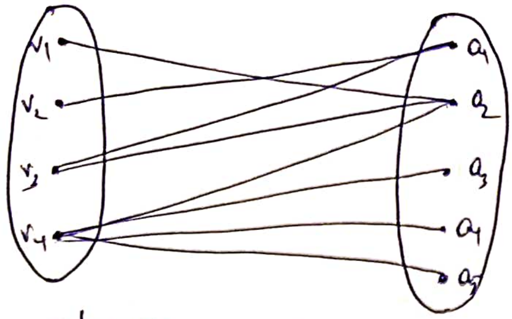
By choosing $m=2$, $\exists$ a complete matching from the set V_1 to V_2.

Problem

Prove that the bipartite graph shown in the following graph does not have a complete matching.



Sol^n: We observe that the three vertices v_1, v_2, v_3 in V_1 are together joined to two vertices a_1, a_2 in V_2. Thus, there is a subset of 3 vertices in V_1 which is collectively adjacent to $2 (< 3)$ vertices in V_2.

Hence, by Hall's theorem, there does not exist a complete matching from V_1 to V_2.

Matching in General Graphs

Definition

A factor of a graph G is a spanning subgraph of G. A k-factor is a spanning k-regular subgraph. An odd component of a graph is a component of odd order; the number of odd components of H is $o(H)$.

Remark

→ A 1-factor and a perfect matching are almost the same thing.

→ The precise distinction is that "1-factor" is a spanning 1-regular subgraph of G, while "prefect matching" is the set of edges in such a subgraph.

→ A 3-regular graph that has a prefect matching ~~decomposition~~ decomposes into a 1-factor and 2-factor.

TUTTE'S 1-FACTOR THEOREM

Tutte found a necessary and sufficient condition for which graph have 1-factor. and we consider a set $S \subseteq V(G)$, then every odd component of $G-S$ has a vertex matched to something outside it, which can only belong to S. Since these vertices of S must be distinct, $o(G-S) \leq |S|$.

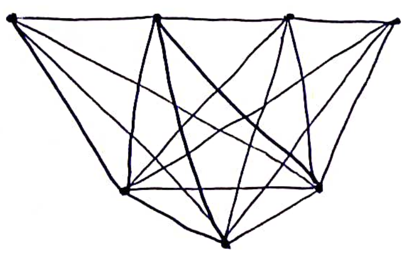
The condition " For all $S \subseteq V(G)$, $o(G-S) \leq |S|$" is Tutte's Condition.

Theorem

A graph G has a 1-factor if and only if $o(G-S) \leq |S|$ for every $S \subseteq V(G)$.

Definition

The join of simple graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union $G+H$ by adding adding the edges $\{xy : x \in V(G), y \in V(H)\}$.



$P_4 \vee K_3$

Berge- Tutte Formula

The largest number of vertices saturated by a a matching in G is $\min_{S \subseteq V(G)} \{n(G) - d(S)\}$, where $d(S) = o(G-S) - |S|$.

Proof: Given $S \subseteq V(G)$, at most $|S|$ edges can match verpices vertices of S to vertices in odd components of $G-S$, so every matching has at least $o(G-S) - |S|$ unsaturated verpices. We want to achieve this bound.

Let $d = \max\{o(G-S) - |S| : S \subseteq V(G)\}$. The case $V = \phi$ gives $d \geq 0$. Let $G' = G \vee K_d$. Since $d(S)$ has the same parity as $n(G)$ for each S, we know that $n(G')$ is even. If G' satisfies Tutte's Condition, then we obtain a matching of the desired size in G from a perfect matching in G', because deleting the d added vertices eliminates edges that saturate at most d vertices of G.

The condition $o(G'-S') \leq |S'|$ holds for $S' = \phi$ beasure $n(G')$ is even. If S' is nonempty but not contain all of K_d, then $G'-S'$ has only one component, and $1 \leq |S'|$. Finally, when $K_d \subseteq S'$, we let $S = S' - V(K_d)$. We have $G' - S' = G - S$, so $o(G'-S') = o(G-S) \leq |S| + d = |S'|$.

Petersen 1-factor Theorem

Every 3-regular graph with no cut-edge has a 1-factor.

Proof: Let G be a 3-regular graph with no cut-edge. We prove that G satisfies Tutte's Cond^n. Given $S \subseteq V(G)$, we count the edges between S and the odd components of $G-S$. Since G is 3-regular, each vertex of S is incident to at most three such edges. If each odd component H of $G-S$ is incident to at least three such edges, then $3\,o(G-S) \le 3|S|$ and hence $o(G-S) \le |S|$.

Let m be the number of edges from S to H. The sum of the vertex degrees in H is $3n(H)-m$. Since H is a graph, the sum of its vertex degrees must be even. Since $n(H)$ is odd, we conclude that m must also be odd. Since G has no cut edge, m can not equal 1. We conclude that there are at least three edges from S to H. Hence every 3-regular graph with no cut-edge has a 1-factor.

(proved)

Petersen 2-factor Theorem

Every regular graph of even degree has a 2-factor.

Proof: Let G be a $2k$-regular graph with vertices $v_1, v_2, \ldots, v_n$. Every component of G is

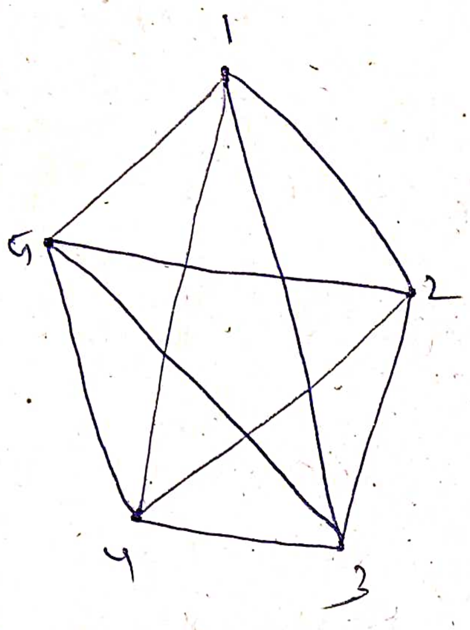
Eulerian, with some Eulerian circuit C. For each component, define a bipartite graph H with vertices $u_1, u_2, \dots, u_n$ and $w_1, w_2, \dots, w_n$ by putting $u_i \leftrightarrow w_j$ if v_j immediately follows v_i somewhere on C. Because C enters and exists each vertex k times, H is k-regular.

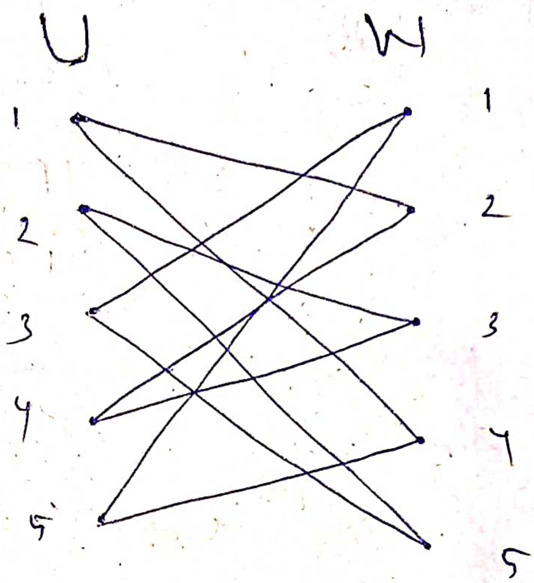
Being a regular bipartite graph, H has a 1-factor M. The edge incident to w_i in H corresponding to an edge entering v_i in C. The edge ~~ed~~ incident to u_i in H corresponding to an edge existing v_i. Thus that 1-factor in H transforms onto 2-regular spanning subgraph of this component of G. Doing this for each component of G yields a 2-factor of G. Hence every ~~component~~ regular graph of ~~ev~~ even degree has a 2-factor.

(proved)

Example (Construction of a 2-factor)

Consider the Eulerian circuit in $G = K_5$ that successively visits 1231425435. The corresponding bipartite graph H is on the right. For the 1-factor whose u, w-pairs are 12, 43, 25, 31, 54, the resulting 2-factor is the cycle $(1, 2, 5, 4, 3)$. The remaining edges from another 1-factor, which corresponds to the 2-factors ~~(1,4,2,3~~, $(1, 4, 2, 3, 5)$ that remains in G.


1
2
3
4
5


U
W
1
2
3
4
5
1
2
3
4
5