the corollary, it is sufficient to show that Hall's matching condition holds here. Therefore we must show that $\forall$ ~~...~~, $S \in X$, $|N(S)| \geq |S|$.

So, the number of edges between S and ~~...~~ ~~...~~ $N(S) = k|S|$ which is less than the number of edge ~~...~~ leaving $N(S)$. Therefore

$$k|S| \leq k|N(S)|$$

$$\Rightarrow |S| \leq |N(S)|, \text{ when } k > 0.$$

Hence k-regular bipartite graph has a perfect matching.

$\square$

Konig-Egervary Theorem

Definition

A vertex cover of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge.

Definition

An edge cover of a graph G is a subset L of the edges of G such that every vertex of $V(G)$ is incident to some edge of L.

Definition

$\alpha(G)$ is the maximum size of an independent set in a graph G.

$\alpha'(G)$ is the maximum ~~minimum~~ size of all of the matchings of a graph G.

$\beta(G)$ is the minimum size of all vertex covers of a graph G.

$\beta'(G)$ is the minimum size of edge covers of a graph G.

König - Egervary Theorem

If a graph G is a bipartite graph, then the maximum size of a matching in G equals to the minimum size of a vertex cover. So,

$$\alpha'(G) = \beta(G).$$

Proof: First we observe that for any matching M and any vertex cover Q we have $|Q| \geq |M|$. Suppose that Q is a minimal vertex cover. Our aim is to construct a matching cover $|Q|$. Let

$$R = Q \cap X \quad \text{and} \quad T = Q \cap Y,$$

where X and Y are the two ~~part~~ bipartitions of the bipartite graph G. Now construct a matching from R to ~~Y~~ $Y \setminus T$. This matching saturates R. Now it is sufficient to show that Hall's condition on the induced subgraph H on $R \cup Y \setminus T$ holds.

Suppose Hall's condition does not hold. If $\exists$ an S such that $|N_H(S)| < |S|$

replace S by $\{N_H(S)\}$ in Q. Now $Q \cup (N(S) \setminus S)$ is a vertex cover. But the size of this vertex cover is less than the size Q. This is a contradiction to the minimality ~~that saturates~~ Q. Therefore Hall's cond^n is verified. Now we can say that there is a matching from R to $T \setminus T$ that saturates R. Similarly there is a matching from T to $X-R$ that saturates T. Let M be the union of these two matchings. Then we know that

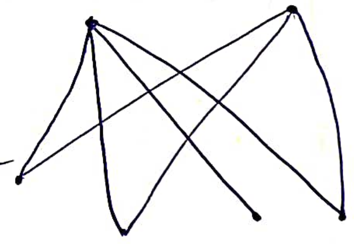
$$|M| = |R| + |T| = |Q|$$

and we have successfully constructed a matching of size $|Q|$. (proved)

Example

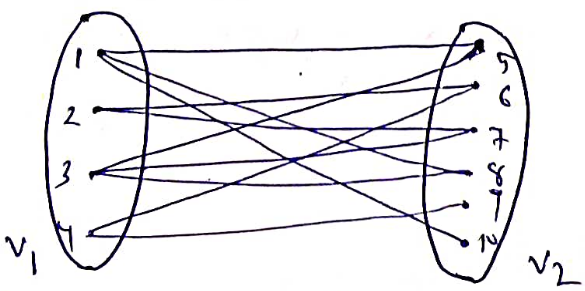
Maximum Matching : 2

Minimum vertex cover : 2



Problem

Show that a complete matching of V_1 into V_2 exists in the following graph.

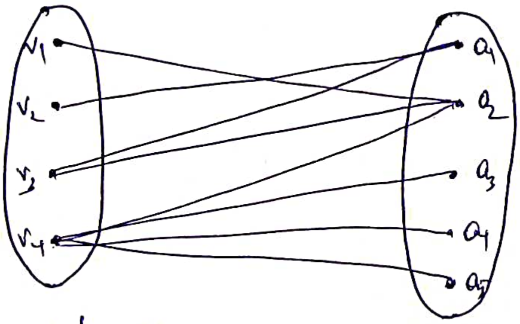


Sol^n: The minimum degree of a vertex of $V_1 = 2 \geq 2$
$=$ Maximum degree of a vertex in V_2

By choosing $m=2$, $\exists$ a complete matching from the set V_1 to V_2.

Problem

Prove that the bipartite graph shown in the following graph does not have a complete matching.



Sol^n: We observe that the three vertices v_1, v_2, v_3 in V_1 are together joined to two vertices a_1, a_2 in V_2. Thus, there is a subset of 3 vertices in V_1 which is collectively adjacent to $2 (< 3)$ vertices in V_2.

Hence, by Hall's theorem, there does not exist a complete matching from V_1 to V_2.

Matching in General Graphs

Definition

A factor of a graph G is a spanning subgraph of G. A k-factor is a spanning k-regular subgraph. An odd component of a graph is a component of odd order; the number of odd components of H is $o(H)$.

Remark

→ A 1-factor and a perfect matching are almost the same thing.

→ The precise distinction is that "1-factor" is a spanning 1-regular subgraph of G, while "perfect matching" is the set of edges in such a subgraph.

→ A 3-regular graph that has a perfect matching ~~decomposition~~ decomposes into a 1-factor and 2-factor.