Verification in HOL of an algorithm for computing SCCs

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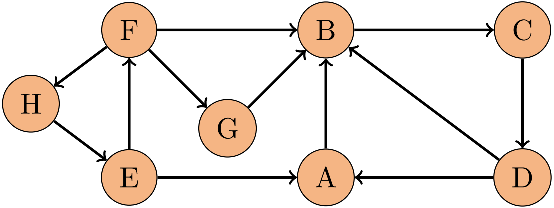
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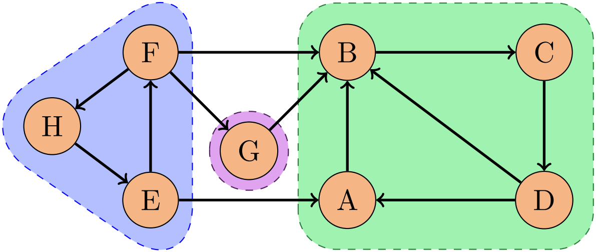
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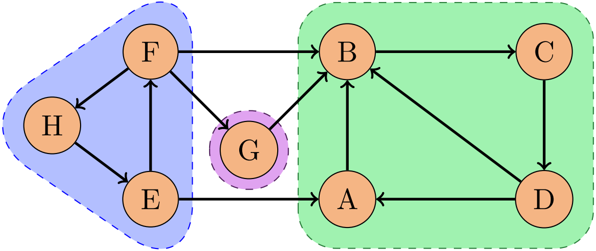
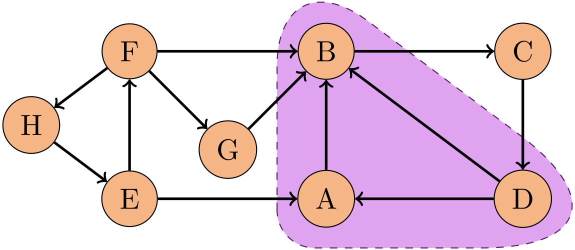


Definition 1

Let $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ be a directed graph and $\mathcal{C} \subseteq \mathcal{V}$. $\mathcal{C}$ is a SCC of $\mathcal{G}$ if:

$$\forall x, y \in \mathcal{C}, (x \Rightarrow y) \wedge (y \Rightarrow x)$$



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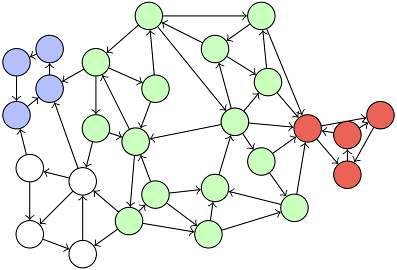
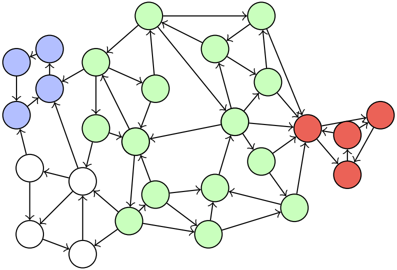


- Networks: connection and data sharing
- Model checking: counter-examples finding



- Networks: connection and data sharing
- Model checking: counter-examples finding

Efficient algorithms (ex: Tarjan)

- Formal verification of correctness is worthwhile
- Parallelization is another challenge

Isabelle / HOL

- Generic proof assistant
- Formalisation of mathematical proofs
- Higher-Order Logic theorem proving environment
- Powerful proof tools and language (Isar)
- Mutual induction, recursion and datatypes, complex pattern matching

(Type definition)

```
datatype 'a list = Empty | Cons 'a "'a list"
```

- Generic / polymorphic and static type
- Implicit constructor definition
- Recursive structure giving an induction principle for that type

(Function definition)

```
fun concat :: "'a list ⇒ 'a list ⇒ 'a list" where
  "concat Empty xs = xs"
| "concat (Cons x xs) ys = Cons x (concat xs ys)"
```

```
fun rev :: "'a list ⇒ 'a list" where
  "rev Empty = Empty"
| "rev (Cons x xs) = concat (rev xs) (Cons x Empty)"
```

(Theorem statement)

```
theorem rev_rev [simp]:  "rev (rev x) = x"
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(Subgoal)

```
⋀ x1 x.
  rev (rev x) = x ⟹
  rev (concat (rev x) (Cons x1 Empty)) = Cons x1 x
```

(Adding a first lemma)

```
 lemma rev_concat [simp]:
"rev (concat xs ys) = concat (rev ys) (rev xs)"
   apply (induction xs)
   apply auto
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 lemma rev_concat [simp]:
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   apply (induction xs)
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(Subgoals)

```
1.  rev ys = concat (rev ys) Empty
2.  ⋀ x1 xs.
  rev (concat xs ys) = concat (rev ys) (rev xs) ⟹
  rev (concat (Cons x1 xs) ys) = concat (rev ys) (rev (Cons x1 xs))
```

(Adding a second lemma)

```
lemma concat_empty [simp]:  "concat xs Empty = xs"
  apply (induction xs)
  apply auto
```

(Adding a third lemma: associative property)

```
lemma concat_assoc [simp]:  "concat (concat xs ys) zs =
concat xs (concat ys zs)"
  apply (induction xs)
  apply auto
```

```
theorem rev_rev [simp]:  "rev (rev x) = x"
  apply (induction x)
  apply auto
```

No subgoals!

1. Introduction
   - Definition
   - Motivation

2. Example of the proof process

3. A sequential set-based SCC algorithm
   - Description of the algorithm
   - Implementation in Isabelle

```
Data: A graph 𝒢 = (𝒱, ℰ), a starting node v0;
Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v0);
function setBased: v ∈ 𝒱 → None
    VISITED := VISITED ∪ {v};
    R.push(v);
    foreach w ∈ POST(v) do
        if w ∈ DEAD then
            continue;
        else if w ∉ VISITED then
            setBased(w);
        else
            while 𝒮(v) ≠ 𝒮(w) do
                r := R.pop();
                UNITE(𝒮, r, R.top());
    if v = R.top() then
        report SCC 𝒮(v);
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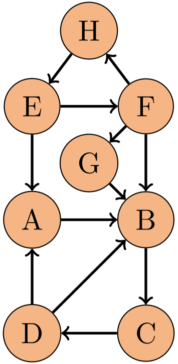

$$R = \{\}$$
$$VISITED = \{\}$$
$$DEAD = \{\}$$

$\mathcal{S} = \{A\} \cup \{B\} \cup \{C\} \cup \{D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$

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        else
            while $\mathcal{S}(v) \neq \mathcal{S}(w)$ **do**
                r := `R.pop()`;
                `UNITE(`$\mathcal{S}, r,$ `R.top())`;
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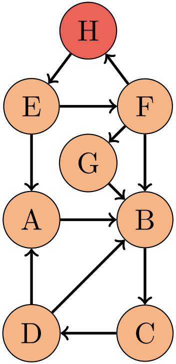

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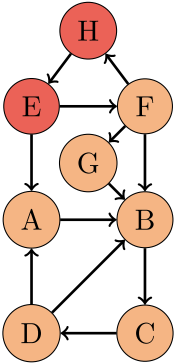

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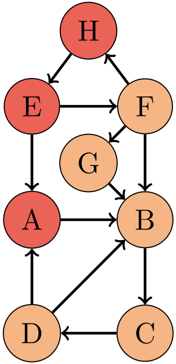

$$R = \{H, E, A\}$$
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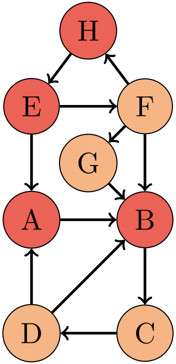

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        else
            while S(v) ≠ S(w) do
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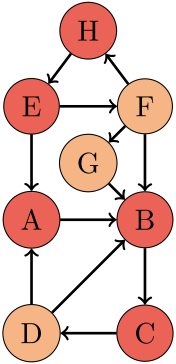

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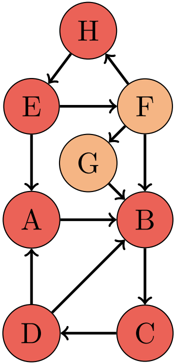

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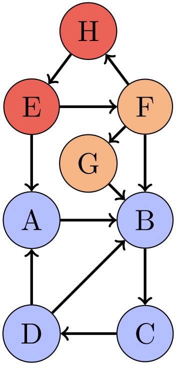

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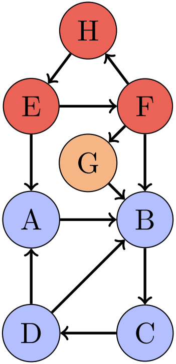

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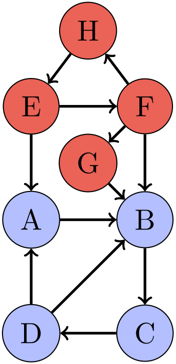

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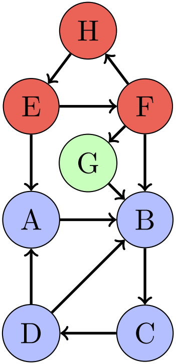

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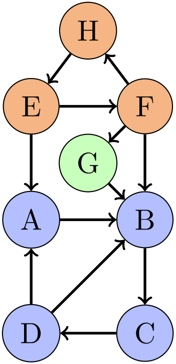

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$$\mathcal{S} = \{A, B, C, D\} \cup \{E, F, H\} \cup \{G\}$$

(Finite directed graphs)

```
locale graph =
  fixes vertices :: "'v set"
  and successors :: "'v ⇒ 'v set"
  assumes vfin: "finite vertices"
  and sclosed: "∀ x ∈ vertices. successors x ⊆ vertices"
```

```
abbreviation edge where
"edge x y ≡ y ∈ successors x"
```

```
inductive reachable where
  reachable_refl[iff]: "reachable x x"
| reachable_succ[elim]:
    "⟦edge x y; reachable y z⟧ ⟹ reachable x z"
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(SCC)

```
definition is_subscc where
  "is_subscc S ≡ ∀x ∈ S. ∀y ∈ S. reachable x y"
```

(Maximal SCC)

```
definition is_scc where
  "is_scc S ≡ S ≠ {}
∧ is_subscc S
∧ (∀ S'. S ⊆ S' ∧ is_subscc S' ⟶ S' = S)"
```

Proof process

(Well-formed environment)

```
definition wf_env where
  "wf_env e ≡"
     distinct (stack e)
  ∧ set (stack e) ⊆ visited e
  ∧ explored e ⊆ visited e
  ∧ explored e ∩ set (stack e) = {}
  ∧ (∀v w.  w ∈ 𝒮 e v ⟷ 𝒮 e v = 𝒮 e w)
  ∧ (∀v ∈ set(stack e).  ∀w ∈ set(stack e).  v≠w ⟶
       𝒮 e v ∩ 𝒮 e w = {})
  ∧ (∀v.  v ∉ visited e ⟶ 𝒮 e v = {v})
  ∧ ⋃ {𝒮 e v | v.  v ∈ set(stack e)} = visited e - explored e
```

```
definition pre_dfs where
  "pre_dfs v e ≡ wf_env e ∧ v ∉ visited e"
definition post_dfs where "post_dfs v e ≡ wf_env e"
```

```
definition pre_dfss where "pre_dfs v vs e ≡ wf_env e"
definition post_dfss where "post_dfs v vs e ≡ wf_env e"
```

```
lemma pre_dfs_pre_dfss:
  assumes "pre_dfs v e"
  shows "pre_dfss v (successors v) (e(|visited:=visited e ∪ {v},
stack:= v # stack e|))"
```

```
lemma pre_dfss_pre_dfs:
  fixes w
  assumes "pre_dfss v vs e" and "w ∉ visited e"
  shows "pre_dfs w e"
```

```
lemma pre_dfs_implies_post_dfs:...
```

```
lemma pre_dfss_implies_post_dfss:...
```

Possible prospects

- Finish the entire proof (with termination and functions domains)
- Parallel algorithm for computing SCC ? Proof ?