Therefore we know that part three of case one must occur. So then $\exists$ a path P where the first edge of the path is an element of M'. P is then M-augmented path. Therefore, $\exists$ an M-augmented path.

Hence, by method of contrapositive if M is maximum matching, then G has no M-augmenting path.

(proved)

Hall's Matching Condition

If a matching M saturates X, then for every $S \subseteq X$ there must be at least $|S|$ vertices that have neighbors in S, because the vertices matched to S must be chosen from that set. We use $N_G(S)$ or simply $N(S)$ to denote the set of vertices having a neighbor in S. Thus $|N(S)| \geq |S|$ is a necessary condition.

Hall's Theorem

An X, Y-bipartite graph G has a matching that saturates X iff $|N(S)| \geq |S|$ for all $S \subseteq X$.

Proof: Let M be a matching on G that saturates X. Then M defined a map f such that

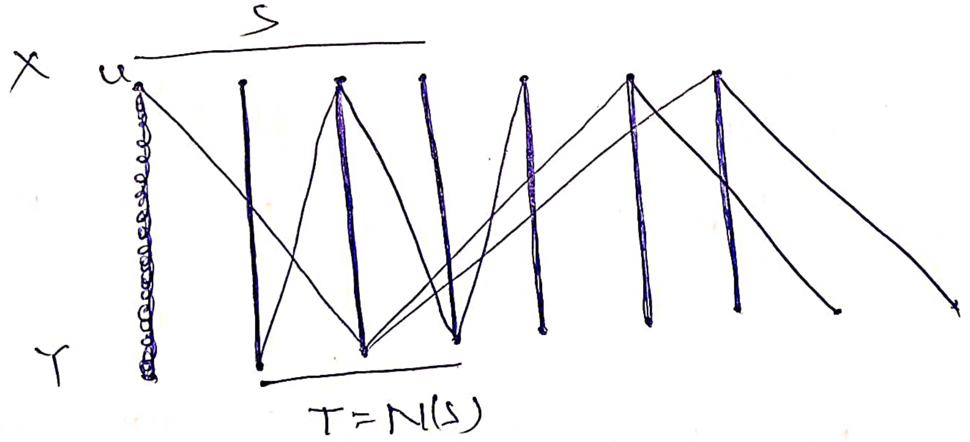
$$f: X \longrightarrow Y$$

is a one-to-one mapping (since M saturates X). Then we know that

$$N(S) \supseteq f(S).$$

$$\Rightarrow |N(S)| \geq |f(S)| = |S| \quad (\text{Since } f \text{ is one-to-one})$$

Conversely, we use contrapositive method to prove sufficient part. If M is a maximum matching in G and M does not saturate X, then we obtain a set $S \subseteq X$ such that $|N(S)| < |S|$. Let $u \in X$ be a vertex unsaturated by M. Among all the vertices reachable from u by M-alternating paths in G, let S consist of those in X, and let T consist of those in Y. Note that $u \in S$



Claim: M matches T with $S-\{u\}$. The M-alternating paths from u reach T along edges not in M and return to X along edges in M. Hence every vertex of $S-\{u\}$ is reached by an edge in M from a vertex in T. Since there is no ~~M-alter~~ M-augmenting path, every vertex of T is saturated, thus an M-alternating path reaching $y \in T$ extends via M to a vertex of S. Hence these edges of M yield a bijection from T to $S-\{u\}$, and we have $|T| = |S-\{u\}|$.

The matching between T and $S-\{u\}$ yield $T \subseteq N(S)$. In fact, $T = N(S)$. Suppose that $y \in Y-T$ has a neighbor $v \in S$. The edge vy can not be in M, since u is unsaturated and the rest of S is matched to T by M. Thus adding vy to an M-alternating path reaching v yields an M-alternating path to y. This contradicts $y \notin T$, and hence vy can not exist.

With $T = N(S)$, we have proved that $|N(S)| = |T| = |S|-1 < |S|$ for this choice of S. Hence by by method of contrapositive the theorem holds true.

(proved)

→ When the sets of the bipartition have the same size, Hall's Theorem is the Marriage Theorem.

<u>Corollary</u>: For $k>0$, every k-regular bipartite graph has a perfect matching.

Proof: Let X and Y be the two partitions of a k-regular bipartite graph. Then we know that

$$k|X| = k|Y|$$

$$\Rightarrow |X| = |Y|$$

Therefore, any matching that saturates X also saturates Y and vice versa. Then to prove the corollary, it is sufficient to show that Hall's matching condition holds before. Therefore we must show that $\forall$ ~~S~~, $S \in X$, $|N(S)| \geq |S|$.

So, the number of edges between S and ~~N(S)~~ ~~N(S)\X~~ $N(S) = k|S|$ which is less than the number or edge ~~betw~~ leaving $N(S)$. Therefore

$$k|S| \leq k|N(S)|$$

$$\implies |S| \leq |N(S)|, \text{ when } k > 0.$$

Hence k-regular bipartite graph has a perfect matching. $\square$