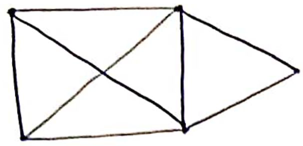
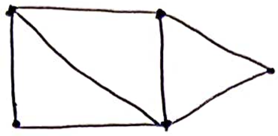
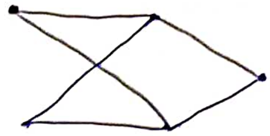


Not K_4- free graph



K_4- free graph but not triangle- free graph



K_3- free graph or, triangle-free graph.

Turan's Theorem

Among n-vertex simple graphs with no K_{r+1}, $T_{n,r}$ has the maximum number of edges. Here, K_{r+1} refers to the $(r+1)$-clique and $T_{n,r}$ refers to the Turan Graph on n vertices having r-partitions.

Proof: Every r-colorable (or r-partite) graph, including Turan graph $T_{n,r}$, has no $r+1$-Clique, since each partite set contributes at-most one vertex to each clique. If we can prove that the maximum edges is achieved by a r-partite graph, then by the above Lemma the required graph is $T_{n,r}$. Thus, it is sufficient to show that for every graph G that has no $r+1$- clique, there is an r-partite graph H with the same vertex set as G i.e $V(H) = V(G)$, and at-least as many edges i.e $e(H) \geq e(G)$.

We prove this by method of induction on r.

Base step: For $r=1$, any simple graph with no 2-clique is a null-graph, and is trivially 1-partite. Thus, in this case, $H=G$.

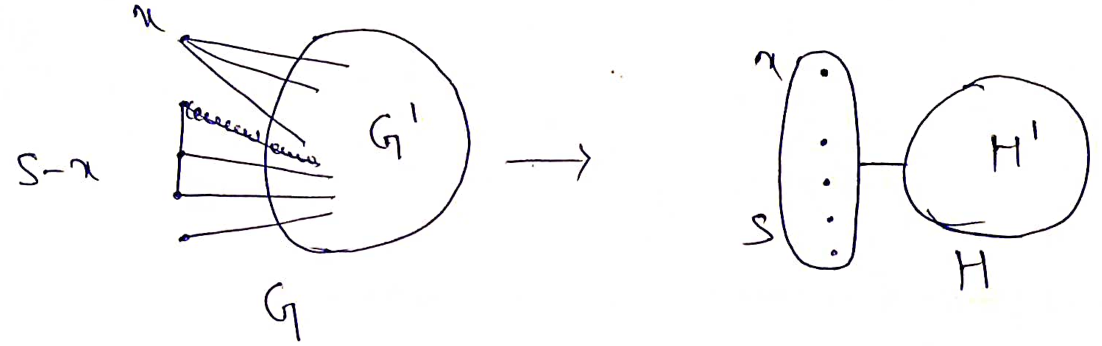
Inductive hypothesis: Suppose the result is true for fewer than r.

Inductive step: Let G is an n-vertex simple graph with no $r+1$-clique, where $r>1$. Let $x \in V(G)$ be a vertex of degree $k = \Delta(G)$. Let the sub-graph G' be the induced sub-graph of G by the set $N(x)$, where $N(x)$ is the neighbours of x.

Claim 1: If G has no $(r+1)$-clique, then G' has no r-clique.

As x is adjacent to every vertex in G', if G' had an r-clique then G would have an $(r+1)$-clique which would be a contradiction.

Thus, we can apply the induction hypothesis to G'. Thus, there exists an $(r-1)$-partite graph H' with $V(H') = V(G') = N(x)$ and $e(H') \geq e(G')$. Here $V(H') = N(x) = k$. Let H be the graph formed from H' by joining all of $N(x)$ to all of $S = V(G) - N(x)$. Since S is an independent set of $n-k$ vertices and H' is $(r-1)$-partite, thus H would be r-partite. Thus, G' has no r-clique



Claim 2: $e(H) \geq e(G)$.

By construction, $e(H) = e(H') + k(n-k)$. We also have $e(G) \leq e(G') + \sum_{v \in S} d_G(v)$ as the difference of edges between G and G' would only consist of those edges that have at-least one end-point on the set $S = V(G) - V(G')$. Note that the edges with both end-points on the set S are counted twice. Since $\Delta(G) = k$, we have $d_G(v) \leq k$ for each $v \in S$. As $|S| = n-k$, we have $\sum_{v \in S} d_G(v) \leq k(n-k)$. Therefore, we have

$$\begin{aligned} e(G) &\leq e(G') + \sum_{v \in S} d_G(v) \\ &\leq e(G') + k(n-k) \\ &\leq e(H') + k(n-k) \\ &= e(H). \end{aligned}$$

$\Rightarrow e(G) \leq e(H)$.

Hence, among the n-vertex simple graphs with no K_{r+1}, $T_{n,r}$ has the maximum number of edges. $\square$

Another statement of Turan's Theorem

Among the n-vertex simple graph

with no $(r+1)$-Clique, $T_{n,r}$ has the maximum number of edges.

Edge-Coloring

An Edge-Coloring of a graph is an assignment of colors to the edges of the graph so that no two incident edges have the same color.

OR, An edge coloring of a graph G is a function $f: E(G) \longrightarrow C$, where C is a set of distinct colors. For any positive integer k, a k-edge coloring is an edge coloring that uses exactly k different colors.

Proper Edge Coloring

A proper edge coloring of a graph is an edge coloring such that no two adjacent edges are assigned the same color. Thus a proper edge coloring f of G is a function

$$f: E(G) \longrightarrow C$$

such that

$$f(e) \neq f(e')$$

whenever edges e and e' are adjacent in G.

Chromatic Index

The chromatic index of a graph G, denoted by $\chi'(G)$, is the minimum number of different colors required for a proper edge colouring of G. G is k-edge-chromatic if $\chi'(G) = k$.

→ Chromatic index of Bipartite graph is always $\Delta(G)$.