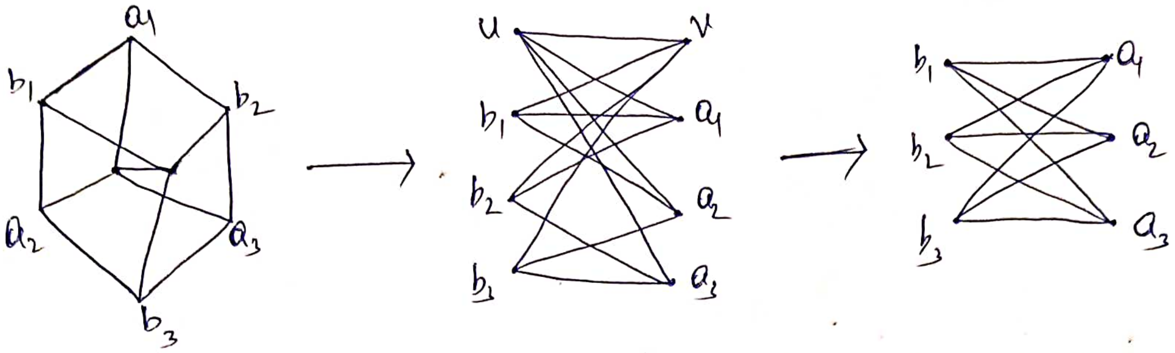
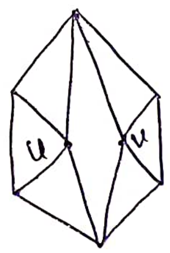
If the neighbours of u and v interleave, then G has a $K_{3,3}$ minor, which is contradiction.



This graph has $K_{3,3}$ as minor.

(iii) <u>Case 3:</u> If the neighbour of u and v does not interleave and $|N(u) \cap N(v)| \leq 2$ is satisfied, then G is planar.



This is a planar graph.

All the cases given contradiction. This proves the theorem. □

<u>List coloring of Planar Graphs</u>

<u>Thomassen Theorem</u>

Planar graphs are 5-choosable.

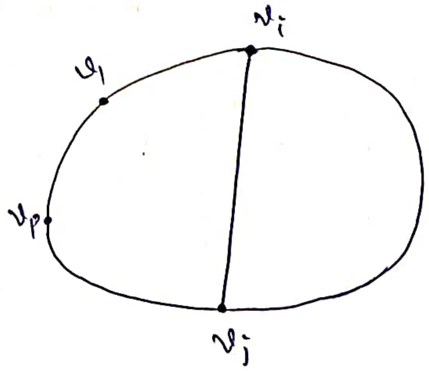
Proof: Adding edges can't reduce the list chromatic number, so ~~may~~ we may restrict our attention to plane graphs where the outer face is a cycle and every bounded face is a triangle. By induction on $n(G)$, we prove the stronger result that a coloring can be chosen even when two adjacent external vertices have distinct lists of size 1 and the other external vertices have lists of size 3. For the basis step $(n=3)$, a color remains available for the third vertex.

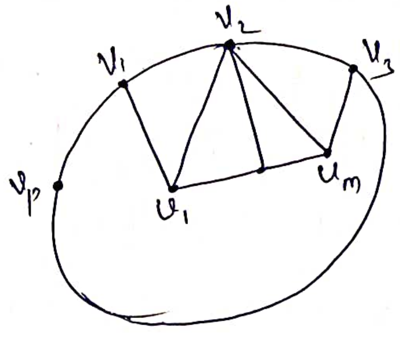
Now consider $n > 3$. Let v_p, v_1 be the vertices with fixed colors on the external cycle C. Let $v_1, \ldots, v_p$ be $V(C)$ in clockwise order.

Case 1: C has a chord $v_i v_j$ with $1 \le i \le j-2 \le p-2$. We apply the inductive hypothesis to the graph consisting of the cycle $v_1, \ldots, v_i, v_j, \ldots, v_p$ and its interior. This selects a proper coloring in which v_i, v_j receive some fixed colors. Next we apply the induction hypothesis to the graph consisting of the cycle $v_i, v_{i+1}, \ldots, v_j$ and its interior to complete the list coloring of G.

Case 2: C has no chord. Let $v_1, u_1, \ldots, u_m, v_3$ be the neighbors of v_2 in order ($3 = p$ is possible). Because bounded faces are triangles, G contains the path P with vertices $v_1, u_1, \ldots, u_m, v_3$. Since C is chordless, $u_1, u_2, \ldots, u_m$ are internal vertices, and the outer face of $G' = G - v_2$ is bounded by a cycle C' in which P replace v_1, v_2, v_3.

Let c be the color assigned to v_1. Since $|L(v_2)| \geq 3$, we may choose distinct colors





$x, y \in L(v_2) - \{c\}$. We reserve x, y for possible use on v_2 by forbidding x, y from $u_1, \dots, u_m$. Since $|L(u_i)| \geq 5$, we have $|L(u_i) - \{x, y\}| \geq 3$. Hence we can apply the induction hypothesis to G', with $u_1, \dots, u_m$ having lists of size at least 3 and other vertices having the same lists as in G. In the resulting coloring, v_1 and $u_1, \dots, u_m$ have colors outside $\{x, y\}$. We extend this coloring to G by choosing for v_2 a color in $\{x, y\}$ that does not appear on v_3 in the coloring of G'. Hence planar graphs are 5-choosable.

□

Dual of a Planar Graph or Planar Duals

Procedure for Planar dual

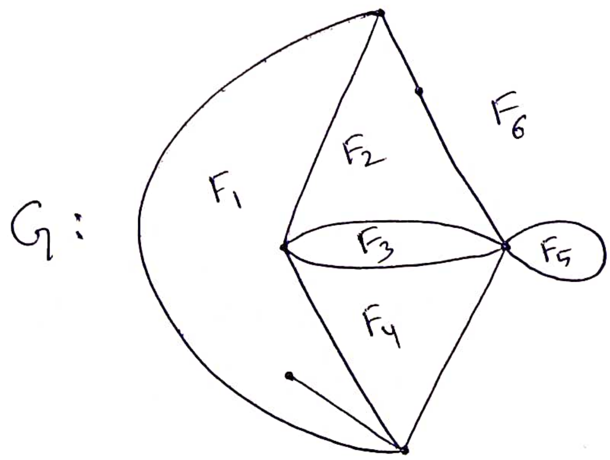
(1) If two regions F_i and F_j are adjacent (i.e have a common edge), draw a line joining points P_i and P_j that intersects the common edge between F_i and F_j exactly once.

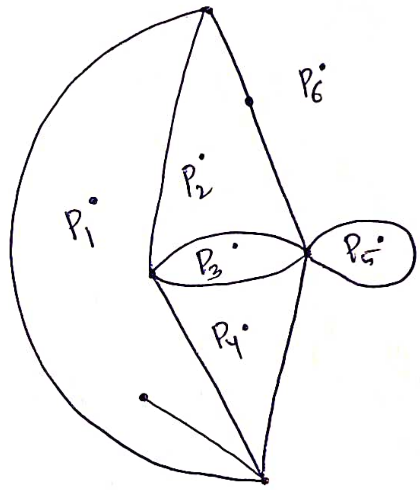
(2) If there is more than one edge common between F_i and F_j draw one line between

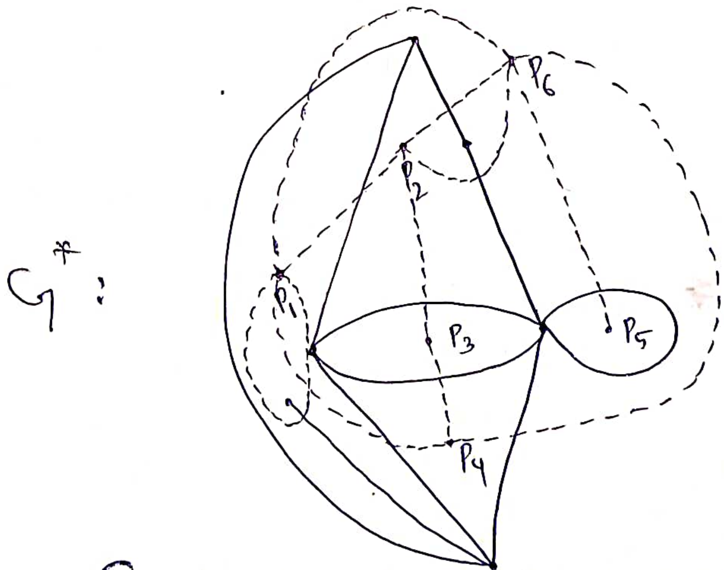
points P_i and P_j for each of the common edges.

(3) For an edge e lying entirely in one region, say F_k, draw a self-loop at point P_k intersecting e exactly once.

Example







Constructions of a dual graph.

Observations:

(a) An edge forming a self loop in G yields a pendant edge in G^*

(b) A pendant edge in G yeilds a self loo in G^*.

(c) Edges that are in series in G produce parallel edge in G^*.

(d) Parallel edges in G produce edges in series on G^*.

By this procedure we obtained a new graph G^* (in broken lines in figure) consisting of six vertices, $P_1, P_2, \ldots, P_6$ and of edges joining these vertices. Such a graph G^* is called dual (a geometrical dual) of G.

Theorem

A graph has a dual iff it is planar.

Proof: Suppose that a graph G is planar. Then G has a geometric dual in G^*. Since G^* is a geometric dual, it is a dual. Thus G has a dual.

Conversely, suppose G has a dual. Assume that G is non planar. Then by Kuratowski's theorem, G contain K_5 and $K_{3,3}$ or a graph homeomorphic to either these as a subgraph.

But K_5 and $K_{3,3}$ have no duals and therefore a graph homeomorphic to either of these also has no dual. Thus, G contains a subgraph which has no dual. Hence, G has no dual. This is a contradiction.

Hence, G is planar if it has a dual

(proved)

Definition

The Ramsey number $R(s,t)$ is the minimum number n such that any graph on n vertices contains either independent set of size 's' or a clique of size 't'.

Properties

(i) $R(s,t) = R(t,s)$

(ii) $R(1,1) = 1$, $R(2,2) = 2$, $R(3,3) = 6$, $R(4,4) = 18$.

(iii) $R(s,1) = R(1,t) = 1$

(iv) $R(2,t) = n$; $R(s,2) = s$

(v) $R(s,t) \leq R(s-1,t) + R(s,t-1)$ $\forall s,t \geq 2$ and are integers.

Upper bound for $R(s,t)$

$$R(s,t) \leq \binom{s+t-2}{s-1} = {}^{s+t-2}C_{s-1}$$ (where C is combination)

Lower bound for $R(s,t)$

$$R(s,t) \geq 2^{\frac{k}{2}}, \quad k = \min\{s,t\}$$

That means

$$\boxed{2^{\frac{k}{2}} \leq R(s,t) \leq {}^{s+t-2}C_{s-1}}$$

Problem

Find the upper bound and Lower bound of Ramsey number $R(4,4)$.

Solⁿ: Upper bound

$$R(4,4) \leq {}^{4+4-2}C_{4-1} = {}^{6}C_{3} = \frac{6!}{(6-3)!\,3!}$$

$$= \frac{6!}{3!\,3!} = \frac{6\times5\times4}{2\times3} = 20$$

Lower bound

$k = \min\{4, 4\} = 4$

$R(4,4) \geq 2^{\frac{4}{2}} = 2^2 = 4.$

Ramsey's Theorem

For any $s, \ell \geq 1$, there $R(s,\ell) < \infty$ such that any graph on $R(s,\ell)$ vertices contains either an independent set of size s or a clique of size ℓ. In particular,

$$R(s,\ell) \leq {}^{s+\ell-2}C_{s-1}.$$

Proof: We show that $R(s,\ell) \leq R(s-1,\ell) + R(s,\ell-1)$.

To see this, let $n = R(s-1,\ell) + R(s,\ell-1)$ and consider any graph G on n vertices. Fix a vertex $v \in V$. We consider two cases:

Case-I

There are at least $R(s,\ell-1)$ edges incident with v. Then we apply induction on the neighbors of v, which implies that either they contain an independent set of size s, or a clique of size $\ell-1$. In the second case, we extend the clique by adding v, and hence G contains either an independent set of size s or a clique of size ℓ.

Case-II

There are at least $R(s-1,\ell)$ non-neighbors of v. Then we apply induction to the non-neighbors of v and we get either an independent set of size