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A FAMILY OF TVD SECOND ORDER SCHEMES OF NONLINEAR SCALAR CONSERVATION LAWS

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Abstract

Two second order finite difference schemes for solving linear hyperbolic conservation laws are presented. An analysis of the total variation diminishing (TVD) constrains for the schemes is carried out. We discuss the extension of the two schemes to nonlinear scalar hyperbolic conservation laws. The performance of the schemes is assessed by solving test problems for the Burgers' equation. We use exact solution to validate the results.

Key words: difference schemes, nonlinear scalar conservation laws, Burgers equation, total variation diminishing (TVD) schemes
65N06, 65M06,

1. Introduction. A major difficulty in the numerical approximation of nonlinear hyperbolic conservation laws is the presence of discontinuities in the solution even the initial condition is smooth. Traditional schemes generate spurious oscillations in the numerical solution near these discontinuities. Standard methods based on central differing together with artificial viscosity during the last few years have often been replaced by the so-called total variation diminishing (TVD) schemes. The main property of a TCD scheme is that, unlike monotone schemes, it can be second order accurate (or higher) and oscillations free across discontinuities (when applied to nonlinear equations).

In [5], a five-point second order explicit difference scheme has been obtained. However, this scheme is only suitable for linear problems with smooth solutions. It is well known that in solving nonlinear problems, such as Burgers' equation, when applying second order schemes it is inevitable that oscillations will be observed in the vicinity of the shocks which might trigger instabilities.

In this paper an oscillations free formulation of the scheme (TVD version) is derived for linear equations.

Another TVD scheme is presented in [6] for linear equations.

The main aim of this paper is to extend the two schemes to nonlinear scalar hyperbolic conservation laws. The paper is organized as follows. In section 2, we construct a five-point second order scheme. The scheme is also reformulated to make its oscillations free (TVD) for linear case. This was achieved by introducing flux limiters. We also review the four-point TVD scheme 6. In section 3, we describe the way to extend the two schemes for nonlinear scalar conservation laws. Finally, in section 4, numerical results for test problems are presented. The numerical results are compared with exact solutions.

2. Family of second order TVD schemes. In this section we first construct and reformulate the conservative second order explicit scheme presented in [5]. Sufficient conditions for this scheme to be TVD are obtained for linear scalar hyperbolic conservation laws. Then we briefly review the second order TVD scheme introduced in [6] for linear case.

The initial value problem (IVP) for the one-dimensional scalar hyperbolic conservation law is considered, namely

$$u_t + f(U)_x = 0 \quad -\infty < x < \infty, \quad t \geq 0, \tag{2.1}$$

$$u(x,0) = u_0(x). \tag{2.2a}$$

Here u is the unknown function and $f(u)$ is the physical flux.

First let us consider the linear case $f(u) = au$, so that $f'(u) = a$ is a constant wave propagation speed.

If the initial data consist of two constant states,

$$u_0 = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases} \tag{2.1b}$$

equation (2.2) is called the usual Riemann problem (RP).

Let u_j^n be the numerical solution of (2.1) at $x = j\Delta x$ and $t = n\Delta t$, with Δx the spatial mesh size and Δt the time step.

The five-point second order scheme introduced in [5] has the form

$$u_j^{n+1} = u_j^n - \frac{c}{8}\left[u_{j+2}^n + 2u_{j+1}^n - 2u_{j-1}^n - u_{j-2}^n\right] + \frac{c^2}{8}\left[u_{j+2}^n - 2u_j^n + u_{j-2}^n\right], \tag{2.3}$$

where $c = \dfrac{\Delta t}{\Delta x}\alpha$ is the Courant number. The scheme (2.3) is stable for $c \leq \sqrt{2}$.

The scheme (2.3) can be written in the conservative form as

$$u_j^{n+1} = u_j^n - \lambda\left[F_{j+1/2}^n - F_{j-1/2}^n\right] \quad \lambda = \frac{\Delta t}{\Delta x}, \tag{2.4}$$

where $F = F(u_{j-k+1}, \dots, u_{j+k})$ is a numerical flux function satisfying

$$F(u, \dots, u) = f(u)$$

for consistency.

The flux F can be written in the form [3]

$$\begin{aligned} F_{j+1/2} = \frac{1}{2}(au_j + au_{j+1}) - \frac{1}{2}|a|\Delta_{j+1/2}u + \\ |a|\{A_0\Delta_{j+1/2}u + A_1\Delta_{j+L+1/2}u + A_2\Delta_{j+M+1/2}u\}, \end{aligned} \tag{2.5}$$

where

$$\begin{gathered} A_0 = \frac{|c|}{2} - \frac{c^2}{4}, \quad A_1 = -\frac{|c|}{8} - \frac{c^2}{8}, \quad A_2 = \frac{|c|}{8} - \frac{c^2}{8}, \\ L = -1, \; M = 1, \text{ for } c > 0 \text{ and } L = 1, \; M = -1 \text{ for } c < 0, \end{gathered} \tag{2.6}$$

where $\Delta_{j+\frac{1}{2}}u = u_{j+1} - u$.

TVD version of the method. Define the total variation $\mathrm{TV}(u_j^n)$ of the mesh function u^n:

$$\mathrm{TV}(u^n) = \sum_{-\infty}^{\infty} |u_{j+1}^n - u_j^n| = \sum_{-\infty}^{\infty} |\Delta_{j+1/2}u^n|. \tag{2.7}$$

The numerical scheme (2.4) is said to be TVD scheme if

$$\mathrm{TV}(u^{n+1}) \leq \mathrm{TV}(u^n) \tag{2.8}$$

which simply states that the total variations do not increase as time evolves, so that $\mathrm{TV}(u^n)$ at any time n is bounded by $\mathrm{TV}(u^0)$ of the initial data.

To apply the TVD concept we use HARTEN'S theorem [1] which states that a scheme written as

$$u_j^{n+1} = u_j^n - B_{j-1/2}\Delta_{j-1/2}u + C_{j+1/2}\Delta_{j+1/2}u, \tag{2.9}$$

is TVD provided that

$$B_{j+1/2} \geq 0, \quad C_{j+1/2} \geq 0, \quad B_{j+1/2} + C_{j+1/2} \leq 1, \tag{2.10}$$

where $B_{j+1/2}$ and $C_{j+1/2}$ are data dependent coefficients, i.e., functions of the set $\{u_j^n\}$.

The method (2.4) and (2.5), being second order accurate, is not TVD. It can be made TVD by replacing (2.5) with the more general flux

$$F_{j+1/2} = \frac{1}{2}(au_j + au_{j+1}) - \frac{1}{2}|a|\Delta_{j+1/2}u + |a|\{A_0\Delta_{j+1/2}u + A_1\Delta_{j+L+1/2}u\}\varphi_j +$$

$$|a|A_2\Delta_{j+M+1/2}u\phi_{j+M}, \tag{2.11}$$

where ϕ_j and ϕ_{j+M} are flux limiter functions.

Theorem. Scheme (2.4), (2.11) is TVD for $|c| \leq 1$ if the limiting function is determined by

$$\phi_j \leq \frac{(1-|c|)r_j}{\eta(A_1 r_j + A_0 - A_2)}, \tag{2.12a}$$

$$\phi_j \leq \frac{1-|c| + \eta A_2/r_j^*}{\eta(A_1 r_j + A_0)}, \tag{2.12b}$$

$$\phi_j \geq \frac{A_2}{(A_1 r_j + A_0)r_j^*}, \tag{2.12c}$$

$$\phi_j \geq 0, \tag{2.12d}$$

where r_j is called the local flow parameter and is defined by

(2.13a)
$$r_j = \frac{\Delta_{j-1/2}u}{\Delta_{j+1/2}u} \quad \text{for } c > 0,$$

(2.13b)
$$r_j = \frac{\Delta_{j+3/2}u}{\Delta_{j+1/2}u} \quad \text{for } c < 0,$$

and r_j^* is called the upwind-downward flow parameter and is given by

(2.14a)
$$r_j^* = \frac{\Delta_{j-1/2}u}{\Delta_{j+3/2}u} \quad \text{for } c > 0,$$

(2.14b)
$$r_j^* = \frac{\Delta_{j+3/2}u}{\Delta_{j-1/2}u} \quad \text{for } c < 0,$$

and η is defined by

(2.15)
$$\eta = \begin{cases} 1-|c| & \text{for} \quad 0 \le |c| < 1/2 \\ |c| & \text{for} \quad 1/2 \le |c| \le 1 \end{cases}.$$

Proof. See [3] and [6]. By applying the last theorem to the scheme (2.4), (2.11), the flux limiter can be defined as

(2.16a)
$$\phi_j = \begin{cases} \dfrac{(1-|c|)r_j}{\eta(A_1 r_j + A_0 - A_2)} & \text{for} \quad 0 \le r_j \le r^L \\ 1 & \text{for} \quad r^L \le r_j \le r^R \\ \dfrac{1-|c|+\eta A_2 \phi_{j+M}/r_j^*}{\eta(A_1 r_j + A_0)} & \text{for} \quad r_j > r^R \\ 0 & \text{for} \quad r_j < 0 \end{cases}$$

(2.16b)
$$\phi_{j+M} = \begin{cases} \eta r_{j+M} & \text{for} \quad 0 \le r_{j+M} < 0.5 \\ 1 & \text{for} \quad r_j > 0.5 \\ 0 & \text{for} \quad r_j = 0 \end{cases}$$

where
$$r^L = \frac{\eta(A_0 - A_2)}{1-|c|-\eta A_1}, \quad r^R = \frac{1-|c|-\eta(A_0 - A_2\phi_{j+M}/r_j^*)}{\eta A_1}.$$

Therefore, the scheme (2.4), (2.11) becomes TVD.

In [6] a four-point second order scheme was presented and the TVD theory was applied.

The scheme is in the form

(2.17)
$$u_j^{n+1} = u_j^n - \frac{c}{2}[u_{j+1} - u_{j-1}^n] - \frac{c}{2}[u_{j-1}^n - u_{j+1}^n - u_{j-2}^n + u_j^n].$$

This scheme can be written in the general form (2.11), where

(2.18) $$A_0 = \frac{1}{2} - \frac{|c|}{4}, \quad A_1 = -\frac{|c|}{4}, \quad A_2 = 0$$

and

(2.19) $$\phi(r) = \mathrm{Max}\{0, \mathrm{Min}(1, 4r), \mathrm{Min}(r, 4)\}.$$

For details see [6].

3. Extension to nonlinear scalar hyperbolic conservation laws. To extend the schemes (2.3) and (2.17) to nonlinear scalar problems, we considered the equation

(3.1) $$u_t + f(u)_x = 0.$$

Define the wave speed

(3.2) $$a_{j+1/2} = \begin{cases} \dfrac{\Delta_{j+1/2} f}{\Delta_{j+1/2} u} & \Delta_{j+1/2} u \neq 0 \\ \left.\dfrac{\partial f}{\partial u}\right|_{u_j} & \Delta_{j+1/2} u = 0 \end{cases}$$

Now we redefine the r_j in (2.13) as

(3.3) $$\begin{aligned} r_j &= \frac{|a_{j-1/2}|\, \Delta_{j-1/2} u}{|a_{j+1/2}|\, \Delta_{j+1/2} u} \quad \text{for} \quad c_{j+1/2} > 0 \\ r_j &= \frac{|a_{j-3/2}|\, \Delta_{j-3/2} u}{|a_{j+1/2}|\, \Delta_{j+1/2} u} \quad \text{for} \quad c_{j+1/2} < 0. \end{aligned}$$

Here $c_{j+1/2} = \dfrac{\Delta t}{\Delta x} a_{j+1/2}$.

Unlike the constant coefficient case, $a_{j+1/2}$ and $a_{j-1/2}$ are not always with the same sign.

Then the numerical flux (2.11) takes the form

(3.4) $$\begin{aligned} F_{j+1/2} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}\left|a_{j+1/2}\right| \Delta_{j+1/2} u + \left|a_{j+1/2}\right| \left\{A_0 \Delta_{j+1/2} u + A_1 \Delta_{j+L+1/2} u\right\} \varphi_j \\ + \left|a_{j+1/2}\right| A_2 \Delta_{j+M+1/2} u \phi_{j+M}. \end{aligned}$$

After considering all the possible combinations of the signs of $a_{j+1/2}$ and $a_{j-1/2}$, a set of sufficient conditions on ϕ (the limiter function of the scheme (2.3)) still can be of the form similar to (2.12) and is

(3.5a) $$\phi_j \leq \frac{(1 - |a_{j+1/2}|) r_j}{\eta(A r_j + A_0 - A_2)},$$

(3.5b) $$\phi_j \le \frac{1 - |a_{j+1/2}| + \eta A_2/r_j^*}{\eta(A_1 r_j + A_0)},$$

(3.5c) $$\phi_j \ge \frac{A_2}{(A_1 r_j + A_0) r_j^*},$$

(3.5d) $$\phi_j \ge 0.$$

The limiter function of the scheme (2.17) is

(3.6) $$\phi(r) = \mathrm{Max}\{0, \mathrm{Min}(1, 4r), \mathrm{Min}(r, 4)\}.$$

where r is defined in (3.3).

Observe that when $a_{j+1/2} = 0$, the schemes have zero dissipation. One way is to approximate $a_{j+1/2}$ by a Lipschitz continuous function [1]. For example, instead of using (3.4), one can use

$$F_{j+1/2} = \frac{1}{2}(f_j + f_{j+1}) - \frac{1}{2}\psi(a_{j+1/2})\Delta_{j+1/2}u + \psi(a_{j+1/2})\left\{A_0\Delta_{j+1/2}u + A_1\Delta_{j+L+1/2}u\right\}\varphi_j$$

(3.7) $$+\psi(a_{j+1/2})A_2\Delta_{j+M+1/2}u\phi_{j+M}.$$

Here ψ is a function of $a_{j-1/2}$ and is defined $\psi(x) = \begin{cases} \dfrac{x^2}{4\varepsilon} + \varepsilon & \text{for } |x| < 2\varepsilon \\ |x| & \text{for } |x| \ge 2\varepsilon \end{cases}$

where $0 \le \varepsilon \le 0.5$.

4. The numerical experiments. In this section some numerical results are presented to show the performance of our methods. We consider the approximate solution of inviscid Burgers' equation

(4.1) $$u_r + \left[\frac{1}{2}u^2\right]_x = 0$$

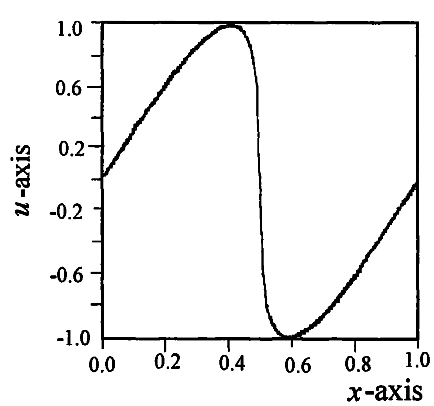

Fig. 1

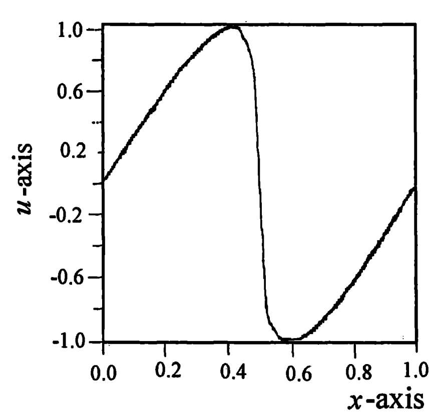

Fig. 2

using the two difference schemes discussed in the last section. They include:

1. The five-point second order scheme (2.4) with the flux (3.4) with (2.6) and the limiter function (3.5). It will be referred to as FD5.

2. The four-point second order scheme (2.4) with the flux (3.4) with (2.18) and limiter function (3.6). It will be referred to as FD4.

Equation (4.1) is solved with two sets of initial conditions.

A) In the first case, we have the smooth periodic initial data [2]

$$u(x,0) = \sin(2\pi x), \tag{4.2a}$$

$$u(0) = u(1). \tag{4.2b}$$

The exact solution of this problem remains smooth unit $t = \dfrac{1}{2\pi}$, after which a shock discontinuity forms. Using $\Delta x = 0.01$ and the Courant number $c = 0.5$ (where $c = \operatorname{Max} c_{j+1/2}$) the numerical solution is displayed at $t = \dfrac{1}{2\pi} \approx 0.16$. Figure 1 shows

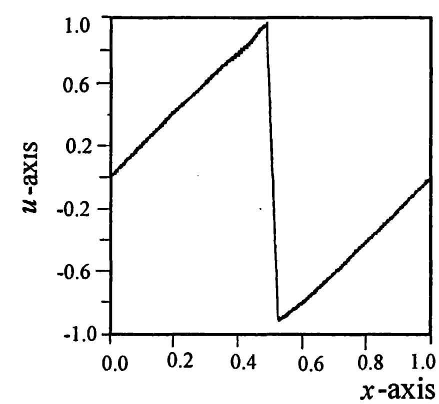

Fig. 3

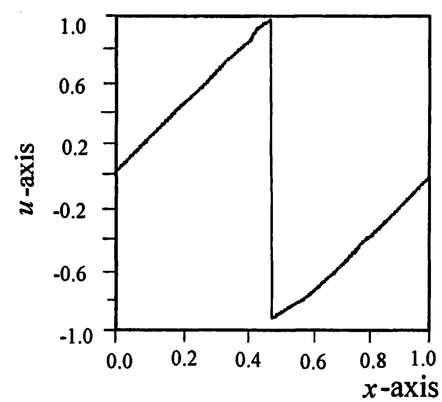

Fig. 4

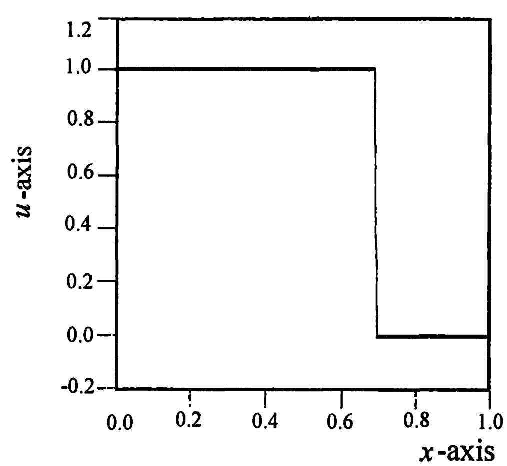

Fig. 5

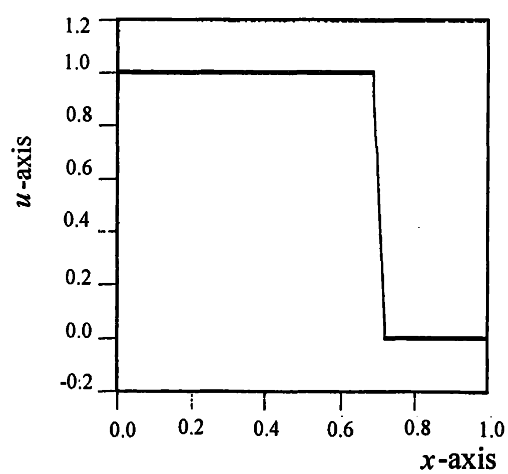

Fig. 6

the computed results using FD5 scheme and Fig. 2 the results using FD4 scheme. The numerical solution is shown in symbols and the exact solution in full lines.

Notice that the numerical solutions given in Fig. 1 and Fig. 2 are very accurate and the numerical solution given by FD5 scheme is better than that given by FD4 scheme.

Figures 3 and 4 show the numerical solutions at $t = 0.32$ using FD5 and FD4, respectively. Note that FD4 scheme (Fig. 4) gives sharp resolution of discontinuity, while the solution obtained by FD5 (Fig. 3) is almost indistinguishable from the exact solution.

B) The second case is the Burgers' equation (4.1) Riemann initial data [4]

$$u(x,0) = \begin{cases} 1, & x \leq 0.2 \\ 0 & \text{otherwise} \end{cases}$$

and the results are taken at time $t = 1.0$.

Figures 5 and 6 illustrate the results obtained by FD5 and FD4, respectively. We note that the method FD4 gives very good approximation while FD5 reproduces exact solution.

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