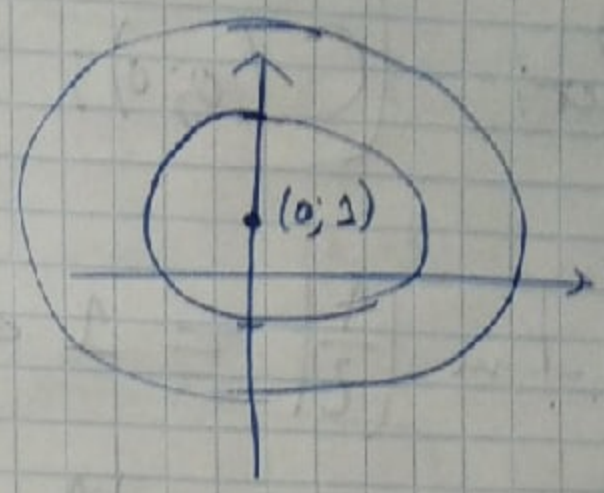
Задание 1

№1 $2 \le |z-i| \le 4$



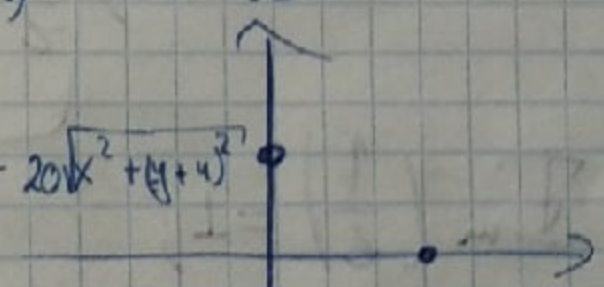
1) $z = i + re^{i\varphi}$; $z - i = re^{i\varphi}$;

$\boxed{\text{Ответ: } C(0;1)}$ $\boxed{S = \pi(4^2 - 2^2) = 12\pi}$

~~№2~~

2) $|z - 4i| + |z + 4i| = 10$;

$$\sqrt{x^2 + (y-4)^2} + \sqrt{x^2 + (y+4)^2} = 10$$



$$\cancel{x^2} + (y-4)^2 = 100 + \cancel{x^2} + (y+4)^2 - 20\sqrt{x^2 + (y+4)^2}$$

$$400\left[x^2 + (y+4)^2\right] = 10000 + \cancel{(y+4)^4}\ 16^2 y^2 + 100 \cdot 16 \cdot 2y$$

$(y+4)^2 - (y-4)^2 = 2y \cdot 8 = 16y$

$$400x^2 + 400y^2 + \cancel{8 \cdot 400y} + 16 \cdot 400 - 16^2 y^2 - \cancel{8 \cdot 400y} = 10.000$$

$$20^2 x^2 + \underbrace{20^2 - 16^2}_{4 \cdot 36} y^2 = 10000 - (4 \cdot 20)^2$$

$$20^2 x^2 + 4 \cdot 36\, y^2 = 20 \cdot 180$$

$$\boxed{\frac{x^2}{9} + \frac{y^2}{25} = 1} \Longrightarrow C(0;0).$$

тут центр в нуле.

$$\left(\frac{x}{a}\right) + \frac{y}{b} = 1; \quad \boxed{a = 3} \quad \boxed{b = 5} \Rightarrow$$ большая полуось.

Ответ: $\boxed{C(0;0);\ a=3.}$

3). $\operatorname{Im}\left(\frac{1}{z}\right) = 1$ ← описывает круг

$\cancel{z = |z|\cdot e^{i\varphi};\ \operatorname{Im}\left(\frac{1}{z}\right) = \frac{1}{|z|}\cdot \operatorname{Im}(e^{-i\varphi}) = 1;}$

$\cancel{\frac{1}{|z|}\cdot(-\sin\theta) = 1;\ -|z| =}$

$z = x+iy.$ $\boxed{\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2};}$

$\operatorname{Im}\left(\frac{1}{z}\right) = 1;\quad \frac{-y}{x^2+y^2} = 1;$

$x^2+y^2+y+\frac{1}{4} = \frac{1}{4};\quad x^2+\left(y+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$ что

$$\boxed{\text{Ответ: } C = \left(0;\ -\frac{1}{2}\right)\quad R = \frac{1}{2}}$$

№2.

$\sum n\varepsilon^{n-1} = ?\quad S_1 = \sum \varepsilon^n = \frac{1}{1-\varepsilon} = \frac{\varepsilon^{n+1}-1}{\varepsilon-1}$

$\frac{\partial S_1}{\partial \varepsilon} = \frac{(n+1)\varepsilon^n(\varepsilon-1) - \varepsilon^{n+1}+1}{(\varepsilon-1)^2} = \varepsilon^{n+1}\cdot n =$

$$= \frac{\varepsilon^{n+1}\cdot n + 1 - \varepsilon^{n}\cdot n - \varepsilon^{n}}{(\varepsilon-1)^2} = \frac{\varepsilon^{n}\cdot n(\varepsilon-1)+1-\varepsilon^{n}}{(\varepsilon-1)^2}$$

$$= \boxed{\frac{(n(\varepsilon-1)-1)\varepsilon^{n}+1}{(\varepsilon-1)^2}} \leftarrow \text{r.g.} \leftarrow \text{order}$$

No 3.

$\cancel{\operatorname{Im}(z^3+3z)=1,\ z=a+ib}$

$\cancel{z=a+ib,\ \operatorname{Im}\{a^3+3a^2ib-3ab^2-ib^3+}$

$\cancel{+3a+3ib-i\}=1,}$

$\cancel{\operatorname{Im}\{3a^2b-b^3+3b-1\}=1}$

$\operatorname{Im}(z)=1 \Rightarrow z=a+i$

$w(z)=(a+i)^3+3(a+i)-i = a^3+3a^2i-\cancel{3a}-i+$

$+\cancel{3a}+3i-i = \boxed{a^3+i\{3a^2+1\}} \rightarrow \operatorname{Re}(w)=a^3$

$a=\operatorname{Re}(w)^{1/3}$

$\operatorname{Im}(w) = 3|\operatorname{Re}(w)|^{2/3}+1$

4. $|z-i|=1$ $\quad z=a+(b+1)i$

$\boxed{a^2+b^2=1}$ $\quad \cancel{|a+ib|=1}$

$w(z)=\frac{1}{a+(b+1)i-2i} = \frac{1}{a+(b-1)i};$ =

$$\frac{a - i(b-1)}{a^2+(b-1)^2}; \quad Re(w) = \frac{a}{1+1-2b} = \frac{a}{2-2b}$$

$$Im(w) = \frac{b-1}{2-2b} =$$

$$a = \sqrt{1-b^2} \qquad Re(w) = \frac{\sqrt{(1-b)(1+b)}}{2(1-b)^2} = \frac{1}{2}\sqrt{\frac{1+b}{1-b}}$$

$$Re(w) = \frac{a}{\underbrace{a^2+b^2}_{1}+1-2b} = \frac{a}{2-2b}$$

$$Im(w) = \frac{1-b}{2-2b} = \frac{1}{2}$$

№4.

1) $w(z) = x^2+y^2; \quad \frac{dw}{\partial x} = \partial$

$w = u + iv;$

$u = x^2+y^2 \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ — нет

$v = 0$

2) $w(z) = z^2 \qquad \frac{\partial w}{\partial x} = \frac{\partial w}{\partial y}$

$u = x^2 - y^2$

$v = 2xy \qquad \frac{\partial w}{\partial}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad 2x = 2x$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \quad -2y = -2y$$

$$w = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}; \quad u = \frac{x}{x^2}; \quad v = -\frac{y}{x^2}; \quad \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \leftarrow \text{да} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}}$$

№5.

1). $|f|$

$\ln(|$

$f =$

$\ln f$

g

$|f|$

$|f|$

2).

№ 5.

1). $|f| = e^{r^2\cos 2\varphi}$; $z = re^{i\varphi}$;

$(\cos\varphi - i\sin\varphi)^2 \Rightarrow$

$\Rightarrow \underbrace{\cos^2\varphi - \sin^2}_{\cos 2\varphi}$

$\ln(|f|) = r^2\cos 2\varphi$;

$f = u + iv$; $\ln(u^2+v^2) = r^2\cos(2\varphi)$

$\cancel{r\cos 2\varphi} = \mathrm{Re}\ \underbrace{r^2\cos 2\varphi}_{u^2+v^2} = \mathrm{Re}(z^2)$

$\cancel{\ln(\sqrt{u^2+v^2})} =$

$\cancel{\ln f = u+iv}$ $f = |f|\cdot e^{i\arg f}$;

$\cancel{\ln|f| = \sqrt{u^2+v^2}}$; $f = e^{r^2\cos 2\varphi}\cdot e^{i\arg f}$

$\ln f = \ln|f|\cdot e^{i\arg f}$;

$g = \ln f$; $= u+iv$; $|f| =$

$\ln|f| = r^2\cos 2\varphi = \cancel{x^2\arg f}\ \mathrm{Re}((x-y)^2)$

$|f| = e^{r^2\cos 2\varphi} = e^{x^2-y^2}$

$r^2(\cos^2\varphi - \sin^2\varphi) = x^2 - y^2$;

$|f|^2 = f\cdot f^* = e^{2(x^2-y^2)}$; $2(x^2-y^2) = z^2 + z^{*2}$

$f\cdot f^* = e^{z^2+z^{*2}}$, $\Rightarrow \boxed{f(z) = e^{z^2}}$ ← ответ.

$= -\frac{\partial u}{\partial x}$

2). $\mathrm{Arg}\, f = xy$; $f = |f|e^{i\cdot\arg f} = |f|\cdot e^{ixy}$;

$\ln f = \ln|f| + i\,xy$

$g = \ln f; = u + iv;$ $\boxed{\arg f = xy = v}$

$\frac{\partial v}{\partial y} = x = \frac{\partial u}{\partial x}, \Rightarrow u = \frac{x^2}{2} + \alpha(y)$

$\frac{\partial v}{\partial x} = y = -\frac{\partial u}{\partial y}; \quad u = -\frac{y^2}{2} + \beta(x)$

$\Rightarrow u = \frac{x^2 - y^2}{2};$

$$\boxed{g = \frac{x^2 - y^2}{2} + i\,xy} = \underbrace{\left[x^2 + 2ixy - y^2\right]}_{z^2}\frac{1}{2}$$

$$f = e^{\frac{1}{2}z^2}$$

№6

$f = u(x;y) + iv(x;y); \quad f(z = x + iy)$

1). $u = \varphi(x^2 - y^2); \varphi = \operatorname{arctg}\left(\frac{y}{x}\right)$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ← Коши-Риман

$\cancel{\Delta u = 0}$ ← гармоничность $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\cancel{\Delta v = 0}$

$$\cancel{\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \end{cases}}$$

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$$

$$\partial^2_{xy} v = \begin{cases} \partial_x \underbrace{(\partial_x u)}_{\partial_y u} = \partial_x^2 u \\ -\partial_y^2 u \end{cases}$$ ← тоже из Коши-Римана

$$\partial_x^2 u = \partial_x^2 (\varphi(x^2-y^2)) = \partial_x (2x\varphi'(x^2-y^2)) =$$

$$= 4x^2\varphi''(x^2-y^2) + 2\varphi'(x^2-y^2)$$

$$\partial_y^2 u = 2\varphi'(x^2-y^2)' - 4y^2\varphi''(x^2-y^2)$$

$$\partial_x^2 u + \partial_y^2 u = 0:$$

$$\cancel{4}\varphi'(x^2-y^2) + \cancel{4}(x^2-y^2)\varphi''(x^2-y^2)^2 = 0$$

$$(x^2-y^2)\cdot\varphi''\cdot(x^2-y^2) = 0; \quad \boxed{t = x^2-y^2}$$

$$t^2\varphi'' = 0 \quad \varphi(t) = c_0 + c_1 t;$$

$$u = c_1(x^2-y^2) + C_0$$

$$\partial_x u = \partial_y v = 2x C_1; \quad v = 2xy\cdot C_1 + C_0'$$

$$f(z) = \cancel{2x}\, C_1(x^2-y^2) + C_0 + C_1 i\cdot 2xy + iC_0' =$$

$$= C_1(x+iy)^2 + C_0 + iC_0' = \boxed{az^2 + b.}$$

2) $u = \varphi\left(\frac{y}{x}\right)$

$\partial_x^2 u + \partial_y^2 u = 0$:

$\frac{\varphi''}{x^4}(y^2+x^2) + \frac{2y}{x^3}\varphi' = 0$;

$\frac{y}{x} = t$;

$\partial_x^2 u = \partial_x\left(\varphi' \cdot \left(\frac{-y}{x^2}\right)\right) =$

$= + \frac{y^2}{x^4}\varphi'' + \frac{2y}{x^3}\varphi'$;

$\partial_y^2 u = \partial_y\left(\frac{1}{x}\varphi'\right) = \frac{\varphi''}{x^2}$

$\varphi''(1+t^2) = -2t\varphi'$

$\frac{d\varphi'}{\varphi'} = -\frac{2t\,dt}{1+t^2}$; $\ln(\varphi') = -2\ln(1+t^2) + C_0$

$\varphi' = \frac{C_0}{(1+t^2)^2}$;

$\frac{d\varphi}{dt} = \frac{C_0}{(1+t^2)^2}$; $d\varphi = \frac{C_0\,dt}{(1+t^2)^2} =$

$= d\varphi = \frac{C_0\,dt}{(1-it)^2(1+it)^2} \Rightarrow \frac{iC_0}{2} \cdot \left\{\ln(1-it) - \ln(1+it)\right\} + C_1$, $\Rightarrow$

$\partial_x(t) = -\frac{y}{x^2}$

$\Rightarrow u = iC_0 \ln\left(\frac{1-it}{1+it}\right) + C_1$

$v = \int [\partial_x u]\,\partial y = \int \partial y \left\{ iC_0 \cdot \left(\frac{1+it}{1-it}\right) \cdot \left[\frac{+iy}{x^2}(1+it) - (1-it)\cdot\frac{iy}{x}\right] / (1+it)^2 \right\} =$

$$\frac{C_0}{1+t^2}\cdot 2\cdot i\cdot i\frac{C_0\cdot x}{x}\cdot\int \partial t\,\frac{1\cdot t}{1+t^2} =$$

$$= -\frac{2C_0}{2}\ln(1+t^2) + C_2 = -C_0\ln\left(1+\frac{y^2}{x^2}\right) + C_2(x);$$

$$\boxed{\partial_y v = -\partial_x v}, \Rightarrow C_2' = \boxed{-C_0\ln(x^2)} = C_2$$

$$\S\S\ f = u+iv = iC_0\ln\left(\frac{x-iy}{x+iy}\right) + C_1 +$$

$$+ i\left\{-C_0\ln\left(\frac{x^2+y^2}{x^2}\right) + C_2\right\} =$$

$$= iC_0\left[\ln\left(\frac{(x-iy)^2}{x^2+y^2}\right) - \ln\left(\frac{x^2+y^2}{x^2}\right)\right] + C_1 + iC_2 =$$

$$= iC_0\ln\left(\frac{z^2}{x^2}\right) + C_1 + i\left(-C_0\ln(x^2)\right) =$$

$$= -iC_0\ln\left(\frac{1}{z^2}\right) + C_1 =$$

$$= \boxed{a + ib(\ln z)}$$

Note; $\boxed{z = e^{i\varphi};\quad dz = ie^{i\varphi}d\varphi}$

$$\int_C z\,dz = \frac{1}{2i}\int_0^{2\pi} e^{2i\varphi}\,d(2i\varphi) = \frac{1}{2}\cdot e^{2i\varphi}\Big|_0^{2\pi} = 0;$$

$$\int_0^{2\pi} e^{-i\varphi} i \cdot e^{i\varphi}\, d\varphi = 2\pi i$$

№ 8.

1) $$\int_C \frac{y\,dx - x\,dy}{x^2+y^2} = \int_0^{2\pi} \frac{-\sin^2\varphi - \cos^2\varphi}{\cos^2\varphi + \sin^2\varphi}\, d\varphi = \boxed{-2\pi}$$

$x = \cos\varphi;\ dx = -\sin\varphi\, d\varphi$

$y = \sin\varphi;\ dy = \cos\varphi\, d\varphi$

г). Внутри нет особых точек $\Rightarrow$ $\boxed{I = 0}$

№ 9.

1). $$p(1) = \frac{1}{2\pi i} \int_C \frac{dz}{z^2} \cdot \prod_{k=1}^{\infty} \frac{1}{1-z^k};$$

$$\boxed{\frac{1}{(1-z)} \cdot \frac{1}{(1-z^2)} \cdots} = (1+z+z^2+\dots) \cdot (1+z^2+z^4+\dots)$$

$$\prod_k \frac{1}{1-z^k} = \frac{1}{1-z} \cdot \frac{1}{1-z^2} \cdots;\quad \frac{1}{1-z} = 1+z+z^2+\dots;$$

то есть $f(z) = \sum_0^{\infty} a_n z^n$

$$p(1) = \frac{1}{2\pi i} \int_C \frac{dz}{z^2} \cdot (1+z) = \cancel{\frac{1}{2\pi i} \int_0^{2\pi} \frac{1}{e^{2i\varphi}} \cdot i e^{i\varphi}\, d\varphi} \to 0 + \frac{1}{2\pi i} \int_0^{2\pi} d(i\varphi) =$$

вклад будет только от z^1 {от $\prod_k$}.

$z = e^{i\varphi}$

$$= \frac{2\pi i}{2\pi i} = 1$$

кол-во способов получить эти степени уже не 1, а нечто вроде $\binom{n}{k}$

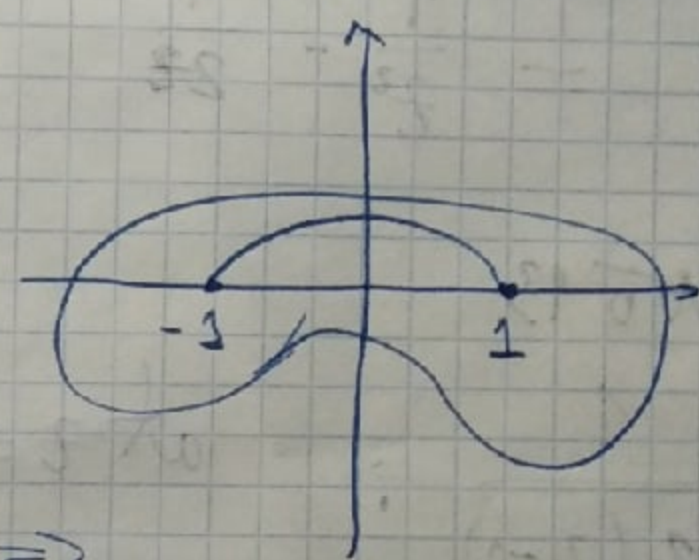
2). $$p(4) = \int_C \frac{dz}{z^{-5}} \cdot (1 + z + \overset{2}{z^2} + \overset{3}{z^3} + \overset{5}{z^4}) =$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} i5\,d\varphi = \frac{2\pi i}{2\pi i}\cdot 5 = \boxed{5}$$

все пред интегралы уходят в 0, т.к. вычет $e^{-ik\varphi}$, $k \in \mathbb{Z}$

№10.

$$y'(z) = \frac{1}{2z}; \quad y(1) = 0;$$



а) $z = e^{i\varphi}$

$$\int_0^{\pi} y'\,dz = y(-1) - y(0), \Rightarrow$$

$$y(-1) = \int_0^{\pi} \frac{1}{2e^{i\varphi}}\cdot i e^{i\varphi}\,d\varphi = i\int_0^{\pi}\frac{d\varphi}{2} = \boxed{\frac{\pi i}{2}}$$

б) $$y(-1) = -\int_{-\pi}^{0} \frac{1}{2e^{i\varphi}}\cdot i e^{i\varphi}\,d\varphi; =$$

$$\cancel{\frac{1}{2}\cdot(-\pi)} - \frac{1}{2}\cdot i\cdot\varphi\Big|_{-\pi}^{0} =$$

$$= \frac{1}{2}\cdot i\cdot(-\pi) = \boxed{-\frac{\pi i}{2}}$$

№11.

$$\frac{1+2z^2}{z^3+z^5} \text{ at } z=0: \quad \frac{1}{z^3}(1+2z^2)\cdot\frac{1}{1+z^2} =$$

$$= \frac{1}{z^3}(1+2z^2)\cdot(1-z^2+z^4+\dots) = \frac{1}{z^3} + \frac{1}{z^2}$$

№12

$$f(z) = \frac{1}{z(e^z-1)} = \frac{1}{z(e^z-1)} =$$

$$= \frac{1}{z\left(1+z+\frac{z^2}{2}+\dots-1\right)} = \frac{1}{z\left(z+\frac{z^2}{2}+\dots\right)} =$$

$$= \frac{1}{z^2} - \frac{1}{2z}$$

№13.

$$\frac{1}{z(z-1)} : \quad \text{at } z=0: \quad -\frac{1}{z} - \sum_{n=0}^{\infty} z^n$$

$$\text{at } z \in (1;+\infty): \quad \frac{1}{z^2} \cdot \frac{1}{\left(1-\frac{1}{z}\right)} =$$

$$= \frac{1}{z^2} \cdot \left\{1+\frac{1}{t}+\sum_{n=0}^{\infty} t^n\right\} = \left[+\frac{1}{z^2}+\frac{1}{z}+\sum_{n=0}^{\infty} z^{-n}\right] =$$

$$\left(\frac{1}{z}=t\to 0\right)$$

$$= \left(\sum_{n=2}^{\infty} z^{-n}\right)$$

№14.

$$\frac{z}{(z-i)(z+i)} = \frac{1}{2(z-i)} + \frac{1}{2(z+i)}$$

$$\left| z-i = t \to 0 \right| = \frac{1}{2t} + \frac{1}{2(t+2i)}$$

$$\frac{1}{2}\cdot\frac{1}{(2i+t)} = -\frac{i}{4} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{i^n t^{n+1}}{2^{n+1}}$$

$|z-i| < 2$ { расстояние до $z=-i$ есть 2 }.

ДЗ №2

№1.

$$f(z) = \frac{1}{\sin z} + \frac{2z}{z^2-\pi^2}; \quad z = \pm\pi:$$

⊙ z_0: особая точка $\underbrace{\odot,\ 0<|z-z_0|<r}_{\text{изолированная сингулярность}}$

$\lim\limits_{z\to z_0} f(z)$

- не существует — существенная особая точка
- $\exists \lim\limits_{z\to z_0} f(z)$ (конечный) — устранимая особая точка
- $\exists \lim\limits_{z\to z_0} f(z) = \infty$: $z=z_0$: полюс

$\to f(z) = \sum\limits_{n=-\infty}^{\infty} a_n (z-z_0)^n$

{ • всех $n<0$: $a_n \neq 0$
• $\forall n<0$: $a_n = 0$
• a_n, $\{n<n_0\} = 0$, полюс порядка n.

$\boxed{\sin(-\pi) = \sin(\pi) = 0}$ at $z=\pi$.