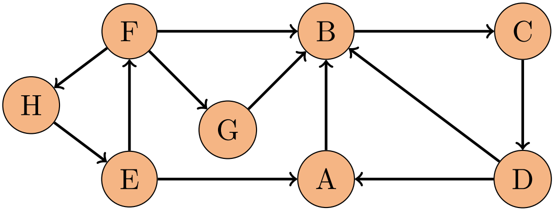
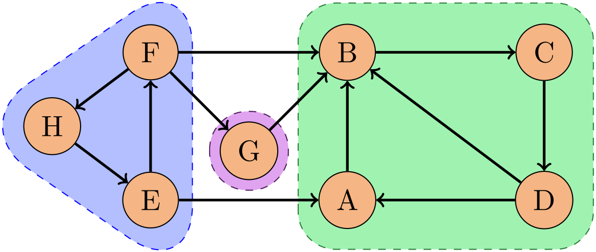
Verification in HOL of an algorithm for computing SCCs

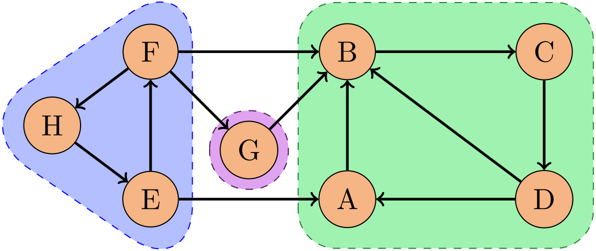
Vincent Trélat

École Nationale Supérieure des Mines de Nancy
Département Informatique

January 14, 2022


F
B
C
H
G
E
A
D

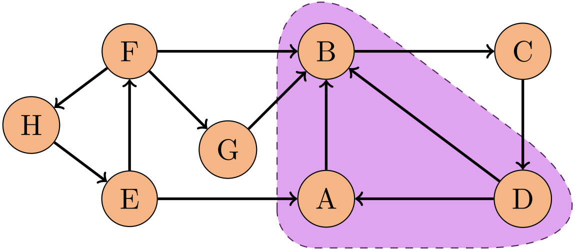

F
B
C
H
G
E
A
D



Definition 1

Let $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ be a directed graph and $\mathcal{C} \subseteq \mathcal{V}$. $\mathcal{C}$ is a SCC of $\mathcal{G}$ if:

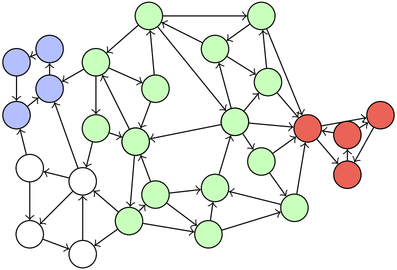
$$\forall x, y \in \mathcal{C}, (x \Rightarrow y) \wedge (y \Rightarrow x)$$



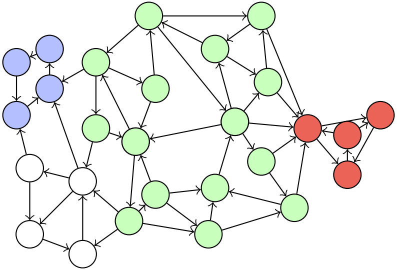
Definition 1

Let $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ be a directed graph and $\mathcal{C} \subseteq \mathcal{V}$. $\mathcal{C}$ is a SCC of $\mathcal{G}$ if:

$$\forall x, y \in \mathcal{C}, (x \Rightarrow y) \land (y \Rightarrow x)$$



- Networks: connection and data sharing
- Model checking: counter-examples finding



- Networks: connection and data sharing
- Model checking: counter-examples finding

Efficient algorithms (ex: Tarjan)

- Formal verification of correctness is worthwhile
- Parallelization is another challenge

Isabelle / HOL

- Generic proof assistant
- Formalisation of mathematical proofs
- Higher-Order Logic theorem proving environment
- Powerful proof tools and language (Isar)
- Mutual induction, recursion and datatypes, complex pattern matching

(Type definition)

```
datatype 'a list = Empty | Cons 'a "'a list"
```

- Generic / polymorphic and static type
- Implicit constructor definition
- Recursive structure giving an induction principle for that type

(Function definition)

```
fun concat ::  "'a list ⇒ 'a list ⇒ 'a list" where
  "concat Empty xs = xs"
| "concat (Cons x xs) ys = Cons x (concat xs ys)"
```

```
fun rev ::  "'a list ⇒ 'a list" where
  "rev Empty = Empty"
| "rev (Cons x xs) = concat (rev xs) (Cons x Empty)"
```

(Theorem statement)

```
theorem rev_rev [simp]:  "rev (rev x) = x"
```

(Theorem statement)

```
theorem rev_rev [simp]:  "rev (rev x) = x"
 apply (induction x)
 apply auto
```

(Theorem statement)

```
theorem rev_rev [simp]:  "rev (rev x) = x"
 apply (induction x)
 apply auto
```

(Subgoal)

```
⋀ x1 x.
  rev (rev x) = x ⟹
  rev (concat (rev x) (Cons x1 Empty)) = Cons x1 x
```

(Adding a first lemma)

```
 lemma rev_concat [simp]:
"rev (concat xs ys) = concat (rev ys) (rev xs)"
   apply (induction xs)
   apply auto
```

(Adding a first lemma)

```
lemma rev_concat [simp]:
"rev (concat xs ys) = concat (rev ys) (rev xs)"
   apply (induction xs)
   apply auto
```

(Subgoals)

```
1.  rev ys = concat (rev ys) Empty
2.  ⋀ x1 xs.
  rev (concat xs ys) = concat (rev ys) (rev xs) ⟹
  rev (concat (Cons x1 xs) ys) = concat (rev ys) (rev (Cons x1 xs))
```

(Adding a second lemma)

```
lemma concat_empty [simp]:  "concat xs Empty = xs"
  apply (induction xs)
  apply auto
```

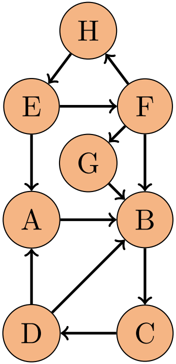
(Adding a third lemma: associative property)

```
lemma concat_assoc [simp]:  "concat (concat xs ys) zs =
concat xs (concat ys zs)"
  apply (induction xs)
  apply auto
```

```
theorem rev_rev [simp]:  "rev (rev x) = x"
  apply (induction x)
  apply auto
```

No subgoals!

```
Data: A graph 𝒢 = (𝒱, ℰ), a starting node v0;
Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v0);
function setBased: v ∈ 𝒱 → None
    VISITED := VISITED ∪ {v};
    R.push(v);
    foreach w ∈ POST(v) do
        if w ∈ DEAD then
            continue;
        else if w ∉ VISITED then
            setBased(w);
        else
            while 𝒮(v) ≠ 𝒮(w) do
                r := R.pop();
                UNITE(𝒮, r, R.top());
    if v = R.top() then
        report SCC 𝒮(v);
        DEAD := DEAD ∪ 𝒮(v);
        R.pop();
```



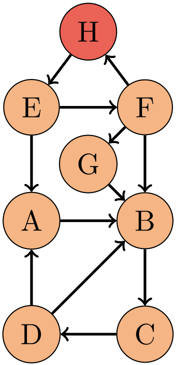
$$\mathrm{R} = \{\}$$

$$\mathrm{VISITED} = \{\}$$

$$\mathrm{DEAD} = \{\}$$

$$\mathcal{S} = \{A\} \cup \{B\} \cup \{C\} \cup \{D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$$

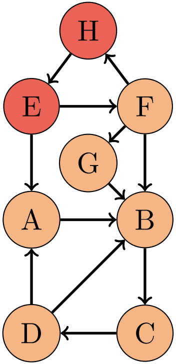
Data: A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a starting node v_0;
Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v_0);
function *setBased*: $v \in \mathcal{V} \to$ *None*
  VISITED := VISITED $\cup \{v\}$;
  R.push(v);
  foreach $w \in$ *POST(v)* **do**
    if $w \in$ *DEAD* **then**
      continue;
    else if $w \notin$ *VISITED* **then**
      setBased(w);
    else
      while $\mathcal{S}(v) \neq \mathcal{S}(w)$ **do**
        r := R.pop();
        UNITE($\mathcal{S}$, r, R.top());
  if v = *R.top()* **then**
    report SCC $\mathcal{S}(v)$;
    DEAD := DEAD $\cup\, \mathcal{S}(v)$;
    R.pop();



$$\mathrm{R} = \{\mathrm{H}\}$$
$$\mathrm{VISITED} = \{\mathrm{H}\}$$
$$\mathrm{DEAD} = \{\}$$

$$\mathcal{S} = \{A\} \cup \{B\} \cup \{C\} \cup \{D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$$

Data: A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a starting node v_0;
Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v_0);
function *setBased*: $v \in \mathcal{V} \rightarrow$ *None*
  VISITED := VISITED $\cup \{v\}$;
  R.push(v);
  foreach $w \in$ *POST(v)* **do**
    if $w \in$ *DEAD* **then**
      continue;
    else if $w \notin$ *VISITED* **then**
      setBased(w);
    else
      while $\mathcal{S}(v) \neq \mathcal{S}(w)$ **do**
        r := R.pop();
        UNITE($\mathcal{S}$, r, R.top());
  if v = *R.top()* **then**
    report SCC $\mathcal{S}(v)$;
    DEAD := DEAD $\cup\, \mathcal{S}(v)$;
    R.pop();



$$R = \{H, E\}$$
$$VISITED = \{H, E\}$$
$$DEAD = \{\}$$

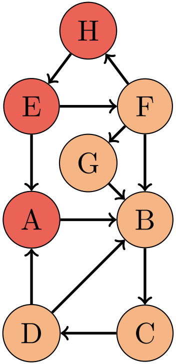
$\mathcal{S} = \{A\} \cup \{B\} \cup \{C\} \cup \{D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$

Data: A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a starting node v_0;

Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v_0);

function *setBased*: $v \in \mathcal{V} \rightarrow None$

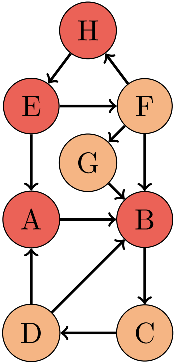
    VISITED := VISITED ∪ {v};
    R.push(v);
    foreach $w \in POST(v)$ **do**
        if $w \in DEAD$ **then**
            continue;
        else if $w \notin VISITED$ **then**
            setBased(w);
        else
            while $\mathcal{S}(v) \neq \mathcal{S}(w)$ **do**
                r := R.pop();
                UNITE($\mathcal{S}$, r, R.top());
    if $v = R.top()$ **then**
        report SCC $\mathcal{S}(v)$;
        DEAD := DEAD ∪ $\mathcal{S}(v)$;
        R.pop();



$$\mathrm{R} = \{\mathrm{H}, \mathrm{E}, \mathrm{A}\}$$
$$\mathrm{VISITED} = \{\mathrm{H}, \mathrm{E}, \mathrm{A}\}$$
$$\mathrm{DEAD} = \{\}$$

$$\mathcal{S} = \{A\} \cup \{B\} \cup \{C\} \cup \{D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$$

```
Data: A graph 𝒢 = (𝒱 , ℰ ), a starting node v0;
Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v0);
function setBased: v ∈ 𝒱 → None
    VISITED := VISITED ∪ {v};
    R.push(v);
    foreach w ∈ POST(v) do
        if w ∈ DEAD then
            continue;
        else if w ∉ VISITED then
            setBased(w);
        else
            while 𝒮(v) ≠ 𝒮(w) do
                r := R.pop();
                UNITE(𝒮, r, R.top());
    if v = R.top() then
        report SCC 𝒮(v);
        DEAD := DEAD ∪ 𝒮(v);
        R.pop();
```



$$R = \{H, E, A, B\}$$

$$VISITED = \{H, E, A, B\}$$

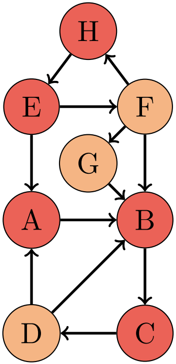
$$DEAD = \{\}$$

$$\mathcal{S} = \{A\} \cup \{B\} \cup \{C\} \cup \{D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$$

Data: A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a starting node v_0;

Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v_0);

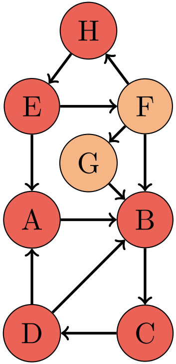
function *setBased*: $v \in \mathcal{V} \to None$
    VISITED := VISITED $\cup \{v\}$;
    R.push(v);
    foreach $w \in POST(v)$ **do**
        if $w \in DEAD$ **then**
            continue;
        else if $w \notin VISITED$ **then**
            setBased(w);
        else
            while $\mathcal{S}(v) \neq \mathcal{S}(w)$ **do**
                r := R.pop();
                UNITE($\mathcal{S}$, r, R.top());
    if $v = R.top()$ **then**
        report SCC $\mathcal{S}(v)$;
        DEAD := DEAD $\cup\, \mathcal{S}(v)$;
        R.pop();



$$\mathrm{R} = \{\mathrm{H, E, A, B, C}\}$$
$$\mathrm{VISITED} = \{\mathrm{H, E, A, B, C}\}$$
$$\mathrm{DEAD} = \{\}$$

$\mathcal{S} = \{A\} \cup \{B\} \cup \{C\} \cup \{D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$

```
Data: A graph 𝒢 = (𝒱 , ℰ ), a starting node v0;
Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v0);
function setBased: v ∈ 𝒱 → None
    VISITED := VISITED ∪ {v};
    R.push(v);
    foreach w ∈ POST(v) do
        if w ∈ DEAD then
            continue;
        else if w ∉ VISITED then
            setBased(w);
        else
            while 𝒮(v) ≠ 𝒮(w) do
                r := R.pop();
                UNITE(𝒮, r, R.top());
    if v = R.top() then
        report SCC 𝒮(v);
        DEAD := DEAD ∪ 𝒮(v);
        R.pop();
```



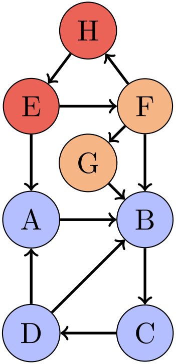
$$R = \{H, E, A, B, C, D\}$$

$$VISITED = \{H, E, A, B, C, D\}$$

$$DEAD = \{\}$$

$$\mathcal{S} = \{A\} \cup \{B\} \cup \{C\} \cup \{D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$$

```
Data: A graph G = (V, E), a starting node v0;
Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v0);
function setBased: v ∈ V → None
    VISITED := VISITED ∪ {v};
    R.push(v);
    foreach w ∈ POST(v) do
        if w ∈ DEAD then
            continue;
        else if w ∉ VISITED then
            setBased(w);
        else
            while S(v) ≠ S(w) do
                r := R.pop();
                UNITE(S, r, R.top());
    if v = R.top() then
        report SCC S(v);
        DEAD := DEAD ∪ S(v);
        R.pop();
```



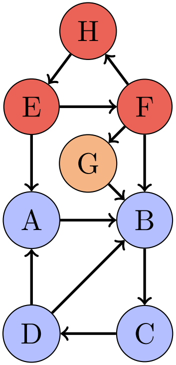
$$\mathrm{R} = \{\mathrm{H}, \mathrm{E}\}$$

$$\mathrm{VISITED} = \{\mathrm{H}, \mathrm{E}, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$$

$$\mathrm{DEAD} = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$$

$$\mathcal{S} = \{A, B, C, D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$$

```
Data: A graph 𝒢 = (𝒱 , ℰ ), a starting node v0;
Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v0);
function setBased: v ∈ 𝒱 → None
    VISITED := VISITED ∪ {v};
    R.push(v);
    foreach w ∈ POST(v) do
        if w ∈ DEAD then
            continue;
        else if w ∉ VISITED then
            setBased(w);
        else
            while 𝒮(v) ≠ 𝒮(w) do
                r := R.pop();
                UNITE(𝒮, r, R.top());
    if v = R.top() then
        report SCC 𝒮(v);
        DEAD := DEAD ∪ 𝒮(v);
        R.pop();
```



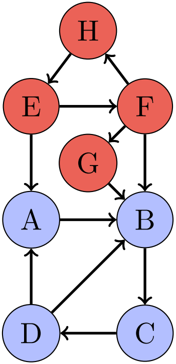
$$R = \{H, E, F\}$$

$$VISITED = \{H, E, A, B, C, D, F\}$$

$$DEAD = \{A, B, C, D\}$$

$$\mathcal{S} = \{A, B, C, D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$$

```
Data: A graph 𝒢 = (𝒱 , ℰ ), a starting node v0;
Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v0);
function setBased: v ∈ 𝒱 → None
    VISITED := VISITED ∪ {v};
    R.push(v);
    foreach w ∈ POST(v) do
        if w ∈ DEAD then
            continue;
        else if w ∉ VISITED then
            setBased(w);
        else
            while 𝒮(v) ≠ 𝒮(w) do
                r := R.pop();
                UNITE(𝒮, r, R.top());
    if v = R.top() then
        report SCC 𝒮(v);
        DEAD := DEAD ∪ 𝒮(v);
        R.pop();
```



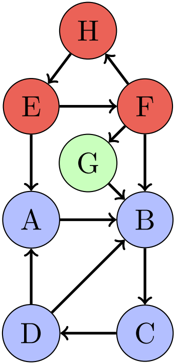
$$R = \{H, E, F, G\}$$

$$VISITED = \{H, E, A, B, C, D, F, G\}$$

$$DEAD = \{A, B, C, D\}$$

$$\mathcal{S} = \{A, B, C, D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$$

Data: A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a starting node v_0;
Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v_0);
function *setBased*: $v \in \mathcal{V} \to$ *None*
    VISITED := VISITED $\cup \{v\}$;
    R.push(v);
    foreach $w \in$ *POST(v)* **do**
        if $w \in$ *DEAD* **then**
            continue;
        else if $w \notin$ *VISITED* **then**
            setBased(w);
        else
            while $\mathcal{S}(v) \neq \mathcal{S}(w)$ **do**
                r := R.pop();
                UNITE($\mathcal{S}$, r, R.top());
    if v = *R.top()* **then**
        report SCC $\mathcal{S}(v)$;
        DEAD := DEAD $\cup\, \mathcal{S}(v)$;
        R.pop();



$$R = \{H, E, F\}$$

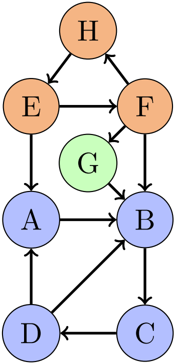
$$VISITED = \{H, E, A, B, C, D, F, G\}$$

$$DEAD = \{A, B, C, D, G\}$$

$$\mathcal{S} = \{A, B, C, D\} \cup \{E\} \cup \{F\} \cup \{G\} \cup \{H\}$$

Data: A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a starting node v_0;

Initialize an empty set DEAD;
Initialize an empty set VISITED;
Initialize an empty stack R;
setBased(v_0);
function *setBased*: $v \in \mathcal{V} \rightarrow$ *None*
  VISITED := VISITED $\cup \{v\}$;
  R.push(v);
  foreach $w \in$ *POST(v)* **do**
    if $w \in$ *DEAD* **then**
      continue;
    else if $w \notin$ *VISITED* **then**
      setBased(w);
    else
      while $\mathcal{S}(v) \neq \mathcal{S}(w)$ **do**
        r := R.pop();
        UNITE($\mathcal{S}$, r, R.top());
  if v = *R.top()* **then**
    report SCC $\mathcal{S}(v)$;
    DEAD := DEAD $\cup\ \mathcal{S}(v)$;
    R.pop();



R = {}

VISITED = {H, E, A, B, C, D, F, G}

DEAD = {A, B, C, D, G, E, F, H}

$\mathcal{S} = \{A, B, C, D\} \cup \{E, F, H\} \cup \{G\}$

(Finite directed graphs)

```
locale graph =
  fixes vertices ::  "'v set"
  and successors ::  "'v ⇒ 'v set"
  assumes vfin:  "finite vertices"
  and sclosed:  "∀ x ∈ vertices. successors x ⊆ vertices"
```

```
abbreviation edge where
"edge x y ≡ y ∈ successors x"
```

```
inductive reachable where
  reachable_refl[iff]:  "reachable x x"
| reachable_succ[elim]:
    "⟦edge x y; reachable y z⟧ ⟹ reachable x z"
```

(Finite directed graphs)

```
locale graph =
  fixes vertices ::  "'v set"
  and successors ::  "'v ⇒ 'v set"
  assumes vfin:  "finite vertices"
  and sclosed:  "∀ x ∈ vertices. successors x ⊆ vertices"
```

```
abbreviation edge where
"edge x y ≡ y ∈ successors x"
```

```
inductive reachable where
  reachable_refl[iff]:  "reachable x x"
| reachable_succ[elim]:
     "⟦edge x y; reachable y z⟧ ⟹ reachable x z"
```

(SCC)

```
definition is_subscc where
  "is_subscc S ≡ ∀x ∈ S. ∀y ∈ S. reachable x y"
```

(Maximal SCC)

```
definition is_scc where
  "is_scc S ≡ S ≠ {}
∧ is_subscc S
∧ (∀ S'. S⊆S'∧ is_subscc S' ⟶ S' = S)"
```

Proof process

```
definition wf_env where
  "wf_env e ≡"
   distinct (stack e)
  ∧ set (stack e) ⊆
```

```
definition wf_env where
  "wf_env e ≡"
    distinct (stack e)
  ∧ set (stack e) ⊆
```