Dirac extended Menger's Theorem to other families of paths.

4.2.22. Definition. Given a vertex x and a set U of vertices, an x, U**-fan** is a set of paths from x to U such that any two of them share only the vertex x.

4.2.23. Theorem. (Fan Lemma, Dirac [1960]). A graph is k-connected if and only if it has at least $k+1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an x, U-fan of size k.

Proof: *Necessity.* Given k-connected graph G, we construct G' from G by adding a new vertex y adjacent to all of U. The Expansion Lemma (Lemma 4.2.3) implies that G' also is k-connected, and then Menger's Theorem yields k pairwise internally disjoint x, y-paths in G'. Deleting y from these paths produces an x, U-fan of size k in G.

Sufficiency. Suppose that G satisfies the fan condition. For $v \in V(G)$ and $U = V(G) - \{v\}$, there is a v, U-fan of size k; thus $\delta(G) \geq k$. Given $w, z \in V(G)$, let $U = N(z)$. Since $|U| \geq k$, we have an w, U-fan of size k; extend each path by adding an edge to z. We obtain k pairwise internally disjoint w, z-paths, so $\lambda(w, z) \geq k$. This holds for all $w, z \in V(G)$, so G is k-connected. ■