Mycielski's construction can be applied to C_5, to produce 4-chromatic graph Grötzsch graph.

Definition

A Mycielski graph M_k of order k is a triangle-free graph with chromatic number k having the smallest possible number of vertices.

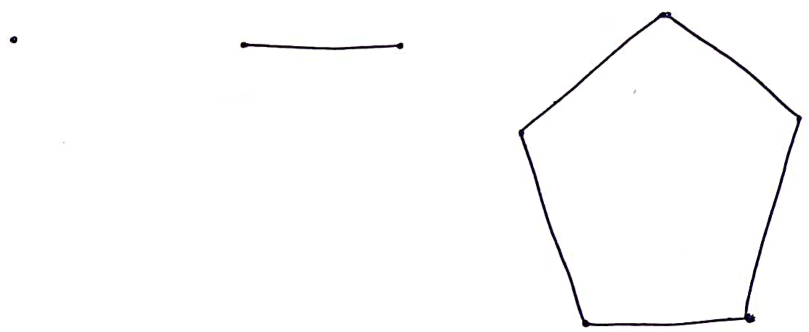
For example, triangle-free graphs with chromatic number $k=4$ is Grötzsch graph having 11 vertices.

M_1 Singleton graph

M_2 2-path graph

M_3 5-cycle graph

M_4 Grötzsch graph



The k-Mycielski graph has vertex count

$$n(M_k) = \begin{cases} 1 & , \text{ for } n=1 \\ 3 \cdot 2^{n-2} - 1 & , \text{ for } n>1 . \end{cases}$$

Extremal Graph Theory

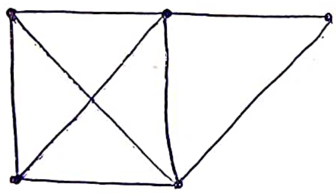
Extremal graph theory is the branch of graph theory that studies extremal (maximal or minimal) graphs which satisfy a certain property.

Proposition

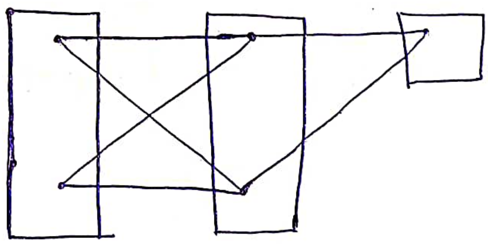
Every k-chromatic graph with n vertices has at least ${}^{k}C_2$ edges. Equality holds for a complete graph plus isolated vertices.

Definition

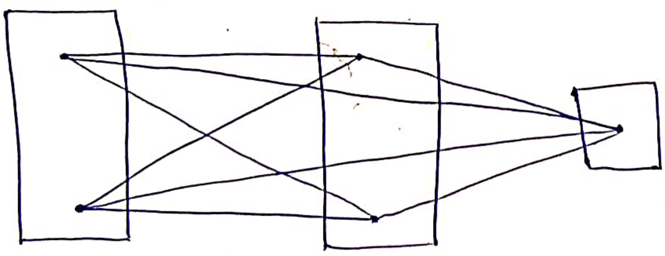
A complete multipartite graph is a simple graph G whose vertices can be partitioned into sets such that two vertices are adjacent if and only if they are not in the same of partitioned sets. Equivalently, every component of $\bar{G}$ is a complete graph. For $k \geq 2$, the complete k-partite graph with partite set of sizes $n_1, n_2, \ldots, n_k$ is written as $K_{n_1, n_2, \ldots, n_k}$.



A graph



A 3-partite graph



A complete 3-partite graph

Turan Graph

The Turan Graph, denoted by $T_{n,r}$ is the complete r-partite graph with n vertices, whose partite sets differ in size by at most 1. By the pigeon-hole principle, every partite set has size eith $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$.

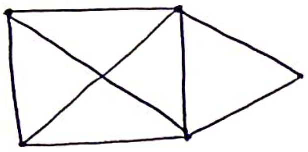
Ceiling $\lceil \frac{9}{5} \rceil = \lceil 1.8 \rceil = 2$

floor $\lfloor \frac{9}{5} \rfloor = \lfloor 1.8 \rfloor = 1$

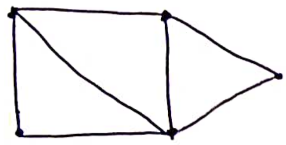
Lemma

Among simple r-partite, that is, r-colorable graphs with n vertices, the Turan graph is the unique graph with the most edges.

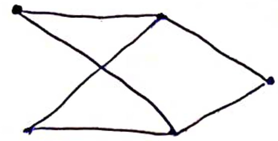
Proof: We can add edges without increasing the Chromatic number untill it becomes a complete multipartite graph. Now, given a complet r-partite graph with partite sets differing by more than 1 in size, we can move a vertex v from the largest partite set size i to the smallest partite set size j. The edges not involving v remain the same as before, but v joins i-1 neighbours in its old partite set, and loses j neighbours in its new partite set. Since $i-j > 1$, the number of edges increases due to this switch. Hence, we maximize the number of edges only by equalizing the size of all partite set, as in $T_{n,r}$. □



Not K_4 free graph



K_4- free graph but not triangle- free graph



K_3- free graph or, triangle-free graph.

Turan's Theorem

Among n-vertex simple graphs with no K_{r+1}, $T_{n,r}$ has the maximum number of edges. Here, K_{r+1} refers to the $(r+1)$-clique and $T_{n,r}$ refers to the Turan Graph on n vertices having r-partitions.

Proof: Every r-colourable (or r-partite) graph, including Turan graph $T_{n,r}$, has no $r+1$-Clique, since each partite set contributes at-most one vertex to each clique. If we can prove that the maximum edges is achieved by a r-partite graph, then by the above Lemma the required graph is $T_{n,r}$. Thus, it is sufficient to show that for every graph G that has no $r+1$-clique, there is an r-partite graph H with the same vertex set as G i.e $V(H) = V(G)$, and at-least as many edges i.e $e(H) \geq e(G)$.