Robust minimization

Non-exhaustive list of “Robust” Objectives

- Worst case [?]:

$$\min_{\theta \in \Theta} \left\{ \max_{u \in \mathbb{U}} J(\theta, u) \right\}$$

- M-robustness [?]:

$$\min_{\theta \in \Theta} \mathbb{E}_U \left[J(\theta, U) \right]$$

- V-robustness [?]:

$$\min_{\theta \in \Theta} \mathbb{V}\mathrm{ar}_U \left[J(\theta, U) \right]$$

- Multiobjective [?]:

Pareto frontier

- Best performance achievable given $u \sim U$

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$$\theta^*(U) = \underset{\theta \in \Theta}{\arg\min}\, J(\theta, U)$$

$\longrightarrow$ estimate its density (how often is the value θ a minimizer)

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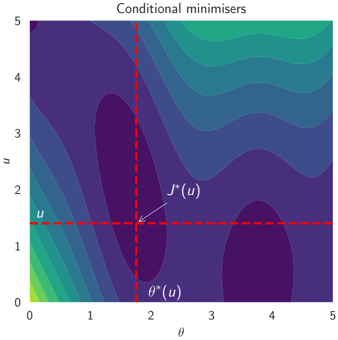
How to take into account values not optimal, but not too far either
$\longrightarrow$ relaxation of the equality with $\alpha > 1$:

$$\Gamma_\alpha(\theta) = \mathbb{P}_U\left[J(\theta, U) \leq \alpha J^*(U)\right]$$

Illustration



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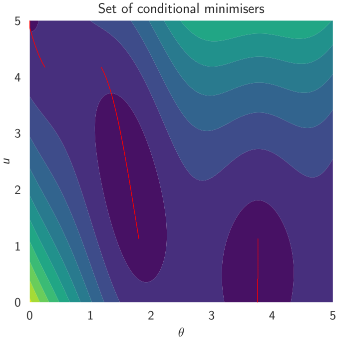

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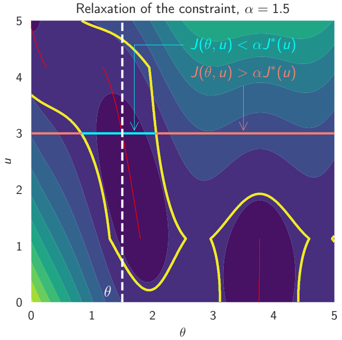

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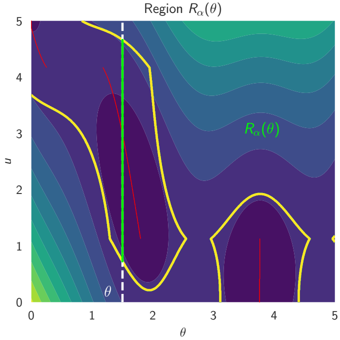


- Sample $u \sim U$, and solve $\theta^*(u) = \arg\min_{\theta \in \Theta} J(\theta, u)$
- Set of conditional minimisers: $\{(\theta^*(u), u) \mid u \in \mathbb{U}\}$
- Set $\alpha \geq 1$
- $R_\alpha(\theta) = \{u \mid J(\theta, u) \leq \alpha J^*(u)\}$
- $\Gamma_\alpha(\theta) = \mathbb{P}_U\left[U \in R_\alpha(\theta)\right]$

Getting an estimator

$\Gamma_\alpha(\theta)$: probability that the cost (thus θ) is α-acceptable

- If α known, maximize the probability that θ gives acceptable values:

$$\max_{\theta\in\Theta} \Gamma_\alpha(\theta) = \max_{\theta\in\Theta} \mathbb{P}_U\left[J(\theta, U) \leq \alpha J^*(U)\right] \tag{1}$$

- Set a target probability $1-\eta$, and find the smallest α.

$$\inf\{\alpha \mid \max_{\theta\in\Theta} \Gamma_\alpha(\theta) \geq 1-\eta\} \tag{2}$$

Relative-regret family of estimators

$$\left\{\hat{\theta} \mid \hat{\theta} = \arg\max_{\theta\in\Theta} \Gamma_\alpha(\theta), \alpha > 1\right\} \tag{3}$$

Interpretation

If we either set α or η

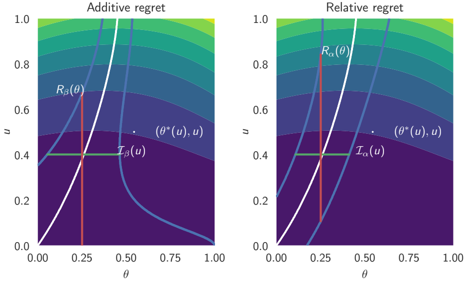
$$\hat{\theta} = \arg\max \Gamma_\alpha \tag{4}$$

$$\max \Gamma_\alpha = \Gamma_\alpha(\hat{\theta}) = 1 - \eta \tag{5}$$

The maximal *relative regret* J/J^* of the function will be α, except for the $100\eta\%$ least favourable cases.

- α and η

Why the relative regret ?



- Relative regret
  - α-acceptability regions large for flat and bad situations ($J^*(u)$ large)
  - Conversely, puts high confidence when $J^*(u)$ is small
  - No units $\rightarrow$ ratio of costs