with no $(r+1)$-Clique, $T_{n,r}$ has the maximum number of edges.

Edge - Coloring

Edge - Coloring of a graph is an assignment of colors to the edges of the graph so that no two incident edges have the same color.

OR, An edge coloring of a graph G is a function $f: E(G) \longrightarrow C$, where C is a set of distinct colors. For any positive integer k, a k-edge coloring is an edge coloring that uses exactly k different colors.

Proper Edge Coloring

A proper edge coloring of a graph is an edge coloring such that no two adjacent edges are assigned the same color. Thus a proper edge coloring f of G is a function

$$f: E(G) \longrightarrow C$$

such that

$$f(e) \neq f(e')$$

whenever edges e and e' are adjacent in G.

Chromatic Index

The chromatic index of a graph G, denoted by $\chi'(G)$, is the minimum number of different colors required for a proper edge colouring of G. G is k-edge-chromatic if $\chi'(G) = k$.

→ Chromatic index of Bipartite graph is always $\Delta(G)$.

Theorem

If G is any simple graph, then $\chi'(G) \geq \Delta$.

Proof: Given that G is any simple graph. We know Δ is the maximum degree of the graph G. Therefore, there exist a vertex v such that

$$d(v) = \Delta.$$

So there are Δ edges incident on this vertex v, i.e, all the Δ edges are adjacent. So we required at least Δ colours to proper coluring of the edge of G.

Thus $\chi'(G) \geq \Delta$.

Lemma: Let $C = (E_1, E_2, \ldots, E_n)$ be the optimal k-edge colouring of G. If there is a vertex v_i in G and colours i and j such that i is not represented at u and j is represented twice at u. Then the component H of $G(E_i \cup E_j)$ that

that contain u is an odd cycle.

Theorem

If G is bipartite, then $\chi'(G) = \Delta$

Proof: Suppose G is bipartite graph with $\chi'(G) > \Delta$

Let $C = (E_1, E_2, \ldots, E_\Delta)$ be an optimal Δ-edge colouring of G.

Since $\chi'(G) > \Delta$, this colouring can not be proper.

Let u be a vertex such that $c(u) < d(u)$. Now,

Consider Δ-edge colouring at u. Some color i is not used and j is represented twice.

By Lemma, G contains an odd cycle, which is a ~~contd~~ contradiction to the fact that G has no odd cycle. We know that for any simple graph $\chi'(G) \geq \Delta$.

Therefore $\chi'(G) = \Delta$.

□

Vizing Theorem

If G is simple, then either $\chi'(G) \geq \Delta$ or $\chi'(G) \leq \Delta + 1$.

Proof: Let G be a simple graph. If G is bipartite, then $\chi'(G) = \Delta$.

We need only to show that $\chi'(G) \leq \Delta + 1$.

Suppose $\chi'(G) > \Delta + 1$. Let $C = (E_1, E_2, \ldots, E_{\Delta+1})$ be an optimal $\Delta+1$-edge colouring of G and u be the vertex such that $c(u) < d(u)$. Then there exist colour i_0 and i_1 such that i_0 not represented at u and i_1 is represented at least twice at u.

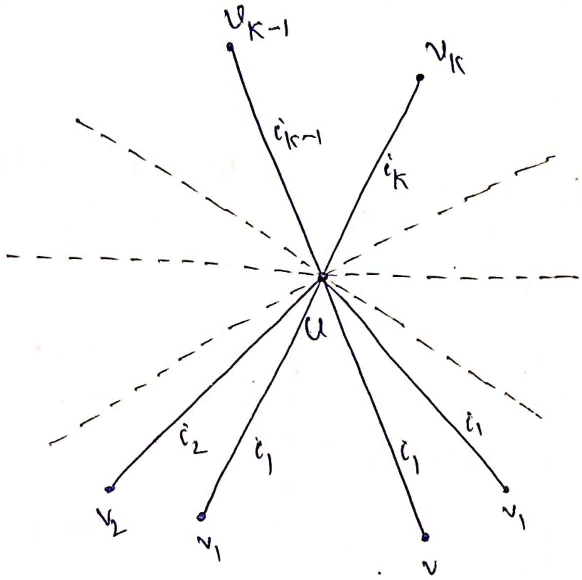


Fig: a

Let uv_1 have colour i_1 as shown on fig (a). Since $d(v_1) < \Delta+1$, some colour i_2 is not represented at v_1. Now i_2 must be represented at u, since otherwise by recolouring uv_1 with i_2 we would obtain an improvement on C. Thus some edge v_2u has colour i_2.

Again $d(v_2) < \Delta+1$, some colour i_3 is not represented at v_2 and i_3 must be represented at u. Since, otherwise by recolouring uv_1 with i_2 and uv_2 with i_3 we would obtain an improved $\Delta+1$ edge colouring

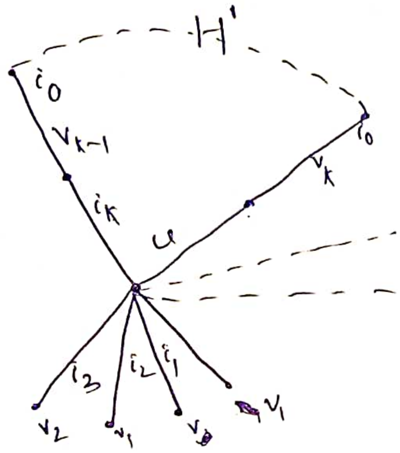
Thus some edge uv_3 has colour i_3, continuing this procedure we construct a sequence $(v_1, v_2, \ldots)$ of vertices and $(i_1, i_2, \ldots)$ of colours such that

(i) uv_j has colour i_j

(ii) i_{j+1} is not represented at v_j (since $d(u)$ is ~~finite~~ finite, there exist a smallest integer l such that for some $k < l$).

(iii) $i_k = i_{l+1}$

Now we recolour G as follows for $1 \le i \le k-1$, uv_j with colour i_{j+1} yielding a new $\Delta+1$-edge colouring $c' = (E_1', E_2', \ldots, E'_{\Delta+1})$ of G.



Clearly, $c'(v) \ge c(v)$ for every $v \in V$ and c' is also an optimal $\Delta+1$ edge colouring of G.

The component H' of $G[E'_{i_k} \cup E'_{i_0}]$ that contains u is an odd cycle.

Now, in addition uv_j with the colour i_{j+1}, $k \le j \le l-1$ and uv_l with the colour i_k to obtain a $\Delta+1$ edge colouring $c'' = (E_1'', E_2'', \ldots, E''_{\Delta+1})$ as the following fig: c.

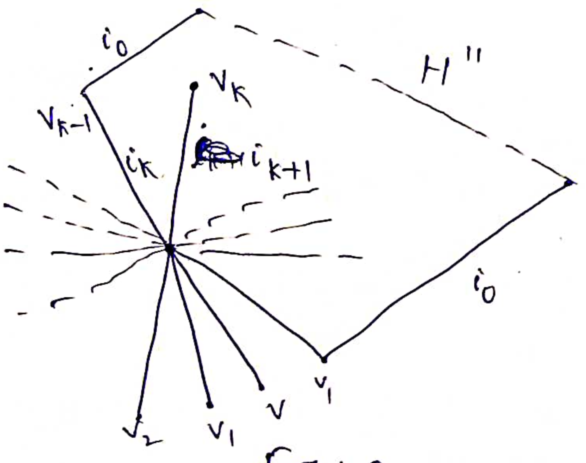


Fig: c

$C''(v) \geq c(v)$ for every $v \in V$ and the component H'' of $G[E''_{i_k} \cup E''_{i_0}]$ that contains u is an odd cycle. But since v_k has degree 2 in H', clearly v_k has degree 1 in H''. But H'' is an odd cycle contains u, which is a contradiction.

This prove that our assumption is wrong.

Thus $\chi'(G) \geq \Delta$ and $\chi'(G) \leq \Delta + 1$.

(proved)