New Lax-Friedrichs Scheme for Convective-Diffusion Equation*

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Abstract. Two different types of generalized Lax-Friedrichs scheme for the convective-diffusion equation $u_t + au_x = \varepsilon u_{xx} (a \in R, \varepsilon > 0)$ are given and analyzed. For the convection term, both of two schemes use generalized Lax-Friedrichs scheme. For the diffusion term, explicit central difference scheme is used. The propagation of chequerboard mode is considered for two schemes and several numerical examples are presented only for the first scheme , which display how different parameters control oscillations. For low and high frequency modes, applying discrete Fourier analysis , some results have been obtained about stability condition of schemes, and clarify the reasons of oscillations and the interrelation among the amplitude error.

Keywords: generalized Lax-Friedrichs (LxF) scheme, discrete Fourier analysis, oscillations, frequency modes.

1 Introduction

In [1], to compute the numerical solution of the hyperbolic conservation laws

$$u_t + f(u)_x = 0, x \in R, t > 0, \tag{1}$$

where $u = (u_1, \cdots, u_m)^T$, and $f(u) = (f_1, \cdots, f_m)^T$, we consider the generalized Lax-Friedrichs(LxF) scheme of the viscosity form

$$u_j^{n+1} = u_j^n - \frac{\nu}{2}[f(u_{j+1}^n) - f(u_{j-1}^n)] + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{2}$$

where the mesh ratio $\nu = \tau/h$ is assumed to be a constant, τ and h are step sizes in time and space, respectively, u_j^n denotes an approximation of $u(jh, n\tau)$, the term $q \in (0, 1]$ is the coefficient of numerical viscosity. When $q = 1$, it is the classical Lax-Friedrichs(LxF) scheme.

With the flux function $f = au$, (1) is the linear advection equation as follow

$$u_t + au_x = 0, \quad x \in R, t > 0, \tag{3}$$

and the scheme (2) turns into the generalized LxF scheme of equation (3)

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$$u_j^{n+1} = u_j^n - \frac{\nu a}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n). \tag{4}$$

By adding a diffusion term εu_{xx} (ε is a positive constant) to the right of (3), we obtain a convective-diffusion equation.

$$u_t + au_x = \varepsilon u_{xx}, \quad a \in R, \varepsilon > 0. \tag{5}$$

There are two different finite difference schemes of the convective-diffusion equation (5). For the convective term, we still use the generalized LxF scheme. Then, we have two following ways to approximate the diffusion term: one uses explicit central difference scheme, i.e.

$$u_j^{n+1} = u_j^n - \frac{\nu a}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \frac{\varepsilon\tau}{h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{6}$$

and the other one uses implicit central difference scheme, i.e.

$$u_j^{n+1} = u_j^n - \frac{\nu a}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{q}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \frac{\varepsilon\tau}{h^2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}). \tag{7}$$

Scheme (6) also can be written in this form

$$u_j^{n+1} = u_j^n - \frac{\nu a}{2}(u_{j+1}^n - u_{j-1}^n) + (\frac{q}{2} + \mu)(u_{j+1}^n - 2u_j^n + u_{j-1}^n), \tag{8}$$

where $\mu = \frac{\varepsilon\tau}{h^2}$. Scheme (8) is the generalized LxF schemes of the convective-diffusion equation (5). As observed in [1], we discussed the discretization of initial data

$$u(x,0) = u_0(x), \qquad x \in [0,1], \tag{9}$$

with M grid points and $h = 1/M$, while M is even, and $u_0(0) = u_0(1)$. The numerical solution value at the grid point x_j is denoted by u_j^0. We express this grid point value u_j^0 by using the usual discrete Fourier sums, as in [5, Page 120], and obtain

$$u_j^0 = \sum_{k=-M/2+1}^{M/2} c_k^0 e^{i\xi j}, \quad i^2 = -1, \quad j = 0, 1, \cdots, M-1, \tag{10}$$

where $\xi = 2\pi kh$. And the coefficients c_k^0 are expressed as

$$c_k^0 = \frac{1}{M}\sum_{k=0}^{M-1} u_j^0 e^{-i\xi j}, \quad k = -M/2+1, \cdots, M/2, \tag{11}$$

First, for the special case that

$$c_k^0 = \begin{cases} 1, & \text{if} k = M/2, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

i.e. the initial datas are taken to be just the highest Fourier mode which is a single chequerboard mode oscillation

$$u_j^0 = e^{i2\pi\frac{M}{2}jh} = e^{i\pi j} = (-1)^j. \tag{13}$$

Second, For single square signal, as follow

$$u(x,0) = \begin{cases} 1, & 0 < x^{(1)} < x < x^{(2)} < 1, \\ 0, & otherwise. \end{cases}$$

we have two different types of discretizations, if they are discretized with an odd number of grid points, the chequerboard mode is present. In contrast, if they are discretized with an even number of grid points, the chequerboard mode is suppressed.

According to the discussions in [1], the solution to (??) also can be expressed in the standard form of discrete Fourier series,

$$u_j^n = \sum_{k=-M/2+1}^{M/2} c_k^n e^{i\xi j}, \tag{14}$$

where $\xi = 2\pi kh$. Combining (14) and (??), the coefficients c_k^n are obtained analogously and expressed as,

$$c_k^n = [1 + q'(\cos\xi - 1) - i\nu a \sin\xi]^n c_k^0. \tag{15}$$

First, if we take initial data as a single chequerboard mode like (13), i.e. $u_j^0 = e^{i2\pi\frac{M}{2}jh} = e^{i\pi j} = (-1)^j$, we have

$$u_j^n = (1 - 2q')^n (-1)^j. \tag{16}$$

2 Discrete Fourier analysis

In this section, we use the method of discrete Fourier analysis to discuss the generalized LxF scheme (8) for the convective-diffusion equation. While using $q' = q + 2\mu$, the variable q' in scheme (??) acts as q in scheme (4). Thus, most important conclusions are obtained easily through the related results about scheme (4) of the linear advection equation in [1].

We denote a Fourier mode using the scaled wave number $\xi = 2\pi kh$ by $e^{i\xi}$. Then using it as initial data for a linear finite difference scheme results after n times steps in the solution

$$u_k^n = \lambda_k^n e^{i\xi} = (\lambda(k))^n e^{i\xi}, \quad i^2 = -1, \tag{17}$$

where λ_k^n is the amplitude. The modulus of the ratio

$$\lambda(k) = \lambda_k^{n+1} / \lambda_k^n$$

is the amplitude of the mode for one time step. For the scheme (8) we have with $\nu = \tau/h$ in particular, where $\mu = \frac{\varepsilon\tau}{h^2}$.

$$\lambda(k) = 1 + (q + 2\mu)(\cos\xi - 1) - i\nu a \sin\xi \tag{18}$$

and

$$|\lambda(k)|^2 = [1 + (q + 2\mu)(\cos\xi - 1)]^2 + (\nu a)^2 \sin^2\xi. \tag{19}$$

In [1], considering $\xi \approx 0$, $\xi \approx \pi$ and $|\lambda| \leq 1$, we obtained the conditions $0 < \nu^2 a^2 \leq q \leq 1$ are necessary and sufficient for stability of scheme (1.4). Therefore, the conditions $0 < \nu^2 a^2 \leq q' \leq 1$, i.e.

$$0 < \nu^2 a^2 \leq q + 2\mu \leq 1, \tag{20}$$

are necessary and sufficient for stability of scheme (8), too.

The exact solution of the Fourier mode $e^{i\xi}$ for $x = h$ after one time step τ is $e^{i(\xi - 2\pi a k \tau)} = e^{-i2\pi a k \tau} e^{i\xi} = \lambda_{exact}(k) e^{i\xi}$. The exact amplitude $\lambda_{exact}(k)$ has modulus 1. If the modulus of $\lambda(k)$ is less than one, the effect of the multiplication of a solution component with $\lambda(k)$ is called *numerical dissipation* and then the amplification error is called *dissipation error*.

Further, comparing the exponents of $\lambda(k)$ and $\lambda_{exact}(k)$ there is a *phase error*

$$\arg\lambda(k) - (-\nu a \xi),$$

where $\nu = \tau/h$, $\xi = 2\pi k h$. The *relative phase error* is then defined as

$$E_p(k) := \frac{\arg\lambda(k)}{-\nu a \xi} - 1.$$

A mode is a low frequency mode if $\xi \approx 0$ and a high frequency mode if $\xi \approx \pi$.

2.1 Low Frequency Modes

We first look at the low frequency modes

$$(U^s)_j^n := \lambda_k^n e^{i\xi j}, \qquad \xi \approx 0$$

Then substituting $(U^s)_j^n$ into (**??**),

$$\lambda(k) = 1 + q'(\cos\xi - 1) - i\nu a \sin\xi \tag{21}$$

and

$$\begin{aligned} |\lambda(k)|^2 &= [1 + q'(\cos\xi - 1)]^2 + (\nu a)^2 \sin^2\xi \\ &= 1 - (1 - \cos\xi)[2q' - q'^2(1 - \cos\xi) - \nu^2 a^2(1 + \cos\xi)], \end{aligned} \tag{22}$$

where $q' = q + 2\mu$. We express (22) by using $\cos\xi = 1 - \frac{1}{2}\xi^2 + O(\xi^4) + \cdots$

$$|\lambda|^2 = 1 - [q' - \nu^2 a^2]\xi^2 + O(\xi^4) \tag{23}$$

So, we can see that the dissipation error is of order $O(\xi)$, if $q' > \nu^2 a^2$.

As $q' = q + 2\mu \le 1$, $\xi \in [0, \pi/2]$, from (2.11) we obtain

$$\frac{d(|\lambda(k)|^2)}{dq'} = 2[1 - q'(1 - \cos\xi)] \cdot (\cos\xi - 1) < 0. \tag{24}$$

This implies that when $0 < \nu^2 a^2 \le q' \le 1$, the dissipation becomes stronger as q' larger. If μ is fixed, we also can say the dissipation becomes stronger as q larger. Thus scheme (8) with $q + 2\mu = 1$ has the largest numerical dissipation for low frequency modes.

The phase of the low frequency modes in approximated by Taylor expansion at $\xi = 0$

$$\begin{aligned} \arg\lambda &= \arctan\left(\frac{-\nu a \sin\xi}{1 - q'(1 - \cos\xi)}\right) = \arctan\left(\frac{-\nu a(\xi - \frac{1}{6}\xi^3 + \frac{1}{120}\xi^5 + \cdots)}{1 - q'(1 - \frac{1}{2}\xi^2 + \cdots)}\right) \\ &= \arctan\left(-\nu a\xi + \frac{1 - 3q'}{6}\nu a\xi^3 + \cdots\right). \end{aligned} \tag{25}$$

According to lemma 4.1 [2, page 97], we obtain

$$\arg\lambda = -\nu a\xi\left(1 + \frac{3q' - 1 - 2\nu^2 a^2}{6}\xi^2 + \cdots\right). \tag{26}$$

Then,

$$E_p(k) = \frac{3q' - 1 - 2a^2\nu^2}{6}\xi^2 + \cdots,$$

so the relative phase error $E_p(k)$ is of order $O(\xi^2)$, at least (in some cases, $3q' - 1 - 2a^2\nu^2 = 0$). Therefore, oscillations caused by this relative phase error can be suppressed by the stronger dissipation of order $O(\xi)$.

2.2 High Frequency Modes

For high frequency modes, i.e. $\xi \approx \pi$, the situation is very different. We use $\xi = \pi + \xi'$, i.e. $\xi' = 2\pi k' h$ with $kh = 1/2 + k'h$, and thus $\xi' \approx 0$. We write the modes in the form

$$(U^h)_j^n = \lambda_k^n e^{i\xi j} = \lambda_k^n e^{i(\pi + \xi')j} = (-1)^{j+n}\lambda_{k'}^n e^{i\xi' j}, \tag{27}$$

with $\lambda_{k'}^n = (-1)^{j+n} e^{i\pi j}\lambda_k^n$ and set

$$(U^o)_j^n := \lambda_{k'}^n e^{i\xi' j}.$$

The factor $(U^o)_j^n$ can be regarded as a perturbation amplitude of the chequer-board modes $(e^{i\pi})^{j+n} = (-1)^{j+n}$. The dissipation (amplitude error) depends only on $\lambda_{k'}^n$. Then substituting $(U^h)_j^n$ into (??) yields

$$\lambda' := \lambda_{k'}^{n+1}/\lambda_{k'}^n = -1 + q'(1 + \cos\xi') - i\nu a \sin\xi'.$$

Therefore, we have

$$\begin{aligned}|\lambda'|^2 &= [-1+q'(1+\cos\xi')]^2+(\nu a)^2\sin^2\xi' \\ &= 1+4(\nu^2a^2-q')\cos^2(\xi'/2)+4(q'^2-\nu^2a^2)\cos^4(\xi'/2). \end{aligned} \tag{28}$$

Similar to low frequency modes, we express (28) by using $\cos\xi = 1-\frac{1}{2}\xi^2+O(\xi^4)+\cdots$

$$\begin{aligned}|\lambda'|^2 &= [-1+q'(1+\cos\xi')]^2+(\nu a)^2\sin^2\xi' \\ &= (1-2q')^2+(q'-2q'^2+\nu^2a^2)\xi'^2+O(\xi'^4) \\ &= 1-[1-(1-2q')^2]+(q'-2q'^2+\nu^2a^2)\xi'^2+O(\xi'^4). \end{aligned} \tag{29}$$

If $0<\nu^2a^2\le q'<1$, the dissipation error is $O(1)$ (particularly if $q'=1$, the dissipation error is of order $O(\xi')$).

Obviously, for $\xi'\approx 0$, i.e. the high frequency modes, we consider

$$\frac{d(|\lambda'|^2)}{dq'} = -2(1+\cos\xi')[1-q'(1+\cos\xi')]. \tag{30}$$

If $q'>1/2$, we obtain $\frac{d(|\lambda'|^2)}{dq'}>0$, which means $|\lambda'|^2$ is an increasing function of q'; on the other hand, if $q'<1/2$, we have $\frac{d(|\lambda'|^2)}{dq'}<0$ and $|\lambda'|^2$ is a decreasing function of q'.

That means, under $0<\nu^2a^2\le q'<1$, if q' is closer to $1/2$, high frequency modes decay stronger, see Figure 2 and Figure 3. By the way, the results are in sharp contrast with the situation for low frequency modes.

Furthermore, let us look at the relative phase error. We compute

$$\begin{aligned}\arg\lambda' &= \arctan\Big(\frac{-\nu a\sin\xi'}{-1+q'(1+\cos\xi')}\Big) = \arctan\Big(\frac{-1}{2q'-1}\nu a\xi' - \frac{q'+1}{6(2q'-1)^2}\nu a\xi'^3-\cdots\Big) \\ &= -\frac{\nu a\xi'}{2q'-1} - \frac{2q'^2+q'-2\nu^2a^2-1}{6(2q'-1)^3}\nu a\xi'^3+O(\xi'^5). \end{aligned} \tag{31}$$

Then for the high frequency modes $(U^h)^n_j$, we have by recalling that $\xi=\pi+\xi'$

$$\begin{aligned}(U^h)^n_j &= (-1)^{j+n}\lambda'^n_k e^{i\xi' j} = |\lambda'|^n e^{in(-\pi+\arg\lambda')}\cdot e^{ij(\pi+\xi')} \\ &= |\lambda'|^n e^{i(j\xi-2\pi kan\tau)}\cdot e^{in(-\pi+\arg\lambda'+\nu a\xi)}. \end{aligned} \tag{32}$$

Therefore, the relative phase error of high frequency modes at each time step is using (31)

$$\begin{aligned}E_p(k) &= -\frac{-\pi+\arg\lambda'+\nu a\xi}{\nu a\xi} \\ &= \frac{\pi(1-\nu a)}{\nu a\xi}+\frac{(2-2q')\xi'}{(2q'-1)\xi}+\frac{(2q'^2+q'-2\nu^2a^2-1)\xi'^3}{6(2q'-1)^3\xi}+O(\xi'^5) \end{aligned} \tag{33}$$

We note that $\xi\approx\pi$. Therefore the relative phase error has $O(1)$, while $1-\nu a>0$. This error is huge, and strong numerical dissipation is needed to suppress it.

When $0 < \nu^2 a^2 \le q' < 1$, the dissipation error is of order $O(1)$. As the parameter q' is closer to $1/2$, the dissipation error becomes larger. The dissipation error can suppress the numerical oscillations caused by the relative phase error when $q' = 1/2$.

3 The Relation of Numerical Oscillations and Several Parameters

In this part, we only discuss high frequency modes. (1)The relation between numerical oscillations and the coefficient of numerical viscosity q: From (30), we have

$$\frac{d(|\lambda'|^2)}{dq} = \frac{d(|\lambda'|^2)}{dq'} \cdot \frac{dq'}{dq} = -2(1+\cos\xi')[1-(q+2\mu)(1+\cos\xi')]. \qquad (34)$$

If $q' = q + 2\mu > 1/2$, $\xi' \approx 0$, thus

$$\frac{d(|\lambda'|^2)}{dq} > 0. \qquad (35)$$

It implies that the dissipation becomes stronger for high frequency modes as the parameter q decreases, and then oscillations become weaker. That is, the numerical dissipation is more effective in controlling numerical oscillations as q decreases. In contrast, if $q' = q+2\mu < 1/2$, $\xi' \approx 0$, we obtain $\frac{d(|\lambda'|^2)}{dq} < 0$. It means the dissipation becomes stronger for high frequency modes as the parameter q increases. (2)The relation between numerical oscillations and the coefficient of physical viscosity ε: Similar to (1) above, since $q' = q + 2\mu = q + \frac{2\varepsilon\tau}{h^2}$, we obtain

$$\frac{d(|\lambda'|^2)}{d\varepsilon} = \frac{d(|\lambda'|^2)}{dq'} \cdot \frac{dq'}{d\varepsilon} = -\frac{4\tau}{h^2}(1+\cos\xi')[1-(q+\frac{2\varepsilon\tau}{h^2})(1+\cos\xi')].$$

If $q' = q + 2\mu > 1/2$, i.e. $q + \frac{2\varepsilon\tau}{h^2} > 1/2$, $\xi' \approx 0$, we obtain

$$\frac{d(|\lambda'|^2)}{d\varepsilon} > 0.$$

The same to the parameter q, the dissipation becomes much more stronger for high frequency modes as ε decreases, and then oscillations become weaker. As ε decreases, the numerical dissipation is effective in controlling numerical oscillations. If $q' = q + 2\mu < 1/2$, the conclusion is opposite. (3) The relation between numerical oscillations and width of mesh h. Now the mesh ratio $\nu = \tau/h$ is assumed to be a constant c and h is step size in space. Similar to (1) above, since $q' = q + 2\mu = q + \frac{2\varepsilon c}{h}$, we have

$$\begin{aligned}\frac{d(|\lambda'|^2)}{dh} &= \frac{d(|\lambda'|^2)}{dq'} \cdot \frac{dq'}{dh} = -2(1+\cos\xi')[1-q'(1+\cos\xi')] \cdot \frac{-2\varepsilon c}{h^2} \\ &= \frac{4\varepsilon c}{h^2}(1+\cos\xi')[1-(q+\frac{2\varepsilon c}{h})(1+\cos\xi')].\end{aligned}$$

If $q' = q + 2\mu > 1/2$, i.e. $q + \frac{2\varepsilon c}{h} > 1/2$, $\xi' \approx 0$, we obtain

$$\frac{d(|\lambda'|^2)}{dh} < 0.$$

This implies that the dissipation becomes much more stronger for high frequency modes as the parameter h increases, and then oscillations become weaker. That is, the numerical dissipation is effective in controlling numerical oscillations as h increases. However, if $q' = q + 2\mu < 1/2$, the conclusion is opposite, too.

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