Cut vertex, cut set and edge or bridge

Sometimes the removal of a vertex and all edges incident with it produces a subgraph with more connected components. A cut vertex of a connected graph G is a vertex whose removal increases the number of components.

Clearly, if v is a cut vertex or articulation vertex of a connected graph G, G−v is disconnected. A cut vertex is also called a cut point.

Cut edge or bridge

An edge whose removal produces a graph with more connected ~~compon~~ components then the edge of the original graph is called a cut edge or bridge.

Cut set

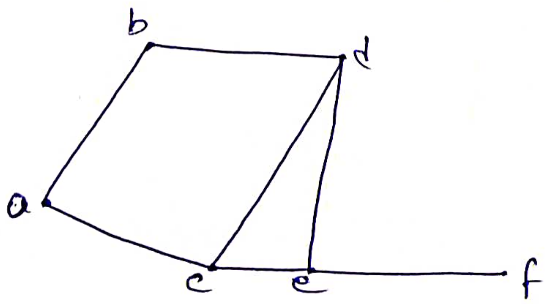
The set of all minimum number of edges of G whose removal disconnects a graph G is called a cut set of G. Thus a cut set S satisfying the following cond^n:

(i) S is a subset of the edge set E of G.

(ii) Removal of edges from a connected graph G disconnects G.

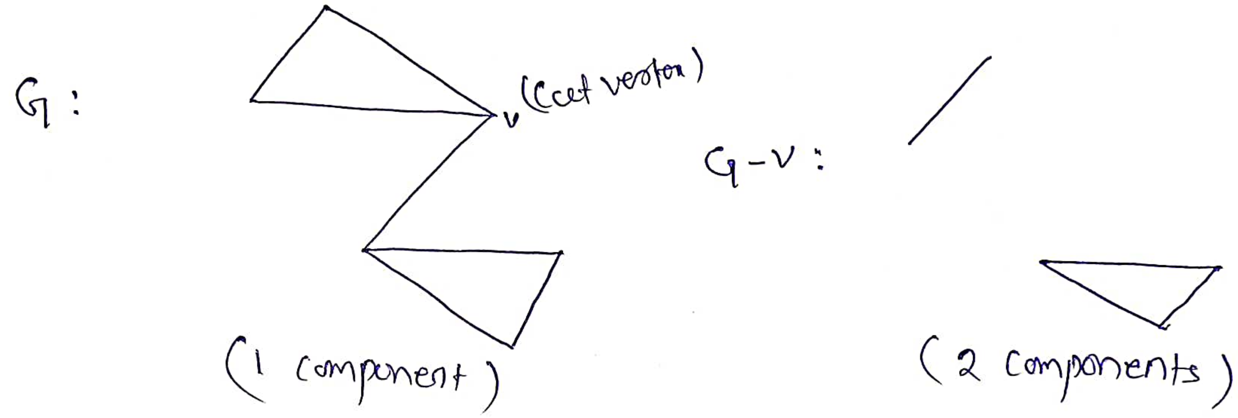
(iii) No proper subset of G satisfy the condition.

Example



In the graph in figure, each of the sets $S_1 = \{(b,d), (c,d), (c,e)\}$ and $S_2 = \{(e,f)\}$ are cut sets. The ~~set~~ edge $\{e, f\}$ is the only bridge.

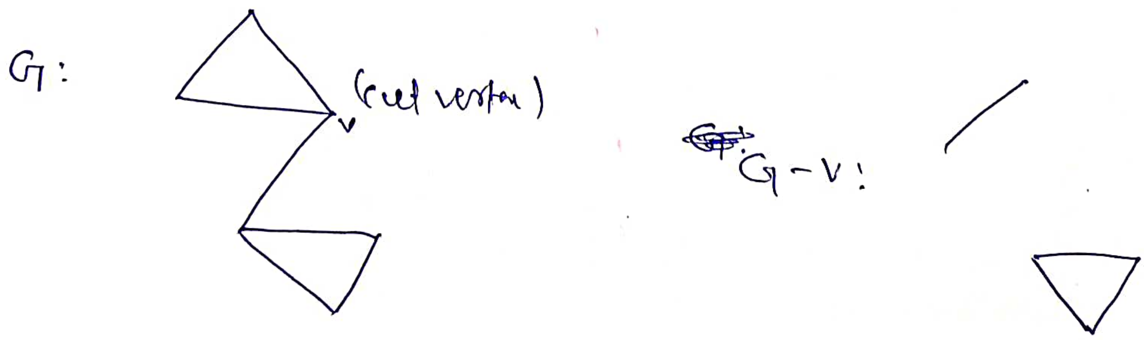
Example



Separable

A graph is said to be separable if it is not connected or if there exist atleast one cut vertex a present in that graph. Otherwise it is not separable.

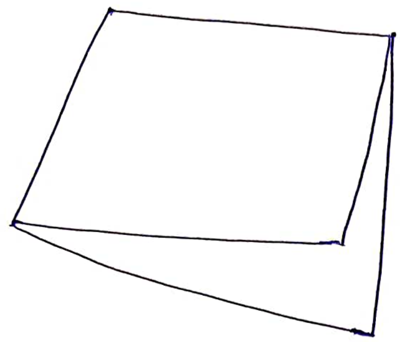
Example



Here G is separable graph.

→ In trivial graph the vertex is neither a cut vertex nor a isolated vertex.

G:



G is not separable, since there is no cut vertex.

Theorem

The vertex v is a cut vertex of the connected graph G iff there exists two vertices u and w in the graph G such that $v \neq u$, $v \neq w$ & $u \neq w$ but v is on every u-w path.

Proof: Let us consider v is a cut vertex in G. Then $G - v$ is not connected. Then there are at least two components say $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. We choose $u \in V_1$ and $w \in V_2$. The u-w path in G.

If v is not in the path then the path also in $G - v$.

If v is on every u-w path then the vertices u and w are not connected in $G - v$. Therefore v is a cut vertex.

Theorem

A nontrivial simple graph has at least two vertices which are not cut vertices.

Proof: We shall prove the theorem by using method of induction.

Base step: For $n=2$ it is obvious. i.e for $n=2$ both are not cut vertices.

Inductive hypothesis:

Assume that the result is true for $n \leq k$ $(k \geq 2)$

Inductive step:

We shall prove this result is true for $n = k+1$. Suppose there are no cut vertex in G. Then the result is obviously true. Otherwise, we consider a cut vertex v of G. Let $G_1, G_2, \dots, G_m$ be the components of $G-v$ $(m \geq 2)$. Every components G_i, $i=1,2,\dots,m$ falls into one of these two cases.

1. G_i is trivial, then the only vertex of G_i is a pendent vertex or an isolated vertex of G but not a cut vertex of G.

2. G_i is nontrivial.

by inductive hypothesis there exists 2 vertices u and w in G_i which are not cut vertices of G_i.

If v and u are not adjacent in G then u is not a cut vertex in G. If v and w are not adjacent in G, then w is not a cut vertex. If both v and u as well as u and w are adjacent in G, then at least 2 of them are not cut vertices. □