Statistical Methods: Lecture 2

Lecture Overview

Probability in statistics

Question about population:

Alice is pregnant. Probability of baby being ≥ 21 days early?

Estimate!

E.g. sample of 300 births, thereof 6 were ≥ 21 days early.
$\Rightarrow$ estimated probability $= \frac{6}{300} = 0.02 = 2\%$.

$\Rightarrow$ Question $\rightarrow$ sample $\rightarrow$ answer.
Problem(?): different sample $\Rightarrow$ different estimate.

Probability in statistics

Testing a claim: Is the coin unbiased?
Throwing coin 100 times, consider the following scenarios:

- 47 times Heads (47H) and 53 Tales (53T)
- 5H and 95T
- 37H and 63T

Idea: does scenario "fit" the claim?
Not "fitting": if claim true $\Rightarrow$ small probability of the observed outcome.

We treat probability theory more extensively than in the book. This is required for the assignments and exam.

3.2 Basic concepts of probability

Probability experiment: production of (random) outcome.
Example: die roll, coin toss.

Sample space Ω: Set of all possible outcomes.
Example (for dies): $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Event $A, B, \ldots$: collection of outcomes.
Example: $A = \{\text{even number is thrown}\} = \{2, 4, 6\}$.
Simple event: consists 1 outcome. E.g. $\{1\}$.

Probability measure: function $P(\cdot)$ assigning values between 0 and 1 to events.
(More properties later.)
Example: $P(A) = P(\{2, 4, 6\}) = \frac{1}{2}$.

Interpretation of probabilities:

- $P(A) = 0$: occurence of A is impossible.
  E.g. $P(\varnothing) = 0$. ($\varnothing$ = empty event: nothing happens)
- $P(A) = 1$: occurence of A is certain. Example: $P(\Omega) = 1$.
- Event A is unlikely when $P(A)$ is small, e.g. < 0.05.

3.2 Basic concepts of probability

Three ways to determine probability $P(A)$ of event A:

1. Estimate with relative frequency:

$$P(A) = \frac{\text{number of times } A \text{ occurred}}{\text{number of times the procedure was repeated}}.$$

2. Classical (theoretical) approach.
   Make probability model (outcome space, probability measure, etc.), compute $P(A)$ (using properties of P).
   Examples: rolling dice, card games, etc.
3. Subjective approach. (Not of importance for this course.)
   Estimate $P(A)$, based on intuition and/or experience.

3.2 Basic concepts of probability

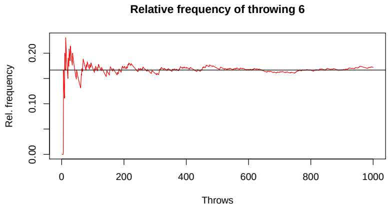
Example: relative frequency

NBA Season 2015/16, Kevin Durant: 644 free throws, 577 scored.
Estimate probability of a hit: $\frac{577}{644} \approx 0.896$.

Many trials: relative frequency $\approx$ real (true) value of $P(A)$.
Supported by Law of large numbers (LLN):

Theorem (LLN: relative frequency)

Suppose a procedure is repeated (independently).
$\Rightarrow$ relative frequency probability of an event A tends towards true $P(A)$.



3.2 Basic concepts of probability

Example: classical (theoretical) approach

Throw a fair (unbiased) coin $3\times$.
Probability of $1\times$ Heads?

Sample space Ω has $2 \times 2 \times 2 = 8$ outcomes (write "H" (Heads), "T" (Tails)):

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Interesting event $A = \{1 \text{ Heads}\} \quad \Rightarrow \quad A = \{HTT, THT, TTH\}.$

Here: outcomes equally likely, hence:

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of different simple events}} = \frac{3}{8}.$$

Determining $P(A)$ if all outcomes are equally likely

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of different simple events}}.$$

3.2 Basic concepts of probability

We used

Counting principle

Suppose 2 probability experiments are performed.
1^{st}: $a \geq 0$ possible outcomes;
2^{nd}: $b \geq 0$ consecutive outcomes.
Combined: $a \times b$ possible outcomes.

Principle extends to any number of experiments.

Example 1

First throw coin, then roll (ordinary) die.
$\Rightarrow$ total number of outcomes of both experiments: $2 \times 6 = 12$.

Example 2

License plate consisting of 3 letters and 3 digits.
$\Rightarrow$ total number of possible license plates $= 26^3 \times 10^3 = 17\ 576\ 000$.

3.2 Basic concepts of probability

We treated all outcomes as being equally likely. In general: not necessarily true.

Example 1: Biased die

Probability of "6" is 2× other numbers' probabilities:

$$P(1) = P(2) = \ldots = P(5) = \frac{1}{7},\ P(6) = \frac{2}{7}$$

Example 2: Vase with 6 balls: red, red, red, orange, orange, blue

Balls of same colour indistinguishable.
$\Rightarrow$ outcome space: $\Omega = \{\text{red}, \text{orange}, \text{blue}\}$ and

$$P(\text{red}) = \frac{1}{2}, \quad P(\text{orange}) = \frac{1}{3}, \quad P(\text{blue}) = \frac{1}{6}.$$

Example 3: Machine failure

Due to 1 out of 3 causes: electrical / mechanical failure / human misuse.
Probabilities:

$$P(\text{electrical}) = 0.2, \quad P(\text{mechanical}) = 0.5, \quad P(\text{human}) = 0.3.$$

3.2 Basic concepts of probability

In all cases of discrete sample spaces (finite/countable):

General probability measure for finite/countable sample space

- Sample space Ω is finite/countable.
- Each outcome $\omega \in \Omega$ has a probability:
  - $P(\omega) \geq 0$ and
  - $\sum_{\omega \in \Omega} P(\omega) = 1$
- Any event A: probability defined by

$$P(A) = \sum_{\omega : \omega \in A} P(\omega).$$

Note: includes case of all outcomes equally likely.

3.2 Basic concepts of probability

General recipe to find P(A) (in discrete case)

- Find sample space Ω.
- Determine probabilities $P(\omega)$ for all ω in Ω.
  Finite case with N equally likely outcomes: $P(\omega) = 1/N$.
- Determine which outcomes belong to A.
- Compute $P(A) = \sum_{\omega:\omega\in A} P(\omega)$.

Example: biased die

See prev. slide. Probability of "even number"?

- $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Outcomes not equally likely; $P(6) = \frac{2}{7}$ and $P(1) = P(2) = \ldots = P(5) = \frac{1}{7}$.
- $A = \{\text{even number}\} = \{2, 4, 6\}$.

$$\Rightarrow \; P(A) = P(\{2, 4, 6\}) = P(2) + P(4) + P(6) = \frac{1}{7} + \frac{1}{7} + \frac{2}{7} = \frac{4}{7}.$$

3.2 Basic concepts of probability: recap

Three approaches to determine probability

- Estimating with relative frequency
- Classical (theoretical) approach
- Subjective probability

General recipe to find P(A) (in discrete case)

- Find sample space Ω.
- Determine probabilities $P(\omega)$ for all ω in Ω.
- Determine: outcomes forming A and compute $P(A) = \sum_{\omega:\omega\in A} P(\omega)$.

3.3 Addition rule: Example

Draw a card from each of two card decks, totally at random. What is the probability of drawing *at least one* spade?

$$? \ P(\text{a spade}) = P(\text{spade from pile 1}) + P(\text{spade from pile 2}) = \frac{1}{2} + \frac{1}{2} = 1 \ ?$$

Wrong, we counted "drawing two spades" twice.

Two solutions:

- Count. Sample space $\Omega = \{SS, SH, HS, HH\}$; outcomes equally likely. Then $\{\text{a spade}\} = \{SS, SH, HS\}$, hence

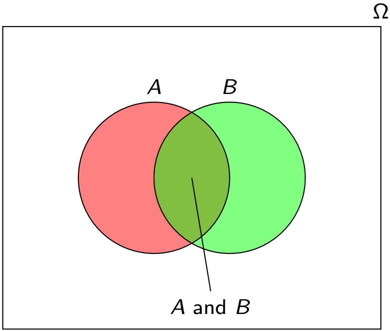
$$P(\text{a spade}) = P(\{SS, SH, HS\}) = 3 \cdot \frac{1}{4} = \frac{3}{4}.$$

- Subtract drawing two spades:

$$\begin{aligned} P(\text{a spade}) &= P(\text{spade from pile 1}) + P(\text{spade from pile 2}) - P(\text{spade twice}) \\ &= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}. \end{aligned}$$

3.3 Addition rule

Second solution: example of addition rule.



Addition rule

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Idea: every outcome is counted only once.

3.3 Addition rule

Notation:

$A \cup B = A$ or B: union, set of outcomes which are in A or B (both allowed!)
$A \cap B = A$ and B: intersection, set of outcomes which are both in A and B.

Addition rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Example: 3 coin tosses (unbiased coin)

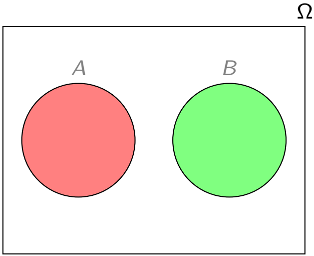
P("Tails twice or Heads in first throw")? discuss and calculate 3 min
$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

$A = \{\text{Tails twice}\} = \{HTT, THT, TTH\}$, so $P(A) = \frac{3}{8}$.
$B = \{\text{Heads in first throw}\} = \{HTT, HHT, HTH, HHH\}$, so $P(B) = \frac{4}{8} = \frac{1}{2}$.
$A \cap B = \{\text{Tails twice and heads in first throw}\} = \{HTT\}$, so $P(A \cap B) = \frac{1}{8}$.

$$\Rightarrow \; P(\text{Tails twice or Heads in first throw}) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{3}{8} + \frac{1}{2} - \frac{1}{8} = \frac{3}{4}$$

3.3 Addition rule

A and B are disjoint if they exclude each other, i.e. $A \cap B = \emptyset$.



Addition rule for 2 disjoint events

If A and B are disjoint then:

$$P(A \cup B) = P(A) + P(B).$$

Different from independence

Example

Roll a fair die once. Probability of "even number or 3"?

$\Omega = \{1, 2, 3, 4, 5, 6\}$.
$A = \{\text{even number}\} = \{2, 4, 6\}$, so $P(A) = \frac{3}{6} = \frac{1}{2}$.
$B = \{3\}$, so $P(B) = \frac{1}{6}$. Furthermore, $A \cap B = \varnothing$, so A and B disjoint. Hence,

$$P(A \cup B) = P(A) + P(B) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}.$$

3.3 Addition rule

General addition rule for disjoint events

Let $A_1, \ldots, A_m$ be disjoint, i.e. $A_i \cap A_j = \varnothing$ for $i \neq j$. Then:

$$P(A_1 \cup \ldots \cup A_m) = \sum_{i=1}^{m} P(A_i)$$

Example: rolling two fair dice

Probability that "sum equals 4, 8, or 9"?

Sample space $\Omega = \{(1,1), \ldots, (1,6), (2,1), \ldots (6,6)\}$ contains $6 \times 6 = 36$ outcomes; equal probabilities.
$A = \{\text{Sum is } 4\} = \{(1,3), (2,2), (3,1)\}$,
$B = \{\text{Sum is } 8\} = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$,
$C = \{\text{Sum is } 9\} = \{(3,6), (4,5), (5,4), (6,3)\}$.
Disjoint events! General addition rule:

$$P(\text{sum is 4, 8 or 9}) = P(A) + P(B) + P(C) = \frac{3}{36} + \frac{5}{36} + \frac{4}{36} = \frac{1}{3}.$$

3.3 Addition rule

$\bar{A}$ (or A^c): complement of A; outcomes which are not in A.

Complement rule

$$P(\bar{A}) = 1 - P(A).$$

Example: 3 fair coin tosses

Probability of at least one Head?
$A = \{\text{at least 1 Heads}\} \quad \Rightarrow \quad \bar{A} = \{\text{no Heads}\}$.
Complement rule:

$$P(A) = 1 - P(\bar{A}) = 1 - P(\text{no Heads}) = 1 - P(TTT) = 1 - \frac{1}{8} = \frac{7}{8}.$$

3.3 Addition rule: recap

Addition rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

General addition rule for disjoint events

Let $A_1, \ldots, A_m$ be disjoint, i.e. $A_i \cap A_j = \varnothing$ for $i \neq j$. Then:

$$P(A_1 \cup \ldots \cup A_m) = \sum_{i=1}^{m} P(A_i)$$

Complement rule

$$P(\bar{A}) = 1 - P(A).$$

3.4 and 3.5 Multiplication rule

Alice rolls a die twice, first score was 3. Probability that sum equals 8?

- Given first die equals 3, only 6 possibilities: $(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)$.
- Given first die equals 3, conditional probability of each such outcome: $\frac{1}{6}$.
- "sum is 8" $\hat{=} (3,5) \Rightarrow$ desired probability $= \frac{1}{6}$.

Let $A = \{\text{first die is 3}\}$ and $B = \{\text{sum is 8}\}$; we computed conditional probability that B occurs given that A has occurred, denoted as "$P(B|A)$".

3.4 and 3.5 Multiplication rule

$P(B|A)$: conditional probability that B occurs given that A has occured.

General formula:

Definition (Conditional probability)

If $P(A) > 0$, then:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Previous example: $P(A \cap B) = P((3,5)) = \frac{1}{36}$, $P(A) = \frac{1}{6}$. Indeed, $P(B|A) = \frac{1}{6}$.

Explanation:

- If A has occured, B only happens if outcome is in both A and B. Hence, in $A \cap B$.
- Sample space reduced to A.
- Hence, given A has occured, compute $P(A \cap B)$ relative to $P(A)$:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

3.4 and 3.5 Multiplication rule

Conditional probability $P(B|A) = \frac{P(A \cap B)}{P(A)}$.

Example: 2 fair coin tosses deliver HH

Conditional probability of "two Heads" given that

1. First flip is Heads?
2. At least one Heads?   discuss and calculate 3 min

Sol. (1.): Outcome space $\Omega = \{HH, HT, TH, TT\}$. $B = \{\text{Twice Heads}\} = \{HH\}$ and $A_1 = \{\text{First flip Heads}\} = \{HH, HT\}$. Then $A_1 \cap B = \{HH\}$, so

$$P(B|A_1) = \frac{P(A_1 \cap B)}{P(A_1)} = \frac{P(\{HH\})}{P(\{HH, HT\})} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Sol. (2.): Here, $A_2 = \{\text{At least one Heads}\} = \{HH, HT, TH\}$, so

$$P(B|A_2) = \frac{P(A_2 \cap B)}{P(A_2)} = \frac{P(\{HH\})}{P(\{HH, HT, TH\})} = \frac{1/4}{3/4} = \frac{1}{3}.$$

3.4 and 3.5 Multiplication rule

Caution: in general, $P(B|A) \neq P(A|B)$.

Example: rolling fair die twice

Recall Alice's experiment ($P(B|A) = \frac{1}{6}$ where $A = \{\text{First die is } 3\}$ and $B = \{\text{Sum is } 8\}$).

However, $B = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$ so $P(B) = \frac{5}{36}$

$$\Rightarrow \quad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{5/36} = \frac{1}{5} \neq \frac{1}{6} = P(B|A).$$

3.4 and 3.5 Multiplication rule

Vase with balls $1, 2, \dots, 9$. Draw two balls, after each other. What is the probability of ball 1, then ball 2?

$$P((1,2)) = \frac{1}{81}?$$

No! Only 8 balls left in 2^{nd} draw. Two solutions:

- Count. $\Omega = \{(1,2),(1,3),\dots,(1,9),(2,1),(2,3),\dots,(2,9),\dots(9,8)\}$. Ω has $9 \times 8 = 72$ elements. So:

$$P((1,2)) = \frac{1}{72} \neq \frac{1}{81}.$$

- Process information: $P((1,2)) = P(\text{first 1, then 2})$

$$= P(\text{first 1}) \cdot P(\text{draw ball 2} \mid \text{ball 1 is drawn}) = \frac{1}{9} \cdot \frac{1}{8} = \frac{1}{72}.$$

3.4 and 3.5 Multiplication rule

Second solution uses conditional probabilities:

$P(B|A)$: probability of B occurring, given that A occurred already.

Multiplication rule

$$P(A \cap B) = P(A) \cdot P(B|A).$$

Previous example:

- $A = \{\text{First ball is } 1\}$. So $P(A) = \frac{1}{9}$.
- $B = \{\text{Second ball is } 2\}$.
- $B|A = \{\text{Second ball is 2, given first ball is } 1\}$. 8 balls left, so $P(B|A) = \frac{1}{8}$.

$$\Rightarrow \quad P(\text{first 1, then 2}) = P(A \cap B) = P(A) \cdot P(B \mid A) = \frac{1}{9} \cdot \frac{1}{8}.$$

- Also, note that $B = \{(1,2), (3,2), \ldots, (9,2)\}$. So

$$P(B) = \frac{8}{72} = \frac{1}{9} \neq \frac{1}{8} = P(B \mid A).$$

3.4 and 3.5 Multiplication rule

Recall previous example. Now: draw two balls with replacement.
Probability of first drawing ball 1, then ball 2?

- $A = \{\text{First ball is } 1\}$. So $P(A) = \frac{1}{9}$.
- $B = \{\text{Second ball is } 2\}$.
- $B|A = \{\text{Second ball is 2, given first ball is } 1\}$. Now 9 balls left, so $P(B|A) = \frac{1}{9}$.

$$\Rightarrow \quad P(\text{first 1, then 2}) = P(A \cap B) = P(A) \cdot P(B \mid A) = \frac{1}{9} \cdot \frac{1}{9} = \frac{1}{81}.$$

- Also, note that $B = \{(1,2), (2,2), \ldots, (9,2)\}$. So

$$P(B) = \frac{9}{81} = \frac{1}{9} = P(B \mid A).$$

- Moreover: $P(A \cap B) = P(A) \cdot P(B)$.

3.4 and 3.5 Multiplication rule

Definition (Independence)

Two events A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B).$$

Thus: $P(B) = P(B|A)$ when A and B are independent.

Example: roll fair die twice

Are $A = \{\text{First throw is } 1\}$ and $B = \{\text{Sum is } 7\}$ independent?
Are A and B independent?

$$P(A) = P(\{(1,1), \ldots, (1,6)\}) = \frac{6}{36} = \frac{1}{6},$$
$$P(B) = P(\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}) = \frac{6}{36} = \frac{1}{6}.$$

3.4 and 3.5 Multiplication rule

Example: roll fair die twice

$A = \{\text{First throw is } 1\}$ and $B = \{\text{Sum is } 7\}$.

$$P(A \cap B) = P(\{\text{First throw is 1 and sum is } 7\}) = P((1, 6)) = \frac{1}{36}.$$

Consequently,

$$P(A \cap B) = \frac{1}{36} = P(A) \cdot P(B);$$

A and B are independent!

Always check independence of events by definition, no vague reasoning.

3.4 and 3.5 Multiplication rule

Caution: independence $\neq$ disjoin
tness.

Example: roll fair die once

$A = \{\text{number is even}\}$ and $B = \{3 \text{ is rolled}\}$.

$\Rightarrow\ A \cap B = \varnothing$, so A and B are disjoint and $P(A \cap B) = 0$.

However, $P(A) = \frac{1}{2}$ and $B = \frac{1}{6}$, so $P(A) \cdot P(B) = \frac{1}{12} \neq 0$.
Hence, A and B are not independent!

Example: roll fair die twice

$A = \{\text{First throw is } 1\}$ and $B = \{\text{Sum is } 7\}$.

Saw earlier:
1. $A \cap B = \{(1, 6)\}$, so A and B are not disjoint.
2. A and B are independent, since $P(A \cap B) = P(A) \cdot P(B)$.

3.4 and 3.5 Multiplication rule

Two different sampling methods (cf. vase example):

- Sampling with replacement: selections are independent events
- Sampling without replacement: selections are dependent events.

However, to simplify calculations:

Small sample rule

Drawing small sample from large population?
Then treat selections as independent events.

Example: lost luggage

Airplane, 300 passengers, each has 1 suitcase; 6 got lost.
P("5 randomly selected passengers have their luggage")?

$P("suitcaseofarandompassengerlost") = \frac{6}{300} = 0.02.$
Use Small sample and complement rule:

$$\begin{aligned} P(\text{all 5 suitcases arrived}) &\approx P(\text{suitcase 1 arrived}) \cdot \ldots \cdot P(\text{suitcase 5 arrived}) \\ &= [1 - P(\text{suitcase 1 lost})] \cdot \ldots \cdot [1 - P(\text{suitcase 5 lost})] \\ &= (1 - 0.02) \cdot \ldots \cdot (1 - 0.02) = 0.98^5 \approx 0.90. \end{aligned}$$

Compare with exact prob.: $\frac{294}{300} \cdot \frac{293}{299} \cdots \frac{289}{295} \approx 0.885.$

3.4 and 3.5 Multiplication rule

Airplane, 300 passengers, each has 1 suitcase; 6 lost.

P("at least one out of 5 random passengers lost the luggage")?

Complement rule:

$$P(\geq 1 \text{ lost}) = 1 - P(\text{none lost}).$$

Previous slide:

$$P(\text{none lost}) = P(\text{all 5 arrived}) \approx 0.90,$$

So:

$$P(\geq 1 \text{ lost}) = 1 - P(\text{none lost}) \approx 0.10.$$

In general:

Complement of at least one

$$P(\geq 1 \text{ occurence of } \dots) = 1 - P(\text{no occurence of } \dots)$$

3.4 and 3.5 Multiplication rule: Recap

Conditional probability

If $P(A) > 0$, then the conditional probability $P(B|A)$ is defined by:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Multiplication rule

$$P(A \cap B) = P(A) \cdot P(B|A).$$

Independence

Two events A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B).$$

Complement of at least one

$$P(\text{at least one occurence of} \dots) = 1 - P(\text{no occurence of} \dots)$$