MODULE-3 MATCHING AND FACTORS

Matching

A matching in a graph G is a set of non-loop edges with no shared endpoints.

Definition

The vertices incident to the edges of a matching M are saturated by M.

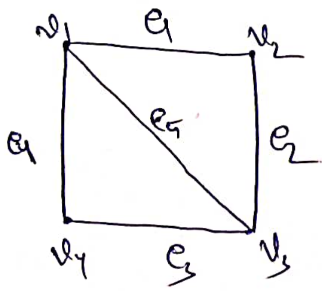
Note: All vertices are said to be unsaturated.

Definition: A perfect matching in a graph G is a matching that saturates every vertex.

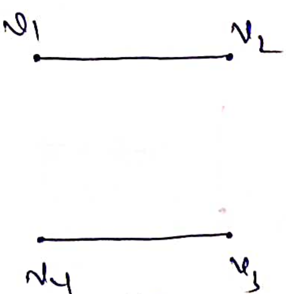
Definition (Another def^n of matching)

A subgraph M of a graph G is said to be matching if every vertex of G is incident with at most one vertex in M, and no two edges are adjacencet. Note: $\deg(v) \leq 1$.

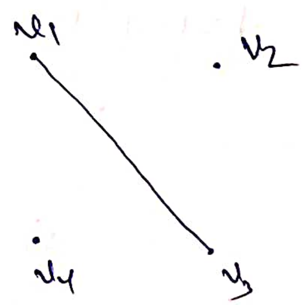
Example

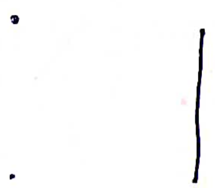


G



M_1

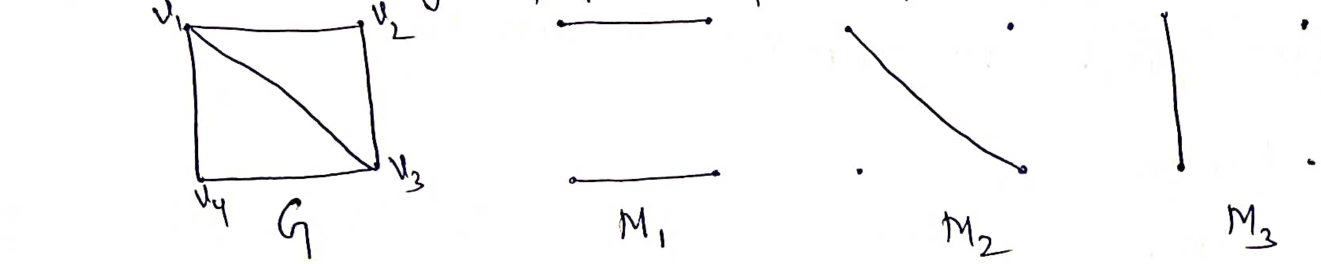



M_3


M_4

Definition

A maximal matching in a graph is a matching that can not be enlarged by adding an edge. Also, M is a maximal matching if $|M| \geq |M'|$ for any matching M'.



Here M_1 & M_2 are maximal matching.

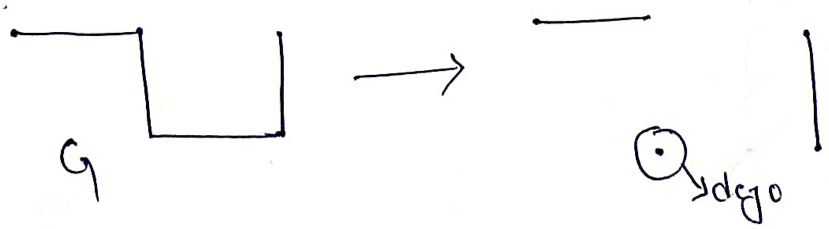
Definition

A maximum matching is a matching of maximum size among all matchings in the graph.
Or, Matching with maximum number of edges.

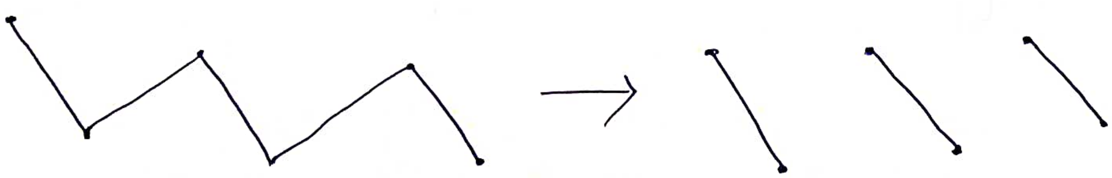
Perfect Matching

A matching in which every vertex is saturated, i.e, has an edge. For $v \in V(G)$, $d(v) = 1$, 1-regular graph.

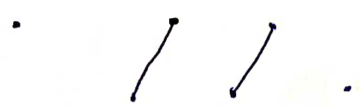
Example



So not a perfect matching as G has odd vertices



perfect matching



not perfect matching.

Observation

(1) All perfect matchings are maximum matchings and also maximal.

(2) Perfect Matching has always even no:s of vertices, but converse is not true.

Number of perfect matching in $K_{n,n}$.

$1 \to n$ choices , $2 \to n-1$ choices , $3 \to n-2$ choices , ..., $n \to 1$ choice.

So total number of choice in $K_{n,n} = n \times (n-1) \times (n-2) \times \dots \times 1$
$= n!$

Number of perfect matching in K_{m} is $\frac{(2n)!}{2^n n!}$

Definition

An M- alternating path is a path that alternates between edges that are in M and edges that are not in M.

Definition

An M-augmenting path is a M- alternating where the end points are unsaturated by M.

→ Given an M-augmenting path P, we can replace the edges of M in P with the other edges of P to obtain a new matching M' which one more edge. Thus when M is a maximal matching, there is no M-augmenting path.

Definition

The symmetric difference of two sets A and B, denoted by $A \Delta B$ is defined as

$$A \Delta B = (A-B) \cup (B-A)$$

Lemma

Every component of the symmetric difference of two matching is a path or an even cycle.

Proof: Let M_1 and M_2 be two matchings. Also let

$$F = M_1 \Delta M_2.$$

Considering F as a subgraph we can see that ~~dp~~ $d_F(v) \leq 2$ for ~~every~~ any vertex v. So, F is a graph that is the disjoint union of cycles and paths. But those cycles can not be odd cycle since odd cycle are not a union of two matching. Therefore, each component of F is either a path or an even cycle.

(proved)

Berge Theorem

A matching M in a graph G is a maximum matching in G iff G has no M-augmenting path.

Proof: We will prove this theorem by method of contrapositive. That means first we need to show that if G has an M-augmented path then M is not maximal matching.

Let P be an M-augmeted path in G (written so that first and final edges in the path are not in M). Also, let

$$M' = M \Delta P.$$

Then M' is a matching.

$$|M'| = |M| + 1.$$

Therefore, M is not a ~~maximal~~ maximum matching.
Hence, by method of contrapositive if G has no M-augmenting path, then M is maximum matching.

Conversely, also we use contrapositive method. That means, we need to show that if M is not a maximum matching, then $\exists$ an M-augmented path M' be a maximal matching.

Considering $M \Delta M'$, we see that $M \Delta M'$ is the ~~m~~ disjoint union of paths and even cycles.

<u>Case 1</u>: If $\exists$ a path in $M \Delta M'$.

(i) $|P|$ is even. Then P's contribution to
$$|M'|-|M| = 0.$$

(ii) $|P|$ is odd and the first edge of the path is an element of M. Then P's contribution to
$$|M'|-|M| = -1.$$

(iii) $|P|$ is odd and the first edge of the path is an element of M. Then P's contribution to
$$|M'|-|M| = 1.$$

<u>Case-2</u>: If $\exists$ an even cycle in $M \Delta M'$. Then the cycle's contribution to
$$|M'|-|M| = 0.$$

If we sum up all contribution then we get that
$$|M'|-|M| = 1.$$

Therefore we know that $\exists$ part three of case one must occur. So then $\exists$ a path P where the first edge of the path is an element of M'. P is then M-augmented path. Therefore, $\exists$ an M-augmented path.

Hence, by method of contrapositive if M is maximum matching, then G has no M-augmenting path.

(proved)