CONNECTIVITY

Vertex Connectivity

The vertex connectivity of a graph G denoted by $\kappa(G)$ is the minimum vertices whose removal from G disconnects G or a trivial graph.

- A graph G is k-connected if its connectivity is at least k.

(i) For any connected graph, $\kappa(G) \geq 1$.

(ii) For any trivial graph, $\kappa(G) = 0$.

(iii) For any disconnected graph, $\kappa(G) = 0$.

(iv) If G is a complete graph of order 'n', then $\kappa(G) = n-1$. For K_4  K_4, $\kappa(G) = 3$.

(v) If G is not a complete graph of order 'n', then $\kappa(G) \leq n-2$.

(vi) If G is a complete bipartite graph, then $\kappa(G = K_{m,n}) = \min\{m, n\}$. The connectivity of $K_{3,3}$ is 3, means the graph is 1-connected, 2-connected, and 3-connected, but not 4-connected.

Definition

~~A graph G is said to be k-connected if $\kappa(G) = k$.~~

Example

(vii) The vertex connectivity of a graph of order at least ~~three~~ there is one if and only if it has a cut vertex.

(viii) Vertex connectivity of a path is one and that of cycle C_n $(n \geq 4)$ is two.

Problem

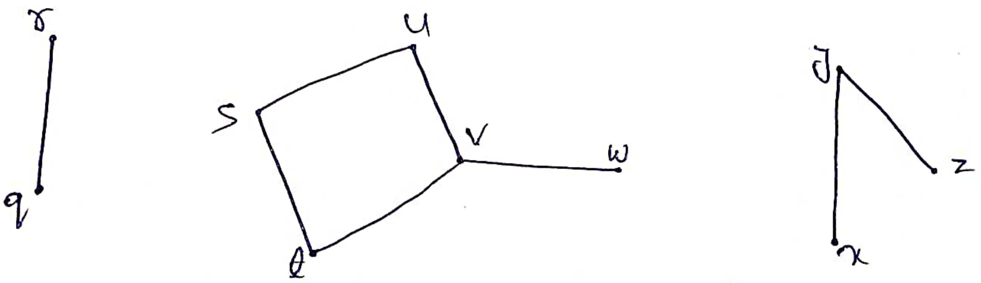
Find the (i) vertex sets of components

(ii) Cut-vertices

(iii) Cut-edges of the graph given

bellow.

G :



Solⁿ: The graph has three components. The vertex set of the components are $\{q, r\}$, $\{s, t, u, v, w\}$ and $\{x, y, z\}$.

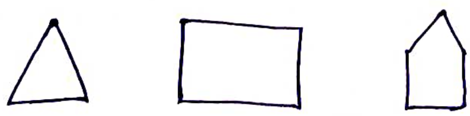
The cut vertices are v and y.

The cut edges are qr, vw, xy and yz.

Definition

A graph G is said to be k-connected if $k(G) = k$.

Example: All cycle graph $C_n, n \geq 3$ are 2-connected

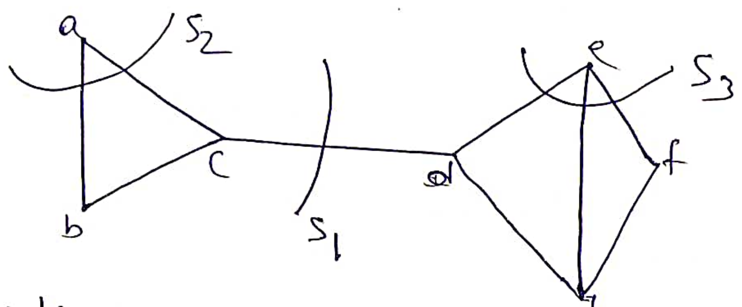


If we remove any two vertex, then graph is disconnected. So it is 2-connected.

Edge Connectivity $k'(G)$

The number of edges in the smallest cutset of a graph G is defined as the edge connectivity of G and it is denoted by $k'(G)$.

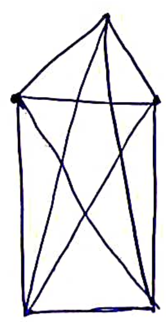
G :



Here $k'(G) = 1$, because if we remove $\{c,d\}$ edge then G is disconnect.

- If G is a trivial graph, then $k'(G) = 0$.
- If G is a disconnected graph, then $k'(G) = 0$.
- In complete graph K_n, $k'(G) = n-1$.

Ex :-



Here $k'(K_5) = 4$.

K_5.

Definition

A graph G is said to be k-edge connected if $k'(G) = k$.

- All nontrivial tree are 1-edge connected.

- All cycle's are 2-edge connected.

Notation

1. $\delta(G)$ is known as minimum degree of a graph G.
2. $\Delta(G)$ is known as maximum degree of a graph G.

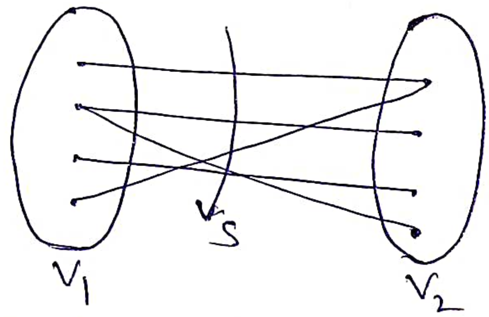
Whitney Inequality

For any simple graph G,

$$K(G) \leq K'(G) \leq \delta(G).$$ Equality holds for complete graph.

Proof:

Let $K'(G) = \alpha$. Then there exists a cutset S having α-edges such that $G-S$ is disconnected.



Let V_1 and V_2 be the partition of $V(G)$ with respect to S. By removing of most α-vertices from V_1 (or V_2) on which the edges in S are incident, we have

$$K(G) \leq \alpha = |S| = K'(G)$$

$$\Rightarrow K(G) \leq K'(G) \quad \text{—— (1)}$$

Next, we have to show that $K'(G) \leq \delta(G)$.

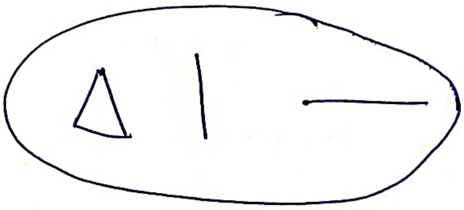
Case-I

If G has no edges, then $K'(G) = 0$ and $\delta(G) = 0$.

$$\therefore K'(G) = \delta(G)$$

Case-II

If G is a disconnected graph having some edges

G : 

Here $k'(G) = 0$ and $\delta(G) \geq 0$.

So $k'(G) \leq \delta(G)$

Case-III

If G is a connected graph with a vertex v having minimum degree $\delta(G)$, then a disconnected graph results when all the incident on v are removed.

$$\therefore \; k'(G) \leq \delta(G) \quad \text{———} \; (2)$$

Combining (1), \$ (2), we get

$$K(G) \leq k'(G) \leq \delta(G)$$

(proved).