Hamiltonian Graphs

Hamiltonian circuit or cycle

A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as Hamiltonian circuit.

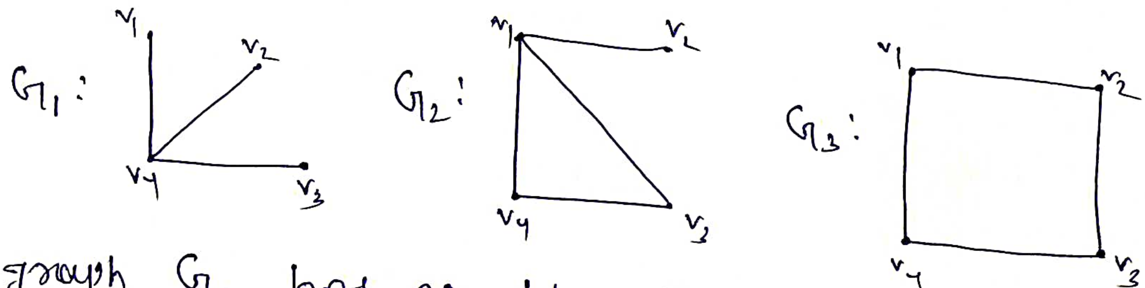
Hamiltonian Graph

A graph G is called a Hamiltonian graph, if it contains a Hamiltonian circuit.

Hamitonian path

A Hamitonian path is a simple path that contains all vertices of G where the end points may be distinct.

In the following figures, Hamiltonian path and cycle are shown:



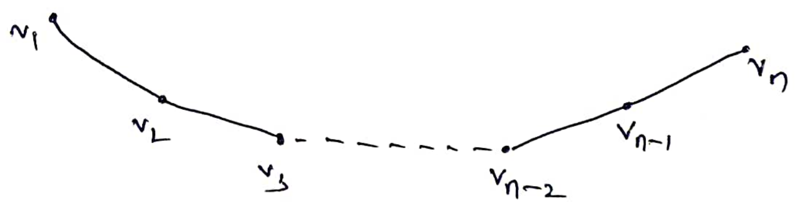
The graph G_1 has no Hamiltonian path and Hamiltonian cycle, G_2 has Hamiltonian path $v_4 v_3 v_1 v_2$ but no Hamiltonian cycle, while G_3 has Hamiltonian cycle $v_1 v_2 v_3 v_4 v_1$.

Dirac's Theorem

Let G be a graph of order $p \geq 3$. If $\deg v \geq \frac{p}{2}$ for every vertex v of G, then G is Hamiltonian.

Proof: If $p = 3$, then the condition on G implies that G is isomorphic to K_3 and hence G is Hamiltonian. We may assume, that $p \geq 4$.

Let $P: v_1, v_2, \ldots, v_n$ be a longest path in G. Then every neighbour of v_1 and every neighbour of v_n is on P.



Otherwise, there must be a longer path P.

Consequently, $n \geq 1 + \frac{p}{2}$.

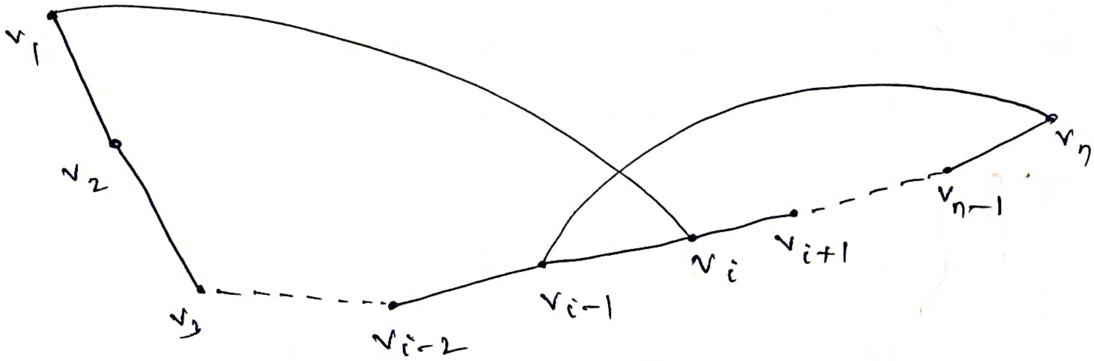
There must be some vertex. v_i where $2 \leq i \leq n$, such that v_1 is adjacent to v_i and v_n is adjacent to v_{i-1}

If this were not the case, then whenever v_1 is adjacent to v_i, the vertex v_n is not adjacent to v_{i-1}.

Since at least $\frac{p}{2}$ of $p-1$ vertices different from v_n are not adjacent to v_n. Hence, ~~deg v_n~~

$\deg v_n \leq (p-1) - \frac{p}{2} < \frac{p}{2}$, which is a

contradiction to the fact that $\deg v_n \geq \frac{p}{2}$.
Therefore, as we claimed, there must be a vertex v_i adjacent to v_1 and v_{i-1} adjacent to v_n



We now see that G has cycle $C: v_1, v_i, v_{i+1}, \ldots, v_{n-1}, v_n, v_{i-1}, v_{i-2}, \ldots, v_2, v_1$ that contains all the vertices of P.

If C contains all the vertices of G (if $n=p$) then C is a Hamiltonian cycle, and the proof of the theorem complete. Otherwise, there is some vertex u of G that is not on C.

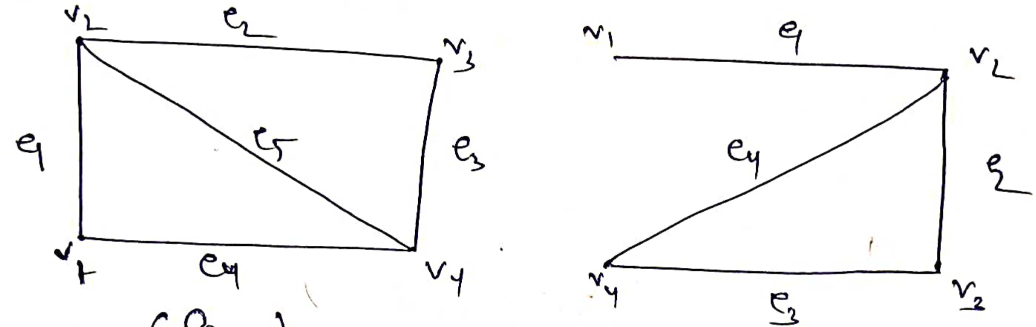
By hypothesis, $\deg u \geq \frac{p}{2}$. Since P contains at least $1+\frac{p}{2}$ vertices, there are fewer that $\frac{p}{2}$ vertices not on C; so u must be adjacent to a vertex v that lies on C.

However, the edge uv plus the cycle C contain a path whose length is greater than that of P, which is impossible.

Thus C contains all vertices of G and G is Hamiltonian. Hence the proof.

□

Q: Which of the graph given in figure below is Hamiltonian circuit. Give the circuits on the graphs that contains them.



(G_1) (G_2)

Sol^n:

~~The graph shown in figure~~ G_1 has a Hamiltonian circuit $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$.

Note that all vertices appear on this a circuit but not all edges.

The edge e_5 is not used in the circuit.

The graph G_2 does not contain circuit since every circuit containing every vertex must contain the e_1 twice.

But the graph have a Hamiltonian path $v_1 - v_2 - v_3 - v_4$.

Q: Show that any k-regular simple graph with $2k-1$ vertices is Hamiltonian.

Sol^n: In a k-regular graph, ~~simple~~ the degree of every vertex is k and $k > k - \frac{1}{2} = \frac{1}{2}(2k-1) = \frac{n}{2}$, where $n = 2k-1$ is the number of vertices. Then by Dirac's Theorem, the graph considered is Hamiltonian.