If we delete e' from the plane representation of G, then it will merge the two region into a new region.

So $G - e'$ is a connected graph with n-number of vertices but $(e-1)$ number of edges and $(r-1)$ regions. So by inductive hypothesis

$$r - 1 = (e-1) - n + 2$$
$$= e - n + 1$$
$$\Rightarrow r = e - n + 2$$

Hence by mathematical induction ~~for~~ the formula is true for all plane graphs.

(proved)

Corollary

If a connected simple planar graph G has $n \geq 3$ vertices, e- edges and r- regions, then show that

(i) $r \leq \frac{2e}{3}$ (ii) $e \leq 3n - 6$.

Proof: Since given $n \geq 3$, any face or region of the given planar simple graph will have at least 3 edges but each edge will have 2 faces or regions so

$$3r \leq 2e \Rightarrow r \leq \frac{2e}{3}$$

~~Now $r \leq \frac{2e}{3}$ by using this in the formula for~~

Using $r \leq \frac{2e}{3}$ in the Euler's formula, we get

$$r = e-n+2 \le \frac{2e}{3}$$

$$\Rightarrow 3e-3n+6 \le 2e$$

$$\Rightarrow e-3n+6 \le 0$$

$$\Rightarrow e \le 3n-6$$

(proved)

Corollary

If G is connected simple planar graph with $n \geq 3$ vertices and e-edges and no circuits of length 3, then $e \le 2n-4$.

Proof: If the graph is planar, then the degree of each region is atleast 4. Hence the total number of edges around all the regions is atleast $4r$. Since every edge borders two regions, the total number of edges around all the regions is $2e$, so we established that $2e \geq 4r$

$$\Rightarrow 2r \le e$$

If we combine this with Euler's formula

$n-e+r=2$, we get

$$\Rightarrow r = e-n+2$$

$$\Rightarrow 2r = 2e-2n+4 \le e$$

$$\Rightarrow e \le 2n-4.$$

(proved)

Problem

Show that the graph K_5 is not planar.

Sol^n: Since K_5 is a simple graph, the smallest possible length for any cycle K_5 is three. We suppose that the graph is planar.

Number of vertices in $K_5 = 5$

Number of edges in $K_5 = \frac{5 \times 4}{2} = 10$.

Now $3n-6 = 3\times5-6 = 9$.

So $e \leq 3n-6 \Rightarrow 10 \leq 9$, which is not possible. So K_5 is not planar.

This may be noted that the inequality $e \leq 3n-6$ is only by a necessary condition but not a sufficient condition for the planarity of the graph.

For example, graph $K_{3,3}$ satisfies the inequality

$$e \leq 3n-6$$

$$\Rightarrow 9 \leq 3\times6-6 = 12,$$

yet the graph is non-planar.

Problem: Show that $K_{3,3}$ is not-planar.

Solⁿ: Since $K_{3,3}$ has no circuits of length 3 and has 6 vertices and 9 edges, i.e,

$$n=6,\ e=9.$$

Now

$$9 \leq 2n-4 = 2\times6-4 = 12-4 = 8$$

$$\Rightarrow 9 \leq 8$$

which is not possible. Hence $K_{3,3}$ is not planar.

Problem: Prove that K_4 and $K_{2,2}$ are planar.

Solⁿ: In K_4, we have $v=4$ and $e=6$.

Obviously, $6 \leq 3 \times 4 - 6 = 6$.

Thus, this relation is satisfied for K_4.

For $K_{2,2}$, we have $v = 4$ and $e = 4$.

Again in this case, the relation $e \leq 3n - 6$,

i.e $4 \leq 3 \times 4 - 6 = 6$ is satisfied.

Hence, both K_4 and $K_{2,2}$ are planar.

KURATOWSKI'S THEOREM

Corollary

If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

Proof: We prove this by method of contradiction.

Now, we know that for a simple planar graph G $(n \geq 3)$, $e \leq 3n - 6$.

This is not true for the graph having one or two vertices, but the degree of vertex would be less than 5.

$$e \leq 3n - 6$$
$$\Rightarrow 2e \leq 6n - 12$$

Let the graph which is connected and planar have a vertex at least degree 6, then by the application of Handshaking theorem

$$2e = \sum_{v_i \in V} deg(v_i)$$

But we have $2e \leq 6n - 12$ which is a contradiction. Thus the degree of each vertex of G must be less than equal to 5. (proved)