Generalized Lax-Friedrichs Scheme for Convective-Diffusion Equation*

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Abstract. A new type of generalized Lax-Friedrichs scheme for the convective-diffusion equation $u_t + au_x = \varepsilon u_{xx}(a \in R, \varepsilon > 0)$ is given and analyzed. For the convection term, the scheme use generalized Lax-Friedrichs scheme. For the diffusion term, it uses implicit central difference scheme. The scheme is discussed by applying modified equation analysis, in order to find the relative phase error, numerical dissipation, numerical viscosity, numerical damping and oscillations.

Keywords: generalized Lax-Friedrichs (LxF) scheme, modified equation analysis, oscillations, numerical dissipation, numerical damping, numerical viscosity.

1 Introduction

In [1], to compute the numerical solution of the hyperbolic conservation laws

$$u_t + f(u)_x = 0, x \in R, t > 0, \tag{1}$$

where $u = (u_1, \cdots, u_m)^T$, and $f(u) = (f_1, \cdots, f_m)^T$, we consider the generalized Lax-Friedrichs(LxF) scheme of the viscosity form

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}[f(u_{j+1}^n) - f(u_{j-1}^n)] + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{2}$$

where the mesh ratio $\nu = \tau/h$ is assumed to be a constant, τ and h are step sizes in time and space, respectively, u_j^n denotes an approximation of $u(jh, n\tau)$, the term $q \in (0, 1]$ is the coefficient of numerical viscosity. When $q = 1$, it is the classical Lax-Friedrichs(LxF) scheme.

With the flux function $f = au$, (1) is the linear advection equation as follow

$$u_t + au_x = 0, \quad x \in R, t > 0, \tag{3}$$

and the scheme (2) turns into the generalized LxF scheme of equation (3)

$$u_j^{n+1} = u_j^n - \frac{\nu a}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n). \tag{4}$$

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By adding a diffusion term εu_{xx} (ε is a positive constant) to the right of (3), we obtain a convective-diffusion equation.

$$u_t + au_x = \varepsilon u_{xx}, \quad a \in R, \varepsilon > 0. \tag{5}$$

There are two different finite difference schemes of the convective-diffusion equation (5). For the convective term, we still use the generalized LxF scheme. Then, we have two following ways to approximate the diffusion term: one uses explicit central difference scheme, i.e.

$$u_j^{n+1} = u_j^n - \tfrac{\nu a}{2}(u_{j+1}^n - u_{j-1}^n) + \tfrac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \tfrac{\varepsilon\tau}{h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{6}$$

and the other one uses implicit central difference scheme, i.e.

$$u_j^{n+1} = u_j^n - \frac{\nu a}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \frac{\varepsilon\tau}{h^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}). \tag{7}$$

Scheme (6) also can be written in this form

$$u_j^{n+1} = u_j^n - \frac{\nu a}{2}(u_{j+1}^n - u_{j-1}^n) + (\frac{q}{2} + \mu)(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{8}$$

where $\mu = \frac{\varepsilon\tau}{h^2}$. Similarly, scheme (7) can be written like this

$$u_j^{n+1} = u_j^n - \frac{\nu a}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \mu(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}). \tag{9}$$

Scheme (8) and scheme (9) are both the generalized LxF schemes of the convective-diffusion equation(1.5). The term q and ε (in term μ) are called the coefficient of numerical viscosity and the coefficient of physical viscosity, respectively.

2 Chequerboard Modes in the Initial Discretization

As observed in [1], we discussed the discretization of initial data

$$u(x, 0) = u_0(x), \qquad x \in [0, 1], \tag{10}$$

with M grid points and $h = 1/M$, while M is even, and $u_0(0) = u_0(1)$. The numerical solution value at the grid point x_j is denoted by u_j^0. We express this grid point value u_j^0 by using the usual discrete Fourier sums, as in [5, Page 120], and obtain

$$u_j^0 = \sum_{k=-M/2+1}^{M/2} c_k^0 e^{i\xi j}, \quad i^2 = -1, \quad j = 0, 1, \cdots, M-1, \tag{11}$$

where $\xi = 2\pi kh$. And the coefficients c_k^0 are expressed as

$$c_k^0 = \frac{1}{M}\sum_{k=0}^{M-1} u_j^0 e^{-i\xi j}, \quad k = -M/2+1, \cdots, M/2, \tag{12}$$

we have two different types of discretizations, if they are discretized with an odd number of grid points, the chequerboard mode is present. In contrast, if they are discretized with an even number of grid points, the chequerboard mode is suppressed.

(i) Discretization with an odd number of grid points:

Take $j_1, j_2 \in N$ such that $j_1 + j_2$ is an even number. We set $x^{(1)} = (\frac{M}{2} - j_1)h$ and $x^{(2)} = (\frac{M}{2} + j_2)h$. We discretize the square signal (1.12) with $p := j_1 + j_2 + 1$ nodes, such that

$$u_j^0 = \begin{cases} 1, & if \quad j = M/2 - j_1, \cdots, M/2 + j_2, \\ 0, & otherwise. \end{cases} \tag{13}$$

In [1], we have

$$u_j^0 = (-1)^{j+j_1+M/2}h + ph + \sum_{k \neq 0, M/2} \frac{(-1)^k e^{i\xi(j+j_1)}(1 - e^{-i\xi p})}{M(1 - e^{-i\xi})} \tag{14}$$

(ii) Discretization with an even number of grid points:

Similarly, we use $p - 1 = j_1 + j_2$ even number of grid points to express the square signal in (13) as follows

$$u_j^0 = \begin{cases} 1, & if \quad j = M/2 - j_1 + 1, \cdots, M/2 + j_2, \\ 0, & otherwise. \end{cases} \tag{15}$$

Then the initial data (15) can be written as follow in [1]

$$u_j^0 = 0 \times (-1)^j + (p-1)h + \sum_{k \neq 0, M/2} \frac{(-1)^k e^{i\xi(j+j_1-1)}[1 - e^{-i\xi(p-1)}]}{M(1 - e^{-i\xi})}. \tag{16}$$

3 Modified Equation Analysis

As the amplitude error and relative phase error of the Fourier modes have a correspondence with dissipation and phase error mechanisms displayed by related partial differential equations, we use the method of modified equation analysis to further investigate the mechanisms of dissipation and phase error of scheme (1.9). The same to the modified equation analysis for scheme(8), we want to see how the dissipation offsets the large phase error of high frequency modes.

We use implicit central difference scheme to difference the diffusion term εu_{xx} in scheme (1.9), which will give rise to further problem in the modified equation analysis as follows. Similarly, we consider a smooth solution $(U^s)_j^n$ and an oscillatory solution $(U^h)_j^n = (-1)^{j+n}(U^o)_j^n$ respectively, where

$$U_j^n = (U^s)_j^n + (-1)^{j+n}(U^o)_j^n$$

.

3.1 Low Frequency Modes

The smooth solution $(U^s)_n^j$ satisfies scheme (1.9). Implying the same labels $D_{+t} := \Delta_{+t}/\tau$, $\Delta_{+t} = e^{\tau\partial t} - 1$, then we have (17), i.e.

$$\partial_t = D_{+t} - \frac{1}{2}\tau D_{+t}^2 + \frac{1}{3}\tau^2 D_{+t}^3 - \frac{1}{4}\tau^3 D_{+t}^4 + \cdots. \tag{17}$$

In accordance with (9),

$$\begin{aligned}(U^s)_j^{n+1} = (U^s)_j^n - \frac{\nu a}{2}[(U^s)_{j+1}^n - (U^s)_{j-1}^n] + \frac{q}{2}[(U^s)_{j+1}^n - 2(U^s)_j^n + (U^s)_{j-1}^n]\\ +\mu[(U^s)_{j+1}^{n+1} - 2(U^s)_j^{n+1} + (U^s)_{j-1}^{n+1}]\end{aligned} \tag{18}$$

Similar to $\Delta_{+t} = e^{\tau\partial t} - 1$, $\Delta_{+x} = e^{h\partial x} - 1$ is used as follows,

$$\begin{aligned}(U^s)_j^{n+1} = (U^s)_j^n + [-\frac{\nu a}{2}(e^{h\partial x} - e^{-h\partial x}) + \frac{q}{2}(e^{h\partial x} + e^{-h\partial x} - 2)](U^s)_j^n\\ +\mu(e^{h\partial x} + e^{-h\partial x} - 2)(U^s)_j^{n+1},\end{aligned}$$

$$\begin{aligned}&[1 - \mu(e^{h\partial x} + e^{-h\partial x} - 2)](U^s)_j^{n+1}\\ &= [1 - \frac{\nu a}{2}(e^{h\partial x} - e^{-h\partial x}) + \frac{q}{2}(e^{h\partial x} + e^{-h\partial x} - 2)](U^s)_j^n,\end{aligned}$$

$$\begin{aligned}&[1 - \mu(e^{h\partial x} + e^{-h\partial x} - 2)]((U^s)_j^{n+1} - (U^s)_j^n)\\ &= [-\frac{\nu a}{2}(e^{h\partial x} - e^{-h\partial x}) + (\frac{q}{2} + \mu)(e^{h\partial x} + e^{-h\partial x} - 2)](U^s)_j^n,\end{aligned}$$

$$\begin{aligned}&[1 - \mu(e^{h\partial x} + e^{-h\partial x} - 2)]\Delta_{+t}(U^s)_j^n\\ &= [-\frac{\nu a}{2}(e^{h\partial x} - e^{-h\partial x}) + (\frac{q}{2} + \mu)(e^{h\partial x} + e^{-h\partial x} - 2)](U^s)_j^n.\end{aligned} \tag{19}$$

From (19), taking the standard Taylor expansion yields, we obtain

$$\begin{aligned}\Delta_{+t} &= \frac{-\frac{\nu a}{2}(e^{h\partial x} - e^{-h\partial x}) + (\frac{q}{2} + \mu)(e^{h\partial x} + e^{-h\partial x} - 2)}{1 - \mu(e^{h\partial x} + e^{-h\partial x} - 2)}\\ &= \frac{-\frac{\nu a}{2}(2h\partial_x + \frac{2h^3}{3!}\partial_x^3 + \cdots) + (\frac{q}{2} + \mu)(\frac{2h^2}{2!}\partial_x^2 + \frac{2h^4}{4!}\partial_x^4 + \cdots)}{1 - \mu(\frac{2h^2}{2!}\partial_x^2 + \frac{2h^4}{4!}\partial_x^4 + \cdots)}\\ &= \frac{-\nu a h\partial_x + (\frac{q}{2} + \mu)h^2\partial_x^2 - \frac{\nu a}{6}h^3\partial_x^3 + \frac{q+2\mu}{24}h^4\partial_x^4 + \cdots}{1 - \mu h^2\partial_x^2 - \frac{\mu}{12}h^4\partial_x^4 - \cdots}\\ &= -\nu a h\partial_x + (\frac{q}{2} + \mu)h^2\partial_x^2 - (\frac{1}{6} + \mu)\nu a h^3\partial_x^3 + \cdots,\end{aligned} \tag{20}$$

then,

$$D_{+t} = \frac{\Delta_{+t}}{\tau} = \frac{1}{\tau}[-\nu a h\partial_x + (\frac{q}{2} + \mu)h^2\partial_x^2 - (\frac{1}{6} + \mu)\nu a h^3\partial_x^3 + \cdots]. \tag{21}$$

Substituting (21) into (20), we obtain

$$\partial_t = -a\partial_x + [\varepsilon + \frac{h^2}{2\tau}(q - \nu^2 a^2)]\partial_x^2 + \frac{ah^2}{6}(3q - 2\nu^2 a^2 - 1)\partial_x^3 + \cdots, \tag{22}$$

where $\mu = \frac{\varepsilon\tau}{h^2}$, $\nu = \tau/h$.

From (22), we derive the modified equation about the smooth part, where $\tilde{U}^s$ is used to express the solution of exact differential equation corresponding to (9),

$$\begin{aligned}\partial_t\tilde{U}^s + a\partial_x\tilde{U}^s - \varepsilon\partial_x^2\tilde{U}^s = & \frac{h^2}{2\tau}(q - \nu^2 a^2)\partial_x^2\tilde{U}^s \\ & + \frac{ah^2}{6}(3q - 2a^2\nu^2 - 1)\partial_x^3\tilde{U}^s + \cdots.\end{aligned} \tag{23}$$

The second order term $\frac{h^2}{2\tau}(q - \nu^2 a^2)\partial_x^2\tilde{U}^s$ in (23) represents the numerical viscosity of (9). Under the condition $q > \nu^2 a^2$, the dissipation effect of the scheme (8) becomes stronger as q increases, which is observed by the Fourier analysis.

3.2 High Frequency Modes

The oscillatory solution $(U^h)_j^n$ is written as

$$(U^h)_j^n = (-1)^{j+n}(U^o)_j^n,$$

where $(U^o)_j^n$ is viewed as the perturbation amplitude of the chequerboard mode. Then the oscillatory term $(U^o)_j^n$ satisfies, in accordance with (1.9),

$$\begin{aligned}& (-1)^{j+n+1}(U^o)_j^{n+1} \\ = & (-1)^{j+n}(U^o)_j^n - \frac{\nu a}{2}[(-1)^{j+n+1}(U^o)_{j+1}^n - (-1)^{j+n-1}(U^o)_{j-1}^n] \\ & + \frac{q}{2}[(-1)^{j+n+1}(U^o)_{j+1}^n - 2(-1)^{j+n}(U^o)_j^n + (-1)^{j+n-1}(U^o)_{j-1}^n] \\ & + \mu[(-1)^{j+n+2}(U^o)_{j+1}^{n+1} - 2(-1)^{j+n+1}(U^o)_j^{n+1} + (-1)^{j+n}(U^o)_{j-1}^{n+1}],\end{aligned}$$

that is ,

$$\begin{aligned}(U^o)_j^{n+1} = & -(U^o)_j^n - \frac{\nu a}{2}[(U^o)_{j+1}^n - (U^o)_{j-1}^n] + \frac{q}{2}[(U^o)_{j+1}^n + 2(U^o)_j^n + (U^o)_{j-1}^n] \\ & -\mu[(U^o)_{j+1}^{n+1} + 2(U^o)_j^{n+1} + (U^o)_{j-1}^{n+1}].\end{aligned}$$

Now using the same methods about low frequency modes, we deduce as follows,

$$\begin{aligned}& (1 + 4\mu)(U^o)_j^{n+1} + \mu[(U^o)_{j+1}^{n+1} - 2(U^o)_j^{n+1} + (U^o)_{j-1}^{n+1}] \\ = & (2q - 1)(U^o)_j^n - \tfrac{\nu a}{2}[(U^o)_{j+1}^n - (U^o)_{j-1}^n] + \tfrac{q}{2}[(U^o)_{j+1}^n - 2(U^o)_j^n + (U^o)_{j-1}^n],\end{aligned}$$

$$\begin{aligned}& [1 + 4\mu + \mu(e^{h\partial x} + e^{-h\partial x} - 2)](U^o)_j^{n+1} \\ = & [2q - 1 - \frac{\nu a}{2}(e^{h\partial x} - e^{-h\partial x}) + \frac{q}{2}(e^{h\partial x} + e^{-h\partial x} - 2)](U^o)_j^n,\end{aligned}$$

$$
\begin{aligned}
&[1+4\mu+\mu(e^{h\partial x}+e^{-h\partial x}-2)]((U^o)_j^{n+1}-(U^o)_j^n)\\
&=[2q-4\mu-2-\frac{\nu a}{2}(e^{h\partial x}-e^{-h\partial x})+(\frac{q}{2}-\mu)(e^{h\partial x}+e^{-h\partial x}-2)](U^o)_j^n,
\end{aligned}
$$

$$
\begin{aligned}
&[1+4\mu+\mu(e^{h\partial x}+e^{-h\partial x}-2)]\Delta_{+t}(U^o)_j^n\\
&=[2q-4\mu-2-\frac{\nu a}{2}(e^{h\partial x}-e^{-h\partial x})+(\frac{q}{2}-\mu)(e^{h\partial x}+e^{-h\partial x}-2)](U^o)_j^n. \quad (24)
\end{aligned}
$$

From (24), taking the standard Taylor expansion yields, we have

$$
\begin{aligned}
\Delta_{+t}&=\frac{2q-4\mu-2-\frac{\nu a}{2}(e^{h\partial x}-e^{-h\partial x})+(\frac{q}{2}-\mu)(e^{h\partial x}+e^{-h\partial x}-2)}{1+4\mu+\mu(e^{h\partial x}+e^{-h\partial x}-2)}\\
&=\frac{2q-4\mu-2-\nu ah\partial_x+\frac{q-2\mu}{2}h^2\partial_x^2-\frac{\nu a}{6}h^3\partial_x^3+\frac{q-2\mu}{24}h^4\partial_x^4+\cdots}{1+4\mu+\mu h^2\partial_x^2+\frac{\mu}{12}h^4\partial_x^4-\cdots}\\
&=\frac{2q-4\mu-2}{1+4\mu}-\frac{\nu ah}{1+4\mu}\partial_x+\frac{q+2\mu}{2(1+4\mu)^2}h^2\partial_x^2-\frac{1-2\mu}{6(1+4\mu)^2}\nu ah^3\partial_x^3+\cdots \quad (25)
\end{aligned}
$$

We write (25) as

$$(e^{\tau\partial_t}-1)\tilde{U}^o=P-H\partial_x+Q\partial_x^2-R\partial_x^3+\cdots,$$

where $P=\frac{2q-4\mu-2}{1+4\mu}$, $H=\frac{\nu ah}{1+4\mu}$,$Q=\frac{q+2\mu}{2(1+4\mu)^2}h^2$, $R=\frac{1-2\mu}{6(1+4\mu)^2}\nu ah^3$. According to the following basic facts and the formal operator expansion

$$\tau\partial_t=\ln((e^{\tau\partial_t}-1)+1)=\sum_{m=1}^{\infty}(-1)^{m+1}\frac{(e^{\tau\partial_t}-1)^m}{m},$$

and the well known power series

$$\frac{1}{(1+z)^2}=\sum_{m=0}^{\infty}(-1)^m(m+1)z^m, \qquad where \quad z\in(-1,1),$$

$$\frac{1}{(1+z)^3}=\sum_{m=0}^{\infty}(-1)^m\frac{(m+1)(m+2)}{2}z^m, \qquad where \quad z\in(-1,1),$$

Let $C_m^l=\frac{m!}{(m-l)!l!}$ denote the binomial coefficients for $l\le m$. We obtain ,by ignoring terms of orders higher than three, that

$$
\begin{aligned}
\tau\partial_t\tilde{U}^o&=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}(e^{\tau\partial_t}-1)^m\tilde{U}^o\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}(P-H\partial_x+Q\partial_x^2-R\partial_x^3+\cdots)^m\tilde{U}^o\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}P^m\tilde{U}^o+\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}C_m^1P^{m-1}(-H\partial_x)\tilde{U}^o
\end{aligned}
$$

$$
\begin{aligned}
&+\{\sum_{m=2}^{\infty}\frac{(-1)^{m+1}}{m}C_m^2P^{m-2}H^2\partial_x^2+\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}C_m^1P^{m-1}Q\partial_x^2\}\tilde{U}^o\\
&+\{\sum_{m=3}^{\infty}\frac{(-1)^{m+1}}{m}C_m^3P^{m-3}(-H\partial_x)^3+\sum_{m=2}^{\infty}\frac{(-1)^{m+1}}{m}C_m^1C_{m-1}^1P^{m-2}(-H\partial_x)Q\partial_x^2\\
&+\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}C_m^1P^{m-1}(-R\partial_x^3)\}\tilde{U}^o+\cdots\\
=&\ln|P+1|\tilde{U}^o-\frac{H}{1+P}\partial_x\tilde{U}^o+\{-\frac{H^2}{2(1+P)^2}+\frac{Q}{(1+P)}\}\partial_x^2\tilde{U}^o\\
&+\{\frac{-H^3}{3(1+P)^3}+\frac{HQ}{(1+P)^2}-\frac{R}{1+P}\}\partial_x^3\tilde{U}^o+\cdots\\
=&\frac{1}{\tau}\ln\left|\frac{1-2q}{1+4\mu}\right|\tilde{U}^o-\frac{a}{2q-1}\partial_x\tilde{U}^o+\frac{h^2}{2\tau}\frac{(2q-1)(q+2\mu)-\nu^2a^2(1+4\mu)}{(2q-1)^2(1+4\mu)}\partial_x^2\tilde{U}^o\\
&+\frac{ah^2}{6}\frac{[(q+1)(2q-1)-2\nu^2a^2]}{(2q-1)^3}\partial_x^3\tilde{U}^o+\cdots. \qquad (26)
\end{aligned}
$$

Thus we derive the modified equation for the oscillatory part:

$$
\begin{aligned}
\partial_t\tilde{U}^o+\frac{a}{2q-1}\partial_x\tilde{U}^o &= \tfrac{1}{\tau}\ln\left|\tfrac{1-2q}{1+4\mu}\right|\tilde{U}^o\\
&+\tfrac{h^2}{2\tau}\tfrac{(2q-1)(q+2\mu)-\nu^2a^2(1+4\mu)}{(2q-1)^2(1+4\mu)}\partial_x^2\tilde{U}^o\\
&+\tfrac{ah^2}{6}\tfrac{(q+1)(2q-1)-2\nu^2a^2}{(2q-1)^3}\partial_x^3\tilde{U}^o+\cdots. \qquad (27)
\end{aligned}
$$

Then, we write (27) as

$$
\begin{aligned}
\partial_t\tilde{U}^o+\partial_x\tilde{U}^o\quad -\varepsilon\partial_x^2\tilde{U}^o &= \tfrac{1}{\tau}\ln\left|\tfrac{1-2q}{1+4\mu}\right|\tilde{U}^o+\tfrac{2a(q-1)}{2q-1}\partial_x\tilde{U}^o\\
&+\tfrac{h^2}{2\tau}\tfrac{(2q-1)(q+2\mu)-[\nu^2a^2+2\mu(2q-1)^2](1+4\mu)}{(2q-1)^2(1+4\mu)}\partial_x^2\tilde{U}^o\\
&+\tfrac{ah^2}{6}\tfrac{(q+1)(2q-1)-2\nu^2a^2}{(2q-1)^3}\partial_x^3\tilde{U}^o+\cdots. \qquad (28)
\end{aligned}
$$

We call the zero term $\frac{1}{\tau}\ln\left|\frac{1-2q}{1+4\mu}\right|\tilde{U}^o$ in (28) a numerical damping term, which exerts dominant dissipation to suppress the oscillations caused by the relative phase error of high frequency modes. The first order term $\frac{2a(q-1)}{2q-1}\partial_x\tilde{U}^o$ can be used to estimate the extent of dispersion.

Now the numerical dissipation is not enough to control oscillations caused by the relative phase errors of high frequency modes, and only the numerical damping can suppress the oscillations. Thus, we need conditions $0<\left|\frac{1-2q}{1+4\mu}\right|<1$, which ensure the coefficient of the numerical damping term negative. Considering the stability condition of the scheme (9), we obtain $0<\nu^2a^2\le q+2\mu<1+4\mu$ and $q\neq\frac{1}{2}$.

4 Remarks and Conclusions

We use the method of modified equation analysis to obtain the mechanisms of dissipation and phase error of scheme (1.9)and The same to the modified equa-

tion analysis for scheme(8). **(1)** The numerical damping becomes stronger as q is closer to $\frac{1}{2}$. Particularly, if $q = \frac{1}{2}$, the oscillation is damped out immediately, by noting that $\lim\limits_{q' \to \frac{1}{2}} \frac{1}{\tau} \ln \left| \frac{1-2q}{1+4\mu} \right| = -\infty$. So the numerical damping becomes infinite for $q = \frac{1}{2}$.
(2) In particular, if $q = 1$, the first order term vanishes; the numerical damping term is $-\frac{\ln|1+4\mu|}{\tau}\tilde{U}^o$, the numerical damping becomes stronger as μ is larger, and the numerical damping term will not vanish at all. when $1-\nu^2 a^2(1+4\mu)-8\mu^2 > 0$, the numerical viscosity term still plays dissipation roles, but it is still weak in comparison with the numerical damping term.

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