50 Solutions

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1.

Set the length of YC (the radius of the first circle) to be m, and the length of BX (the radius of the second circle) to be n. Because the two circles are tangent and the distance between the two centers is 9, m + n = 9. Additionally, using the big circle's radius, 5 + m = 6 + n. Therefore, m = 5 and n = 4. AX equals 6 + 4 = **10**.

2.

Because AC = CD, <CAD = <CDA. Set <CAD as x and <DAB as y. <CAB = x + y, and <ABC = 180 - <ACD - <CAB = 180- (180-2x) – (x+y) = 2x – x – y = x – y. Therefore, <CAB - <ABC = 2y = 30 deg. Y = 15 deg, so <DAB = y = **15 deg**.

3.

The diameter of the semicircle is 2, so the radius is 1. Set the midpoint of AB to be M. Drop a perpendicular line from M to EC and call the intersection F (FM is perpendicular to EC). FM equals the radius = 1. Set EA to be x, so ED is 2-x. Triangle EFM and triangle EAM are congruent due to HL symmetry (FM = AM, EM = EM, <EFM = <EAM = 90 deg). EF = x. Triangle FCM and triangle CMB are congruent due to HL symmetry (FM = FM, CM = CM, <CFM = <CBM = 90 deg). FC = 2. EC = FC + EF = 2 + x. ED = 2-x and DC = 2. Using the Pythagorean Theorem, x = 0.5, so CD = **2.5**.

4.

Set <AMD = <CMD = x. <ADM = 90 – x, so <MDC = x. <MCD also equals x, so MC = CD = 6. Therefore, triangle MBC is a 30-60-90 triangle, so <BMC = 30 deg, and x = **75 deg.**

5.

Triangle AEB and DFC are congruent via HL congruence. AE = FC as a result, so BF = ED, so triangle EDF and EBF are congruent via SSS congruence. Set the side of the square to be x. FC = sqrt (900 – x^2). BF = sqrt(30^2 – x^2) * 2, and FC + BF = x. Solving, we get x^2 = **810**.

6.

Quadrilateral EDBC is cyclic because <DEC and <CBD are 90 degrees. Therefore, the angles <DBE and <DCE are equal because they both encompass the same arc, DE. **Q.E.D**

7.

The points are positioned such so they form a equilateral triangle with one of the sides being extended to 2a. Assume the shortest side length is 1. The distance is sqrt(1.5^2 + (sqrt(3)/2)^2) = **sqrt(3).**

8.

ANLC and AMBL are cyclic. We will use the fact that <BAM and <BAC = 90 deg. <NLC is 90 degrees and <MLB = <MLC = 90 deg. Because <NLC and <MLC = 90 deg, L, M, and N are collinear. **Q.E.D**

9.

Set the tangent length from A to the incircle to be a, B to the incircle to be b, and the C to the incircle be c. The distance from x to the incircle is equal to a because A and X are the same distance from the incircle, and Y also follows. Therefore, XY is equal to 2a, which is equal to a + b + a + c – b- c = AB + AC – BC = **1186.**

10.

Arcs AB and AC are equal, as well as arcs AD and EC. This means that arcs DB and AE are equal, so <ABE = <BAD, so AD and BE are parallel. **Q.E.D**

11.

When a shape is stretched by x, the shape's perimeter increases by x, but it's area increases by x^2, because both height and width are increasing. Therefore, as a shape is dilated, the ratio between perimeter and area changes and can be changed to equal any positive number. Therefore, there must be a point where the ratio is 1. **Q.E.D**

12.

<BEA = 90 - <CEF = <CFE. Therefore, triangle AEB and ECF are similar via AA similarity. Set the side length of ABCD to be x. x / 4 = EC / 3, so x^2 + (x/4)^2 = 16, so x^2 = **256/17.**

13.

NXYM is cyclic from the perpendiculars. Additionally, note that ABMN are all on one circle. Therefore, <ABC = <AMN = <YMN = 180 - <NXY = <CXY. AB and XY are parallel. **Q.E.D**

14.

Triangles AEB and DCF are right triangles. Extend AE, EB, FC, and DF to form a square. The four side triangle are congruent, so the large square has a length = 12 + 5 = 17. EF is a diagonal, so it has length 17sqrt(2), so the answer is **578**.

15.

Set the intersection of AE and DF to be X. Set arc AB = a, arc BC = b, and arc CA = c. <AXF = 180 - <EAF - <AFD = 180 – b – c – a. <AXD = 180 - <DAE - <FDA = 180 – a – b – c. <AXF = <AXD = 90 deg. **Q.E.D**

16.

Triangle ABD has an area of 12, so the altitude from D to AB is 8. Therefore, the altitude from D to ABC is 8 * sin(30 deg) = 4, and the volume is 4 * 15/3 = **20.**

17.

In this problem, the quirk of a / sinA = 2R = D for any triangle will be used (law of Sines).

XP1^2 + XP2^2 + XP3^2 + XP4^2 = P1P2^2 + P3P4^2. P1P2 = D * sin(<P1P3P2), while P3P4 = D * sin(<P3P2P4). sin(<P3P2P4) = sin(90 – P1P3P2) = cos(<P1P3P2). Using the substitution and the identity sinx ^ 2 + cosx ^ 2 = 1, we get D^2. **Q.E.D**

18.

The center of the two circles, the midpoint of AB, and the center of the circle is a rectangle. (If this is non intuitive, think of symmetry and the fact that the planes are perpendicular).

The center of the sphere is O, while the center of the horizontal circle is C, the center of the vertical plane is D, and the center of AB is X.

The distance from C and D to X equals sqrt(54^2 – 21^2) and sqrt(66^2 – 21^2) respectively. The reason is because the C/D, X and A/B are a right triangle. Once this is found, all that is left is the find the distance between O and X, which is the square root of the sum of the squares of the previously computed two distances. The final right triangle is the triangle with side lengths of R, the distance between A and X (21), and the distance between O and X. The answer is **6831**.

19.

Extend AB and CD to intersect at X. <AXD is 90 deg from A = 37 deg and D = 53 deg. M is the circumcenter of BXC, and N is the circumcenter of AXD. XM = BM = MC = 500, while XN = AN = 1004. Their difference is MN is **504**.

20.

Triangles CEA and CFB are congruent via symmetry. Set the intersection of FB and EA to be X. <FXE = <AXB, and because of symmetry, XB / XF = XA / XE. This means that triangles XFE and XAB are similar, so FE is paralle to AB.

CFBD is a cyclic quadrilateral. <CFD = <CBD = <CAB = <CFE. Therefore, F, E, D are collinear. **Q.E.D.**

21.

It can be proven that angle formed by the center of two circles is 60 degrees (use congruent SSS triangles and symmetry). Set the radius of the tangent circles to be r. Of one of those 60 deg triangles, one side is r+1, one side is 4-r, and one side is 2r (tangency). Therefore, using Law of Cosines gives us **126.**

22.

<CBD = <CAB. Draw the center of the circle, and call it O. <BOC = 2<BAC. Therefore, <OCB = 90 - <BAC = 90 - <CBD. <OBD = <OBC + <CBD = <OCB + <CBD = 90. **Q.E.D**

23.

Triangle BNQ and QBC are congruent due to AAS, and also is triangles ACP and MAP. This means that in the triangle QNC, <QNC = <QCN. Set <A = x and <B = y. <AQN = 90-x, so <NQC = 90 + x. Therefore, <NCQ = (90-x)/2. Similarily, <MCB = (90-y)/2. Remember that x + y = 90. Therefore, <MCN = 90 – (180 – 90)/2 = **45.**

24.

Triangle AND and AMB are similar via AA similarity. <CAB = <ACD. Because of right angles, AMCN is cyclic, so <AMN = <ACN = <ACD = <CAB. Also due to the cyclic quadrilateral, <ANM = <ACM, and thus via AA similarity, MAN is similar to ABC. **Q.E.D**

25.

A: Define D as the base of the altitude from A. <HAB = <DAB = 90 - <B. <OAC = 90 - <AOC = 90 - <B (inscribed angles). **Q.E.D**

B: <HAO = <HAB + <OAC - <A = 180 – 2<B - <A = (<A + <B + <C) – 2<B = <C - <B. Note that depending on which angle is larger, an absolute value sign is needed, so the correct result would be |<B-<C|. **Q.E.D**

26.

This problem was written with an error. APD and BPC must be collinear for the proof to be possible. <APC = <BPD, and side length ratios prove SAS similarity. **Q.E.D**

27.

Define A as the intersection of XY and ZO. Triangles XAZ and AOY are similar due to vertical and inscribed angles, which gives us AO/XA = AY/AZ = r/13. XAO and ZAY are also similar, so r/7 = XA/ZA = AO/AY.

XA = 13AO/r = r*AZ/7. This means that AO/AZ = r^2/91. AO + AZ = 11. AO = 11 – AZ. This means that 11/AZ – 1 = r^2/91. Additionally, from the equations, we can derive that AZ * 20r/91 = XY. Using Ptolemy's theorem, XY * 11 = 20r. Then, AZ * 20r/91 = 20r/11. AZ = 91/11. **r = sqrt(30).**

28.

Set the intersection of GM and EK to be X. <CDA+ <CBA = 180 deg due to cyclic. Therefore, <CDA = 180-<CBA = <KBA. Because <BKF = <EKD, triangle FBK and EKD are similar. This means <DEF = <BFK = <EFA, so EM = FM. Therefore, triangle EGM and GFM are congruent via SAS congruence. This means that EG = GF. Similarly, FH = HE.

Also, this means that triangle EMX and XMF are congruent due to SAS congruence. This means <EXM = <FXM = 90 deg. Repeating this, we can determine that all interior angles of X inside EGFH are 90 degrees. Therefore, we can use SAS to prove that EG = EH. Repeating this, we can prove that EGFH is a rhombus. **Q.E.D**

29.

Law of cosines on triangle ABC gives us 225 + 196 – 420cos C = 169. cosC = 252/420 = 3/5. This means that BM^2 = BM^2 + BC^2 – 2 * BM * BC * 3/5. 49 + 225 – 210 * 3/5 = 148, so BM = 2 sqrt(37).

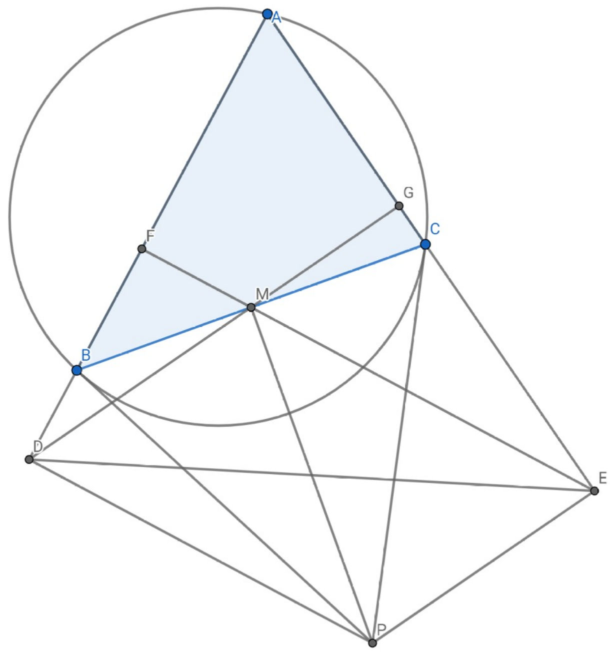
Set AP = X, PC = Y, PM = Z. X^2 + Y^2 = 196. X^2 = Z^2 + 49 – 14Z cos<PMA. Y^2 = Z^2 + 49 – 14Zcos<PMC. X^2 + Y^2 = 196 = 2Z^2 + 98 + 0 (because cos(180-x) + cos(x) = 0, so cos<PMC = -cos<PMA). Z = 7.

Triangle APC area = PM/BM * 84 (area of ABC) = 7/(2sqrt(37)) * 84 = 294/sqrt(37)= 294sqrt(37)/37. **368.**

30.

Set M as the midpoint of BC. Draw lines from D and E through M so they intersection AE and AD respectively, and mark the intersection points G and F. ADPE is a cyclic quadrilateral from right angles. <CBP = <BCP = <A. <DPE = 180 - <A due to cyclicity. <PMC = 90 deg because triangles BMP and MCP are congruent (SSS). Therefore, quadrilaterals DBMP and MCEP are both cyclic. <MEP = <MCP = <A. Since <DPE = 180-A, EF and DP are parallel.

Therefore, <DFE = 360 – 90 - 180 = 90 deg. Repeat to prove that <DGE is 90 deg, so M is the orthocenter of ABC. **Q.E.D**



31.

<HH_bH_c = <HAH_c because the quadrilateral is cyclic (90 degrees). <Hh_bH_c = <HAH_c = 90 – <AHH_c = 90 - <H_aHC = <HCB = <HH_bH_a (cyclic quadrilateral). Repeat to prove that H is the incenter. **Q.E.D**

32.

We will use power of the point. 86 is the radius of the circle. Using power of the point on C gives us (97 - 86)* (97 + 86) = CX * CB = 11 * 183. CX clearly is smaller than 86 + 97, so 183 is out. CX has to be 33 then, so BC = **61.**

33.

BO = OC, and BT = TC, so triangle BOT and OTC are congruent (SSS). Therefore, <BTO = <OTC, so <BTN = <CTN. Therefore, triangle BTN is congruent to traingle CTN (SAS). Arc BN = arc CN, so.

<A = <BOC / 2 = arc BNC / 4 = arc BN/2 = <BON.

<A = arc BNC/2 – arc PQ/2 (external angle theorem). <BON = A, so <BOC = 2A, so arc BNC = 4A. Therefore, arc PQ/2 = <A, so <PNQ = <A. Additionaly, <BPN = <NQC, so <APN = <AQN, and APNQ is a parallelogram. **Q.E.D**

34.

Symmetry tells us that all small triangles are equal in this diagram.

The leg length of each of the small triangles is equal to x. 2x + xsqrt(2) = 1, so x = 1 – sqrt(2)/2. This means that the area of one of the small triangles is equal to ¾ – sqrt(2)/2. The shaded area is part of a sector with center E, and the sector has an area of 1/8 * ½ * pi = 1/16 pi. Finally, the sector also has a quadrilateral which can be broken into two triangles. They both have heights of ½, and their bases are both xsqrt(2)/2.

The shaded area is equal to pi/16 - (¾ – sqrt(2)/2) –(sqrt(2) – 1)/4 = pi/16 – ½ + sqrt(2)/4. The final area is 4 * shaded area + 4 * small triangle area + 1 = **pi/4 + 2 – sqrt(2).**

35.

Make the center of AB be O. Triangles OMS and OMT are congruent via SSS congruence, so MOPS is cyclic because <SMO = <SPO = 90 degrees. <SPM = <SOM. SO is obviously constant, and OM is constant because SM is constant and SMO is a right triangle. Therefore, the sine value of SOM is constant. Clearly, <SOM is less than 90 degrees (or else SMO couldn't be a triangle), so <SOM must be constant. **Q.E.D**

36.

<A = <BDA_1 because of tangent inscribed arc. Similarily, <C = <BDC_1. <BDA_1 + <BDC_1 = 180 - <B= <C_1DA_1. Therefore, C_1DA_1B is cyclic, and < BDC_1 = <C = C_1A_1B. Therefore, A_1C_1 is parallel to AC. **Q.E.D**

37.

AD = 12 from area. From right triangles and Pythagorean triples, we determine that CD = 9 and DB = 5. Again using area, DE = 36/5. Triangles ECD and EAD are similar via AA similarity, so CE / ED = ED / AE = 9/12 = ¾. Because CED is a right triangle, CE^2 + (4/3)^2 * CE^2 = 81. Therefore, 25/9 CE^2 = 81 and CE = 27/5, which means that AE = 48/5. Triangles AED and AFB are similar because AFDB is cyclic (AA similarity). (48/5)/AF = (36/5) / BF = 12 / 13. Therefore, AF = 52/5. Using Pythagorean triples, we can find that EF = 4. Therefore, FD = 16/5. **21.**

38.

Law of sines tells us A/sinA = B/sinB = C/sinC = 2R. We need to minimize AXB's radius squared, and AXC's radius squared. Set <AXB = x. We need to minimize: (5/sin(x))^2 + (6/sin(180-x))^2 = 25/sin(x)^2 + 36/sin(x)^2 = 61/sin(x)^2. Clearly, this occurs when sin(x) = 1, so x = 90 deg. Therefore, <AXB = 90 deg.

Using Heron's formula, we find that ABC has an area of 6 sqrt(6), so AX = 12sqrt(6)/7. Using the Pythagorean theorem on ABX, we find that BX = **19/7.**

39.

KBLD is cyclic because of right angles. <BDL = <BKL. Also, <BDL = <BDC = <BAC. Therefore, <BAC = <BKL, so ABFK is cyclic. Because <A = 90 deg, BFK = 90 deg. **Q.E.D**

40.

Power of the point on C tells us CD * CA = CE * CB. Set AD = X, BE = Y. 2X * 3X = 3Y * 4Y. X^2 = 2Y^2.

Additionally, triangle ADB tells us X^2 + DB^2 = 900. Triangle CDB tells us 4X^2 + DB^2 = 16Y^2. .Solving, we get X = 6sqrt(5), Y = 3sqrt(10), DB = 12sqrt(5). Area is AC * DB/2 = 3X * DB/2 = 18sqrt(5) * 12sqrt(5)/2 = **540.**

41.

NOTE: The question should be <DAP and <BCP. <APD + <BPC = 180. Draw two duplicates of the rectangle on the left and right side of ABCD. Shifted points to the left will have one ‘, while shifted points to the right have two ‘’. P’APD is cyclic. Similarily, PBP’’C is cyclic. <DAP = <PP’D, while <BCP = <BP’’P. Triangle PP’D is congruent to PDC via SSS congruence. Similarily, triangle BP’’P is congruent to PAB. So, <PP’D = <PCD and <BP’’P = <PAB. Therefore, <PAD = <PCD and <PCB = <PAB. <PAD + <PAB = 90 deg, so <PAD + <PBC = **90 deg**.

42.

<Set the intersection of the circumcircles of RCQ and PQB to be X. <RXQ = 180 - <C and <PXB = 180- <B. Therefore, <RXP = <C + <B = 180 - <A. This means that APXR is cyclic, so the Miquel Point exists. **Q.E.D**

43.

QC = BQ = BC = 1. Using law of Sines on triangle PAF gives us 1/ sin(75) = AP / sin(45) = FP/ sin(60). Therefore, AP = sqrt(3) – 1 and FP = (3sqrt(2) – sqrt(6))/2. Since <RCF = 60 deg, riangle RFC is similar to RPQ. This gives us RF/(RF + FP) = FC/PQ = RC/RQ. RF / (RF + (3sqrt(2) – sqrt(6))/2 = 2 / (1 + sqrt(3) = RC / (RC + 1). Then, RP = sqrt(6)/2 + 3sqrt(2)/2. RQ = 2 + sqrt(3). Area = ½ * sin(60) * RQ * PQ = sqrt(3) / 4 * (2 + sqrt(3) * (sqrt(3) + 1) = (9+5sqrt(3))/4. **21**.

44.

AI bisects <BAC, so AI extended hits M (midpoint of arc BC). Therefore, AIM is a straight line. BM = CM because of equal arc inscribed. <ICM = <BCM + <ICB = arc MC/2 + <ICA = <CAM + <ICA = 180 - <AIC = <CIM. IM = CM = BM. M is circumcenter. **Q.E.D**

45.

<CAT = <TAB = 30 deg. <CTA = 75 deg, <ACT = 75 deg, so AC = AT = 24. Using law of Sines on triangle ABC gives AB / sin(75) = 24 / sin(45). AB = 12 + 12sqrt(3).

(NOTE: If you don’t know sin75, try solving via law of Cosines and law of Sines. Use law of Cosines on angle A to produce an expression with AB and AC, and use law of Cosines on angle B to produce another expression with AB and BC, and solve AB. It’s difficult but will work).

Area of ABC = ½ * AC * AB * sin(A) = ½ * 24 * (12sqrt(3) + 12) * sqrt(3)/2 = 216 + 72sqrt(3). **291**.

46.

BADE is cyclic, so <BDE = <BAE. <OMB = <OMC = 90 deg because O is the circumcenter and M is the midpoint of BC. <BME = <MED + <MDE = <BOE (BOME is cyclic). Extend AE to the circle, X. <BOE = <BOX = 2 * <BAE = 2*<BDE. <MDE + <MED = 2<MDE, so <MED = <MDE, so MD = ME. **Q.E.D**

47.

Draw the perpendicular lines from D and F respective to BC and AB, respectively. The intersection is O, the circumcenter of ABC. OB = circumradius of ABC. NODB is cyclic, and ND is a diameter because it is opposite of a 90 degree angle. OB is also a diameter, so ND = OB = circumradius. **Q.E.D**

48.

BP and AC are parallel because of 90 degree angles. Set the intersection of the perpendicular bisector of P with AC to be X. MX is parallel to AB because of midpoints. Set the intersection of MX and BP to be N. BP is parallel to AC and AB is parallel to NX so AXNB is a parallelogram. BN = AX = XC and BP is parallel to AC so BNCX is also a parallelogram.

This means that BX is parallel to NC, and BX = NC. Because M is the midpoint of BC, it is also the midpoint of XN (bisected diagonals of a parallelogram). <NPX = 90 deg, and M is the midpoint of XN, so M is the circumcenter of PXN and PM = MX. Finally, <QBP = <A. <QPB = 180 - <BPM = 90 - <XPM = 90 - <XMP = <MXC = <A, so <QBP = <QPB. **Q.E.D**

49.

If ABC is equilateral ZY = YX = ZX and UV = UW = VW by symmetry. <ZAU = 30 deg. <ZIU = 60 deg and ZI = IU, so <ZUI = 60 deg and <ZUA = 120 deg. This means that <UZA = 30 deg. Repeat to prove that <BZV = 30 deg, so <UZV = 120 deg.

<UZV = 120 deg, so <ZVU = 30 deg = <ZAU. Similarily, <AUV = <AZV, so AZVU is a parallelogram. This means that AZ = UV. AZ = ½ AB from symmetry, so YZ = 1/2AB, so UV = YZ. Repeat to prove that the two triangles are congruent. (Note that there are many ways to use symmetry to solve this part)

If UVW and ZYX are congruent then ZY = VW, VU = YX, and UW = ZX. Both triangles are concyclic. <UWV = <UIV/2 = 180 - <UIW/2 - <VIW/2. <XZY = ½ * <YIX = ½ * (180 - <C) = 90 - <C/2. <UWV = <XZY, so 180 - <UIW/2 - <VIW/2 = 90 - <C/2. 90 = <UIW/2 + <VIW/2 - <C/2. = 180 - <AIB/2 - <C/2. 90 = 3<C/2, so <C = 60 deg. Repeat to prove that ABC is equilateral. **Q.E.D**

50.

Set the center of the incircle to be I. <DBI = 1/2<B. <DIB + <DBI = <D = <B, so <DIB = <DBI and DI = DB. Therefore, the perimeter of ADE is 43, so DE = 43/63 * 20 = 860/63. **923**