MODULE : 2

EULERIAN GRAPHS

Euler path

A path on a graph G is called Euler path if it includes every edges exactly once. To find such a path vertices may be ~~repa~~ repeated but edges are not repeated.

→ Since the path contains every edge exactly once, it is also called Euler trail.

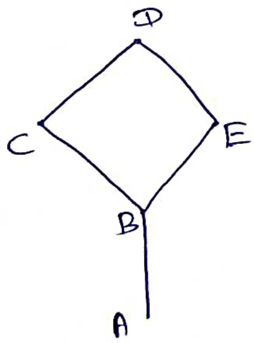
Euler Circuit

A circuit on a graph that includes all the edges of the graph is called an Euler circuit. OR, A Euler path that is circuit is called Euler circuit.

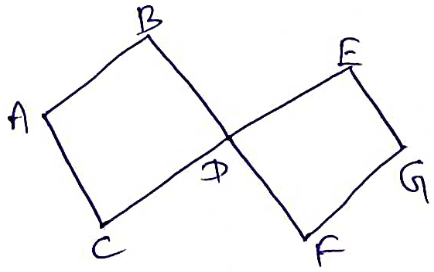
→ In Euler circuit internal vertices should be distinct may be starting vertices repeated.

Eulerian Graph

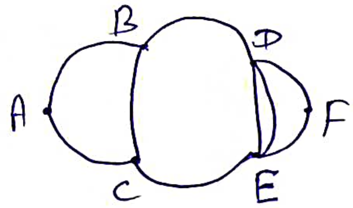
A graph G is said to be Eulerian or Euler graph if it contains at least one Euler circuit.



G: 1



G: 2



G: 3

The graph of figure G:1 has an Euler path but no Euler circuit. Note that two vertices A and B are of odd degree 1 and 3. That means AB can be used to either arrive at vertex A or leave vertex A but not for both.

→ ABCDEB and BCDEBA are two Euler path. starting from any vertex no Euler circuit can be found.

→ In G:2 ABDEGFDCA is an Euler path and circuit. Note that all vertices of even degree.

→ In G:3 there is no Euler path or circuit.

→ G:2 is Eulerian graph.

Note: To determine whether a graph G has an Euler circuit, we note the following points:

(i) List the degree of all vertices in the graph.

(ii) If any value is zero, the graph is not connected and hence it can't have Euler path or Euler circuit.

(iii) If all the degree are even, then G has both Euler path and Euler circuit.

(iv) If exactly two vertices are odd degree, then G has Euler path but not Euler circuit.

Theorem

A non empty connected graph G is Eulerian or Euler graph if and only if its vertices are all of even degree.

Proof: Let a non empty connected graph G be Eulerian.

Claim: All the vertices of G are of even degree.

Since G is an Euler graph G contains an Euler circuit, starting from a vertex say v_1 in G. The circuit traversing all the edges of G is $v_1, e_1, v_2, e_2, \ldots, v_n, e_n, v_{n+1}$ where $v_{n+1} = v_1$. In this Euler circuit all the edges are distinct but some of the vertices may be repeated. It is clear that the pair of successive edges e_k and e_{k+1}, $1 \leq k \leq n-1$ contributes 2 to the degree of the vertex v_{k+1}.

Therefore the vertices $v_2, v_3, \ldots, v_n$ are of even degree. Besides the vertex v_1 gets a contribution of 2 to its degree from the initial and final edge e_1 & e_n.

Thus all the vertices are of even degree.

Conversely, suppose all the vertices of a connected graph is of even degree.

Claim: G is an Eulerian or Euler graph.

Let v be the any vertex in G. Construct a walk starting at v and going through the edges of G such that no edges is traced more than once. Since v is of even degree we shall eventually arrive at v when the tracing comes to an end. So it is a closed walk say 'K'.

There are two cases arrises

Case-1

If this K contains all the edges of G then G is an Euler graph, which completes the prove.

Case-2:

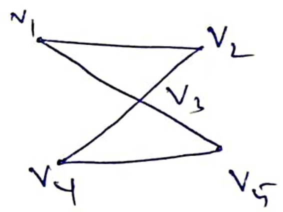
If K does not contain all the edges of G then we delete all the ~~de~~ edge of K from G to obtain a new subgraph $K' = G - K$.

Since all the vertices of G & K are of even degree, it follows that the vertices of K' are also of even degree. Since G is connected K' must touch K at least one vertex say 'u'. We again construct a new walk in the graph K' starting at 'u' as well as the vertices K' are of even degree.

Now this walk in K' can be combined with K to from a new walk starting and edgs out V. This argument can be repeated untill we obtained an Euler circuit which travels all the edge of 'G'.

Hence G is an Eulerian or Euler graph.

<u>Example</u> : Let G be a graph of Figure. Verify that G has an Eulerian circuit.



<u>Sol^n</u>: We observe that G is connected and all the degree are having even degree

$$\deg(v_1) = \deg(v_2) = \deg(v_4) = \deg(v_5) = 2.$$

Thus G has a Eulerian circuit.

By inspection, we find the Euler circuit

$$v_1 - v_3 - v_5 - v_4 - v_3 - v_2 - v_1.$$

<u>Problem</u>: Find all positive integers n such that the complete graph K_n is Eulerian.

Theorem

A connected graph G has an Eulerian trail if and only if has at most two odd vertices, i.e, it has either no vertices of odd degree or exactly two vertices of odd degree.

Proof: Suppose G has an Eulerian trail which is not closed. Since each vertex in the middle of the trail is associated with two edges and since there is only one edge associated with each end vertex of the trail, these end vertices must be odd and other vertices must be even.

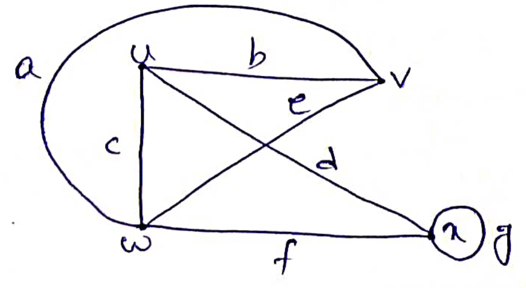
Conversely, suppose that G is connected with at most two odd vertices.

If G has no odd vertices then G is Euler and so has Eulerian trail.

Also ~~since~~ G can not have just one odd vertex ~~since~~ because in any graph there is an even number of vertices with odd degree.

Hence G has two odd vertices. Consequently, G has an Eulerian trail.

Problem Show that the graph shown in Figure has no Eulerian circuit but has a Eulerian trail.



Sol^n: Here $deg(u) = deg(v) = 3$ and

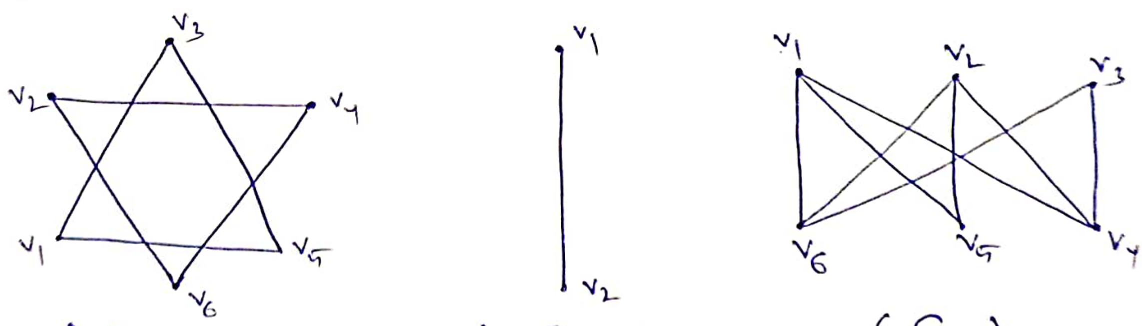
$deg(w) = deg(x) = 4$

Since u and v have only two vertices of odd degree, the graph shown in Figure, does not contain Eulerian circuit, but the path

b - a - c - d - g - f - e is an Eulerian path

or Eulerian trail.

Problem: Show that the graphs in Figure below contain no Eulerian circuit.



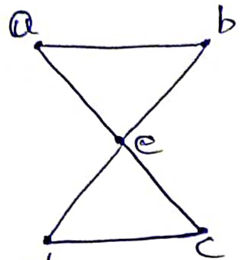
(G_1) (G_2) (G_3)

Sol^n: The graph shown in Figure G_1 does not contain Eulerian circuit since it is not connected.

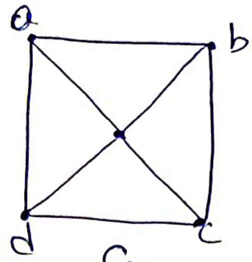
The graph shown in Figure G_2 is connected but vertices v_1 and v_2 are of degree 1. Hence it does not contain Eulerian circuit.

All the vertices of the graph shown in Figure G_3 are of degree 3. Hence it does not contain Eulerian circuit.

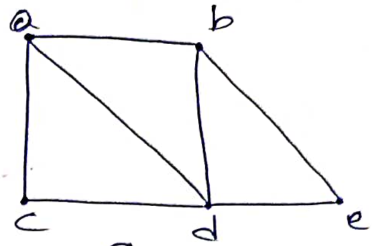
Problem: Which of the undirected graph in Figure have an Euler circuit? Of those that do not, which have an Euler path?



G_1

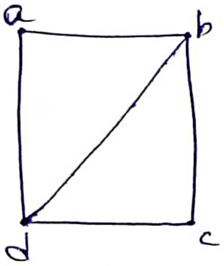


G_2

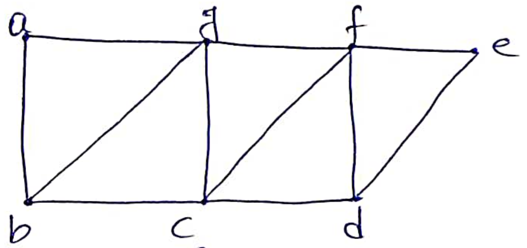


G_3

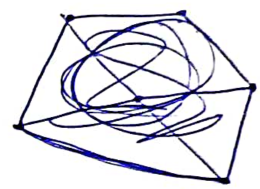
<u>Problem</u>: Which graphs shown in Figure have an Euler path?

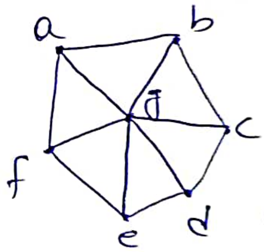


G_1



G_2





G_3