Handshaking Theorem (For simple graph):

The sum of the degree of the vertices of a graph is equal to twice of the number of edges

Proof: Let G be a graph with 'm' number of edges and 'n' number of vertices. Therefore,

$$|V| = n, \quad |E| = m.$$

Let $V = \{v_1, v_2, ..., v_n\}$ be the set of vertices. We have to find degree of v_i for $i = 1, 2, 3, ..., n$, i.e., $\deg(v_i)$ is the number of edge incident on v_i.

Let e be an edge, then e is either a loop or incident with two distinct vertices. If e is a loop then e contributes degree of vertex '2'. If e is incident with two distinct vertices it also contributes sum of degree of vertices is two. Similarly, for m numbers of edge we have sum of degree of vertices is $2 + 2 + ... + 2$ (up to m-times) $= 2m$,

$$\text{i.e} \quad \sum_{i=1}^{n} \deg(v_i) = 2m = 2 \times \text{number of edge}.$$

~~Hence~~ This completes the proof of the theorem.

Theorem

In a graph, the number of odd degree vertices is always even.

Proof: Let $G = (V, E)$ be a graph with 'n' number of vertices.

Let us assume that the degree of first K vertices $v_1, v_2, \ldots, v_k$ be even and the rest $(n-k)$ vertices $v_{k+1}, v_{k+2}, \ldots, v_n$ are of odd degree.

Then

$$\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{k} \deg(v_i) + \sum_{i=k+1}^{n} \deg(v_i) = 2m,$$

(By Handshaking Theorem)

where m is the number of edges.

$$\Rightarrow 2\ell + \sum_{i=k+1}^{n} \deg(v_i) = 2m$$

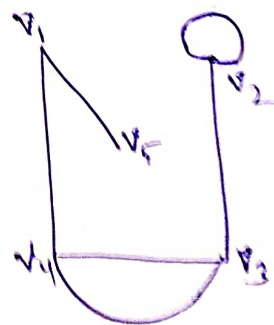$$\Rightarrow \sum_{i=k+1}^{n} \deg(v_i) = 2(m-\ell),$$

which is an even number.

Therefore, in a graph the number of odd degree vertices is always even.

Ex: Here $\deg(v_1) = 2$, $\deg(v_2) = 3$
$\deg(v_3) = 3$, $\deg(v_4) = 3$
$\deg(v_5) = 1$.

∴ The number of odd degree vertices in the above ~~poop~~ graph is 4 which is even number.
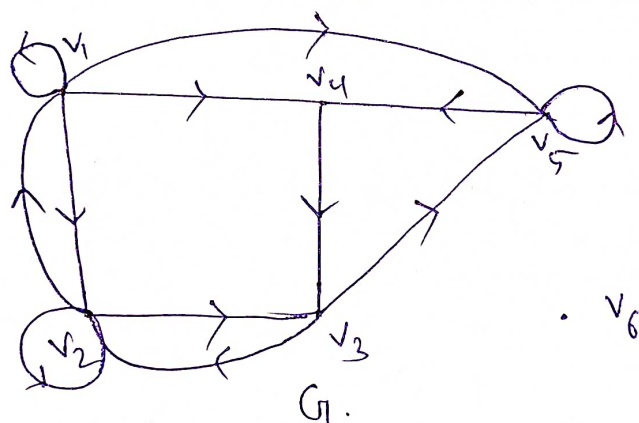
<u>In degree</u> and <u>Out degree</u> of a vertex:
In a directed graph the indegree of a

vertex v, denoted by deg⁻(v) is the number of edges as their terminal vertex.

The out degree of v denoted by deg⁺(v) is number of edges with v as their initial vertex.

Note: A loop ⟲ at a vertex contributes 1 to both indegree & out degree of this vertex.

Ex: Find the in degree and out degree vertices of the following graph.

G.

Soln: The in degree of G are

$$deg^-(v_1) = 2, \quad deg^-(v_2) = 3, \quad deg^-(v_3) = 2,$$
$$deg^-(v_4) = 2, \quad deg^-(v_5) = 3, \quad deg^-(v_6) = 0.$$

The out degree of G are

$$deg^+(v_1) = 4, \quad deg^+(v_2) = 2, \quad deg^+(v_3) = 2, \quad deg^+(v_4) = 1$$
$$deg^+(v_5) = 3, \quad deg^+(v_6) = 0.$$

Q→ Find the in degree and out degree