Theorem

Every connected graph with n vertices and (n-1) edges is a tree.

Proof: Given G is a connected graph with n vertices and (n-1) edges.

Claim: G is a tree.

In order to prove that G is a tree we only show that G has no cycles.

Suppose if possible G has at least one cycle. Since deleting any edge from the cycle does not disconnect the graph, we may delete edges but not vertices from G untill we get the resulting graph G_1 which does not contain any cycles.

Now G_1 is a connected graph with n vertices without cycle, then G_1 is a tree. Therefore G_1 has (n-1) edges, which is contradiction to the fact that the graph G has n vertices having (n-1) edges.

Thus G has no cycle. Consequently G is a tree.

$\square$

Theorem

A graph G with n vertices, (n-1) edges and having no cycle is a tree.

Proof: Since a tree is a connected graph without cycles. Therefore in order to prove G is a tree, we have to show G is connected.

Given G has n vertices, & (n-1) edges. We have to prove the theorem by method of contradiction. Suppose if possible G is disconnected. Then G will have two or more components without cycle. Let G_1 and G_2 are two components of G having no cycle. Lets add an edge 'e' between a vertex 'u' of G_1 to a vertex 'v' of G_2. Since G_1 and G_2 are different components in G, there is no path between 'u' and 'v'. Then addition of an edge 'e' will not forming cycle. Then $G \cup \{e\}$ is a connected graph without cycle. So $G \cup \{e\}$. This implies $G \cup \{e\}$ has n vertices and n edges. which is not possible. so our assumption is wrong.

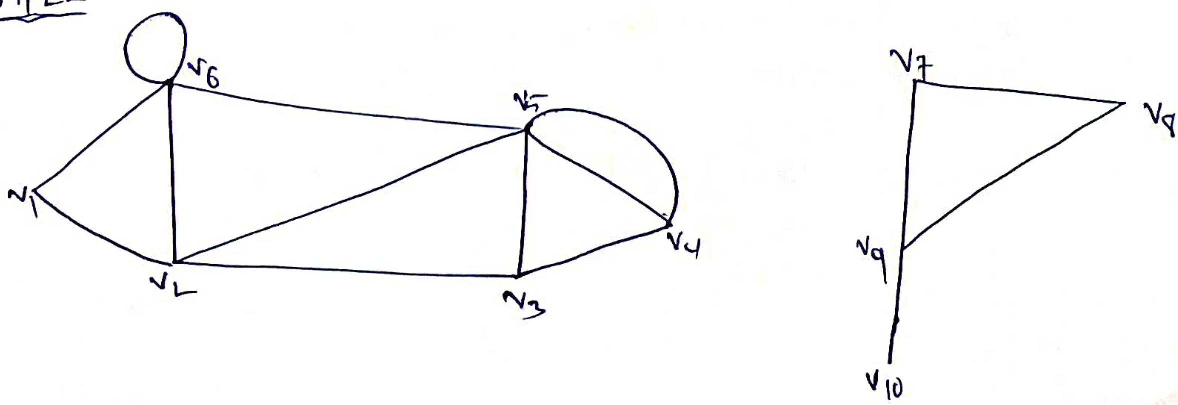
Therefore, G is a connected graph, consequently G is a tree.

Distance and Center in a Tree:

~~Let G~~ If G has a u, v-path, then the distance from u to v, written as $d_G(u,v)$ or simply $d(u,v)$, is a least length of a u, v-path.

→ If G has no such path, then $d(u,v) = \infty$.

EXAMPLE



$d(v_1, v_4) = 3$

$d(v_5, v_4) = 1$

$d(v_1, v_{10}) = \infty$

$d(v_7, v_{10}) = 2$

$d(v_6, v_6) = 0$

Eccentricity of a vertex:

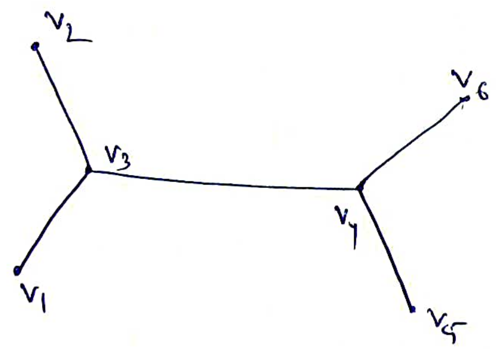
The Eccentricity of a vertex v of a graph G is denoted by $e(v)$ and it is the distance between v and the vertex v_i farthest from v in G.

$$e(v) = \max d(v, v_i) \text{ for } v_i \in G.$$

Center of Graph:

A vertex of a graph G whose eccentricity is minimum is called the center of the graph G.

EXAMPLE



$e(v_1) = \max\{d(v_1, v_2), d(v_1, v_3), d(v_1, v_4), d(v_1, v_5), d(v_1, v_6)\}$

$= \max\{2, 1, 2, 3, 3\}$

$= 3$

$e(v_2) = \max\{d(v_2, v_1), d(v_2, v_3), d(v_2, v_4), d(v_2, v_5), d(v_2, v_6)\}$

$= \max\{2, 1, 2, 3, 3\} = 3$

$e(v_3) = \max\{d(v_3, v_1), d(v_2, v_2), d(v_3, v_4), d(v_3, v_5), d(v_3, v_6)\}$

$= \max\{1, 1, 1, 2, 2\} = 2$

$e(v_4) = \max\{d(v_4, v_1), d(v_4, v_2), d(v_4, v_3), d(v_4, v_5), d(v_4, v_6)\}$
$= \max\{2, 2, 1, 1, 1\} = 2$

$e(v_5) = \max\{d(v_5, v_1), d(v_5, v_2), d(v_5, v_3), d(v_5, v_4),$
$d(v_5, v_6)\}$
$= \max\{3, 3, 2, 1, 2\} = 3$

$e(v_6) = \max\{d(v_6, v_1), d(v_6, v_2), d(v_6, v_3), d(v_6, v_4), d(v_6, v_5)\}$
$= \max\{3, 3, 2, 1, 2\} = 3$

Here $\min e(v_i) = e(v_3)$ & $e(v_4)$. So v_3 & v_4 are the center of the tree.

Remark: Every tree has either one or two centers.

Radius of a Tree

The eccentricity of the center in a tree is called the radius of the tree and denoted by rad (T).

Ex: In the above example,

rad (T) = 2.

Diameter of a Tree

The diameter of a tree is the maximum eccentricity of its vertices is called diameter and denoted by diam (T).

Ex: In the above example,

diam (T) = 3