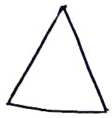
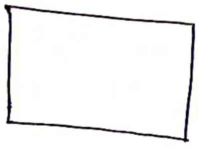
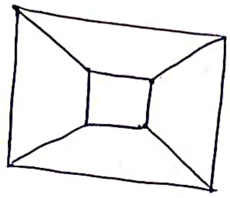
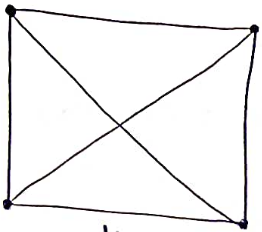
Definition (plane graph)

A graph that can not be drawn on a plane without a crossover between its crossing is called a plane graph.

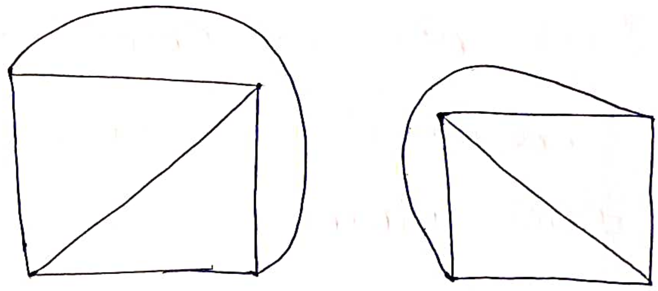




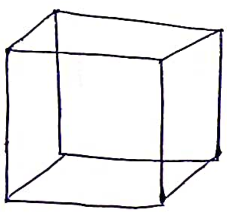




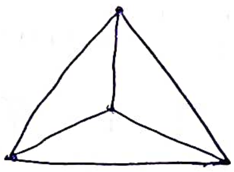
K_4

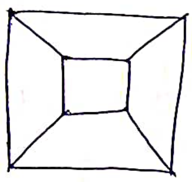


Embeding of K_4 i.e planar representation of K_4 ~~on plane~~



Planar representation

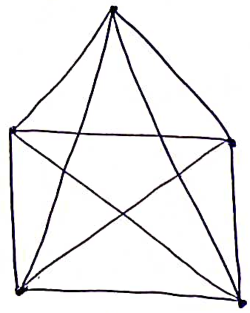




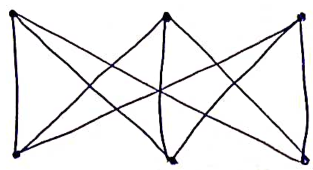
Kuratowski's Graphs

A complete graph with 5-vertices, i.e, K_5 is known as first kuratowski's graph, where as the complete bipartite graph $K_{3,3}$ is known as second kuratowski's graph.

~~Obse~~



K_5



$K_{3,3}$

Observations:

(i) Both are regular graphs.

(ii) Both are non-planar graphs

(iii) Removal of one vertex or one edge makes the graph planar.

(iv) First graph is non-planar with smallest number of vertices and second graph is non-planar with smallest number of edges. Thus, both are simplest non-planar graphs.

The first and second graphs of Kuratowski are represented as K_5 and $K_{3,3}$. The letter K being for Kuratowski.

Region / Faces

A plane representation of a planar graph G divides the plane into regions or faces. A region of a planar graph is an area of the plane, i.e, bounded by edges and is not further divided into sub area.

- If it is bounded by finite area, the it is finite region.
- If it is not bounded, then it is infinite region.
- Every planar graph ~~has~~ has exactly one infinite region.

Euler's Formula

A connected planar graph with n-vertices and e-edges has e-n+2 regions.

OR, A connected planar graph with n-vertices and e-edges and f-regions or r-regions, then prove that
$f = e-n+2$. or $r = e-n+2$

Proof: Let G be a connected and planar graph with n-no. of vertices and e-number of edges. Let r be the no. of regions in G.

Claim: $r = e-n+2$ regions.

We will prove this theorem by using method of induction on number of edges.

~~Basis step~~

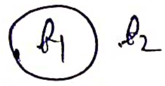
Basis step: Let $e=0$, then G must have just one vertex, i.e, $n=1$ and one infinite region i.e, $r=1$

Then $n-e+r = 1-0+1 = 2 \Rightarrow r = e-n+2$

Let $e=1$, then $n=1$ & $n=2$. We have

the following figure

$e=1, n=2$



$e=1, n=1$

When $n=1, e=1$, i.e, for loop

$$n-e+r = 1-1+2 = 2 \quad \Rightarrow r = e-n+2$$

When $n=2$, $e=1$, then

$$n-e+r = 2-1+1 = 2 \quad \Rightarrow r = e-n+2$$

Therefore basis step holds true.

Inductive hypothesis

Now, we assume that the result is true for all graph with at most $e-1$ edges.

Inductive step:

Let G be a connected graph with e-edges and r- regions. There are two cases.

Case -I: If G is a tree. Then number of edges 'e' $= n-1$ and number of regions is 1, which is infinite. So

$$\begin{aligned} 1 = r &= e-(e+1)+2 \\ &= e-e-1+2 \\ &= 1 \end{aligned}$$

So $r = e-n+2$.

Case -II: If G is not a tree, then G has some circuit. Let e' be an edge is some circuit

If we delete e' from the plane representation of G, then it will merge the two region in to a new region.

So $G - e'$ is a connected graph with n - number of vertices but $(e-1)$ number of edges and $(r-1)$ regions. So by inductive hypothesis

$$r - 1 = (e-1) - n + 2$$

$$= e - n + 1$$

$$\Rightarrow r = e - n + 2$$

Hence by mathematical induction ~~for~~ the formula is true for all plane graphs.

(proved)