 \equiv 5 \pmod{8} \text{ has no solutions}$$

Proposition 3.3. Suppose b is odd. If $x^2 \equiv b \pmod{8}$ has solutions then $x^2 \equiv b \pmod{2^r}$ has solutions for all r

Proof. Suppose $a_0 \equiv b \pmod{8}$, this implies a_0 is odd.

Let $f(x) = x^2 - b$

$\therefore c = V_2(f'(a_0)) = V_2(2a_0) = 1$ because a_0 is odd

$\therefore 2^{2c+1} = 8$

$\therefore a_0$ is a root of f modulo p^{2c+1}

By Hensel's lemma, there are solutions modulo all powers of 2.

□

E.g. For which n does the congruence $x^2 \equiv 5 \pmod{5}$ have solutions?

First consider the case $n \equiv p^r$ (p prime)

If $p \neq 2, 5$ then by the first proposition, there are solutions $p^n \iff \left(\frac{5}{p}\right) = 1$

$\left(\frac{5}{p}\right) = +\left(\frac{p}{5}\right)$ depends on $p \pmod{5}$

x	s	
1	1	(different x)
2	-1	The congruence $x^2 \equiv 5 \pmod{p}$ has solutions
3	-1	$\iff p \equiv 1, 4 \pmod{5}$ (in the cases $p \neq 2, 5$)
4	1	

For $p = 2$, $x^2 \equiv 5 \pmod{2}$ has a solution, $x = 1$

$x^2 \equiv 5 \pmod{4}$ has a solution, $x = 1$

But the only odd square mod 8 is 1. So $x^2 \equiv 5 \pmod{8}$ has no solutions.

\therefore no solutions mod 2^n if $n \geq 3$

For $p = 5$ $x^2 \equiv 5 \pmod{5}$ has solutions, here's how we check. Assume:

$$x^2 \equiv 5 \pmod{25}$$

$$\therefore x^2 \equiv 0 \pmod{5}$$

$$\text{So } 5|x^2$$

$$\text{So } 5|x$$

$$\therefore x^2 \equiv 0 \pmod{25} \quad \nexists$$

So there are solution modulo n if $n = 2^a * 5^b * \prod p_i^{c_i}$ where $a \leq 2, b \leq 1, p_i \equiv 1 \pmod{5}, c_i \in \mathbb{N}$

E.g. For which n does $x^2 \equiv -7 \pmod{n}$ have solutions?

Assume p is a prime $\neq 2, 7$

$$\begin{aligned}
\left(\frac{-7}{p}\right) &= \left(\frac{-1}{p}\right)\left(\frac{7}{p}\right) \\
&= (-1)^{\frac{p-1}{2}} (-1)^{\frac{(7-1)(p-1)}{4}} \left(\frac{p}{7}\right) \\
&= (-1)^{\frac{p-1}{2} + \frac{3(p-1)}{2}} \left(\frac{p}{7}\right) \\
&= (+1)\left(\frac{p}{7}\right) \text{ depends on } p \bmod 7
\end{aligned}$$

x	$\left(\frac{x}{7}\right)$
1	1
2	1
3	-1
4	1
5	-1
6	-1

$$\begin{aligned}
3^2 &= 9 \equiv 2 \pmod{7} \\
x^2 &\equiv -7 \pmod{p^r} \text{ has solutions} \\
&\implies p \equiv 1, 2, 4 \pmod{7}
\end{aligned}$$

For $p = 2$: $-7 \equiv 1 \pmod{8}$ so -7 is a square modulo 8 by the proposition.
 $x^2 \equiv -7 \pmod{2^r}$ has solutions for all r .

For $p = 7$: $x^2 \equiv -7 \pmod{7}$ has a solution $x = 0$ but $x^2 \equiv -7 \pmod{7^2}$ has no solutions. Suppose

$$\begin{aligned}
x^2 &\equiv -7 \pmod{7^2} \\
\therefore x^2 &\equiv 0 \pmod{7} \\
\therefore 7|x^2 \\
\implies 7|x \\
\implies x^2 &\equiv 0 \pmod{49} \quad \nexists
\end{aligned}$$

So $x^2 \equiv -7 \pmod{n}$ has solutions $\iff n = 7^a * \prod p_i^{b_i}$ where $a \leq 1$, $p_i \equiv 1, 2, 4 \pmod{7}$, $b_i \in \mathbb{N}$

3.3 P-adic congruence

Suppose we have a series $\sum_{n=1}^{\infty} x_n$ for $x_n \in \mathbb{Z}_{(p)}$. We'll say that the series converges **p-adically** if for every a , there are only finitely many terms x_n with $x_n \not\equiv 0 \pmod{p^a}$. We can add up the series in \mathbb{Z}/p^a because only finitely many terms are non zero.

Lemma 3.4. $\sum x_n$ converges *p-adically* $\iff V_p(x_n) \rightarrow \infty$

Proof. If $V_p(x_n) \rightarrow \infty$ then for n significantly large, $V_p(x_n) \geq a$, i.e., $x_n \equiv 0 \pmod{p^a}$

□

E.g. $p=3$

$$(1 + 3x)^{\frac{1}{2}} = 1 + \frac{1}{2}(3x) + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)(3x)^2}{2!} + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)(3x)^3}{3!}$$

if $x \in \mathbb{Z}_{(3)}$ then this series converge 3-adically.

$$\begin{aligned}
(1 + 3x)^{\frac{1}{2}} &\equiv 1 \quad (3) \\
&\equiv 1 + \frac{3x}{2} \quad (9) \\
&\equiv 1 + \frac{3x}{2} + \frac{9}{8}x^2 \quad (27) \\
&\equiv 1 + \frac{3x}{2} + \frac{9}{8}x^2 + \frac{27}{16}x^3 \quad (27)
\end{aligned}$$

We can write these polynomials with integer coefficients.

$$\begin{aligned}
(1 + 3x)^{\frac{1}{2}} &\equiv 1 + 15x + 9x^2 \quad (27) \\
&\equiv 1 + 42x + 9x^2 + 27x^3 \quad (81)
\end{aligned}$$

Important point; these polynomials play the same role in number theory $(1 + 3x)^{\frac{1}{2}}$ does in analysis $\sqrt{1 + 3x}$

E.g.

$$\begin{aligned}
(1 + 15x + 9x^2)^2 &= 1 + (30)x + (18 + 15^2)x^2 + (2 * 9 * 15)x^3 + (81)x^4 \\
&\equiv 1 + 3x \quad (27)
\end{aligned}$$

E.g. Find a square root of 7 in $\mathbb{Z}/81$

$$\begin{aligned}
7^{\frac{1}{2}} &= (1 + 3 * 2)^{\frac{1}{2}} \\
&\equiv 1 + 42 * 2 + 9 * 2^2 + 27 * 2^3 \quad (81) \\
&\equiv 1 + 84 + 36 - 27 \quad (81) \\
&\equiv 13 \quad (81)
\end{aligned}$$

Check $13^2 = 169 \equiv 7 \quad (81)$

This works because of a result called the power series trick.

Notation We'll write $\mathbb{Z}_{(p)}[[x]]$ for the set of power series in x with coefficient in $\mathbb{Z}_{(p)}$.

$\mathbb{Z}_{(p)}[[x]]$ is a ring with addition and multiplication of power series as operations. We can often compose two power series $f, g \in \mathbb{Z}_{(p)}[[x]]$ to get a new power series $f \circ g$.

$(f \circ g)(x) = f(g(x))$.

We can define $f \circ g$ as long as either f is a polynomial or g has zero constant term. Suppose

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n x^n \\
g(x) &= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}$$

We'll see that $f \circ g$ is a power series

$$\begin{aligned} f(g(x)) &= \sum_{n=0}^{\infty} a_n \left(\sum_{m=1}^{\infty} b_m x^m \right)^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} b_{m_1} \cdots b_{m_n} x^{m_1 + \cdots + m_n} \end{aligned}$$

so $f(g(x)) = \sum c_d x^d$ where

$$c_d = \underbrace{\sum_{m_1, \dots, m_n=1}^{\infty} a_n b_{m_1} \cdots b_{m_n}}_{\text{finite sum in } \mathbb{Z}_{(p)}}$$

Note $f \circ g$ is not defined otherwise.

E.g

$$f(x) = 1 + x + x^2 + \dots$$

$$g(x) = 1 + x$$

$$\implies f(g(x)) = 1 + (1 + x) + (1 + x)^2$$

This has constant term $1 + 1 + 1 + 1 + \dots$, so $f \circ g$ is not defined.

3.4 Power Series Trick

Suppose f, g, h are power series with coefficients in $\mathbb{Z}_{(p)}$. Assume either f is a polynomial or g has no constant term. Also assume:

- For small real numbers x , $f(x), g(x), h(x)$ converge and $f(g(x)) = h(x)$
- For all $x \in \mathbb{Z}_{(p)}$, $f(x), g(x)$ and $h(x)$ converge p-adically

Then for all $x \in \mathbb{Z}_{(p)}$, $f(g(x)) \equiv h(x) \pmod{p^n}$

In the example $f(x) = x^2, g(x) = (1 + 3x)^{\frac{1}{2}}, h(x) = 1 + 3x$.

For small real x , $f(g(x)) = h(x)$, so as long as we know that $g(x)$ converges 3-adically ($\forall x \in \mathbb{Z}_{(3)}$) the power series trick implies $g(x)^2 \equiv 1 + 3x \pmod{3^n}$

How do we check for p-adic convergence?

Lemma 3.5. $\sum x_n$ converge p-adically if and only if $V_p(x_n) \rightarrow \infty$

We need a way of calculating valuations of n^t term of a square.

Proposition 3.6. $V_p(n!) = \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \cdots \leq \frac{n}{p-1}$

We'll prove this later, first use the properties to show that $(1 + 3x)^{\frac{1}{2}}$ converge 3-adically for all $x \in \mathbb{Z}_{(3)}$

$$(1 + 3x)^{\frac{1}{2}} = 1 + \frac{1}{2}(3x) + \frac{(\frac{1}{2})(\frac{-1}{2})(3x)}{2!} + \dots$$

$$n^{th} \text{ term} = \frac{(\frac{1}{2})(\frac{1}{2} - 1)(\frac{1}{2} - 2) \dots (\frac{1}{2} - n + 1)}{n!} (3x)^n$$

$$\begin{aligned} V_3(n^{th} \text{ term}) &= V_3\left(\frac{1}{2}(\frac{1}{2} - 1) \dots (\frac{1}{2} - n + 1)\right) - V_3(n!) + V_3((3x)^n) \\ &\geq 0 - \frac{n}{3-1} + n \\ &\geq \frac{n}{2} \rightarrow \infty \text{ as } n \rightarrow \infty \\ &\implies \text{series converges 3-adically} \end{aligned}$$

E.g. Assume p is an odd prime

Let $\exp(px) = 1 + px + \frac{(px)^2}{2!} + \frac{(px)^3}{3!} + \dots$

We'll see that this converges for all $x \in \mathbb{Z}_{(p)}$

$$n^{th} \text{ term} = \frac{(px)^n}{n!}$$

$$\begin{aligned} V_p(n^{th} \text{ term}) &= V_p((px)^n) - V_p(n!) \\ &= (n * V_p(p)) + (n * V_p(x)) - V_p(n!) \\ &\quad 1 \qquad \qquad \geq 0 \qquad \leq \frac{n}{p-1} \\ &\geq n - \frac{n}{p-1} \\ &\geq \left(\frac{p-2}{p-1}\right)n \rightarrow \infty \text{ for } p \neq 2 \end{aligned}$$

E.g. $\log(1 + px)$ converges p-adically for all $x \in \mathbb{Z}_{(p)}$

$$\begin{aligned} V_p\left(\pm \frac{(px)^n}{n}\right) &= V_p((px)^n) - V_p(n) \\ &\quad nV_p(px) < V_p(n!) \\ &\geq n - \frac{n}{p-1} \\ &\geq \left(\frac{p-2}{p-1}\right)n \rightarrow \infty \text{ if } p \neq 2 \end{aligned}$$

Remark A quick way to remember the series for $\log(1 + px)$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots \quad \text{geometric series}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\log(1+px) = x - \frac{(px)^2}{2} + \frac{(px)^3}{3} - \frac{(px)^4}{4} + \dots$$

Proof. Calculating $V_p(n!)$

$$n! = 1 * 2 * \dots * n$$

$$V_p(n) = \sum_{i=1}^n V_p(i) \quad (*)$$

The number of i between 1 & n which are multiples of p is $\lfloor \frac{n}{p} \rfloor$.

There are $\frac{n}{p^2}$ values of i which are multiples of p^2 , etc.

$\lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor$ values of i are multiples of p , but not of p^2 , i.e. $V_p(i) = 1$

So $\lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor$ terms in the sum $(*)$ are equal to 1.

Similarly $\lfloor \frac{n}{p^2} \rfloor - \lfloor \frac{n}{p^3} \rfloor$ terms in the sum $(*)$ are equal to 2.

In general there are exactly $\lfloor \frac{n}{p^a} \rfloor - \lfloor \frac{n}{p^{a+1}} \rfloor$ terms in $(*)$ which are equal to a

$$\therefore V_p(n!) = 1 * \text{no of terms equal to 1} + 2 * \text{number of terms equal to 2} + \dots$$

$$\begin{aligned} V_p(n!) &= 1 * (\lfloor \frac{n}{p} \rfloor - \lfloor \frac{n}{p^2} \rfloor) \\ &\quad + 2 * (\lfloor \frac{n}{p^2} \rfloor - \lfloor \frac{n}{p^3} \rfloor) \\ &\quad + 3 * (\lfloor \frac{n}{p^3} \rfloor - \lfloor \frac{n}{p^4} \rfloor) \\ &\quad + \dots \\ &= \lfloor \frac{n}{p} \rfloor + (2-1)\lfloor \frac{n}{p^2} \rfloor + (3-2)\lfloor \frac{n}{p^3} \rfloor + \dots \\ &= \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \dots \end{aligned}$$

Using this we can prove the upper bound.

$$\begin{aligned}
V_p(n!) &\leq \frac{n}{p} + \frac{n}{p^2} + \dots \\
&\leq \frac{n}{p} \underbrace{\left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots\right)}_{\text{geometric series } \frac{1}{1-\frac{1}{p}}} \\
&\leq \frac{n}{p-1}
\end{aligned}$$

□

3.4.1 P-adic log & exp

Let p be an odd prime. Use the notation

$$p\mathbb{Z}/p^n = \{px : x \in \mathbb{Z}/p^n\}$$

E.g.

$$3\mathbb{Z}/27 = \{0, 3, 6, 9, 12, 15, 18, 21, 24\}$$

$p\mathbb{Z}/p^n$ is closed under $+$, so it is a subgroup of $(\mathbb{Z}/p^n)^\times$

$$1 + p\mathbb{Z}/p^n = \{1 + px : x \in \mathbb{Z}/p^n\}$$

E.g.

$$1 + 3\mathbb{Z}/27 = \{1, 4, 7, 10, 13, 16, 19, 22, 25\}$$

$1 + p\mathbb{Z}/p^n$ is closed under $*$, so $1 + p\mathbb{Z}/p^n$ is a subgroup $(\mathbb{Z}/p^n)^\times$. Both subgroups have p^{n-1} elements, but one is additive and the other is multiplicative. But actually there are isomorphic. This isomorphism is \exp & \log .

Theorem 3.7. *Let p be an odd prime. Then there is an isomorphism:*

$$p\mathbb{Z}/p^n \xleftrightarrow[\exp]{\log} 1 + p\mathbb{Z}/p^n$$

$$px \longmapsto \exp(px)$$

$$1 + px \longmapsto \log(1 + px)$$

E.g. $\mathbb{Z}/27$ ($p = 3$) We'll find the isomorphisms in this case.

$$\exp(3x) \equiv 1 + 3x + \frac{3^2 x^2}{2!} + \frac{3^3 x^3}{3!} \quad (27)$$

$$\equiv 1 + 3x + 18x^2 + 18x^3 \quad (27)$$

$$\log(1 + 3x) \equiv 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} \quad (27)$$

$$\equiv 3x + 9x^2 + 9x^3 \quad (27)$$

Check:

$$\begin{aligned}
 \log(\exp(3x)) &\equiv \log(1 + 3(x + 6x^2 + 6x^3)) \\
 &\equiv 3(x + 6x^2 + 6x^3) + 9(x + 6x^2 + 6x^3)^2 + 9(x + 6x^2 + 6x^3) \\
 &\equiv 3x + 18x^2 + 18x^3 + 9x^2 + 9x^3 \\
 &\equiv 3x
 \end{aligned}$$

Similarly $\exp(\log(1 + 3x)) \equiv 1 + 3x$ (27)

We can use the theorem to solve congruences.

E.g. Solve $7^x \equiv 13$ (27)

7 and 13 are in $1 + 3\mathbb{Z}/27$, so we can take their logarithms.

$$x \log(7) \equiv \log(13)$$

Using the formula for $\log(1 + 3x)$, we get:

$$\begin{aligned}
 \log(7) &= \log(1 + 6) \\
 &\equiv 6 - \frac{6^2}{2} + \frac{6^3}{3} - \frac{6^4}{4} \\
 &\equiv 6 - 18 + 72 \\
 &\equiv 6 \quad (27)
 \end{aligned}$$

$$\begin{aligned}
 \log(13) &\equiv \log(1 + 12) \\
 &\equiv 12 - \frac{12^2}{2} + \frac{12^3}{3} \quad (27) \\
 &\equiv 12 - 72 + 3^2 * 4^3 \quad (27) \\
 &\equiv 12 - 72 + 9 \quad (27) \\
 &\equiv 3 \quad (27)
 \end{aligned}$$

So $7^x \equiv 13$ (27) reduces to:

$$\begin{aligned}
 7^x &\equiv 13 \quad (27) \\
 \implies 6x &\equiv 3 \quad (27) \\
 \implies 2x &\equiv 1 \quad (9) \\
 \implies x &\equiv 5 \quad (9)
 \end{aligned}$$

Proof.

We'll use the power series trick. We've shown that $\log(1 + px), \exp(px)$ converge p-adically for $x \in \mathbb{Z}_{(p)}$ and they converge for small real numbers and for small real x

$$\begin{aligned}\log(\exp(px)) &= px \\ \exp(\log(1 + px)) &= 1 + px\end{aligned}$$

By the power series trick:

$$\begin{aligned}\exp(\log(1 + px)) &\equiv 1 + px \pmod{p^n} \\ \log(\exp(px)) &\equiv px \pmod{p^n}\end{aligned}$$

$\therefore \log$ and \exp are inverse functions, so they are bijective.

Remains to show that $\exp(px + py) \equiv \exp(px) * \exp(py) \pmod{p^n}$

For any $a \in \mathbb{N}$: $\exp(pax) = \exp(px)^a$ for small real x

By the power series trick with:

$$\begin{aligned}f(x) &= x^a \\ g(x) &= \exp(px) \\ h(x) &= \exp(pax) \\ \exp(pax) &\equiv (\exp(px))^a \pmod{p^a}\end{aligned}$$

Take $x = 1$

$$\begin{aligned}\exp(pa) &\equiv \exp(p)^a \pmod{p^a} \\ \therefore \exp(pa + pb) &\equiv \exp(p)^{a+b} \pmod{p^a} \\ &\equiv \exp(p)^a * \exp(p)^b \\ &\equiv \exp(pa) * \exp(pb) \pmod{p^n}\end{aligned}$$

We've proved this when a & b are positive integers, but every element of \mathbb{Z}/p^n can be written as a positive integer. □

3.5 Teichmüller Lifts

Let p be an odd prime. We saw that $(\mathbb{Z}/p^n)^\times$ has a big subgroup $1 + p\mathbb{Z}/p^n$ and we can easily do calculations in the subgroup. Teichmüller lifts is another subgroup.

$$(\mathbb{Z}/p^n)^\times = \text{Teichmüller lifts} * (1 + p\mathbb{Z}/p^n)$$

Let $x \in \mathbb{Z}_{(p)}$ and assume $x \not\equiv 0 \pmod{p}$:

$$x, x^p, x^{p^2}, x^{p^3}, \dots$$

All these terms are constant mod p :

$$\begin{aligned} x^{p-1} &\equiv 1 \pmod{p} \\ x^p &\equiv x \pmod{p} \end{aligned}$$

The sequence is constant mod p^2 , but all terms after the 2^{nd} are constant mod p^2 .

E.g. $p = 3, x = 2$

We'll look at the sequence mod 9:

$$\begin{aligned} 2^3 &\equiv 8 \pmod{9} \\ 2^9 &\equiv 8^3 \equiv 8 \pmod{9} \\ 2^{27} &\equiv 8 \pmod{9} \quad \text{etc} \end{aligned}$$

The sequence is eventually constant modulo p^n

Definition 3.8. *The Teichmüller lift of x modulo p^n is:*

$$T(x) \equiv x^{p^{n-1}} \pmod{p^n}$$

To calculate Teichmüller lifts, we use:

Lemma 3.9. *Suppose $x \equiv y \pmod{p^n}$ then $x^p \equiv y^p \pmod{p^{n+1}}$*

Proof. Let $x \equiv y + p^n \implies$

$$x^p \equiv (y + p^n)^p$$

$$x^p \equiv y^p + py^{p-1}p^n + \text{multiples of } p^{2n}$$

$$x^p \equiv y^p \pmod{p^{n+1}}$$

□

E.g. Calculate $T(12) \pmod{125}$

Definition is $12^{25} \pmod{125}$. Using the lemma:

$$12 \equiv 2 \pmod{5}$$

$$\begin{aligned} 12^5 &\equiv 2^5 \pmod{5^2} \\ &\equiv 32 \equiv 7 \pmod{25} \end{aligned}$$

$$12^5 \equiv 7 \pmod{25}$$

$$12^{25} \equiv 7^5 \pmod{5^3}$$

$$\begin{aligned} T(12) &\equiv (2 + 5)^5 \pmod{125} \\ &\equiv 2^5 + 5(2^4) * 5 + 10 * 2^3 * 5^2 + \text{multiples of } 125 \\ &\equiv 2^5 + 5^2 * 2^4 \pmod{125} \\ &\equiv 2^5 + 25 * 16 \pmod{125} && \text{note } 16 \equiv 1 \pmod{5} \\ &\equiv 2^5 + 25 * 1 \pmod{125} \\ &\equiv 32 + 25 \pmod{125} \\ &\equiv 57 \pmod{125} \end{aligned}$$

x	$T(x) \pmod{125}$
1	1
2	57
3	$T(-1) * T(2) = -1 * 57 \equiv 68 \pmod{125}$
4	$(-1)^{25} \equiv -1$

Theorem 3.10.

1. If $r > n - 1$ then $x^{p^r} \equiv T(x) \pmod{p^n}$
2. $T(x)^{p-1} \equiv 1 \pmod{p^n}$
3. $T(x)$ depends only on $x \pmod{p}$ and $T(x) \equiv x \pmod{p}$
4. $T: \mathbb{F}_p^\times \mapsto (\mathbb{Z}/p^n)^\times$ is an injective homomorphism

Proof.

By Euler's theorem, $\phi(p^n) = (p-1)p^{n-1}$

$$\begin{aligned} &\implies \underbrace{x^{(p-1)p^{n-1}}}_{T(x)^{p-1}} \equiv 1 \pmod{p^n} \\ &\implies T(x)^{p-1} \equiv 1 \pmod{p^n} \end{aligned}$$

This proves **2**.

$$\therefore T(x)^p \equiv T(x) \pmod{p^n}$$

Doing this several times we get $T(x) \equiv T(x)^p \equiv T(x)^{p^2} \equiv \dots \pmod{p^n}$

This proves **1**.

Suppose:

$$\begin{aligned} x &\equiv y \pmod{p} \\ x^p &\equiv y^p \pmod{p^2} && \text{by the lemma} \\ x^{p^2} &\equiv y^{p^2} \pmod{p^3} && \text{by the lemma} \\ &\vdots \\ T(x) &\equiv T(y) \pmod{p^n} && \text{by Fermat's Little Theorem} \\ x &\equiv x^p \equiv x^{p^2} \equiv \dots \equiv T(x) \pmod{p} \end{aligned}$$

This proves **3**.

$$\begin{aligned} T(xy) &\equiv (xy)^{p^{n-1}} \equiv x^{p^{n-1}} y^{p^{n-1}} \\ &\equiv T(x)T(y) \pmod{p} \end{aligned}$$

So T is a homomorphism, suppose:

$$\begin{aligned} T(x) &\equiv T(y) \pmod{p^n} \\ \therefore T(x) &\equiv T(y) \pmod{p} \\ x &\equiv y \pmod{p} && \text{by 3.} \end{aligned}$$

$$\therefore T: \mathbb{F}_p^\times \mapsto (\mathbb{Z}/p^n)^\times \text{ is injective}$$

□

Corollary 3.11. *Let p be an odd prime, every element in $(\mathbb{Z}/p^n)^\times$ can be written uniquely in the form:*

$$\begin{aligned} &T(x) * \exp(py) \text{ with } x \in \mathbb{F}_p^\times \\ &py \in p\mathbb{Z}/p^n \end{aligned}$$

E.g $22 \in (\mathbb{Z}/125)^\times$

$$22 = T(x) \exp(5y) \pmod{125}$$

$$\equiv 2 \pmod{5}$$

$$\equiv x \pmod{5}$$

$$\implies x \equiv 2 \pmod{5}$$

$$22 = T(2) \exp(5y) \pmod{125}$$

$$22 * T(2^{-1}) \equiv \exp(5y) \pmod{125}$$

from the table $T(3) = 68$

$$\exp(5y) \equiv 22 * 68 \pmod{125}$$

$$\equiv 121 \pmod{125}$$

$$\equiv -4 \pmod{125}$$

$$\therefore 5y \equiv \log(-4) \pmod{125}$$

$$\equiv \log(1 - 5) \pmod{125}$$

$$\equiv -5 - \frac{25}{2} - \frac{125}{3} + \dots$$

$$\equiv -5 - 75$$

$$\equiv 45 \pmod{125}$$

$$\therefore 22 = T(2) \exp(45) \pmod{125}$$

E.g Calculate $22^{37} \pmod{125}$

$$22^{37} \equiv (T(2) \exp(45))^{37}$$

$$\equiv T(2^{37}) * \underbrace{\exp(45 * 37)}_{\equiv 40} \pmod{125}$$

$$\equiv 2 \pmod{5}$$

$$\therefore 22^{37} \equiv \underbrace{T(2)}_{\equiv 57} * \underbrace{\exp(40)}_{\equiv 841}$$

$$\equiv 57 * 91 \pmod{125}$$

$$\equiv 62 \pmod{125}$$

E.g. Calculate $T(23) \pmod{7^3}$

$$\begin{aligned} 23 &\equiv 2 \pmod{7} \\ 23^7 &\equiv 2^7 \equiv 128 \equiv 30 \pmod{7^2} \\ 23^{7^2} &\equiv 30^7 \pmod{7^3} \end{aligned}$$

Using the binomial theorem:

$$\begin{aligned} 23^{7^2} &\equiv (2 + 4 \cdot 7)^7 \pmod{7^3} \\ &\equiv 2^7 + 7 \cdot 2^6 \cdot 4 + \text{multiples of } 7^3 \pmod{7^3} \end{aligned}$$

Since $2^6 \cdot 4 \equiv 4 \pmod{7}$, it follows that $(7^2 \cdot 2^6 \cdot 4) \equiv (49 \cdot 4) \equiv 196 \pmod{7^3}$. This shows that $T(23) \equiv 128 + 196 \equiv 324 \pmod{7^3}$

The following corollary was stated but not proved:

Corollary 3.12. *Let p be an odd prime number. Every element of $(\mathbb{Z}/p^n)^\times$ can be written uniquely in the form $T(x) \cdot \exp(py)$ where $x \in \mathbb{F}_p^\times$ and $py \in p\mathbb{Z}/p^n$. This is an isomorphism of groups:*

$$(\mathbb{Z}/p^n)^\times \cong \mathbb{F}_p^\times * p\mathbb{Z}/p^n$$

Proof. Take any $a \in (\mathbb{Z}/p^n)^\times$ and let $x \equiv a \pmod{p}$. We have $a \equiv x \equiv T(x) \pmod{p}$. Therefore $aT(x)^{-1} \equiv 1 \pmod{p}$. This implies that $\log(\frac{a}{T(x)})$ converges p -adically. Let $py = \log(aT(x)^{-1})$. Then obviously $a = T(x) \exp(py)$.

For uniqueness, suppose $T(x) \cdot \exp(py) \equiv T(x') \cdot \exp(py') \pmod{p^n}$. Since the image of \exp is congruent to 1 \pmod{p} , we have $T(x) \equiv T(x') \pmod{p}$.

This implies $x \equiv x' \pmod{p}$. Therefore $T(x) \equiv T(x') \pmod{p^n}$. From this we get $\exp(py) \equiv \exp(py') \pmod{p^n}$. Taking logs we get $py \equiv py' \pmod{p^n}$

□

This corollary can be used to solve the following types of equations:

E.g $x^{21} \equiv 71 \pmod{81}$

Note 21 is not coprime to $\phi(81) = 54$, so previous methods cannot be used to solve this equation. Also $(1 + 70)^{\frac{1}{21}}$ does not converge 3-adically. Start with $71 \equiv 2 \pmod{3}$:

$$\implies 71 \equiv T(2) \exp(3y) \pmod{81}$$

$$T(2) \equiv -1 \pmod{81},$$

$$\implies \exp(3y) \equiv -71$$

$$\implies 3y \equiv \log(1 + 9) = 9 - \frac{81}{2} + \dots \equiv 9 \pmod{81}$$

$$\implies 71 = T(2) \exp(9) \pmod{81}$$

Suppose we also decompose $x = T(u) \exp(3v)$. Then

$$x^{21} = T(u^{21}) \exp(3 * 21v) = T(2) \exp(9)$$

Since such a representation is unique, this gives us two simultaneous equations:

$$u^{21} \equiv 2 \pmod{3} \implies u \equiv -1 \equiv 2 \pmod{3}$$

$$\begin{aligned} 63v &\equiv 9 \pmod{81} \implies 7v \equiv 1 \pmod{9} \\ v &\equiv 4 \pmod{9} \\ 3v &\equiv 12 \pmod{27} \end{aligned}$$

$$x \equiv T(2) \exp(12) \pmod{27}$$

$$\begin{aligned} \exp(12) &\equiv 1 + 12 + \frac{12^2}{2} + \frac{12^3}{6} + \frac{12^4}{4!} + \dots \\ &\equiv 1 + 12 + 72 + 288 \\ &\equiv 1 + 12 + 18 + 18 \\ &\equiv 22 \pmod{27} \end{aligned}$$

$$T(2) \equiv -1 \pmod{27}$$

So $x \equiv 5 \pmod{27} \implies x \equiv 5, 32, 59 \pmod{81}$

3.6 Fractional Powers

If p is an odd prime n and $a \equiv 1 \pmod{p}$ and $b \in \mathbb{Z}_{(p)}$ then a^b modulo p^n is:

$$a^b \equiv \exp(b \log(a)) \pmod{p^n}$$

The usual rules hold for powers:

- $(ab)^c \equiv a^c b^c \pmod{p^n}$
- $a^{b+c} \equiv a^b a^c \pmod{p^n}$
- $a^{bc} \equiv (a^b)^c \pmod{p^n}$

E.g $4^{\frac{1}{2}} \pmod{27}$ First find $\log(4) \pmod{27}$

$$\begin{aligned}\log(4) &\equiv (1 + 3) \pmod{27} \\ &\equiv 3 - \frac{9}{2} + \frac{27}{3} \pmod{27} \\ &\equiv 3 + 9 + 9 \pmod{27} \\ &\equiv 6 \pmod{27}\end{aligned}$$

So

$$\begin{aligned}4^b &\equiv \exp(-6b) \pmod{27} \\ 1 - 6b + \frac{36b^2}{2} - \frac{6^3b^3}{6} &\pmod{27} \\ 1 - 6b + 18b^2 - 36b^3 &\pmod{27} \\ 1 - 6b - 9b^2 - 9b^3 &\pmod{27}\end{aligned}$$

$$\begin{aligned}4^{\frac{1}{2}} &\equiv 1 - 3 - 9\left(\frac{1}{4} + \frac{1}{8}\right) \\ &\equiv 1 - 3\frac{27}{8} \\ &\equiv 1 - 3 \\ &\equiv -2 \pmod{27}\end{aligned}$$

3.7 P-adic integers

This section is slightly more highbrow way of looking at the results of the previous lectures. We've defined several congruency classes such as $\exp(px) \pmod{p^n}$, $T(a) \pmod{p^n}$, $\log(1 + px) \pmod{p^n}$. It's a little bit more convenient to be able to write down just $\exp(px)$, $T(a)$, etc ... without needing to write modulo p^n everywhere. The problem is that there is no integer (or even an element of the local ring) which is congruent $T(a) \pmod{p^n}$ for all n . Instead, we work in a bigger ring, the ring \mathbb{Z}_p of p-adic integers. In this ring, the expression $T(a)$, $\exp(px)$ etc all make sense.

Let p be any prime number. By a p-adic integer, we shall mean a p-adically convergent series

$$\sum_{i=1}^{\infty} a_i \qquad a_i \in \mathbb{Z}_{(p)}, V_p(a_i) \longrightarrow \infty$$

Recall that any series represents an element of \mathbb{Z}/p^n for every n . We call two p-adic integers equal if they are congruent modulo p^n for every n . The set of all p-adic integers is denoted \mathbb{Z}_p (without the brackets around the p). Note that we can add, subtract and multiply p-adically convergent series, so in fact \mathbb{Z}_p is a ring.

The advantage of this kind of notation is that we can write (for example) $\log(1 + px)$ to mean a p-adic integer, without having to reduce modulo p^n . This allows us to state many of the recent theorems more simply. If $a \in \mathbb{Z}$ or $\in \mathbb{Z}_{(p)}$, then we can regard a as the series $a = a + 0 + 0 + 0 + \dots$ and so a is a p-adic integer as well. Therefore $\mathbb{Z} \subset \mathbb{Z}_{(p)} \subset \mathbb{Z}$.

However, it turns out that there are many more p-adic integers than there are elements in the local ring $\mathbb{Z}_{(p)}$. For example consider the following 5-adic integer:

$$\begin{aligned} a &= (1 + 5)^{\frac{1}{2}} \\ &= 1 + \frac{1}{2} * 5 + \frac{\frac{1}{2} * \frac{-1}{2}}{2} + 5^2 + \dots \end{aligned}$$

In fact a is a square root of 6. We've shown earlier that $a^2 \equiv 6 \pmod{5^n}$ for all n and therefore $a^2 \equiv 6 \in \mathbb{Z}_5$. However, the local ring $\mathbb{Z}_{(5)}$ has no square roots of 6 since its elements are rational numbers. This shows that a is in \mathbb{Z}_5 but not $\mathbb{Z}_{(5)}$.

Proposition 3.13. *Every p-adic integer can be written uniquely in the form:*

$$\sum_{i=0}^{\infty} a_i p^i$$

with coefficients $a_i \in \{0, 1, \dots, p-1\}$

Proof.

Let x be a p-adic integer, so x is defined modulo p^n for all n . There is a unique choice of a_0 such that $a_0 \equiv x \pmod{p}$.

This means that $x - a_0$ is a multiple of p . There is a unique choice of a_1 such that $a_1 \equiv \frac{x - a_0}{p} \pmod{p}$.

This implies $pa_1 \equiv x - a_0 \pmod{p^2}$, so $x \equiv a_0 + a_1 p \pmod{p^2}$. This implies $x - a_0 - a_1 p$ is a multiple of p^2 and there is a unique a_2 such that $p^2 a_2 \equiv x - a_0 - a_1 p \pmod{p^3}$, etc. \square

We've already seen what it means for a series to converge p-adically. We'll now make a corresponding definition for sequences.

Definition 3.14. *Let a_n be a sequence for elements of $\mathbb{Z}_{(p)}$. We'll say that this sequence converges p-adically if the corresponding series:*

$$a_0 + (a_1 - a_0) + (a_2 - a_1) + \dots$$

If this is the case, then we define the limit of the sequence to be this series, regarded as an element of \mathbb{Z}_p . Note that the partial sums of the series above are exactly the terms of the sequence a_n . In fact, we have already seen many examples of p-adic limits.

Suppose $a_0 \in \mathbb{Z}_{(p)}$ satisfies the conditions of Hensel's lemma for a polynomial $f(x)$, i.e. $f(a_0) \equiv 0 \pmod{p^{2c+1}}$, where $c = V_p(f'(a_0))$. Consider the series:

$$a = a_0 + (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \dots$$

We'll show that this series converges p-adically. Recall that $a_{n+1} - a_n = \frac{f(a_n)}{f'(a_n)}$

When proving Hensel's lemma, we showed that $f(a_n) \equiv 0 \pmod{p^{2c+2^n}}$, $V_p(f'(a_n)) = c$

Therefore $V_p(a_{n+1} - a_n) \geq 2c + 2^n - c = c + 2^n \rightarrow \infty$

Hence a is a p -adic integer, and is congruent to $a_n \pmod{p^{c+2^n}}$. We can re-interpret Hensel's lemma as saying the following:

Proposition 3.15. *Let a_0 and f satisfy the conditions of Hensel's lemma and let $a \in \mathbb{Z}_p$ be the p -adic integer defined above. Then $f(a) = 0$*

Proof.

We just need to prove that $f(a) \equiv 0 \pmod{\text{any power of } p}$. But we have $f(a) \equiv f(a_n) \equiv 0 \pmod{p^{c+2^n}}$ □

Next consider Teichmüller lifts. For an odd prime p and an element $a \in \mathbb{Z}_{(p)}$ such that $p \nmid a$ let:

$$T(a) = a + (a^p - a) + (a^{p^2} - a^p) + (a^{p^3} - a^{p^2}) + (a^{p^4} - a^{p^3}) + \dots$$

We've shown that $a^{p^n} - a^{p^{n-1}} \equiv 0 \pmod{p^n}$, and therefore the valuation of the n -th term is at least n . This shows that the series converges p -adically, so $T(a) \in \mathbb{Z}_p$. Now the properties of Teichmüller lifts can be restated as follows:

Proposition 3.16. *The p -adic integer $T(a)$ depends only on the congruence class of a modulo p , and the map $T: \mathbb{F}_p^\times \mapsto \mathbb{Z}_p^\times$ is an injective group homomorphism.*

Proof.

Since $T(x) \equiv x \pmod{p}$, it follows that T is injective. For every n , we have $T(xy) \equiv T(x)T(y) \pmod{p^n}$. Therefore $T(xy) = T(x)T(y) \in \mathbb{Z}_p$ □

4 Quadratic rings

An integer d is called square-free if d is not a multiple of a square (apart from 1^2). Let d be a square-free integer with $d \neq 1$. Define a complex number α by:

$$\alpha = \begin{cases} \sqrt{d} & \text{when } d \not\equiv 1 \pmod{4} \\ \frac{1+\sqrt{d}}{2} & \text{when } d \equiv 1 \pmod{4} \end{cases}$$

Consider the set $\{x + y\alpha : x, y \in \mathbb{Z}\}$. This is called a "Quadratic Ring".

Lemma 4.1. *Every quadratic ring is a ring, i.e. closed under $+$, \times .*

Proof. Clearly closed under $+$

$$(x + y\alpha)(r + s\alpha) = xr + (xs + yr)\alpha + ys\alpha^2$$

Sufficient to show that α^2 is in the quadratic ring.

Case 1:

$$\alpha = \sqrt{d} \implies \alpha^2 = d \text{ which is in the quadratic ring}$$

Case 2:

$$\alpha = \frac{1 + \sqrt{d}}{2} \qquad d \equiv 1 \pmod{4}$$

$$\left(\alpha - \frac{1}{2}\right)^2 = \frac{d}{4}$$

$$\alpha^2 - \alpha + \frac{1}{4} = \frac{d}{4}$$

$$\alpha \equiv \alpha + \frac{d-1}{4} \qquad d-1 \equiv 0 \pmod{4} \implies \frac{d-1}{4} \in \mathbb{Z}$$

$\therefore \alpha^2$ is in the quadratic ring. □

We call $\mathbb{Z}[\alpha] = \{x + y\alpha : x, y \in \mathbb{Z}\}$:

- A real quadratic ring if $d > 0$
- A complex quadratic ring if $d < 0$

E.g $d = -1$

$$-1 \equiv 1 \pmod{4}$$

$$\therefore \alpha = \sqrt{-1} = i$$

$\mathbb{Z}[i] = \{x + iy : x, y \in \mathbb{Z}\}$ is the ring of Gaussian integers.

E.g $d = -3$

$$-3 \equiv 1 \pmod{4} \text{ so } \alpha = \frac{1+\sqrt{3}}{2}$$

This is the ring of Eisenstein integers. It is the same as $\mathbb{Z}[\zeta_3] = \zeta = e^{\frac{2\pi i}{3}}$.

Definition 4.2. Let $\mathbb{Z}[\alpha]$ be a quadratic ring. The elements all have the form $A = x + y\sqrt{d}$ where x, y are rational. The conjugate of such an element $\bar{A} = x - y\sqrt{d}$

4.0.1 Properties of conjugates

1 $\bar{\alpha} =$

$$1. \bar{\alpha} = \begin{cases} -\alpha & d \not\equiv 1 \pmod{4} \\ 1 - \alpha & d \equiv 1 \pmod{4} \end{cases}$$

$$2. \overline{A+B} = \bar{A} + \bar{B} \\ \overline{AB} = \bar{A} \bar{B}$$

$$3. \bar{A} \in \mathbb{Z}[\alpha] \text{ if } A \in \mathbb{Z}[\alpha]$$

$$4. \bar{\bar{A}} = A$$

Proof.

$$1. \text{ If } d \not\equiv 1 \pmod{4} \text{ then } \alpha = \sqrt{d} \\ \implies \bar{\alpha} = -\sqrt{d} = -\alpha$$

$$\text{If } d \equiv 1 \pmod{4} \text{ then } \alpha = \frac{1+\sqrt{d}}{2} \\ \implies \bar{\alpha} = \frac{1-\sqrt{d}}{2} = 1 - \alpha$$

2. Suppose:

$$A = x + y\sqrt{d}$$

$$B = r + s\sqrt{d}$$

Clearly $\overline{A+B} = \bar{A} + \bar{B}$

$$\begin{aligned} \overline{A+B} &= \overline{(x + y\sqrt{d})(r + s\sqrt{d})} \\ &= \overline{(xr + dys) + (xs + yr)\sqrt{d}} \\ &= \overline{(xr + dys) - (xs + yr)\sqrt{d}} \end{aligned}$$

$$\begin{aligned} \bar{A} \cdot \bar{B} &= (x - y\sqrt{d})(r - s\sqrt{d}) \\ &= (xr + dys) + (-xs - yr)\sqrt{d} \end{aligned}$$

3. Let $A = x + y\alpha$ $x, y \in \mathbb{Z}$

by **2.** $\bar{A} = x + y\alpha$

by **1.** $\bar{\alpha} \in \mathbb{Z}[\alpha]$

$\therefore \bar{A} \in \mathbb{Z}[\alpha]$

4. Trivial □

Definition 4.3. For an element $A \in \mathbb{Z}$ we define $N(A) = A\bar{A}$ - The norm of A

Remark: If $\mathbb{Z}[\alpha]$ is a complex quadratic ring then \bar{A} is the complex conjugate of A . This means $N(A) = |A|^2$

E.g. $d = -1 \implies \mathbb{Z}[\alpha] = \mathbb{Z}[i]$

The elements have the form $x + iy$, $x, y \in \mathbb{Z}$

$$\begin{aligned} N(x + iy) &= (x + iy)(x - iy) \\ &= x^2 + y^2 \end{aligned}$$

E.g. $d = -3 \implies -3 \equiv 1 \pmod{4}$, so $\alpha = \frac{1+\sqrt{-3}}{2}$

$$\begin{aligned} N(x + y\alpha) &= (x + y\alpha)(x + y\bar{\alpha}) & \bar{\alpha} &= 1 - \alpha \\ &= x^2 + xy \underbrace{(\alpha + 1 - \alpha)}_{=1} + y^2 \underbrace{(\alpha(1 - \alpha))}_{=1} \end{aligned}$$

Note that:

$$\begin{aligned} \alpha &= \frac{1 + \sqrt{d}}{2} \\ \implies \left(\alpha - \frac{1}{2}\right)^2 &= \frac{d}{4} \\ \implies \alpha^2 - \alpha + \frac{1}{4} &= \frac{d}{4} \\ \implies \alpha(1 - \alpha) &= \frac{1 - d}{4} \end{aligned}$$

In this case:

$$N(x + y\alpha) = x^2 + xy + y^2$$

4.0.2 Formula for norms

The general formula for norms is given by:

$$N(x + y\alpha) = \begin{cases} x^2 - dy^2 & \text{if } d \not\equiv 1 \pmod{4} \\ x^2 + xy + \frac{1-d}{4}y^2 & \text{if } d \equiv 1 \pmod{4} \end{cases}$$

Case 1: $d \neq 1$ (4)

$$\implies \bar{\alpha} \equiv -\alpha$$

$$N(x + y\alpha) = (x + y\alpha)(x - y\alpha) = x^2 - dy^2 \quad \alpha^2 = d$$

Case 2: $\bar{\alpha} = 1 - \alpha$

$$\implies \alpha = \frac{1+\sqrt{d}}{2}$$

$$\begin{aligned} N(x + y\alpha) &= (x + y\alpha)(x + y(1 - \alpha)) \\ &= x^2 + xy + y^2(\underbrace{\alpha - \alpha^2}_{\frac{1-d}{4}}) \end{aligned}$$

4.0.3 Properties of norms

1. $N(A) \in \mathbb{Z}$
2. $N(AB) = N(A)N(B)$
3. If $N(A) = 0$ then $A = 0$

Proof.

1. Follows from formulas for norms
2. $N(AB) = AB\overline{AB} = AB\bar{A}\bar{B}$
3. If $N(A) = 0$ then $A\bar{A} = 0$
 \therefore either $A = 0$ or $\bar{A} = 0$
 $\therefore \bar{A} = 0$ then $A = \bar{A} = \bar{0} = 0$

□

Recall - A unit in a ring R is an element with an inverse in R .

E.g. $1 + \sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}] \implies (1 + \sqrt{2})(\sqrt{2} - 1) = 1$

Corollary 4.4. *An element $A \in \mathbb{Z}[\alpha]$ is a unit if and only if $N(A) = \pm 1$*

Proof. If $N(A) = \pm 1$ then $A\bar{A} = \pm 1$

$$\implies A^{-1} = \pm A \in \mathbb{Z}[\alpha]. \text{ So } A \text{ is a unit.}$$

□

If A is a unit with inverse B , $AB = 1 \implies N(A)N(B) = N(AB) = N(1) = 1$

Using this proposition, it's easy to find all the units in any complex quadratic ring.

E.g The units in $\mathbb{Z}[i]$ are $\pm 1, \pm i$

Proof. Since $N(x + iy) = x^2 + y^2$, the units correspond to the solutions to $x^2 + y^2 = 1$.

These solutions are $x = \pm 1, y = 0$ and $x = 0, y = \pm 1$

□

E.g. The units in the Eisenstein integers $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ are $\pm 1, \pm \alpha, \pm(\alpha - 1)$. Equivalently these are $\pm 1, \pm \zeta_3, \pm \zeta_3^2$

Proof. Find all solutions to $x^2 + xy + y^2 = 1$. We can complete the square to get:

$$(x + \frac{1}{2}y)^2 + \frac{3}{4}y^2 = 1$$

$y^2 < \frac{4}{3}$ and since y is an integer $|y| < 1$
Similarly $|x| < 1$.

If $y = 0$ then $x = \pm 1$

If $y = \pm 1$ then $x^2 + xy + 1 = 1 \implies x = 0, -y$ are solutions.

So the 6 solutions are $(1, 0), (-1, 0), (0, 1), (-1, 1), (0, -1), (1, -1)$

□

Corollary 4.5. If $d < 0$ and $d \neq -1, -3$ then the units in $\mathbb{Z}[\alpha]$ are $\{1, -1\}$

Proof. Assume first that $d \not\equiv 1 \pmod{4}$

$$\implies N(x + y\alpha) = x^2 - dy^2 \quad x, y \in \mathbb{Z} \text{ and } -d > 1 \text{ so } y = 0 \implies x = \pm 1$$

So $(1, 0), (-1, 0)$ give us the two units $1, -1$

Assume now $d \equiv 1 \pmod{4} \implies -d \geq -7$ and need to find solutions to the equation:

$$x^2 + xy + \frac{1-d}{4}y^2 = 1$$

$$\implies (x + \frac{1}{2}y)^2 - \frac{d}{4}y^2 = 1$$

Since $\frac{d}{4} > 1, y^2 < 1 \implies y = 0 \implies x = \pm 1$

□

4.1 Norm-Euclidean quadratic rings

Definition 4.6. A quadratic ring $\mathbb{Z}[\alpha]$ is norm-Euclidean if $\forall A, B \in \mathbb{Z}[\alpha]$ with $B \neq 0$ $\exists Q, R \in \mathbb{Z}[\alpha]$ such that:

- $A = QB + R$
- $|N(R)| < N(B)$

Finitely many of the quadratic rings are norm-Euclidean.

Theorem 4.7. If $\mathbb{Z}[\alpha]$ is norm-Euclidean then every non-zero element of $\mathbb{Z}[\alpha]$ can be factorised as $UQ_1 \dots Q_r$

Q_i are irreducible elements of $\mathbb{Z}[\alpha]$, U is a unit.

This factorisation is unique in the sense that if $U_1Q_1 \dots Q_r = U_2R_1 \dots R_s$ then $r = s$ and (after reordering Q_r, R_i is a unit for each U

E.g Let $d = -7$ so $\alpha = \frac{1+\sqrt{-7}}{2}$. This is norm-Euclidean.

Suppose $z = x + y\sqrt{-7}$ $x, y \in \mathbb{Q}$. We'll show that there is an element $Q \in \mathbb{Z}[\alpha]$ such that $|N(Z - Q)| < 1$.

Choose $b \in \mathbb{Z}$ such that $|y - \frac{b}{2}| < \frac{1}{4}$

Note that $z - b\alpha = (x - \frac{b}{2}) + (y - \frac{b}{2})\sqrt{-7}$

Then choose $a \in \mathbb{Z}$ so that $|x - \frac{b}{2} - a| \leq \frac{1}{2}$. Also note that the maximum distance to the closest integer is $\frac{1}{2}$.

We let $Q = a + b\alpha$ and we have:

$$Z - Q = (x - \frac{b}{2} - a) + (y - \frac{b}{2})\sqrt{-7}$$

$$N(Z - Q) \leq (\frac{1}{2})^2 + 7(\frac{1}{4})^2 = \frac{11}{16} < 1$$

Now to show the ring is norm-Euclidean. Choose $A, B \in \mathbb{Z}[\alpha]$ with $B \neq 0$. By what we've shown there is an element $Q \in \mathbb{Z}[\alpha]$ such that $|N(\frac{A}{B} - Q)| \leq 1$

Let $R = A - QB$ then $A = QB + R$

$$\begin{aligned} |N(R)| &= |N(A - QB)| \\ &= |N(\frac{A}{B} - Q)N(B)| \\ &< |N(B)| \end{aligned}$$

E.g Let $d = 3$. In this case $\alpha = \sqrt{3}$. We'll show that the quadratic ring $\mathbb{Z}[\sqrt{3}]$ is norm-Euclidean.

Let $z = x + y\sqrt{3}$ with $x, y \in \mathbb{Q}$. Need to show there is an element $Q = r + s\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$ such that $|N(Z - Q)| < 1$. Choose $r, s \in \mathbb{Z}$ such that:

$$|x - r| \leq \frac{1}{2} \qquad |y - s| \leq \frac{1}{2}$$

$$\begin{aligned} \implies N(Z - A) &= (x - r)^2 - 3(y - s)^2 \\ \implies -\frac{3}{4} &\leq N(Z - Q) \leq \frac{1}{4} \\ \implies |N(Z - Q)| &< 1 \end{aligned}$$

Now to show that $\mathbb{Z}[\sqrt{3}]$ is norm-Euclidean. Suppose $A, B \in \mathbb{Z}[\sqrt{3}]$ with $B \neq 0$, there is already a $Q \in \mathbb{Z}[\sqrt{3}]$ such that $|N(\frac{A}{B} - Q)| < 1$.

Let $R = A - QB$. This implies $A = QB + R$

$$\begin{aligned} \implies |N(R)| &= |N(A - QB)| \\ &= |N(\frac{A}{B} - Q)N(B)| \\ &< |N(B)| \end{aligned}$$

Hence $\mathbb{Z}[\sqrt{3}]$ is norm-Euclidean.

Theorem 4.8. *The disappointing theorem - The quadratic rings with $d = -1, -2, -3, -7, 1, 2, 3, 5, 13$ are norm-Euclidean.*

Definition 4.9. *Suppose $A, B \in \mathbb{Z}[\alpha]$. A highest common factor of A and B is an element $C \in \mathbb{Z}[\alpha]$ with the following properties:*

- C is a factor of both A and B i.e. $\frac{A}{C}$ and $\frac{B}{C}$ are both in $\mathbb{Z}[\alpha]$
- If D is a factor of both A and B then D is a factor of C (and hence $|N(D)| \leq |N(C)|$)

If C is a highest common factor of A and B , then so is UC for every unit U , but these are all the highest common factors. Hence highest common factors, if they exist are unique up to multiplication by a unit.

Lemma 4.10. *Bezout's Lemma - Let $\mathbb{Z}[\alpha]$ be norm-Euclidean ring and let $A, B \in \mathbb{Z}[\alpha]$ not both 0. Then there is a highest common factor C of A, B and there exist $H, K \in \mathbb{Z}[\alpha]$, such that $HA + KB = C$*

Proof. The proof goes similarly to in the ring \mathbb{Z} . We prove by induction on $\min(|N(A)|, |N(B)|)$. The induction step consists of writing $A = QB + R$ with $|N(R)| < |N(B)|$ and using the lemma. To prove the start of the induction, we assume $B = 0$. But then it's easy to check that A is a highest common factor. □

Definition 4.11. *An element $P \in \mathbb{Z}[\alpha]$ is called irreducible if:*

- P is not a unit
- If $P = AB$ with $A, B \in \mathbb{Z}[\alpha]$ then either A or B is a unit

Definition 4.12. *We'll say that a quadratic ring $\mathbb{Z}[\alpha]$ has unique factorisation if the following is true:*

- For every non-zero element $A \in \mathbb{Z}[\alpha]$ there is a factorisation $A = UP_1 \dots P_r$ with U a unit and each P_i irreducible
- If we have another factorisation $A = U'Q_1 \dots Q_s$, then $r = s$ and we can reorder Q_1, \dots, Q_s so that each P_i/Q_i is a unit.

Lemma 4.13. *Let $\mathbb{Z}[\alpha]$ be norm-Euclidean. Let p be irreducible and suppose $P|AB$ in $\mathbb{Z}[\alpha]$. Then $P|A$ or $P|B$*

Proof. Suppose P does not divide A . Then the highest common factor of P and A is not P , so it must be 1. Therefore we can find $H, K \in \mathbb{Z}[\alpha]$ such that $HP + KA = 1$. This implies $B = HPB + KPB$ which is a multiple of P . \square

Theorem 4.14. *If $\mathbb{Z}[\alpha]$ is norm-Euclidean, then $\mathbb{Z}[\alpha]$ has unique factorisation.*

Proof.

The proof is exactly as for \mathbb{Z} (using the previous lemma for the uniqueness part), except that we prove by induction on $N(A)$. \square

In fact, there are many examples when $\mathbb{Z}[\alpha]$ has unique factorisation, even though it is not norm-Euclidean. It's known that a complex quadratic ring has unique factorisation for exactly the following values of d and no more:

$$d = -1, -2, -3, -7, -11, -19, -43, -67, -163$$

In contrast, it is much more common for a real quadratic ring to have unique factorisation. In fact, the following is believed (but not proved):

Conjecture: There are infinitely many positive square-free integers d such that $\mathbb{Z}[\alpha]$ has unique factorisation. On the other hand, there are many quadratic rings which do not have unique factorisation.

E.g In the ring $\mathbb{Z}[\sqrt{-5}]$ we have non-unique factorisation. For example $6 = 2 * 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.

The elements $2, 3, 1 \pm \sqrt{-5}$ are all irreducible. To see this, note that they have norms 2, 9, 6&6. Hence any proper factors would have norm 2 and 3. However the ring $\mathbb{Z}[\sqrt{-5}]$ has no elements of norm 2 and 3 since $x^2 + 5y^2$ is never equal to 2 or 3 for integers x, y .

4.2 The Decomposition Theorem

Assume that $\mathbb{Z}[\alpha]$ is a quadratic ring with unique factorisation into irreducible elements, e.g. $\mathbb{Z}[\alpha]$ could be norm-Euclidean.

Lemma 4.15. *If Q is an irreducible element in $\mathbb{Z}[\alpha]$ then there exists a unique prime number p such that $Q|p$*

Proof.

$$Q|N(Q) = \pm p_1 p_2 \dots p_r \quad (p_i \text{ prime})$$

By uniqueness of factorisation $Q|p_i$ for some i if $Q|p$ and $Q|q$ where p, q are distinct primes, $hcf(p, q) = 1 = hp + kq \quad (h, k \in \mathbb{Z}) \implies Q|1 \nmid$ \square

The lemma means that to find all the irreducible elements, we just need to factorise all the primes in $\mathbb{Z}[\alpha]$. Suppose $Q|p$, where Q is irreducible in $\mathbb{Z}[\alpha]$, p prime:

$$\begin{aligned} &\implies N(Q)|N(p) = p^2 \\ &\implies N(Q) = \pm p \text{ or } \pm p^2 \end{aligned}$$

If $N(Q) = \pm p^2$ then $Q = \text{unit} * p$ so p is irreducible.

- If $P = Q_1 Q_2$ where $\frac{Q_1}{Q_2}$ is not a unit, then we say P is **split** in $\mathbb{Z}[\alpha]$
- If $P = U Q^2$ (U a unit, Q irreducible) then we say P is **ramified** in $\mathbb{Z}[\alpha]$
- If P is irreducible in $\mathbb{Z}[\alpha]$ then we say that P is **inert** in $\mathbb{Z}[\alpha]$

E.g $d = -1$ $\alpha = \sqrt{-1} = i$ $N(x + iy) = x^2 + y^2$

A prime number p factorises in $\mathbb{Z}[i] \implies$ there is an element with norm $\pm p$

- $2 = 1^2 + 1^2 = N(1 + i) = (1 + i)(1 - i) = -i(1 + i)^2 \implies 2$ is ramified
- 3 is inert
- $5 = 2^2 + 1^2 = (2 + i)(2 - i)$ 5 is split
- 7 is inert
- 11 is inert
- $13 = 3^2 + 2^2 = (3 + 2i)(3 - 2i)$ 13 is split

Check if a number is ramified by dividing one by the other and checking if the result is in the ring.

E.g $d = -3$ $\alpha = \frac{1+\sqrt{3}}{2}$ $N(x + y\alpha) = x^2 + xy + y^2$

- 2 inert
- $3 = -\sqrt{-3}^2 = -(1 - 2\alpha)^2$ 3 ramified
- 5 inert
- $7 = N(2 + \alpha) = (2 + \alpha)(3 - \alpha)$
- 11 inert

Assume $\mathbb{Z}[\alpha]$ is a quadratic ring with unique factorisation. Let p be an odd prime number.

$$\begin{array}{ll} p \text{ is ramified} & \iff p|d \\ p \text{ is split} & \iff \left(\frac{d}{p}\right) = 1 \\ p \text{ is inert} & \iff \left(\frac{d}{p}\right) = -1 \end{array} \quad \begin{array}{ll} 2 \text{ splits} & \iff d \equiv 1 \pmod{8} \\ 2 \text{ inert} & \iff d \equiv 5 \pmod{8} \\ \text{in other cases } 2 & \text{ is ramified} \end{array}$$

Idea of proof:

Assume $d \not\equiv 1 \pmod{4}$, $N(x + y\alpha) = x^2 - dy^2$

If p factorises then $\exists x, y \in \mathbb{Z}, x^2 - dy^2 = \pm p$

$$\begin{aligned} &\implies x^2 \equiv dy^2 \pmod{p} \\ &\implies \left(\frac{x}{y}\right)^2 \equiv d \pmod{p} \end{aligned}$$

If d is a quadratic residue then $x^2 \equiv d \pmod{p} \implies p \mid (x + \sqrt{d})(x - \sqrt{d})$.

If p were inert then $p \mid x + \sqrt{d}$ or $x - \sqrt{d} \nmid$ (number not in ring)
 \implies factorises

4.3 Solving $|N(A)| = n$

Assume that $\mathbb{Z}[\alpha]$ has unique factorisation, does the equation $|N(A)| = n$ have solutions?

E.g $d = -1 \quad \mathbb{Z}[\alpha] = \mathbb{Z}[i] \quad N(x + iy) = x^2 + y^2$

$2 = 1^2 + 1^2$	$8 = 2^2 + 2^2$
$3 \times$	$9 = 3^2 + 0^2$
$4 = 2^2 + 0^2$	$10 = 3^2 + 1^2$
$5 = 2^2 + 1^2$	$11 \times$
$6 \times$	$12 \times$
$7 \times$	$13 = 3^2 + 2^2$

The answer is a corollary to the Decomposition Theorem.

Corollary 4.16. Assume $\mathbb{Z}[\alpha]$ has unique factorisation and let n be a positive integer. Then the following are equivalent:

1. $\exists A \in \mathbb{Z}[\alpha] : |N(A)| = n$
2. \forall inert primes $p \mid n$, $V_p(n)$ is even

Proof.

1 \implies 2 Assume $|N(A)| = n$

$$\begin{aligned} A &= Q_1^{a_1} \dots Q_r^{a_r} \text{ for } Q_i \text{ irreducible in } \mathbb{Z}[\alpha] \\ &\quad Q_i \mid P_i \quad (p_i \text{ prime}) \end{aligned}$$

$$|N(Q_i)| = \begin{cases} P_i & \text{if } P_i \text{ splits or is ramified} \\ P_i^2 & P_i \text{ inert} \end{cases}$$

$$n = |N(A)| = \left(\prod_{\substack{P_i \text{ split} \\ \text{or ramified}}} P_i^{a_i} \right) * \left(\prod_{P_i \text{ inert}} P_i^{2a_i} \right)$$

So powers of inert primes are even.

2 \implies **1** Let

$$n = \left(\prod_{\substack{P_i \text{ split} \\ \text{or ramified}}} P_i^{a_i} \right) * \left(\prod_{P_i \text{ inert}} P_i^{2a_i} \right)$$

Choose an element Q_i with norm $\pm P_i$ if P_i is split or ramified:

$$n = N \left(\prod_{\substack{P_i \text{ ramified} \\ \text{or split}}} Q_i^{a_i} \times \prod_{P_i \text{ inert}} P_i \right)$$

□

E.g Solve $x^2 + y^2 = 585$ i.e. $N(x + iy) = 585$

Note $585 = 3^2 * 5 * 13$

- 3 is inert because $\left(\frac{-1}{3}\right) = -1$
- 5 is split because $\left(\frac{-1}{5}\right) = +1$
- 13 is split because $\left(\frac{-1}{13}\right) = +1$

The only inert prime factor of 585 is 3 and its power is even so $x^2 + y^2 = 585$ will have solutions.

$$\begin{aligned} 5 &= 2^2 + 1^2 = N(2 + i) = (2 + i)(2 - i) \\ 13 &= 3^2 + 2^2 = N(3 + 2i) = (3 + 2i)(3 - 2i) \end{aligned}$$

$$\begin{aligned} 585 &= 3^2 * 5 * 13 \\ &= N(3 * (2 + i)(3 + 2i)) \\ &= N(3(6 + 7i - 2)) \\ &= N(12 + 21i) \\ &= 12^2 + 21^2 \end{aligned}$$

The other elements of norm 585 are unit multiples of it:

- $3(2+i)(3-2i) = 24 - 3i$
- $3(2-i)(3+2i) = 24 + 3i$
- $3(2-i)(3-2i) = 12 - 21i$

E.g. $x^2 + xy + y^2 = 84$

Note that $84 = 2^2 * 3 * 7$

- 2 is inert in $\mathbb{Z}[\alpha]$
 - $3 = -(\sqrt{3})^2 = -(1 - 2\alpha)^2 \implies$ ramified
 - $7 \left(\frac{-3}{7} \right) = \left(\frac{4}{7} \right) = 1 \implies 7$ splits
- $$7 = N(2 + 2\alpha) = (2 + \alpha)(3 - \alpha)$$

So the elements with norm 84 are:

- $2(1 - 2\alpha)(2 + \alpha) * \text{unit}$
- $2(1 - 2\alpha)(3 - \alpha) * \text{unit}$

There are 6 units in this ring, therefore there are 12 solutions to $x^2 + xy + y^2 = 84$

One possible solution is:

$$\begin{aligned} 2(1 - 2\alpha)(2 + \alpha) &= 2(2 - 3\alpha + 3\alpha^2) \\ &= 2(2 - 3\alpha - 2\alpha + 2) \\ &= 8 - 10\alpha \end{aligned}$$

so $N(8 - 10\alpha) = 84$

Super cool trick

Using $\alpha = \frac{1+\sqrt{d}}{2}$

$$\begin{aligned} \left(\alpha - \frac{1}{2} \right)^2 &= \frac{d}{4} \\ \alpha^2 - \alpha + \frac{1}{4} &= \frac{d}{4} \\ \alpha^2 &= \frac{d-1}{4} + \alpha \end{aligned}$$

4.4 Continued Fractions

A finite continued fraction is $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots \frac{1}{a_n}}}}$ where $a_0 \in \mathbb{Z}$

and $a_1, \dots, a_n \in \mathbb{Z} > 0$.

We'll use the notation $[a_0, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots \frac{1}{a_n}}}}$

$$\text{E.g. } [1, 2, 3, 2] = 1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{2}}} = 1 + \frac{1}{2 + \frac{1}{7/2}} = 1 + \frac{1}{2 + \frac{2}{7}} = 1 + \frac{7}{16} = \frac{23}{16}$$

More generally if $\alpha \in \mathbb{R}$, $\alpha > 0$

$$[a_0, \dots, a_n, \alpha] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots \frac{1}{a_n + \frac{1}{\alpha}}}}}$$

and as a consequence $[a_0, \dots, a_n] = [a_0, \dots, a_r, [a_{r+1}, \dots, a_n]]$

$$\text{E.g. } [1, 2, 3, 2] = [1, 2, [3, 2]] = [1, 2, \frac{7}{2}] = [1, 2 + \frac{2}{7}] = [1, \frac{16}{7}] = 1 + \frac{7}{16} = \frac{23}{16}$$

Clearly every finite continued fraction is in \mathbb{Q} . Conversely if $\frac{n}{m} \in \mathbb{Q}$, then we can write $\frac{n}{m}$ as a finite continued fraction.

E.g. By Euclid's algorithm:

$$\begin{array}{ll}
89 = 2 * 39 + 11 & 89/39 = 2 + \frac{11}{39} \\
39 = 3 * 11 + 6 & 39/11 = 3 + \frac{6}{11} \\
11 = 1 * 6 + 5 & 11/6 = 1 + \frac{5}{6} \\
6 = 1 * 5 + 1 & 6/5 = 1 + \frac{1}{5} \\
5 = 1 * 5 + 0 & 5/1 = 5
\end{array}$$

Therefore $\frac{89}{39} = 2 + \frac{1}{3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5}}}} = [2, 3, 1, 1, 5]$

Now suppose we have a sequence $a_n \in \mathbb{Z}$ for all $n, a_1, a_2, \dots > 0$. For any n we have a finite continued fraction $[a_0, \dots, a_n] = \frac{h_n}{k_n} \in \mathbb{Q}$

We define $[a_0, a_1, \dots] = \lim_{n \rightarrow \infty} [a_0, \dots, a_n] = \lim_{n \rightarrow \infty} \frac{h_n}{k_n}$

Definition 4.17. $[a_0, a_1, \dots]$ is called an infinite continued fraction

Theorem 4.18. For any sequence of integers $a_n > 0$ for $n > 0$, the limit $[a_0, \dots]$ exists. If $\alpha = [a_0, a_1, \dots]$ then $\left| \alpha - \frac{h_n}{k_n} \right| < \frac{1}{k_n^2}$ (will be proved later)

Sometimes we can calculate infinite continued fractions.

E.g. $\alpha = [1, 2, 1, 2, 1, 2, 1, 2, \dots]$ Which real number is α ?

$$\begin{aligned}
\alpha &= [1, 2, \alpha] \\
&= \left[1, 2 + \frac{1}{\alpha} \right] \\
&= \left[1, \frac{2\alpha + 1}{\alpha} \right] \\
&= 1 + \frac{\alpha}{2\alpha + 1} \\
&= \frac{3\alpha + 1}{2\alpha + 1}
\end{aligned}$$

$$\begin{aligned} 2\alpha^2 + \alpha &= 3\alpha + 1 \\ 2\alpha^2 - 2\alpha - 1 &= 0 \end{aligned}$$

$$\alpha = \frac{1 \pm \sqrt{3}}{2}$$

Since $\alpha = 1 + \frac{2}{1 + \dots} > 1$, $\alpha = \frac{1 + \sqrt{3}}{2}$.

Every infinite continued fraction converges to a real number. Conversely if α is an irrational real number, then we can write α as an infinite continued fraction.

Method: We define a sequence $\alpha_n \in \mathbb{R}$, $a_n \in \mathbb{Z}$ such that $\alpha_0 = \alpha$ and $a_n = \lfloor \alpha_n \rfloor$

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n} > 1 \qquad a_n > 0$$

From this definition:

$$\begin{aligned} \alpha &= \alpha_0 \\ &= a_0 + \frac{a_0}{\alpha_1} \\ &= a_0 + \frac{1}{a_1 + \frac{1}{\alpha_2}} \\ &= [a_0, a_1, a_2, \alpha_3] \text{ etc} \end{aligned}$$

Using this we can show that $\alpha = [a_0, a_1, \dots]$

E.g. Write $\sqrt{2}$ as an infinite continued fraction

$$\begin{aligned} \alpha_0 &= \sqrt{2} & a_0 &= \lfloor \sqrt{2} \rfloor = 1 \\ \alpha_1 &= \frac{1}{\alpha_0 - a_0} & a_1 &= \lfloor \sqrt{2} + 1 \rfloor = 2 \\ &= \frac{1}{\sqrt{2} - 1} \\ &= \frac{\sqrt{2} + 1}{(\sqrt{2} - 1)(\sqrt{2} + 1)} \\ &= \frac{\sqrt{2} + 1}{2 - 1} \\ &= \sqrt{2} + 1 \end{aligned}$$

$$\begin{aligned}
\alpha_2 &= \frac{1}{\alpha_1 - a_1} \\
&= \frac{1}{(\sqrt{2} + 1) - 2} \\
&= \frac{1}{\sqrt{2} - 1} \\
&= \alpha_1
\end{aligned}$$

$$a_2 = \lfloor \alpha_2 \rfloor = \lfloor \alpha_1 \rfloor = 2$$

$$\begin{aligned}
\alpha_3 &= \frac{1}{\alpha_2 - a_2} \\
&= \frac{1}{\alpha_1 - a_1} \\
&= \alpha_2
\end{aligned}$$

$$a_3 = \lfloor a_1 \rfloor = 2$$

So $\alpha_2 = \alpha_3 = \alpha_4 = \dots = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$ and $a_3, a_4, a_5 = 2$

Therefore $\sqrt{2} = [a_0, a_1, a_2, \dots] = [1, 2, 2, 2, \dots]$.

Using this method we can write \sqrt{d} for any +ve d as an infinite continued fraction.

Recall that an element in $\mathbb{Z}[\sqrt{2}]$ is a unit if its norm is ± 1 , i.e. elements $x + y\sqrt{2}$ where $x^2 - 2y^2 = \pm 1$ i.e. $\left| \left(\frac{x}{y} \right)^2 - 2 \right| = 1$ so $\frac{x}{y}$ is close to $\sqrt{2}$.

Let $\frac{h_n}{k_n} = \underbrace{[1, 2, 2, \dots, 2]}_{n \text{ terms}}$

This is close to $\sqrt{2}$

$[1] = 1/1 = 1$	$1^2 - 2 * 1^2 = -1$
$[1, 2] = 1 + 1/2 = 3/2$	$3^2 - 2 * 2^2 = +1$
$[1, 2, 2] = 7/5$	$7^2 - 2 * 5^2 = -1$
$[1, 2, 2, 2] = 17/12$	$17^2 - 2 * 12^2 = +1$

In this case when $\frac{h}{k} = [1, 2, \dots, 2]$, we always have $h^2 - 2 * k^2 = \pm 1$, so $h + k\sqrt{2}$ is a unit in $\mathbb{Z}[\sqrt{2}]$

4.5 Pell's equation and units in real quadratic rings

Let $d > 1$ be a square free integer. Pell's equation is $x^2 - dy^2 = 1$. We'll see how to find the solutions (x, y) in integers.

Let $A = x + y\sqrt{d}$. Pell's equation $\leftrightarrow N(A) = 1$. Therefore A is a unit in $\mathbb{Z}[\sqrt{d}]$ with norm 1.

There are obvious solutions $x = \pm 1, y = 0$. We'll call these the trivial solutions, these correspond to the units $A = \pm 1$

Theorem 4.19. *For any d , there are non-trivial solutions*

Definition 4.20. *The smallest solution (x, y) with $x, y > 0$ is called the fundamental solution*

E.g. $d = 2$

$x^2 - 2y^2 = 1$ and so $(x, y) = (3, 2)$ is the fundamental solution

A^n is also a unit with norm 1. This gives an infinite sequence of solutions to Pell's equations.

$$A^2 = (3 + 2\sqrt{2})^2 = 9 + 12\sqrt{2} + 8 = 17 + 12\sqrt{2}$$

$$A^3 = (3 + 2\sqrt{2})(17 + 12\sqrt{2}) = 99 + 70\sqrt{2}$$

So $(17, 12)$ and $(99, 70)$ are also the solutions to $x^2 - 2y^2 = 1$

Proposition 4.21. *If (x, y) is the fundamental solution, then any other solution in positive integers will be (x_n, y_n) where $(x_n + y_n\sqrt{d}) = (x + y\sqrt{d})^n$*

Proof. Let $A = x + y\sqrt{d}$

A is the smallest unit with norm 1 such that $A > 1$

Let B be any unit in $\mathbb{Z}[\sqrt{d}]$ which is bigger than 1, and has norm 1.

Want to show $B = A^n$ for some n

$$1 < A < A^2 < A^3 < \dots \rightarrow \infty$$

There exists n such that $A^n \leq B < A^{n+1} \implies 1 \leq A^{-n}B < A$

$A^{-n}B$ is a unit with norm 1. By choice of A , $A^{-n}B = 1$ and $B = A^n$ □

Once we have the fundamental solution, we've solved the equation. Sometimes the fundamental solution is big so it's difficult to find, for example:

$$d = 151 \implies x^2 - 151y^2 = 1$$

Fundamental solution $x = 1728148040, y = 140634693$

So we need a fast way of finding the fundamental solution.

If $\alpha \in \mathbb{R}$ and α is irrational then it has an infinite continued fraction expansion.

$$\alpha = [a_0, a_1, \dots] = \lim_{n \rightarrow \infty} a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots \frac{1}{\ddots a_n}}}$$

The limit above converges to \sqrt{d} and $[a_0, \dots, a_n]$.

Let $\frac{h_n}{k_n} = [a_0, \dots, a_n]$.

Definition 4.22. The rational numbers $\frac{h_n}{k_n}$ are called **convergents** of α

Theorem 4.23. If $\frac{h}{k} \in \mathbb{Q}$ with $\left| \alpha - \frac{h}{k} \right| < \frac{1}{2k^2}$ then $\frac{h}{k}$ must be one of the convergents of α

Corollary 4.24. If (x, y) is a solution to Pell's equation $x^2 - dy^2 = 1$ for $x > y > 0$, then $\frac{x}{y}$ is a convergent in the continued fraction of \sqrt{d}

Proof. (Corollary)

$$\begin{aligned} x^2 - dy^2 &= 1 \\ (x + y\sqrt{d})(x - y\sqrt{d}) &= 1 \end{aligned}$$

$$\begin{aligned} |x - y\sqrt{d}| &= \frac{1}{x + y\sqrt{d}} < \frac{1}{2y} \\ \left| \frac{x}{y} - \sqrt{d} \right| &< \frac{1}{2y^2} \end{aligned}$$

$\frac{x}{y}$ is a convergent. □

E.g. $d = 7 \implies x^2 - 7y^2 = 1$

To find the fundamental solution we find $\sqrt{7}$ as a continued fraction.

$$\begin{aligned}
\alpha_0 &= \sqrt{7} & a_n &= \lfloor \alpha_n \rfloor \\
\alpha_{n+1} &= \frac{1}{\alpha_n - a_n} \\
\alpha_1 &= \frac{1}{\sqrt{7} - 2} = \frac{\sqrt{7} + 2}{(\sqrt{7} - 2)(\sqrt{7} + 2)} = \frac{\sqrt{7} + 2}{3} & a_1 &= 1 \\
\alpha_2 &= \frac{1}{\frac{\sqrt{7} + 2}{3} - 1} = \frac{3}{\sqrt{7} - 1} = \frac{3(\sqrt{7} + 1)}{(\sqrt{7} - 1)(\sqrt{7} + 1)} = \frac{\sqrt{7} + 1}{2} & a_2 &= 1 \\
\alpha_3 &= \frac{1}{\frac{\sqrt{7} + 1}{2} - 1} = \frac{2}{\sqrt{7} - 1} = \frac{2(\sqrt{7} + 1)}{6} = \frac{\sqrt{7} + 1}{3} & a_3 &= 1 \\
\alpha_4 &= \frac{1}{\frac{\sqrt{7} + 1}{3} - 1} = \frac{3}{\sqrt{7} - 2} = \frac{3(\sqrt{7} + 2)}{3} = \sqrt{7} + 2 & a_4 &= 4 \\
\alpha_5 &= \frac{1}{\sqrt{7} + 2 - 4} = \frac{1}{\sqrt{7} - 2} = \alpha_1 & a_5 &= 1
\end{aligned}$$

So $\sqrt{7} = [2, 1, 1, 1, 4, 1, 1, 1, 4, \dots]$

$$\begin{aligned}
[2] &= 2/1 & 2^2 - 7 * 1^2 &= -3 \\
[2, 1] &= 3/1 & 3^2 - 7 * 1^2 &= 2 \\
[2, 1, 1] &= 5/2 & 5^2 - 7 * 2^2 &= -3 \\
[2, 1, 1, 1] &= 8/3 & 8^2 - 7 * 3^2 &= 1
\end{aligned}$$

Therefore $(8, 3)$ is fundamental solution.

Proposition 4.25. *If (x, y) is any solution in integers to $x^2 - dy^2 = 1$, with $x > y > 0$ then $\frac{x}{y}$ is a convergent.*

E.g. If $d = 13$, find the fundamental solution to $x^2 - 13y^2 = 1$

We define sequences α_n, a_n by $\alpha_0 = \sqrt{d}$, $a_n = \lfloor \alpha_n \rfloor$, $\alpha_{n+1} = \frac{1}{\alpha_n - a_n} = \frac{1}{\alpha_n - \lfloor \alpha_n \rfloor}$

$$\alpha_0 = \sqrt{13}$$

$$a_0 = 3$$

$$\alpha_1 = \frac{1}{\sqrt{13} - 3}$$

$$a_1 = 4$$

$$= \frac{\sqrt{13} + 3}{(\sqrt{13} + 3)(\sqrt{13} - 3)}$$

$$= \frac{\sqrt{13} + 3}{4}$$

$$\alpha_2 = \frac{1}{\frac{\sqrt{13} + 3}{4} - 1}$$

$$a_2 = 1$$

$$= \frac{4}{\sqrt{13} - 1}$$

$$= \frac{4(\sqrt{13} + 1)}{(\sqrt{13} - 1)(\sqrt{13} + 1)}$$

$$= \frac{4(\sqrt{13} + 1)}{12}$$

$$= \frac{\sqrt{13} + 1}{3}$$

$$\alpha_3 = \frac{1}{\frac{\sqrt{13} + 1}{3} - 1}$$

$$a_3 = 1$$

$$= \frac{3}{\sqrt{13} - 2}$$

$$= \frac{3(\sqrt{13} + 2)}{9}$$

$$= \frac{\sqrt{13} + 2}{3}$$

$$\alpha_4 = \frac{1}{\frac{\sqrt{13}+2}{3}-1} \qquad a_4 = 1$$

$$= \frac{3}{\sqrt{13}-1}$$

$$= \frac{3(\sqrt{13}+1)}{12}$$

$$= \frac{\sqrt{13}+1}{4}$$

$$\alpha_5 = \frac{1}{\frac{\sqrt{13}+1}{4}-1} \qquad a_5 = 6$$

$$= \frac{4}{\sqrt{13}-3}$$

$$= \frac{4(\sqrt{13}+3)}{4}$$

$$= \sqrt{13}+3$$

$$\alpha_6 = \frac{1}{\sqrt{13}-3} = \alpha_1$$

$$\alpha_7 = \alpha_2 \text{ etc}$$

$$\text{So } \sqrt{13} = [3, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, \dots] = [3, \overline{1, 1, 1, 1, 6}]$$

$$[3] = 3/1$$

$$[3, 1] = 4/1$$

$$[3, 1, 1] = 7/2$$

$$[3, 1, 1, 1] = 11/3$$

$$[3, 1, 1, 1, 1] = 18/5$$

$$3^2 - 13 * 1^2 = -4$$

$$4^2 - 13 * 1^2 = +3$$

$$7^2 - 13 * 2^2 = -3$$

$$11^2 - 13 * 3^2 = +4$$

$$18^2 - 13 * 5^2 = -1$$

$$N(18 + 5\sqrt{13}) = -1$$

$$\implies N((18 + 5\sqrt{13})^2) = 1$$

$$(18 + 5\sqrt{13})^2 = 324 + 180\sqrt{13} + 325$$

$$= 649 + 180\sqrt{13}$$

This means that $649^2 - 13 * 180^2 = 1$. If we find a unit of norm -1 , before any unit of norm $+1$, then its square will be the fundamental solution.

In general if $\sqrt{d} = [a_0, \overline{a_1, \dots, a_n, 2a_0}]$ and if $[a_0, \dots, a_n] = \frac{h_n}{k_n}$ then $h_n^2 - dk_n^2 = (-1)^{n+1}$

4.5.1 Convergence of continued fractions

Let $[a_0, a_1, \dots]$ be a continuous fraction. Then $x_n = [a_0, \dots, a_n]$ is the n^{th} convergent. Want a formula for numerator and denominator of x_n

$$x_0 = \frac{a_0}{1} \qquad x_1 = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1}$$

Define sequences of integers h_n, k_n by:

$$\begin{aligned} h_0 &= a_0 & k_0 &= 1 \\ h_1 &= a_1 a_0 + 1 & k_1 &= a_1 \\ h_n &= a_n h_{n-1} + h_{n-2} & k_n &= a_n k_{n-1} + k_{n-2} \end{aligned}$$

Lemma 4.26. $x_n = \frac{h_n}{k_n}$

Proof. By induction on n . True for $n = 0, 1$.

$$\begin{aligned} x_n &= [a_0, \dots, a_n] \\ &= \left[a_0, \dots, a_{n-1} + \frac{1}{a_n} \right] \\ &= \frac{h_{n-1}}{k_{n-1}} \end{aligned}$$

By the inductive hypothesis:

$$\begin{aligned} h'_{n-1} &= (a_{n-1} + \frac{1}{a_n})h_{n-2} + h_{n-3} \\ k'_{n-1} &= (a_{n-1} + \frac{1}{a_n})k_{n-2} + k_{n-3} \end{aligned}$$

This means that:

$$\begin{aligned}
x_n &= \frac{(a_{n-1} + \frac{1}{a_n})h_{n-2} + h_{n-3}}{(a_{n-1} + \frac{1}{a_n})k_{n-2} + k_{n-3}} \\
&= \frac{a_n a_{n-1} h_{n-2} + h_{n-2} + a_n h_{n-3}}{a_n a_{n-1} k_{n-2} + k_{n-2} + a_n k_{n-3}} \\
&= \frac{a_n \overbrace{(a_{n-1} h_{n-2} + h_{n-3})}^{h_{n-1}} + h_{n-2}}{a_n \underbrace{(a_{n-1} k_{n-2} + k_{n-3})}_{k_{n-1}} + k_{n-2}} \\
x_n &= \frac{\overbrace{a_n h_{n-1} + h_{n-2}}^{h_n}}{\underbrace{a_n k_{n-1} + k_{n-2}}_{k_n}}
\end{aligned}$$

Therefore $x_n = \frac{h_n}{k_n}$

Since $k_0 = 1 > 0$, $k_1 = a_1 > 0$, $k_n = a_n k_{n-1} + k_{n-2} > k_{n-1}$, the denominators are an increasing sequence of positive integers.

□

Lemma 4.27. h_n and k_n are coprime and $h_{n+1}k_n - h_n k_{n+1} = (-1)^n$

Proof. By induction on n . Check in cases $n = 0, 1$. Assume true for $n - 1 > 1$ and prove for n .

$$\begin{aligned}
h_{n+1}k_n - h_n k_{n+1} &= (\overbrace{a_{n+1}h_n + h_{n-1}}^{h_{n+1}})k_n - h_n(\overbrace{a_{n+1}k_n + k_{n-1}}^{k_{n+1}}) \\
&= -(h_n k_{n-1} - h_{n-1} k_n) \\
&= -(-1)^{n-1} \\
&= (-1)^n
\end{aligned}$$

□

Theorem 4.28. The continued fraction $[a_0, \dots]$ converges to a real number α and

$$\left| \alpha - \frac{h_n}{k_n} \right| < \frac{1}{k_n^2}$$

Alternating Series Test Suppose y_n is decreasing and $y_n \rightarrow 0$. Then $\sum_{n=1}^{\infty} (-1)^n y_n$ converges if $S = \sum_{n=1}^{\infty} (-1)^n y_n$ then S is between $\sum_{n=1}^N (-1)^n y_n$ and $\sum_{n=1}^{N+1} (-1)^n y_n$

Proof. Let $x_n = \frac{h_n}{k_n}$

$$x_{n+1} - x_n = \frac{h_{n+1}}{k_{n+1}} - \frac{h_n}{k_n} = \frac{h_{n+1}k_n - h_nk_{n+1}}{k_nk_{n+1}} = \frac{(-1)^n}{k_nk_{n+1}}$$

$$\begin{aligned} x_n &= x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \\ &= x_0 + \frac{1}{k_0k_1} - \frac{1}{k_1k_2} + \frac{1}{k_2k_3} - \cdots + \frac{-1}{k_{n-1}k_n} \end{aligned}$$

Therefore x_n converges to some $\alpha \in \mathbb{R}$ by the alternating series test. Also α is between x_n and x_{n+1}

$$|x_n - \alpha| < |x_n - x_{n+1}| \implies \frac{1}{k_nk_{n+1}} < \frac{1}{k_n^2}$$

□

Using the theorem we'll prove:

Theorem 4.29. *For any square-free $d > 1$, Pell's equation has non trivial solutions in integers. Equivalently, every real quadratic ring has non trivial units.*

Proof. \sqrt{d} has a continued fraction expansion. For any convergent $\frac{h}{k}$ we have

$$\begin{aligned} \left| \frac{h}{k} - \sqrt{d} \right| &< \frac{1}{k^2} \\ |h - k\sqrt{d}| &< \frac{1}{k} \\ |h^2 - k^2d| &= |h + k\sqrt{d}| * |h - k\sqrt{d}| \\ &< \left| \frac{h}{k} + \sqrt{d} \right| < 2\sqrt{d} + 1 \end{aligned}$$

This shows that for the convergents $\frac{h}{k}$ to \sqrt{d} , $h^2 - dk^2$ takes only finitely many values.

There exists n which can be written as $h^2 - dk^2$ in infinitely many ways. The values of h and $k \bmod n$ have only finitely many possibilities but we have infinitely many pairs (h, k) such that $h^2 - dk^2 = n$

Choose two solutions $(h, k), (h', k')$ where $h \equiv h' \pmod{n}$ and $k \equiv k' \pmod{n}$.

Let $A = \frac{h + k\sqrt{d}}{h' + k'\sqrt{d}}$. Claim A is a unit in $\mathbb{Z}[\sqrt{d}]$.

Clearly $N(A) = \frac{N(h + k\sqrt{d})}{N(h' + k'\sqrt{d})} = \frac{n}{n} = 1$. Remains to show that $A \in \mathbb{Z}[\sqrt{d}]$.

$$A = \frac{h + k\sqrt{d}}{h' + k'\sqrt{d}} = \frac{(h + k\sqrt{d})(h' - k'\sqrt{d})}{h'^2 - dk'^2} = \frac{(hh' - dk k') + (kh' - hk')\sqrt{d}}{n}$$

Recall $h = h' \pmod{n}$ and $k = k' \pmod{n}$.

Therefore

$$\begin{aligned} hh' - dk k' &= h^2 - dk^2 = n \equiv 0 \pmod{n} \\ kh' - hk' &\equiv kh - hk \equiv 0 \pmod{n} \end{aligned}$$

So $A \in \mathbb{Z}[\sqrt{d}]$ and A is a unit with norm 1 in $\mathbb{Z}[\sqrt{d}]$. □

Theorem 4.30. *Let $\alpha \in \mathbb{R}$ be irrational. If $\frac{a}{b} \in \mathbb{Q}$ with $\left| \frac{a}{b} - \alpha \right| < \frac{1}{2b^2}$ then $\frac{a}{b}$ is a convergent of α*

In order to solve this, we will state and prove the following lemma:

Lemma 4.31. *Let α be an irrational real number $\frac{h_n}{k_n}$ and the n^{th} convergent of α . If $\frac{a}{b}$ is any rational number with $b > 0$ and $b < k_{n+1}$ and $\frac{a}{b}$ is not a convergent then*
 $|a - b\alpha| > |h_n - k_n\alpha|$

Proof. Consider these simultaneous equations:

$$\begin{aligned} h_n x + h_{n+1} y &= a \\ k_n x + k_{n+1} y &= b \end{aligned}$$

The matrix $\begin{pmatrix} h_n & h_{n+1} \\ k_n & k_{n+1} \end{pmatrix}$ has determinant ± 1 which means the solutions of x, y are integers
 $x, y \neq 0$ because $\frac{a}{b} \neq \frac{h_n}{k_n}, \frac{h_{n+1}}{k_{n+1}}$. Plug $x = 0$ or $y = 0$ for a contradiction.

Also x, y have opposite signs because $b < k_{n+1}$ and α is between $\frac{h_n}{k_n}$ and $\frac{h_{n+1}}{k_{n+1}}$.

Therefore $\frac{h_n}{k_n} - \alpha$ and $\frac{h_{n+1}}{k_{n+1}} - \alpha$ have opposite signs.

Therefore $h_n - k_n\alpha$ and $h_{n+1} - k_{n+1}\alpha$ have opposite signs.

Therefore $x(h_n - k_n\alpha)$ and $y(h_{n+1} - k_{n+1}\alpha)$ have the same sign.

$$\begin{aligned} |a - b\alpha| &= |(h_n x + h_{n+1} y) - (k_n x + k_{n+1} y)\alpha| \\ &= |x(h_n - k_n\alpha) + y(h_{n+1} - k_{n+1}\alpha)| \\ &= |x| * |h_n - k_n\alpha| + |y| * |h_{n+1} - k_{n+1}\alpha| \\ &> |h_n - k_n\alpha| \end{aligned}$$

□

Proof. Assume $\frac{a}{b}$ is not a convergent to α , choose an a such that $k_n \leq b < k_{n+1}$. By the

$$\text{lemma } |h_n - k_n \alpha| < \underbrace{|a - b\alpha|}_{< \frac{1}{2b}} = |b| * \underbrace{\left| \frac{a}{b} - \alpha \right|}_{< \frac{1}{2b^2}}$$

This means that $\left| \frac{h_n}{k_n} - \alpha \right| < \frac{1}{2bk_n}$

$$\frac{a}{b} \neq \frac{h_n}{k_n} \implies \left| \frac{a}{b} - \frac{h_n}{k_n} \right| \geq \frac{1}{bk_n}$$

$$\begin{aligned} \therefore \frac{1}{bk_n} &\leq \left| \frac{a}{b} - \frac{h_n}{k_n} \right| = \left| \left(\frac{a}{b} - \alpha \right) + \left(\alpha - \frac{h_n}{k_n} \right) \right| \underbrace{\leq}_{\frac{1}{bk_n} < \frac{1}{bk_n} \text{ } \curvearrowright} \frac{1}{2b^2} + \underbrace{\frac{1}{2bk_n}}_{\frac{1}{2bk_n} + \frac{1}{2bk_n} = \frac{1}{bk_n}} \\ &= \frac{1}{bk_n} \end{aligned}$$

So $\frac{a}{b}$ must be a convergent.

□