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Article *in* PAMM · December 2004

DOI: 10.1002/pamm.200410299

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PAMM · Proc. Appl. Math. Mech. **4**, 636–637 (2004) / **DOI** 10.1002/pamm.200410299

About the Lax-Friedrichs scheme for the numerical approximation of hyperbolic conservation laws

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We discuss the numerical stability of the classical Lax-Friedrichs method. The scheme features well-established properties, especially it is TVD and monotone. However, it turns out that oscillations can occur at data extrema which seems to be in severe contrast to the large numerical diffusion the scheme exhibits. We briefly explain the phenomenon by use of a simple model problem and we give a short theoretical discussion.

1 Introduction

Monotone schemes like the Lax-Friedrichs (LF) scheme investigated here serve as an important construction tool in the devicing of higher-order methods: Usually, higher order schemes incorporate a switch to a first order scheme in order to guarantee the stability of the method as well as to damp oscillations occuring by the application of the higher order part. Moreover, modern finite volume or finite element methods for the approximation of hyperbolic problems make use of a low-order local numerical flux for which the local version of the LF method is the prototype, see e.g. [4] and the references therein.

As indicated, the LF method features many well-established attributes, the most important ones are that it is a consistent, conservative and monotone method, and hence it is also TVD. The scheme is very easy to use since it does not involve the knowledge about the structure of the solution of Riemann problems and because it uses only flux evaluations. It is of general interest to investigate the LF scheme since it forms the basis of other (central) numerical approaches not requiring a sophisticated Riemann solver as an ingredient, see e.g. [4, 5] for useful discussions.

Within the next sections, we show that the LF scheme may yield oscillatory solutions and we briefly discuss this phenomenon. Finally, we state some acknowledgements.

2 The Lax-Friedrichs scheme

The LF scheme proposed in [2] for the approximation of hyperbolic conservation laws

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f\left(u(x,t)\right) = 0$$

which we discuss here reads

$$U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[f\left(U_{j+1}^n\right) - f\left(U_{j-1}^n\right)\right], \tag{1}$$

where j denotes the spatial index at $j\Delta x$, $k \in \{n,\, n+1\}$ denotes the temporal level $k\Delta t$, and where we employ U as a notation for discrete data. The scheme is stable under a CFL condition which we assume to hold.

2.1 Discussion of a linear model problem

An important tool to analyze the actual behaviour of a numerical approxiamtion of a conservation law is the so-called modified equation. In the case of the LF scheme and, as an example, for the approximation of the linear advection equation

$$u_t + au_x = 0,$$

we obtain the modified equation in the form

$$u_t + au_x = \frac{(\Delta x)^2}{2\Delta t}\left(1 - \left(a\frac{\Delta t}{\Delta x}\right)^2\right)u_{xx}.$$

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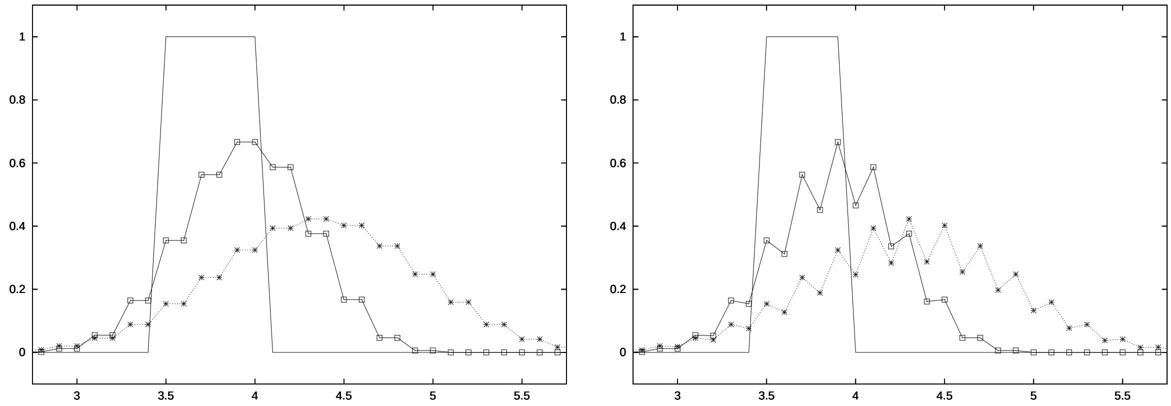

Fig. 1 This Figure shows (a) on the left hand side numerical results based on an even number of non-zero initial data, and (b) on the right hand side numerical results based on an odd number. In both cases, the initial condition together with the numerical solutions after 10 (lines with boxes) and 30 (dotted lines with stars) time steps are displayed, respectively.

This is an advection-diffusion equation suggesting that the scheme yields a better approximation of a diffusion process than of the original transport process. Thus, it is a widely accepted point of view that the Lax-Friedrichs scheme gives very stable, non-oscillatory (i.e., diffusive) approximations.

However, already a simple example using the linear advection equation and a square signal as initial condition gives the following evolutions, depending on the number of discretization points used for the initial signal, see Figure 1: We clearly observe oscillatory states appearing in the numerical solution for an odd number of discretization points.

If we write the LF scheme for the linear case in the form

$$U_j^{n+1} = \frac{1}{2}\left(1 - \lambda a\right) U_{j+1}^n + \frac{1}{2}\left(1 + \lambda a\right) U_{j-1}^n,$$

it becomes evident that we observe in Figure 1(b) a failure of the interpolation procedure inherently involved with the scheme in the linear case.

2.2 Theoretical discussion

Using the interpolation idea, it quickly becomes evident that the occurence of a local extremum during the simulation is responsible for oscillatory results.

In fact, it is possible to prove rigorously, that the LF scheme conserves the variation of the data at an extremum *exactly* if an oscillation occurs. It is also possible to identify a wide range of situations where such oscillations occur. Since one of these situations is exactly the one featuring already an oscillation, it is clear that once an oscillation appears it is preserved by the scheme and will not simply vanish by application of the scheme. This is also an astonishing property of the method having in mind the modified equation showing the excessive amount of numerical viscosity incorporated in the scheme.

Let us emphasize, that the discussed behaviour not only holds for the linear case, it can also be verified rigorously for a wide range of nonlinear fluxes (including convex and non-convex ones). Experimentally, the phenomenon also occurs for scalar conservation laws in more than one spatial dimension as well as for systems of equations.

Let us note, that the occurence of new extrema is not an unknown phenomenon, see [3]. However, our work represents the first rigorous analysis of the underlying mechanisms, see [1] for more details.

Acknowledgements The work of the author was conducted during a research stay at the University of Bordeaux in France. The author would like to thank the Deutsche Forschungsgemeinschaft (DFG) for supporting this stay under grant No. BR 2245/1-1 and the laboratory MAB (Mathématiques Appliquées de Bordeaux) for hosting it.

References

[1] Breuß, M. (2003): On the correct use of the Lax-Friedrichs method. Accepted for publication in M2AN.
[2] Lax, P.D. (1954): Weak solutions of nonlinear hyperbolic equations and their numerical approximation. Comm. Pure and Appl. Math., **7**, pp. 159-193
[3] LeFloch, P.G., Liu, J.-G. (1999): Generalized Monotone Schemes, Discrete Paths of Extrema, and Discrete Entropy Conditions. Math. Comp., **68**, No. 227, pp. 1025-1055
[4] LeVeque, R.J. (2002): Finite Volume Methods for Hyperbolic Problems. Cambridge University Press
[5] Nessyahu, H., Tadmor, E. (1990): Non-oscillatory central differencing for hyperbolic conservation laws. J. Comput. Phys., **87**, pp. 408-436