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A COMPARISON OF TVD LAX–WENDROFF METHODS

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SUMMARY
This paper compares two formulations of Lax–Wendroff TVD methods. The basis of comparison is to the results of several test problems providing both qualitative and quantitative results. Results show that using an upwind biased limiter provides higher resolution of both smooth and discontinuous solutions from a lower amount of induced numerical viscosity. The conclusion is that a limiter should have as small a support as possible in order to limit its effects if high resolution is the object.

1. INTRODUCTION

In recent years, a number of high-resolution numerical methods have been developed to solve hyperbolic conservation laws. Among the most important of these methods has been the total variation diminishing (TVD) methods that have provided both excellent results as well as theoretical support for the techniques.

Several different varieties of TVD methods have been introduced: Harten's modified flux formulations,[1,2] and several 'symmetric' TVD schemes. Roe introduced the symmetric form of the TVD scheme.[3] Davis[4] also presents a similar method. Sweby[5] and Roe[6] present another similar method, but the limiters are of an upwind biased nature rather than symmetric in support. Yee[7] christened these schemes as symmetric TVD schemes in her paper. The general form of symmetric TVD schemes can be looked at in several different ways: as an advanced form of artificial diffusion, a Lax–Wendroff method[8] with an additional dissipative flux to ensure a TVD solution, or a TVD method that is symmetric in its stencil whenever the limiter is not present. Similar statements can be made concerning the upwind-biased schemes.

The methods introduced as being symmetric TVD schemes are differentiated by their flux limiters, which are centred in support about the cell edges. The other methods like those introduced by Sweby and Roe are upwind biased in the support for their limiters. Both methods, however, are closely related to the Lax–Wendroff method. The symmetric schemes have been favourably viewed because of their lower operation count and an increased convergence rate.[9]

This paper is organized into four Sections. The second Section presents the methods used in this study. The following Section gives results using these methods. The results are discussed and the final Section of the paper gives conclusions and closing remarks.

2. METHODS

The methods compared in this paper are of almost identical construction. They differ by the limiters used to ensure that the overall scheme is TVD. The starting point of these schemes is

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the Lax–Wendroff method. In a previous paper, I have shown how these two schemes can be related via geometric arguments.[10] The Lax–Wendroff method can be described by a simple linear interpolation scheme with time-averaged cell fluxes. The TVD modifications of this scheme modify the interpolation to introduce non-linear dissipation. Geometrically this can be interpreted as reducing the slope of the linear interpolation. This general concept is shown in Figure 1.

In one dimension, we are interested in solving the following basic equation:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0 \tag{1}$$

where $\mathbf{U}$ can be either a scalar or a vector and $\mathbf{F}(\mathbf{U})$ is a scalar or vector of equal length. A conservative discretization of this equation with explicit forward Euler time differencing is

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n - \lambda(\mathbf{F}(\mathbf{U}_{j+1/2}) - \mathbf{F}(\mathbf{U}_{j-1/2})) \tag{2}$$

where n refers to the time level, $\lambda = \Delta t/\Delta x$, with Δt and Δx being the time and spatial grid size, respectively. The values of $\mathbf{U}$ at the cell interfaces is determined by

$$\mathbf{U}_j(x) = \begin{cases} \mathbf{U}_j + \tilde{\mathbf{s}}_{j+1/2}(x - x_j); & x \in [x_j, x_{j+1/2}] \\ \mathbf{U}_j + \tilde{\mathbf{s}}_{j-1/2}(x - x_j); & x \in [x_{j-1/2}, x_j] \end{cases} \tag{3a}$$

which can be time-centred by taking

$$x_{j+1/2}^{n+1/2} = x_{j+1/2} - \frac{a\Delta t}{2} \tag{3b}$$

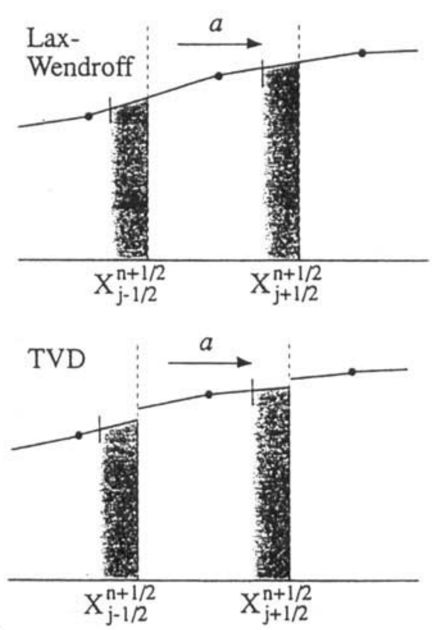

Figure 1. Construction of TVD Lax–Wendroff and Lax–Wendroff methods. The action of the limiters will introduce discontinuities at cell interfaces in order to assure oscillation-free solutions. The Lax–Wendroff scheme is continuous at the cell interfaces. The shaded region denotes the region traced by the characteristic at the cell edge yielding time-centred fluxes

and

$$x_{j-1/2}^{n+1/2} = x_{j-1/2} - \frac{a\Delta t}{2} \tag{3c}$$

where a is the appropriate characteristic velocity measured at the cell edge. The modified slopes $\tilde{s}_{j\pm 1/2}$ are to be functions of the local gradients

$$\tilde{s}_{j\pm 1/2} = Q(r) s_{j\pm 1/2} \tag{4}$$

where $s_{j+1/2} = \Delta_{j+1/2}u/\Delta_{j+1/2}x$ and $r = \Delta_{j-1/2}u/\Delta_{j+1/2}u$ or $r = \Delta_{j+3/2}u/\Delta_{j+1/2}u$. The term $\Delta_{j+1/2}u = u_{j+1} - u_j$.

In the Lax–Wendroff method, the value of $\tilde{s}_{j+1/2} = s_{j+1/2}$. Where this is not true with the TVD methods, a discontinuity exists at cell interfaces. To compute the fluxes in these cases, I will use Roe's approximate Riemann solver,[11] which uses a local characteristic decomposition for systems of equations, but is equivalent to the Murman–Cole[12] scheme for scalar equations.

The limiting functions used are fairly standard. The schemes are differentiated in this respect. The upwind scheme biases the limiter's support with the direction of the flow locally. In other words, if a limiter is needed at $j+\frac{1}{2}$ and the local velocity is positive, the limiter uses the derivative evaluated at $j+\frac{1}{2}$ and $j-\frac{1}{2}$ in the limiter. The symmetric scheme's limiters are centred about the cell edge as shown in Figure 2. When implemented in one dimension this gives the upwind-biased limiter with two gradients as arguments and the symmetric limiter has three gradients as arguments. Typical upwind-biased limiters are:

$$Q_1(1,r) = \max[0, \min(1,r)] \tag{5a}$$

$$Q_{vl}(1,r) = \frac{r+|r|}{1+|r|} \tag{5b}$$

$$Q_2(1,r) = \max[0, \min(2, 2r, \tfrac{1}{2}(1+r))] \tag{5c}$$

and

$$Q_{SB}(1,r) = \max[0, \min(2,r),\ \min(1,2r)] \tag{5}$$

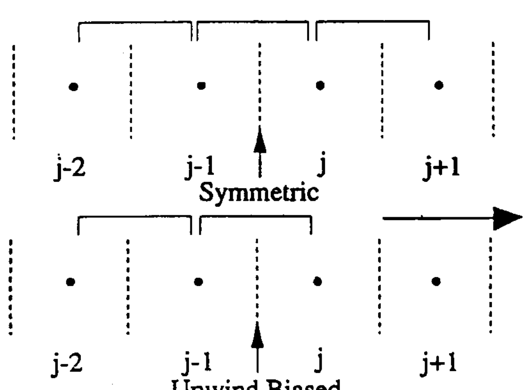

Figure 2. Stencils used in constructing symmetric and upwind-biased limiters. These demonstrate the support used by each limiter category. Limiters of the form of Q_l^i have different behaviours near discontinuities and extrema. Roe[3] noted this behaviour by defining this type of function as a 'separable Q function'

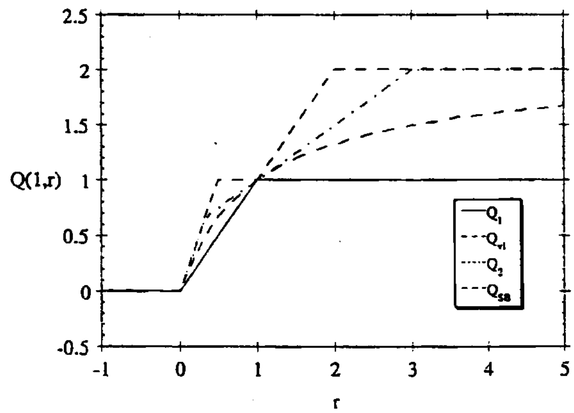

Figure 3. Behaviour of two parameter flux limiters. The second-order TVD region, as defined by Sweby,[5] is given by the area bounded by the curves defined by Q_1 and Q_{SB}

Figure 3 shows the behaviour of these limiters for a range of local gradient ratios. Yee (following Roe[3]) redefined these limiters as symmetric three parameter functions such as

$$Q_1(r^-, 1, r^+) = \max[0, \min(r^-, 1, r^+)] \tag{6a}$$

$$Q_2(r^-, 1, r^+) = \max\{0, \min[2r^-, 2, 2r^+, \tfrac{1}{2}(r^- + r^+)]\} \tag{6b}$$

and

$$Q_1'(r^-, 1, r^+) = \max[0, \min(r^-, 1)] + \max[0, \min(1, r^+)] - 1 \tag{6c}$$

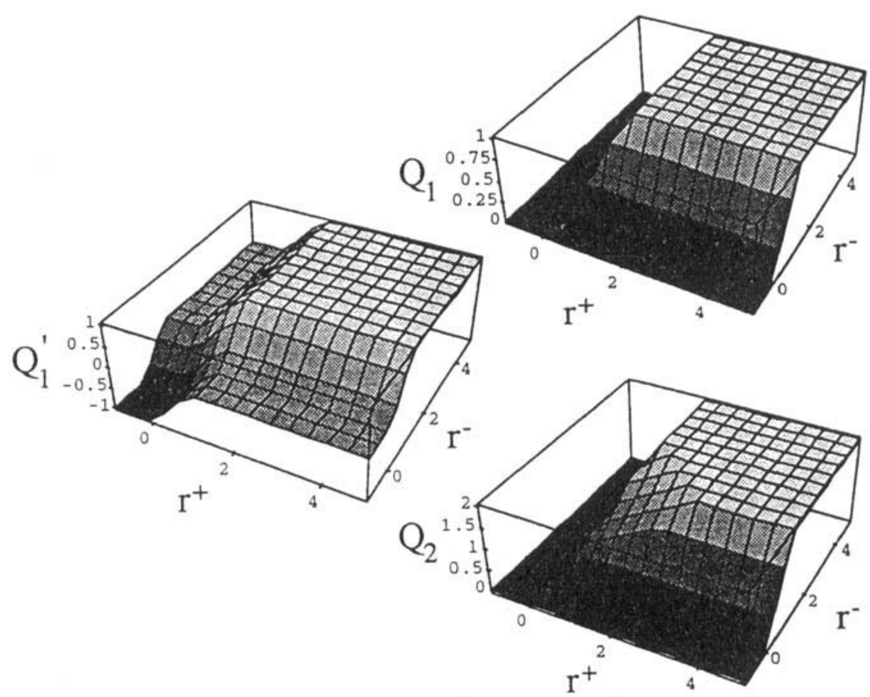

Figure 4. Behaviour of the three parameter limiters

where $r^- = \Delta_{j-1/2}u/\Delta_{j+1/2}u$ and $r^+ = \Delta_{j+3/2}u/\Delta_{j+1/2}u$. These limiters will be used in the symmetric TVD schemes. Figure 4 shows these limiters.

These limiters are representative of a much larger class of limiters applicable to these schemes.[13] For any grid resolution, the use of these limiters should produce a scheme that is TVD. It should be noted that the proof that these schemes are TVD is limited to scalar conservation laws.

3. RESULTS

In considering the performance of these schemes, six test problems will be completed: two for the scalar wave equation, one for Burger's equation and three for the Euler equations. The two problems for the scalar wave equation are the advection of a square wave and of a 'teepee' function across a periodic domain. Each structure is ten cells wide on a domain of 100 equidistant cells. Each test will run for 300 time steps with a Courant–Friedrichs–Lewy (CFL) number of $\frac{1}{2}$. The Burger equation problem is simply a sin x initial condition on a periodic domain with length of 2π (used in Reference 14). Results will be given at $t = 0\cdot 2$ where the solution is smooth and at $t = 1\cdot 0$ where a shock exists in the solution. The CFL number in this problem is approximately $0\cdot 08$.

The three Euler equation problems are Sod's problem,[15] Lax's problem[16] and a blast wave problem.[17] Both Sod's and Lax's problem are solved on grids of 100 equidistant grid points ($\Delta x = 1\cdot 0$) at a CFL number of $0\cdot 90$. The solution of Sod's problem is given at $t = 20\cdot 0$ and $t = 15\cdot 0$ for Lax's problem. The blast wave problem is solved on a 400-point equidistant grid ($\Delta x = 0\cdot 25$) with a CFL number of $0\cdot 95$, with a solution given at $t = 3\cdot 80$. The combination of these problems will highlight the strengths and weaknesses of these algorithms.

The initial condition for Sod's problem consists of two semi-infinite states separated at $t = 0$, and the left and right states are set to the following conditions:

(i) for $X < 50\cdot 0$

$$\begin{bmatrix} \tau_L \\ u_L \\ p_L \end{bmatrix} = \begin{bmatrix} 1\cdot 0 \\ 0\cdot 0 \\ 1\cdot 0 \end{bmatrix}$$

(ii) for $X \geqslant 50\cdot 0$,

$$\begin{bmatrix} \tau_R \\ u_R \\ p_R \end{bmatrix} = \begin{bmatrix} 8\cdot 0 \\ 0\cdot 0 \\ 0\cdot 1 \end{bmatrix}$$

with $\gamma = 1\cdot 4$. An ideal gas equation of state is used, and τ is the specific volume, u is the velocity and p is the pressure. The subscripts L and R refer to the left and right states, respectively. The initial condition for Lax's problem consists of two semi-infinite states separated at $t = 0$; the left and right states are set to the following conditions:

(i) $X < 50\cdot 0$,

$$\begin{bmatrix} \tau_L \\ u_L \\ p_L \end{bmatrix} = \begin{bmatrix} 2\cdot 247 \\ 0\cdot 698 \\ 3\cdot 528 \end{bmatrix}$$

(ii) for $X \geqslant 50 \cdot 0$,

$$\begin{bmatrix} \tau_R \\ u_R \\ p_R \end{bmatrix} = \begin{bmatrix} 2 \cdot 0 \\ 0 \cdot 0 \\ 0 \cdot 571 \end{bmatrix}$$

with $\gamma = 1 \cdot 4$. The initial conditions for the blast wave problem consist of the following:

(i) for $X \leqslant 10 \cdot 0$,

$$\begin{bmatrix} \tau_L \\ u_L \\ p_L \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 \\ 0 \cdot 0 \\ 1000 \cdot 0 \end{bmatrix}$$

(ii) for $10 \cdot 0 > X > 90 \cdot 0$,

$$\begin{bmatrix} \tau_L \\ u_L \\ p_L \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 \\ 0 \cdot 0 \\ 0 \cdot 01 \end{bmatrix}$$

(iii) for $X \geqslant 90 \cdot 0$

$$\begin{bmatrix} \tau_R \\ u_R \\ p_R \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 \\ 0 \cdot 0 \\ 100 \cdot 0 \end{bmatrix}$$

with $\gamma = 1 \cdot 4$.

Both algorithms will always use the limiter denoted by Q_2 in the preceding Section for all problems except the Burger equation problem, where Q_1 is used. Because the Burger equation problem only contains a shock, the added compression given by the Q_2 limiter is not necessary, and the Q_1 limiter suffices.

Figure 5 shows the solutions to the scalar wave equation. The symmetric scheme obviously provides lower resolution in both cases. The difference is also fairly great in terms of both peak preservation as well as signal width. The symmetric scheme also has problems with signal shape

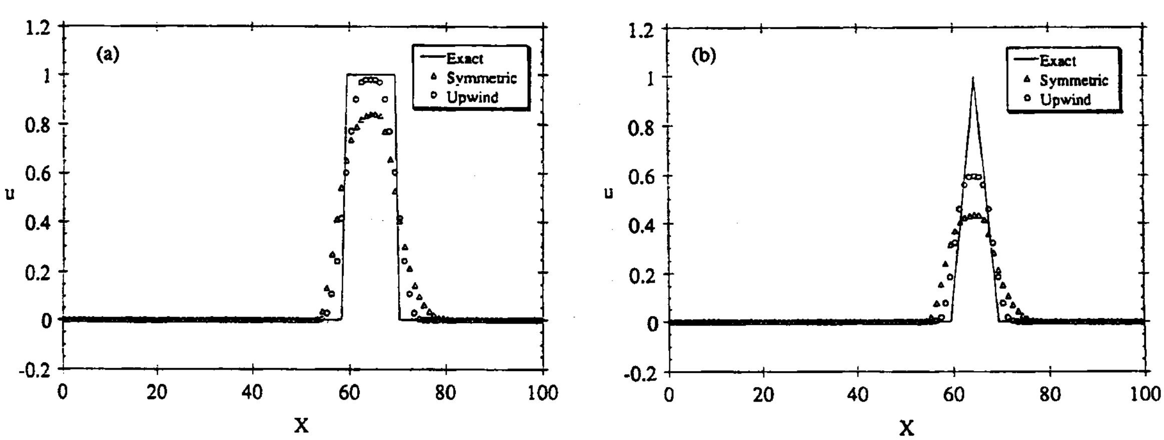

Figure 5. Solution of the scalar wave equation by both methods for two test problems. In both cases, the upwind method provides superior performance

Table 1. Order of accuracy in several norms for the schemes solving Burger's equation

Scheme	L_1	L_2	L_∞
Symmetric ($t = 0 \cdot 2$)	1·83	1·58	1·19
Upwind ($t = 0 \cdot 2$)	1·90	1·65	1·28
Symmetric ($t = 1 \cdot 0$)	1·48	1·19	0·78
Upwind ($t = 1 \cdot 0$)	1·41	1·14	0·74

as it is somewhat distorted. This distortion can be attributed to the use of anti-upwind data by the symmetric scheme's limiter. A notable feature of the upwind-biased scheme is that for the scalar wave equation the solution is identical to that obtained by the modified flux TVD scheme if the same limiters are used. This can be explained by the support of the limiter used and the resulting interpolation on the upwind side of each cell interface. For non-linear problems this will not hold.

In Table I the rates of convergence are given for Burger's equation. When the solution is smooth, the upwind method is evidently superior in every error norm. After a shock forms, the symmetric scheme is slightly more convergent; however, for all test cases (up to 1000 grid cells) the actual error is lower for the upwind scheme. In addition, as time progresses after $t = 1 \cdot 0$, the upwind scheme recovers its initially higher rate of convergence.

The solutions for the Euler equations echo the results with the previous three problems. Across the board, the resolution afforded by the upwind scheme is superior. The major flow structures–shocks rarefactions and contact discontinuities–are all noticeably better resolved with the upwind method. The results from Sod's problem demonstrate this to some degree. In Figure 6, each of the features is sharper with the upwind method. This is probably most noticeable at the contact discontinuity. In Figure 7, the noted behaviour for the contact discontinuity and shock are clearly shown. Also evident from this Figure are the symmetry problems exhibited by the symmetric scheme. The shape of the density peak is more consistent with the exact solution with the upwind-biased method.

The blast wave problem (see Figure 8) accentuates each of these issues. This is particularly true with respect to the right density peak, which is significantly closer to the converged

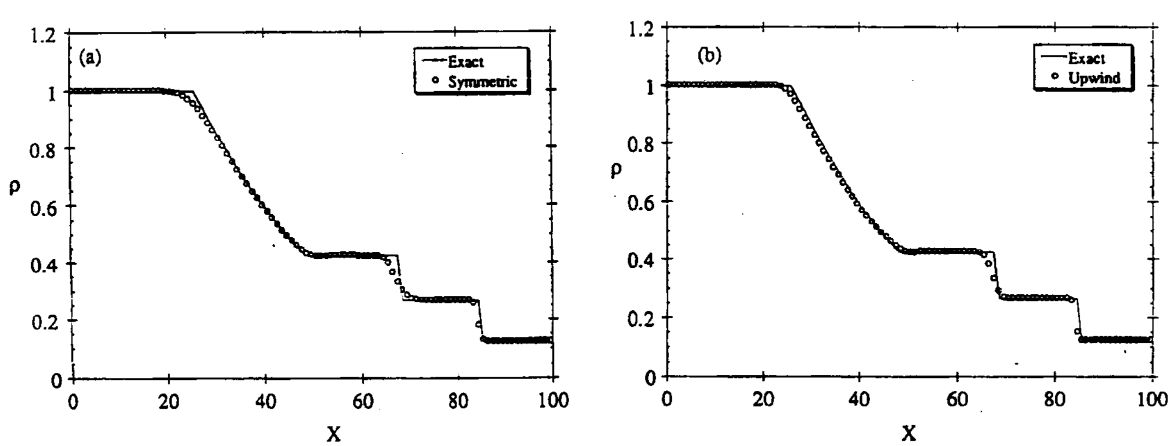

Figure 6. Solution to Sod's problem. Improved resolution is given by the upwind-biased scheme

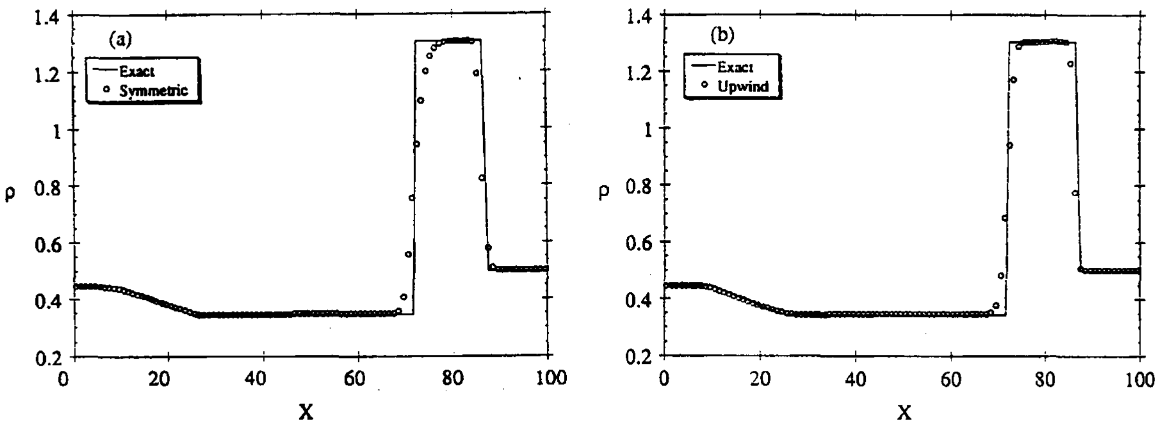

Figure 7. Solution to Lax's problem. Highlights the resolution of both shocks and contact discontinuities as well as the symmetry properties of the solution methods

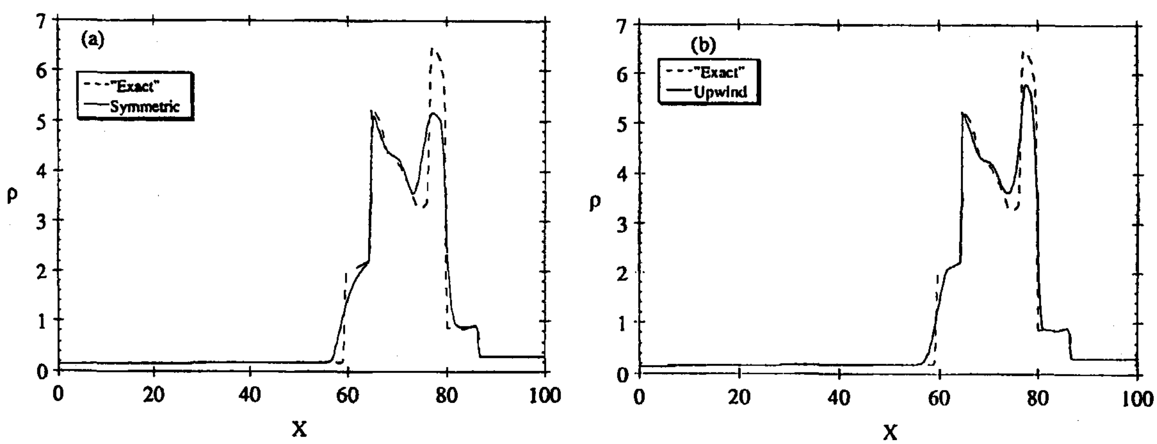

Figure 8. Blast wave problem. The deficiencies of both methods are most clearly shown. The difficulty of the problem is due to the large amount of structure confined to a small physical space

solution with the upwind method. Two other key features of the solution are the degree of 'fill-in' between the peaks and the contact discontinuity to the left of the left density peak. The fill-in regions are both smeared nearly equally, but the shape of the upwind computed solution is better. The left-most contact discontinuity is much more smeared by the symmetric scheme.

The computer time used by the upwind-biased scheme is slightly greater, but the differences are not great enough to be an issue.

4. CONCLUSIONS

The results of Section 3 show conclusively that the upwind scheme produces results of higher resolution when compared with the symmetric scheme. This raises the issue of cause. These schemes are second-order-accurate when the solution is smooth. The limiters are based on minimum principles, and increasing their support will lower the value returned by the function.

The subsequent 'flattening' of the slope is akin to increasing the numerical viscosity of the scheme, thus lowering the accuracy.

Interpreted on a more physical basis, the upwind scheme takes data from a more physically meaningful location on the grid. The support for the limiter can be perceived to affect the solution at that point, whereas the symmetric limiters are centred by taking both upwind and anti-upwind data. Both arguments lead to a conclusion that, if resolution is of primary concern, the limiter should have as small a support as possible in order to limit its induced viscosity. This of course should be within the limitations of providing physically meaningful oscillation-free (or nearly so) results.

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