

Number Theory

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Abstract

What did the number theorist say as he drowned?

Log, log, log, log....

For an up to date version of this pdf, check my GitHub :)

<https://github.com/vrvinny/number-theory>

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1 Introduction/Review

1.1 Introduction

Number Theory is the theory of the ring \mathbb{Z} and other related rings. A ring (in this course) is a set R with two binary operations $+$ and $*$ such that:

- $(R, +)$ is an abelian group
- $*$ is associative, commutative and has an identity element 1
- $x(y + z) = xy + xz \quad \forall x, y, z \in R$

Examples of rings:

- \mathbb{Z} is a ring
- Every field is a ring, (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{Q}$)
- \mathbb{Z}/n \mathbb{Z} modulo $n = \{0, \dots, n-1\}$
- $\mathbb{F}[X] = \{ \text{polynomials } f(x) \text{ with coefficients in } \mathbb{F} \}$

1.2 Review

1.2.1 Congruences

Let n be a positive integer. Given $x, y \in \mathbb{Z}$, we say x is congruent to y modulo n if $x - y$ is a multiple of n .

$$x \equiv y(n) \quad \text{or} \quad x \equiv y \pmod{n}$$

E.g
$$\begin{aligned} 2 &\equiv 12 \pmod{10} \\ &\equiv -8 \pmod{10} \end{aligned}$$

We write \mathbb{Z}/n for the ring of congruency classes modulo n , i.e. the elements are integer, with two of them regarded as the same if they are congruent modulo n .

Since every integer is congruent to a unique integer in the set $\{0, \dots, n-1\}$, we have $\mathbb{Z}/n = \{0, \dots, n-1\}$.

An element x of \mathbb{Z}/n is called "invertible" or a "unit" if $\exists y \in \mathbb{Z}/n$ such that $xy \equiv 1(n)$.

Theorem 1.1. x is invertible modulo n iff x and n are coprime

Recall Two numbers are coprime if their highest common factor is 1.

Here's how we find the inverse of x in \mathbb{Z}/n . Since X and n are coprime we can find $h, k \in \mathbb{Z}$ such that $hx + kn = 1 \implies hx \equiv 1 \pmod{n}$. So h is the inverse of x modulo n .

E.g We'll find the inberse of 7 modulo 25 using Euclid's algorithm

$$25 = 3 \times 7 + 4$$

$$7 = 1 \times 4 + 3$$

$$4 = 1 \times 3 + 1$$

$$1 = 4 - 1(3)$$

$$1 = 4 - 1(7 - 1(4)) = 2(4) - 1(7)$$

$$1 = 2(25 - 3(7)) - 1(7) = 2(25) - 7(7)$$

$$2(25) - 7(7) = 1$$

$$-7(7) = 1 \pmod{25}$$

$$(7^{-1}) = -7 = 18 \pmod{25}$$

$$7 \times 18 = 126 = 1 \pmod{25}$$

We'll write $(\mathbb{Z}/n)^\times$ for the invertible elements in \mathbb{Z}/n

E.g

$$(\mathbb{Z}/3)^\times = \{ \emptyset, 1, 2 \}$$

$$(\mathbb{Z}/6)^\times = \{ \emptyset, 1, \cancel{2}, \cancel{3}, \cancel{4}, 5 \}$$

Theorem 1.2. $(\mathbb{Z}/n)^\times$ is a group with the operation of multiplicity.

1.2.2 Solving Linear Congruences

Suppose we want to solve $ax \equiv b \pmod{n}$ (given a, b and n).

Case 1: If a is coprime to n then we can find a^{-1} modulo n by Euclid's algorithm,
 $x \equiv a^{-1}b \pmod{n}$

Case 2: If a is a factor of n , then there are two possibilities:

2a) if a is also a factor of b then $ax \equiv b \pmod{n}$ is equivalent to $x = \frac{b}{a} \pmod{\frac{n}{a}}$

2b) if a is not a factor of b then there are no solutions

E.g. Solve $5x = 11 \pmod{13}$

This is case 1 because 5 and 13 are coprime

$$13 = 2 \times 5 + 3$$

$$5 = 1 \times 3 + 2$$

$$3 = 1 \times 2 + 1$$

$$1 = (3) - 1(2)$$

$$1 = (3) - 1(5 - 1(3)) = 2(3) - (5)$$

$$1 = 2(13 - 2(5)) - (5) = 2(13) - 5(5)$$

$$1 \equiv -5(5) \pmod{13}$$

$$5^{-1} \equiv -5 \equiv 8 \pmod{13}$$

$$5x \equiv 11 \pmod{13}$$

$$x \equiv 8 \times 11 \equiv 88 \pmod{13}$$

$$x \equiv 10 \pmod{13}$$

E.g. Solve $7x \equiv 84 \pmod{490}$

7 is a factor of 490 so case 2)

7 is a factor of 84 so case 2a)

$$7x \equiv 84 \pmod{490}$$

$$x \equiv 12 \pmod{70}$$

E.g. Solve $7x \equiv 85 \pmod{490}$

This is case 2b (7 is a factor of 490 but not of 85) \therefore No solutions

$$7x \equiv 85 \pmod{490}$$

$$\implies 7x = 85 + 490y \text{ for some } y \in \mathbb{Z}$$

$$\implies 0 \equiv 1 \pmod{7}$$

E.g. Solve $6x \equiv 3 \pmod{21}$

This is neither case 1 nor case 2 but we can rewrite as:

$$3(2x) \equiv 3 \pmod{21}$$

$$\text{By case 2 we can solve for } 2x \equiv 1 \pmod{7}$$

but now 2 is invertible modulo 7 so now solve by case 1

$$\therefore x \equiv 4 \pmod{7}$$

1.3 Chinese Remainder Theorem

Suppose we know the congruency class of x modulo 10. Then we can work out its congruency class mod 2 and mod 5.

E.g. if $x \equiv 7 \pmod{10}$, then $x \equiv 1 \pmod{2}$ and $x \equiv 2 \pmod{5}$

Then the Chinese Remainder Theorem allows us to do the opposite, i.e. if we know x modulo 2 and modulo 5, then we can work out the value of x modulo 10.

Suppose n & m are coprime positive integers, let $a \in (\mathbb{Z}/n)$ and $b \in (\mathbb{Z}/m)$ then there is a unique

$$x \in (\mathbb{Z}/nm) \text{ such that } \begin{aligned} x &\equiv a \pmod{n} \\ x &\equiv b \pmod{m} \end{aligned}$$

Proof of existence part:

Since n & m are coprime, we can find $h, k \in \mathbb{Z}$ such that $hn + km = 1$.

Let $x = hnb + kma$

Check that this a solution to both congruences:

$$\begin{aligned} x &\equiv kma \pmod{n} \\ x &\equiv (1 - hn)a \pmod{n} \\ x &\equiv (1)a \pmod{n} \\ x &\equiv a \pmod{n} \end{aligned}$$

Similarly, this holds for $x \equiv b \pmod{m}$.

E.g. Solve the simultaneous congruence:

$$\begin{aligned} x &\equiv 3 \pmod{8} \\ x &\equiv 4 \pmod{5} \end{aligned}$$

By the Chinese Remainder Theorem, there is unique solution modulo 40. To find the solution we let $x = hnb + kma$.

First find h, k by Euclid's algorithm.

$$\begin{aligned} 8 &= 1 \times 5 + 3 & 1 &= (3) - 1(2) \\ 5 &= 1 \times 3 + 2 & 1 &= (3) - 1(5 - 1(3)) = 2(3) - (5) \\ 3 &= 1 \times 2 + 1 & 1 &= 2(8 - 2(5)) - (5) = 2(8) - 5(5) \end{aligned}$$

$$\begin{aligned} \therefore x &= (2 * 8 * 4) - (3 * 5 * 3) \\ x &= 64 - 45 \end{aligned}$$

$$\implies x \equiv 19 \pmod{40}$$

Remark: We can use the Chinese Remainder Theorem to solve a congruence modulo nm , by first solving mod n and then mod m and then combining the results.

E.g. Solve $x^2 \equiv 2 \pmod{119}$. Note $119 = 7 * 17$.

By CRT this is equivalent to:

$$\begin{aligned} x^2 &\equiv 2 \pmod{7} & \implies x &\equiv \pm 3 \pmod{7} \\ x^2 &\equiv 2 \pmod{17} & \implies x &\equiv \pm 6 \pmod{17} \end{aligned}$$

Now we combine the solutions:

$$\begin{aligned} 17 &= 2 * 7 + 3 & 1 &= (7) - 2(3) \\ 7 &= 2 * 3 + 1 & 1 &= (7) - 2(17 - 2(7)) \\ & & 1 &= 5(7) - 2(17) \end{aligned}$$

Since

$$\begin{array}{ll} x \equiv \pm 3 \pmod{7} & \text{We get } x \equiv 5 * 7 * (\pm 6) - 2 * 17 * (\pm 3) \\ x \equiv \pm 6 \pmod{17} & x \equiv \pm 11 \text{ or } \pm 45 \pmod{119} \end{array}$$

1.4 Prime numbers

Defintion 1.3. An integer $p \geq 2$ is a prime number if the only factors of p are $\pm 1, \pm p$

We'll write \mathbb{F}_p for \mathbb{Z}/p . This is because:

Theorem 1.4. If p is prime, then \mathbb{F}_p is a field

Proof. Need to check that the non-zero elements of \mathbb{F}_p all have inverses.

Let $x \in \mathbb{F}_p$ with $x \not\equiv 0 \pmod{p}$ i.e. x is not a multiple of p

$$\therefore \text{hcf}(x, p) = 1$$

$\therefore x$ & p coprime

□

1.5 Fermat's Little Theorem

Theorem 1.5. Let p be a prime number. If x is not a multiple of p then $x^{p-1} \equiv 1 \pmod{p}$

Proof. $x \in \mathbb{F}_p^\times = \{1, 2, \dots, p-1\}$ a group with $p-1$ elements.

Let n be the order of x in this group.

(order of x is smallest $n > 0$ such that $x^n \equiv 1 \pmod{p}$)

By corollary to Lagrange's Theorem, $p-1$ is a multiple of n

$$x^n \equiv 1 \pmod{p}$$

$$x^{p-1} \equiv 1 \pmod{p}$$

□

Theorem 1.6. Lagrange's Theorem: If H is a subgroup of a finite group G , then $|H|$ is a factor of $|G|$.

Corollary 1.7. Order of an element is a factor of $|G|$

We can use Fermat's Little Theorem to do calculations.

E.g. Calculate 10^{100} modulo 19

By Fermat's Little Theorem: $10^{18} \equiv 1 \pmod{19}$

$$\begin{aligned} 10^{100} &\equiv (10^{18})^5 * 10^{10} \pmod{19} \\ &\equiv 100^5 \pmod{19} \\ &\equiv 5^5 \pmod{19} \\ &\equiv 25 * 125 \equiv 6 * 11 \equiv 9 \pmod{19} \end{aligned}$$

Also using Fermat's Little Theorem we can solve congruence of the form $x^a \equiv b \pmod{p}$ as long as p prime and a inverible modulo $p-1$

1.5.1 General method to solve $x^a \equiv b \pmod{p}$

Let

$$\begin{aligned}c &= a^{-1} \pmod{p-1} \\ac &= 1 + (p-1)r\end{aligned}$$

Raise both sides of the congruence to power c :

$$\begin{aligned}\therefore x^{ac} &\equiv b^c \pmod{p} \\x^{1+(p-1)r} &\equiv b^c \pmod{p} \\x &\equiv b^c\end{aligned}$$

So the solution is $x \equiv b^c \pmod{p}$

E.g. Solve $x^5 \equiv 2 \pmod{19}$

19 is prime and 5 is coprime to 18.

Find $c = 5^{-1} \pmod{18}$

$$\begin{array}{ll}18 = 3 * 5 + 3 & 1 = 2 * 3 - 5 \\5 = 2 * 3 - 1 & 1 = 2(18 - 3 * 5) - 5 \\& 1 = 2 * 18 - 7 * 5\end{array}$$

$$\begin{aligned}\therefore 5^{-1} &\equiv -7 \pmod{18} \\&\equiv 11 \pmod{18}\end{aligned}$$

$$\begin{aligned}\therefore x &\equiv 2^{11} \pmod{19} \\&\equiv 2048 \pmod{19} \\&\equiv 15 \pmod{19}\end{aligned}$$

1.6 Fundamental Theorem of Arithmetic

If n is a positive integer then there is a unique factorisation, $n = p_1 p_2 \dots p_r$ with p_i prime. "Unique" means up to reordering the primes p_1, \dots, p_r . Showing that a factorisation exists is easy. For the uniqueness part we use:

1.6.1 Euclid's Lemma

Lemma 1.8. Suppose p prime, and $p|ab$. Then $p|a$ or $p|b$.

To prove Euclid's lemma we use Bezout's lemma.

Proof. Assume $p|ab$ but $p \nmid a$. Then $\text{hcf}(a, p) = 1$

By Bezout's lemma, $\exists h, k$ such that:

$$1 = ha + kp$$

$$b = hab + kpb$$

Both hab and kpb are multiples of p .

$\therefore p|b$

□

If n is composite then the smallest factor of n is (apart from 1) is a prime number $p \leq \sqrt{n}$, i.e. to show that n is prime, we just need to show that none of the primes up to \sqrt{n} are factors of n .

$$\sqrt{199} < 15 \text{ since } 15^2 = 225$$

The primes up to 15 are ~~2~~, ~~3~~, ~~5~~, ~~7~~, ~~11~~, ~~13~~

$199 \equiv 3 \pmod{2}$ (7)

$199 \equiv 4 \pmod{3}$ (13)

$\therefore 199$ is prime

Proof. Suppose p_1, \dots, p_n are all the primes.

$\therefore N$ has no prime factors \nmid

Proof. Assume there are only finitely many primes, call them p_1, p_2, \dots, p_r . All other primes are either 3 or are congruent to 1 mod 3.

$$\therefore N \equiv 1 \pmod{3} \implies \text{because clearly } N \equiv 2 \pmod{3}$$

2 Elementary Number Theory

2.1 Euler Totient Function

Recall $(\mathbb{Z}/n)^\times$ is the group of invertible elements in \mathbb{Z}/n .

E.g. $(\mathbb{Z}/6)^\times = \{1, 5\}$

$(\mathbb{Z}/8)^\times = \{1, 3, 5, 7\}$

These are groups with the multiplication operation, $*$. The multiplication table for $(\mathbb{Z}/8)^\times$ is given below.

$*$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Definition 2.1. The Euler Totient function is $\phi(n) = |(\mathbb{Z}/n)^\times|$

E.g. $\phi(6) = 2$

$\phi(8) = 4$

If p prime then $(\mathbb{Z}/p)^\times = \{1, \dots, p-1\}$ so $\phi(p) = p-1$

Theorem 2.2. Euler's Theorem- Let $x \in (\mathbb{Z}/n)^\times$ then $x^{\phi(n)} \equiv 1 \pmod{n}$

In the case $n = p$ is prime, this is just Fermat's Little Theorem.

Proof. Let d be the order of x , i.e. $x^d \equiv 1 \pmod{n}$. By a corollary to Lagrange's Theorem, d is a factor of $\phi(n) \implies x^{\phi(n)} \equiv 1 \pmod{n}$ \square

We can use Euler's theorem to solve congruences and calculate powers mod n . To use the theorem, we need a quick way of calculating $\phi(n)$.

Lemma 2.3. Let $n = p^a$ where p is prime $a > 0$. Then $\phi(n) = (p-1)p^{a-1}$

E.g. $\phi(8) = \phi(2^3) = (2-1)2^{3-1} = 4$

Proof. An integer is coprime to p^a as long as it's not a multiple of p .

\therefore The elements of \mathbb{Z}/p^a which are not invertible are the multiples of p . $0, p, 2p, \dots, p^a - p$.

There are $p^a - 1$ of these:

$$\therefore |(\mathbb{Z}/p^a)^\times| = p^a - p^{a-1} = (p-1)p^{a-1} \quad \square$$

Theorem 2.4. Let n and m be coprime. Then there is an isomorphism:

$$(\mathbb{Z}/nm)^\times \cong (\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$$

We'll use the theorem before we prove it.

Remark: If G and H are groups, $G \times H = \{(x, y) : x \in G, y \in H\}$, then $G \times H$ is a group with the operation $(x, y)(x', y') = (xx', yy')$ and $G \times H$ is the "direct product" of G and H

Corollary 2.5. *If n and m are coprime then $\phi(nm) = \phi(n)\phi(m)$*

Proof.

$$\begin{aligned}\phi(nm) &= |(\mathbb{Z}/nm)^\times| = |(\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times| \\ &= |(\mathbb{Z}/n)^\times| * |(\mathbb{Z}/m)^\times| \\ &= \phi(n)\phi(m)\end{aligned}$$

□

Corollary 2.6. *(Corollary of the corollary): Suppose $n = p_1^{a_1} \dots p_r^{a_r}$ with p_1, \dots, p_r distinct primes and $a_i > 0$. Then*

$$\phi(n) = (p_1 - 1)p_1^{a_1-1} * \dots * (p_r - 1)p_r^{a_r-1}$$

Proof. Since $p_1^{a_1}, \dots, p_r^{a_r}$ are coprime,

$$\begin{aligned}\phi(n) &= \phi(p_1^{a_1}) \dots \phi(p_r^{a_r}) && \text{by the corollary} \\ &= (p_1 - 1)p_1^{a_1-1} \dots (p_r - 1)p_r^{a_r-1} && \text{by the lemma}\end{aligned}$$

□

E.g. Calculate $\phi(200)$

$$\begin{aligned}\phi(200) &= \phi(2^3 * 5^2) \\ &= (2 - 1)2^{3-1} * (5 - 1)5^{2-1} \\ &= 4 * 4 * 5 \\ &= 80\end{aligned}$$

Theorem 2.7. *Suppose n and m are coprime, then $(\mathbb{Z}/nm)^\times \cong (\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$. The isomorphism is the map $x \mapsto (x \bmod n, x \bmod m)$*

E.g. $n = 4, m = 5$

$$\begin{aligned}(\mathbb{Z}/4)^\times &= \{1, 3\} \\ (\mathbb{Z}/5)^\times &= \{1, 2, 3, 4\} \\ \therefore (\mathbb{Z}/4)^\times * (\mathbb{Z}/5)^\times &= \{(1, 1), (1, 2), (1, 3), (1, 4), \\ &\quad (3, 1), (3, 2), (3, 3), (3, 4)\} \\ (\mathbb{Z}/20)^\times &= \{1, 3, 7, 9, 11, 13, 17, 19\}\end{aligned}$$

The isomorphism is:

$$\begin{array}{ll} 1 \mapsto (1, 1) & 11 \mapsto (3, 1) \\ 3 \mapsto (3, 3) & 13 \mapsto (1, 3) \\ 7 \mapsto (3, 2) & 17 \mapsto (1, 2) \\ 9 \mapsto (1, 4) & 19 \mapsto (3, 4) \end{array}$$

Proof. Let $\Phi : \mathbb{Z}/nm \mapsto \mathbb{Z}/n * \mathbb{Z}/m$

$$\Phi(x) = (x \bmod n, x \bmod m)$$

This is a bijection by the Chinese Remainder Theorem.

We'll next show that x is invertible mod $nm \iff x$ is invertible mod n and mod m

(\implies) Suppose x is invertible mod nm

$$\text{Let } xy \equiv 1 \pmod{nm}$$

$$\therefore xy \equiv 1 \pmod{n}$$

$$xy \equiv 1 \pmod{m}$$

$$\therefore x \text{ invertible mod } n \text{ and } m$$

(\impliedby) Suppose x invertible mod n and m

$$xa \equiv 1 \pmod{n}$$

$$xb \equiv 1 \pmod{m}$$

By the Chinese Remainder Theorem, $\exists y$ such that $y \equiv a \pmod{n}$

$$y \equiv b \pmod{m}$$

$$\left. \begin{array}{l} \therefore xy \equiv xa \equiv 1 \pmod{n} \\ \equiv xb \equiv 1 \pmod{m} \end{array} \right\} \implies xy \equiv 1 \pmod{nm} \text{ by the Chinese Remainder Theorem}$$

We've shown that Φ gives a bijection between $(\mathbb{Z}/nm)^\times$ and $(\mathbb{Z}/n)^\times * (\mathbb{Z}/m)^\times$. We'll next check that $\Phi(xy) = \Phi(x)\Phi(y)$.

$$\begin{aligned} \Phi(xy) &= (xy \bmod n, xy \bmod m) \\ &= (x \bmod n, x \bmod m) * (y \bmod n, y \bmod m) \\ &= \Phi(x)\Phi(y) \end{aligned}$$

□

2.2 Euler's Theorem

If $x \in (\mathbb{Z}/n)^\times$ then $x^{\phi(n)} \equiv 1 \pmod{n}$ and $\phi(p_1^{a_1} \dots p_r^{a_r}) = (p_1 - 1)p_1^{a_1-1} \dots (p_r - 1)p_r^{a_r-1}$

E.g. Calculate $7^{135246872002} \bmod 10000$

$$7 \text{ coprime to } 10000 \text{ so } 7^{\phi(10000)} \equiv 1 \pmod{10000}$$

$$10000 = 2^4 * 5^4$$

$$\therefore \phi(10000) = (2-1)2^3 * (5-1) * 5^3 = 8 * 500$$

$$7^{4000} \equiv 1 \pmod{10000} \implies 7^n \text{ depends only on } n \bmod 4000$$

$$135246872002 \equiv 2 \pmod{4000}$$

$$\therefore 7^{135246872002} \equiv 7^2 \equiv 49 \pmod{10000}$$

We can also use Euler's THEorem to solve congruence with powers

2.2.1 Solving equations of the form $x^a \equiv b \pmod{n}$

Suppose we want to solve $x^a \equiv b \pmod{n}$ where b is coprime to n and a is coprime to $\phi(n)$.

Clearly any solution x must be coprime to n by Euler's Theorem $x^{\phi(n)} \equiv 1 \pmod{n}$.

\therefore The congruency class of $x^y \pmod{n}$ depends only $y \pmod{\phi(n)}$

Let

$$c = a^{-1} \pmod{\phi(n)}$$

Raise both sides of the congruence to power c :

$$x^{ac} \equiv x^1 \equiv b^c \pmod{n}$$

\therefore The solution is $x \equiv b^c \pmod{n}$

E.g. $x^7 \equiv 3 \pmod{50}$

3 is coprime to 50,

$$\begin{aligned} 50 &= 2 * 5^2 \\ \implies \phi(50) &= 1 * 4 * 5 = 20 \end{aligned}$$

7 is coprime to $\phi(50)$. To solve, we need to find

$$\begin{aligned} c &\equiv 7^{-1} \pmod{\phi(50)} \\ &\equiv 3 \pmod{20} \end{aligned}$$

$$x \equiv 3^3 \equiv 27 \pmod{50}$$

E.g. $x^{27} \equiv 5 \pmod{123}$

5 is coprime to 123,

$$\begin{aligned} 123 &= 3 * 41 \\ \implies \phi(123) &= 2 * 40 = 80 \end{aligned}$$

27 is coprime to 80

To solve, we find $27^{-1} \pmod{80}$

$$\begin{aligned} 80 &= 3 * 27 - 1 \\ \implies 1 &= 3 * 27 - 80 \end{aligned}$$

$$27^{-1} = 3$$

$$\begin{aligned} x &= 5^3 \\ x &= 125 \equiv 2 \pmod{123} \end{aligned}$$

2.3 Primitive roots

Recall, let G be a finite group. G is called a cyclic group if $\exists x \in G$ such that, every element in G has the form x^n for some $n \in \mathbb{Z}$, i.e. $G = \{1, x, x^2, \dots, x^{n-1}\}$ where n is the order of x , equivalentl the order of x is $|G|$. The element x is called a generator of G .

Theorem 2.8. (Gauss' Theorem), For ever prime number p , the group \mathbb{F}_p^\times is cyclic

Defintion 2.9. A generator of \mathbb{F}_p^\times is called a primitive root. Equivalently, this is an element of order $p - 1$

E.g. $p = 7, x = 3$ We'll see that 3 is a primitive root modulo 7

$$\begin{array}{llll} \text{Powers of 3 in } F_7^\times : & 3^0 = 1 & 3^3 \equiv 6 \pmod{7} & 3^6 \equiv 1 \pmod{7} \\ & 3^1 = 3 & 3^4 \equiv 4 \pmod{7} & \\ & 3^2 \equiv 2 \pmod{7} & 3^5 \equiv 1 \pmod{7} & \end{array}$$

so 3 is a primitive root modulo 7. There is a quicker way to check whether x is a primitive root.

Proposition 2.10. Let $x \in \mathbb{F}_p^\times$, then x is a primitive root modulo p if and only if for every prime factor q of $p - 1$:

$$x^{\frac{p-1}{q}} \not\equiv 1 \pmod{p}$$

Proof. Assume the second statement is false, so \exists prime factor q of $p - 1$ such that:

$$\begin{aligned} x^{\frac{p-1}{q}} &\equiv 1 \pmod{p} & \therefore \text{order of } x &\leq \frac{p-1}{q} < p-1 \\ & & \therefore x &\text{ is not a primitive root} \end{aligned}$$

Conversely, assume x is not a primitive root, so x doe not have order $p - 1$. But the order of x is a factor of $p - 1$.

Suppose the order of x is $\frac{p-1}{d}$, $d > 1$.

Let q be a prime factor of $d \implies q|p-1$

$$\frac{p-1}{q} \text{ is a multiple of } \frac{p-1}{d} \text{ but } x^{\frac{p-1}{q}} \equiv 1 \pmod{p} \implies x^{\frac{p-1}{d}} \equiv 1 \pmod{p}$$

□

E.g. $p = 29$

By the proposition x is a primitive root mod 29 $\iff x^{28/2} \not\equiv 1 \pmod{29}$ and $x^{28/7} \not\equiv 1 \pmod{29}$

$$\iff x^{14} \not\equiv 1 \pmod{29} \text{ and } x^4 \not\equiv 1 \pmod{29}$$

$$\begin{aligned} \text{Try } x = 2 : & & 2^4 &\equiv 16 \not\equiv 1 \pmod{29} \\ & & 2^{14} &\equiv 128^2 \equiv 12^2 \equiv 144 \equiv -1 \pmod{29} \end{aligned}$$

$\therefore 2$ is a primitive root mod 29

Another trick to speed up the calculation:

\mathbb{F}_p is a field \therefore every polynomial of d has no more than d in \mathbb{F} (proved in 2201).

\therefore if $x^2 \equiv 1 \pmod{p}$ then $x \equiv \pm 1 \pmod{p}$

This means that checking whether $x^{14} \equiv 1 \pmod{29}$ is equivalent to checking whether $x^7 \equiv \pm 1 \pmod{29}$.

E.g 3 is also a primitive root modulo 29

$$3^2 \equiv 9 \not\equiv \pm 1 \pmod{29}$$

$$3^4 \equiv 1 \pmod{29}$$

$$3^7 \equiv 27^2 * 3 \pmod{29}$$

$$\equiv (-2)^2 * 3 \equiv 12 \pmod{29}$$

$$\equiv \pm 1 \pmod{29}$$

$$\therefore 3^{14} \not\equiv 1 \pmod{29}$$

2.4 Roots of unity and Cyclotomic Polynomials

A complex number ζ is called an n^{th} root of unity if $\zeta^n = 1$. The n^{th} roots of unity are $e^{2\pi i \frac{a}{n}}$ for $a = \{0, 1, \dots, n-1\}$

We call ζ a primitive n^{th} root of unity if n smaller power than ζ^n is equal to 1, i.e. ζ has order n in \mathbb{C}^\times if ζ is not a primitive n^{th} root of unity $\zeta = e^{2\pi i \frac{b}{d}}$ where $b = \{0, \dots, d-1\}$ for $d < n$

$$\therefore \frac{a}{n} = \frac{b}{d}$$

The cancellation happens when a is not coprime to n . This shows that the primitive n^{th} of unity are $e^{2\pi i \frac{a}{n}}$, $a \in (\mathbb{Z}/n)^\times$.

Corollary 2.11. *There are exactly $\phi(n)$ primitive n^{th} roots of unity*

We'll actually prove a more precise version of Gauss' Theorem.

Theorem 2.12. *For every factor d of $p-1$ there are $\phi(d)$ elements in \mathbb{F}_p^\times of order d .*

Defintion 2.13. *The n^{th} cyclotomic polynomial is:*

$$\Phi_n(x) = \prod_{\substack{\text{primitive} \\ n^{th} \text{ roots} \\ \text{of unity } \zeta}} (X - \zeta)$$

i.e $\zeta^n = 1$ and no smaller power of ζ is 1, $\zeta = e^{2\pi i \frac{a}{n}}$, $a \in (\mathbb{Z}/n)^\times$

This has degree $\phi(n)$.

E.g. $n=4$

Primitive 4^{th} roots of unity are $i, -i$:

$$\begin{aligned}\Phi_4(x) &= (x - i)(x - (-i)) \\ &= x^2 + 1\end{aligned}$$

Lemma 2.14. For every $n > 0$:

$$x^n - 1 = \prod_{\substack{d \text{ factors} \\ d \text{ of } n}} \Phi_d(x)$$

E.g. Calculate $\Phi_6(x)$

$$\begin{aligned}\text{By the lemma} \quad x^6 - 1 &= \Phi_1 \Phi_2 \Phi_3 \Phi_6 & x^6 - 1 &= (x^3 - 1) \Phi_2 \Phi_6 \\ x^3 - 1 &= \Phi_1 \Phi_3\end{aligned}$$

$$\therefore \Phi_6 = \frac{x^6 - 1}{(x^3 - 1)(x + 1)} = \frac{x^3 + 1}{x + 1} = x^2 - x + 1$$

Let p be a prime number. A primitive root mod p is an $x \in \mathbb{F}_p^\times$, such that x generates \mathbb{F}_p^\times .
Equivalently order = $p - 1$

2.4.1 How to calculate $\Phi_n(x)$

Lemma 2.15. $x^n - 1 = \prod_{d|n} \Phi_d(x)$

E.g. $n = 4$

$$\begin{aligned}x^4 - 1 &= \Phi_1 \Phi_2 \Phi_4 & \Phi_1 &= x - 1 \\ & & \Phi_2 &= (x - (-1)) = x + 1 \\ & & \Phi_4 &= (x - i)(x - (-i)) = x^2 + 1 \\ &= (x - 1)(x + 1)(x^2 + 1)\end{aligned}$$

Proof.

$$x^n - 1 = \prod_{\substack{\zeta \text{ is an} \\ n^{th} \text{ root of} \\ \text{unity}}} (x - \zeta)$$

but every n^{th} root of unity is a primitive d^{th} root of unity for some $d|n$.

$$x^n = \prod_{d|n} (\prod_{\substack{\text{primitive} \\ d^{th} \text{ roots} \\ \text{of unity}}} (x - \zeta)) = \prod_{d|n} \Phi_d(x)$$

□

E.g. Calculate $\Phi_5(x)$

$$\begin{aligned} x^5 - 1 &= \Phi_1(x)\Phi_5(x) \\ &= (x - 1)\Phi_5(x) \end{aligned}$$

$$\Phi_5(x) = \frac{x^5 - 1}{x - 1} = 1 + x + x^2 + x^3 + x^4$$

More generally if p prime then $x^p - 1 = (x - 1)\Phi_p(x) \implies \Phi_p(x) = 1 + x + \dots + x^{p-1}$

E.g. Calculate $\Phi_8(x)$

$$x^8 - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x)\Phi_8(x)$$

$$x^4 - 1 = \Phi_1(x)\Phi_2(x)\Phi_4(x) \implies \Phi_8(x) = \frac{x^8 - 1}{x^4 - 1} = x^4 + 1$$

Corollary 2.16. $\Phi_n(x)$ has coefficients in \mathbb{Z}

$$\text{Proof. } \Phi_n(x) = \frac{x^n - 1}{\prod_{\substack{d|n \\ d \neq n}} \Phi_d(x)}$$

We'll prove the corollary by induction on n , clearly true when $n = 1$. Assume Φ_d has integer coefficients $\forall d < n$.

It is proved in Algebra 3 (MATH2201) that, if $f, g \in \mathbb{Z}[X]$ and g monic then $f = qg + r$ where $\deg(r) < \deg(g)$ and $g, r \in \mathbb{Z}[x]$.

Using this, we get that the denominator $\prod_{\substack{d|n \\ d \neq n}} \Phi_d(x)$ is a monic polynomial with coefficients in $\mathbb{Z} \implies \Phi_n \in \mathbb{Z}[X]$. □

2.4.2 Gauss' Theorem

Theorem 2.17. Let n be a factor of $p - 1$, where p is prime. Then there are exactly $\phi(n)$ elements of order n in \mathbb{F}_p^\times . These are the roots of Φ in \mathbb{F}_p^\times . In particular there are $\phi(p - 1)$ primitive roots.

Proof. Let $f(x) = x^{p-1} - 1$

By Fermat's Little theorem, $f(x) = 0 \pmod{p}$ for $x = 1, \dots, p - 1$ for $(x \neq 0)$

$$\begin{aligned} \therefore f(x) &= (x - 1)(x - 2) \dots (x - (p - 1)) \\ &= \prod_{n|p-1} \Phi_n(x) \end{aligned}$$

This implies that:

- Each Φ_n (for $n|p - 1$) factorises completely into linear factors with no repeated roots $\therefore \Phi_n$ has $\phi(n)$ roots in \mathbb{F}_p
- Every element of \mathbb{F}_p^\times is a root of exactly one of the polynomials Φ_n with $n|p - 1$

It remains to show that the roots of $\Phi_n(x)$ in \mathbb{F}_p has order of exactly n .
 Suppose $\Phi_n(x) \equiv 0 \pmod{p}$

By the lemma $\Phi_n(x)$ is a factor $x^n - 1$
 $\therefore x^n - 1 \equiv 0 \pmod{p}$
 $\therefore x^n \equiv 1 \pmod{p}$

Suppose $x^m \equiv 1 \pmod{p}$ for some $m|n, m < n$
 $\implies x^m - 1 \equiv 0 \pmod{p}$

By the lemma $\Pi_{d|m} \Phi_d(x) \equiv 0 \pmod{p}$
 $\implies \Phi_d(x) \equiv 0 \pmod{p}$ for some $d \nmid n$

We already know that x is only a root of 1 of the cyclotomic polynomials, therefore x has order n . \square

2.5 Quadratic reciprocity (Quadratic equations modulo prime numbers)

Recall we can solve $x^a \equiv b \pmod{p}$ as long as a is coprime to $p - 1$. This won't work if $a = 2$ because a will not be invertible mod $p - 1$. An easier question to ask is, which quadratic equations have solutions modulo p ?

E.g. Does $x^2 \equiv 37 \pmod{149}$ have solutions?

Notation: We always let p be an odd prime (i.e. $p \neq 2$)

An element $a \in \mathbb{F}_p^\times$ is a quadratic residue if $x^2 \equiv a \pmod{p}$ has solutions.

An element $a \in \mathbb{F}_p^\times$ is a quadratic non-residue if there are no solutions.

The quadratic residue symbol is defined for $a \in \mathbb{F}_p^\times$ by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{a quadratic residue} \\ -1 & \text{a quadratic non-residue} \end{cases}$$

Lemma 2.18. *Let g be a primitive root modulo p (p odd prime). Then g^r is a quadratic residue iff r even.*

Proof.

(\Leftarrow) Assume r even

Clearly g^r is a square in \mathbb{F}_p^\times

So g^r is a quadratic residue

(\Rightarrow) Assume $g^r \equiv x^2 \pmod{p}$

$x \equiv g^s \pmod{p}$ ($s \in \mathbb{Z}$) since g primitive roots

$\therefore g^r \equiv g^{2s} \pmod{p}$

$g^{r-2s} \equiv 1 \pmod{p}$

g has order $p - 1$, so $r - 2s$ is a multiple of $p - 1$

p odd $\implies p - 1$ is even $\implies r$ is even

\square

E.g. $p = 7$

x	$x^2 \pmod{7}$		a	$\left(\frac{a}{7}\right)$
± 1	1	\implies	1	1
± 2	4		2	1
± 3	2		3	-1
			4	1
			5	-1
			6	-1

So 1,2,4 are quadratic residues; 3,4,6 are quadratic non-residues

Corollary 2.19. *There are exactly $\frac{p-1}{2}$ quadratic residues and $\frac{p-1}{2}$ quadratic non-residues mod p*

Defintion 2.20. *Euler's criterion: Let p be an odd prime and $a \in \mathbb{F}_p^\times \implies \left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$*
Also $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

Proof. $(a^{\frac{p-1}{2}})^2 \equiv 1 \pmod{p}$ by Fermat's Little theorem.

$$\therefore a^{\frac{p-1}{2}} \equiv \pm 1 \pmod{p}$$

Let $a = g^r$ where g is a primitive root $\implies a^{\frac{p-1}{2}} \equiv g^{(p-1)\frac{r}{2}}$

$$\begin{aligned} a \text{ is a quadratic residue} &\iff r \text{ is even} \\ &\iff (p-1)\frac{r}{2} \text{ is a multiple of } p-1 \\ &\iff g^{(p-1)\frac{r}{2}} \equiv 1 \pmod{p} \\ &\iff a^{\frac{p-1}{2}} \equiv 1 \pmod{p} \end{aligned}$$

□

To calculate $\left(\frac{a}{p}\right)$, we'll use three theorems:

2.5.1 Quadratic Reciprocity Law

Let p, q be distinct odd prime numbers. Then $\left(\frac{p}{q}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}$

$$\text{i.e. } \left(\frac{p}{q}\right) = \begin{cases} \left(\frac{q}{p}\right) & \text{if } p \equiv 1 \pmod{4} \text{ or } q \equiv 1 \pmod{4} \\ -\left(\frac{q}{p}\right) & \text{if } p \equiv q \equiv -1 \pmod{4} \end{cases}$$

2.5.2 First Nebensatz

If p is an odd prime, then $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$

$$\text{i.e. } \left(\frac{-1}{p}\right) = \begin{cases} 1 & p \equiv 1 \pmod{4} \\ -1 & p \equiv -1 \pmod{4} \end{cases}$$

2.5.3 Second Nebensatz

Let p be an odd prime, then $(\frac{2}{p}) = (-1)^{\frac{p^2-1}{8}}$

$$\text{i.e. } (\frac{2}{p}) = \begin{cases} 1 & p \equiv \pm 1 \pmod{8} \\ -1 & p \equiv \pm 3 \pmod{8} \end{cases}$$

We'll prove the theorems later.

E.g. Does the congruence $x^2 \equiv 37 \pmod{199}$ have solutions?

$$\begin{aligned} 199 \text{ is an odd prime } (\frac{37}{199}) &= +(\frac{199}{37}) && \text{by quadratic reciprocity} \\ &\equiv (\frac{14}{37}) && \text{because } 199 \equiv 14 \pmod{37} \\ &\equiv (\frac{2}{37})(\frac{7}{37}) && \text{by the corollary} \\ &\equiv (-1)(\frac{7}{37}) && \text{by the 2}^{nd} \text{ Nebensatz} \\ &\equiv (-1)(+1)(\frac{37}{7}) && \text{by the quadratic reciprocity law} \\ &\equiv -(\frac{2}{7}) && \text{because } 37 \equiv 2 \pmod{7} \\ &\equiv -(+1) && \text{by the 2}^{nd} \text{ Nebensatz} \\ &\equiv -1 && \therefore x^2 \equiv 37 \pmod{199} \text{ has no solutions} \end{aligned}$$

E.g. $x^2 \equiv 47 \pmod{53}$ have solutions?

$$(\frac{47}{53}) = +(\frac{53}{47}) = (\frac{6}{47}) = (\frac{2}{47})(\frac{3}{47}) = (+1)(-1)(\frac{47}{3}) = -(-\frac{1}{3}) = -(-1) = +1$$

This shows that 47 is a quadratic residue mod 53, so $x^2 \equiv 47 \pmod{53}$ does have solutions. ($x = 10$)

We can speed up the test for primitive roots using quadratic reciprocity,

$$x \text{ is a primitive root mod } p \iff \forall q|p-1, q \text{ prime } x^{\frac{p-1}{q}} \not\equiv 1 \pmod{p}$$

This means we need to calculate $x^{\frac{p-1}{q}} \pmod{p}$ for primes $q|p-1$, the biggest power of x to calculate is $x^{\frac{p-1}{2}}$. But we can calculate this, because it is $(\frac{x}{p})$ by Euler's criterion.

E.g. Is 35 a primitive root modulo 83?

The primes q dividing 82 are 2, 41, need to check $35^2, 35^{41}$
 $35^2 \not\equiv 1 \pmod{83}$ because $35 \not\equiv \pm 1 \pmod{83}$, a quadratic equation cannot have more than 2 roots.
 $35^{41} \equiv (\frac{35}{83}) \pmod{83} = (\frac{5}{83})(\frac{7}{83}) = (\frac{83}{5})(-1)(\frac{83}{7}) = (\frac{3}{5})(-1)(\frac{-1}{7}) = (\frac{5}{3})(-1)(-1) = (\frac{2}{3})$
 $= -1 \not\equiv 1 \pmod{83}$

So 35 is a primitive root modulo 83.

Proof. First Nebensatz:

By Euler's criterion, $\left(\frac{-1}{p}\right) \equiv (-1)^{\frac{p-1}{2}} \pmod{p}$.

Both sides are ± 1 , and $+1 \not\equiv -1 \pmod{p}$ because $p \geq 3 \implies$ they are equal. □