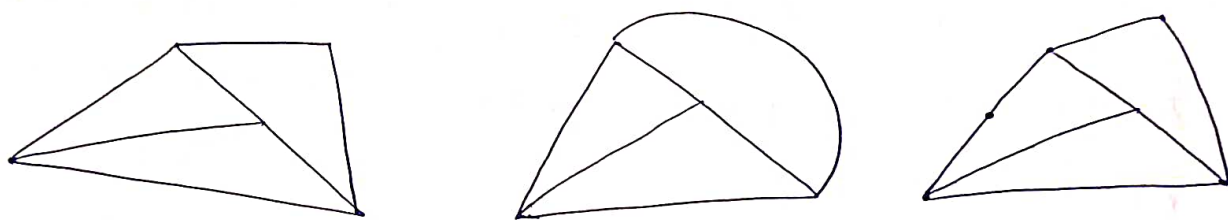# Homeomorphic Graph

Two graph G_1 and G_2 are said to be homeomorphic to each other if one of the graph is obtained by other after inserting vertices of degree 2 or merging vertices of degree 2 to other vertices.
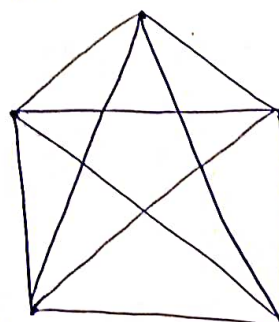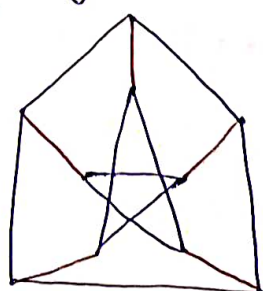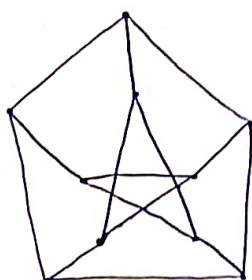
Example

Definition

A minimal non-planar graph is a non-planar graph G such that every proper subgraph of G is planar.

Definition

An undirected graph H is a minor of another undirected graph G if H can be obtained from G by.

(i) Contracting some edges

(ii) Deleting some edges

(iii) Deleting some vertices

Petersen graph has k_5 minor. Red edges are contracted

to get K_5.

Lemma

For every face of a given plane graph G, there is a drawing of G for which the face is exterior.

Lemma

Every minimal non-planar graph is 2-connected.

Lemma

Let $G = (V, E)$ be a graph with fewest edges among all non-planar graphs without K_5 or $K_{3,3}$ as minor. Then G is 3-connected.

Kuratowski's Theorem

A graph is planar iff it does not have K_5 or $K_{3,3}$ as minors.

Proof: We know that if a graph contains K_5 or $K_{3,3}$ as a minor graph, then it is not planar. It remains to prove that every non-planar graph contains K_5 or $K_{3,3}$ as minor. That is, we have to show that
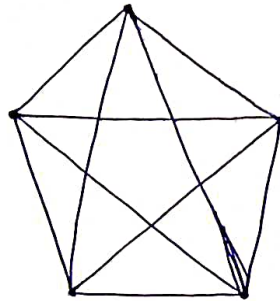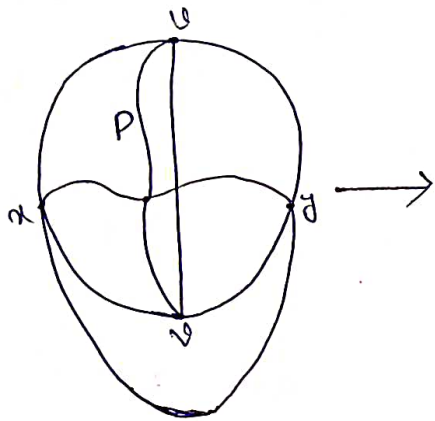
(i) It suffices to prove this only for minimal non-planar graphs.

(ii) We will show that every minimal non-planar graph with no K_5 or $K_{3,3}$ as minor must be 3-connected.

(iii) We then show that every 3-connected

graph with no K_5 or $K_{3,3}$ as minor is planar. But we started with a non-planar graph, which is contradiction. So a non-planar graph must contain K_5 or $K_{3,3}$ as minor graph.

This graph has K_5 as minor.

We need to show that if a graph is non-planar then it must contain a K_5 or $K_{3,3}$ as minor graphs. Let G be the smallest graph in the set of non planar graph which is 3-connected. Then we prove the contra-positive of statement " G does not have K_5 and $K_{3,3}$ as minor and G is non-planar", i.e, " G is planar or G has K_5 or $K_{3,3}$ as minor ".
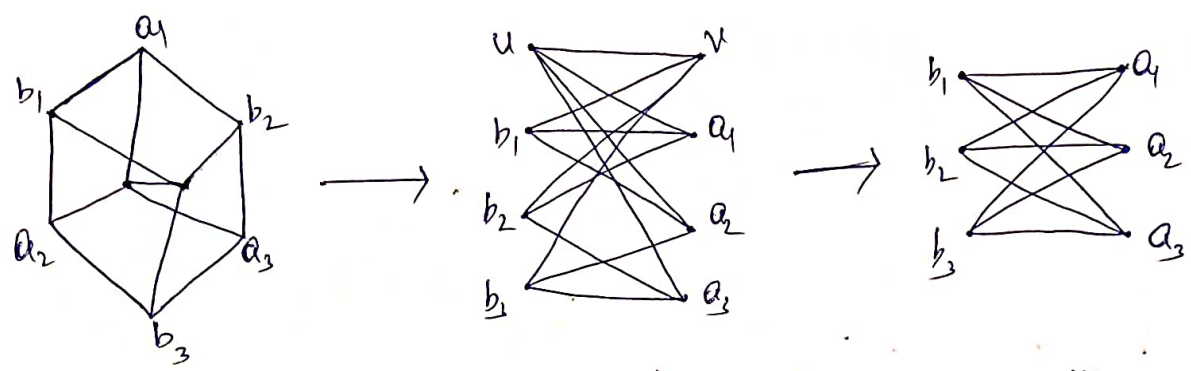
(i) Case 1: Let denote $N(v)$ as set of all vertices neighbours to V. Then if $|N(u) \cap N(v)| \geq 3$, then G has a K_5 a minor. Which is contradiction.

*→ Draw the above graph here.

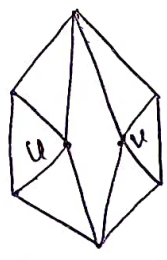(ii) Case 2: Here $|N(u) \cap N(v)| \leq 2$ is satisfied but it is also non planar. Let

$$N(u) = a_1, a_2, a_3, \ldots, a_n , \quad N(v) = b_1, b_2, \ldots, b_n$$

If the neighbour of u and v interleave, then G has a $K_{3,3}$ minor. Which is contradiction.

This graph has $K_{3,3}$ as minor.

(iii) **Case 3:** If the neighbours of u and v does not interleave and $|N(u) \cap N(v)| \leq 2$ is satisfied, then G is planar.

This is a planar graph.

All the cases given contradiction. This proves the theorem.

□