## Corollary

Let $G$ be a graph with $p$-vertices. If $\deg v \geq \frac{p-1}{2}$ for every vertex $v$ of $G$, then $G$ contain a Hamiltonian path.

**Proof:** If $p=1$, then $G \cong K_1$ and $G$ contains a (trivial) Hamiltonian path.

Suppose $p \geq 2$ and define $H = G + K_1$. Let $v$ denote the vertex of $H$ that is not in $C$. Since $H$ has a vertex $p+1$, it follows that $\deg v \geq p$. Moreover, for every vertex $u$ of $G$

$$\deg_H u = \deg_G u + 1 \geq \frac{p-1}{2} + 1 = \frac{p+1}{2} = \frac{V(H)}{2}.$$

By Dirac's theorem, $H$ contains a Hamiltonian cycle. By removing the vertex $v$ from $C$, we obtain a Hamiltonian path in $G$.

## ORE'S THEOREM

If $G$ is a simple graph with $n \geq 3$ number of vertices and if $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices $u \neq v$, then $G$ is Hamiltonian.

**Proof:** We use the method of contradiction to prove this theorem. Suppose there exist a non-Hamilton graph with $n$ vertices satisfying

$$\deg(u) + \deg(v) \geq n$$

for ~~for~~ non-adjacent vertices u & v. Among all such non-Hamilton graph, let G be a non-Hamilton graph with maximum number of edges.

Since G is maximal non-hamilton graph, there exists two non-adjacent vertices u and v in G such that addition of any edg joining u and v will result a Hamilton graph.

Thus there ~~must~~ exist a Hamilton path in G, namely $u = u_1, u_2, \ldots, u_n = v$ with u and v are end vertices.

Let $A = \{ i \mid u_i \text{ is adjacent to } u \text{ in } G \}$

$B = \{ i \mid v_i \text{ is adjacent to } v \text{ in } G \}$

Now $|A| = \deg(u)$, $|B| = \deg(v)$.

Claim: (i) $A \cap B = \Phi$

(ii) $|A \cup B| \leq n-1$

If possible $A \cap B \neq \Phi$, then say $i \in A \cap B$

$\Rightarrow (u, u_i)$ and $(u_i, v)$ could be in G,

and then G could form a Hamilton circuit, which is a contradiction. Thus

$$A \cap B = \Phi$$

Now $A \cup B = \{1, 2, \ldots, n\}$. Since $u_1 = u$ is ~~neith~~ neither adjacent to u nor adjacent to v,

$1 \notin A \cup B$.

So $|A \cup B| \leq n-1$

We know that

$$|A\cup B| = |A| + |B| - |A\cap B|$$
$$= |A| + |B| \qquad [\because |A\cap B| = 0]$$

$$\Rightarrow |A| + |B| \leq n-1$$

$$\Rightarrow deg(u) + deg(v) \leq n-1$$

which is a contradiction to the given fact that $deg(u) + deg(v) \geq n$.

Therefore, $G$ is a Hamilton graph.
(proved)

Ramsey's theorem for graphs

Proposition: Among any six people, there are three any two of whom are friends, or there are three such that no two of them are friends.

Proof: Let $G = (V, E)$ be a graph and $|V| = 6$. Fix a vertex $v \in V$. We consider two cases.

Case-I: If the degree of $v$ is at least 3, then consider three neighbors of $v$, call them $x, y, z$. If any two among $\{x, y, z\}$ are friends, we are done because they form a triangle together with $v$. If not, no two of $\{x, y, z\}$ are friend and we are done as well.

Case-II: If degree of $v$ is at most 2, then there are at least three other vertices which are not neighbors of $v$, call them $x, y, z$. In this case, the argument is complementary to the previous one. Either $\{x, y, z\}$ are mutual friends, in which case we are done. Or there are two among $\{x, y, z\}$ who are not friends, for example $x$ and $y$, and then no two of $\{v, x, y\}$ are friends.

More generally, we color the edge $K_n$ (a complete graph on $n$ vertices) with a certain number of colors and we as whether there is a complete subgraph (a clique) of certain size such that all its edges have same color.

<u>Definition</u>

A clique of size $\ell$ is a set of $\ell$ vertices such that all pairs among them are edges.

An independent set of size $s$ is a set $s$ vertices such that there is no edge between them.

Ramsey's therom states that for any large enough graph, there is an independent set of size $s$ or a clique of size $\ell$. The smallest number of vertices required to achieve this is called a Ramsey number.

## Definition

The Ramsey number $R(s, \ell)$ is the minimum number $n$ such that any graph on $n$ vertices contains either independent set of size 's' or a clique of size '$\ell$'.

## Properties

(i) $R(s, \ell) = R(\ell, s)$

(ii) $R(1,1) = 1$, $R(2,2) = 2$, $R(3,3) = 6$, $R(4,4) = 18$.

(iii) $R(s, 1) = R(1, \ell) = 1$

(iv) $R(2, \ell) = n$ ; $R(s, 2) = s$

(v) $R(s, \ell) \le R(s-1, \ell) + R(s, \ell-1)$ $\forall s, \ell \geq 2$ and are integers.

## Upper bound for $R(s, \ell)$

$$R(s, \ell) \le \binom{s+\ell-2}{s-1} = {}^{s+\ell-2}C_{s-1}$$ (where C is combination)

## Lower bound for $R(s, \ell)$

$$R(s, \ell) \geq 2^{\frac{k}{2}}, \quad k = \min\{s, \ell\}$$

That means

$$\boxed{2^{\frac{k}{2}} \le R(s, \ell) \le {}^{s+\ell-2}C_{s-1}}$$

## Problem

Find the upper bound and lower bound of Ramsey number $R(4, 4)$.

Sol$^n$: Upper bound

$$R(4,4) \le {}^{4+4-2}C_{4-1} = {}^{8}C_{3} = \frac{8!}{(8-3)!\,3!}$$

$$= \frac{8!}{5!\,3!} = \frac{6 \times 7 \times 8}{2 \times 3} = 56$$

## Lower bound

$k = \min\{4, 4\} = 4$

$R(4,4) \geq 2^{\frac{4}{2}} = 2^2 = 4.$

## Ramsey's Theorem

For any $s, \ell \geq 1$, there $R(s, \ell) < \infty$ such that any graph on $R(s,\ell)$ vertices contains either an independent set of size $s$ or a clique of size $\ell$. In particular,

$$R(s, \ell) \leq {}^{s+\ell-2}C_{s-1}.$$

**Proof:** We show that $R(s, \ell) \leq R(s-1, \ell) + R(s, \ell-1)$.

To see this, let $n = R(s-1, \ell) + R(s, \ell-1)$ and consider any graph $G$ on $n$ vertices. Fix a vertex $v \in V$. We consider two cases:

### Case-I

There are at least $R(s, \ell-1)$ edges incident with $v$. Then we apply induction on the neighbors of $v$, which implies that either they contain an independent set of size $s$, or a clique of size $\ell-1$. In the second case, we extend the clique by adding $v$, and hence $G$ contains either an independent set of size $s$ or a clique of size $\ell$.

### Case-II

There are at least $R(s-1, \ell)$ non-neighbors of $v$. Then we apply induction to the non-neighbors of $v$ and we get either an independent set of size

$s-1$, or a clique of size $\ell$. Again, the independent set can be extended by adding $v$ and hence we are done.

In particular, $R(s,\ell) \leq R(s-1,\ell) + R(s,\ell-1)$, it followes ~~that~~ by induction that these Ramsey numbers are finite. Moreover, we get an explicit bound. First, $R(s,\ell) \leq {}^{s+\ell-2}C_{s-1}$ holds for the base case, where $s=1$, or $\ell=1$ since every graph contain a clique or an independent set of size 1. The inductive step is as follows:

$$R(s,\ell) \leq R(s-1,\ell) + R(s,\ell-1)$$

$$\leq {}^{s+\ell-3}C_{s-2} + {}^{s+\ell-3}C_{s-1}$$

$$= {}^{s+\ell-2}C_{s-1}$$ , where C is combination.

(proved)