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2 authors:

V. S. Borisov
Ben-Gurion University of the Negev

Michael Mond
Ben-Gurion University of the Negev

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ON STABILITY, MONOTONICITY, AND CONSTRUCTION OF DIFFERENCE SCHEMES II: APPLICATIONS*

V. S. BORISOV† AND M. MOND†

Abstract. In this paper, we focus on the application and illustration of the approach developed in part I. This approach is found to be useful in the construction of stable and monotone central difference schemes for hyperbolic systems. A new modification of the central Lax–Friedrichs scheme is developed to be of second-order accuracy. The stability of several versions of the developed central scheme is proved. Necessary conditions for the variational monotonicity of the scheme are found. A monotone piecewise cubic interpolation is used in the central schemes to give an accurate approximation for the model in question. The monotonicity parameter introduced in part I is found to be important. That parameter, along with the Courant–Friedrichs–Lewy (CFL) number, plays a major role in the criteria for monotonicity and stability of the central schemes. The modified scheme is tested on several conservation laws taking a CFL number equal or close to unity, and the scheme is found to be accurate and robust.

Key words. stability, monotonicity, difference equations, difference schemes, central schemes, schemes in variations, hyperbolic equations, systems of partial differential equations, source terms, numerical solution, spurious oscillations, monotone piecewise cubics

AMS subject classifications. 39A30, 65L20, 65M12, 65N12

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1. Introduction. In part I [8] we have developed the theoretical background to investigate stability and monotonicity of nonlinear difference schemes for, e.g., systems of partial differential equations (PDEs). We have proven that a non-linear scheme is stable (and monotone) if the corresponding scheme in variations (or variational scheme [9]) is stable (and, respectively, monotone), and that a nonlinear explicit scheme will be stable *iff* (if and only if) its scheme in variations will be stable. Since the variational scheme will always be linear, the stability theory (see, e.g., [15], [17], [30], [42], [44], [45]) for linear difference equations may be applied to establish stability of the nonlinear scheme. The nonlinear PDE system to be investigated is represented in part I [8] by the following Cauchy problem:

$$\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial}{\partial x}\mathbf{f}(\mathbf{u})=\frac{1}{\tau}\mathbf{q}(\mathbf{u}), \quad x\in\mathbb{R},\ 0<t\le T_{\max}; \quad \mathbf{u}(x,t)|_{t=0}=\mathbf{u}^0(x), \tag{1.1}$$

where $\mathbf{u}=\{u_1,u_2,\ldots,u_M\}^T$ is a vector-valued function from $\mathbb{R}\times[0,+\infty)$ into an open subset $\Omega_{\mathbf{u}}\subseteq\mathbb{R}^M$, $\mathbf{f}(\mathbf{u})=\{f_1(\mathbf{u}),f_2(\mathbf{u}),\ldots,f_M(\mathbf{u})\}^T$ is a smooth function (flux-function) from $\Omega_{\mathbf{u}}$ into $\mathbb{R}^M$, $\mathbf{q}(\mathbf{u})=\{q_1(\mathbf{u}),q_2(\mathbf{u}),\ldots,q_M(\mathbf{u})\}^T$ denotes the source term, $\tau>0$ denotes the stiffness parameter, and $\mathbf{u}^0(x)$ is either of compact support or periodic. We will, in general, assume that $\tau=$ const without loss of generality. System (1.1) is assumed to be hyperbolic and, hence, the Jacobian matrix of $\mathbf{f}(\mathbf{u})$ has M real eigenvalues and M linearly independent corresponding eigenvectors. In

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†The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, Beer-Sheva, Israel (viatslav@bgu.ac.il, mond@bgu.ac.il).

addition, it is assumed in this paper that

$$\sup_{\mathbf{u}\in\Omega_{\mathbf{u}}} \|\mathbf{A}\|_2 \le \lambda_{\max} < \infty, \quad \mathbf{A} = \frac{\partial \mathbf{f}(\mathbf{u})}{\partial \mathbf{u}}, \tag{1.2}$$

and all eigenvalues, $\xi_k = \xi_k(\mathbf{u})$, of the Jacobian matrix $\mathbf{G}$ $(=\partial \mathbf{q}(\mathbf{u}) / \partial \mathbf{u})$ have nonpositive real parts, i.e.,

$$R_e \xi_k(\mathbf{u}) \le 0 \quad \forall k, \quad \forall \mathbf{u} \in \Omega_{\mathbf{u}}. \tag{1.3}$$

Here and in what follows the notation of part I [8] is used.

In this paper we are mainly concerned with application and exemplification of the approach developed in part I [8]. Aiming to demonstrate potentialities of this approach in construction numerical schemes, we will develop a novel version of the central Lax–Friedrichs (LxF) scheme (see, e.g., [16, p. 170]) with the second-order accuracy in space and time. The stability of this scheme is proved in section 3 on the basis of [8, Thm. 2.4]. We restrict our proof mainly to the case of vanishing-viscosity (see, e.g., [16], [30]) solutions to (1.1). Notice that a property such as smoothness is peculiar to not only vanishing-viscosity solutions of a hyperbolic system. Since the stiffness parameter, τ, in the system (1.1) is like a viscosity [30], a solution to the system of conservation laws (1.1) is expected to be smooth provided that input data are sufficiently smooth. This specific type of system is interesting in itself, as such systems are not uncommon in practice. For a detailed discussion on this subject, including the sufficient conditions for the global existence of smooth solutions, see [19].

An extensive literature is devoted to central schemes, since these schemes are attractive for the following various reasons: no Riemann solvers, characteristic decompositions, complicated flux splittings, etc., must be involved in construction of a central scheme (see, e.g., [6], [26], [27], [28], [30], [38], [39], and references therein), and hence such schemes can be implemented as black-box solvers for general systems of conservation laws [26]. Let us, however, note that the numerical domain of dependence [30, p. 69] for a central scheme approximating, e.g., a scalar transport equation, coincides with the numerical domain of dependence for a standard explicit scheme approximating diffusion equations [30, p. 67]. Such a property is inherent to central schemes in contrast to, e.g., the first-order upwind schemes [30, p. 73]. Hence, central schemes do not satisfy the well-known principle (see, e.g., [2, p. 304]) that derivatives must be correctly treated using type-dependent differences, and hence there is the risk of a central scheme exhibiting spurious solutions. The results of simulations in [35] can be seen as an illustration of the last assertion. Notice that all versions of the so-called Nessyahu–Tadmor (NT) central scheme, in spite of a sufficiently small CFL (Courant–Friedrichs–Lewy [30]) number ($Cr = 0.475$), exhibit spurious oscillations in contrast to the second-order upwind scheme ($Cr = 0.95$). On the other hand, the first-order, $O(\Delta t + \Delta x)$, LxF scheme exhibits the excessive numerical viscosity. Thus, the central scheme should be chosen with great care to reflect the true solution and to avoid significant but spurious peculiarities in numerical solutions.

As shown in [9], a conservative scheme being H-monotone (hence, consistent with the entropy condition [20]), total-variation-diminishing (TVD), and S-monotone, can produce spurious oscillations comparable with the size of jump-discontinuity. In this connection, the notion of GOS-monotonicity [9] is introduced to ensure the monotonicity (i.e., absence of spurious oscillations) of numerical schemes. Let us note that the LxF scheme—the forerunner of central schemes [6], [26]—does not produce spurious oscillations, while, from the pioneering works of Nessyahu and Tadmor [35] and on, the higher-order versions of the LxF scheme can produce spurious oscillations. The reason has to do with a negative numerical viscosity introduced to obtain a higher-order accurate scheme. Let us illustrate it with the scheme developed in [8, sect. 3], which is $O(\Delta t + (\Delta x)^2)$ accurate. We rewrite this scheme to read

$$\frac{1}{2}\frac{\mathbf{v}_{i+0.5}^{n+0.25} - \mathbf{v}_{i+0.5}^{n}}{0.25\Delta t} + \frac{\mathbf{f}\left(\mathbf{v}_{i+1}^{n}\right) - \mathbf{f}\left(\mathbf{v}_{i}^{n}\right)}{\Delta x}$$

$$= \frac{\Delta x^2}{\Delta t}\frac{\mathbf{v}_{i}^{n} - 2\mathbf{v}_{i+0.5}^{n} + \mathbf{v}_{i+1}^{n}}{\Delta x^2} - \frac{\Delta x^2}{4\Delta t}\frac{\mathbf{d}_{i+1}^{n} - \mathbf{d}_{i}^{n}}{\Delta x}, \tag{1.4}$$

where $\mathbf{d}_i^n$ denotes the derivative of the interpolant at $x = x_i$. Notice that the second term in the right-hand side of (1.4) is, in fact, the negative numerical viscosity. Without this term, scheme (1.4) would be of the first order, $O(\Delta t + \Delta x)$, namely, the LxF scheme. Let us note that there is the possibility of increasing the scheme's order of accuracy, up to $O((\Delta t)^2 + (\Delta x)^2)$, by introducing into scheme (1.4) an additional nonnegative numerical viscosity (see section 2). Such an approach is similar to the vanishing viscosity method [16], [30], and, hence, possesses its advantages (e.g., the scheme satisfies the entropy condition), yet it appears to be free of the disadvantages of this method, since the additional viscosity term is not artificial.

A stable numerical scheme may yield spurious results when applied to a stiff ($\tau \ll 1$) hyperbolic system with relaxation (see, e.g., [1], [5], [7], [10], [11], [22], [36], [37], [40], [41]). Specifically, spurious numerical solution phenomena may occur when underresolved numerical schemes (i.e., insufficient spatial and temporal resolution) are used (see, e.g., [1], [22], [24], [34]). However, during a computation, the stiffness parameter may be very small, and, hence, to resolve the small stiffness parameter, we need a huge number of time and spatial increments, making the computation impractical. Hence, we are interested in solving the system (1.1) with underresolved numerical schemes. It is significant that for relaxation systems a numerical scheme must possess a discrete analogy to the continuous asymptotic limit, because any scheme violating the correct asymptotic limit leads to spurious or poor solutions (see, e.g., [10], [22], [23], [34], [38]). Most methods for solving such systems can be described as operator-splitting methods [11], or methods of fractional steps [7]. After operator splitting, one solves the advection homogeneous system, and then solves the ordinary differential equations associated with the source terms. As reported in [18], this approach is well suited for the stiff systems. Let us, however, note that the schemes based on the operator-splitting techniques are, generally, of the first order in time, excluding rare cases such as, e.g., the Strang splitting [30]. Nevertheless, as applied to system (1.1), the second-order schemes can be constructed on the basis of operator-splitting techniques with ease [8]. This scheme contains, in fact, (1.4) approximating the advection homogeneous system with the accuracy $O(\Delta t + (\Delta x)^2)$ as well as an implicit scheme approximating the differential equations associated with the source terms. Notice that the scheme incorporating (1.4) can produce spurious oscillations even if the CFL number is sufficiently small (see section 4.1). In this connection, a new nonoscillatory second-order scheme for the advection system is developed in section 2. The sufficient condition for stability as well as the necessary condition for GOS-monotonicity [8] of the scheme is found in section 3. The developed scheme is tested on several conservation laws in section 4. As for the differential equations associated with the source terms, it is developed in section 2, and tested in section 4, a second-order implicit Runge–Kutta scheme.

2. Construction of first- and second-order central schemes. Central difference schemes with first- and second-order accuracy are introduced in this section. The construction of the central schemes is based on (i) the variational GOS-monotonicity notion, (ii) monotone piecewise cubic interpolation (see, e.g., [12], [25]), and (iii) operator-splitting techniques (see also LOS in [44]).

By virtue of the operator-splitting idea, the following chain of equations corresponds to the problem (1.1):

$$\frac{1}{2}\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial}{\partial x}\mathbf{f}\left(\mathbf{u}\right)=0, \quad t_n<t\le t_{n+0.5}, \quad \mathbf{u}\left(x,t_n\right)=\mathbf{u}^n\left(x\right), \tag{2.1}$$

$$\frac{1}{2}\frac{\partial \mathbf{u}}{\partial t}=\frac{1}{\tau}\mathbf{q}\left(\mathbf{u}\right), \quad t_{n+0.5}<t\le t_{n+1}, \quad \mathbf{u}\left(x,t_{n+0.5}\right)=\mathbf{u}^{n+0.5}\left(x\right). \tag{2.2}$$

Using central differencing, we approximate (2.1) on the cell $[x_i, x_{i+1}]\times[t_n, t_{n+0.25}]$ by the following difference equation [8]:

$$\mathbf{v}_{i+0.5}^{n+0.25}=\mathbf{v}_{i+0.5}^{n}-\frac{\Delta t}{2\Delta x}\left(\mathbf{g}_{i+1}^{n+0.125}-\mathbf{g}_{i}^{n+0.125}\right). \tag{2.3}$$

Considering that (2.3) approximates (2.1) with the accuracy $O((\Delta x)^2+(\Delta t)^2)$, the next problem is to approximate $\mathbf{v}_{i+0.5}^{n}$ and $\mathbf{g}_{i}^{n+0.125}$ in such a way as to retain the accuracy of the approximation as well as to obtain a stable and monotone difference scheme. It is suggested in [8] to use componentwise monotone C^1 piecewise cubics, $\mathbf{p}=\mathbf{p}\left(x\right)$, in construction of central schemes. The derivatives, $\mathbf{p}_i'$, of the interpolant at $x=x_i$ are represented in the following form [8]:

$$\mathbf{p}_i'=\mathbb{A}_i\cdot\frac{\Delta \mathbf{p}_i}{\Delta x}, \quad \mathbf{p}_{i+1}'=\mathbb{B}_i\cdot\frac{\Delta \mathbf{p}_i}{\Delta x}, \quad \Delta\mathbf{p}_i=\mathbf{p}_{i+1}-\mathbf{p}_i, \tag{2.4}$$

where $\mathbb{A}_i$ and $\mathbb{B}_i$ are the diagonal matrices such that

$$0\le\mathbb{A}_i\le 4\aleph\mathbf{I}, \quad 0\le\mathbb{B}_i\le 4\aleph\mathbf{I}, \quad 0\le\aleph\le 1 \quad \forall i; \tag{2.5}$$

$\mathbf{I}$ denotes the identity matrix. The procedure for the estimation of $\mathbf{p}_i'$ (and hence $\mathbb{A}_i$ and $\mathbb{B}_i$) [8] is based on the algorithm suggested in [12] (see also [25]). The following interpolation formula is obtained [8] by virtue of the interpolant, $\mathbf{p}=\mathbf{p}\left(x\right)$:

$$\mathbf{p}_{i+0.5}=0.5\left(\mathbf{p}_i+\mathbf{p}_{i+1}\right)-\frac{\Delta x}{8}\left(\widetilde{\mathbf{p}}_{i+1}'-\widetilde{\mathbf{p}}_{i}'\right)+O\left(\left(\Delta x\right)^r\right), \tag{2.6}$$

where $\widetilde{\mathbf{p}}_i'=\mathbf{p}_i'+O\left(\left(\Delta x\right)^s\right)$, and $r=\min\left(4,s+1\right)$ if $\mathbf{p}\left(x\right)$ has a continuous fourth derivative. By virtue of the interpolation formula (2.6), the central scheme (see section 3 in [8]) with the accuracy $O(\Delta t+(\Delta x)^r\big/\Delta t+(\Delta t)^2+(\Delta x)^2)$ is developed. Notice that, if the ratio $\Delta t\big/\Delta x$ is fixed, then this scheme will be of the first order [8].

Let us note that instead of point values, employed in the construction of the scheme (2.3), cell averages (see, e.g., [6], [26], [30]), calculated on the basis of the monotone C^1 piecewise cubics, can be used. In such a case we obtain, instead of (2.6), the following interpolation formula:

$$\mathbf{p}_{i+0.5}=0.5\left(\mathbf{p}_i+\mathbf{p}_{i+1}\right)-\varkappa\frac{\Delta x}{8}\left(\widetilde{\mathbf{p}}_{i+1}'-\widetilde{\mathbf{p}}_{i}'\right), \tag{2.7}$$

where $\varkappa = 2/3$. The region of monotonicity in this case will also be (2.5). Notice that the interpolation formula (2.7) coincides with (2.6) under $\varkappa = 1$. Thus, in view of the interpolation formula (2.7), the staggered scheme (2.3) is written as

$$\mathbf{v}_{i+0.5}^{n+0.25} = 0.5\left(\mathbf{v}_{i+1}^{n} + \mathbf{v}_{i}^{n}\right) - \varkappa\frac{\Delta x}{8}\left(\mathbf{d}_{i+1}^{n} - \mathbf{d}_{i}^{n}\right) - \frac{\Delta t}{2}\frac{\mathbf{f}\left(\mathbf{v}_{i+1}^{n}\right) - \mathbf{f}\left(\mathbf{v}_{i}^{n}\right)}{\Delta x}, \tag{2.8}$$

where $\mathbf{d}_i^n$ denotes the derivative of the interpolant at $x = x_i$, and the range of values for the parameter $\varkappa$ is the segment $0 \leq \varkappa \leq 1$. As usual, the mathematical treatments for the second step of the staggered scheme will not, in general, be included in the text, because the second step is quite similar to (2.8). If $\varkappa = 1$ (or $\varkappa = 0$), then scheme (2.8) coincides with the scheme developed in [8, sect. 3] (or with the LxF scheme, respectively). As it was shown above, the scheme (2.8) is of the first order provided $\Delta t = O(\Delta x)$. In such a case, since the source terms can be, in general, stiff (i.e., $\tau \ll 1$), it is natural to use the following first-order implicit scheme for (2.2):

$$\mathbf{v}_{i}^{n+1} = \mathbf{v}_{i}^{n+0.5} + \frac{\Delta t}{\tau}\mathbf{q}\left(\mathbf{v}_{i}^{n+1}\right). \tag{2.9}$$

The first-order, $O(\Delta t + (\Delta x)^2)$, central scheme (2.8) will be abbreviated to COS1.

Using Taylor series expansion, we write

$$\mathbf{g}_{i}^{n+0.125} = \mathbf{f}\left(\mathbf{v}_{i}^{n}\right) + \left.\frac{\partial \mathbf{f}\left(\mathbf{v}_{i}^{n}\right)}{\partial t}\right|_{t=t_n}\frac{\Delta t}{8} + O\left(\Delta t^2\right). \tag{2.10}$$

By virtue of the PDE system (2.1), we find

$$\frac{\partial \mathbf{f}}{\partial t} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\cdot\frac{\partial \mathbf{u}}{\partial t} = -2\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\cdot\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\cdot\frac{\partial \mathbf{u}}{\partial x} = -2\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)^2\cdot\frac{\partial \mathbf{u}}{\partial x} = -2\mathbf{A}^2\cdot\frac{\partial \mathbf{u}}{\partial x}. \tag{2.11}$$

Notice that, since system (2.1) is assumed to be hyperbolic, the operator $\mathbf{D} \equiv \mathbf{A}^2$ is nonnegative [44, p. 43], i.e., $\mathbf{v}^T \cdot \mathbf{D} \cdot \mathbf{v} \geq 0\ \forall\, \mathbf{v} \in \mathbb{R}^M$. Using the interpolation formula (2.7) and the formulae (2.10)–(2.11), we obtain from (2.3) the following second-order central scheme:

$$\begin{aligned} \mathbf{v}_{i+0.5}^{n+0.25} &= 0.5\left(\mathbf{v}_{i+1}^{n} + \mathbf{v}_{i}^{n}\right) - \varkappa\frac{\Delta x}{8}\left(\mathbf{d}_{i+1}^{n} - \mathbf{d}_{i}^{n}\right) \\ &\quad + \varsigma\frac{(\Delta t)^2}{8\Delta x}\left[\left(\mathbf{A}_{i+1}^{n}\right)^2\cdot\mathbf{d}_{i+1}^{n} - \left(\mathbf{A}_{i}^{n}\right)^2\cdot\mathbf{d}_{i}^{n}\right] - \frac{\Delta t}{2}\frac{\mathbf{f}\left(\mathbf{v}_{i+1}^{n}\right) - \mathbf{f}\left(\mathbf{v}_{i}^{n}\right)}{\Delta x}, \end{aligned} \tag{2.12}$$

where ς is introduced by analogy with $\varkappa$ in (2.8), and hence $0 \leq \varsigma \leq 1$. Scheme (2.12) coincides with (2.8) provided that $\varsigma = 0$. Since $\mathbf{d}_i^n$ is the derivative of the interpolant at $x = x_i$, the third term in the right-hand side of (2.12) can be seen as the positive numerical viscosity introduced into the first-order scheme (2.8). Notice that, if $\varkappa < 1$, then, without loss of generality, it can be assumed that $\mathbf{D}_i^n \equiv \left(\mathbf{A}_i^n\right)^2$ is positive definite [44, p. 44], i.e., $\mathbf{v}^T \cdot \mathbf{D}_i^n \cdot \mathbf{v} \geq a^2\mathbf{v}^T \cdot \mathbf{v}\ \forall\, \mathbf{v} \in \mathbb{R}^M$, $a^2 > 0$. Actually, by adding to and subtracting from the right-hand side of scheme (2.12) the quantity

$$\varsigma\frac{(\Delta t)^2}{8\Delta x}a^2\left(\mathbf{d}_{i+1}^{n} - \mathbf{d}_{i}^{n}\right), \tag{2.13}$$

where a^2 is such that

$$\widetilde{\varkappa} \equiv \varkappa + \varsigma \frac{(\Delta t)^2 a^2}{(\Delta x)^2} \le 1, \tag{2.14}$$

we represent (2.12) in the form

$$\begin{aligned} \mathbf{v}_{i+0.5}^{n+0.25} &= 0.5\left(\mathbf{v}_{i+1}^{n} + \mathbf{v}_{i}^{n}\right) - \widetilde{\varkappa}\frac{\Delta x}{8}\left(\mathbf{d}_{i+1}^{n} - \mathbf{d}_{i}^{n}\right) - \frac{\Delta t}{2}\frac{\mathbf{f}\left(\mathbf{v}_{i+1}^{n}\right) - \mathbf{f}\left(\mathbf{v}_{i}^{n}\right)}{\Delta x} \\ &+ \varsigma\frac{(\Delta t)^2}{8\Delta x}\left[\left(\left(\mathbf{A}_{i+1}^{n}\right)^2 + a^2\mathbf{I}\right)\cdot\mathbf{d}_{i+1}^{n} - \left(\left(\mathbf{A}_{i}^{n}\right)^2 + a^2\mathbf{I}\right)\cdot\mathbf{d}_{i}^{n}\right]. \end{aligned} \tag{2.15}$$

Thus, in fact, the vanishing viscosity method [16], [30] is used, and hence (see, e.g., [16, pp. 27–32]) the scheme (2.12) satisfies the entropy condition. To make this point a little bit more clear, we consider a viscous perturbation of the system (2.1). We associate with (2.1) the parabolic system

$$\frac{1}{2}\frac{\partial \mathbf{u}_\epsilon}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}\left(\mathbf{u}_\epsilon\right) = \epsilon\frac{\partial}{\partial x}\left(\mathbf{A}^2 \cdot \frac{\partial \mathbf{u}_\epsilon}{\partial x}\right), \quad \epsilon > 0, \tag{2.16}$$

where the right-hand side can be viewed as a viscosity term. It is assumed that Mock's assumption of admissibility [16, p. 32] is valid, i.e., the matrix $U'' \cdot \mathbf{A}^2$ is positive definite. Here $U = U\left(\mathbf{u}_\epsilon\right)$ denotes a strictly convex positive entropy, and U'' denotes the Hessian matrix of U. Assuming that (2.16) has sufficiently smooth solutions [16, p. 27], we recover solutions of (2.1) as the limits of solutions to (2.16) as $\epsilon \to 0$. If we multiply the scheme COS1, (2.8), by $2/\Delta t$ and group all the terms of this scheme together in its left-hand side, then this group, in view of the above, is the finite difference approximation of the left-hand side of (2.16) on the cell $[x_i, x_{i+1}] \times [t_n, t_{n+0.25}]$. The right-hand side of (2.16) is approximated as

$$\epsilon\frac{\partial}{\partial x}\left(\mathbf{A}^2 \cdot \frac{\partial \mathbf{u}_\epsilon}{\partial x}\right) = \epsilon\frac{\left(\mathbf{A}_{i+1}^{n}\right)^2 \cdot \mathbf{d}_{i+1}^{n} - \left(\mathbf{A}_{i}^{n}\right)^2 \cdot \mathbf{d}_{i}^{n}}{\Delta x} + O\left((\Delta x)^2\right). \tag{2.17}$$

If $\epsilon = \varsigma\Delta t/4$, then we obtain the scheme (2.12) as the approximation of (2.16) on the cell $[x_i, x_{i+1}] \times [t_n, t_{n+0.25}]$. Thus, owing to the third term in the right-hand side of (2.12), the grid function $\mathbf{v}_i^n$ in (2.12) can be considered Lipschitz-continuous. Moreover, owing to this term, the scheme (2.12) is $O((\Delta x)^2 + (\Delta t)^2)$ accurate provided that $\varkappa = \varsigma = 1$.

Since the source terms in (1.1) can be, in general, stiff (i.e., $\tau \ll 1$), it is natural to use the following second-order implicit Runge–Kutta scheme for (2.2), since this scheme possesses a discrete analogy to the continuous asymptotic limit:

$$\begin{aligned} \mathbf{v}_i^{n+0.75} &= \mathbf{v}_i^{n+1} - \frac{\gamma}{2}\mathbf{q}\left(\mathbf{v}_i^{n+1}\right), \\ \mathbf{v}_i^{n+1} &= \mathbf{v}_i^{n+0.5} + \gamma\mathbf{q}\left(\mathbf{v}_i^{n+0.75}\right), \quad \gamma \equiv \frac{\Delta t}{\tau}. \end{aligned} \tag{2.18}$$

The second-order, $O((\Delta x)^2 + (\Delta t)^2)$, central scheme (2.12) will be abbreviated to COS2.

Before proceeding to the proof of stability of the developed schemes, the following remark is necessary. Since the Jacobian matrix $\mathbf{A}$ in (1.2) possesses M linearly independent eigenvectors, $\mathbf{A}_i^n$ $(= \mathbf{A}(x_i, t_n))$ is similar to a diagonal matrix [32]; i.e., there exists a nonsingular matrix $\mathbf{S}_i^n = \mathbf{S}(x_i, t_n)$ such that

$$(\mathbf{S}_i^n)^{-1} \cdot \mathbf{A}_i^n \cdot \mathbf{S}_i^n = \operatorname{diag}\left\{\lambda_i^{n,1}, \lambda_i^{n,2}, \ldots, \lambda_i^{n,M}\right\} \quad \forall i, n. \tag{2.19}$$

The right and left eigenvectors of $\mathbf{A}_i^n = \mathbf{A}(x_i, t_n)$ can be defined in such a way that $\mathbf{S}_i^n = \mathbf{S}(x_i, t_n)$ will be the matrix having the right eigenvectors as its columns, and the rows of $(\mathbf{S}_i^n)^{-1} = \mathbf{S}^{-1}(x_i, t_n)$ will be the left eigenvectors [29, p. 62]. For [8, Thm. 2.10] to be used, it must be proven that the functions

$$\left[(\mathbf{S}_i^n)^{-1} - (\mathbf{S}^n)^{-1}(x)\right] \cdot \mathbf{S}^n(x), \quad \left[(\mathbf{S}_i)^{-1}(t) - (\mathbf{S}_i^n)^{-1}\right] \cdot \mathbf{S}_i^n \tag{2.20}$$

will be Lipschitz-continuous in space and, respectively, time $\forall i, n$. Since each of the above functions depends only on one parameter (x or t), it can be done with ease for strictly hyperbolic systems, i.e., when the eigenvalues of the operator $\mathbf{A}(= \partial \mathbf{f}(\mathbf{u}) \diagup \partial \mathbf{u})$ in (1.2) are all distinct [30, p. 2]. In this case the proof follows from the well-known results of perturbation theory for simple eigenvalues [46, p. 67] (see also Theorem 2.1 in [3]). For a detailed discussion on the derivatives (sensitivities) of eigenvectors of matrix-valued functions depending on several parameters when the eigenvalues are multiple, see [3], [47]. In what follows, it will be assumed that the functions (2.20) are Lipschitz-continuous in space and time.

3. Stability and monotonicity of the developed schemes. The sufficient conditions for stability as well as the necessary conditions for S-monotonicity of the schemes COS2 and COS1 are found in this section. It is assumed that the bounded operator $\mathbf{A}(= \partial \mathbf{f}(\mathbf{u}) \diagup \partial \mathbf{u})$ in (1.2) is Fréchet-differentiable on the set $\Omega_{\mathbf{u}} \subseteq \mathbb{R}^M$, and its derivative is bounded on $\Omega_{\mathbf{u}}$.

First, let us find the conditions for stability of the scheme COS2, (2.12). In view of [8, Thm. 2.4], the stability of (2.12) will be investigated on the basis of its variational scheme. It is assumed, for the sake of simplicity, that $\varkappa, \varsigma = \text{const}$ in (2.12). By virtue of (2.4), the second term in the right-hand side of (2.12) can be written in the form

$$\varkappa \frac{\Delta x}{8}\left(\mathbf{d}_{i+1}^n - \mathbf{d}_i^n\right) = \frac{\varkappa}{8}\left(\mathbb{B}_i^n - \mathbb{A}_i^n\right) \cdot \left(\mathbf{v}_{i+1}^n - \mathbf{v}_i^n\right). \tag{3.1}$$

In view of (2.4) and (3.1), the variational scheme corresponding to (2.12) is the following:

$$\begin{aligned}
&\delta \mathbf{v}_{i+0.5}^{n+0.25} - 0.5\left(\delta \mathbf{v}_i^n + \delta \mathbf{v}_{i+1}^n\right) - \frac{\varkappa}{8}\mathbb{D}_i^n \cdot \left(\delta \mathbf{v}_i^n - \delta \mathbf{v}_{i+1}^n\right) \\
&- \varsigma \frac{(\Delta t)^2}{8\Delta x^2}\left[\left(\mathbf{A}_{i+1}^n\right)^2 \cdot \mathbb{B}_i^n - \left(\mathbf{A}_i^n\right)^2 \cdot \mathbb{A}_i^n\right] \cdot \left(\delta \mathbf{v}_{i+1}^n - \delta \mathbf{v}_i^n\right) \\
&- \frac{\Delta t}{2\Delta x}\left(\mathbf{A}_i^n \cdot \delta \mathbf{v}_i^n - \mathbf{A}_{i+1}^n \cdot \delta \mathbf{v}_{i+1}^n\right) = \frac{\varkappa}{8}\left[\left(\mathbf{v}_i^n - \mathbf{v}_{i+1}^n\right)^T \cdot \delta \mathbb{D}_i^n\right]^T \\
&+ \varsigma \frac{(\Delta t)^2}{8\Delta x^2}\left\{\delta\left[\left(\mathbf{A}_{i+1}^n\right)^2 \cdot \mathbb{B}_i^n - \left(\mathbf{A}_i^n\right)^2 \cdot \mathbb{A}_i^n\right]\right\} \cdot \left(\mathbf{v}_{i+1}^n - \mathbf{v}_i^n\right),
\end{aligned} \tag{3.2}$$

where $\mathbb{D}_i^n = \operatorname{diag}\left\{D_{i,1}^n, D_{i,2}^n, \ldots, D_{i,M}^n\right\} \equiv \mathbb{B}_i^n - \mathbb{A}_i^n$. By virtue of (1.2) and (2.5), we find that

$$\left\| \left(\mathbf{A}_{i+1}^n\right)^2 \cdot \mathbb{B}_i^n - \left(\mathbf{A}_i^n\right)^2 \cdot \mathbb{A}_i^n \right\|_2 \leq 8\lambda_{\max}^2 \aleph. \tag{3.3}$$

Thus, we may write that

$$\left\| \delta \left[\left(\mathbf{A}_{i+1}^n\right)^2 \cdot \mathbb{B}_i^n - \left(\mathbf{A}_i^n\right)^2 \cdot \mathbb{A}_i^n \right] \right\|_2 \leq 16\lambda_{\max}^2 \aleph. \tag{3.4}$$

By virtue of (2.5), we find that $-4\aleph\mathbf{I} \leq \mathbb{D}_i^n \leq 4\aleph\mathbf{I}$, and hence $-8\aleph\mathbf{I} \leq \delta\mathbb{D}_i^n \leq 8\aleph\mathbf{I}$. Thus, we may write that

$$\left\|\delta\mathbb{D}_i^n\right\|_2 \leq 8\aleph. \tag{3.5}$$

Considering that $\mathbf{v}_i^n$ in (2.12) is Lipschitz-continuous, we write

$$\left\|\mathbf{v}_i^n - \mathbf{v}_{i+1}^n\right\| \leq C_v \Delta x, \quad C_v = \text{const}. \tag{3.6}$$

It is assumed [8] that there exists $\alpha_0 = \text{const}$ such that $\Delta x \leq \alpha_0 \Delta t$ for a sufficiently small Δt. Then, by virtue of (3.4)–(3.6), and since $0 \leq \varkappa, \varsigma, \aleph \leq 1$, we find the following estimate for the right-hand side of (3.2):

$$\begin{aligned}
&\left\| \frac{\varkappa}{8} \left[\left(\mathbf{v}_i^n - \mathbf{v}_{i+1}^n\right)^T \cdot \delta\mathbb{D}_i^n \right]^T \right\|_2 \\
&+ \left\| \varsigma \frac{(\Delta t)^2}{8\Delta x^2} \left\{ \delta \left[\left(\mathbf{A}_{i+1}^n\right)^2 \cdot \mathbb{B}_i^n - \left(\mathbf{A}_i^n\right)^2 \cdot \mathbb{A}_i^n \right] \right\} \cdot \left(\mathbf{v}_{i+1}^n - \mathbf{v}_i^n\right) \right\|_2 \\
&\leq \left\{ \frac{\varkappa}{8} \left\|\delta\mathbb{D}_i^n\right\|_2 + \varsigma \frac{(\Delta t)^2}{8\Delta x^2} \left\| \delta \left[\left(\mathbf{A}_{i+1}^n\right)^2 \cdot \mathbb{B}_i^n - \left(\mathbf{A}_i^n\right)^2 \cdot \mathbb{A}_i^n \right] \right\|_2 \right\} \left\|\mathbf{v}_i^n - \mathbf{v}_{i+1}^n\right\|_2 \\
&\leq \left(1 + 2C_r^2\right) \alpha_0 C_v \Delta t, \quad C_r = \frac{\Delta t \lambda_{\max}}{\Delta x}.
\end{aligned} \tag{3.7}$$

Since the uniform stability with respect to the initial data implies the stability of the scheme (see section 5), we conclude, in view of (3.7), that scheme (3.2) will be stable if the following scheme, i.e., (3.2) without the right-hand side, is stable:

$$\begin{aligned}
\delta\mathbf{v}_{i+0.5}^{n+0.25} &= 0.5\left(\delta\mathbf{v}_i^n + \delta\mathbf{v}_{i+1}^n\right) - \frac{\varkappa}{8}\mathbb{D}_i^n \cdot \left(\delta\mathbf{v}_{i+1}^n - \delta\mathbf{v}_i^n\right) \\
&+ \varsigma \frac{(\Delta t)^2}{8\Delta x^2} \left[\left(\mathbf{A}_{i+1}^n\right)^2 \cdot \mathbb{B}_i^n - \left(\mathbf{A}_i^n\right)^2 \cdot \mathbb{A}_i^n \right] \cdot \left(\delta\mathbf{v}_{i+1}^n - \delta\mathbf{v}_i^n\right) \\
&- \frac{\Delta t}{2\Delta x} \left(\mathbf{A}_{i+1}^n \cdot \delta\mathbf{v}_{i+1}^n - \mathbf{A}_i^n \cdot \delta\mathbf{v}_i^n\right).
\end{aligned} \tag{3.8}$$

Notice that, for the sake of convenience, we use the same notation in (3.2) and in (3.8) in spite of the fact that these schemes are different. We rewrite (3.8) to read

$$\delta\mathbf{v}_{i+0.5}^{n+0.25} = 0.5\left(\mathbf{I} + \mathbf{E}_{i,i}^n\right) \cdot \delta\mathbf{v}_i^n + 0.5\left(\mathbf{I} - \mathbf{E}_{i,i+1}^n\right) \cdot \delta\mathbf{v}_{i+1}^n, \tag{3.9}$$

where

$$\mathbf{E}_{i,i}^{n}=\frac{\varkappa}{4}\mathbb{D}_{i}^{n}-\varsigma\frac{(\Delta t)^{2}}{4(\Delta x)^{2}}\left[\left(\mathbf{A}_{i+1}^{n}\right)^{2}\cdot\mathbb{B}_{i}^{n}-\left(\mathbf{A}_{i}^{n}\right)^{2}\cdot\mathbb{A}_{i}^{n}\right]+\frac{\Delta t}{\Delta x}\mathbf{A}_{i}^{n}, \tag{3.10}$$

$$\mathbf{E}_{i,i+1}^{n}=\frac{\varkappa}{4}\mathbb{D}_{i}^{n}-\varsigma\frac{(\Delta t)^{2}}{4(\Delta x)^{2}}\left[\left(\mathbf{A}_{i+1}^{n}\right)^{2}\cdot\mathbb{B}_{i}^{n}-\left(\mathbf{A}_{i}^{n}\right)^{2}\cdot\mathbb{A}_{i}^{n}\right]+\frac{\Delta t}{\Delta x}\mathbf{A}_{i+1}^{n}. \tag{3.11}$$

We write, in view of (1.2) and (2.5), that $s\left(\mathbf{E}_{i,i}^{n}\right), s\left(\mathbf{E}_{i,i+1}^{n}\right)\subseteq[-\lambda_{E},\lambda_{E}]$, where

$$\lambda_{E}=\begin{cases}\varkappa\aleph-\varsigma C_{r}^{2}\aleph+C_{r}, & \varsigma C_{r}^{2}\leq\varkappa,\\ \varsigma C_{r}^{2}\aleph-\varkappa\aleph+C_{r}, & \varsigma C_{r}^{2}\geq\varkappa,\end{cases}\qquad C_{r}=\frac{\Delta t\lambda_{\max}}{\Delta x}. \tag{3.12}$$

Hence, by virtue of [8, Thm. 2.10] we find that the scheme (3.9) will be stable if

$$\max_{\lambda\in[-\lambda_{E},\lambda_{E}]}0.5\left(|1+\lambda|+|1-\lambda|\right)\leqslant 1\quad\forall i,n. \tag{3.13}$$

We obtain from (3.13) the following sufficient condition for the stability of the variational scheme (3.2):

$$\begin{array}{ll}\left(\varkappa-\varsigma C_{r}^{2}\right)\aleph+C_{r}\leq 1, & C_{r}\sqrt{\varsigma}\leq\sqrt{\varkappa},\\ \left(\varsigma C_{r}^{2}-\varkappa\right)\aleph+C_{r}\leq 1, & C_{r}\sqrt{\varsigma}\geq\sqrt{\varkappa},\end{array}\qquad C_{r}=\frac{\Delta t\lambda_{\max}}{\Delta x}. \tag{3.14}$$

Thus, in view of [8, Thm. 2.4], the scheme COS2, (2.12), will be stable if (3.14) is valid.

Let us note that the parameters $\varkappa$ and ς are taken as constant in scheme (2.12). However, in practice, it can be convenient to take that $\varkappa_{i}^{n}=\varkappa(\mathbf{v}_{i}^{n})$ and $\varsigma_{i}^{n}=\varsigma(\mathbf{v}_{i}^{n})$. In such a case, the condition (3.14) for the stability of (2.12) can be grounded in perfect analogy to the above, since $0\leq\varkappa,\varsigma\leq 1$ and, hence, $|\delta\varkappa|,|\delta\varsigma|\leq 1$.

Assuming $\varsigma\to 0$ in (3.14), we obtain from (3.14) the following sufficient condition for the stability of the scheme COS1, (2.8):

$$\varkappa\aleph+C_{r}\leqslant 1,\quad C_{r}=\frac{\Delta t\lambda_{\max}}{\Delta x}. \tag{3.15}$$

Let us note that the monotonicity parameter, $\aleph$, introduced in part I [8] plays, along with the CFL number, a major role in the criteria, (3.14) and (3.15), for the stability of the central schemes.

Notice that (3.14) and (3.15) were obtained on the basis that $\mathbf{E}_{i,i}^{n}$ is diagonalizable. Such an assumption should be verified for a concrete problem. This assumption has a rigorous basis if $\mathbf{A}_{i}^{n}$ is diagonalizable and $\mathbb{D}_{i}^{n}$ is a scalar matrix. In such a case $\mathbf{E}_{i,i}^{n}$ will be diagonalizable. Thus, for instance, the LxF scheme will be stable if $C_{r}\leqslant 1$. In the case of the scheme COS1 with a nonscalar matrix, $\mathbb{D}_{i}^{n}$, and symmetric Jacobian matrix, $\mathbf{A}_{i}^{n}$, the condition (3.15) for stability of (2.8) can be found with ease by virtue of [8, Prop. 2.12]. In a more general case, the stability of (2.8) can be investigated by virtue of [8, Thm. 2.13].

Let us find the necessary conditions for the variational S-monotonicity (see [8, Def. 1.1]) of the schemes (2.8), (2.9) and (2.12), (2.18), assuming that $\mathbf{v}_{i}^{n}=\mathbf{C}^{n}=$ const. The variational scheme corresponding to the scheme COS2, (2.12), under (2.4), (3.1) is

$$\delta\mathbf{v}_{i+0.5}^{n+0.25} = 0.5\left(\delta\mathbf{v}_i^n + \delta\mathbf{v}_{i+1}^n\right) - \frac{\varkappa}{8}\mathbb{D}^n \cdot \left(\delta\mathbf{v}_{i+1}^n - \delta\mathbf{v}_i^n\right) + \frac{\varsigma\,(\Delta t)^2}{8\,(\Delta x)^2}\left(\mathbf{A}_c^n\right)^2 \cdot \mathbb{D}^n \cdot \left(\delta\mathbf{v}_{i+1}^n - \delta\mathbf{v}_i^n\right) - \frac{\Delta t}{2\Delta x}\mathbf{A}_c^n \cdot \left(\delta\mathbf{v}_{i+1}^n - \delta\mathbf{v}_i^n\right), \tag{3.16}$$

where $\mathbf{A}_c^n = \mathbf{A}(\mathbf{C}^n)$, $\mathbf{A}(\mathbf{u}) = \partial\mathbf{f}(\mathbf{u})\,/\,\partial\mathbf{u}$. Here, in (3.16), $\mathbf{A}_c^n$ is a diagonalizable matrix such that its spectrum $s(\mathbf{A}_c^n) \subset [-\lambda_{\max}, \lambda_{\max}]$; see (1.2). In view of (3.1), $\mathbb{D}^n$ in (3.16) can be taken at will, as $\mathbf{v}_i^n = \mathbf{C}^n = \text{const}$, $\mathbf{v}_{i+1}^n - \mathbf{v}_i^n = 0$, and $\mathbf{d}_i^n = \mathbf{d}_{i+1}^n = 0$. Aiming to obtain the necessary condition for (3.16) to be variationally S-monotone, it is assumed that $\mathbb{D}^n = \zeta^n\mathbf{I}$ is a scalar matrix with $\zeta^n \in [-4\aleph, 4\aleph]$. In view of [8, Thm. 2.11], the scheme (3.16) will be S-monotone *iff*

$$\max_{|\zeta|\le 4\aleph,\ \lambda\in s(\mathbf{A}_c^n)}\left(\left|\frac{1}{2} + \frac{\varkappa\zeta}{8} - \frac{\varsigma\,(\Delta t)^2}{8\,(\Delta x)^2}\zeta\lambda^2 + \frac{\Delta t}{2\Delta x}\lambda\right| + \left|\frac{1}{2} - \frac{\varkappa\zeta}{8} + \frac{\varsigma\,(\Delta t)^2}{8\,(\Delta x)^2}\zeta\lambda^2 - \frac{\Delta t}{2\Delta x}\lambda\right|\right) \leqslant 1. \tag{3.17}$$

By virtue of (3.17) we find that (3.14) will be the necessary condition for the variational S-monotonicity [8, Def. 1.1] of the scheme COS2, (2.12). Hence, (3.15) will be the necessary condition for the variational S-monotonicity of the scheme COS1, (2.8).

Notice that, using (3.16), the necessary condition for the variational GOS-monotonicity of the scheme COS2 can be found with ease, by virtue of [8, Thm. 2.8]. One may readily check that this condition coincides with (3.14).

The variational scheme corresponding to (2.9) reads

$$\delta\mathbf{v}_i^{n+1} = \delta\mathbf{v}_i^{n+0.5} + \frac{\Delta t}{\tau}\mathbf{G}\left(\mathbf{v}_i^{n+1}\right) \cdot \delta\mathbf{v}_i^{n+1}, \tag{3.18}$$

where $\mathbf{G}(\mathbf{V}) = \partial\mathbf{q}(\mathbf{V})\,/\,\partial\mathbf{V}$. Let us rewrite (3.18) to read

$$\delta\mathbf{v}_i^{n+1} = \left\{\mathbf{I} - \frac{\Delta t}{\tau}\mathbf{G}\left(\mathbf{v}_i^{n+1}\right)\right\}^{-1} \cdot \delta\mathbf{v}_i^{n+0.5}. \tag{3.19}$$

Let $\mathbf{G}$ be a normal matrix. In such a case, in view of [9, Thm. 3.3] the variational scheme (3.19) will be S-monotone if

$$\max_k \frac{1}{\left|1 - \frac{\Delta t}{\tau}\xi_k\left(\mathbf{v}_i^{n+1}\right)\right|} \le 1 \quad \forall i, n, \tag{3.20}$$

where ξ_k denotes the *kth* eigenvalue of the matrix $\mathbf{G}$. In view of (1.3), the inequality (3.20) is valid, and hence the variational scheme (3.18) will be unconditionally S-monotone.

Thus, the scheme (2.8)–(2.9) will be variationally S-monotone only if (3.15) will be valid.

PROPOSITION 3.1. *By virtue of* [14, Thm. 1.2.1] *the stability of the scheme in variations,* (3.19)*, can be proved with ease, and hence the stability of the original scheme,* (2.9)*, can be proved.*

The variational scheme corresponding to (2.18) (under $\mathbf{v}_i^n = \mathbf{C} = \text{const}$) reads

$$\delta \mathbf{v}_i^{n+0.75} = \delta \mathbf{v}_i^{n+1} - \frac{\gamma}{2} \mathbf{G}_c \cdot \delta \mathbf{v}_i^{n+1}, \quad \gamma \equiv \frac{\Delta t}{\tau},$$

$$\delta \mathbf{v}_i^{n+1} = \delta \mathbf{v}_i^{n+0.5} + \gamma \mathbf{G}_c \cdot \delta \mathbf{v}_i^{n+0.75}, \quad \mathbf{G}_c = \mathbf{G}\left(C\right), \tag{3.21}$$

where $\mathbf{G}\left(\mathbf{u}\right) = \partial \mathbf{q}\left(\mathbf{u}\right) \big/ \partial \mathbf{u}$. Let $\mathbf{G}$ be a normal matrix. In such a case, since all eigenvalues of $\mathbf{G}_c$ have nonpositive real parts (see (1.3)), the first step of the scheme (2.18) will be unconditionally S-monotone (see [9, Thm. 3.3], [8, Thm. 2.6]). It remains to prove that the scheme (3.21), taken as a whole, will be S-monotone under the same condition. Eliminating $\delta \mathbf{v}_i^{n+0.75}$, we obtain that

$$\delta \mathbf{v}_i^{n+1} = \left[\mathbf{I} - \gamma \mathbf{G}_c \cdot \left(\mathbf{I} - \frac{\gamma}{2} \mathbf{G}_c\right)\right]^{-1} \cdot \delta \mathbf{v}_i^{n+0.5}. \tag{3.22}$$

In view of [9, Thm. 3.3], the scheme (3.22) will be S-monotone if

$$\max_k \left| \frac{1}{1 - \gamma \xi_k\left(\mathbf{C}\right)\left(1 - \frac{\gamma}{2} \xi_k\left(\mathbf{C}\right)\right)} \right| \le 1 \quad \forall \mathbf{C} \in \Omega_{\mathbf{u}}, \tag{3.23}$$

where $\xi_k\left(\mathbf{C}\right)$ denotes the kth eigenvalue of the matrix $\mathbf{G}_c$. Since all eigenvalues of $\mathbf{G}_c$ have nonpositive real parts, the variational scheme (3.22) will be unconditionally S-monotone.

Thus, the scheme (2.12), (2.18) will be variationally S-monotone only if (3.14) is valid.

4. Exemplification and discussion. In this section we are mainly concerned with verification of the second-order central scheme COS2, (2.12). To demonstrate the monotonicity (i.e., absence of spurious oscillations) of this scheme in contrast to the central scheme COS1, (2.8), being $O(\Delta t + (\Delta x)^2)$ accurate, we will use the inviscid Burgers equation. The scheme (2.12), (2.18) is verified using Pember's rarefaction test problem [41]. Euler equations of gas dynamics, namely Sod's problem, as well as a 3-D axial symmetric problem, are also used for the verification of the second-order central scheme COS2, (2.12).

4.1. Scalar nonlinear equation. As the first stage in the verification, we will focus on the following scalar version of the problem (1.1):

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f\left(u\right) = 0, \quad x \in \mathbb{R},\ 0 < t \le T_{\max}; \quad u\left(x,t\right)\big|_{t=0} = u^0\left(x\right). \tag{4.1}$$

To test the first-order scheme COS1, (2.8), we will solve the inviscid Burgers equation (i.e., $f\left(u\right) \equiv u^2 \big/ 2$) with the following initial condition:

$$u\left(x,0\right) = \begin{cases} u_0, & x \in \left(h_L, h_R\right), \\ 0, & x \notin \left(h_L, h_R\right), \end{cases} \qquad h_R > h_L,\ u_0 = \text{const} \ne 0. \tag{4.2}$$

The exact solution to (4.1), (4.2) is given by

$$u\left(x,t\right) = \begin{cases} u_1\left(x,t\right), & 0 < t \le T, \\ u_2\left(x,t\right), & t > T, \end{cases} \tag{4.3}$$

where $T = 2S/u_0$, $S = h_R - h_L$,

$$u_1(x,t) = \begin{cases} \frac{x-h_L}{b-h_L}u_0, & h_L < x \le b,\ b = u_0 t + h_L, \\ u_0, & b < x \le 0.5u_0 t + h_R, \\ 0, & x \le h_L \ or\ x > 0.5u_0 t + h_R, \end{cases} \tag{4.4}$$

$$u_2(x,t) = \begin{cases} \frac{2S(x-h_L)}{(L-h_L)^2}u_0, & h_L < x \le L, \\ 0, & x \le h_L \ or\ x > L, \end{cases} \tag{4.5}$$

$$L = 2\sqrt{S^2 + 0.5u_0 S(t-T)} + h_L. \tag{4.6}$$

The numerical solutions were computed on a uniform grid with spatial increments of $\Delta x = 0.01$, the velocity $u_0 = 1$ in (4.2), $h_L = 0.2$, $h_R = 1$, the monotonicity parameter $\aleph = 0.5$, the CFL number $Cr \equiv u_0\Delta t/\Delta x = 0.5$, and the parameter $\varkappa = 1$ in (2.8). The results of simulation are depicted with the exact solution in Figure 4.1.

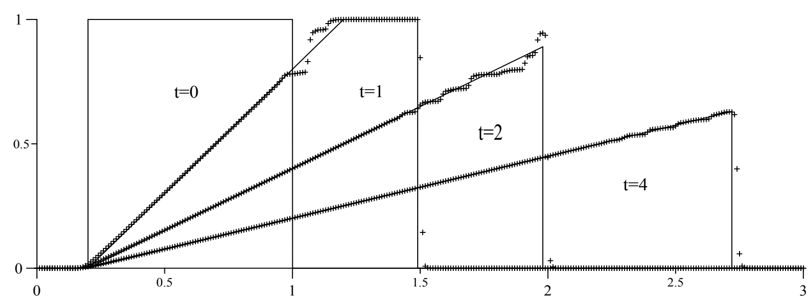

FIG. 4.1. *Inviscid Burgers equation. The scheme COS*1 *($\varkappa = 1$) versus the analytical solution. Crosses: numerical solution. Solid line: analytical solution and initial data.* $C_r = \aleph = 0.5$, $\Delta x = 0.01$.

We note (Figure 4.1) that spurious solutions are produced by the first-order scheme, (2.8), in spite of the fact that the condition (3.15) being sufficient for the stability as well as necessary for S-monotonicity of the scheme is not violated. To demonstrate that (3.15) is the necessary condition for the variational GOS-monotonicity and the variational H-monotonicity (see [8, Def. 1.1]) of the scheme COS1, (2.8), we will use the variational scheme (3.16). As applied to the scheme COS1, the variational scheme (3.16) can be written in the form

$$\delta v_{i+0.5}^{n+0.5} = 0.5(1+E^n)\,\delta v_i^n + 0.5(1-E^n)\,\delta v_{i+1}^n, \quad E^n = \frac{\zeta^n}{4} + \frac{\Delta t}{\Delta x}A_c^n. \tag{4.7}$$

Since all of the coefficients of (4.7) will be nonnegative under (3.15), the scheme in question, (2.8), will be GOS-monotone [8, Thm. 2.8] as well as H-monotone [8] (and hence TVD and G-monotone [8]) only if (3.15) is valid. Notice that the numerical simulations were performed with such values of the parameter $\varkappa$, CFL number Cr, and monotonicity parameter $\aleph$, so that (3.15) was not violated. As can be seen in Figure 4.1, the boundary maximum principle is not violated by the scheme; i.e., the maximum positive values of the dependent variable, v, occur at the boundary $t = 0$. It is interesting that the spurious solution (see Figure 4.1) produced by the scheme

COS1 has the monotonicity property [21], since no new local extrema in x are created. Also the value of a local minimum is nondecreasing and the value of a local maximum is nonincreasing.

Let us note that the problem of building free-of-spurious-oscillations schemes is, in general, unsettled up to the present. Even the most modern high-resolution schemes can produce spurious oscillations, and these oscillations are often of ENO type (see, e.g., [39] and references therein). We found that the oscillations produced by the COS1 scheme, (2.8), are of ENO type; namely, their amplitude decreases rapidly with decreasing the time-increment Δt, and the oscillations virtually disappear under a relatively low CFL number, $Cr \leq 0.15$. However, the reduction of the CFL number causes some smearing of the solution. The spurious oscillations (see Figure 4.1) can be eradicated without reducing CFL number, but rather by decreasing the parameter $\varkappa$. Particularly, the spurious oscillations disappear if $\varkappa = 2/3$, $C_r = 0.5$; however, this introduces more numerical smearing than in the case of the CFL number reduction. Satisfactory results are obtained under $\varkappa = 0.82$ ($C_r = 0.5$). The results of simulations are not depicted here.

To gain insight into why the scheme COS1, (2.8), can exhibit spurious solutions, let us consider the so-called *first differential approximation* of this scheme (see [13, p. 45], [45, p. 376]; see also "modified equations" in [13, p. 45], [30], [33]). As reported in [13, p. 45], [45, p. 376], this heuristic method was originally presented by Hirt (1968) as well as by Shokin and Yanenko (1968), and it has since been widely employed in the development of stable difference schemes for PDEs.

We found that the local truncation error, ψ, for the scheme COS1 can be written in the form

$$\psi = \frac{(1-\varkappa)\,(\Delta x)^2}{4\Delta t}\frac{\partial^2 u\,(x,t)}{\partial x^2} + \frac{\Delta t}{4}\frac{\partial^2 f\,(u)}{\partial t \partial x}$$

$$+\, O\left(\frac{(\Delta x)^4}{\Delta t} + (\Delta t)^2 + (\Delta x)^2\right). \tag{4.8}$$

By virtue of (4.8), we find the first differential approximation of the scheme COS1,

$$\frac{\partial u}{\partial t} + \frac{\partial f\,(u)}{\partial x} = \frac{\Delta t}{4}\frac{\partial}{\partial x}\left(B\frac{\partial u\,(x,t)}{\partial x}\right), \tag{4.9}$$

where $B = (1-\varkappa)\,(\Delta x/\Delta t)^2 - A^2$. The term in the right-hand side of (4.9) will be dissipative if

$$(1-\varkappa)\left(\frac{\Delta x}{\Delta t}\right)^2 - A^2 > 0 \implies C_r^2 < 1-\varkappa. \tag{4.10}$$

Thus, the scheme COS1, (2.8), is nondissipative under $\varkappa = 1$, and hence it can produce spurious oscillations. Notice that, if $\varkappa = 0.82$, then we obtain from (4.10) that $C_r < 0.42$. Nevertheless, as is reported above, satisfactory results can be obtained under $C_r = 0.5$ as well. So then the notion of first differential approximation has enabled us to understand that the spurious solutions exhibited by the scheme COS1, (2.8), are mainly associated with the negative numerical viscosity introduced to obtain the scheme of second order in space, i.e., $O(\Delta t + (\Delta x)^2)$.

Let us consider the scheme COS2, (2.12), approximating (4.1) with the accuracy $O((\Delta t)^2 + (\Delta x)^2)$. To test the scheme COS2, (2.12), the inviscid Burgers equation was solved under the initial condition (4.2). The numerical solutions were computed under the same values of parameters as in the case of the scheme COS1, but $C_r = 1$. The results of the simulation are depicted with the exact solution in Figure 4.2.

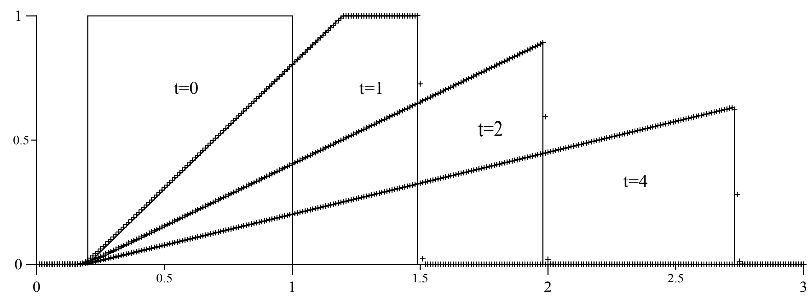

FIG. 4.2. *Inviscid Burgers equation. The scheme COS2 ($\varkappa = \varsigma = 1$) versus the analytical solution. Crosses: numerical solution. Solid line: analytical solution and initial data. $C_r = 1$, $\aleph = 0.5$, $\Delta x = 0.01$.*

We note (see Figure 4.2) that the scheme COS2, (2.12), exhibits a typical second-order nature without any spurious oscillations. Assuming $\aleph = 1$, we obtain a minor improvement of the numerical solutions, whereas decreasing the value of Cr up to 0.5 leads to a mild smearing of the solutions. The results of the simulations are not depicted here.

Let us estimate the computational cost for the scheme COS2, (2.12), in comparison with the one for the NT-scheme [35, eqs. (2.16a)–(2.16b)]. To estimate the derivative, d_i^n, of the interpolant in (2.12), we use the monotone piecewise cubic interpolation [8, sect. 3] together with the Thomas algorithm, which requires 5 floating-point operations per 1 grid node. Besides, we need 4 floating-point operations to calculate and modify the derivative [8, sect. 3]. To calculate $f(v_i^n)$ we need 1 floating-point operation, since the inviscid Burgers equation is considered. To calculate $(A_i^n)^2 d_i^n$ we need 2 floating-point operations. All coefficients of the scheme are constant and must be preassigned. In addition, we need 4 multiplications to estimate the right-hand side in (2.12). Hence, the computational cost for the scheme COS2 will be 16 floating-point operations per 1 grid node. In the case of the NT-scheme [35, eqs. (2.16a)–(2.16b)] we have to calculate, in addition to the scheme COS2, the value $f\left(v_i^{n+0.125}\right)$ as well as the derivative f_i' (the notation of this paper is used) by virtue of the monotone piecewise cubic interpolation [8, sect. 3]; however, we have no need of the value $(A_i^n)^2 d_i^n$. Thus, the computational cost for the NT-scheme will be 24 floating-point operations per 1 grid node, and, hence, it is 3 times more expensive than the scheme COS2 in regard to the fact that the NT-scheme is used with a CFL number less than 0.5.

In general, the flux-function is an M-vector $\mathbf{f}(\mathbf{u}) = \{f_1(\mathbf{u}), f_2(\mathbf{u}), \ldots, f_M(\mathbf{u})\}^T$. Let K be the number of floating-point operations required for the calculation of a vector's component. Then, in the case of the scheme COS2, the computational cost will be $(13 + K)M + (K + 1) M^2$ floating-point operations per 1 grid node, and $(22+2K)M$ in the case of the NT-scheme. For instance, in the case of Euler equations ($M = 3$, $K = 2$), the cost for the scheme COS2 will be almost 72, while for the NT-scheme it will be approximately 78 floating-point operations per 1 grid node. Hence, the NT-scheme will be at least 2 times more expensive than the scheme COS2, as

Table 4.1

N	1280	640	320	160	80
ε	1.0×10^{-6}	4.1×10^{-6}	1.8×10^{-5}	7.6×10^{-5}	3.5×10^{-4}

Table 4.2

Δx	6.25×10^{-4}	1.25×10^{-3}	2.5×10^{-3}	5.0×10^{-3}	0.01
ε	7.1×10^{-4}	1.5×10^{-3}	3.2×10^{-3}	6.5×10^{-3}	1.3×10^{-2}

the NT-scheme is usually used with a CFL number less than 0.5. Obviously, for the sufficiently large values of M, the NT-scheme will be cheaper than the scheme COS2. Hence, if we need to use the latter in practice, this scheme should be modified in such a way as to reduce the computational cost for calculating the third term in the right-hand side of (2.12).

To demonstrate that the scheme COS2, (2.12), is of the second order, we consider the linear transport equation, i.e., (4.1) with $f(u) \equiv u$, subject to the initial data: $u^0(x) = \sin(\pi x)$. The numerical solutions were computed under $\varkappa = \varsigma = 1$. Since the numerical solution under the CFL number $Cr = 1$ practically coincides with the analytical one (L_1 error $\sim 10^{-13}$), it is assumed that $Cr = 0.95$. In such a case, by virtue of the condition (3.14), we find that the scheme COS2 will be S-monotone only if $\aleph \leq 1/1.95 \approx 0.51$. Thus, we have taken $\aleph = 0.5$. L_1 errors (ε), at $t = 10$, $0 \leq x \leq 2$, versus the number of nodes (N), are depicted in Table 4.1.

Let us, however, note that the scheme COS2, being second-order accurate for smooth solutions, will, in general, be closer to the first than to the second order in the case of discontinuous solutions. As an example of such a case, let us consider the forgoing numerical solution to the inviscid Burgers equation by application of the scheme COS2. L_1 errors (ε), at $t = 2$, $0 \leq x \leq 3$, versus the spatial increments (Δx), are depicted in Table 4.2.

4.2. Hyperbolic conservation laws with relaxation. Let us consider the model system of hyperbolic conservation laws with relaxation developed in [41]:

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + aw\right) = 0, \tag{4.11}$$

$$\frac{\partial z}{\partial t} + \frac{\partial}{\partial x}az = \frac{1}{\tau}Q(w, z), \tag{4.12}$$

where

$$Q(w, z) = z - m(u - u_0), \quad u = w - q_0 z, \tag{4.13}$$

τ denotes the relaxation time of the system, and q_0, m, a, and u_0 are constants. The Jacobian, $\mathbf{A}$, can be written in the form

$$\mathbf{A} = \left\{ \begin{array}{cc} w - q_0 z + a & -q_0 (w - q_0 z) \\ 0 & a \end{array} \right\}. \tag{4.14}$$

The system (4.11)–(4.12) has the frozen [41] characteristic speeds $\lambda_1 = a$, $\lambda_2 = u + a$. The equilibrium equation for (4.11)–(4.12) is

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u_*^2 + aw\right) = 0, \tag{4.15}$$

where

$$u_* = w - q_0 z_*, \quad z_* = \frac{m}{1 + m q_0} (w - u_0) . \tag{4.16}$$

The equilibrium characteristic speed λ_* can be written in the form

$$\lambda_* (w) = \frac{u_* (w)}{1 + m q_0} + a. \tag{4.17}$$

Pember's rarefaction test problem is to find the solution $\{w, z\}$ to (4.11)–(4.12), and hence the function $u = u(x, t)$, under $\tau \to 0$, and where

$$\{w, z\} = \begin{cases} \{w_L, z_* (w_L)\}, & x < x_0, \\ \{w_R, z_* (w_R)\}, & x > x_0, \end{cases} \tag{4.18}$$

$$0 < u_L = w_L - q_0 z_* (w_L) < u_R = w_R - q_0 z_* (w_R) . \tag{4.19}$$

The analytical solution of this problem can be found in [41]. The parameters of the model system are assumed as follows: $q_0 = -1$, $m = -1$, $u_0 = 3$, $a = \pm 1$, $\tau = 10^{-8}$. The initial conditions of the rarefaction problem are defined by

$$u_L = 2 \implies z_L = m (u_L - u_0) = 1, \; w_L = u_L + q_0 z_L = 1, \tag{4.20}$$

$$u_R = 3 \implies z_R = m (u_R - u_0) = 0, \; w_R = u_R + q_0 z_R = 3. \tag{4.21}$$

The position of the initial discontinuity, x_0, is set according to the value of a so that the solutions of all the rarefaction problems are identical [41]. Let a position, x_R^t, of leading edge or a position, x_L^t, of trailing edge of the rarefaction be known (e.g., see $x_R^t = 0.85$, $x_L^t = 0.7$ in [41]); then

$$x_0 = x_R^t - \left(\frac{u_R}{1 + m q_0} + a \right) t = x_L^t - \left(\frac{u_L}{1 + m q_0} + a \right) t. \tag{4.22}$$

At $t = 0.3$, under (4.20)–(4.21) we have [41]

$$u = \begin{cases} 2, & x \le 0.7, \\ 2 + \frac{x - 0.7}{0.85 - 0.7}, & 0.7 < x < 0.85, \\ 3, & x \ge 0.85. \end{cases} \tag{4.23}$$

The results of the simulations, based upon the scheme (2.12), (2.18) under different values of the parameter a ($a = 1$, $a = -1$) and different values of a grid spacing, Δx, are depicted in Figure 4.3.

One can clearly see (Figure 4.3) that the scheme COS2 is free from spurious oscillations. Let us also note that the results generated by the scheme COS2 are less accurate in the case of negative value of a than those in the case of positive value of a. Specifically, in the numerical solutions produced under $a = -1$, the representations of the trailing and leading edges of the rarefaction are more smeared than those in the solutions produced under $a = 1$. Notice that, under some negative value of a, the frozen and the equilibrium characteristic speeds do not all have the same sign.

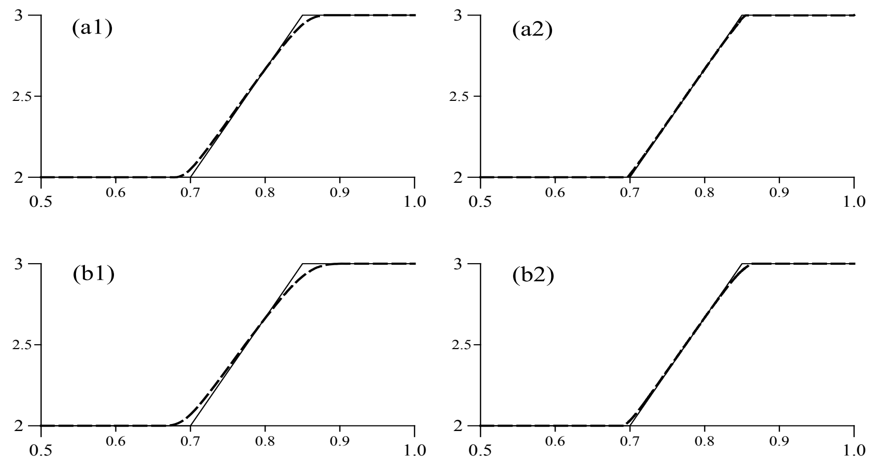

FIG. 4.3. *Pember's rarefaction test problem. The second-order scheme COS2 ($\varkappa = \varsigma = 1$) unified with the second-order implicit Runge–Kutta scheme versus the analytical solution for u. Dashed line: numerical solution. Solid line: analytical solution. Time $t = 0.3$, Courant number $C_r = 1$, monotonicity parameter $\aleph = 1$.* (a1): $\Delta x = 10^{-3}$, $a = 1$. (a2): $\Delta x = 2.5 \times 10^{-4}$, $a = 1$. (b1): $\Delta x = 10^{-3}$, $a = -1$. (b2): $\Delta x = 2.5 \times 10^{-4}$, $a = -1$.

4.3. 1-D Euler equations of gas dynamics. In this section we apply the second-order scheme COS2, (2.12), to the Euler equations of gamma-law gas:

$$\frac{\partial \mathbf{u}(x,t)}{\partial t} + \frac{\partial}{\partial x}\mathbf{F}(\mathbf{u}) = 0, \quad x \in \mathbb{R},\ t > 0; \quad \mathbf{u}(x,0) = \mathbf{u}^0(x), \tag{4.24}$$

$$\mathbf{u} \equiv \{u_1, u_2, u_3\}^T = \{\rho, \rho v, e\}^T, \quad \mathbf{F}(\mathbf{u}) = \left\{\rho v, \rho v^2 + p, (e+p)v\right\}^T, \tag{4.25}$$

$$e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2, \quad \gamma = \text{const}, \tag{4.26}$$

where ρ, v, p, e denote the density, velocity, pressure, and total energy, respectively. We consider the Cauchy problem subject to Riemann initial data

$$\mathbf{u}^0(x) = \begin{cases} \mathbf{u}_L & x < x_0, \\ \mathbf{u}_R & x > x_0, \end{cases} \quad \mathbf{u}_L, \mathbf{u}_R = \text{const}. \tag{4.27}$$

The analytic solution to this so-called Riemann problem can be found in [30, sect. 14].

We solve the shock tube problem (see, e.g., [6], [30], [31]) with Sod's initial data:

$$\mathbf{u}_L = \left\{\begin{array}{c} 1 \\ 0 \\ 2.5 \end{array}\right\}, \quad \mathbf{u}_R = \left\{\begin{array}{c} 0.125 \\ 0 \\ 0.25 \end{array}\right\}. \tag{4.28}$$

Following Balaguer and Conde [6], as well as Liu and Tadmor [31], we assume that the computational domain is $0 \le x \le 1$; the point x_0 is located in the middle of the interval $[0,1]$, i.e., $x_0 = 0.5$; the equations (4.24) are integrated up to $t = 0.16$ on a spatial grid with 200 nodes as in [6] and in [31]. By virtue of (3.14), the CFL number is taken to be $Cr = 0.95$ in contrast to [6] and [31], where the simulations

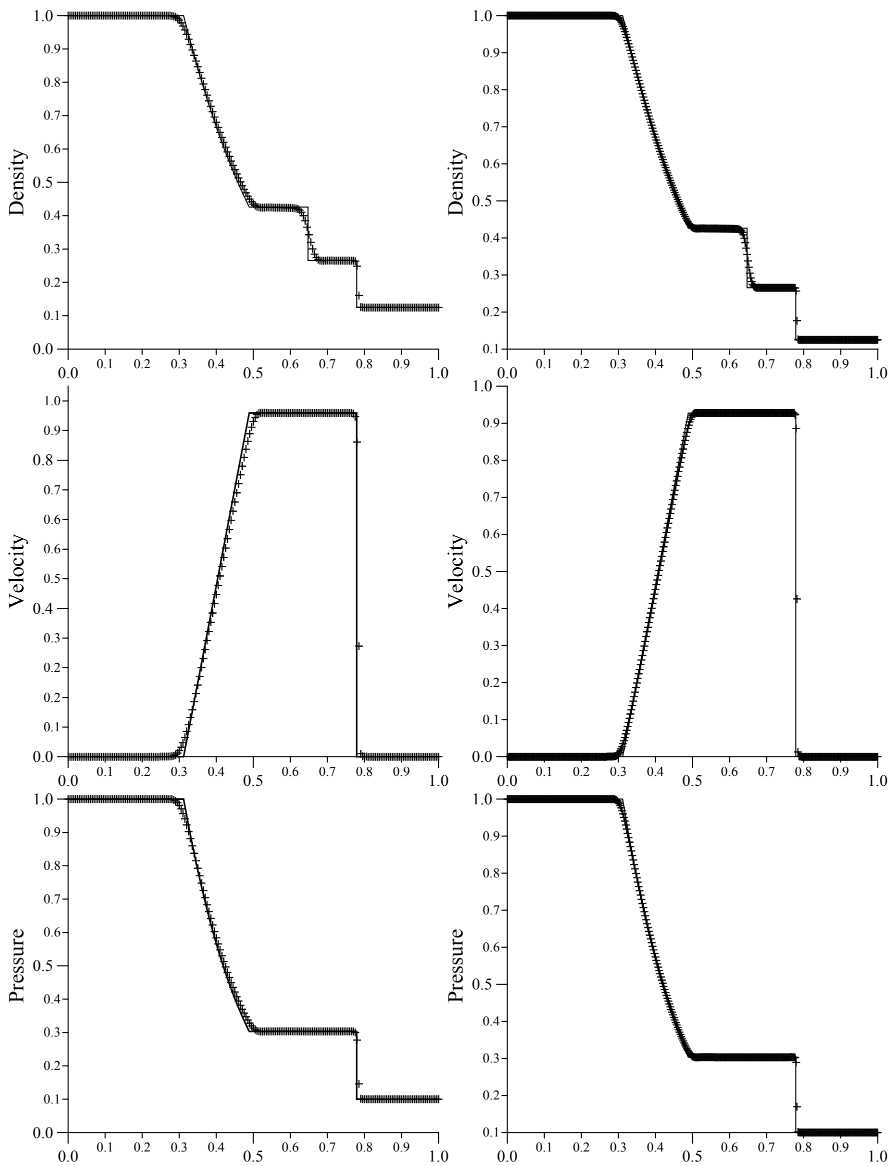

FIG. 4.4. *Sod's problem, time $t = 0.16$. The scheme COS2 under $\aleph = 0.5$, $\varsigma = 1$ versus the analytical solution. The scheme parameters: $\varkappa = 0.8$, CFL number $C_r \approx 0.95$, spatial increment $\Delta x = 0.005$ (left column) and $\varkappa = 0.83$, CFL number $C_r \approx 0.96$, spatial increment $\Delta x = 0.0025$ (right column).*

were done under $\Delta t = 0.1\Delta x$ (i.e., $0.13 \lesssim Cr \lesssim 0.22$). The results of the simulations are depicted in Figure 4.4.

The results depicted in Figure 4.4 (left column) are not worse in comparison to the corresponding third-order central results of [31, p. 418] or to the results obtained by the fourth-order nonoscillatory scheme in [6, p. 472]. Notice that the fourth-order scheme [6, p. 472] gives a slightly better resolution but, in contrast to the scheme COS2, can produce spurious oscillations.

Let us also note that the floating-point operations per one grid node in scheme COS2, (2.12), and in the second-order scheme considered in [31], is approximately the same, but less than, this number in the third- and fourth-order schemes developed in [31, p. 418], [6, p. 472], respectively. Hence, the results depicted in Figure 4.4 (right column) are rather more chipper (in terms of CPU time) than the ones demonstrated in [31, p. 418], [6, p. 472]. Along with loss of computational efficiency, simulations with low CFL number can, in general, lead to excessive numerical smearing. As demonstrated above, scheme COS2 is, in general, free from such drawbacks.

To gain insight into why the NT-scheme [35, eqs. (2.16a)–(2.16b)] can exhibit spurious oscillations, let us find the necessary conditions for the variational S-monotonicity [8, Def. 1.1] of the NT-scheme. As with the scheme COS2 (see section 3), it is assumed that $\mathbf{v}_i^n = \mathbf{C}^n$, $\mathbf{v}_i^{n+0.125} = \mathbf{C}^{n+0.125}$, where $\mathbf{C}^n$, $\mathbf{C}^{n+0.125} = \text{const}$. In view of (2.4), the variational scheme corresponding to the predictor step of the NT-scheme is

$$\delta\mathbf{v}_i^{n+0.125} = \delta\mathbf{v}_i^n - \frac{\Delta t}{4\Delta x}\mathbf{A}_c^n \cdot \mathbb{A}_i^n \cdot \left(\delta\mathbf{v}_{i+1}^n - \delta\mathbf{v}_i^n\right), \quad \mathbf{A}_c^n = \mathbf{A}\left(\mathbf{C}^n\right), \ \mathbf{C}^n = \text{const}. \tag{4.29}$$

By virtue of (4.29), the variational scheme corresponding to the corrector step of the NT-scheme is

$$\delta\mathbf{v}_{i+0.5}^{n+0.25} = 0.5\left(\delta\mathbf{v}_i^n + \delta\mathbf{v}_{i+1}^n\right) - \frac{1}{8}\mathbb{D}^n \cdot \left(\delta\mathbf{v}_{i+1}^n - \delta\mathbf{v}_i^n\right) \tag{4.30}$$

$$+ \frac{(\Delta t)^2}{8(\Delta x)^2}\mathbf{A}_c^{n+0.125} \cdot \mathbf{A}_c^n \cdot \mathbb{D}^n \cdot \left(\delta\mathbf{v}_{i+1}^n - \delta\mathbf{v}_i^n\right) - \frac{\Delta t}{2\Delta x}\mathbf{A}_c^{n+0.125} \cdot \left(\delta\mathbf{v}_{i+1}^n - \delta\mathbf{v}_i^n\right),$$

where $\mathbb{D}^n = \zeta^n\mathbf{I}$ is a scalar matrix with $\zeta^n \in [-4\aleph, 4\aleph]$. Notice that the main difference between (4.30) and (3.16) is in the third terms. Let us rewrite (4.30) to read

$$\delta\mathbf{v}_i^{n+0.125} = 0.5\left(\mathbf{I} + \mathbf{E}^n\right) \cdot \mathbf{v}_i^n + 0.5\left(\mathbf{I} - \mathbf{E}^n\right) \cdot \delta\mathbf{v}_{i+1}^n, \tag{4.31}$$

where

$$\mathbf{E}^n = \frac{\mathbb{D}^n}{4} + \frac{\Delta t}{\Delta x}\mathbf{A}_c^{n+0.125} - \frac{(\Delta t)^2}{4(\Delta x)^2}\mathbf{A}_c^{n+0.125} \cdot \mathbf{A}_c^n \cdot \mathbb{D}^n. \tag{4.32}$$

Assuming that $\mathbf{E}^n$ is a diagonalizable matrix, we find, in view of [8, Thm. 2.11], that the NT-scheme [35, eqs. (2.16a)–(2.16b)] will be variationally S-monotone only if

$$\left(1 + C_r^2\right)\aleph + C_r \leq 1, \quad C_r = \frac{\Delta t \lambda_{\max}}{\Delta x}. \tag{4.33}$$

Obviously, if $\aleph \to 1$ in (4.33), then $C_r \to 0$. Notice that, if $\aleph = 0$, then, in view of (2.5) and (2.4), the scheme will be of the first order. Let us assume, by way of example, that $\aleph = 0.5$. Then, in view of (4.33), the NT-scheme will be free of spurious oscillations only if $C_r \leq \sqrt{2} - 1$.

4.4. 3-D axial symmetric gas dynamics. To illustrate the extension of the scheme COS2 to more space dimensions, we will consider an adiabatic expansion of a gas plume into a vacuum [4]. Taking into account the symmetry of the plume with respect to the axis z, the gas-dynamic equations can be written (for $0 < r$, $z < \infty$) as

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho v_r)}{\partial r} + \frac{\partial (\rho v_z)}{\partial z} = 0, \tag{4.34}$$

$$\frac{\partial}{\partial t}(\rho v_r) + \frac{1}{r}\frac{\partial}{\partial r}\left[r\rho (v_r)^2\right] + \frac{\partial}{\partial z}(\rho v_z v_r) + \frac{\partial p}{\partial r} = 0, \tag{4.35}$$

$$\frac{\partial}{\partial t}(\rho v_z) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_z v_r) + \frac{\partial}{\partial z}\left[\rho (v_z)^2\right] + \frac{\partial p}{\partial z} = 0, \tag{4.36}$$

$$\frac{\partial \rho E}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left[r v_r (\rho E + p)\right] + \frac{\partial}{\partial z}\left[v_z (\rho E + p)\right] = 0. \tag{4.37}$$

$$\rho E = \frac{p}{\gamma - 1} + 0.5\rho v^2, \quad v^2 = v_r^2 + v_z^2, \quad p/\rho^\gamma = \text{const}, \quad \gamma = \text{const}. \tag{4.38}$$

The initial conditions are the following (for details, see [4]):

$$\rho = \rho_0(r,z), \quad v_r = v_z = 0, \quad r, z \geq 0, \quad t = 0. \tag{4.39}$$

We assume that the axis z is a reflection line. It prohibits any normal flux of mass through the boundary $r = 0$, i.e.,

$$v_r = 0, \quad r = 0, \quad z \geq 0. \tag{4.40}$$

Moreover, it is assumed that the pressure (p), density (ρ), and tangential velocity (v_z) are even functions of normal distance to the axis z, while the normal velocity (v_r) is an odd function of r. It is also assumed that the plane $z = 0$ is a reflection surface, i.e., the pressure (p), density (ρ), and tangential velocity (v_r) are even functions of normal distance above the target surface while the normal velocity (v_z) is an odd function of z. The analytic solution to the problem (4.34)–(4.40) can be found in [4].

Notice that every point on the axis $r = 0$ is a singular point for system (4.34)–(4.37). Assuming that all terms in (4.34) are bounded values at a vicinity of $r = 0$, we find that $v_r \to 0$ as $r \to 0$, and hence $(\rho v_r)/r \to \partial(\rho v_r)/\partial r$, $\left(\rho v_r^2\right)/r \to \partial\left(\rho v_r^2\right)/\partial r$, $(\rho v_z v_r)/r \to \partial(\rho v_z v_r)/\partial r$, $v_r(\rho E + p)/r \to \partial v_r(\rho E + p)/\partial r$. Thus, we obtain the following conditions at $r = 0$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(2\rho v_r) + \frac{\partial}{\partial z}(\rho v_z) = 0, \tag{4.41}$$

$$\frac{\partial}{\partial t}(\rho v_r) + \frac{\partial}{\partial r}\left[2\rho (v_r)^2 + p\right] + \frac{\partial}{\partial z}(\rho v_z v_r) = 0, \tag{4.42}$$

$$\frac{\partial}{\partial t}(\rho v_z) + \frac{\partial}{\partial r}(2\rho v_z v_r) + \frac{\partial}{\partial z}\left[\rho (v_z)^2 + p\right] = 0, \tag{4.43}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial r}\left[2 v_r (\rho E + p)\right] + \frac{\partial}{\partial z}\left[v_z (\rho E + p)\right] = 0. \tag{4.44}$$

In the analytic solution [4] of the problem (4.34)–(4.40) the following input data are required: the initial dimensions of the plume, R_0 and Z_0, its mass M_P, and the initial energy E_P. We will use the following values as the reference quantities: $l_* = R_0$, $v_* = \sqrt{(5\gamma - 3)\, E_P / M_P}$, $t_* = l_* / v_*$, $\rho_* = M_P / \left(R_0^2 Z_0\right)$, $p_* = \rho_* v_*^2$.

We rewrite the system of balance equations, (4.34)–(4.37), to read

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial r}\mathbf{f}_r(\mathbf{u}) + \frac{\partial}{\partial z}\mathbf{f}_z(\mathbf{u}) = \mathbf{q}(\mathbf{u}),\ 0 < t \le\ t_{\max},\ \mathbf{u}(r,z,t)|_{t=0} = \mathbf{u}^0(r,z), \tag{4.45}$$

where $\mathbf{u} = \{\rho, \rho v_r, \rho v_z, \rho E\}^T$, $\mathbf{f}_z(\mathbf{u}) = \{\rho v_z, \rho v_z v_r, \rho (v_z)^2 + p, v_z(\rho E + p)\}^T$,

$$\mathbf{f}_r(\mathbf{u}) = \begin{cases} \{\rho v_r, \rho (v_r)^2 + p, \rho v_z v_r, v_r(\rho E + p)\}^T, & r \ne 0, \\ \{2\rho v_r, 2\rho (v_r)^2 + p, 2\rho v_z v_r, 2v_r(\rho E + p)\}, & r = 0, \end{cases} \tag{4.46}$$

$\mathbf{q}(\mathbf{u}) = -\{\rho v_r / r, \rho (v_r)^2 / r, \rho v_z v_r / r, v_r(\rho E + p) / r\}^T$ if $r \ne 0$, and $\mathbf{q}(\mathbf{u}) = 0$ if $r = 0$. After operator splitting, we obtain the following chain of equations corresponding to the problem (4.45):

$$\frac{1}{4}\frac{\partial \mathbf{U}_r}{\partial t} + \frac{\partial}{\partial r}\mathbf{f}_r(\mathbf{U}_r) = 0,\ t_n < t \le t_{n+0.25},\ \mathbf{U}_r(r,z,t_n) = \mathbf{U}^n,\ \mathbf{U}^0 = \mathbf{u}^0(r,z), \tag{4.47}$$

$$\frac{1}{4}\frac{\partial \mathbf{U}_z}{\partial t} + \frac{\partial}{\partial z}\mathbf{f}_z(\mathbf{U}_z) = 0,\ t_{n+0.25} < t \le t_{n+0.5},\ \mathbf{U}_z(r,z,t_{n+0.25}) = \mathbf{U}_r^{n+0.25}, \tag{4.48}$$

$$\frac{1}{2}\frac{\partial \mathbf{U}}{\partial t} = \mathbf{q}(\mathbf{U}), \quad t_{n+0.5} < t \le t_{n+1}, \quad \mathbf{U}(r,z,t_{n+0.5}) = \mathbf{U}_z^{n+0.5}. \tag{4.49}$$

The scheme COS2 will be used for the numerical solution of the PDEs, (4.47) and (4.48). To solve the system (4.49), the second-order implicit Runge–Kutta scheme, (2.18), will be used. The equations (4.47)–(4.49) are integrated up to $t = 0.4$ with $\sigma \equiv Z_0 / R_0 = 0.1$. The CFL number is taken to be $Cr = 1$. The results of simulations as well as the analytical solution are depicted in Figures 4.5 and 4.6.

We observe (Figures 4.5 and 4.6) that the numerical and analytical solutions practically coincide, except for the solutions in the vicinity of the front, namely, those for relatively small values of the dependent variables. Thus, these results reinforce LeVeque's statement [30] that the splitting error is often no worse than the one introduced by the numerical method.

4.5. Concluding remarks. By virtue of the approach developed in [8], the necessary condition (3.14) for GOS-monotonicity of the scheme COS2, (2.12), was easily found. However, this condition, being sufficient for the stability of the scheme COS2, is, in general, not sufficient for this scheme to be free of spurious oscillations. In such a case we have to consider the parameters $\varkappa$, $\aleph$ in (3.14) as experimental. This leads to the reasonable, but laborious, task of estimating $\varkappa$ together with $\aleph$. Although this problem is beyond the scope of our paper, nevertheless we make several remarks on it in this section.

Considering the scheme COS2, and, hence, the case $\varsigma = 1$ in (3.14), we note that (3.14) will be fulfilled under $\varkappa = \aleph = C_r = 1$, but not under $\varkappa = \aleph = 1$, $C_r < 1$. If $C_r < 1$, then (under $\varkappa = 1$) the monotonicity parameter $\aleph \le 1 / (1 + C_r)$. With the proviso that $0.5 \le \aleph \le 1$, let us find a condition such that (3.14) will be fulfilled for all $C_r \in [0, 1]$, and in addition such that the parameter $\aleph$ does not depend on C_r. Hence, in the case $C_r \le \sqrt{\varkappa}$ in (3.14), the discriminant of the quadratic polynomial in C_r must be negative. We find

$$0 \le \varkappa < \frac{4\aleph - 1}{4\aleph^2}, \quad 0.5 \le \aleph \le 1. \tag{4.50}$$

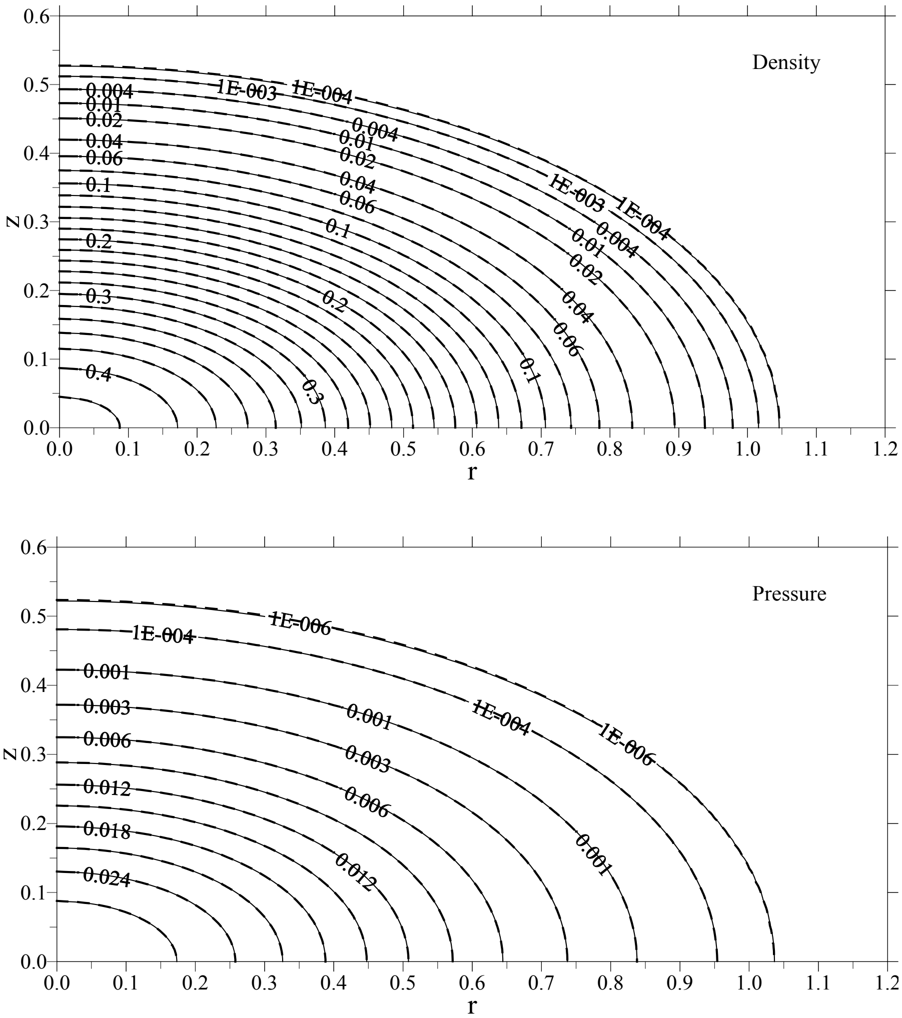

FIG. 4.5. *Expansion of a gas plume into a vacuum; density and pressure distribution. The scheme COS2 unified with the second-order implicit Runge–Kutta scheme versus the analytical solution.* $\sigma \equiv Z_0/R_0 = 0.1$, *time* $t = 0.4$, *CFL number* $C_r = 1$, *monotonicity parameter* $\aleph = \varkappa = \varsigma = 1$. *Spatial increments:* $\Delta r = 2.5 \times 10^{-3}$, $\Delta z = 2.5 \times 10^{-4}$ *if* $0 < t \leq 0.05$; $\Delta r = 2.5 \times 10^{-3}$, $\Delta z = 5 \times 10^{-4}$ *if* $0.05 < t \leq 0.1$; $\Delta r = 5 \times 10^{-3}$, $\Delta z = 10^{-3}$ *if* $0.1 < t \leq 0.2$; $\Delta r = 0.01$, $\Delta z = 2 \times 10^{-3}$ *if* $0.2 < t \leq 0.4$. *Dashed lines: numerical solution. Solid lines: analytical solution.*

In view of (4.50), $\varkappa$ is a monotonically decreasing function of $\aleph$, and this function ranges from 1 to 0.75. Hence, increasing $\aleph$, to improve a quality of the monotone piecewise cubic interpolant [8, sect. 3], leads to a lower order of accuracy for the scheme. Excluding the unique condition, $\varkappa = \aleph = C_r = 1$, which is mainly used for smooth solutions, it is usually assumed that $1/3 \leq \aleph \leq 0.5$ and $0.75 \leq \varkappa \leq 1$; see sections 4.1 and 4.3. Notice that in section 4.3 it is assumed that $\aleph = 0.5$; however, $\varkappa = 0.8, 0.83$ in order to suppress spurious oscillations.

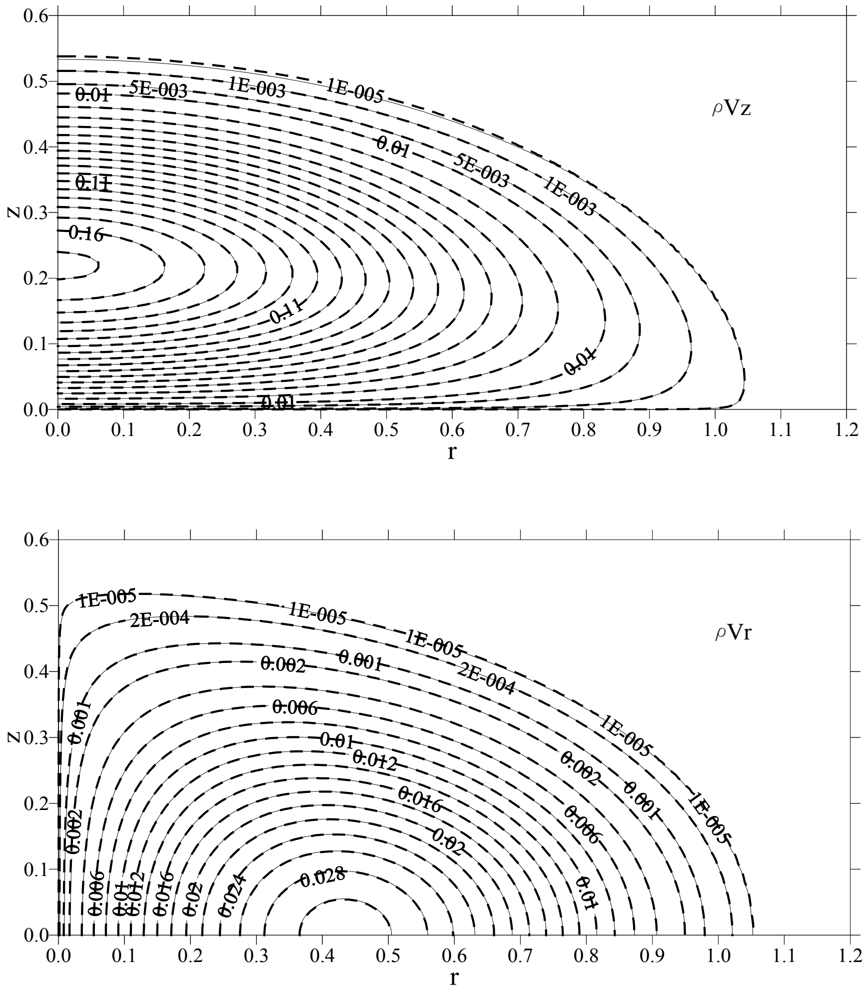

FIG. 4.6. *Expansion of a gas plume into a vacuum; momenta (ρV_z and ρV_r) distribution. The scheme COS*2 *unified with the second-order implicit Runge–Kutta scheme versus the analytical solution.* $\sigma \equiv Z_0 / R_0 = 0.1$*, time* $t = 0.4$*, CFL number* $C_r = 1$*, monotonicity parameter* $\aleph = \varkappa = \varsigma = 1$*. Spatial increments:* $\Delta r = 2.5 \times 10^{-3}$, $\Delta z = 2.5 \times 10^{-4}$ *if* $0 < t \leq 0.05$*;* $\Delta r = 2.5 \times 10^{-3}$, $\Delta z = 5 \times 10^{-4}$ *if* $0.05 < t \leq 0.1$*;* $\Delta r = 5 \times 10^{-3}$, $\Delta z = 10^{-3}$ *if* $0.1 < t \leq 0.2$*;* $\Delta r = 0.01$, $\Delta z = 2 \times 10^{-3}$ *if* $0.2 < t \leq 0.4$*. Dashed lines: numerical solution. Solid lines: analytical solution.*

To gain insight into why the scheme COS2 can exhibit spurious oscillations, and why the spurious oscillations can be suppressed by decreasing $\varkappa$, let us consider the first differential approximation [13, p. 45], [45, p. 376] (see also "modified equations" in [13], [30], [33]) of the scheme COS2:

$$\frac{1}{\kappa}\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \frac{\Delta t}{4}\frac{\partial}{\partial x}\left(\mathbf{B}\cdot\frac{\partial \mathbf{u}}{\partial x}\right) - \frac{\Delta t^2}{16\kappa}\frac{\partial^2}{\partial t \partial x}\left(\mathbf{B}\cdot\frac{\partial \mathbf{u}}{\partial x}\right) + \frac{\Delta t^2}{24\kappa^3}\frac{\partial}{\partial t}\left[\left(\frac{\kappa \Delta x}{\Delta t}\right)^2 \frac{\partial^2 \mathbf{u}}{\partial x^2} - \frac{\partial^2 \mathbf{u}}{\partial t^2}\right], \quad \kappa = \text{const} \in \mathbb{N}, \tag{4.51}$$

where κ depends on "depth of splitting," e.g., $\kappa = 4$ in (4.47), and $\kappa = 2$ in (2.1),

$$\mathbf{B} = (1-\varkappa)\,\frac{\Delta x^2}{\Delta t^2}\mathbf{I} - (1-\varsigma)\,\mathbf{A}^2. \tag{4.52}$$

If $\varkappa = \varsigma = 1$, then the scheme COS2 will be oscillatory, since the first two terms in the right-hand side of (4.51) will be equal to zero, and, hence, the third-order PDE, (4.51), without dissipative terms will be its first differential approximation. The first term in the right-hand side of (4.51) will be dissipative if $\varkappa < 1$ and $\varsigma = 1$ in (4.52). It is apparent that $1-\varkappa$ in (4.52) should be reasonably large to suppress spurious oscillations, while it should be as close to zero as possible to lose less in the order of accuracy of the scheme. Since the scheme is convergent, we can conclude that the smaller Δt is, the smaller $1-\varkappa$ can be. It can often be assumed that $1-\varkappa = a_\varkappa \Delta x$, where $a_\varkappa$ is an experimental parameter, and in its turn can be a function of Δx. In such a case, considering that Δt and Δx are related in a fixed manner, we find, by virtue of (4.51), (4.52), that the scheme COS2 will be of the second order, in spite of the fact that $\varkappa < 1$.

5. Appendix. For the sake of convenience, we will demonstrate (following [44, pp. 390–392]) that the uniform stability of an explicit two-layer scheme with respect to the initial data implies the stability of the scheme with respect to the right-hand side and, hence, the stability of the scheme. We will consider the following explicit difference scheme:

$$\mathbf{y}_{n+1} - \mathbf{S}_n \cdot \mathbf{y}_n = \Delta t\, \mathbf{f}_n, \quad n, n+1 \in \omega_t, \quad \mathbf{y}_0, \mathbf{f}_n \in \mathbb{Y}, \quad \mathbf{S}_n : \mathbb{Y} \to \mathbb{Y}, \tag{5.1}$$

where $\omega_t \equiv \{0, 1, \ldots, N\}$, $\Delta t = t_{\max}/N$ denotes the time increment, $t_{\max}$ denotes some finite time over which we wish to compute, $\mathbb{Y}$ denotes the Banach space, $\mathbf{S}_n$ is the transition operator, and n denotes the time level, $t_n\,(= n\Delta t)$.

Scheme (5.1) is said to be stable [44, p. 388] if there exist positive constants C_1 and C_2 such that a solution to the scheme satisfies the estimate

$$\|\mathbf{y}_{n+1}\| \le C_1 \|\mathbf{y}_0\| + C_2 \max_{0\le k\le n} \|\mathbf{f}_k\|, \quad n, n+1 \in \omega_t. \tag{5.2}$$

Scheme (5.1) is said to be stable with respect to the initial data if a solution to (5.1), under $\mathbf{f}_n = 0$, $n = 0, 1, \ldots, N-1$, satisfies the estimate

$$\|\mathbf{y}_{n+1}\| \le C_1 \|\mathbf{y}_0\|, \quad n, n+1 \in \omega_t. \tag{5.3}$$

Scheme (5.1) is said to be stable with respect to the right-hand side if a solution to (5.1), under $\mathbf{y}_0 = 0$, satisfies the estimate

$$\|\mathbf{y}_{n+1}\| \le C_2 \max_{0\le k\le n} \|\mathbf{f}_k\|, \quad n, n+1 \in \omega_t. \tag{5.4}$$

It is evident that a stable scheme will be stable with respect to the right-hand side as well as with respect to the initial data.

Scheme (5.1) is said to be uniformly stable with respect to the initial data [44, p. 392] if the Cauchy problem

$$\mathbf{y}_{n+1} = \mathbf{S}_n \cdot \mathbf{y}_n, \quad n = k, k+1, \ldots, N-1, \quad \mathbf{S}_n : \mathbb{Y} \to \mathbb{Y}, \tag{5.5}$$

is stable for any $k = 0, 1, \ldots, N-1$. That is

$$\|\mathbf{y}_n\| \le C_1 \|\mathbf{y}_k\| \quad \forall k:\ 0 \le k < n \le N, \quad C_1 = \text{const}. \tag{5.6}$$

By virtue of the recurrence relation (5.1), we find [44, p. 390] that

$$\mathbf{y}_{n+1} = \mathbf{T}_{n+1,0} \cdot \mathbf{y}_0 + \sum_{k=0}^{n} \Delta t \, \mathbf{T}_{n+1,k+1} \cdot \mathbf{f}_k, \tag{5.7}$$

where

$$\mathbf{T}_{n+1,j} = \mathbf{S}_n \cdot \mathbf{S}_{n-1} \cdot \ldots \cdot \mathbf{S}_{j+1} \cdot \mathbf{S}_j, \; j = 0, 1, \ldots, n, \quad \mathbf{T}_{n+1,n+1} = \mathbf{I}. \tag{5.8}$$

The triangular inequality yields

$$\|\mathbf{y}_{n+1}\| \le \|\mathbf{T}_{n+1,0}\| \, \|\mathbf{y}_0\| + \sum_{k=0}^{n} \Delta t \, \|\mathbf{T}_{n+1,k+1}\| \, \|\mathbf{f}_k\| \,. \tag{5.9}$$

Notice that scheme (5.1) will be stable if for any $0 \le k \le n \le N-1$

$$\|\mathbf{T}_{n,k}\| \le C_1. \tag{5.10}$$

Actually, by virtue of (5.9)–(5.10) we obtain (5.2), where $C_2 = t_{\max} C_1$.

If the condition of uniform stability for (5.1) is satisfied, i.e., (5.5) is stable, then the estimate (5.10) holds true for the operator $\mathbf{T}_{n,k}$ and, hence, scheme (5.1) will be stable.

Let us note that scheme (5.1) will be uniformly stable if the following estimate for the norm of its transition operator $\mathbf{S}_k$ is valid for all $k = 0, 1, \ldots, N-1$:

$$\|\mathbf{S}_k\| \le 1 + \alpha \Delta t, \quad \alpha = \text{const}, \quad \alpha \ge 0. \tag{5.11}$$

Actually, the estimate (5.11) implies the estimate (5.10). The proof is obvious (see, e.g., [44, p. 392]).

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