Vérification formelle en Isabelle(HOL) d'un algorithme calculant les composantes fortement connexes d'un graphe

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Dépôt git

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- Review de code en équipe (GitHub, etc.)
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$\Longrightarrow$ Responsabilité du programmeur !

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Kenneth Appel

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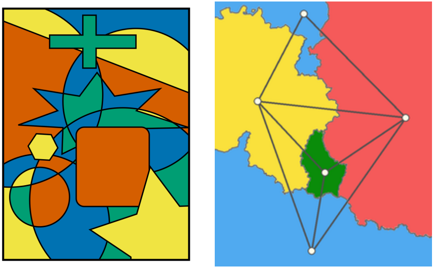

Figure: Illustrations du problème 4-COL

En méthodes formelles, on considère un programme comme une structure mathématique, ce qui permet de raisonner dessus de manière formelle.

- Rigueur
- Concepts de logique
- Validité par rapport à des spécifications
- Sémantique

Preuve papier vs. preuve formelle

- L'utilisateur fournit la structure de la preuve
- Automatisation des preuves
- Différents outils informatiques : assistants à la preuve (Isabelle(HOL), Coq, Why3, B, etc.) et model checkers (TLC, etc.)

Quelles applications ?

Éviter les “bugs” : applications aux systèmes critiques

Ex: Ariane 5, Entreprise Clearsy

Application au milieu ferroviaire : Certification du logiciel de pilotage automatique des lignes 1 et 14 de métro à Paris, bientôt la ligne 4

Composante fortement connexe

Definition 1 (SCC)

Soient $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ un graphe orienté et $\mathcal{C} \subseteq \mathcal{V}$. $\mathcal{C}$ est une SCC de $\mathcal{G}$ si:

$$\forall x, y \in \mathcal{C}, (x \Rightarrow^* y) \wedge (y \Rightarrow^* x)$$

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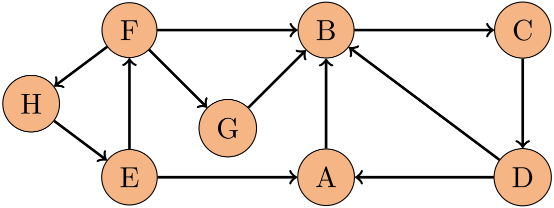

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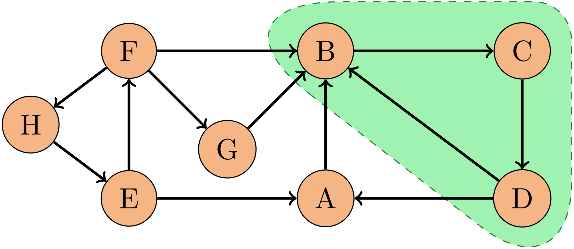

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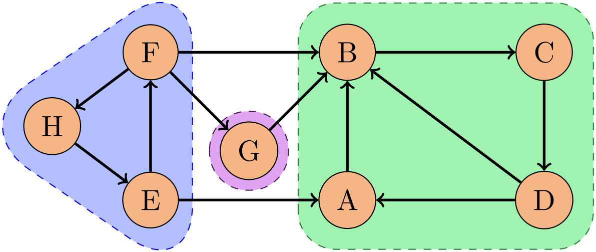

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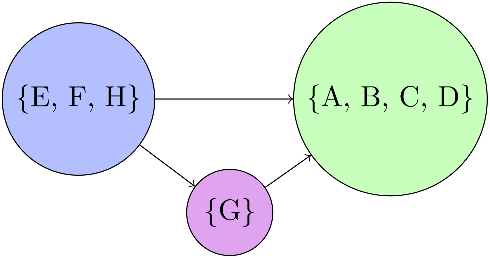
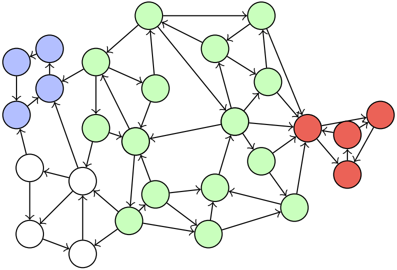
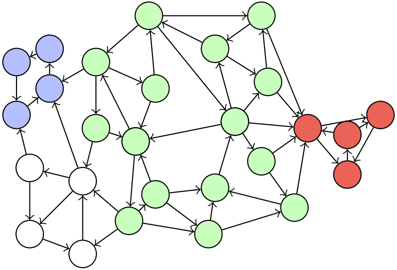




- Réseaux: interconnectivité et partage de données
- Model checking: recherche de contre-exemples



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Algorithmes efficaces (ex: Tarjan)

- La vérification formelle de leur correction est utile
- Un autre défi : la parallélisation

```
Data: A graph G = (V , E ), a starting node v0;
Initialize an empty set VISITED;
Initialize an empty set EXPLORED;
Initialize an empty stack R;
setBased(v0);
function setBased: v ∈ V → None
    VISITED := VISITED ∪ {v};
    R.push(v);
    foreach w ∈ POST(v) do
        if w ∈ EXPLORED then
            continue;
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            setBased(w);
        else
            while S(v) ≠ S(w) do
                r := R.pop();
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    if v = R.top() then
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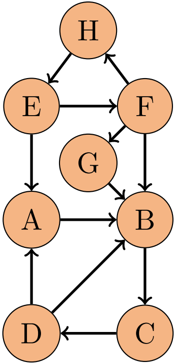

$$\mathrm{R} = []$$

$$\mathrm{VISITED} = \{\}$$

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$$\mathcal{S} = \{A\} \cup \{B\} \cup \{C\} \cup \{D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$$

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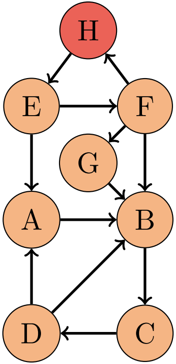

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$$VISITED = \{H\}$$

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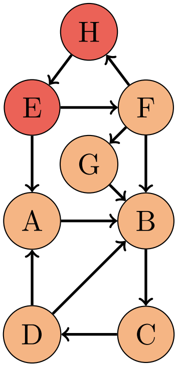

$$\mathrm{R} = [\mathrm{H}, \mathrm{E}]$$

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function *setBased:* $v \in \mathcal{V} \rightarrow$ *None*
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  R.push(v);
  foreach $w \in$ *POST(v)* **do**
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      continue;
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        r := R.pop();
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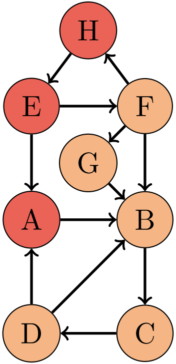

$$R = [H, E, A]$$
$$VISITED = \{H, E, A\}$$
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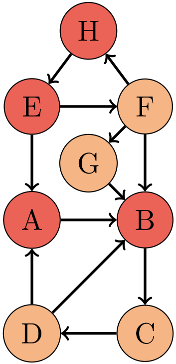

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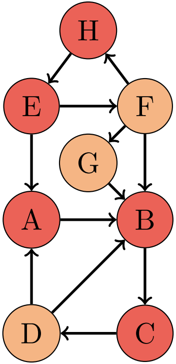

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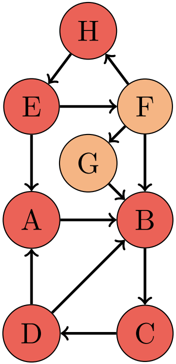

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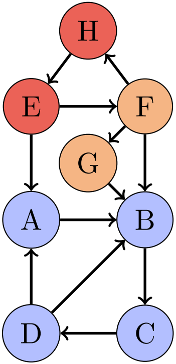

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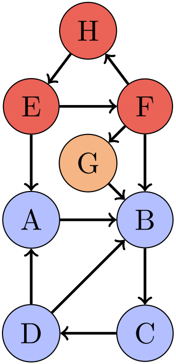

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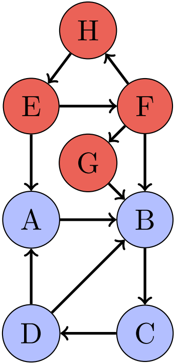

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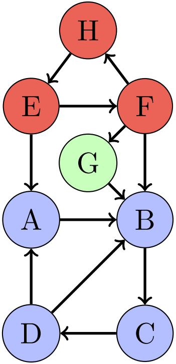

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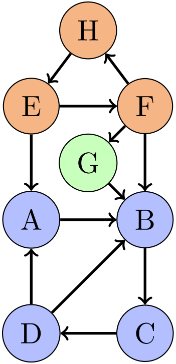

$$R = []$$

$$\text{VISITED} = \{H, E, A, B, C, D, F, G\}$$

$$\text{EXPLORED}=\{A, B, C, D, G, E, F, H\}$$

$$\mathcal{S} = \{A, B, C, D\} \cup \{E, F, H\} \cup \{G\}$$

- Modélisation et implémentation de l'algorithme
  - Définition de l'environnement
  - Fonctions `unite`, `dfs` et `dfss`
- Définition des invariants
  - définitions : `reachable` et `is_scc`
  - well-formed environment
  - pré-conditions et post-conditions sur `dfs` et `dfss`
- Écriture et preuve des lemmes
  - `pre_dfs` $\Longrightarrow$ `pre_dfss`
  - `pre_dfss` $\Longrightarrow$ `pre_dfs`
  - `pre_dfs` $\Longrightarrow$ `post_dfs`
  - `pre_dfss` $\Longrightarrow$ `post_dfss`
  - théorème final

- Ce qui a été fait :
  - Modèle cohérent et stable
  - Preuve de tous les lemmes intermédiaires (ou presque)
  - Implémentation de l'algorithme : environ 10 lignes de code
  - Preuve : presque 2000 lignes de code

- Ce qui manque :
  - Quelques propriétés intermédiaires sont à démontrer
  - Théorème final
  - Montrer la terminaison