Graph Colouring

A colouring of a simple graph is the assignment of colour to each vertex of that graph so that no two adjacent vertices are assigned two same colour.

OR,

Let $G=(V,E)$ be a simple graph and

$C=\{C_1, C_2, \ldots, C_n\}$ be a set of colours

(i) A vertex colours of G using the colours of C is a function $f: V \longrightarrow C$

(ii) Let $f: V \longrightarrow C$ be a vertex colors of G. If for every adjacent ~~vertex~~ vertices $u, v \in V$

$f(u) \neq f(v)$ then f is called

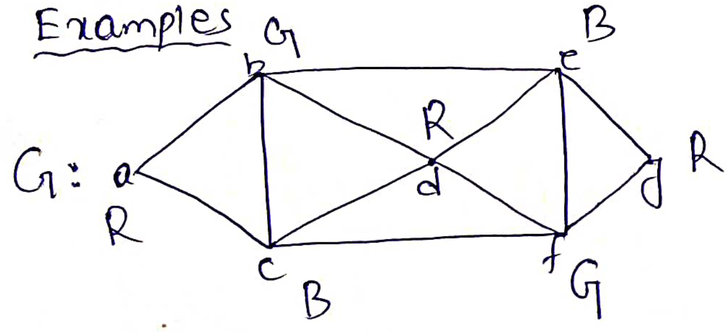
proper vertex coloring.

Chromatic Number

The minimum number of colors required for a proper vertex colouring is known as chromatic number and is denoted by $\chi(G)$.

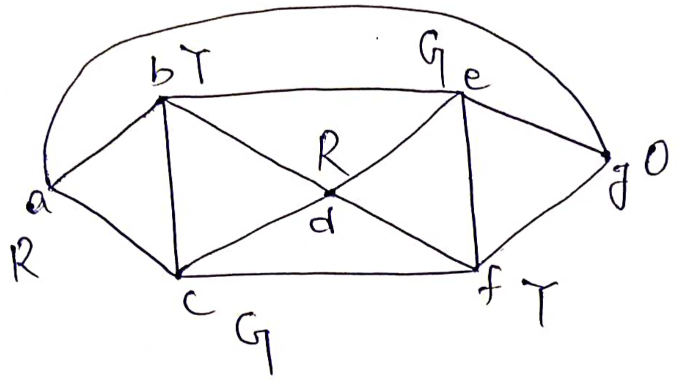
(Kie of G)

Examples

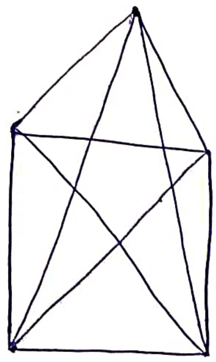


Here, we required minimum 3 colors to coloring the vertices. So $\chi(G)=3$.

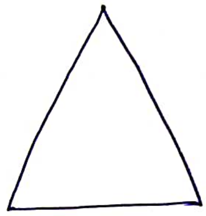
G:



$\chi(G) = 4$



K_5

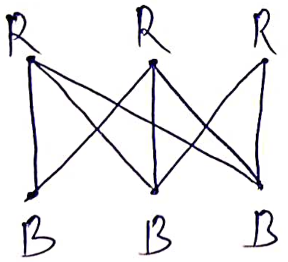


K_3

$\chi(K_3) =$

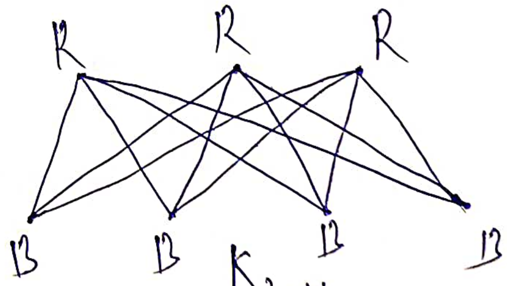
$\chi(K_5) =$

In general $\chi(K_n) = n$.



$K_{3,3}$

$\chi(K_{3,3}) = 2$



$K_{3,4}$

$\chi(K_{3,4}) = 2$

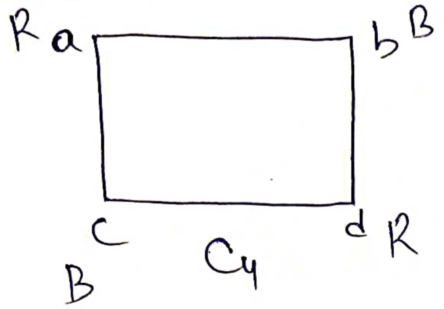
In general $\chi(K_{m,n}) = 2$.

Q: What is the chromatic number for C_n, $n \geq 3$.

Sol^n:

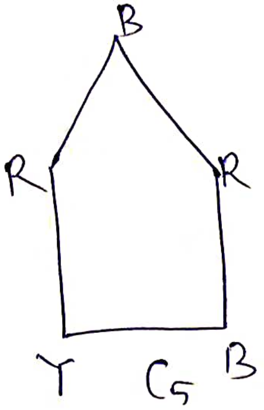
$$\chi(C_n) = \begin{cases} 3, & \text{when } n \text{ is odd} \\ 2, & \text{when } n \text{ is even} \end{cases}$$

Example



C_4

$\chi(C_4) = 2$



C_5

$\chi(C_5) = 3$

Definition

A graph G is said to be K-colourable if we can properly colour it with K number of colours.

Notes:

(1) $\chi(G) \leq |V|$, where $|V|$ is the number of vertices of G.

(2) If some ~~go~~ subgraph of G requires K colours then ~~$\chi(K_3)$~~ $\chi(G) \geq K$.

(4) If degree$(v) = d$, then atmost d colours are required to colour the vertices adjacent to v.

(5) For any graph G, $\chi(G) \leq 1 + \Delta(G)$, where $\Delta(G)$ is the largest degree of any vertex of G.

Theorem

Let G be a nontrivial simple graph. Then $\chi(G) = 2$ if and only if or iff G is a bipartite graph.

Proof: Let $G = (V, E)$ be a bipartite graph. Then the vertex set V can be partition into two ~~non-trivial~~ non-empty subsets V_1 and V_2 such that each edge of G joining one vertex of V_1 and other vertices of V_2. There is no edge joining either two vertices of V_1 or two vertices of V_2.

Claim: $\chi(G) = 2$.

Let $C = \{c_1, c_2\}$ be set of colour defined. a function $f: V \longrightarrow C$ such that

$f(u) = c_1$ if $u \in V_1$
$f(v) = c_2$ if $v \in V_2$

because $V_1 \cap V_2 = \phi$, it follows that the function is ~~well~~ well defined.

Now no two vertices of V_1 are adjacent therefore all vertices have same colour. Similarly all the vertices of V_2 are also not adjacent to each other, so all vertices have same colour.

~~From~~ From the definition of f it follows that two adjacent vertices of G has different colours. Therefore

$$\chi(G) \leq 2.$$

Since G contains at least one edge, i.e

$$\chi(G) > 1.$$

Hence $\chi(G) = 2$.

Conversely, suppose $\chi(G) = 2$.

Claim: G is bipartite.

Let $\chi(G) = 2$. Therefore the graph contains one edge also there exists a function

$$f : V \longrightarrow C$$

such that no two vertices has same image

Let $V_1 = \{ v \in V \mid f(v) = c_1 \}$

and $V_2 = \{ v \in V \mid f(v) = c_2 \}$.

It follows that $V_1 \cap V_2 = \phi$ and $V_1 \cup V_2 = V$.

Let e be an edge with end vertices between V_1 and V_2 because V_1 & V_2 can't have same colour.

$$\Rightarrow v_1 \in V_1 \quad \text{iff} \quad v_2 \in V_2$$

Similarly, $v_2 \in V_1$ iff $v_1 \in V_2$.

Thus G is bipartite graph.

(proved)