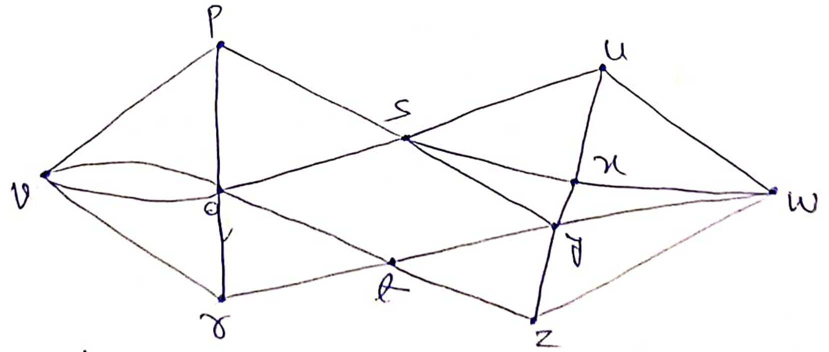
Edge-disjoint paths

The number of paths ~~cont~~ connecting two given vertices v and w in a graph G. We may ask for the maximum number of paths from v to w, no two of which have an edge in common, such paths are called edge-disjoint paths.

Vertex-disjoint paths

The number of paths connecting two given vertices v and w in a graph G. We may ask for the maximum number of paths from v to w, no two of which have a vertex ~~edge~~ in common, such paths are called vertex ~~edge~~-disjoint paths.

For example, in the following graph, there are four edge-disjoint paths and two vertex-disjoint paths



vw-disconnecting ~~disjointing~~ set

Assuming that G is a connected graph and that v and w are disjoint vertices of G. A vw-~~disjoint~~ disconnecting set of G is a set E of edges of G such that each path from v to w includes an edge of E. We note that a vw-disconnecting set is a disconnecting set of G.

vw - separating set

Assum that G is connected graph and that v and w are distinct vertices of G. A vw-separating set of G is a set S of vertices, others than v or w, such that each path from v to w passes through a vertex of S.

In the above figure, the set

$$E_1 = \{ps, qs, ty, tz\}$$

and $E_2 = \{uw, xw, yw, zw\}$

are vw - disconnecting sets, and

$$V_1 = \{s, t\}$$

and $V_2 = \{p, q, y, z\}$

are vw - separating sets.

Menger's Theorem

The maximum number of edge - disjoint paths connecting two distinct vertices v and w of a connected graph is equal to the minimum number of edges in a vw - disconnected set.

Proof: The maximum number of edge - disjoint paths connecting v and w can not exceed the minimum number of edges in a vw - disconnecting set.

We use induction on the number of edges of the graphs G to prove that these numbers are equal.

Suppose that the number of edges of the graph G is m, and that the theorem is true for all graphs with fewer than m edges. There are two cases to consider.

Case-I: Suppose first that there exists a vw-disconnecting set E of minimum size k, such that not all of its edges are incident to v, and not all are incident to w. For example, in the above the set E_1 is such a vw- disconnecting set.

The removal from G of the edges in E leaves two disjoint subgraphs V and W containing v and w, respectively.

We now define two new graphs G_1 and G_2 are follows: G_1 is obtained from G by contracting every edge of V, that is, by shrinking V down to v, and G_2 is obtained by similarly contracting every edge of W; the graph G_1 and G_2 obtained from above figure are shown in bellow, with dashed lines denoting the edges E_1.

Since G_1 and G_2 have fewer edges than G, and since E is a vw-disconnecting set of minimum size for both G_1 and G_2, the induction hypothesis gives us k-edge-disjoint paths in G_1 from v to w, and similarly for G_2.

The required k edge-disjoint paths in G are obtained by combining these paths ~~in G_1 from v to w~~ the obvious way.

Case-II: Now suppose that each vw-disconnecting set of minimum size k consists only of edges that are all incident to v or all incident to w.

Without loss of generality, assume that each edge of G is contained in a vw-disconnecting set of size k, since otherwise its removal would not affect the value of k and we could use the induction hypothesis to obtain k edge-~~disjoint~~ disjoint paths.

If p is and path from v to w, then P must consist one or two edges and can ~~one~~ thus contain atmost one edge of any vw- ~~disjoint~~ disjonnecting set of size k.

By removing from G the edges of P, we obtain ~~with~~ a graph with at least $k-1$ edge-disjoint paths, by the induction hypothesis. These paths, together with P, give the required k paths in G. □