Error Analysis of Generalized LxF Schemes for Linear Advection Equation with Damping

Yun Wu and Dong Qu

College of Science, Jiujiang University
Jiujiang, China, 332005
wuyun19821231@sohu.com

Abstract. Local oscillations existing in the generalized Lax-Friedrichs (LxF) schemes are proposed and analyzed on computing of the linear advection equation with damping. For the discretization of some special initial data under stable conditions, local oscillations in numerical solutions are observed. Three propositions are also raised about how to control those oscillations via some numerical examples. In order to further explain this, discrete Fourier analysis and the modified equation analysis is used to distinguish the dissipative and dispersive effects of numerical schemes for low frequency and high frequency modes, respectively.

Keywords: finite difference schemes, oscillations, discrete Fourier analysis, modified equation analysis.

1 Introduction

Oscillation is an unsurprising phenomenon in numerical computations, but it is a very surprising phenomenon for computing of hyperbolic conservation laws using the Lax-Friedrichs(LxF) scheme. In this thesis, we consider the linear advection equation with damping.

$$u_t + au_x = -\alpha u, \quad x \in R, \alpha > 0. \tag{1}$$

In ([6],[12]), to compute the numerical solution of the hyperbolic conservation laws

$$u_t + f(u)_x = 0, \quad x \in R, t > 0, \tag{2}$$

where $u = (u_1, \cdots, u_m)^T$, and $f(u) = (f_1, \cdots, f_m)^T$, we consider the generalized LxF scheme of the viscosity form

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}[f(u_{j+1}^n) - f(u_{j-1}^n)] + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{3}$$

where the mesh ratio $\nu = \tau/h$ is assumed to be a constant, τ and h are step sizes in time and space, respectively, u_j^n denotes an approximation of $u(jh, n\tau)$, the term $q \in (0, 1]$ is the coefficient of numerical viscosity. When $q = 1$, it is the classical LxF scheme.

C. Liu, L. Wang, and A. Yang (Eds.): ICICA 2012, Part I, CCIS 307, pp. 313–320, 2012.

With the flux function $f = au$, 1 is the linear advection equation accordingly

$$u_t + au_x = 0, \quad x \in R, t > 0, \tag{4}$$

and the scheme 3 turns into the generalized Lax-Friedrichs scheme of 4

$$u_j^{n+1} = u_j^n - \frac{a\nu}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n). \tag{5}$$

By adding a damping term $-\alpha u(\alpha$ is a positive constant) to the right of 4, we obtain the linear advection equation with damping 1. We can take account of the discretization of the damping term in the form

$$u_j^{n+1} = u_j^n - \frac{a\nu}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) - \alpha\tau u_j^n \tag{6}$$

2 Analysis of Numerical Dissipation and Phase Error

In this section we attempt to analyze the numerical dissipation and phase error mechanisms of 6. We apply discrete Fourier analysis and the method of modified equation analysis. Both of the methods give complimentary results, which are consistent. We use the method of discrete Fourier analysis (as in [1,2]) to discuss the dissipation and phase error mechanisms of the generalized LxF scheme 6. Denote a Fourier mode using the scaled wave number $\xi = 2\pi kh$ by $e^{i\xi}$. Then using it as initial data for a linear finite difference scheme results after n times steps in the solution

$$u_k^n = \lambda_k^n e^{i\xi} = (\lambda(k))^n e^{i\xi}, \quad i^2 = -1, \tag{7}$$

where λ_k^n is the amplitude. The modulus of the ratio

$$\lambda(k) = \lambda_k^{n+1}/\lambda_k^n$$

is the amplitude of the mode for one time step. For the scheme 6 we have with $\nu = \tau/h$ in particular.

$$\lambda(k) = 1 - \alpha\tau - q(1 - \cos\xi) - ia\nu\sin\xi \tag{8}$$

and

$$|\lambda(k)|^2 \tag{9}$$

$$\begin{aligned} &= [1 - \alpha\tau - q(1-\cos\xi)]^2 + (\nu a)^2\sin^2\xi \\ &= (1-\alpha\tau)^2 + 4[a^2\nu^2 - q(1-\alpha\tau)]\sin^2(\xi/2) \\ &\quad +4(q^2 - a^2\nu^2)\sin^4(\xi/2) \end{aligned} \tag{10}$$

Under the condition $0 < \frac{a^2\nu^2}{1-\alpha\tau} \le q \le 1 - \alpha\tau$, we have from 8 the estimate

$$|\lambda(k)|^2 \tag{11}$$

$$\begin{aligned} &= (1-\alpha\tau)^2 + 4[a^2\nu^2 - q(1-\alpha\tau)](\sin^2(\xi/2) \\ &\quad - \sin^4(\xi/2)) + 4q[q - (1-\alpha\tau)]\sin^4(\xi/2) \\ &\le 1 \end{aligned} \tag{12}$$

Thus these condition imply that the schemes are linearly stable. Conversely, assume that $|\lambda(k)|^2 \leq 1$ then from 8 we have $[a^2\nu^2 - q(1-\alpha\tau)] + (q^2 - a^2\nu^2)\sin^2(\xi/2) \leq 0$. Now for $\xi = 0$ we obtain $\frac{a^2\nu^2}{1-\alpha\tau} \leq q$. For $\xi = \pi$ this gives $q[q-(1-\alpha\tau)] \leq 0$ or since we have $0 < \frac{a^2\nu^2}{1-\alpha\tau} \leq q$ we obtain $q \leq 1-\alpha\tau$. Therefore, GLxF scheme is conditionally stable and $0 < \frac{a^2\nu^2}{1-\alpha\tau} \leq q \leq 1-\alpha\tau$ is necessary and sufficient for stability.

The exact solution of the Fourier mode $e^{i\xi}$ for $x = h$ after one time step τ is $e^{i(\xi - 2\pi ak\tau)} = e^{-i2\pi ak\tau}e^{i\xi} = \lambda_{exact}(k)e^{i\xi}$. We see from 8 that the *amplification error*, i.e. the error in amplitude modulus is of order $O(1)$. Further, comparing the exponents of $\lambda(k)$ and $\lambda_{exact}(k)$ there is a *phase error* $\arg\lambda(k) - (-2\pi ak\tau)$. The *relative phase error* is then defined as

$$E_p(k) := \frac{\arg\lambda(k)}{-2\pi ak\tau} - 1 = -\frac{\arg\lambda(k)}{\nu a\xi} - 1$$

A mode is a low frequency mode if $\xi \approx 0$ and a high frequency mode if $\xi \approx \pi$.

We first look at the low frequency modes

$$(U^s)_j^n := \lambda_k^n e^{i\xi j}, \qquad \xi \approx 0$$

Then substituting $(U^s)_j^n$ into 6 yields,

$$\lambda(k) = 1 - \alpha\tau - q(1-\cos\xi) - ia\nu\sin\xi \tag{13}$$

and

$$\begin{aligned} |\lambda(k)|^2 &= (1-\alpha\tau - q(1-\cos\xi))^2 + (\nu a)^2\sin^2\xi \\ &= (1-\alpha\tau)^2 - (1-\cos\xi)[2q(1-\alpha\tau) - \\ &\qquad q^2(1-\cos\xi) - a^2\nu^2(1+\cos\xi)] \end{aligned} \tag{14}$$

We express 14 by using $\cos\xi = 1 - \frac{1}{2}\xi^2 + o(\xi^4) + \cdots$

$$|\lambda|^2 = (1-\alpha\tau)^2 + [a^2\nu^2 - q(1-\alpha\tau)]\xi^2 + \frac{1}{4}(q^2 - a^2\nu^2)\xi^2 + \cdots \tag{15}$$

So, we can see that the dissipation error is of order $O(1)$. From 8 we obtain, as $\frac{a^2\nu^2}{1-\alpha\tau} \leq q \leq a\nu$

$$\frac{d(|\lambda(k)|^2)}{dq} = 2(\cos\xi - 1)[(1-\alpha\tau) - q(1-\cos\xi)] < 0 \tag{16}$$

for fixed $\xi \in [0, \pi/2]$. This implies that when $\frac{a^2\nu^2}{1-\alpha\tau} \leq q \leq a\nu$, the dissipation becomes stronger as q larger, and then oscillations become weaker.

Also, we use the method of modified equation analysis to further investigate the mechanisms of dissipation and phase error of (6). This method of modified equation analysis was introduced in [2,6]. We attempt to analyze how the dissipation offsets the oscillations caused by the large phase error.

As in [2], we split the solution $(U)_j^n$ into a smooth part $(U^s)_j^n$ and an oscillatory term $(U^h)_j^n$.

The smooth solution $(U^s)_n^j$ satisfies (6). Here the notation D_{+t} is a forward difference operator in time.

$$\Delta_{+t} = U_j^{n+1} - U_j^n = e^{\tau \partial t} \tag{17}$$

As in [2,Page 170], define $D_{+t} := \Delta_{+t}/\tau$, then we obtain

$$\begin{aligned} \partial_t &= \tfrac{\ln(1+\Delta_{+t})}{\tau} \\ &= D_{+t} - \tfrac{1}{2}\tau D_{+t}^2 + \tfrac{1}{3}\tau^2 D_{+t}^3 - \tfrac{1}{4}\tau^3 D_{+t}^4 + \cdots \end{aligned} \tag{18}$$

In accordance with 6, taking the standard Taylor expansion yields,

$$D_{+t} = \Delta_{+t}/\tau = -\alpha - a\partial_x + \frac{qh^2}{2\tau}\partial_x^2 - \frac{ah^2}{6}\partial_x^3 + \cdots \tag{19}$$

Substituting (19) into (18) yields,

$$\begin{aligned} \partial_t = & (-\alpha - a\partial_x + \tfrac{qh^2}{2\tau}\partial_x^2 - \tfrac{ah^2}{6}\partial_x^3) \\ & -\tfrac{\tau}{2}(-\alpha - a\partial_x + \tfrac{qh^2}{2\tau}\partial_x^2 - \tfrac{ah^2}{6}\partial_x^3)^2 \\ & +\tfrac{\tau^2}{3}(-\alpha - a\partial_x + \tfrac{qh^2}{2\tau}\partial_x^2 - \tfrac{ah^2}{6}\partial_x^3)^3 + \cdots \\ = & \tfrac{\ln|1-\alpha\tau|}{\tau} - \tfrac{a}{1-\alpha\tau}\partial_x + \tfrac{qh^2(1-\alpha\tau)-a^2\tau}{2\tau(1-\alpha\tau)^2}\partial_x^2 \\ & + \tfrac{ah^2[(1-\alpha\tau)(3q-1+\alpha\tau)-2a^2\nu^2]}{6(1-\alpha\tau)^3}\partial_x^3 + \cdots \end{aligned} \tag{20}$$

Here the notation $\widetilde{U}^s$ is used to express the solution of exact differential equation corresponding to (1.6), we derive the modified equation about the smooth part:

$$\begin{aligned} & \partial_t \widetilde{U}^s + \frac{a}{1-\alpha\tau}\partial_x \widetilde{U}^s \\ = & \frac{\ln|1-\alpha\tau|}{\tau}\widetilde{U}^s + \frac{h^2}{2\tau}\frac{q(1-\alpha\tau)-a^2\nu^2}{(1-\alpha\tau)^2}\partial_x^2 \widetilde{U}^s \\ & + \frac{ah^2}{6}\frac{(1-\alpha\tau)(3q-1+\alpha\tau)-2a^2\nu^2}{(1-\alpha\tau)^3}\partial_x^3 \widetilde{U}^s + \cdots \end{aligned} \tag{21}$$

In the modified equation, the second order term $\frac{h^2}{2\tau}\frac{q(1-\alpha\tau)-a^2\nu^2}{(1-\alpha\tau)^2}\partial_x^2 \widetilde{U}^s$ is the numerical viscosity. When $0 < \frac{a^2\nu^2}{1-\alpha\tau} \leq q \leq 1-\alpha\tau$, we have $q(1-\alpha\tau) - a^2\nu^2 \geq 0$, thus the numerical viscosity becomes stronger as q is larger. The conclusion is the same as Fourier analysis for low frequency modes.

Now we use the modified equation analysis to study the oscillatory part.The oscillatory solution $(U^h)_j^n$ is written as

$$(U^h)_j^n = (-1)^{j+n}(U^o)_j^n,$$

where $(U^o)_j^n$ is viewed as the perturbation amplitude of the chequerboard mode. Then the oscillatory term $(U^o)_j^n$ satisfies ,in accordance with (1.6),

$$\begin{aligned}&(-1)^{j+n+1}(U^o)_j^{n+1}\\&=(-1)^{j+n}(U^o)_j^n-\frac{a\nu}{2}[(-1)^{j+n+1}(U^o)_{j+1}^n\\&-(-1)^{j+n-1}(U^o)_{j-1}^n]+\frac{q}{2}[(-1)^{j+n+1}(U^o)_{j+1}^n\\&-2(-1)^{j+n}(U^o)_j^n+(-1)^{j+n-1}(U^o)_{j-1}^n]\\&-\alpha\tau(-1)^{j+n}(U^o)_j^n\end{aligned}\tag{22}$$

that is ,

$$\begin{aligned}&(U^o)_j^{n+1}-(U^o)_j^n\\&=(q+\alpha\tau-2)(U^o)_j^n-\tfrac{a\nu}{2}[(U^o)_{j+1}^n-(U^o)_{j-1}^n]\\&+\tfrac{q}{2}[(U^o)_{j+1}^n+(U^o)_{j-1}^n]\end{aligned}\tag{23}$$

We use the notation $\widetilde{U}^o(jh,n\tau)$ to express $(U^o)_j^n$ inserted into (23) and apply the standard approach. That is , taking the standard Taylor expansion yields

$$\Delta_{+t}\widetilde{U}^o=(2q+\alpha\tau-2)\widetilde{U}^o-a\tau\partial_x\widetilde{U}^o+\frac{qh^2}{2}\partial_x^2\widetilde{U}^o-\frac{a\nu h^3}{6}\partial_x^3\widetilde{U}^o+\cdots\tag{24}$$

where $D_{+t}=\frac{\Delta_{+t}}{\tau}=\frac{e^{\tau\partial_t}-1}{\tau}$. Next we write (24) as

$$(e^{\tau\partial_t}-1)\widetilde{U}^o=\beta\widetilde{U}^o-a\tau\partial_x\widetilde{U}^o+\frac{qh^2}{2}\partial_x^2\widetilde{U}^o-\frac{a\tau h^2}{6}\partial_x^3\widetilde{U}^o+\cdots$$

where $\beta=2q+\alpha\tau-2$. Note the following basic facts, namely the formal operator expansion

$$\tau\partial_t=\ln((e^{\tau\partial_t}-1)+1)=\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(e^{\tau\partial_t}-1)^m}{m},$$

and the well known power series

$$\frac{1}{(1+z)^2}=\sum_{m=0}^{\infty}(-1)^m(m+1)z^m,\qquad where\quad z\in(-1,1),$$

$$\frac{1}{(1+z)^3}=\sum_{m=0}^{\infty}(-1)^m\frac{(m+1)(m+2)}{2}z^m,\qquad where\quad z\in(-1,1),$$

Let $C_m^l=\frac{m!}{(m-l)!l!}$ denote the binomial coefficients for $l\leq m$. We obtain ,by ignoring terms of orders higher than three, that

$$
\begin{aligned}
&\tau\partial_t\widetilde{U}^o\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}[(e^{\tau\partial_t}-1)\widetilde{U}^o]^m\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}(\beta\widetilde{U}^o-a\tau\partial_x\widetilde{U}^o\\
&+\tfrac{qh^2}{2}\partial_x^2\widetilde{U}^o-\tfrac{a\tau h^2}{6}\partial_x^3\widetilde{U}^o+\cdots)^m\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\beta^m\widetilde{U}^o+\\
&\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}C_m^1\beta^{m-1}(-a\tau\partial_x)\widetilde{U}^o\\
&+\Big\{\sum_{m=2}^{\infty}\frac{(-1)^{m+1}}{m}C_m^2\beta^{m-2}a^2\tau^2\partial_x^2\\
&+\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}C_m^1\beta^{m-1}\tfrac{qh^2}{2}\partial_x^2\Big\}\widetilde{U}^o\\
&+\Big\{\sum_{m=3}^{\infty}\frac{(-1)^{m+1}}{m}C_m^3\beta^{m-3}(-a\tau\partial_x)^3\\
&+\sum_{m=2}^{\infty}\frac{(-1)^{m+1}}{m}C_m^1C_{m-1}^1\beta^{m-2}(-a\tau\partial_x)\tfrac{qh^2}{2}\partial_x^2\\
&+\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}C_m^1\beta^{m-1}\left(-\tfrac{a\tau h^2}{6}\partial_x^3\right)\Big\}\widetilde{U}^o+\cdots\\
&=\ln|\beta+1|\widetilde{U}^o-\tfrac{a\tau}{1+\beta}\partial_x\widetilde{U}^o\\
&+\Big\{-\tfrac{a^2\tau^2}{2(1+\beta)^2}+\tfrac{qh^2}{2(1+\beta)}\Big\}\partial_x^2\widetilde{U}^o\\
&+\Big\{\tfrac{-a^3\tau^3}{3(1+\beta)^3}+\tfrac{a\tau}{(1+\beta)^2}\tfrac{qh^2}{2}-\tfrac{1}{1+\beta}\\
&\tfrac{a\tau h^2}{6}\Big\}\partial_x^3\widetilde{U}^o+\cdots\\
&=\ln|2q+\alpha\tau-1|\widetilde{U}^o-\tfrac{a\tau}{2q+\alpha\tau-1}\partial_x\widetilde{U}^o\\
&+\tfrac{1}{2}\tfrac{[qh^2(2q+\alpha\tau-1)-a^2\tau^2]}{(2q+\alpha\tau-1)^2}\partial_x^2\widetilde{U}^o\\
&+\tfrac{a\tau}{6}\tfrac{[h^2(2q+\alpha\tau-1)(q+1-\alpha\tau)-2a^2\tau^2]}{(2q+\alpha\tau-1)^3}\partial_x^3\widetilde{U}^o+\cdots
\end{aligned}
\tag{25}
$$

Thus we derive the modified equation for the oscillatory part:

$$
\begin{aligned}
&\partial_t\widetilde{U}^o+\tfrac{a}{2q+\alpha\tau-1}\partial_x\widetilde{U}^o\\
&=\tfrac{\ln|2q+\alpha\tau-1|}{\tau}\widetilde{U}^o\\
&+\tfrac{h^2}{2\tau}\tfrac{[q(2q+\alpha\tau-1)-a^2\nu^2]}{(2q+\alpha\tau-1)^2}\partial_x^2\widetilde{U}^o\\
&+\tfrac{ah^2}{6}\tfrac{[(2q+\alpha\tau-1)(q+1-\alpha\tau)-2a^2\nu^2]}{(2q+\alpha\tau-1)^3}\partial_x^3\widetilde{U}^o+\cdots
\end{aligned}
\tag{26}
$$

where $(1-\alpha\tau)/2<q<1-\alpha\tau$,

We call the zero term in (26) $\frac{\ln|2q+\alpha\tau-1|}{\tau}\widetilde{U}^o$ a numerical damping term and the second order term $\frac{h^2}{2\tau}\frac{[q(2q+\alpha\tau-1)-a^2\nu^2]}{(2q+\alpha\tau-1)^2}\partial_x^2\widetilde{U}^o$ a numerical viscosity. They play distinct dissipation roles in controlling the amplitude of high frequency modes. But this dissipation is not enough to control numerical oscillations caused by the relative phase errors of high frequency modes, and only the numerical damping can suppress the oscillation.

With the stability condition of the scheme 6, we obtain $(1-\alpha\tau)/2 < q < 1-\alpha\tau$, several conclusions are in order.
(1)As $(1-\alpha\tau)/2 < q < 1-\alpha\tau$, there is a strong damping term $\frac{\ln|2q+\alpha\tau-1|}{\tau}$ $(ln(2q+\alpha\tau-1) < 0)$ to suppress the oscillation. The numerical damping becomes stronger as q decrease.
(2)In particular , if $q = \frac{1-\alpha\tau}{2}$, the oscillation is damped out immediately, by noting that

$$\lim_{q\to(1-\alpha\tau)/2+0} \frac{\ln|2q+\alpha\tau-1|}{\tau} = -\infty$$

So the numerical damping becomes infinite for $q = \frac{1-\alpha\tau}{2}$.

3 Discussion and Conclusion

In the present paper, we are discussing the local oscillations in the particular generalized LxF scheme. We have individually analyzed the resolution of the low and high frequency modes $u_j^n = \lambda_k^n e^{ij\xi}$, $\xi = 2\pi kh$ in numerical solutions. Our approach is the discrete Fourier analysis and the modified equation analysis, which are applied to investigating the numerical dissipative and dispersive mechanisms as well as relative phase errors.

1. Relative phase error. For the low frequency modes, the error is of order $O(1)$, while for high frequency modes the error is of order $O(1)$ after each time step, which is generally independent of the parameter q.

2. Numerical dissipation. For the low frequency modes, the dissipation is usually of order $O(1)$ for the scheme (1.6), which closely depends on the parameter q. For high frequency modes, the scheme usually has the numerical damping of order $O(1)$ that becomes stronger as q is closer to $\frac{1-\alpha\tau}{2}$, unless it vanishes for the limit case $(q = 1 - \frac{\alpha\tau}{2})$.

Thus we obtain that the relative phase errors should be at least offset by the numerical dissipation of the same order. Otherwise the oscillation could be caused.

We also get the following conclusions.

1. The GLxF scheme 6 is conditionally stable and $0 < \frac{a^2\nu^2}{1-\alpha\tau} \le q \le 1-\alpha\tau$ is necessary and sufficient for stability.

2. Under the stable condition, the oscillation is connected with these factors:If $q > \frac{1-\alpha\tau}{2}$, the oscillation becomes weaker as q decrease. If $q > \frac{1-\alpha\tau}{2}$, the oscillation becomes weaker as q increase; The oscillation becomes weaker as α decrease; The oscillation becomes weaker as h decrease; If the initial data can be discretized as square signal, which be discretized with an odd number of grid points, the chequerboard mode (i.e. the oscillation)is present. In contrast, the chequerboard mode (i.e. the oscillation)is suppressed.

3. Once adding the damping into linear advection equation, it is clear from the figures that the damping has resulted in a slight reduction of the modes' height; We also can find even large damping, the oscillation becomes weaker as time goes by, that is to say the chequerboard mode decay.

4. When $q = \frac{1-\alpha\tau}{2}$, the oscillation vanish. When $q = 1 - \frac{\alpha\tau}{2}$, the oscillation is the strongest.

Acknowledgment. The authors would like to thank Prof. Jiequan Li for his helpful remarks and suggestions.

References

1. Li, J., Yang, Z., Zheng, Y.: Characteristic decompositions and interactions of rarefaction waves of 2-D Euler Equations. Journal of Differential Equations 250, 782–798 (2011)
2. Li, J., Li, Q., Xu, K.: Comparison of the Generalized Riemann Solver and the Gas-Kinetic scheme for Compressible Inviscid Flow Simulations. Journal of Computational Physics 230, 5080–5099 (2011)
3. Dou, L., Dou, J.: The Grid: Time-domain analysis of lossy multiconductor transmission lines based on the LaxCWendroff technique. Analog Integrated Circuits and Signal Processing 68, 85–92 (2011)
4. Li, J., Liu, T., Sun, Z.: Implementation of the GRP scheme for computing spherical compressible fluid flows. Journal of Computational Physics 228, 5867–5887 (2009)
5. Zhu, P., Zhou, S.: Relaxation Lax-Friedrichs sweeping scheme for static Hamilton-Jacobi equations. Numer. Algor. 54, 325–342 (2010)
6. Li, J.-Q., Tang, H.-Z., Warnecke, G., Zhang, L.-M.: Local Oscillations in Finite Difference Solutions of Hyperbolic Conservation Laws. Mathematics of Computation, S0025-5718(09)02219-4
7. Morton, K.W., Mayers, D.F.: Numerical Solution of Partial Differential Equations, 2nd edn. Cambridge University Press (2005)
8. Breuss, M.: The correct use of the Lax-Friedrichs method. M2AN Math. Model. Numer. Anal. 38, 519–540 (2004)
9. Breuss, M.: An analysis of the influence of data extrema on some first and second order central approximations of hyperbolic conservation laws. M2AN Math. Model. Numer. Anal. 39, 965–993 (2005)
10. Thomas, J.W.: Numerical Partial Differential Equations: Finite Difference Methods. Springer (1995)
11. Warming, R.F., Hyett, B.J.: The modified equation approach to the stability and accuracy of finite difference methods. J. Comput. Phys. 14, 159–179 (1974)
12. Tadmor, E.: Numerical viscosity and the entropy condition for conservative difference schemes. Math. Comp. 43, 369–381 (1984)