Securitization & CDOs: modeling default behavior using copula methods

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Preface

This paper is written as a part of the Master's degree program *Business Mathematics and Informatics* at the *Vrije Universiteit Amsterdam*. The objective of the paper, as described in the study guide, is to demonstrate the student's ability to describe a problem in a clear manner for the benefit of an expert manager.

The aim of this particular paper is to demonstrate what securitization is, to show how credit ratings agencies contributed to its (partly unjustified) popularity, and to present and compare different copulas which can be used to model default behavior within a collateral pool. To do the comparison, a simulation model is constructed which simulates payoffs and default rates of CDO and CDO^2 tranches.

I would like to thank my supervisor dr. ir. E.M.M. Winands for his help and support and for sharing his thoughts on this subject.

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Management summary

In this paper, the process of securitization, why it is done, and how tranched securitization leads to the issuance of (CDO and CDO^2) tranches with different risk characteristics is investigated. In addition, one gets introduced to copula methods and it is shown how they can be used to model default behavior. This is done mainly because CRAs have been accused of (ab)using the copula approach to rate tranches.

Using a simulation model, it is shown that an increase in default correlation affects both the default rate and the expected payoff of tranches. This is done by applying the industry-standard Gaussian copula approach. Without judging whether or not the use of the Gaussian copula is appropriate, it becomes clear that increasing default correlation has a negative effect on the default rate (and thus the credit rating) of senior CDO tranches; it has a positive effect on the default rate of the junior CDO tranches.

To illustrate the impact of increasing default correlation, the default rate and the corresponding credit rating (according to the *Fitch's CDO matrix* presented in TABLE 1) for different values of market correlation ρ is presented in TABLE 2 below. For example, the senior CDO tranche has lost its AAA rating when $\rho = 0.4$. In addition, plots show that the default rate of junior and mezzanine CDO^2 tranches are very sensitive to collateral performance, while its senior tranche is still AAA rated when $\rho = 0.4$. As expected, expected payoff is inversely related to default rate.

Rating at issuance	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB
5-year default probability (%)	0.05	0.19	0.26	0.36	0.56	0.62	0.92	1.20	1.89

Rating at issuance	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC
5-year default probability (%)	3.63	5.74	8.11	12.50	17.09	21.36	27.08	33.64	37.64

Table 1.: Part of Fitch's CDO default matrix (Source: [8])

		$\rho = 0.2$		$\rho = 0.4$		$\rho = 0.6$		$\rho = 0.8$	
CDO	Junior	97.48%	-	88.27%	-	69.66%	-	43.25%	-
	Mezzanine	5.31%	BB+	12.19%	BB-	14.68%	B+	13.33%	B+
	Senior	<0.01%	AAA	0.24%	AA	2.10%	BBB-	4.79%	BB+
CDO^2	Junior	88.84%	-	99.49%	-	99.83%	-	99.67%	-
	Mezzanine	<0.01%	AAA	17.51%	B	65.15%	-	78.24%	-
	Senior	<0.01%	AAA	<0.01	AAA	1.05%	BBB+	5.37%	BB+

Table 2.: The influence of default correlation on credit ratings

The introduction of tail dependence, which is often seen as a way to model dependency among extreme events, has also been investigated. As the Gaussian copula does not exhibit any tail dependence, the *t*-copula is introduced. For fixed values of market correlation ρ, an increase of tail

dependence also affects the expected payoff and default rate of both CDO and CDO^2 tranches.

In general, it seems that the lower ρ, the greater the influence of an increase in tail dependence is. Furthermore, the presence of tail dependence is beneficial for the junior CDO tranche (both expected payoff and default rate), while it is clearly detrimental for the default rate of the senior CDO tranche. However, the severity of this impairment seems to be bounded since its expected payoff remains fairly constant. The expected payoff of the mezzanine CDO tranche only benefits from tail dependence for relatively high values of ρ.

To illustrate the impact of tail dependence, the default rate and the corresponding credit rating for $\rho = 0.2$ is presented in TABLE 3 below.

$\rho = 0.2$		$\nu = 50\,(\lambda < 0.01)$		$\nu = 10\,(\lambda = 0.02)$		$\nu = 5\,(\lambda = 0.09)$		$\nu = 1\,(\lambda = 0.37)$	
CDO	Junior	95.78%	-	87.29%	-	76.06%	-	30.20%	CCC+
	Mezzanine	7.51%	BB	12.57%	B+	15.12%	B+	15.34%	B+
	Senior	<0.01%	AAA	0.17%	AA+	0.81%	A-	6.30%	BB
CDO^2	Junior	95.71%	-	99.50%	-	99.86%		99.87%	-
	Mezzanine	0.13%	AA+	16.83%	B+	51.06%	-	89.87%	-
	Senior	<0.01%	AAA	<0.01%	AAA	0.16%	AA+	15.89%	B+

Table 3.: The influence of tail dependence on credit ratings

The believe that there exists dependency among the occurrence of extreme events makes it plausible to assume that the use of the t-copula is more appropriate (compared to the Gaussian copula). It is however hard to estimate the corresponding parameter ν, since relevant (i.e. extreme event) data is scarce.

Finally, for illustrative purposes only, the effect of *only* upper or *only* lower dependence is visualized. This is done using the Gumbel copula and the Clayton copula (and their complements Gumbel^c and Clayton^c). For example, it can be observed that the expected payoff of junior CDO tranches benefits from the presence of lower tail dependence.

Contents

1. Introduction

The process of securitization has become a vital funding source since the 1970s, giving financial institutions the possibility to manufacture securities that were widely considered to be safe. However, these securities seemed to be far more risky than expected. Overconfidence concerning the ability to model default behavior of collateral assets had a disastrous effect on the risk assessment of the manufactured securities, and thus led to credit rating agencies assigning inaccurate ratings to them. Investors placed too much faith in the credit rating agencies which, to put it mildly, failed to get it right.

1.1. Problem statement

This paper is mainly concerned with securitization and the risk characteristics of the CDO tranches that result from this structured finance activity. Furthermore, the influence of crediting rating agencies on the popularity of CDOs will be discussed. Therefore, the theoretical part of the paper will be dealing with the following research questions:

- What is securitization?

- What has been the influence of credit rating agencies on the popularity of CDOs?

The more practical part of the paper is concerned with the modeling of default behavior using copula methods. The following research questions are treated:

- What are copulas and how can they be used to model default behavior?

- To what extent does the use of different copulas (and copula parameters) affect the payoff/default rate of CDO tranches?

1.2. Structure

This paper is organized as follows. In *2. Securitization and CDOs* the process of securitization, why it is done, and how tranched securitization leads to the issuance of CDO tranches with different risk characteristics is described. Then, in *3. CDOs and credit ratings*, it is described how CDO tranches are rated in general, and how credit rating agencies have contributed to the (partly unjustified) popularity of these securities. In *4. Copulas* it is explained what copulas are and how they can be applied to model default behavior. The simulation model is presented in *5. Simulating the payoff of CDO tranches*, while some interesting results can be found in *6. Results*. In *7. Conclusion* the most important conclusions concerning the research questions can be found. The remainder of this paper consists of an appendix and a bibliography.

2. Securitization and CDOs

In this chapter I will describe what securitization is and explain why it can be beneficial as well as detrimental for financial institutions. I will also give some basic examples to demonstrate how securities with different risk characteristics can be manufactured.

2.1. What is securitization?

Securitization can best be described as the process in which certain types of assets are pooled so that they can be repackaged into interest-bearing securities. The interest and principal payments from the assets are passed through to the purchasers of the securities [11]. For example, a bank might want to securitize a number of mortgages it has originated. To do so, these mortgages are pooled together. This pool of assets, known as the reference portfolio, is then sold to an off-balance sheet trust called a special-purpose vehicle (SPV).

An SPV, often called a special purpose entity, is a legal entity usually set up by a financial institution (known as the originator) to carry out some specific purpose, or circumscribed activity, or a series of such transactions [9]. It is the SPV that actually issues the tradeable, interest-bearing securities. The proceeds of these issues are used to finance the acquisition of the pooled assets. Once transferred to an SPV, assets in the reference portfolio are removed from the balance sheet of the originator. As a result, the SPV is bankruptcy remote from the originator. On the other hand, the originator does not suffer from potential underperformance of assets in the reference portfolio anymore.

In concept, any type of asset with a stable cash flow can be securitized. To keep things clear, several main types of structured finance instruments can be identified [2]:

- *Asset-backed securities (ABS)* is the general term for securities backed by a pool of assets rather than a single corporation or government. Common types of collateral for ABS are auto loan receivables, student loan receivables, and so on.

- *Mortgage-backed securities (MBS)* are asset-backed securities whose cash flows are backed by the principal and interest payments of a set of mortgage loans.

- *Collateralized debt obligations (CDOs)* are structured finance securities that are pooled and tranched. CDOs are backed by a pool of assets, like other structured finance securities, but its SPV issues multiple tranches with varying levels of seniority. When assets in the collateral pool miss payments or default, subordinate tranches absorb losses first. More senior tranches only suffer losses once the cushion below them has been depleted.

As can be seen in FIGURE 2.1.1, the popularity of CDOs has been increasing dramatically up to 2006. As a result of the late 2000s financial crisis it declined again, rapidly. The CDO is a perfect example of *tranched* securitization. Another basic type of securitization is pass-through

securitization, where only a single type of security is issued. It has to be noted that among the types of securities mentioned above, there exist a number of variations based on the type of collateral. However, since this paper is mainly concerned with CDOs, they are not necessarily worth mentioning.
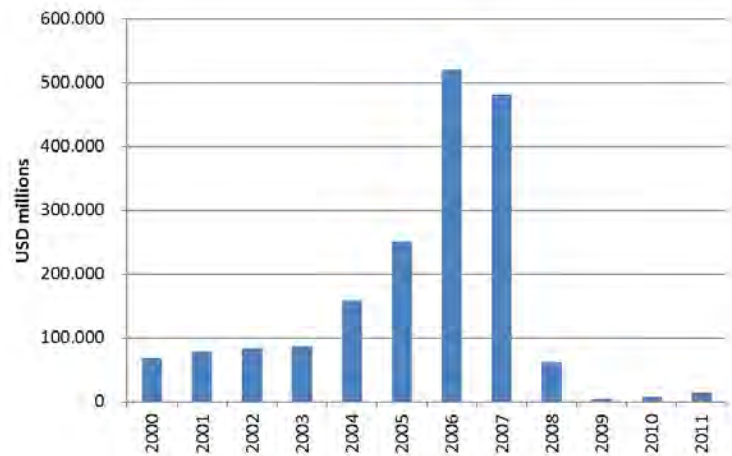

Figure 2.1.1.: Global annual CDO issuance in USD millions (Source: Securities Industry and Financial Markets Association)

2.2. Benefits and disadvantages

Securitization can be beneficial in more than one way [14, 17, 18]. Below, some important benefits (i.e. motivations to securitize) are listed.

- Corporations with weak, declining, or no credit rating can use securitization to access capital market funds that would otherwise require higher ratings. Hence, through securitization, funding costs can be reduced. For example, Moody's downgraded Ford Motor Credit's rating in January 2002, but senior automobile backed securities issued by Ford Motor Credit in January 2002 and April 2002 continue to be rated AAA, because of the strength of the underlying collateral, and other credit enhancements.

- Securitizing long-term and illiquid assets improves balance sheet liquidity as it converts these assets into cash earlier than would have happened in the original situation. For example, a bank might issue mortgage-backed securities instead of holding the mortgages on its balance sheet until maturity. Through securitization, liquidity can be obtained from illiquid assets.

- As financial institutions can use securitization to convert (long-term) loans into cash, it enables the lender to originate more loans. Obviously, these loans can again be securitized. Typically, the originator continues to service the securitized assets. Therefore, it benefits from the revenues generated by servicing and originating the underlying assets, underwriting and structuring the transaction and providing credit and liquidity enhancements for certain structures.

- Securitization can transform cash flows and risks of the collateral pool into those of the issued securities, and can therefore be used to transfer risk to the capital markets. A bank, for example, might consider securitizing a pool of risky assets, such as (subprime) mortgages, if it no longer wishes to bear the credit risk associated with the mortgages.

- For securitization can be used to transfer risk, financial institutions can use it to achieve a reduction in regulatory and economic capital requirements.

- Securitization provides the possibility to tailor securities to meet investor needs. Originators can sell particular assets with greater liquidity if these assets can be transformed into securities demanded by investors. Through tranching in particular, a great amount of securities with varying risk and return characteristics can be manufactured.

While securitization clearly is capable of providing several benefits, it can create additional risks as well [14, 18]. Below, some important disadvantages are listed.

- When the repayment on assets in the reference pool is significantly worse than expected, investors might be concerned about moral hazard. It might raise the presumption that assets were cherry-picked (i.e. the originator securitized only the assets it expected to underperform). As a result, investors may require the originator to provide an explicit guarantee or to take back poorly performing assets.

- As securitization can be used to transfer risk to the capital market, it can create an incentive to over-originate loans to less creditworthy borrowers. The originate-to-distribute model, in which loans are only originated to be securitized again, might reduce the incentive to monitor and screen borrowers properly.

- The structural flexibility in transforming collateral pool characteristics into very different security characteristics, which provides the possibility to tailor securities to meet investor needs, also has the potential to create great risks. This flexibility can lead to securities with very complex structures, which are therefore harder to analyze; forecasting their performance (and therefore, to determine their rating) will become difficult.

2.3. CDO tranches and their risk characteristics

As mentioned earlier, *tranched* securitization leads to the issuance of prioritized structured claims backed by a pool of assets. In this section I will elaborate on these tranches and their risk characteristics using the prototypical structured finance security: the CDO. The examples that are used in this section are based on work by Coval et al. [4]. The conventional securitization structure assumes a three-tier security design— resulting in junior, mezzanine, and senior tranches [11]. The key concept of this structure of tranches is that the different tranches receive coupon payments from the underlying assets in order of priority. This way, securities with varying cash flow risk are manufactured. FIGURE 2.3.1 depicts the conventional three-tier security design.

In case of defaults in the underlying pool of assets, the first loss is borne by the *junior* tranche holders. These defaults result in a reduction of coupons payments to the junior tranche holders directly. When the amount of defaults increases further, the coupon payments to *mezzanine* tranche holders are reduced. Finally, when the mezzanine tranche holders no longer receive payments and even more defaults occur, the coupon payments to *senior* tranche holders are reduced.
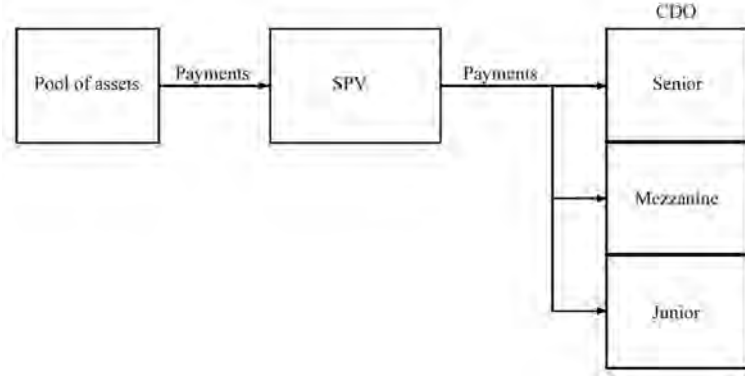
Figure 2.3.1.: Tranching of a CDO into three tranches (Source: [5])

The so called attachment-detachment points $\{[a_i, b_i] | a_i < b_i,\, a_{i+1} = b_i\}$ of a tranche i determine how the payments made by the SPV are distributed over the different tranches; where the index i represents the seniority of a tranche. Let us define L_{ref} as the cumulative loss of the reference portfolio. Given L_{ref}, the cumulative loss of tranche i, $L_{[a_i, b_i]}$, is calculated as follows:

$$L_{[a_i, b_i]} = max\{L_{ref} - a_i, 0\} - max\{L_{ref} - b_i, 0\}.$$

As long as $L_{ref} < a_i$, tranche i will not face value impairment. On the other hand, when $L_{ref} \geq b_i$, tranche i becomes worthless. As the notional (cash flow) value of a tranche equals $b_i - a_i$, the cash flow that is allocated to a tranche i is equal to $b_i - a_i - L_{[a_i, b_i]}$.

To illustrate this so called payment waterfall, an example is given below. For the sake of simplicity, a CDO with a pool of three mortgages as underlying assets is considered. Suppose that each of the mortgages pays 1 unit and defaults with probability $p_{d|mortgage} = 0.1$ over a certain time period. Furthermore, the recovery rate is set to 0 (i.e. no cash flow can be recovered in case of default). It is assumed that defaults occur independently (i.e. are uncorrelated). The CDO issues three tranches; the attachment-detachments points are $[0, 1]$, $[1, 2]$ and $[2, 3]$ for the *junior*, *mezzanine* and *senior* tranche respectively. When one of the underlying mortgages defaults, this results in a default of the junior tranche. When at least two of the underlying mortgages default, the mezzanine tranche defaults. Finally, in the case of all the mortgages defaulting, the senior tranche defaults. Calculating the corresponding probabilities is a simple probability exercise. The senior tranche will default with probability $p_{d|senior} = 0.1^3 = 0.001$. The mezzanine tranche will default when at least two mortgages default, this happens with probability

$$p_{d|mezzanine} = \binom{3}{2} \cdot 0.1^2 \cdot 0.9 + p_{d|senior} = 0.028.$$

For only one mortgage needs to default to make the junior tranche default, its probability of default is

$$p_{d|junior} = \binom{3}{1} \cdot 0.1 \cdot 0.9^2 + p_{d|mezzanine} = 0.271.$$

Obviously, this equals $1 - 0.9^3$.

As can be seen in TABLE 2.1, both the senior and the mezzanine tranche default with a probability smaller than $p_{d|mortgage} = 0.1$; the manufactured securities are less risky than the underlying mortgages.

	Default probability
Mortgage	0.1
Senior CDO tranche	0.001
Mezzanine CDO tranche	0.028
Junior CDO tranche	0.271

Table 2.1.: Default probabilities

In the example described above, the CDO has a pool mortgages as underlying assets. In practice, CDOs are created using all sorts of other types of underlying financial assets. A construction specifically worth mentioning is the so called CDO-squared (CDO^2). This type of CDO is created using the tranches of other collateralized debt obligations. FIGURE 2.3.2 depicts how a typical CDO^2 is constructed. Obviously, the tranches of the CDO^2 can again be used to create a new CDO in a similar way, and so on, leading to increasingly complex structures (CDO^n).
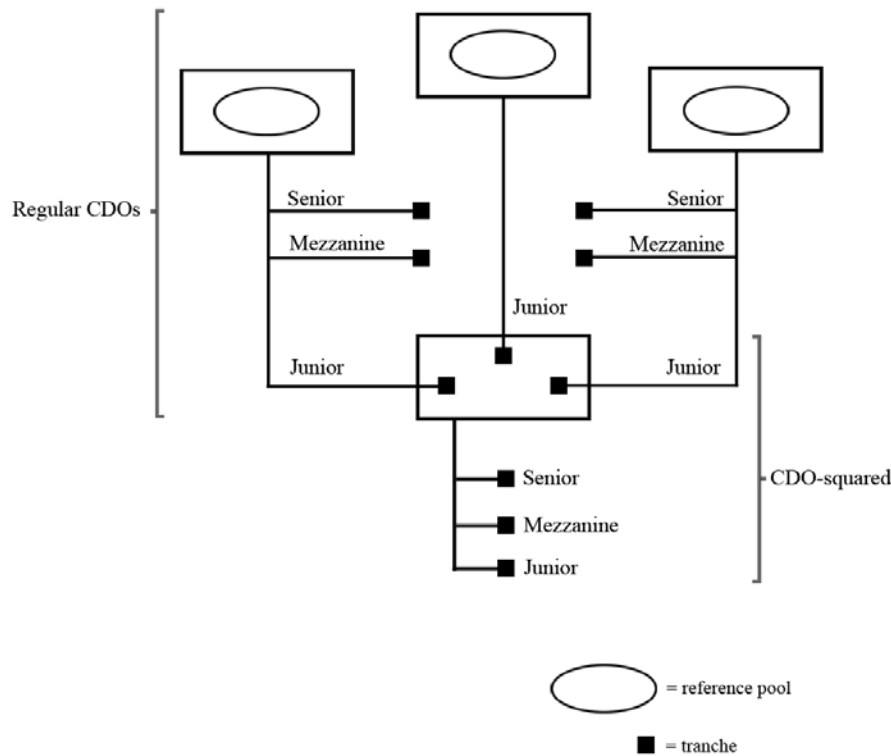

Figure 2.3.2.: Structure of a typical CDO^2

Now consider a CDO that has three junior tranches similar to those in the previous example as underlying assets. Let us call these tranches the *underlying tranches*. Again, assume that this CDO^2 has three tranches. It is known that the underlying tranches will default with probability $p_{d|underlying} = 0.271$. For the CDO^2, $p_{d|senior}$, $p_{d|mezzanine}$ and $p_{d|junior}$ are calculated in a similar way as for the CDO. The results can be found in TABLE 2.2.

	Default probability
Underlying CDO tranche	0.271
Senior CDO^2 tranche	0.0199
Mezzanine CDO^2 tranche	0.181
Junior CDO^2 tranche	0.613

Table 2.2.: Default probabilities

In the two examples given above, it becomes clear that it is possible to manufacture securities that are less risky than the separate assets in the underlying pool when assumed that asset defaults occur independently. However, this is hardly ever the case. As default correlation increases, the more likely it is that assets default simultaneously, which results in the more senior claims becoming less safe. In case of perfect correlation, the underlying assets of a CDO will either survive or default simultaneously, which results in the senior tranche being equally prone to value impairment as the junior tranche. To summarize, the default probability of the senior tranche will be maximized when defaults among assets are perfectly correlated; the default probability will be minimized in case of perfect negative correlation.

3. CDOs and credit ratings

As for all debt and debt related obligations, credit rating agencies (CRAs) such as Standard and Poor's, Moody's and Fitch have also been assigning credit ratings to CDO tranches over the years. In this chapter I will describe how securities are rated in general, how credit rating agencies contributed to the (partly unjustified) popularity of CDOs and why many ratings turned out to be inaccurate.

3.1. Credit rating agencies

In essence, a credit rating reflects a rating agency's opinion, as of a specific date, of the credit-worthiness of a particular company, security, or obligation [22]. The importance of these opinions to investors and other market participants, and the influence of these opinions on the securities markets, has increased significantly over the past decades. This is due in part to the advent of new and complex financial products, such as CDOs.

CRAs roughly base their ratings either on the likelihood of observing a default or on the basis of the expected economic loss – the likelihood of observing default multiplied by the severity of the loss conditional on default [4]. Amongst credit market participants, it is well known that Moody's ratings are based on the concept of expected loss, while S&P and Fitch base their ratings on probability of default [7, 10]. Figure 3.1.1 depicts the rating scales of Standard and Poor's and Fitch. CRAs stress that their ratings are only designed to provide an ordinal ranking of securities' long-run payoff prospects, whereas the expected cash flow interpretation takes a cardinal view of ratings [10]. For example, an AAA rating should carry less risk than a AA+ or AA rating, but the quantitative difference in credit risk between AAA/AA+ and AA+/AA ratings is not necessarily equal.

3.2. Rating CDOs

From the beginning, the CDO market, much as other markets for securitized products, has been a rated market; issuers of structured finance products were eager to let these products get rated according to scales that were comparable to those for bonds, so that investors would feel confident purchasing these structures [4, 7]. Having the securities rated this way created the illusion that structured finance securities were comparable with existing securities such as corporate bonds. As can be seen in Figure 3.2.1 (p. 11), the majority of issued CDO tranches bore an AAA rating.

In [1] it is suggested that CDO structures, and the preponderance of AAA rated tranches in particular, are driven by investor demand. This demand is in turn driven by rating-dependent regulation, asymmetric information, and investor guidelines and heuristics. Below I will briefly

Figure 3.1.1.: Rating scales of Standard and Poor's and Fitch

summarize the arguments (presented in [1]) which support this statement:

- Regulation and restrictions concerning minimum rating requirements and limits on portfolio distribution by rating class creates demand for highly rated securities. Given the fact that highly rated CDO securities incur smaller capital charges, may be used as collateral, and provide a higher yield than single-name securities with the same rating, it is not surprising that AAA-rated CDO securities became attractive to many financial institutions.

- Securities with AAA rating alleviate any concerns about adverse selection in the securities issued by the CDO issuer. The presence of asymmetric information between issuers and investors results in investors being more cautious. When investors use heuristics to classify assets, and only AAA rated securities are perceived to be riskless, issuers will cater to investor demand by carving out large portions of their collateral pools as AAA.

- The uniformity of CDO structures and the fact that the rating model used by S&P was known to issuers made it easy to target the highest possible rating at the lowest cost.

Before the late 2000s financial crisis, a big part of the structured finance securities were highly
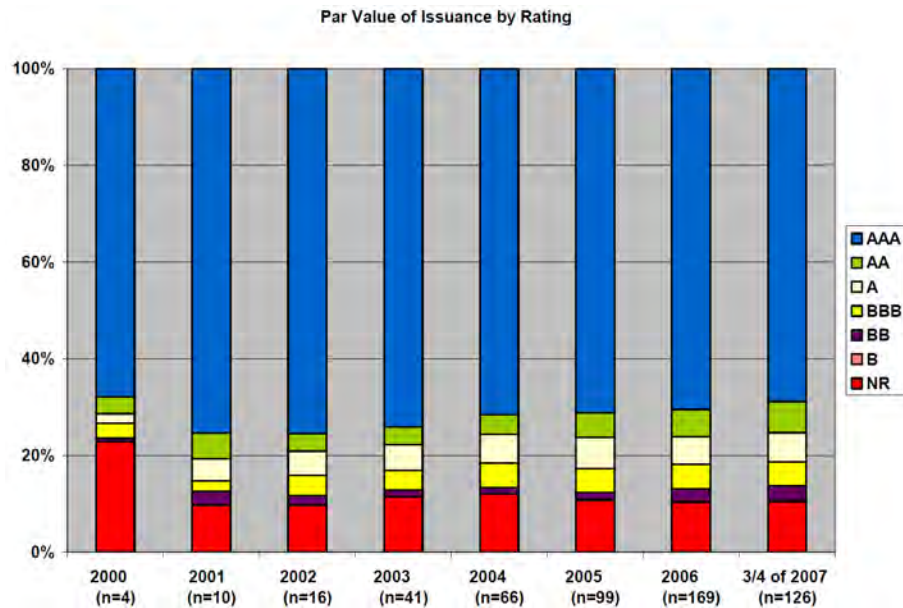
Figure 3.2.1.: CDO rating structure (Source: [1])

rated. However, the performance of many structured finance products, and thus its rating, turned out to be overestimated. As stated in [2]:

> *In 2007 and 2008, the creditworthiness of structured finance securities deteriorated dramatically: 36,346 tranches rated by Moody's were downgraded. Nearly one third of downgraded tranches bore the AAA rating.*

A summary of events concerning these downgrades is given in the research summary of [2][1] , where the word 'notch' is used to indicate the difference between adjacent ratings (i.e. AAA and AA+):

> *.. the deterioration in the creditworthiness of structured finance products began in 2007. There were more than 8,000 downgrades in 2007 - an eightfold increase over the previous year. In the first three quarters of 2008, there were 36,880 downgrades, overshadowing the cumulative number of downgrades since 1990. Downgrades were not only more common in 2007 and 2008 but also more severe. The average downgrade was 4.7 notches in 2007 and 5.8 notches in 2008, compared to 2.5 notches in both 2005 and 2006. Meanwhile, upgrades were less frequent and smaller in magnitude on average. Many of the downgrades in 2007 and 2008 were tied to collateralized debt obligations (CDOs) backed by assets that are themselves structured (ABS CDOs). While initially ABS CDOs were diversified and collateralized by assets from a variety of sectors, they became more concentrated over time. Since 2003 the primary asset classes backing them were subprime and non-conforming residential mortgage-backed securities. Many of these ABS CDOs were downgraded during the crisis, leading to large selloffs of these securities and losses at financial institutions. [..] early 2009, financial institutions around the world wrote down more than half a trillion dollars, out of which more than 200 billion dollars resulted from exposure to ABS CDOs that were severely downgraded.*

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[1]http://www.nber.org/reporter/2010number1/benmelech.html

How could it be that a big part of the ratings assigned by CRAs did not reflect the actual credit risk of a structured finance security? In [2], several causes are mentioned, under which the practice of rating shopping. It is not in the scope of this paper to elaborate on this practice extensively, but it is described as when:

> .. an issuer chooses the rating agency that will assign the highest rating or that has the most lax criteria for obtaining a desired rating.

Obviously, rating shopping did not contribute to the accuracy of credit ratings in general. Seen the scope of this paper, the failure of the black box, as described in [1], is more interesting. The failure of the black box refers to the model risk that the models used by the CRAs were subject to. As stated in [7]:

> .. as CDOs are based on portfolios of credits rather than a single obligor, rating such structures not only requires attributing a probability of default to each obligor within the portfolio. It also involves assumptions concerning recovery rates and correlated defaults of pool assets, thus combining credit risk assessments of the individual assets in the collateral pool with estimates about default correlations by way of credit risk modeling pool with estimates about default correlations by way of credit risk modeling.

Given the complexity of structured finance products, these 'model requirements' could be violated fairly easy, resulting in inaccurate ratings. The ability to model default behavior of assets within a collateral pool seemed to be overestimated by the CRAs. In the article *The Formula That Felled Wall St* by Jones [12], CRAs are more or less are accused of putting too much confidence in the so-called Gaussian copula approach proposed by David X. Li [13]; both Moody's and Standard & Poor's quickly started using this Gaussian copula methodology to rate CDO tranches.

As stated in [4], popular CDO rating toolkits offered by credit rating agencies, such as Fitch's Default VECTOR models, Moody's CDOROM and Standard and Poor's CDO Evaluator, all employed versions of this Gaussian copula model. Providing CDO issuers with these toolkits, as suggested in [1], gave them the ability to achieve the highest possible rating at the lowest possible cost. S&P's CDO evaluator, for example, would indicate to issuers:

> .. the percentage of assets notional [that] needs to be eliminated (added) in order for at a given rating level.

Given the above, it seems plausible to believe that the CRAs may have served not just as monitors and evaluators of existing structures, but contributed effectively to the increase in issuance of (highly rated) CDO securities. Modeling errors embedded in the black boxes employed by CRAs seem to have lead to the mispricing of CDO securities on a large scale, leading to the inevitable rating downgrades the financial world experienced in 2007-2008.

4. Copulas

As was stated in the previous section, the *black boxes* used by the CRAs failed to mimic the default behavior of assets within collateral pools of structured finance securities. In particular, the use of the Gaussian copula approach proposed by David X. Li was criticized. Before elaborating on this approach, I will explain what copulas are and describe their characteristics.

4.1. What is a copula?

In probability theory, the joint cumulative distribution function (cdf)

$$F(x_1, x_2, .., x_n) = F(X_1 \le x_1, X_2 \le x_2, .., X_n \le x_n)$$

fully captures the marginal distributions (marginals) together with the dependence structure of the random variables. However, it does not allow for the separation of marginals from their dependence structure, and vice versa. Copulas do allow for this separation, and therefore provide several important benefits.

A good way to describe what a copula is, is given by Donnely and Embrechts [5]:

> *A copula specifies a dependency structure between random variables $X_1, X_2, .., X_n$, that is, how $X_1, X_2, .., X_n$ behave jointly.*

and

> *A copula allows us to separate the individual behavior of the marginal distributions from their joint dependency on each other.*

More formally, a copula can be defined as follows [21]:

Definition 4.1 A n-dimensional copula is a function $C : [0,1]^n \rightarrow [0,1]$, which satisfies the following conditions:

- $C(1, 1, ..., u_i, ..., 1, 1) = u_i$ for every $i \le n$

- $C(u_1, ..., u_n) = 0$ if $u_i = 0$ for any $i \le n$

- C is n-increasing

For example, the independence or product copula $C^\perp$ is defined as follows:

$$C^\perp(u_1, ..., u_n) = u_1 \cdot u_2 \cdot ... \cdot u_n, \quad \forall (u_1, ..., u_n) \, \epsilon \, [0,1]^n$$

Obviously, Definition 4.1 does not provide us with any insight concerning the benefits of using copulas. The theorem that describes the relationship between a joint cdf, marginals and a copula is part of Sklar's Theorem. It is stated as follows [16, 20]:

Theorem 4.1. Sklar's theorem. *Let H be a n-dimensional joint cdf with one-dimensional marginals $F_1, \ldots, F_n$. Then there exists a copula C such that*

$$H(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)), \quad \forall (x_1, ..., x_n) \, \epsilon \, \mathbb{R}^n \tag{4.1.1}$$

If the marginals are continuous, then C is unique; otherwise, C is uniquely determined on $Ran(F_1) \times ... \times Ran(F_n)$, where $Ran(F_i)$ denotes the range of the cdf F_i. Conversely, if C is a copula and $F_1, \ldots, F_n$ are marginals, then the function H in (4.1.1) is a joint cdf with marginals $F_1, \ldots, F_n$.

Effectively, a copula captures the dependence structure of a joint cdf. For it is known that u_i is $U(0,1)$-distributed when $u_i = F(x_i)$, the following holds

$$C(u_1, ..., u_n) \quad = \quad H(F_1^{-1}(u_1), ..., F_n^{-1}(u_n)) \quad \forall (u_1, ..., u_n) \epsilon [0,1]^n$$

Therefore, a copula can be described as an n-dimensional joint cdf with all univariate marginals being $U(0,1)$-distributed [21]. Which copula one uses depends on the desired dependency structure, and not on the the desired marginals.

4.2. Copulas and copula families

As one might expect, there exist many copulas. In this section I will elaborate on four important copulas, which can be divided into two groups: elliptical and Archimedean copulas. The two elliptical copulas are:

- Gaussian copula

- t-copula (i.e. Student-copula)

The Archimedean copulas on which I will elaborate are:

- Gumbel copula

- Clayton copula

4.2.1. Elliptical copulas

The two elliptical copulas can easily be obtained by applying Sklar's theorem. In case of the Gaussian copula, a multivariate normal distribution, $X \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$, with standard normally distributed marginals, $(Z_1, ... Z_n) \sim \mathcal{N}(0,1)$, is used. Applying Sklar's theorem gives:

$$\begin{aligned} \Phi_\Sigma(x_1, ..., x_n) &= C_\Sigma^{gau}(\Phi(x_1), ..., \Phi(x_n)) \\ &\downarrow \\ C_\Sigma^{gau}(u_1, ..., u_n) &= \Phi_\Sigma(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)) \end{aligned} \tag{4.2.1}$$

where Φ_Σ and Φ represent the (joint) cdf of the specified multivariate normal distribution and the standard normal distribution respectively. In a similar the t-copula is obtained:

$$\begin{aligned} t_{\nu,\Sigma}(x_1, ..., x_n) &= C_{\nu,\Sigma}^t(t_\nu(x_1), ..., t_\nu(x_n)) \\ &\downarrow \\ C_{\nu,\Sigma}^t(u_1, ..., u_n) &= t_{\nu,\Sigma}(t_\nu^{-1}(u_1), ..., t_\nu^{-1}(u_n)) \end{aligned}$$

Both copulas require the input parameter Σ, which represents a correlation matrix:

$$\Sigma = \begin{pmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,n} \\ \rho_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho_{n-1,n} \\ \rho_{n,1} & \cdots & \rho_{n,n-1} & 1 \end{pmatrix}$$

In case of the t-copula, the parameter ν represents the degrees of freedom.

4.2.2. Archimedean copulas

Archimedean copulas are, compared to elliptical copulas, constructed in a completely different way. They use a so-called *generator*, which is a function $\varphi : [0,1] \to [0,\infty]$. In the following theorem the general form of an Archimedean copula is given [3]:

Theorem 4.2. *Let φ be a continuous and strictly decreasing function, with its pseudo-inverse φ^{-1} completely monotonic on $[0,\infty]$. Then an Archimedean copula is defined as*

$$C_\theta(u_1, ..., u_n) = \varphi_\theta^{-1}(\varphi_\theta(u_1) + ... + \varphi_\theta(u_n)) \tag{4.2.2}$$

where

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & 0 \leq \theta \leq \varphi(0) \\ 0 & \varphi(0) \leq \theta \leq \infty \end{cases}$$

As is described in THEOREM 4.2, a generator as well as its pseudo-inverse has to meet several properties for (4.2.2) to be an actual copula. All generators presented in this section meet these properties. In Table 4.1 the generator (and its pseudo-inverse) of the Gumbel and Clayton copula are denoted [3].

Name	$\varphi(t)$	$\varphi^{-1}(t)$	θ
Gumbel	$(-log(t))^\theta$	$e^{-t^{\frac{1}{\theta}}}$	$\theta \geq 1$
Clayton	$t^{-\theta} - 1$	$(1+t)^{-\frac{1}{\theta}}$	$\theta > 0$

Table 4.1.: Generator functions φ (and their pseudo-inverse φ^{-1})

One can easily observe that the strength of the dependence in Archimedean copulas is controlled by only one parameter, even in higher dimensions.

4.3. Dependence concepts

As copulas are used to *glue together* marginal distributions and manage the dependency between them, they provide a natural way to study and measure dependence between random variables.

To give insight in what way and to what extent copulas manage the dependence between marginal distributions, so called dependence concepts are introduced. There are several; some important ones are explained in this section. All definitions and theorems in this section are taken from [3, 6].

4.3.1. Linear correlation

Linear correlation is the most popular measure of dependence. However, it is not a copula-based measure, and can therefore be misleading. The definition of the linear correlation coefficient can be found below.

Definition 4.2. *Let* $(X, Y)^T$ *be a vector of random variables. The linear correlation coefficient for* $(X, Y)^T$ *is given by*

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

where $Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$ *is the covariance of* $(X, Y)^T$, *and* $Var(X)$ *and* $Var(Y)$ *are the variances of* X *and* Y.

In [6], some important properties of linear correlation are listed, being:

- In case of perfect correlation between X and Y, we find that $\{Y = \alpha X + \beta \,|\, \alpha \,\epsilon\, \mathbb{R} \setminus \{0\}, \, \beta \,\epsilon\, \mathbb{R}\}$. In that case, $|\rho(X, Y)| = 1$. The converse also holds. Otherwise, $1 < \rho < 1$.

- Linear correlation is invariant under strictly increasing *linear* transformations.

- The variance of a linear combination of random variables is fully determined by pairwise covariances between the components, a property which is crucial in portfolio theory.

In the same paper, it is stated that when dealing with non-elliptical distributions, the use of linear correlation as a dependence measure does not make sense in most cases. Therefore, some copula-based measures will be presented in the subsections below.

4.3.2. Concordance and rank correlation

Let us consider a vector of two random variables $(X, Y)^T$. Loosely speaking, concordance indicates that large values of X are accompanied by large values of Y. In case of discordance, large values of X are accompanied by small values of Y, and vice versa. It can be described as co-movement of two random variables. Hence, it is defined as:

Definition 4.3. *Let* $(x_1, y_1)^T$ *and* $(x_2, y_2)^T$ *be two observations from a vector* $(X, Y)^T$. *Then* $(x_1, y_1)^T$ *and* $(x_2, y_2)^T$ *are said to be concordant if* $(x_1 - x_2)(y_1 - y_2) > 0$ *and discordant if* $(x_1 - x_2)(y_1 - y_2) < 0$.

Two important measures of concordance are Kendall's tau and Spearman's rho.

Definition 4.4. *Kendall's tau* (τ) *for the random vector* $(X, Y)^T$ *is defined as*

$$\tau(X, Y) = \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) > 0\} - \mathbb{P}\{(X - \tilde{X})(Y - \tilde{Y}) < 0\},$$

where $(\tilde{X}, \tilde{Y})^T$ *is an independent copy of* $(X, Y)^T$.

Therefore, Kendall's tau can be interpreted as the probability of concordance minus the probability of discordance. Spearman's rho is defined below.

Definition 4.5. *Spearman's rho (ρ_S) for the random vector $(X, Y)^T$ is defined as*

$$\tau(X, Y) = \mathbb{P}\{(X - \tilde{X})(Y - Y') > 0\} - \mathbb{P}\{(X - \tilde{X})(Y - Y') < 0\},$$

where $(X, Y)^T$, $(\tilde{X}, \tilde{Y})^T$ and $(X', Y')^T$ are independent copies.

Therefore, Spearman's rho can be interpreted as the probability of concordance minus the probability of discordance, with respect to the independent case.

Both Kendall's tau and Spearman's rho increase (decrease) as concordance (discordance) increases. In TABLE 4.2 expressions for Kendall's tau can be found; closed form expressions for Spearman's

Copula	τ	ρ/θ
Gaussian	$\frac{2}{\pi} \arcsin \rho$	$\frac{\pi}{2} sin(\tau)$
t	$\frac{2}{\pi} \arcsin \rho$	$\frac{\pi}{2} sin(\tau)$
Gumbel	$1 - \frac{1}{\theta}$	$\frac{1}{1-\tau}$
Clayton	$\frac{\theta}{\theta+2}$	$\frac{2\tau}{1-\tau}$

Table 4.2.: Expressions for Kendall's tau

rho are omitted since they either do not exist, or are fairly complicated.

4.3.3. Tail dependence

The concept of tail dependence is concerned with the amount of dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a distribution; it looks at concordance (i.e. co-movement) in the tail values of X and Y. Tail dependence is a copula-based measure. It is defined below.

Definition 3.6. *Let $(X, Y)^T$ be a vector of random variables with marginals F and G respectively. The coefficient of upper tail dependence of $(X, Y)^T$ is*

$$\lim_{u \to 1^-} \mathbb{P}\{Y > G^{-1}(u) | X > F^{-1}(u)\} = \lambda_U$$

The coefficient of lower tail dependence of $(X, Y)^T$ is

$$\lim_{u \to 0^+} \mathbb{P}\{Y < G^{-1}(u) | X < F^{-1}(u)\} = \lambda_L$$

The following theorems state how the coefficient of upper and lower tail dependence can be computed, given a copula.

Theorem 4.5.1. *If a bivariate copula C is such that $\lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1-u} = \lambda_U$ exists, then C has upper tail dependence if $\lambda_U \, \epsilon(0, 1]$. $\lambda_U = 0$ indicates upper tail independence.*

Theorem 4.5.2. *If a bivariate copula C is such that $\lim_{u \to 0} \frac{C(u,u)}{u} = \lambda_L$ exists, then C has lower tail dependence if $\lambda_U \, \epsilon (0, 1]$. $\lambda_L = 0$ indicates lower tail independence.*

For the copulas presented in this chapter, the coefficients of upper and lower tail dependence can be found in TABLE 4.2. These values correspond to the values found in [19].

Copula	λ_L	λ_U
Gaussian	$\begin{array}{ccc} 0 & if & \rho < 1 \\ 1 & if & \rho = 1 \end{array}$	
t	$2t_{\nu+1}\left(-\sqrt{\dfrac{(\nu+1)(1-\rho)}{(1+\rho)}}\right)$	
Gumbel	0	$2 - 2^{1/\theta}$
Clayton	$2^{-1/\theta}$	0

Table 4.3.: Coefficients of upper and lower tail dependence

For both the Gaussian- and t-copula it holds that $\lambda_U = \lambda_L$, since these copulas are, as any elliptical copula, radial symmetric. For the t-copula it holds that the lower the degrees of freedom, the higher the amount of tail dependence. This can be observed in FIGURE 4.3.1.. Comparing (c) with (b) and (d), the values in (c) are more concentrated in the tails even though ρ is equal in all cases. As ν gets big (i.e. 50), the amount of tail dependence gets relatively small, resulting in the plots (b) and (d) looking similar.
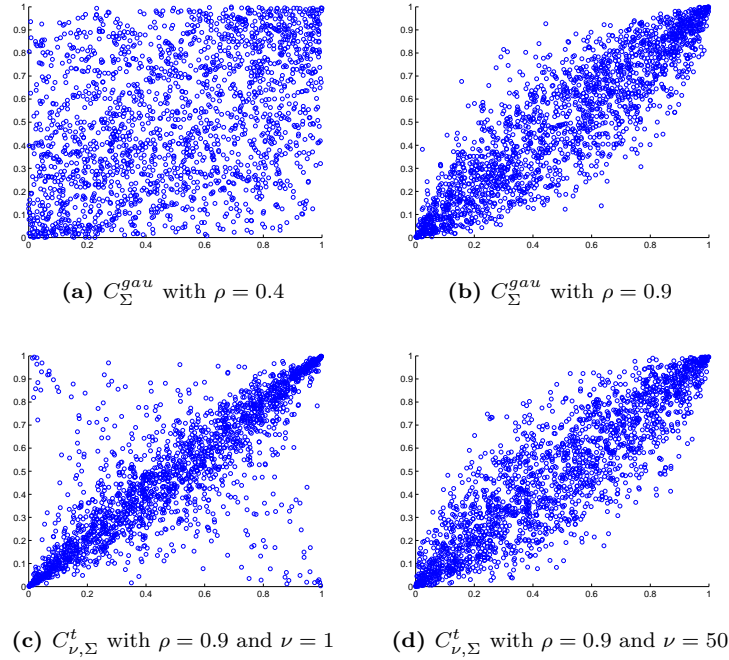

(a) C^{gau}_{Σ} with $\rho = 0.4$ **(b)** C^{gau}_{Σ} with $\rho = 0.9$

(c) $C^{t}_{\nu,\Sigma}$ with $\rho = 0.9$ and $\nu = 1$ **(d)** $C^{t}_{\nu,\Sigma}$ with $\rho = 0.9$ and $\nu = 50$

Figure 4.3.1.: Figures displaying 2000 sample points from the Gaussian- and t-copula

(a) C_θ^{Gumbel} with $\theta = 2$

(b) C_θ^{Gumbel} with $\theta = 5$

(c) $C_\theta^{Clayton}$ with $\theta = 2$
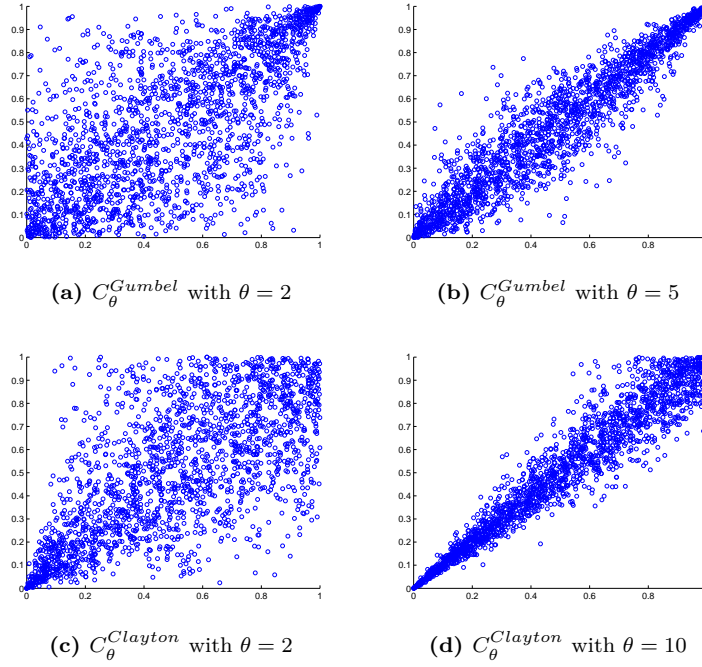
(d) $C_\theta^{Clayton}$ with $\theta = 10$

Figure 4.3.2.: Figures displaying 2000 sample points from the Gumbel- and Clayton-copula

In FIGURE 4.3.2 it can be observed that the higher the value of θ, the higher the amount of tail dependence and concordance for both the Gumbel and the Clayton copula.

4.4. From copulas to default behavior

In this section it is shown how copulas can be used to model default behavior. Basically, the challenge of modeling default behavior is to find a suitable joint default distribution. So, when we consider a collateral pool of n assets, and define T_i the time-until-default of the ith asset, we are looking for $\mathbb{P}(T_1 \leq t_1, ..., T_n \leq t_n)$. When we define F_i as the distribution of the time-until-default of the ith asset, we can rewrite this as:

$$\mathbb{P}(T_1 \leq t_1, ..., T_n \leq t_n) = C(F_1(t_1), ..., F_n(t_n)) \quad \forall (t_1, ..., t_n) \, \epsilon \, [0, \infty]^n \qquad (4.4.1)$$

where the copula C represents the desired dependency structure. Let us now look at Li's one-factor Gaussian model. Li defines the factor X_i as:

$$X_i = \rho_i Z + \sqrt{1 - \rho_i^2} \xi_i, \quad for \, i = 1, ..., n \qquad (4.4.2)$$

where

- $\rho_i \, \epsilon \, (0, 1)$ can be interpreted as the correlation coefficient between X_i and the common (i.e market) factor Z

- $Z \sim \mathcal{N}(0,1)$ represents the market factor which is common to all assets

- $\xi_i \sim \mathcal{N}(0,1)$ is the factor specific to asset i

It is assumed that $\rho_i = \rho$, therefore the vector $(X_1, ... X_n)$ is multivariate normally distributed with $\mu = 0$ and correlation matrix:

$$
\Sigma = \begin{pmatrix} 1 & \rho^2 & \cdots & \rho^2 \\ \rho^2 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \rho^2 \\ \rho^2 & \cdots & \rho^2 & 1 \end{pmatrix}
\tag{4.4.3}
$$

When we introduce the relationship:

$$
\begin{aligned}
\Phi(X_i) = F_i(t_i) &\rightarrow X_i = \Phi^{-1}(F_i(t_i)) \\
&\rightarrow t_i = F_i^{-1}(\Phi(X_i))
\end{aligned}
$$

the copula C in (4.4.1) is actually the Gaussian copula C_Σ^{gau} as stated in (4.2.1), with a correlation matrix Σ as in (4.4.3), leading to:

$$
\mathbb{P}(T_1 \leq t_1, ..., T_n \leq t_n) = C_\Sigma^{gau}(F_1(t_1), ..., (F_n(t_n)) \quad \forall (t_1, ..., t_n) \, \epsilon \, [0, \infty]^n
$$

which is often referred to as *Li's formula*. When the time-until-default distributions $F_1, ..., F_n$ have been estimated, the one-factor model is fully specified. The relationship between the one-factor model described in (4.4.2) and the Gaussian copula will be used to simulate default behavior. When we assume that asset i defaults with probability p_d, it defaults when $F_i(t_i) \leq p_d$. Therefore, we can simulate the default behavior of assets in a collateral pool of size n by generating a random vector $(u_1, ... u_n) \sim C_\Sigma^{gau}$, to subsequently check for how much assets it holds that $u_i \leq p_d$. Note that $u_i \sim U(0,1)$.

Obviously, we are not restricted to the use of the Gaussian copula. For example, the t-copula can be used when one assumes that extreme events (i.e. defaults) in particular are somehow likely to occur simultaneously.

5. Simulating the payoff of CDO tranches

In this chapter a Monte Carlo simulation model presented. This model is used to provide insight in the expected payoff and the default rate of CDO tranches given the default behavior of assets in its collateral pool. Each run, the default characteristics of 40 pools, each consisting of 100 assets will be considered. The model and its setup is based on work by of Coval et al. [4], which is in turn based on the one-factor Gaussian copula approach as described in SUBSECTION 4.4. However, this model allows for the use of three other copulas.

5.1. Input, output and assumptions

In this subsection the input and output parameters of the model are listed. In addition, the assumptions that were made are stated.

5.1.1. Input

The model requires the following input parameters to be specified:

- *Default probability (p_d):* the default probability of an asset in the reference pool in case of independence

- *Copula* and *copula parameters:* the copula which is used to model default behavior given the following parameters:

    - ρ, ν in case of an elliptical copula. ρ represents the off-diagonal elements of the correlation matrix Σ; ν represents the degrees freedom

    - θ in case of an Archimedean copula

- *Recovery rate:* the fraction of cash flow that can be recovered in case of default

- *Tranche structure:* the attachment-detachments points of the CDO and CDO^2 tranches. For example: `0, 0.15, 0.30, 1`. Note that both the CDO and CDO^2 have the same tranche structure and are given as fractions

- *Collateral tranche:* the attachment-detachments points of the CDO tranche that is used as collateral for the CDO^2. For example: `0.15, 0.30`.

- *Runs:* the amount of simulation runs

5.1.2. Output

For both the CDO and CDO^2, the following is computed:

- *Expected payoff of collateral:* fraction of the notional (cash flow) value generated by the reference pool. Note that this should be $1 - (1 - recovery\,rate) \cdot p_d$ in the CDO case

- *Expected payoff per tranche:* fraction of a tranche's notional (cash flow) value that will be allocated to a tranche, including recovered cash flow in case of default

- *Default rate per tranche:* the fraction of times that a tranche faces value impairment. It has to be noted that the default rate does not necessarily tells us anything about the size of the losses

5.1.3. Assumptions

It is assumed that:

- Each asset pays 1 unit in the normal situation; it pays $1 \cdot r$ in case of default

- Defaults within a reference pool can be correlated; defaults of assets belonging to different pools are assumed to occur independently

- Assets in a reference pool have the same default probability and mature after n years

- The time-until-default of assets in a reference pool is standard normally distributed

- All parameters remain constant during a run

5.2. Simulation procedure

Each run, the following procedure is carried out:

- Generate[1] 40 vectors with 100^2 random values. Each vector is distributed according to the specified copula: $(u_1, ..., u_{100}) \sim C_\Theta$; where $u_i \sim U(0,1)$. When dealing with the Gumbel^c of Clayton^c copula, it holds that $(1 - u_1, ..., 1 - u_{100}) \sim C_\Theta$

- Given the default probability p_d, determine the amount of defaults in reference pool i:

$$d_i = \Sigma_{j=1}^{100} \mathbb{1}_{\{u_j \leq p_d\}} \quad for\, i = 1, ..., 40$$

- Given d_i and the recovery rate r, determine the cumulative loss of reference portfolio i:

$$L_{ref,i} = (1 - r) \cdot d_i$$

---

[1]The algorithms which are used to sample from Archimedean copulas (Gumbel, Clayton) are implemented as described in [15]; sampling from elliptical copulas (Gaussian, t) is done using the built-in Matlab function `copularnd.m` (see Appendix)

[2]40 pools each consisting of 100 assets

Then, given the attachment-detachments points and $L_{ref,i}$, compute the:

- Default rate and the expected payoff per tranche of the ith CDO for $i = 1, ..., 40$

- Default rate and the expected payoff of the CDO^2 tranches given the expected payoff of the 40 underlying CDO tranches

Finally, the results of all the runs are averaged together. To make everything clear, two basic examples are given below.

Example 5.2.1 (CDO) Let us consider one pool of 100 assets. Let the attachment-detachment points of the CDO tranches be [0,5], [5,15], [15,100]. Set the recovery rate to $r = 0.5$. When 20 out the 100 assets default, the cumulative loss of the reference portfolio equals:

$$L_{ref} = 20 \cdot 0.5 = 10$$

The loss per tranche is as follows:

$$
\begin{array}{rclcl}
L_{[0,5]} & = & \max\{10 - 0, 0\} - \max\{10 - 5, 0\} & = & 5 \\
L_{[5,15]} & = & \max\{10 - 5, 0\} - \max\{10 - 15, 0\} & = & 5 \\
L_{[15,100]} & = & \max\{10 - 15, 0\} - \max\{10 - 100, 0\} & = & 0
\end{array}
$$

As as result, the junior and mezzanine tranches default while the senior tranche does not. The expected payoff (EP) of the collateral equals $\frac{(100-10)}{100} = 0.9$. The expected payoff of the tranches is as follows:

$$
\begin{array}{rclcl}
EP_{[0,5]} & = & \frac{5-5}{5} & = & 0 \\
EP_{[5,15]} & = & \frac{10-5}{10} & = & 0.5 \\
EP_{[15,100]} & = & \frac{85-0}{85} & = & 1
\end{array}
$$

Example 5.2.1 (CDO^2) Let us consider the payoff of 40 mezzanine ($[5-15]$) CDO tranches which serve as collateral for a CDO^2; denote the expected payoff of such a tranche of the ith CDO as $EP_{[5,15],i}$. The average expected payoff (i.e. expected payoff of the collateral) is equal to

$$\overline{EP}_{[5,15]} = \frac{1}{40}\Sigma_{i=1}^{40} EP_{[5,15],i}$$

From this it follows that the cumulative loss of the CDO^2 reference portfolio equals:

$$L_{ref} = 100 \cdot (1 - \overline{EP}_{[5,15]})$$

Since $\overline{EP}_{[5,15]}$ is a fraction, we can multiply by 100. This is done so that all required computations can be done in the same way as for the CDO case.

6. Results

In this chapter I will present some interesting results that were obtained after simulating different scenarios. Unless stated otherwise, it is assumed that the 5-year default probability $p_d = 0.05$; the *attachment-detachments* points are $[0, 5]$, $[5, 15]$, $[15, 100]$; and the CDO^2 uses the $[5, 15]$-tranches as collateral. Each data point is an average over 100.000 runs.

6.1. Increasing default correlation

In this section I will quantify and visualize the influence of increasing default correlation on default rates and expected payoff of CDO and CDO^2 tranches. Defaults are modeled using the one-factor Gaussian copula approach.

6.1.1. Payoff

In FIGURE 6.1.1. the relation between default correlation and a tranche's expected payoff is visualized. On the x-axis we find the market correlation ρ, leading to a pairwise correlation parameter of ρ^2 (see SECTION 4.4). An increase in ρ can be interpreted as default risk becoming less diversified. The expected payoff is *scaled;* all results are divided by the values corresponding to $\rho = 0$. This way all results can be easily compared to the case in which defaults occur independently.

Looking at the CDO plot, it can be observed that the expected payoff of the collateral is insensitive to changes in default correlation. However, the expected payoff of the three tranches does change as default correlation changes. As default risk becomes less diversified, the junior tranche benefits; risk shifts from the junior to the two more senior tranches. The expected payoff of the junior tranche can be described as a monotonic increasing function of ρ. While the junior tranche benefits, the expected payoff of the mezzanine tranches declines. It does so until it has lost nearly 10% of its payoff (compared to the $\rho = 0$ situation), which is the case around $\rho = 0.8$. As $\rho > 0.8$, the performance of the mezzanine tranches improves again, at the expense of the senior tranche.

The expected payoff of the CDO^2 tranches, which is depicted in FIGURE 6.1.1B, shows that a decline if performance of the collateral results in extreme deterioration of the junior tranche's expected payoff. This also holds for the mezzanine tranche, but to a lesser extent. Finally, the expected payoff of the senior tranche seems to be fairly insensitive to changes in default correlation. This is because the minimum expected payoff of the collateral (tranche) is 0.9168 (0.9556 on average), while the senior CDO^2 tranche will only absorb losses when the payoff of the collateral falls below 0.85. Apparently, this is hardly ever the case.
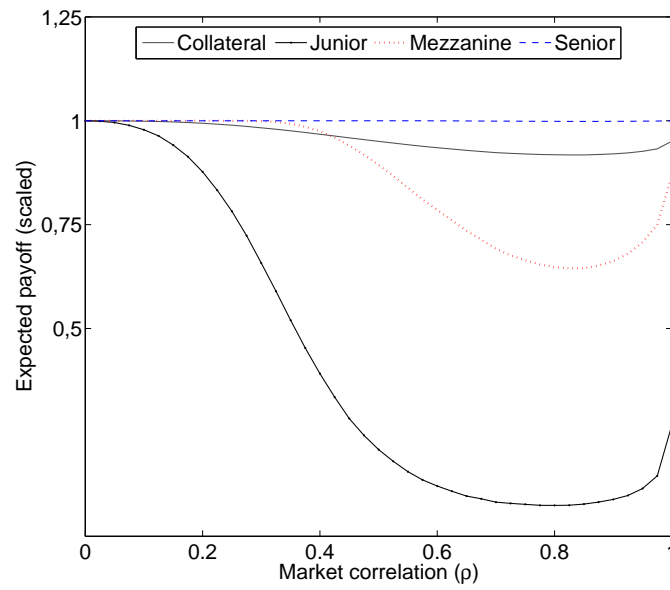
6.1.2. Default rate

In FIGURE 6.1.2. the relation between default correlation and a tranche's default rate is visualized. Obviously, there exists a relation between the expected payoff and the default rate of a tranche. The expected payoff of the junior tranche is a monotonic increasing function of ρ, while the default rate of the junior tranche is monotonic decreasing. Increasing ρ directly leads to the mezzanine tranche becoming more risky. Eventually, this is also the case for the senior tranche. The default rate of the mezzanine tranche reaches its maximum of 0.15 around $\rho = 0.65$. As $\rho > 0.65$, more risk is shifted to the senior tranche. In case of perfect correlation, all tranches have equal default rates. The great decrease in default rates as market correlation (ρ) approaches 1 is caused by the fact that the tail dependence changes from 0 to 1 as $\rho = 1$; either all assets default, or none will default.

The default rate curves of the CDO^2 tranches, as can be seen in FIGURE 6.1.2, are more or less equally shaped. Again, the junior tranche is most sensitive to the performance of the collateral; as soon as $\rho = 0.25$ it defaults with a probability greater than 0.95. The default rate of the mezzanine tranche increases the fastest between $\rho = 0.4$ and $\rho = 0.5$ and reaches it maximum around $\rho = 0.8$. It then declines again. The default rate of the senior tranche is at most 0.0567.
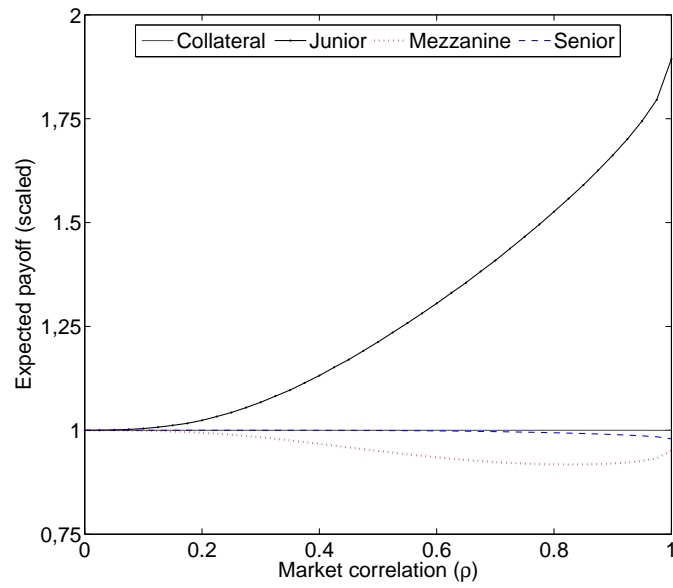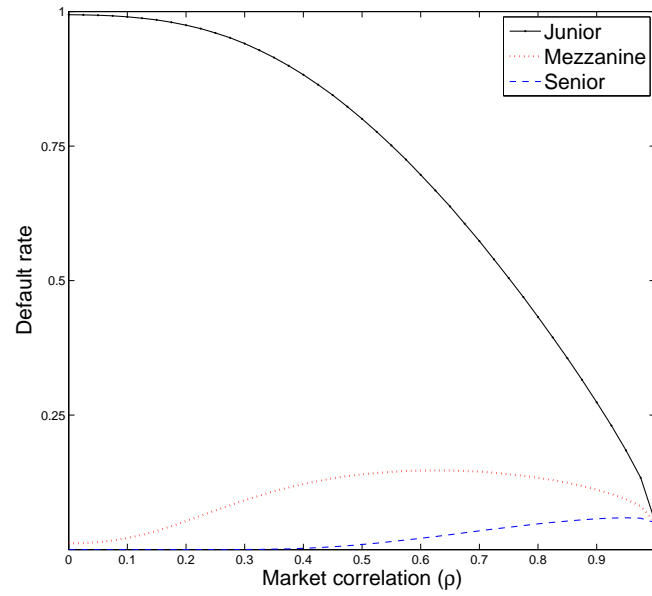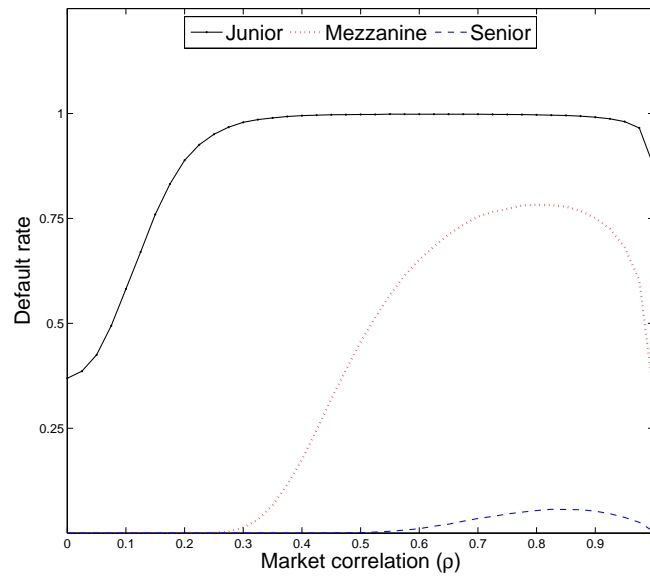
(a) CDO

(b) CDO^2

Figure 6.1.1.: Expected payoff (scaled) of CDO and CDO^2 tranches as a function of ρ

(a) CDO

(b) CDO^2

Figure 6.1.2.: Default rates of CDO and CDO^2 tranches as a function of ρ

6.2. The impact of tail dependence

In this section the influence of increasing tail dependence will be quantified and visualized. To do so, the t-copula is applied. In TABLE 6.1 some coefficients of tail dependence for the t-copula can be found. Again, ρ represents the market correlation coefficient, so the corresponding t-copula parameters equal (ρ^2, ν).

$\sqrt{\rho}/\nu$	50	10	5	4	3	2	1
1	1	1	1	1	1	1	1
0.8	0.0211	0.2925	0.4454	0.4896	0.5415	0.6042	0.6838
0.4	<0.001	0.0527	0.1599	0.2031	0.2606	0.3393	0.4523
0.2	<0.001	0.0204	0.0924	0.1275	0.1778	0.2522	0.3675
0	<0.001	0.0069	0.0498	0.0756	0.1161	0.1817	0.2929

Table 6.1.: Coefficient of tail dependence (t-copula) for different values of ρ and ν

6.2.1. Payoff

In FIGURE 6.2.1 the expected payoff of CDO and CDO^2 tranches as a function of (ν, ρ) is depicted; this is done for $\rho = 0.1$, $\rho = 0.4$ and $\rho = 0.8$. The expected payoff is again *scaled*; all results are divided by the values corresponding $\nu = 50$ (i.e. little tail dependence, see TABLE 6.1). Looking at the CDO plots, it can be observed that an increase in tail dependence is beneficial for the junior tranches in all cases. However, in case of $\rho = 0.1$ and $\rho = 0.4$, the mezzanine tranches' expected payoff declines a bit. As these tranches serve as collateral, this affects the expected payoff of several CDO^2 tranches. Looking at the upper two CDO^2 plots, the expected payoff of the junior tranches decreases rapidly. This is also the case for the mezzanine tranches, though to a lesser extent. In case of $\rho = 0.8$, an increase in tail dependence seems to affect the expected payoff of the mezzanine CDO tranche in a positive way: all corresponding CDO^2 tranches benefit. Finally, the senior tranches of both CDO and CDO^2's seem to be fairly insensitive to an increase in tail dependence. It seems that an increase in tail dependence hardly ever causes the cumulative loss in the reference portfolio to be so high that the payoff of the senior tranche is affected. In general, it seems that the lower ρ, the greater the influence of an increase in tail dependence is.

6.2.2. Default rate

In FIGURE 6.2.2 the default rate of CDO and CDO^2 tranches as a function of (ν, ρ) is depicted. After reading the previous subsections, the plots are more or less self explanatory. Again, an increase in tail dependence is beneficial for the junior CDO tranches. However, more importantly, the default rates of both the senior CDO and CDO^2 tranches clearly increase as tail dependence increases for $\rho = 0.1$ and $\rho = 0.4$. This seems strange, as previously was stated that the expected payoff appeared to be fairly insensitive to an increase in tail dependence. It can however be expained: the amount of times the senior tranche faces value impairment (i.e. defaults) increases, but the severity of this impairment seems to be bounded. The influence of tail dependence is relatively small in case of $\rho = 0.8$.
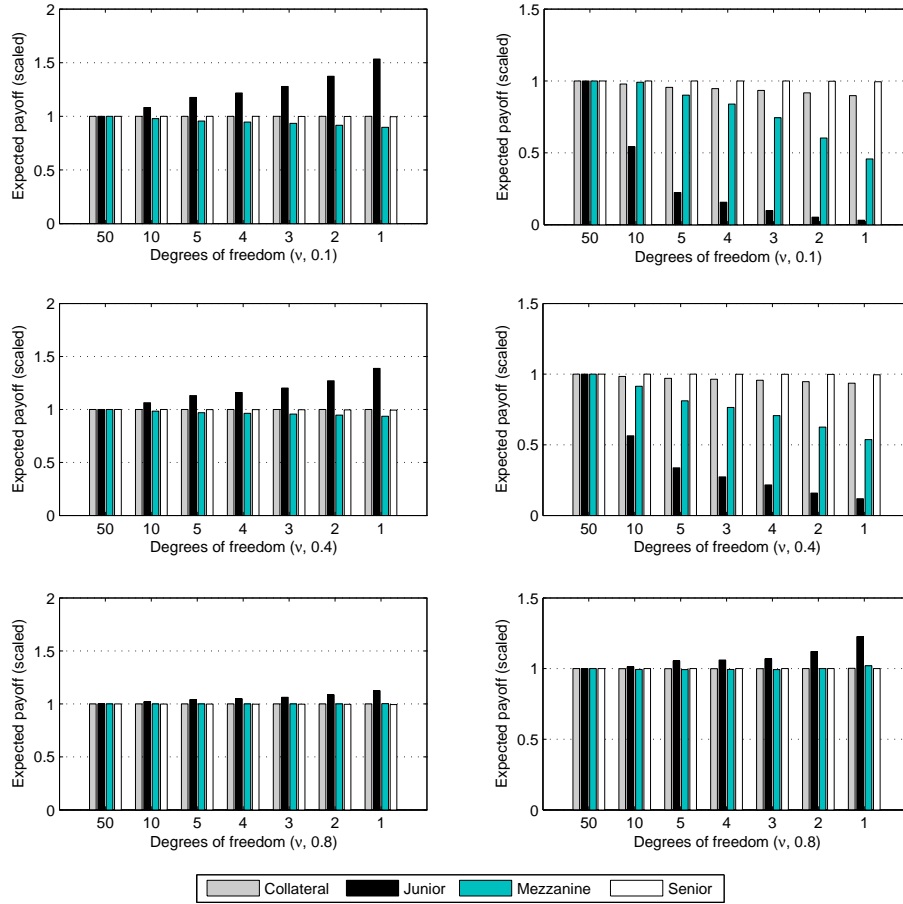
Figure 6.2.1.: Expected payoff (scaled) of CDO tranches (left) and CDO^2 tranches (right) as a function of (ν, ρ)

6.3. The Gumbel and Clayton copula: lower and upper tail dependence

Up to this point, the applied copulas are radial symmetric, and as a result exhibit equal upper and lower tail dependence. As was stated before, this is not the case for the Gumbel and Clayton copula. In FIGURE 6.3.1 the expected payoff of CDO tranches as a function of τ is plotted for the Gumbel and Clayton copula. The results are *scaled*; all results are divided by the values corresponding to $\tau = 0$. Note that the circle-marked lines correspond to the 'complement' copula $((1-u_1, ..., 1-u_{100}) \sim C_\Theta))$, which changes the location of the tail dependence. This line is plotted so that one can get an impression of the influence of *only* upper or *only* lower tail dependence given a certain amount of correlation τ. Remember that the Gumbel copula exhibits upper tail dependence, while the Clayton copula exhibits lower tail dependence. The amount of tail dependence as a
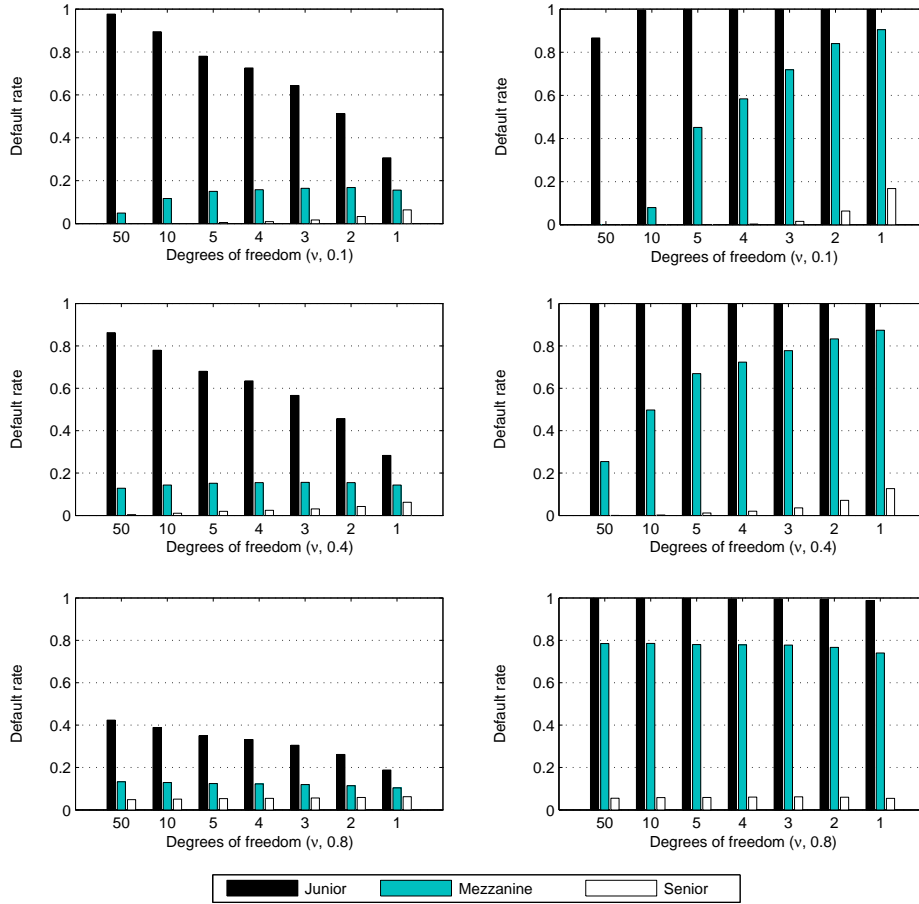
Figure 6.2.2.: Default rates of CDO tranches (left) and CDO^2 tranches (right) as a function of (ν, ρ)

function of τ is depicted in FIGURE 6.3.2.

As expected, the junior tranche benefits from an increase in correlation in general. Furthermore, the presence of lower tail dependence seems to be *more* beneficial in comparison to the presensce of upper tail dependence. The reference portfolio's expected loss distribution for different values of τ is depicted in FIGURE 6.3.3 and FIGURE 6.3.4. Note that the expected payoff of the junior tranche is > 0 only when the expected loss of the collateral is < 5 (given a tranche structure as specified earlier). When the distribution of the collateral's expected loss is translated to the junior tranche's expected payoff distribution, one will find expected payoffs as displayed in TABLE 6.2.

FIGURE 6.3.3 and FIGURE 6.3.4 show that the presence of lower tail dependence will increase the likeliness of large amounts of defaults. Furthermore, the impact of tail dependence will diminish as $\tau \to 1$; a result which was also obtained in SECTION 6.2.
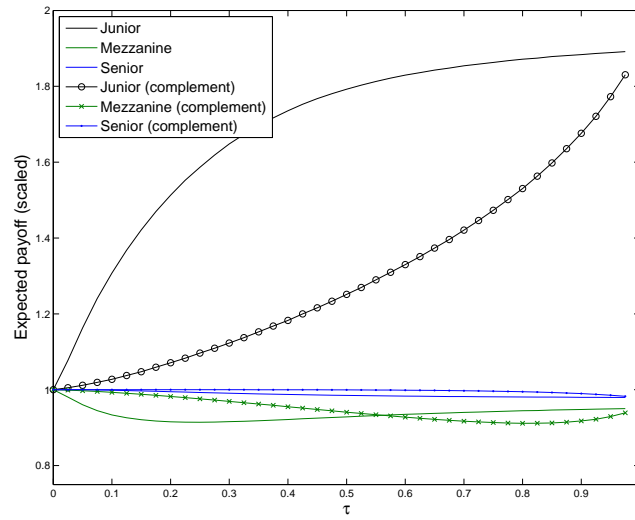
τ	Gumbel	Gumbel^c	Clayton	Clayton^c
0.2	0.59	0.68	0.76	0.54
0.4	0.72	0.79	0.87	0.59
0.8	0.89	0.92	0.94	0.77

Table 6.2.: Expected payoff of the junior tranche

One can observe that the expected payoff of the senior tranche seems to suffer a little from the presence of lower tail dependence. In case of the mezzanine tranche, the presence of lower tail dependence is detrimental up to $\tau \sim 0.20$ for the Gumbel copula and $\tau \sim 0.58$ for the Clayton copula.
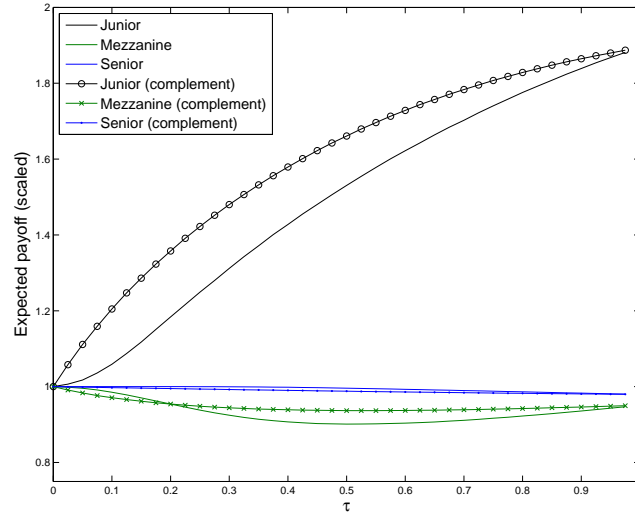
(a) Gumbel and Gumbel^c

(b) Clayton and Clayton^c

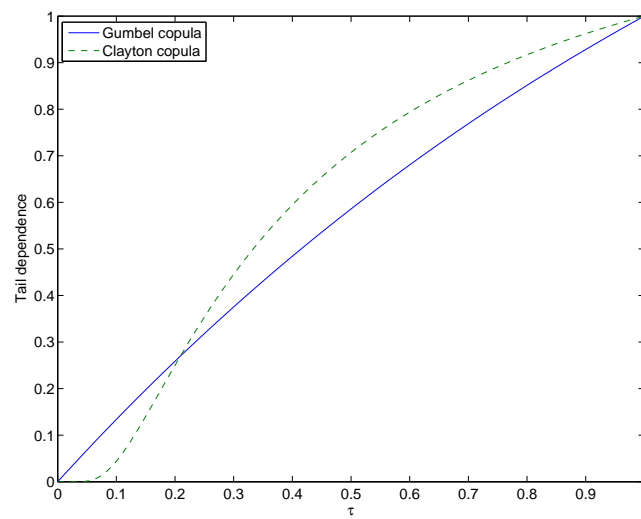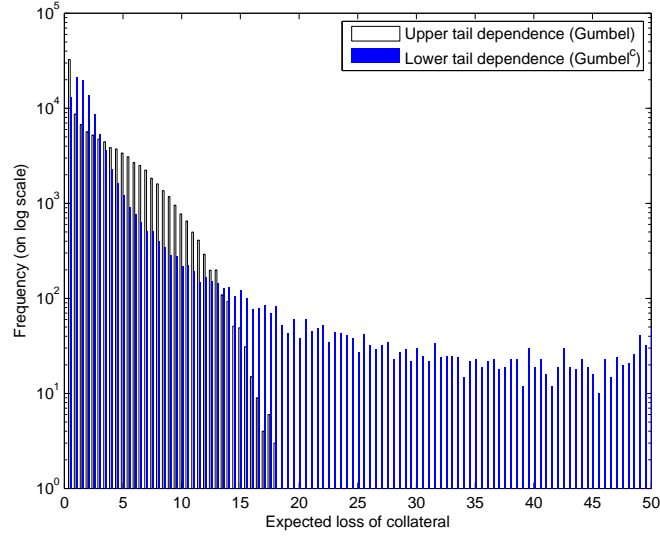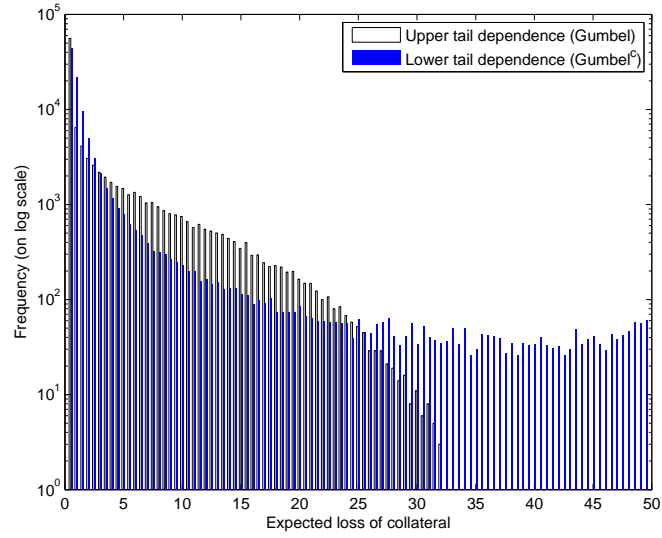**Figure 6.3.1.:** Expected payoff (scaled) of CDO tranches as a function of τ

Figure 6.3.2.: Tail dependence coefficients as a function of τ
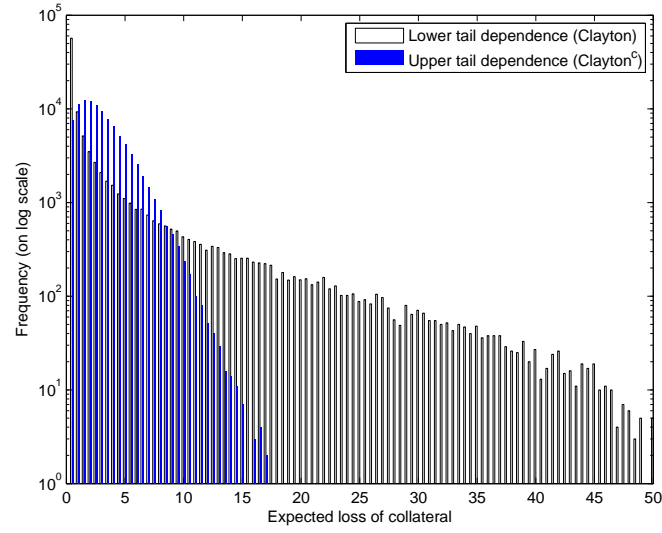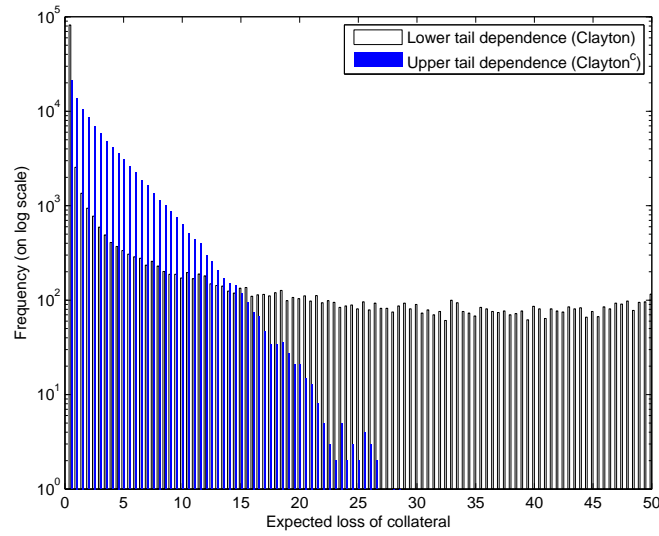
(a) $\tau = 0.2$

(b) $\tau = 0.4$

Figure 6.3.3.: Expected loss of collateral given τ for the Gumbel (and Gumbel^c) case

(a) $\tau = 0.2$

(b) $\tau = 0.4$

Figure 6.3.4.: Expected loss of collateral given τ for the Clayton (Clayton^c) case

7. Conclusion

In this paper I have investigated the process of securitization, why it is done, and how tranched securitization leads to the issuance of tranches with different risk characteristics. In addition, I have elaborated on copula methods and how they can be used to model default behavior. This is done mainly because CRAs have been accused of (ab)using the copula approach to rate tranches.

Using a simulation model, it is shown that an increase in default correlation affects both the default rate and the expected payoff of tranches. This is done by applying the industry-standard Gaussian copula approach. Without judging whether or not the use of the Gaussian copula is appropriate, it becomes clear that increasing default correlation has a negative effect on the default rate (and thus the credit rating) of senior CDO tranches; it has a positive effect on the default rate of the junior CDO tranches. The default rate of the junior CDO tranche is a monotonic decreasing function of ρ, while the default rate of the senior CDO tranche is a monotonic increasing function of ρ (except from $\rho = 1$, since tail dependence suddenly changes from 0 to 1). The default rate of the mezzanine CDO tranche is non-monotonic: it starts to decline again after a certain level of ρ.

To illustrate the impact of increasing default correlation, the default rate and the corresponding credit rating (according to the *Fitch's CDO matrix* presented in TABLE 7.1) for different values of market correlation ρ is presented in TABLE 7.2 below. For example, the senior CDO tranche has lost its AAA rating when $\rho = 0.4$. In addition, plots show that the default rate of junior and mezzanine CDO^2 tranches are very sensitive to collateral performance, while its senior tranche is still AAA rated when $\rho = 0.4$. As expected, expected payoff is inversely related to default rate.

Rating at issuance	AAA	AA+	AA	AA-	A+	A	A-	BBB+	BBB
5-year default probability (%)	0.05	0.19	0.26	0.36	0.56	0.62	0.92	1.20	1.89

Rating at issuance	BBB-	BB+	BB	BB-	B+	B	B-	CCC+	CCC
5-year default probability (%)	3.63	5.74	8.11	12.50	17.09	21.36	27.08	33.64	37.64

Table 7.1.: Part of Fitch's CDO default matrix (Source: [8])

		$\rho = 0.2$		$\rho = 0.4$		$\rho = 0.6$		$\rho = 0.8$	
CDO	Junior	97.48%	-	88.27%	-	69.66%	-	43.25%	-
	Mezzanine	5.31%	BB+	12.19%	BB-	14.68%	B+	13.33%	B+
	Senior	<0.01%	AAA	0.24%	AA	2.10%	BBB-	4.79%	BB+
CDO^2	Junior	88.84%	-	99.49%	-	99.83%	-	99.67%	-
	Mezzanine	<0.01%	AAA	17.51%	B	65.15%	-	78.24%	-
	Senior	<0.01%	AAA	<0.01	AAA	1.05%	BBB+	5.37%	BB+

Table 7.2.: The influence of default correlation on credit ratings

The introduction of tail dependence, which is often seen as a way to model dependency among

extreme events, has also been investigated. As the Gaussian copula does not exhibit any tail dependence, the t-copula is introduced. For fixed values of market correlation ρ, an increase of tail dependence also affects the expected payoff and default rate of both CDO and CDO^2 tranches.

In general, it seems that the lower ρ, the greater the influence of an increase in tail dependence is. Furthermore, the presence of tail dependence is beneficial for the junior CDO tranche (both expected payoff and default rate), while it is clearly detrimental for the default rate of the senior CDO tranche. However, the severity of this impairment seems to be bounded since its expected payoff remains fairly constant. The expected payoff of the mezzanine CDO tranche only benefits from tail dependence for relatively high values of ρ.

To illustrate the impact of tail dependence, the default rate and the corresponding credit rating for $\rho = 0.2$ is presented in TABLE 7.3 below.

$\rho = 0.2$		$\nu = 50\,(\lambda < 0.01)$		$\nu = 10\,(\lambda = 0.02)$		$\nu = 5\,(\lambda = 0.09)$		$\nu = 1\,(\lambda = 0.37)$	
CDO	Junior	95.78%	-	87.29%	-	76.06%	-	30.20%	CCC+
	Mezzanine	7.51%	BB	12.57%	B+	15.12%	B+	15.34%	B+
	Senior	<0.01%	AAA	0.17%	AA+	0.81%	A-	6.30%	BB
CDO^2	Junior	95.71%	-	99.50%	-	99.86%		99.87%	-
	Mezzanine	0.13%	AA+	16.83%	B+	51.06%	-	89.87%	-
	Senior	<0.01%	AAA	<0.01%	AAA	0.16%	AA+	15.89%	B+

Table 7.3.: The influence of tail dependence on credit ratings

The believe that there exists dependency among the occurrence of extreme events makes the use of the t-copula more appropriate in my opinion (compared to the Gaussian copula).

Finally, for illustrative purposes only, the effect of *only* upper or *only* lower dependence is visualized. This is done using the Gumbel copula and the Clayton copula (and their complements Gumbel^c and Clayton^c). For example, it can be observed that the expected payoff of junior CDO tranches benefits from the presence of lower tail dependence. Other relevant results concerning this part can be found in SECTION 6.3.

A. Appendix

The simulation procedure was implemented using Matlab. The function `monte.m` is used to compute *default rates* and *expected payoffs* of CDO and CDO^2 tranches; sampling from copulas is done using the function `coprnd.m`.

A.1. monte.m

The function `monte.m` can, for example, be called as follows:

```
[ settings cdo_data csq_data ] = monte( 0.05, 'Gaussian', 0.2^2, 0.5, [0 .05 .15 1], [.05 .15], 100000 )
```

Its code is as follows:

```
function [ settings cdo_data csq_data ] = monte( p_d, copula, para, rr, att, sq_att, runs )
% p_d     = cumulative default prob
% copula  = used copula
% para    = copula parameters
% rr      = recovery rate in case of default
% att     = CDO attachments points (as fraction, include 0 and 1)
% sq_att  = attachments points of CDO^2 tranche
% runs    = # simulation runs (optional, 1000 is standard)

tic;

%optional parameters
if ~exist('runs')
    runs=1000;
end

%fixed parameters
a_pool  = 40;  %amount of pools
n_pool  = 100; %pool size
%.

if length(para) == 1
    para(2) = NaN;
end

n_tran  = length(att) - 1; %amount of tranches
sq_tran = find(sq_att(1) == att); %find 'index' of 'corresponding' sq tranche
%.

absolute = @(notional, payoff, a, b) (b-a) - (max(notional-payoff-a,0) - max(notional-payoff-b,0));
fraction = @(payoff, a, b) payoff / (b-a);
```

```matlab
%monte carlo
cdo_pool_payoff = zeros(a_pool, runs);
defaults        = zeros(1,runs);

for r = 1:runs %runs

        % generate samples
        uni = coprnd( copula, para, a_pool, n_pool );
        %.
        default = sum((uni<=p_d)')';

        cdo_pool_payoff(:, r) = n_pool-(1-rr)*default;        %determine actual payoff

        if mod(r, 1000) == 0                                  %runtime information
        %fprintf('%d runs done..\n', r);
        end
end
%.

settings    = [p_d para(1) para(2)]';
cdo_data    = zeros(1+n_tran,2);
csq_data    = zeros(1+n_tran,2);

cdo_tran_default            = zeros(a_pool, runs, n_tran);
cdo_tran_payoff             = zeros(a_pool, runs, n_tran);
cdo_tran_payoff_fraction    = zeros(a_pool, runs, n_tran);

notional = n_pool;

%determine CDO loss
cdo_pool_loss   = notional-cdo_pool_payoff;
%.

%evaluate CDO tranche performance
for i=1:n_tran
cdo_tran_default(:,:,i) = (cdo_pool_loss>(att(i)*notional));
cdo_tran_payoff(:,:,i)  = absolute(notional, cdo_pool_payoff, att(i)*notional, att(i+1)*notional);
cdo_tran_payoff_fraction(:,:,i) = fraction(cdo_tran_payoff(:,:,i), att(i)*notional, att(i+1)*notional);

cdo_data(1+i, 1) = mean(mean(cdo_tran_default(:,:,i)));
cdo_data(1+i, 2) = mean(mean(cdo_tran_payoff_fraction(:,:,i)));
end
cdo_data(1,1)    = NaN; %trivial
cdo_data(1,2)    = mean(mean(cdo_pool_payoff))/notional;
%.

csq_tran_default            = zeros(1, runs, n_tran);
csq_tran_payoff             = zeros(1, runs, n_tran);
csq_tran_payoff_fraction    = zeros(1, runs, n_tran);

%notional = 1;
notional = a_pool*n_pool*(sq_att(2) - sq_att(1));

%determine CDO^2 loss
csq_pool_payoff = sum(cdo_tran_payoff(:,:,sq_tran));
csq_pool_loss   = notional - csq_pool_payoff;

%.
%evaluate CDO^2 tranche performance
for i=1:n_tran
csq_tran_default(:,:,i) = (csq_pool_loss>(att(i)*notional));
csq_tran_payoff(:,:,i)  = absolute(notional, csq_pool_payoff, att(i)*notional, att(i+1)*notional);
csq_tran_payoff_fraction(:,:,i) = fraction(csq_tran_payoff(:,:,i), att(i)*notional, att(i+1)*notional);

csq_data(1+i, 1) = mean(mean(csq_tran_default(:,:,i)));
csq_data(1+i, 2) = mean(mean(csq_tran_payoff_fraction(:,:,i)));
end
csq_data(1,1)    = NaN; %trivial
csq_data(1,2)    = mean(csq_pool_payoff)/notional;
%.
```

```
fprintf('\n----------------------------------------------------------------\n');
fprintf('Default prob.: %0.4f | Recovery rate: %0.2f\n', p_d, rr);
fprintf('Copula: %s | Parameters: %0.4f , %0.4f\n\n', copula, para(1), para(2));
fprintf('Attachments points: '); fprintf('%0.2f ', att); fprintf('Runs: %d ', runs)
fprintf('\n----------------------------------------------------------------\n\n');

toc;

end
```

A.2. coprnd.m

The function `coprnd.m` can, for example, be called as follows:

```
[ u ] = coprnd( 't', [0.2^2, 5], 10, 2 )
```

Its code is as follows:

```
function [ u ] = coprnd( copula, para, m, n )
% This function returns M vectors of length N with random values generated
% from a COPULA given its parameters PARA (as vector)

%
% The algorithms which are used to sample from Archimedean copulas (Gumbel, Clayton)
% are implemented as described in:
% [15] M. R. Melchiori, "Tools for sampling Multivariate Archimedean Copulas,"
% YieldCurve, April 2006
%

%
% Sampling from elliptical copulas (Gaussian & t) is done using the
% built-in Matlab-function 'copularnd.m'
%

if strcmp(copula,'Gumbel')
    theta = para(1);

    if theta > 1 && n > 1
    alpha   = 1/theta;
    beta    = 1;
    gamma   = (cos(pi/(2*theta)))^theta;
    delta   = 0;

    o = pi.*(rand(m, 1)-0.5);
    w = exprnd(1,m,1);

    t0 = atan(beta*tan(pi*alpha/2))/alpha;

    z1 =  sin (alpha * ( t0 + o ) ) ./  ((cos(alpha*t0) * cos(o) )).^(1/alpha);
    z2 =  (cos( alpha*t0 + (alpha-1)*o ) ./ w).^((1-alpha)/alpha);
    z  = z1 .* z2;
    Y  = gamma*z + delta;

    u      = rand(m,n);
    s      = bsxfun(@rdivide, -log(u),Y);

    u      = exp(-s.^(1/theta));
    else %independence
    u = rand(m,n);
    end
end
```

```matlab
if strcmp(copula,'Gumbel-inv')
    u = 1-coprnd('Gumbel', para, m, n);
end

if strcmp(copula,'Clayton')
    theta = para(1);

    if theta > 0 && n > 1
    theta = para(1);

    u        = rand(m,n);
    Y        = gamrnd(1/theta,1,m,1);
    s        = bsxfun(@rdivide, -log(u),Y);

    u        = (1+s).^(-1/theta);
    else %independence
    u        = rand(m,n);
    end
end

if strcmp(copula,'Clayton-inv')
    u = 1-coprnd('Clayton', para, m, n);
end

if strcmp(copula,'Gaussian')

    rho     = para(1);              %pair-wise correlation coeffiecient

    if rho == 1
        rho = 1-10^(-10);
    end

    R(1:n,1:n)      = rho;
    R(1:n+1:n*n)    = 1;

    u = copularnd('Gaussian',R,m); %built-in
end

if strcmp(copula,'t')

    rho     = para(1);              %pair-wise correlation coeffiecient
    nu      = para(2);              %degrees of freedom

    if rho == 1
        rho = 1-10^(-10);
    end

    R(1:n,1:n)      = rho;
    R(1:n+1:n*n)    = 1;

    u = copularnd('t',R,nu,m); %built-in
end

end
```

Bibliography

[1] E. Benmelech and J. Dlugosz, "The alchemy of CDO credit ratings," *Journal of Monetary Economics*, vol. 56, pp. 617–634, 2009.

[2] ——, "The Credit Rating Crisis," *NBER Macroeconomics Annual*, vol. 24, pp. 161–207, 2009.

[3] U. Cherubini, E. Luciano, and W. Vecchiato, *Copula Methods in Finance*, ser. The Wiley Finance Series. John Wiley & Sons, 2004.

[4] J. Coval, J. Jurek, and E. Stafford, "The Economics of Structured Finance," *Journal of Economic Perspectives*, vol. 23, no. 1, pp. 3–25, 2008.

[5] C. Donnelly and P. Embrechts, "The devil is in the tails: actuarial mathematics and the subprime mortgage crisis," January 2010.

[6] P. Embrechts, F. Lindskog, and A. McNeil, "Modelling Dependence with Copulas and Applications to Risk Management," September 2001.

[7] I. Fender and J. Kiff, "CDO rating methodology: some thoughts on model risk and its implications," August 2005.

[8] Fitch Ratings, "Global Rating Criteria for Collateralised Debt Obligations," 2004.

[9] G. Gorton and N. S. Souleles, "Special Purpose Vehicles and Securitization," 2007.

[10] J. P. Hunt, *Capital Structure and Corporate Financing Decisions*, 2011, ch. 17: Rating Agencies and Credit Insurance, pp. 299–314.

[11] A. A. Jobst, "What is securitization?" *Finance & Development*, vol. 45, no. 3, pp. 48–49, September 2008.

[12] S. Jones, "Of couples and copulas: the formula that felled Wall St." *Financial Times*, April 2009.

[13] D. X. Li, "On default correlation: a copula function approach," *Journal of Fixed Income*, vol. 9, pp. 43–54, 2000.

[14] A. Martín-Oliver and J. Saurina, "Why do banks securitize assets?" November 2007.

[15] M. R. Melchiori, "Tools for sampling Multivariate Archimedean Copulas," *YieldCurve*, April 2006.

[16] R. Nelsen, *An Introduction to Copulas. 2nd edition.* New York: Springer, 2006.

[17] T. Sabarwal, "Common structures of asset-backed securities and their risks," *Corporate Ownership & Control*, vol. 4, pp. 258–265, 2006.

[18] ——, "Securitization: understanding the risks and rewards," *Q Finance: The Ultimate Resource*, pp. 576–578, 2009.

[19] T. Schmidt, "Coping with Copulas," December 2006.

[20] A. Sklar, "Fonctions de répartition á n dimensions et leurs marges," *Publications de l'Institut de Statistique de l'Université de Paris*, vol. 8, pp. 229–231, 1959.

[21] P. K. Trivedi and D. M. Zimmer, "Copula Modeling: An Introduction for Practitioners," *Foundations and Trends R® in Econometrics*, vol. 1, no. 1, pp. 1–111, 2005.

[22] U.S. Securities and Exchange Commission, "Report on the Role and Function of Credit Rating Agencies in the Operation of the Securities Markets," January 2003.