Application of Ordinary Differential Equations: Pendulums

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1 Second-Order Homogeneous ODEs

This article covers the application of ordinary differential equations, or *ODES*, to the motion of a simple pendulum. We discuss oscillations with and without dampening. In general, oscillatory behavior can be modeled by the second-order homogeneous ODE

$$ay'' + by' + cy = 0\,. \tag{1}$$

The general solution to this ODE is

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}\,,$$

where C_1 and C_2 are constants, and r_1 and r_2 are roots of the *auxiliary equation*

$$ar^2 + br + c = 0\,. \tag{2}$$

If $b^2 - 4ac > 0$, then the roots of Equation (2) are real and distinct, and

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}\,.$$

If $b^2 - 4ac = 0$, then the roots are real and repeated, and

$$r_1 = r_2 = \frac{-b}{2a}\,.$$

For complex roots, which arise when $b^2 - 4ac < 0$, we get

$$r = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}\,.$$

These roots correspond to the form of complex numbers

$$\alpha \pm i\beta\,,$$

from which the general solution to Equation (1) is in the form

$$y = C_1 e^{\alpha t}\cos\beta t + C_2 e^{\alpha t}\sin\beta t\,.$$

2 Simple Harmonic Motion Representation

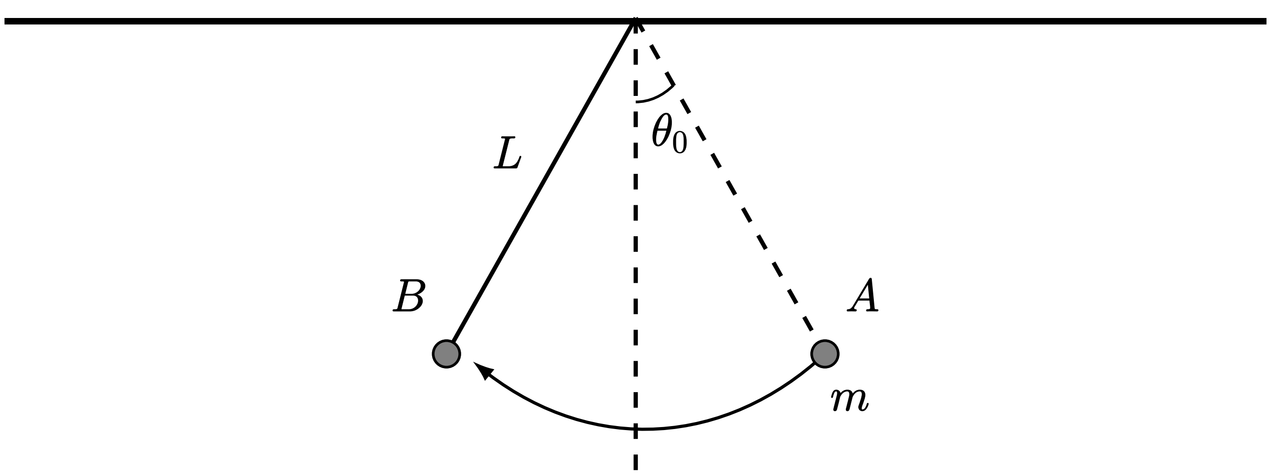

FIGURE 1: A pendulum is displaced by an angle θ_0 and oscillates, exhibiting simple harmonic motion.

Let us first consider pendulums without dampening, which opposes movement. A main example of a dampening force is drag from air resistance. In many instances these factors are negligible, but placing a pendulum in a windy or aqueous medium may require inclusion of these forces. *Simple harmonic motion* represents periodic motion in which the restoring force of a moving object is directly proportional to its displacement from equilibrium. These models lack any other restrictive forces, which are covered in Section 3.

When a pendulum is displaced by an angle θ_0, it swings back and forth. See Figure 1, in which an object of mass m is pivoted at the end of a string of length L.

This pendulum is displaced at point A by an angle θ_0 from the vertical. It is then released, swings to point B, and repeats this cyclic motion. This oscillation is caused by the force of the gravitational field exerting a torque on the swinging pendulum; specifically, the component $\vec{F}_g \sin\theta$ applies a torque to the mass. (Consider Figure 2, in which a free-body diagram of the mass is shown.)

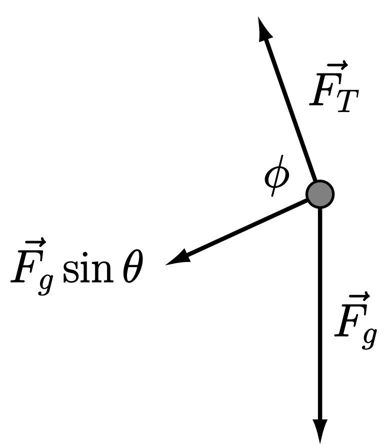

FIGURE 2: A free-body diagram representing the forces on the mass m as the pendulum oscillates in simple harmonic motion.

Newton's Second Law for rotation is

$$\sum \tau = I\ddot{\theta}\,,$$

where $\sum \tau$ is the net torque and I is the rotational inertia of the pendulum, mL^2. The expression of torque is

$$\tau = rF \sin\phi\,,$$

where ϕ is the angle between $\vec{F}_g \sin\theta$ and the axis of the string. The force $\vec{F}_g \sin\theta$ is perpendicular to the axis, so $\phi = 90°$. We also choose the counterclockwise direction to be the direction of positive torque. Thus, the torque on the pendulum is

$$\tau = -LF_g \cdot 1 = -Lmg\,.$$

We then have

$$\begin{aligned} -F_g \sin\theta L &= I\ddot{\theta} \\ -mg \sin\theta L &= mL^2\ddot{\theta} \\ \ddot{\theta} + \frac{g}{L}\sin\theta &= 0\,. \end{aligned}$$

For small values of θ, especially values less than $15°$, the small-angle approximation ($\sin\theta \approx \theta$) provides

$$\ddot{\theta} + \frac{g}{L}\theta = 0\,.$$

Defining the angular frequency as $\omega = \sqrt{g/L}$ gives

$$\ddot{\theta} + \omega^2\theta = 0\,, \tag{3}$$

the defining ODE for simple harmonic motion. Comparing this equation to Equation (1), we have $a = 1$, $b = 0$, and $c = \omega^2$. Our auxiliary equation is then

$$r^2 + \omega^2 = 0\,.$$

Solving for r gives

$$r_1 = i\omega \qquad \text{and} \qquad r_2 = -i\omega\,.$$

These roots are in the form $\alpha \pm \beta i$, where $\alpha = 0$ and $\beta = \omega$. Because these roots are complex, the solution to Equation (3) is

$$\begin{aligned}\theta(t) &= C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t \\ &= C_1 \cos \omega t + C_2 \sin \omega t \,.\end{aligned}$$

We now solve for the constants C_1 and C_2. At $t = 0$, $\theta = \theta_0$. Thus,

$$\theta(0) = C_1 + 0 = \theta_0 \implies C_1 = \theta_0 \,.$$

Additionally, after one-fourth of a revolution—modeled by a fourth of the *period*, $t = (2\pi/\omega)/4 = \pi/2\omega$—the pendulum reaches the trough of its path: $\theta = 0$. Therefore,

$$\theta\left(\frac{\pi}{2\omega}\right) = 0 + C_2 = 0 \implies C_2 = 0 \,.$$

We then have, for our solution,

$$\boxed{\theta(t) = \theta_0 \cos(\omega t) \,.}$$

The key feature to $\theta(t)$ in simple harmonic motion is its uniform amplitude. In cases of dampened oscillations, however, the amplitude gradually decreases as the pendulum slows down.

3 Pendulums with Dampening

During a pendulum swing, the first phase is descending. We model the dampening as a retarding force that is proportional to the linear speed of the pendulum. We then have $\vec{F_R} = b\vec{v}$, where b is a positive quantity called the *retarding constant.* This force acts in the opposite direction to the movement of the pendulum. When descending, the component of gravity $\vec{F_g} \sin\theta$ accelerates the pendulum, while $\vec{F_r}$ decelerates the movement. Therefore, $\vec{F_g} \sin\theta$ and $\vec{F_R}$ act in different directions (see Figure 4). To find the units of b, we consider $b = F_R/v$. The units of F_R are N, and the units of v are m/s . Thus, we have

$$\frac{\mathrm{N}}{\mathrm{m/s}}$$

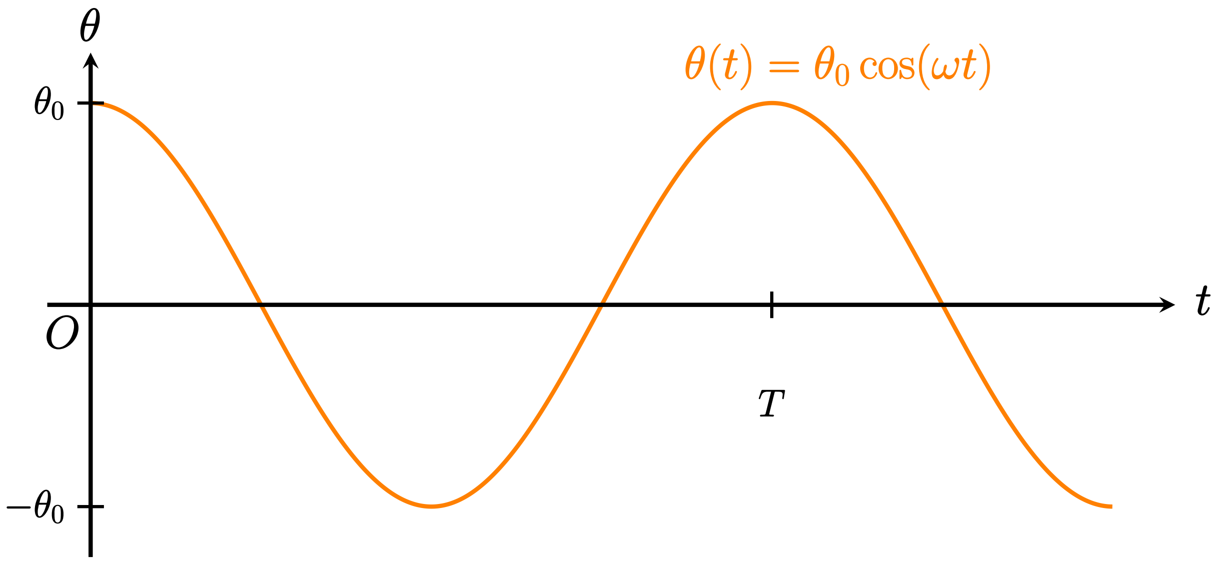

FIGURE 3: The graph of $\theta(t)$ in simple harmonic motion.

as the units of b.

Thus, by Newton's Second Law for rotation, the ODE when ascending is

$$-bvL - mg\sin\theta L = mL^2\ddot{\theta}\ .$$

The relationship between linear speed v and angular speed $\dot{\theta}$ is $v = L\dot{\theta}$. We have, therefore,

$$-bL^2\dot{\theta} - mg\sin\theta L = mL^2\ddot{\theta}$$
$$\ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0\,.$$

Using the small-angle approximation again, we obtain

$$\ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{g}{L}\theta = 0\,. \tag{4}$$

The corresponding auxiliary equation is

$$r^2 + \frac{b}{m}r + \frac{g}{L} = 0\,,$$

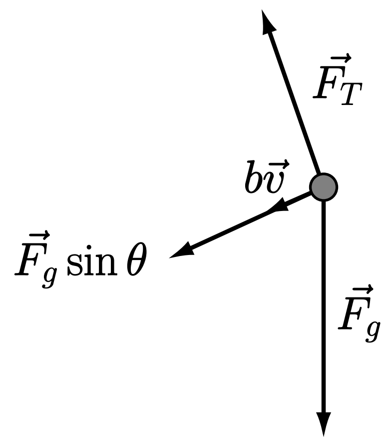

FIGURE 4: The ascending phase of a pendulum oscillating with a retarding force $b\vec{v}$. The retarding force and the component of gravity $\vec{F}_g \sin\theta$ are in the same direction.

for which the zeros are

$$r = \frac{-b}{2m} \pm \frac{1}{2}\sqrt{\frac{b^2}{m^2} - \frac{4g}{L}}\,. \tag{5}$$

We continue using the fact that

$$\frac{b^2}{m^2} - \frac{4g}{L} < 0$$

because the retarding constant b is assumed to be small. The retarding force increases by much less than 1 N for every increase in speed of 1 m/s. We conclude that $b \ll 1$, therefore justifying our assumption. Equation (5) then becomes

$$r = \frac{-b}{2m} \pm \frac{i}{2}\sqrt{\frac{4g}{L} - \frac{b^2}{m^2}} = \frac{-b}{2m} \pm i\sqrt{\frac{g}{L} - \frac{b^2}{4m^2}}\,.$$

Defining the angular frequency as $\omega = \sqrt{\frac{g}{L} - \frac{b^2}{4m^2}}$ gives

$$r = \frac{-b}{2m} \pm i\omega\,.$$

Comparing these zeros to the form $\alpha + i\beta$, we have $\alpha = -b/2m$ and $\beta = \omega$. The solution

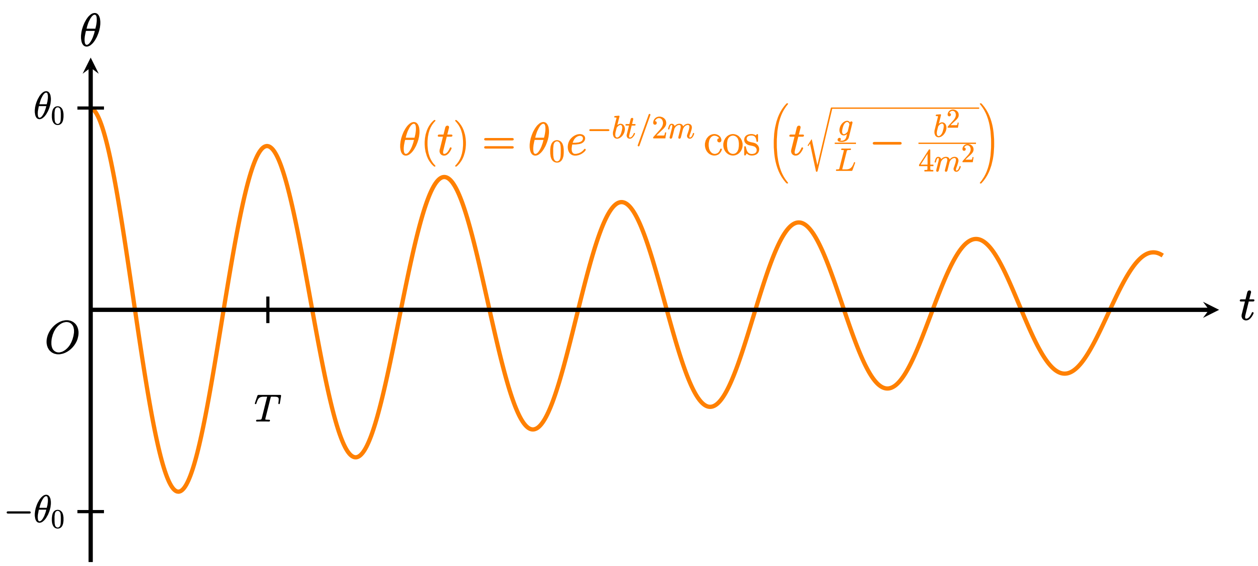

FIGURE 5: The graph of $\theta(t)$, in which the motion is dampened by the retarding force $\vec{F_R} = b\vec{v}$.

to Equation (4) is then

$$\begin{aligned} \theta &= C_1 e^{\alpha t} \cos\beta t + C_2 e^{\alpha t} \sin\beta t \\ &= C_1 e^{-bt/2m} \cos\omega t + C_2 e^{-bt/2m} \sin\omega t\,. \end{aligned}$$

We now solve for C_1 and C_2. At $t = 0$, $\theta = 0$. Therefore,

$$\theta(0) = C_1 + 0 = \theta_0 \implies C = \theta_0\,.$$

Moreover, at $t = (2\pi/\omega)/4 = \pi/2\omega$, we have $\theta = 0$. Thus,

$$\theta\left(\frac{\pi}{2\omega}\right) = 0 + C_2 e^{-bt\pi/4m\omega} = 0 \implies C_2 = 0\,.$$

Our final solution for Equation (4) is then

$$\boxed{\theta(t) = \theta_0 e^{-bt/2m} \cos\left(t\sqrt{\frac{g}{L} - \frac{b^2}{4m^2}}\right)\,.}$$

We note that this function exhibits oscillatory behavior in which the amplitude decreases with each cycle (see Figure 4). Eventually, θ converges to zero due to the dampening of the pendulum. The model when the pendulum decreases involves a different ODE, but we can assume the same behavior during those stages.