Optimization

Minimally Connected Graph

A connected graph G is said to be minimally connected if deletion of any edge from G disconnect the graph G.

Theorem

if and only if or

A graph G is a tree iff G is minimally connected.

Proof: Necessary Part:

Suppose graph G is a tree.

Claim: G is minimally connected.

~~Suppose G is not~~ We will prove necessary part by method of contradiction.

Suppose, if possible G is not minimally connected. Then there exists an edge $e \in E(G)$ such that $G-e$ is connected.

$\Rightarrow e$ is in some cycle.

Which is a contradiction to the fact that G is a tree. Hence, our assumption is wrong.

Thus G is minimally connected.

Sufficient Part: Conversely, suppose G is minimally connected.

Claim: G is a tree.

Since G is minimally connected, then G is connected and cycle less. Otherwise, we would remove one of the edges on the cycle and still the graph is connected, which is a contradiction to the fact that G is minimally connected.

Thus G is a tree.

Hence, G is a tree iff G is minimally connected.

(proved)

Theorem

Prove that every non-trivial tree has at least two vertices of degree one (pendant vertex).

OR

In a tree (with two or more vertices), there are at least two vertices of degree one.

Proof: Let T be a tree with $n \geq 2$ vertices. Then T has $(n-1)$ edges. Since each edge contributes 2 degrees, therefore the sum of degree of all the vertices is $2(n-1)$.

Now $2(n-1)$ degree are to be divided n vertices on T.

Let m be the number of vertices of degree one.

Claim: $m \geq 2$

Since no vertex of T can be of zero degree we have

$$\frac{2(n-1)-m}{n-m} \geq 2$$

$$\Rightarrow \quad 2(n-1)-m \geq 2n-2m$$

$$\Rightarrow \quad 2n-2-m \geq 2n-2m$$

$$\Rightarrow \quad 2m-m \geq 2$$

$$\Rightarrow \quad m \geq 2$$

Thus we must have at least two vertices of degree one in a tree.

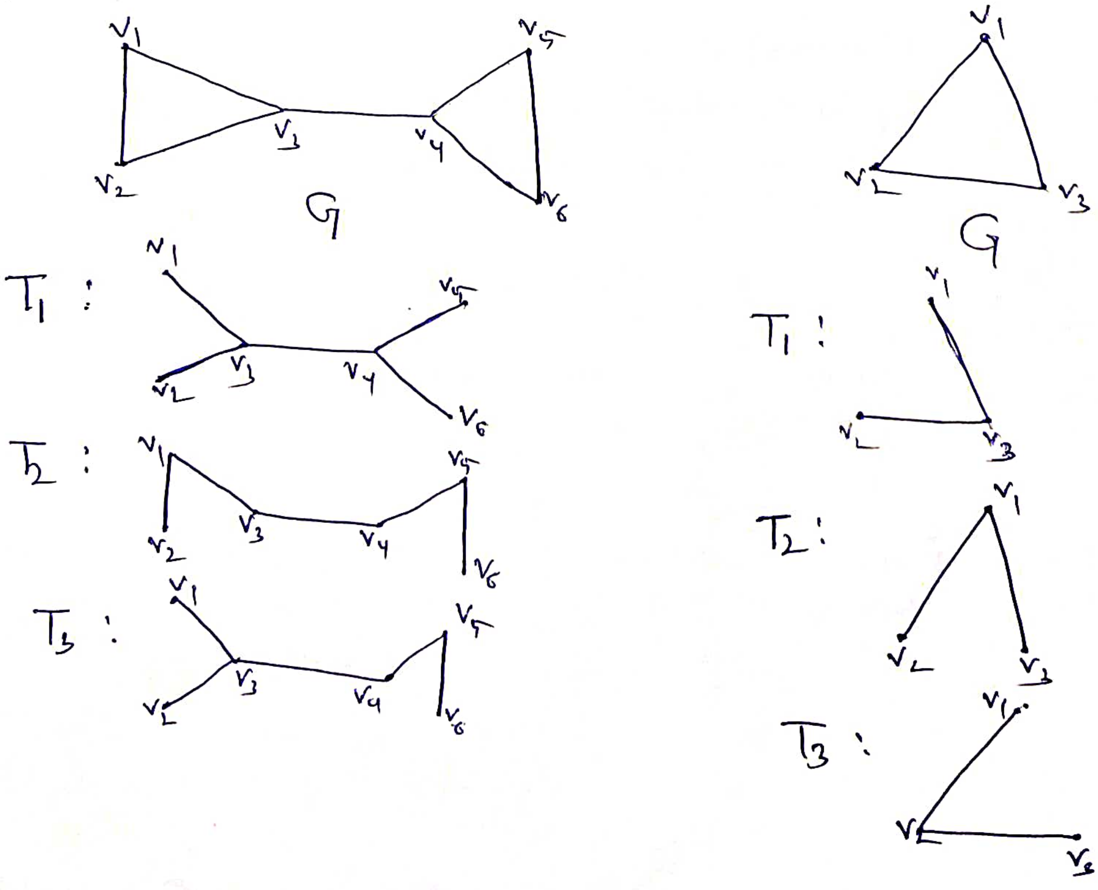
(proved)

Spanning Trees and Enumeration

Spanning Tree

A tree is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all the vertices of G.

EXAMPLE



<u>Theorem</u>

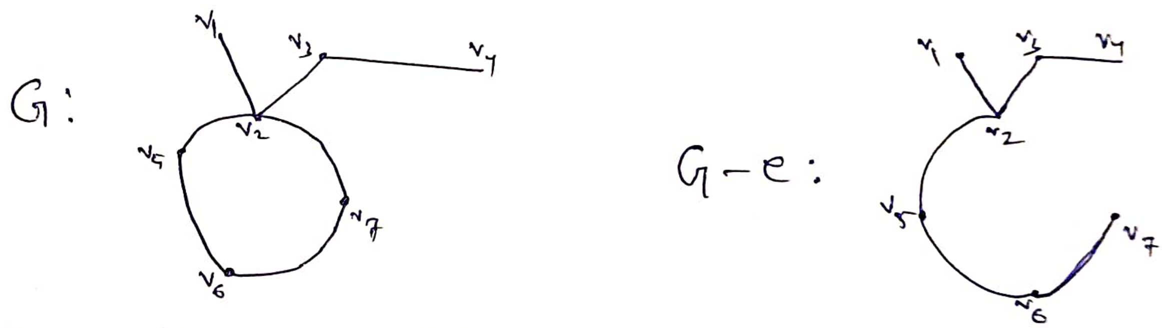
A graph G is connected if and only if (iff) it has a spanning tree.

Proof: Let G be a connected graph.

Claim: G has a spanning tree.

If G has no cycles, then G is a tree and it is a spanning tree of it self.

If G has a cycle say C, then delete an edge 'e' from C as shown in figure.



Then this is still result a connected graph. Since G-e has no cycle, then G-e is a spanning subgraph of G.

If G has two or more cycles, then repeat the operation till an edge from the last cycle deleted, leaving a connected acyclic graph that contain all the vertices of G.

Thus we have a spanning tree T in G

Conversely, suppose a graph G has a spanning tree T on G.

Claim: G is connected

In order to prove G is connected we have to show that every two vertices of G are joined by a path.

Let $u, v \in V(G)$

$\Rightarrow u, v \in T$ (since T is a spanning tree)

Since T is a tree in G, there exists a unique path i.e 'u-v' path P on T.

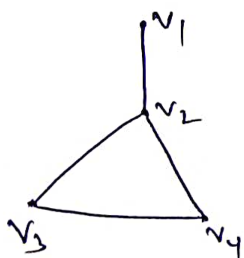
As $T \subseteq G$, u-v path P also appears in G. Since u,v are arbitrary vertices of G follows that every pair of vertices of G are joined by path. Consequently G is connected. □

Enumeration of Trees

Counting of Label tree

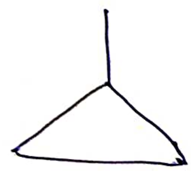
A graph in which each vertex is assigned a unique ~~lable~~ label (name) is called a label graph.

G :



(Label graph)

G_1 :



(Non-label graph)

Q: Find the possible numbers of distinct label tree of order 2.

Sol^n:

or

The possible number of distinct label tree of order 2 is 1.