$C''(v) \geq C(v)$ for every $v \in V$ and the component H'' of $G[E''_{ik} \cup E''_{i0}]$ that contains u is an odd cycle. But since v_k has degree 2 in H', clearly v_k has degree 1 in H''. But H'' is an odd cycle contains u, which is a contradiction.

This prove that our assumption is wrong.

Thus $\chi'(G) \geq \Delta$ and $\chi'(G) \leq \Delta+1$.

(proved)

LIST COLORING AND CHOOSABILITY

List coloring is a more general version of the vertex coloring problem. We still pick a single color for each vertex, but the set of colors available at each vertex may be restricted.

Definition

For each vertex v in a graph G, let $L(v)$ denote a list of colors available at v. A list coloring or choice function is a proper coloring f such that $f(v) \in L(v)$ for all v.

→ A graph G is k-choosable or list k-colorable if every assignment of k-element lists to the vertices permits a proper list coloring.

→ The list chromatic number, choice number, or choosability $\chi_\ell(G)$ is the minimum k such that G is k-choosable.

Definition

Let $L(e)$ denote the list of colors available for e. A list edge-coloring is a proper edge-coloring f with $f(e)$ chosen from $L(e)$ for each e.

→ The edge-choosability $\chi'_\ell(G)$ is the minimum k such that every assignment of lists of size k yields a proper list edge-coloring. Equivalently,

$$\chi'_\ell(G) = \chi_\ell(L(G)),$$

where $L(G)$ is the line graph of G.

Definition

In a simple digraph, we write uv for and edge with tail u and head v. If there is an edge from u to v, then v is a successor of u, and u is a predecessor of v.

Definition

A kernel of a digraph is an independent set S having a successor of every vertex outside S. A digraph is kernel-perfect if every induced subdigraph has a kernel.

→ Given a function $f : V(G) \rightarrow \mathbb{N}$, the graph G is f-choosable if a proper coloring can be chosen from the lists at the vertices whenever $|L(x)| = f(x)$ for each x.

List Coloring Conjecture

$$\chi'_\ell(G) = \chi'(G) \text{ for all } G.$$

In 1995, Galvin proved list coloring conjecture for bipartite graphs.

Kernel Lemma

If graph G has an orientation D such that D is kernel-perfect and $d^+(x) < f(x)$ for all $x \in V(G)$, then G is f-choosable.

Galvin Theorem

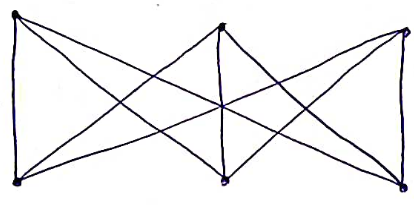
For every simple bipartite graph G,

$$\chi_\ell'(G) = \chi'(G) = \Delta(G).$$

PLANAR GRAPH

In this section we will study the question of whether a graph can be drawn in the plane without edges crossing.

Consider the problem of joining three houses to each of three separate utilities, as shown in figure. Is it possible to join these houses and utilities so that none of the connections cross?



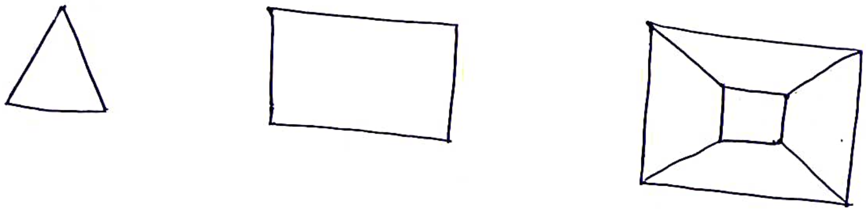
Definition (Planar Graph)

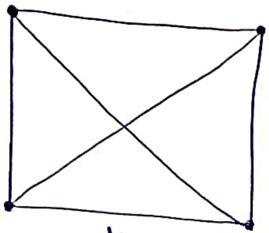
A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect. The points of intersection are called crossovers.

→ The representation of planar graph is embeded.

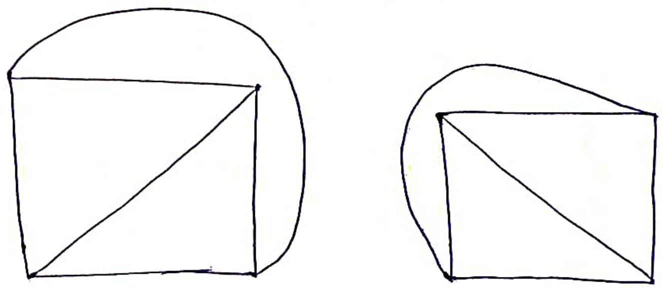
<u>Definition</u> (plane graph)

A graph that can not be drawn on a plane without a crossover between its crossing is called a plane graph.

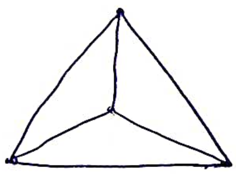


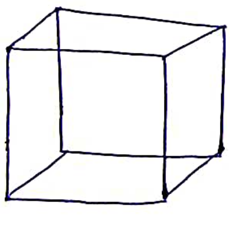


K_4



Embedding of K_4 i.e planar representation of K_4 ~~on plane~~





Planar representation