Circuits: Equivalent Quantities for Resistors, Capacitors, and Inductors

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1 Introduction

Resistors, capacitors, and inductors are circuit components that each provide various functions. In circuit analysis, it is useful to re-express components as one equivalent component. For example, a circuit of two resistors can be redrawn as an equivalent circuit of one resistor. The resistance of the one resistor in this equivalent circuit is called the *equivalent* resistance. This simplification process is fundamental to examining the behaviors of each circuit component.

Circuit components can be connected in two ways: in *series* and in *parallel*. Components in series with each other are connected by one pathway. Figure 1 shows a circuit in which four circuit elements, resistors (R_1, R_2, R_3, and R_4), are connected in series with each other. In contrast, components in parallel are located across from each other, in separate branches connected by the same pathway. See Figure 2, in which a circuit has three resistors (R_1, R_2, and R_3) connected in parallel. In each circuit, a battery of emf $\mathscr{E}$ provides current to the resistors.

In this article, we derive expressions for equivalent quantities of components placed in series and in parallel.

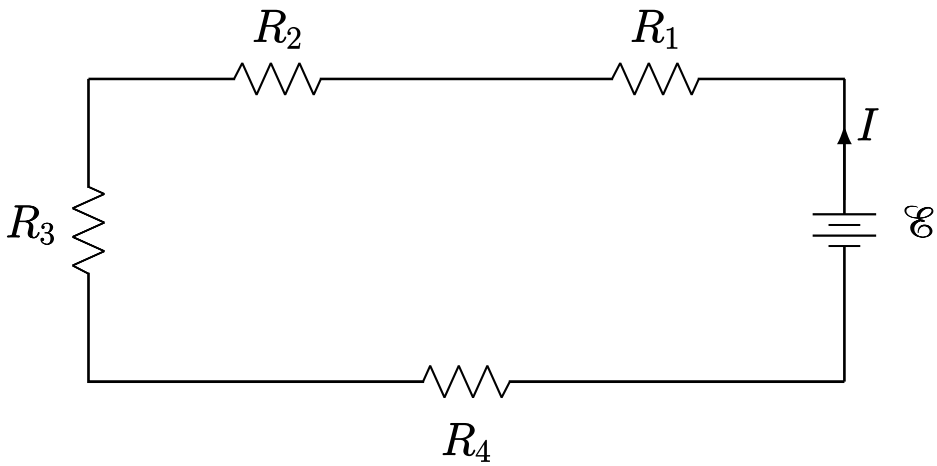

FIGURE 1: Resistors R_1, R_2, R_3, and R_4 are placed in series.

2 Elements of Circuits

In this section, let us discuss *electric potential* (also known as *voltage*, V) and *current*, I. Current is the rate at which charge (Q) in the circuit changes, as modeled by

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t}\,. \tag{1}$$

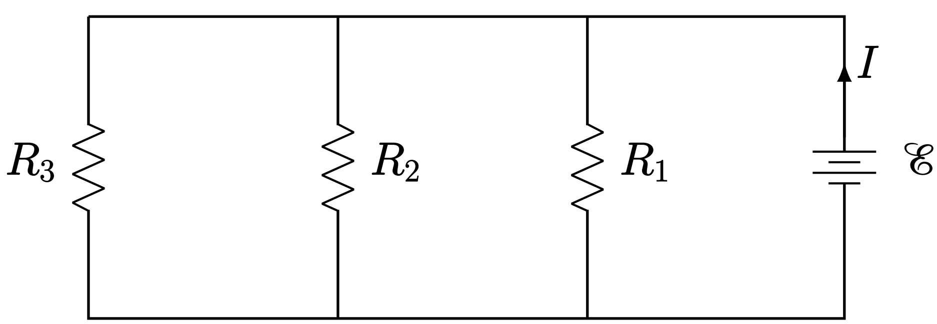

FIGURE 2: Resistors R_1, R_2, and R_3 are placed in parallel.

Voltage can be thought of as a force driving the current. This quantity is similar to to water pressure in water pipes, with the water flow rate being an analog to electrical current. Potential difference, ΔV, is the change in voltage between two points. We will use the terms "electrical potential" and "voltage" interchangeably throughout this article.

3 Resistors

3.1 Ohm's Law

In this subsection, we discuss the relationship between potential difference (ΔV), current (I), and resistance (R).

We will follow conventional current, in which we assume that positive charges are moving through the positive terminal of a battery. (In reality, electrons move through the negative terminal of a battery).

Resistance measures the opposition to the flow of current. A high resistance significantly diminishes current, rapidly transforming electrical energy into heat energy.

The relationship between potential difference, resistance, and current is linear, as given by *Ohm's Law*:

$$\Delta V = IR\,. \tag{2}$$

3.2 Resistors in Series

Resistors are *in series* if they are placed in a single path. Thus, current only flows through this one path. It then follows that I is constant between resistors in series. See Figure 3, in which two resistors of resistances R_1 and R_2 are placed in series. From Ohm's Law (Equation (2)), the potential difference across the R_1 resistor (the drop in potential from point A to point B) is

$$\Delta V_1 = IR_1\,.$$

Similarly, the potential difference across the R_2 resistor (the drop in potential from point B to point C) is

$$\Delta V_2 = IR_2 \,.$$

The potential difference from point A to point C, which we will denote ΔV_{eq}, is the sum of ΔV_1 and ΔV_2. We then have

$$\begin{aligned} \Delta V_{\text{eq}} &= \Delta V_1 + \Delta V_2 \\ IR_{\text{eq}} &= IR_1 + IR_2 \\ R_{\text{eq}} &= R_1 + R_2 \,. \end{aligned}$$

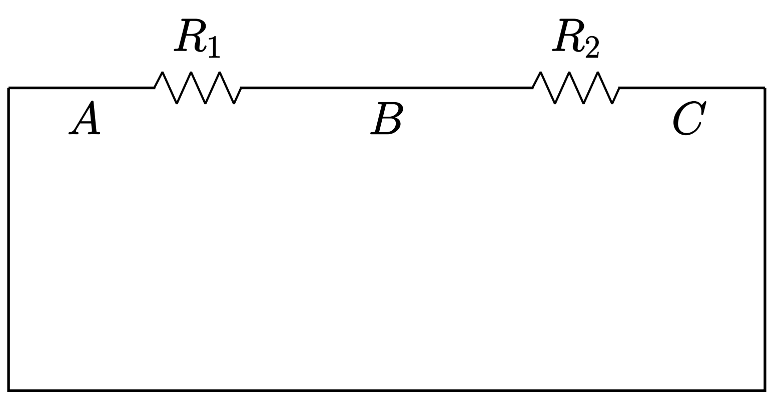

FIGURE 3: Resistors of resistances R_1 and R_2 are placed in series.

For a system of n resistors of resistances $R_1, R_2, R_3, \ldots, R_n$, the pattern generalizes to

$$R_{\text{eq}} = R_1 + R_2 + R_3 + \cdots + R_n \,, \tag{3}$$

which shows that if we have resistors in series, we can add their resistances to find the equivalent resistance of the circuit.

3.3 Resistors in Parallel

We now discuss the equivalent resistance for resistors placed in parallel. Resistors are said to be in parallel if they are in branches connected by the same wire pathway. Consider Figure 4, in which resistors of resistances R_1 and R_2 are placed in parallel.

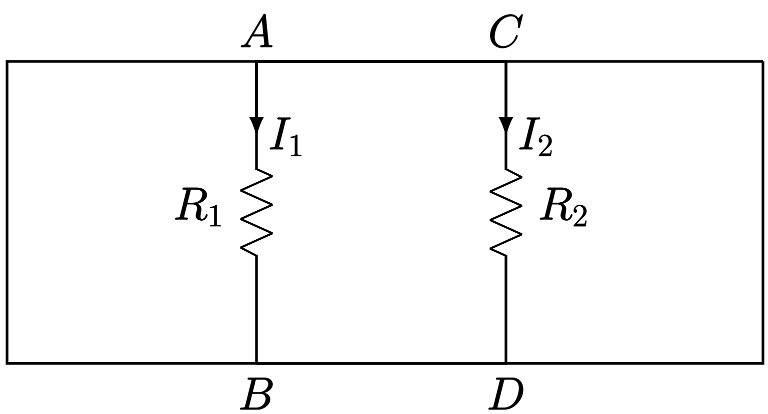

FIGURE 4: Resistors of resistances R_1 and R_2 are placed in parallel.

For resistors in parallel, the potential difference across each branch is equivalent. In Figure 4, the potential difference across points A and B (ΔV_{AB}) is equal to the potential difference across points C and D (ΔV_{CD}). In contrast, current varies across each branch. Let us denote the current that runs through the R_1 resistor to be I_1 and the current that runs through the R_2 resistor to be I_2. The total current of the circuit, I_{eq}, is therefore $I_1 + I_2$. The equivalent circuit to Figure 4 has resistance R_{eq}, potential difference ΔV_{eq}, and current I_{eq}. It then follows that

$$\begin{aligned} I_{\text{eq}} &= I_1 + I_2 \\ \frac{\Delta V_{\text{eq}}}{R_{\text{eq}}} &= \frac{\Delta V_{AB}}{R_1} + \frac{\Delta V_{CD}}{R_2} \, . \end{aligned}$$

But remember that $\Delta V_{\text{eq}} = \Delta V_{AB} = \Delta V_{CD}$. Thus, we have

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2}\,.$$

Similar to Equation (3), the pattern of reciprocating applies for any quantity of resistors in parallel. For a system of resistors of resistances $R_1, R_2, R_3, \ldots, R_n$, the expression involving equivalent resistance is

$$\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots + \frac{1}{R_n}\,. \tag{4}$$

4 Capacitors

4.1 Capacitance

Let us define *capacitance* and understand its interpretations. A *capacitor* is a device that stores charge, and capacitance (denote C) is defined as the ability of a capacitor to store charge. A capacitor that holds charge Q on the positive plate, when hooked up to a circuit, features a potential drop ΔV across its terminals. We relate these quantities as

$$Q = C\Delta V\,, \tag{5}$$

where C is interpreted as a constant of proportionality.

4.2 Capacitors in Series

Capacitors in series run through one line of current. Thus, each capacitor has the same charge on its positive plate. In contrast, the potential difference across each capacitor in series differs. Consider Figure 5, in which capacitors with capacitances C_1 and C_2 are placed in series. The potential difference across points A to C (ΔV_{AC}) is the sum of the potential difference across points A and B (ΔV_{AB}) and the potential difference across points B and C (ΔV_{BC}). From Equation (5), we have

$$Q = C\Delta V \implies \Delta V = \frac{Q}{C}.$$

Let us the charge on each capacitor to be Q_1 and Q_2 and the equivalent charge to be Q_{eq}. Therefore, we get

$$\begin{aligned}\Delta V_{AC} &= \Delta V_{AB} + \Delta V_{BC}\\ \frac{Q_{\text{eq}}}{C_{\text{eq}}} &= \frac{Q_1}{C_1} + \frac{Q_2}{C_2}.\end{aligned}$$

Remember, however, that $Q_{\text{eq}} = Q_1 = Q_2$. Thus, we get

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}.$$

The general pattern for a system of capacitors of capacitances C_1, C_2, C_3, …, C_n in series is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_n}. \tag{6}$$

An analysis of Equation (6) reveals that the equivalent capacitance is always less than the capacitance of any of the capacitors in the circuit.

4.3 Capacitors in Parallel

Each capacitor in parallel exhibits the same potential difference. In Figure 6, capacitors of capacitors C_1 and C_2 (with charges Q_1 and Q_2, respectively) are hooked in parallel. The voltage drop across the C_1 capacitor, ΔV_{AB}, is equal to the voltage drop across the C_2 capacitor, ΔV_{CD}. Conversely, the charge on each capacitor differs. The equivalent circuit features charge Q_{eq}, capacitance C_{eq}, and potential difference ΔV_{eq}. Note that the equivalent charge is the sum of the charges on all capacitors. Thus, we have

$$\begin{aligned}Q_{\text{eq}} &= Q_1 + Q_2\\ C_{\text{eq}}\Delta V_{\text{eq}} &= C_1\Delta V_{AB} + C_2\Delta V_{CD}.\end{aligned}$$

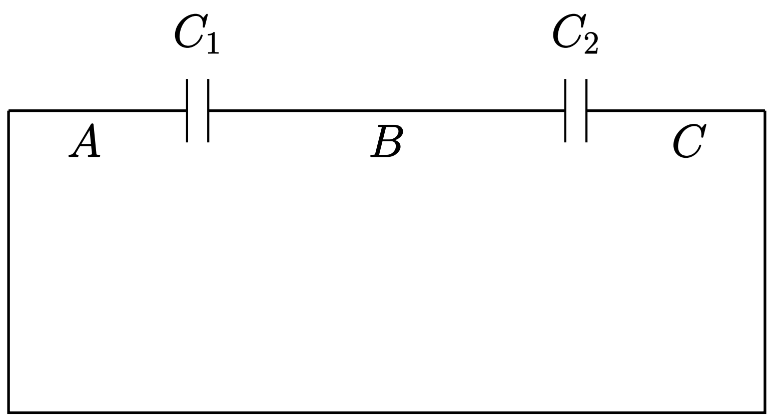

FIGURE 5: Capacitors of capacitances C_1 and C_2 are placed in series. ΔV_{AB} is the potential difference across the C_1 capacitor, and ΔV_{BC} is the potential difference across the C_2 capacitor. The C_1 capacitor holds charge Q_1, and the C_2 capacitor holds charge Q_2.

Remember, however, that $\Delta V_{\text{eq}} = \Delta V_{AB} = \Delta V_{CD}$. Therefore, we find

$$C_{\text{eq}} = C_1 + C_2\,,$$

the general form of which is

$$C_{\text{eq}} = C_1 + C_2 + C_3 + \cdots + C_n\,. \tag{7}$$

5 Inductors

5.1 Inductance

An *inductor* is a device that generates an electromotive force (emf, $\mathscr{E}$) in opposition to a change in current. *Inductance*, L, is the measure of an inductor to generate an opposing emf. We define

$$\mathscr{E} = L\frac{\mathrm{d}I}{\mathrm{d}t}\,. \tag{8}$$

This opposing emf can be thought of as a voltage drop. Throughout these sections, we will use the notations $\mathscr{E}$ and ΔV interchangeably to denote the potential difference across an

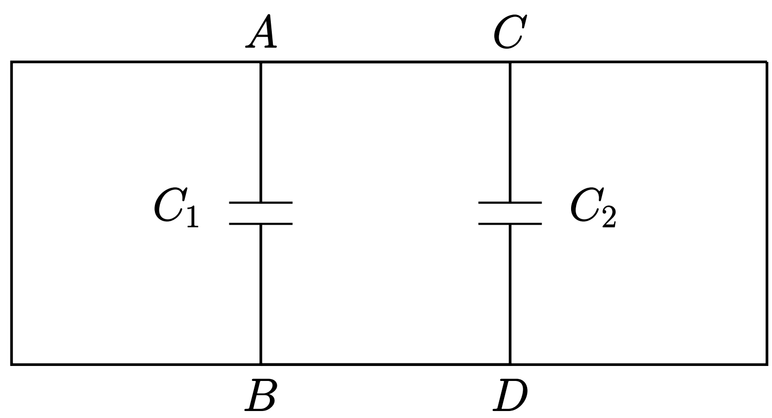

FIGURE 6: Capacitors of capacitances C_1 and C_2 are placed in parallel. The potential difference between points A and B is ΔV_{AB}, and the potential difference between points C and D is ΔV_{CD}.

inductor.

5.2 Inductors in Series

Because single-path circuits have the same current, inductors in series experience the same change in current. Figure 7 shows a circuit with two inductors of inductances L_1 and L_2.

The potential difference across the L_1 inductor, ΔV_1, is

$$\mathscr{E}_1 = L_1 \frac{\mathrm{d}I}{\mathrm{d}t} \,.$$

Likewise, the potential difference across the L_2 inductor, ΔV_2, is

$$\mathscr{E}_2 = L_2 \frac{\mathrm{d}I}{\mathrm{d}t} \,.$$

The total potential difference across all inductors is $\mathscr{E}_{\mathrm{eq}}$, the sum of $\mathscr{E}_1$ and $\mathscr{E}_2$:

$$\begin{aligned} \mathscr{E}_{\mathrm{eq}} &= \mathscr{E}_1 + \mathscr{E}_2 \\ L_{\mathrm{eq}} \frac{\mathrm{d}I}{\mathrm{d}t} &= L_1 \frac{\mathrm{d}I}{\mathrm{d}t} + L_2 \frac{\mathrm{d}I}{\mathrm{d}t} \,, \end{aligned}$$

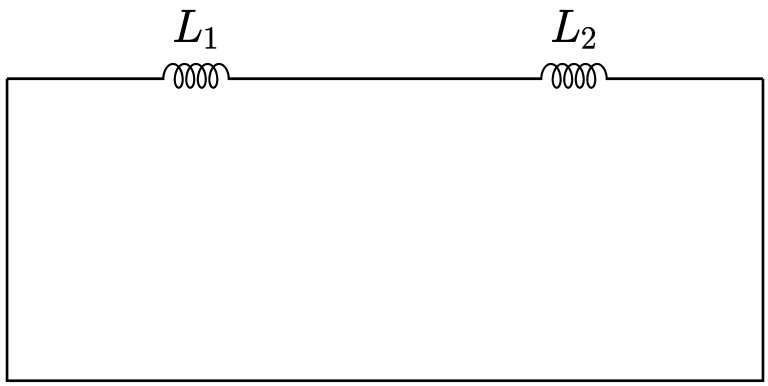

FIGURE 7: Inductors of inductances L_1 and L_2 are placed in series. The voltage drop across the L_1 inductor, ΔV_1, is the emf generated by the L_1 inductor. Likewise, the emf generated by the L_2 inductor is the voltage drop across it.

from which we find

$$L_{\text{eq}} = L_1 + L_2 \,.$$

The general pattern is

$$L_{\text{eq}} = L_1 + L_2 + L_3 + \cdots + L_n \,. \tag{9}$$

5.3 Inductors in Parallel

In this section, we derive the expression for the equivalent inductance of inductors placed in parallel. Figure 8 shows a circuit with inductors of inductances L_1 and L_2 placed in parallel.

The total current in Figure 8 is the equivalent current, given by $I_1 + I_2$. Thus, we have

$$\begin{aligned} I_{\text{eq}} &= I_1 + I_2 \\ \frac{\mathrm{d}I_{\text{eq}}}{\mathrm{d}t} &= \frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{\mathrm{d}I_2}{\mathrm{d}t} \,. \end{aligned}$$

From Equation (8),

$$\mathcal{E} = L\frac{\mathrm{d}I}{\mathrm{d}t} \implies \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathcal{E}}{L} \,.$$

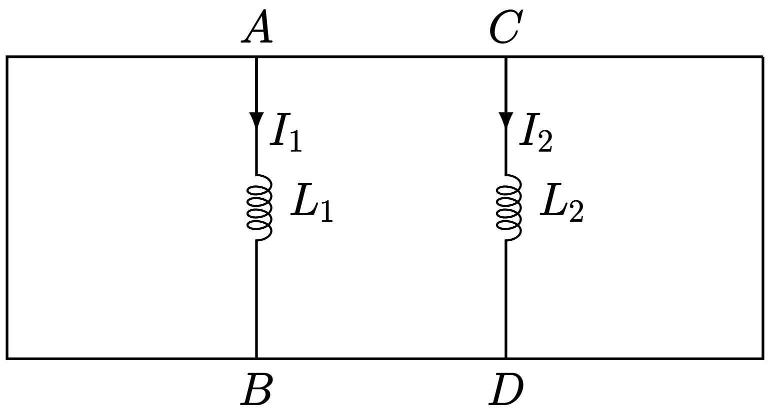

FIGURE 8: Inductors of inductances L_1 and L_2 are placed in parallel. The potential difference between points A and B is $\Delta V_{AB} = \mathscr{E}_1$. Likewise, the voltage drop between points C and D is $\Delta V_{CD} = \mathscr{E}_2$.

Thus, we have

$$\frac{\mathscr{E}_{\text{eq}}}{L_{\text{eq}}} = \frac{\mathscr{E}_1}{L_1} + \frac{\mathscr{E}_2}{L_2}\,.$$

But $\mathscr{E}_{\text{eq}} = \mathscr{E}_1 = \mathscr{E}_2$, so we get

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2}\,,$$

with the general pattern being

$$\frac{1}{L_{\text{eq}}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_n}\,. \tag{10}$$