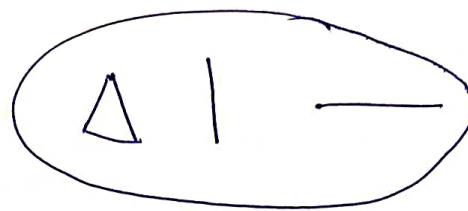


Case-II

If G is a disconnected graph having some edges

$G:$



Here $k'(G) = 0$. and $\delta(G) \geq 0$.

So $k'(G) \leq \delta(G)$

Case-III

If G is a connected graph with a vertex v having minimum degree $\delta(G)$, then a disconnected graph results when all the incident on v are removed.

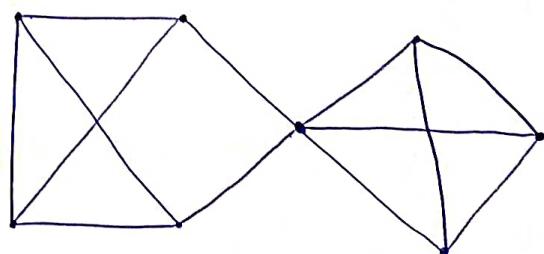
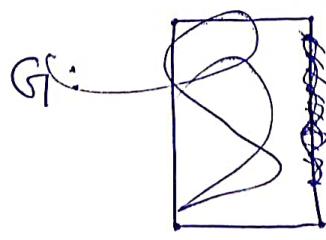
$$\therefore k'(G) \leq \delta(G) \quad \text{--- (2)}$$

Combining (1), & (2), we get

$$k(G) \leq k'(G) \leq \delta(G) \quad (\text{proved})$$

Example

Consider the graph G



For the graph G

$$k(G) = 1, \quad k'(G) = 2, \quad \delta(G) = 3$$

so here $k(G) < k'(G) < \delta(G)$.

Problem

Let $G(n,m)$ be a graph. Show that

$$\delta(G) \leq \frac{2m}{n} \leq \Delta(G)$$

Sol: Let $V(G) = \{v_1, v_2, \dots, v_n\}$

We have $\delta(G) \leq \deg(v_i) \leq \Delta(G)$

$$\Rightarrow \sum_{i=1}^n \delta(G) \leq \sum_{i=1}^n \deg(v_i) \leq \sum_{i=1}^n \Delta(G)$$

$$\Rightarrow n \delta(G) \leq 2m \leq n \Delta(G)$$

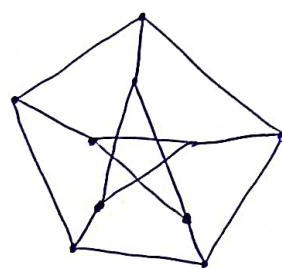
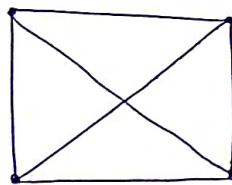
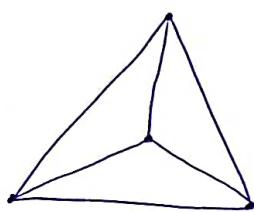
$$\Rightarrow \delta(G) \leq \frac{2m}{n} \leq \Delta(G)$$

□

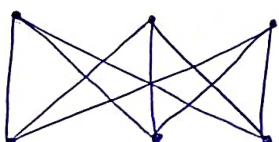
Cubic Graph

A cubic graph is a graph in which all vertices have degree three. In other words, cubic graph is a 3-regular graph.

- Cubic graphs are also called trivalent graphs.
- A bicubic graph is a cubic bipartite graph.



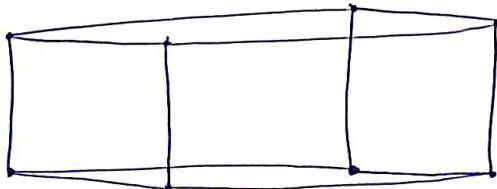
The Petersen graph is a cubic graph.



The complete bipartite graph $K_{3,3}$ is a bicubic graph.

Hypercube graph (α_n)

The hypercube graph is denoted α_n and is the graph formed from the vertices and edges of an n -dimensional hypercube.



α_3

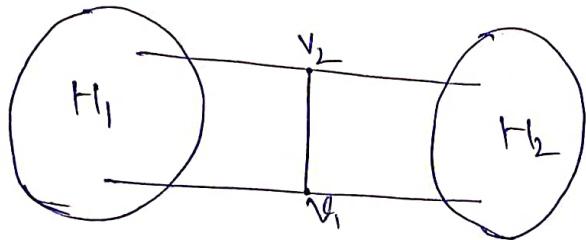
- It has 2^n vertices and $n2^{n-1}$ edges.
- The graph α_0 consists of a single vertex, while α_1 is the complete graph on two vertices.
- Every hypercube graph is bipartite.
- Every hypercube α_n , $n > 1$ has a Hamiltonian ~~cycle~~ circuit.

Theorem

If G is a 3-regular graph, then $k(G) = k'(G)$.

Proof: Let T be a minimum cut vertex $\Leftrightarrow |T| = k(G)$. Since $k(G) \leq k'(G)$ always, we need only provide a cut edge of size $|T|$. Let H_1 and H_2 be two components of $G - T$. Since T is a minimum cut vertex, each $v \in T$ has a neighbor in H_1 and a neighbor in H_2 . Since G is 3-regular, it can not have two neighbors in H_1 and two in H_2 . For each $v \in T$, delete the edge from v to a member of $\{H_1, H_2\}$ where v has only one neighbor.

These $k(G)$ edges break all paths from H_1 to H_2 except in the case below, where a path can enter T via v_1 and leave via v_2 .



In this case we delete the edge to H_1 for both v_1 and v_2 to break all paths from H_1 to H_2 through $\{v_1, v_2\}$. Hence $k(G) = k'(G)$.
(proved)

2-Connected Graphs

Definition

A graph G is said to be 2-connected if $\kappa(G) = 2$.

Definition

Two paths from u to v are internally disjoint if they have no common internal vertex.

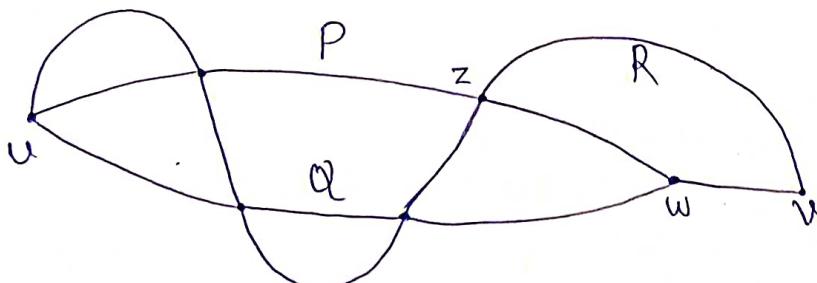
Whitney's Theorem

A graph G having at least three vertices is 2-connected if and only if for each pair $u, v \in V(G)$ there exist internally disjoint u, v -path in G .

Proof: Necessary Part: Suppose that G is 2-connected.
We have to show that for each pair $u, v \in V(G)$ there exist internally disjoint u, v -path in G . We prove by induction on $d(u, v)$ that G has internally disjoint u, v -path.

Base-Step: Let $d(u, v) = 1$. When $d(u, v) = 1$, the graph $G - uv$ is connected, since $\kappa'(G) \geq \kappa(G) \geq 2$. A u, v -path in $G - uv$ is internally disjoint in G from the uv -path formed by the edge uv itself.

Inductive hypothesis: Let w be the vertex before v on a shortest u, v -path, i.e., let $d(u, w) = k-1$.



(27)

Inductive step: $d(u,v) > 1$. Let $d(u,v) = k$. By inductive hypothesis, G has internally disjoint u,w -path P and Q . If $v \in V(P) \cup V(Q)$, then we find the desired paths on the cycle $P \cup Q$. Suppose this is not hold.

Since G is 2-connected, $G-w$ is connected and contain a u,u -path R . If R avoids P or Q , then we are done, but R may share internal vertices with both P and Q . Let z be the last vertex of R (before v) belonging to $P \cup Q$. We may assume that $z \notin P$. We combine the u,z -~~path~~ subpath of P with the z,v -subpath of R to obtain a u,v -path internally disjoint from $Q \cup wv$.

Sufficient Part: When ~~G has internally disjoint u,v -paths~~, deletion of one vertex can not separate u from v . Since this condition is given for every pair u,v , deletion of one vertex can not make any vertex unreachable from any other. We conclude that G is 2-connected.

(proved)

Expansion Lemma

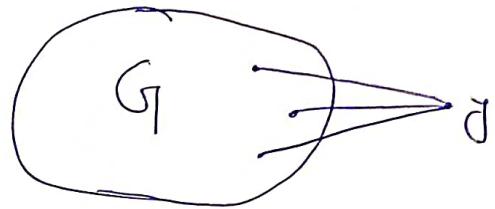
If G is a k -connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G , then G' is k -connected.

Proof: Given that G is a k -connected graph and G' is obtained from G by adding a new

vertex j with at least k neighbors in G . We have to show that G' is k -connected. That is we prove that a separating set S of G' must have size at least k . ~~if~~

If $j \in S$, then $S - \{j\}$ separates G , so $|S| \geq k+1$.

If $j \notin S$ and $N(j) \subseteq S$, then $|S| \geq k$.



Otherwise, j and $N(j) - S$ lie in a single component of $G' - S$. Thus again S must separate G and $|S| \geq k$. Hence G' is k -connected. \square