

## 50 Solutions

1.

Set the length of YC (the radius of the first circle) to be  $m$ , and the length of BX (the radius of the second circle) to be  $n$ . Because the two circles are tangent and the distance between the two centers is 9,  $m + n = 9$ . Additionally, using the big circle's radius,  $5 + m = 6 + n$ . Therefore,  $m = 5$  and  $n = 4$ . AX equals  $6 + 4 = \mathbf{10}$ .

2.

Because  $AC = CD$ ,  $\angle CAD = \angle CDA$ . Set  $\angle CAD$  as  $x$  and  $\angle DAB$  as  $y$ .  $\angle CAB = x + y$ , and  $\angle ABC = 180 - \angle ACD - \angle CAB = 180 - (180 - 2x) - (x + y) = 2x - x - y = x - y$ . Therefore,  $\angle CAB - \angle ABC = 2y = 30$  deg.  $y = 15$  deg, so  $\angle DAB = y = \mathbf{15}$  deg.

3.

The diameter of the semicircle is 2, so the radius is 1. Set the midpoint of AB to be M. Drop a perpendicular line from M to EC and call the intersection F (FM is perpendicular to EC). FM equals the radius = 1. Set EA to be  $x$ , so ED is  $2 - x$ . Triangle EFM and triangle EAM are congruent due to HL symmetry (FM = AM, EM = EM,  $\angle EFM = \angle EAM = 90$  deg). EF =  $x$ . Triangle FCM and triangle CMB are congruent due to HL symmetry (FM = FM, CM = CM,  $\angle CFM = \angle CBM = 90$  deg). FC = 2. EC = FC + EF =  $2 + x$ . ED =  $2 - x$  and DC = 2. Using the Pythagorean Theorem,  $x = 0.5$ , so CD = **2.5**.

4.

Set  $\angle AMD = \angle CMD = x$ .  $\angle ADM = 90 - x$ , so  $\angle MDC = x$ .  $\angle MCD$  also equals  $x$ , so MC = CD = 6. Therefore, triangle MBC is a 30-60-90 triangle, so  $\angle BMC = 30$  deg, and  $x = \mathbf{75}$  deg.

5.

Triangle AEB and DFC are congruent via HL congruence. AE = FC as a result, so BF = ED, so triangle EDF and EBF are congruent via SSS congruence. Set the side of the square to be  $x$ . FC =  $\sqrt{900 - x^2}$ . BF =  $\sqrt{30^2 - x^2} * 2$ , and FC + BF =  $x$ . Solving, we get  $x^2 = \mathbf{810}$ .

6.

Quadrilateral EDBC is cyclic because  $\angle DEC$  and  $\angle CBD$  are 90 degrees. Therefore, the angles  $\angle DBE$  and  $\angle DCE$  are equal because they both encompass the same arc, DE. **Q.E.D**

7.

The points are positioned such so they form an equilateral triangle with one of the sides being extended to 2a. Assume the shortest side length is 1. The distance is  $\sqrt{1.5^2 + (\sqrt{3}/2)^2} = \mathbf{\sqrt{3}}$ .

8.

ANLC and AMBL are cyclic. We will use the fact that  $\angle BAM$  and  $\angle BAC = 90$  deg.  $\angle NLC$  is 90 degrees and  $\angle MLB = \angle MLC = 90$  deg. Because  $\angle NLC$  and  $\angle MLC = 90$  deg, L, M, and N are collinear. **Q.E.D**

9.

Set the tangent length from A to the incircle to be  $a$ , B to the incircle to be  $b$ , and the C to the incircle to be  $c$ . The distance from  $x$  to the incircle is equal to  $a$  because A and X are the same distance from the incircle, and Y also follows. Therefore, XY is equal to  $2a$ , which is equal to  $a + b + a + c - b - c = AB + AC - BC = \mathbf{1186}$ .

10.

Arcs AB and AC are equal, as well as arcs AD and EC. This means that arcs DB and AE are equal, so  $\angle ABE = \angle BAD$ , so AD and BE are parallel. **Q.E.D**

11.

When a shape is stretched by  $x$ , the shape's perimeter increases by  $x$ , but its area increases by  $x^2$ , because both height and width are increasing. Therefore, as a shape is dilated, the ratio between perimeter and area changes and can be changed to equal any positive number. Therefore, there must be a point where the ratio is 1. **Q.E.D**

12.

$\angle BEA = 90 - \angle CEF = \angle CFE$ . Therefore, triangle AEB and ECF are similar via AA similarity. Set the side length of ABCD to be  $x$ .  $x / 4 = EC / 3$ , so  $x^2 + (x/4)^2 = 16$ , so  $x^2 = \mathbf{256/17}$ .

13.

NXYM is cyclic from the perpendiculars. Additionally, note that ABMN are all on one circle. Therefore,  $\angle ABC = \angle AMN = \angle YMN = 180 - \angle NXY = \angle CXY$ . AB and XY are parallel. **Q.E.D**

14.

Triangles AEB and DCF are right triangles. Extend AE, EB, FC, and DF to form a square. The four side triangle are congruent, so the large square has a length  $= 12 + 5 = 17$ . EF is a diagonal, so it has length  $17\sqrt{2}$ , so the answer is **578**.

15.

Set the intersection of AE and DF to be X. Set arc AB =  $a$ , arc BC =  $b$ , and arc CA =  $c$ .  $\angle AXF = 180 - \angle EAF - \angle AFD = 180 - b - c - a$ .  $\angle AXD = 180 - \angle DAE - \angle FDA = 180 - a - b - c$ .  $\angle AXF = \angle AXD = 90$  deg. **Q.E.D**

16.

Triangle ABD has an area of 12, so the altitude from D to AB is 8. Therefore, the altitude from D to ABC is  $8 * \sin(30 \text{ deg}) = 4$ , and the volume is  $4 * 15/3 = \mathbf{20}$ .

17.

In this problem, the quirk of  $a / \sin A = 2R = D$  for any triangle will be used (law of Sines).

$XP1^2 + XP2^2 + XP3^2 + XP4^2 = P1P2^2 + P3P4^2$ .  $P1P2 = D * \sin(\angle P1P3P2)$ , while  $P3P4 = D * \sin(\angle P3P2P4)$ .  $\sin(\angle P3P2P4) = \sin(90 - \angle P1P3P2) = \cos(\angle P1P3P2)$ . Using the substitution and the identity  $\sin^2 x + \cos^2 x = 1$ , we get  $D^2$ . **Q.E.D**

18.

The center of the two circles, the midpoint of AB, and the center of the circle is a rectangle. (If this is non intuitive, think of symmetry and the fact that the planes are perpendicular).

The center of the sphere is O, while the center of the horizontal circle is C, the center of the vertical plane is D, and the center of AB is X.

The distance from C and D to X equals  $\sqrt{54^2 - 21^2}$  and  $\sqrt{66^2 - 21^2}$  respectively. The reason is because the C/D, X and A/B are a right triangle. Once this is found, all that is left is the find the distance between O and X, which is the square root of the sum of the squares of the previously computed two distances. The final right triangle is the triangle with side lengths of R, the distance between A and X (21), and the distance between O and X. The answer is **6831**.

19.

Extend AB and CD to intersect at X.  $\angle AXD$  is 90 deg from  $A = 37$  deg and  $D = 53$  deg. M is the circumcenter of BXC, and N is the circumcenter of AXD.  $XM = BM = MC = 500$ , while  $XN = AN = 1004$ . Their difference is MN is **504**.

20.

Triangles CEA and CFB are congruent via symmetry. Set the intersection of FB and EA to be X.  $\angle FXE = \angle AXB$ , and because of symmetry,  $XB / XF = XA / XE$ . This means that triangles XFE and XAB are similar, so FE is parallel to AB.

CFBD is a cyclic quadrilateral.  $\angle CFD = \angle CBD = \angle CAB = \angle CFE$ . Therefore, F, E, D are collinear. **Q.E.D.**

21.

It can be proven that angle formed by the center of two circles is 60 degrees (use congruent SSS triangles and symmetry). Set the radius of the tangent circles to be r. Of one of those 60 deg triangles, one side is  $r+1$ , one side is  $4-r$ , and one side is  $2r$  (tangency). Therefore, using Law of Cosines gives us **126**.

22.

$\angle CBD = \angle CAB$ . Draw the center of the circle, and call it O.  $\angle BOC = 2\angle BAC$ . Therefore,  $\angle OCB = 90 - \angle BAC = 90 - \angle CBD$ .  $\angle OBD = \angle OBC + \angle CBD = \angle OCB + \angle CBD = 90$ . **Q.E.D**

23.

Triangle BNQ and QBC are congruent due to AAS, and also is triangles ACP and MAP. This means that in the triangle QNC,  $\angle QNC = \angle QCN$ . Set  $\angle A = x$  and  $\angle B = y$ .  $\angle AQN = 90-x$ , so  $\angle NQC = 90 + x$ . Therefore,  $\angle NCQ = (90-x)/2$ . Similarly,  $\angle MCB = (90-y)/2$ . Remember that  $x + y = 90$ . Therefore,  $\angle MCN = 90 - (180 - 90)/2 = \mathbf{45}$ .

24.

Triangle AND and AMB are similar via AA similarity.  $\angle CAB = \angle ACD$ . Because of right angles, AMCN is cyclic, so  $\angle AMN = \angle ACN = \angle ACD = \angle CAB$ . Also due to the cyclic quadrilateral,  $\angle ANM = \angle ACM$ , and thus via AA similarity, MAN is similar to ABC. **Q.E.D**

25.

A: Define D as the base of the altitude from A.  $\angle HAB = \angle DAB = 90^\circ - \angle B$ .  $\angle OAC = 90^\circ - \angle AOC = 90^\circ - \angle B$  (inscribed angles). **Q.E.D**

B:  $\angle HAO = \angle HAB + \angle OAC - \angle A = 180^\circ - 2\angle B - \angle A = (\angle A + \angle B + \angle C) - 2\angle B = \angle C - \angle B$ . Note that depending on which angle is larger, an absolute value sign is needed, so the correct result would be  $|\angle B - \angle C|$ . **Q.E.D**

26.

This problem was written with an error. APD and BPC must be collinear for the proof to be possible.  $\angle APC = \angle BPD$ , and side length ratios prove SAS similarity. **Q.E.D**

27.

Define A as the intersection of XY and ZO. Triangles XAZ and AOY are similar due to vertical and inscribed angles, which gives us  $AO/XA = AY/AZ = r/13$ . XAO and ZAY are also similar, so  $r/7 = XA/ZA = AO/AY$ .

$XA = 13AO/r = r \cdot AZ/7$ . This means that  $AO/AZ = r^2/91$ .  $AO + AZ = 11$ .  $AO = 11 - AZ$ . This means that  $11/AZ - 1 = r^2/91$ . Additionally, from the equations, we can derive that  $AZ \cdot 20r/91 = XY$ . Using Ptolemy's theorem,  $XY \cdot 11 = 20r$ . Then,  $AZ \cdot 20r/91 = 20r/11$ .  $AZ = 91/11$ .  **$r = \sqrt{30}$** .

28.

Set the intersection of GM and EK to be X.  $\angle CDA + \angle CBA = 180^\circ$  due to cyclic. Therefore,  $\angle CDA = 180^\circ - \angle CBA = \angle KBA$ . Because  $\angle BKF = \angle EKD$ , triangle FBK and EKD are similar. This means  $\angle DEF = \angle BFK = \angle EFA$ , so  $EM = FM$ . Therefore, triangle EGM and GFM are congruent via SAS congruence. This means that  $EG = GF$ . Similarly,  $FH = HE$ .

Also, this means that triangle EMX and XMF are congruent due to SAS congruence. This means  $\angle EXM = \angle FXM = 90^\circ$ . Repeating this, we can determine that all interior angles of X inside EGFH are 90 degrees. Therefore, we can use SAS to prove that  $EG = EH$ . Repeating this, we can prove that EGFH is a rhombus. **Q.E.D**

29.

Law of cosines on triangle ABC gives us  $225 + 196 - 420\cos C = 169$ .  $\cos C = 252/420 = 3/5$ . This means that  $BM^2 = BM^2 + BC^2 - 2 \cdot BM \cdot BC \cdot 3/5$ .  $49 + 225 - 210 \cdot 3/5 = 148$ , so  $BM = 2\sqrt{37}$ .

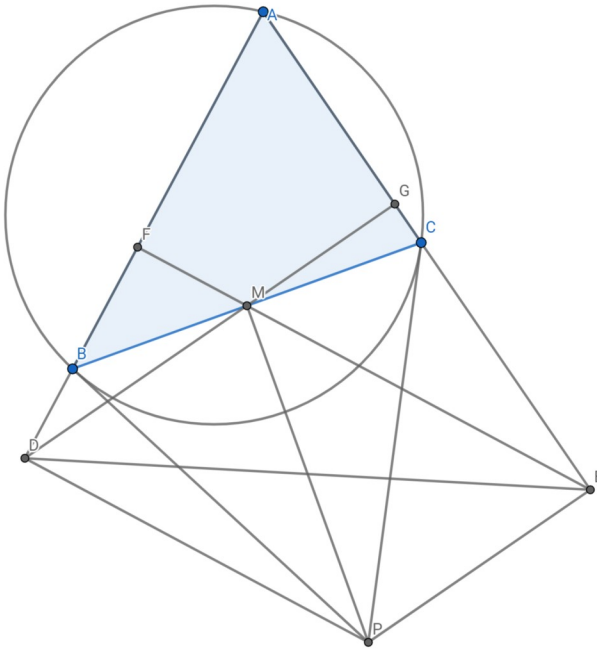
Set  $AP = X$ ,  $PC = Y$ ,  $PM = Z$ .  $X^2 + Y^2 = 196$ .  $X^2 = Z^2 + 49 - 14Z \cos \angle PMA$ .  $Y^2 = Z^2 + 49 - 14Z \cos \angle PMC$ .  $X^2 + Y^2 = 196 = 2Z^2 + 98 + 0$  (because  $\cos(180^\circ - x) + \cos(x) = 0$ , so  $\cos \angle PMC = -\cos \angle PMA$ ).  $Z = 7$ .

Triangle APC area =  $PM/BM \cdot 84$  (area of ABC) =  $7/(2\sqrt{37}) \cdot 84 = 294/\sqrt{37} = 294\sqrt{37}/37$ . **368**.

30.

Set M as the midpoint of BC. Draw lines from D and E through M so they intersect AE and AD respectively, and mark the intersection points G and F. ADPE is a cyclic quadrilateral from right angles.  $\angle CBP = \angle BCP = \angle A$ .  $\angle DPE = 180^\circ - \angle A$  due to cyclicity.  $\angle PMC = 90^\circ$  because triangles BMP and MCP are congruent (SSS). Therefore, quadrilaterals DBMP and MCEP are both cyclic.  $\angle MEP = \angle MCP = \angle A$ . Since  $\angle DPE = 180^\circ - \angle A$ , EF and DP are parallel.

Therefore,  $\angle DFE = 360 - 90 - 180 = 90$  deg. Repeat to prove that  $\angle DGE$  is 90 deg, so M is the orthocenter of ABC. **Q.E.D**



31.

$\angle HH_bH_c = \angle HAH_c$  because the quadrilateral is cyclic (90 degrees).  $\angle Hh_bH_c = \angle HAH_c = 90 - \angle AHH_c = 90 - \angle H_aHC = \angle HCB = \angle HH_bH_a$  (cyclic quadrilateral). Repeat to prove that H is the incenter. **Q.E.D**

32.

We will use power of the point. 86 is the radius of the circle. Using power of the point on C gives us  $(97 - 86) * (97 + 86) = CX * CB = 11 * 183$ . CX clearly is smaller than  $86 + 97$ , so 183 is out. CX has to be 33 then, so  $BC = 61$ .

33.

$BO = OC$ , and  $BT = TC$ , so triangle BOT and OTC are congruent (SSS). Therefore,  $\angle BTO = \angle OTC$ , so  $\angle BTN = \angle CTN$ . Therefore, triangle BTN is congruent to triangle CTN (SAS). Arc BN = arc CN, so.

$$\angle A = \angle BOC / 2 = \text{arc BNC} / 4 = \text{arc BN} / 2 = \angle BON.$$

$\angle A = \text{arc BNC} / 2 - \text{arc PQ} / 2$  (external angle theorem).  $\angle BON = A$ , so  $\angle BOC = 2A$ , so arc BNC =  $4A$ . Therefore,  $\text{arc PQ} / 2 = \angle A$ , so  $\angle PNQ = \angle A$ . Additionally,  $\angle BPN = \angle NQC$ , so  $\angle APN = \angle AQN$ , and APNQ is a parallelogram. **Q.E.D**

34.

Symmetry tells us that all small triangles are equal in this diagram.

The leg length of each of the small triangles is equal to  $x$ .  $2x + x\sqrt{2} = 1$ , so  $x = 1 - \sqrt{2}/2$ . This means that the area of one of the small triangles is equal to  $\frac{3}{4} - \sqrt{2}/2$ . The shaded area is part of a sector with center E, and the sector has an area of  $1/8 * \frac{1}{2} * \pi = 1/16 \pi$ . Finally, the sector also

has a quadrilateral which can be broken into two triangles. They both have heights of  $\frac{1}{2}$ , and their bases are both  $x\sqrt{2}/2$ .

The shaded area is equal to  $\pi/16 - (\frac{3}{4} - \sqrt{2}/2) - (\sqrt{2} - 1)/4 = \pi/16 - \frac{1}{2} + \sqrt{2}/4$ . The final area is  $4 * \text{shaded area} + 4 * \text{small triangle area} + 1 = \pi/4 + 2 - \sqrt{2}$ .

35.

Make the center of AB be O. Triangles OMS and OMT are congruent via SSS congruence, so MOPS is cyclic because  $\angle SMO = \angle SPO = 90$  degrees.  $\angle SPM = \angle SOM$ . SO is obviously constant, and OM is constant because SM is constant and SMO is a right triangle. Therefore, the sine value of SOM is constant. Clearly,  $\angle SOM$  is less than 90 degrees (or else SMO couldn't be a triangle), so  $\angle SOM$  must be constant. **Q.E.D**

36.

$\angle A = \angle BDA_1$  because of tangent inscribed arc. Similarly,  $\angle C = \angle BDC_1$ .  $\angle BDA_1 + \angle BDC_1 = 180 - \angle B = \angle C_1DA_1$ . Therefore,  $C_1DA_1B$  is cyclic, and  $\angle BDC_1 = \angle C = \angle C_1A_1B$ . Therefore,  $A_1C_1$  is parallel to AC. **Q.E.D**

37.

AD = 12 from area. From right triangles and Pythagorean triples, we determine that CD = 9 and DB = 5. Again using area, DE = 36/5. Triangles ECD and EAD are similar via AA similarity, so  $CE / ED = ED / AE = 9/12 = \frac{3}{4}$ . Because CED is a right triangle,  $CE^2 + (4/3)^2 * CE^2 = 81$ . Therefore,  $25/9 CE^2 = 81$  and  $CE = 27/5$ , which means that  $AE = 48/5$ . Triangles AED and AFB are similar because AFDB is cyclic (AA similarity).  $(48/5)/AF = (36/5) / BF = 12 / 13$ . Therefore,  $AF = 52/5$ . Using Pythagorean triples, we can find that EF = 4. Therefore, FD = 16/5. **21.**

38.

Law of sines tells us  $A/\sin A = B/\sin B = C/\sin C = 2R$ . We need to minimize AXB's radius squared, and AXC's radius squared. Set  $\angle AXB = x$ . We need to minimize:  $(5/\sin(x))^2 + (6/\sin(180-x))^2 = 25/\sin(x)^2 + 36/\sin(x)^2 = 61/\sin(x)^2$ . Clearly, this occurs when  $\sin(x) = 1$ , so  $x = 90$  deg. Therefore,  $\angle AXB = 90$  deg.

Using Heron's formula, we find that ABC has an area of  $6\sqrt{6}$ , so  $AX = 12\sqrt{6}/7$ . Using the Pythagorean theorem on ABX, we find that  $BX = 19/7$ .

39.

KBLD is cyclic because of right angles.  $\angle BDL = \angle BKL$ . Also,  $\angle BDL = \angle BDC = \angle BAC$ . Therefore,  $\angle BAC = \angle BKL$ , so ABFK is cyclic. Because  $\angle A = 90$  deg,  $\angle BFK = 90$  deg. **Q.E.D**

40.

Power of the point on C tells us  $CD * CA = CE * CB$ . Set AD = X, BE = Y.  $2X * 3X = 3Y * 4Y$ .  $X^2 = 2Y^2$ .

Additionally, triangle ADB tells us  $X^2 + DB^2 = 900$ . Triangle CDB tells us  $4X^2 + DB^2 = 16Y^2$ . Solving, we get  $X = 6\sqrt{5}$ ,  $Y = 3\sqrt{10}$ ,  $DB = 12\sqrt{5}$ . Area is  $AC * DB/2 = 3X * DB/2 = 18\sqrt{5} * 12\sqrt{5}/2 = 540$ .

41.

NOTE: The question should be  $\angle DAP$  and  $\angle BCP$ .  $\angle APD + \angle BPC = 180$ . Draw two duplicates of the rectangle on the left and right side of ABCD. Shifted points to the left will have one ' , while shifted points to the right have two ' '.  $P'APD$  is cyclic. Similarly,  $PBP''C$  is cyclic.  $\angle DAP = \angle PP'D$ , while  $\angle BCP = \angle BP''P$ . Triangle  $PP'D$  is congruent to  $PDC$  via SSS congruence. Similarly, triangle  $BP''P$  is congruent to  $PAB$ . So,  $\angle PP'D = \angle PCD$  and  $\angle BP''P = \angle PAB$ . Therefore,  $\angle PAD = \angle PCD$  and  $\angle PCB = \angle PAB$ .  $\angle PAD + \angle PAB = 90$  deg, so  $\angle PAD + \angle PBC = 90$  deg.

42.

$\angle$ Set the intersection of the circumcircles of  $RCQ$  and  $PQB$  to be  $X$ .  $\angle RXQ = 180 - \angle C$  and  $\angle PXB = 180 - \angle B$ . Therefore,  $\angle RXP = \angle C + \angle B = 180 - \angle A$ . This means that  $APXR$  is cyclic, so the Miquel Point exists. **Q.E.D**

43.

$QC = BQ = BC = 1$ . Using law of Sines on triangle  $PAF$  gives us  $1/\sin(75) = AP/\sin(45) = FP/\sin(60)$ . Therefore,  $AP = \sqrt{3} - 1$  and  $FP = (3\sqrt{2} - \sqrt{6})/2$ . Since  $\angle RCF = 60$  deg, triangle  $RFC$  is similar to  $RPQ$ . This gives us  $RF/(RF + FP) = FC/PQ = RC/RQ$ .  $RF / (RF + (3\sqrt{2} - \sqrt{6})/2) = 2 / (1 + \sqrt{3}) = RC / (RC + 1)$ . Then,  $RP = \sqrt{6}/2 + 3\sqrt{2}/2$ .  $RQ = 2 + \sqrt{3}$ . Area =  $\frac{1}{2} * \sin(60) * RQ * PQ = \sqrt{3} / 4 * (2 + \sqrt{3}) * (\sqrt{3} + 1) = (9 + 5\sqrt{3})/4$ . **21.**

44.

$AI$  bisects  $\angle BAC$ , so  $AI$  extended hits  $M$  (midpoint of arc  $BC$ ). Therefore,  $AIM$  is a straight line.  $BM = CM$  because of equal arc inscribed.  $\angle ICM = \angle BCM + \angle ICB = \text{arc } MC/2 + \angle ICA = \angle CAM + \angle ICA = 180 - \angle AIC = \angle CIM$ .  $IM = CM = BM$ .  $M$  is circumcenter. **Q.E.D**

45.

$\angle CAT = \angle TAB = 30$  deg.  $\angle CTA = 75$  deg,  $\angle ACT = 75$  deg, so  $AC = AT = 24$ . Using law of Sines on triangle  $ABC$  gives  $AB/\sin(75) = 24/\sin(45)$ .  $AB = 12 + 12\sqrt{3}$ .

(NOTE: If you don't know  $\sin 75$ , try solving via law of Cosines and law of Sines. Use law of Cosines on angle  $A$  to produce an expression with  $AB$  and  $AC$ , and use law of Cosines on angle  $B$  to produce another expression with  $AB$  and  $BC$ , and solve  $AB$ . It's difficult but will work).

Area of  $ABC = \frac{1}{2} * AC * AB * \sin(A) = \frac{1}{2} * 24 * (12\sqrt{3} + 12) * \sqrt{3}/2 = 216 + 72\sqrt{3}$ . **291.**

46.

$BADE$  is cyclic, so  $\angle BDE = \angle BAE$ .  $\angle OMB = \angle OMC = 90$  deg because  $O$  is the circumcenter and  $M$  is the midpoint of  $BC$ .  $\angle BME = \angle MED + \angle MDE = \angle BOE$  ( $BOME$  is cyclic). Extend  $AE$  to the circle,  $X$ .  $\angle BOE = \angle BOX = 2 * \angle BAE = 2 * \angle BDE$ .  $\angle MDE + \angle MED = 2\angle MDE$ , so  $\angle MED = \angle MDE$ , so  $MD = ME$ . **Q.E.D**

47.

Draw the perpendicular lines from  $D$  and  $F$  respective to  $BC$  and  $AB$ , respectively. The intersection is  $O$ , the circumcenter of  $ABC$ .  $OB = \text{circumradius of } ABC$ .  $NODB$  is cyclic, and  $ND$  is a diameter because it is opposite of a 90 degree angle.  $OB$  is also a diameter, so  $ND = OB = \text{circumradius}$ . **Q.E.D**

48.

BP and AC are parallel because of 90 degree angles. Set the intersection of the perpendicular bisector of P with AC to be X. MX is parallel to AB because of midpoints. Set the intersection of MX and BP to be N. BP is parallel to AC and AB is parallel to NX so AXNB is a parallelogram.  $BN = AX = XC$  and BP is parallel to AC so BNCX is also a parallelogram.

This means that BX is parallel to NC, and  $BX = NC$ . Because M is the midpoint of BC, it is also the midpoint of XN (bisected diagonals of a parallelogram).  $\angle NPX = 90$  deg, and M is the midpoint of XN, so M is the circumcenter of PXN and  $PM = MX$ . Finally,  $\angle QBP = \angle A$ .  $\angle QPB = 180 - \angle BPM = 90 - \angle XPM = 90 - \angle XMP = \angle MXC = \angle A$ , so  $\angle QBP = \angle QPB$ . **Q.E.D**

49.

If ABC is equilateral  $ZY = YX = ZX$  and  $UV = UW = VW$  by symmetry.  $\angle ZAU = 30$  deg.  $\angle ZIU = 60$  deg and  $ZI = IU$ , so  $\angle ZUI = 60$  deg and  $\angle ZUA = 120$  deg. This means that  $\angle UZA = 30$  deg. Repeat to prove that  $\angle BZV = 30$  deg, so  $\angle UZV = 120$  deg.

$\angle UZV = 120$  deg, so  $\angle ZVU = 30$  deg =  $\angle ZAU$ . Similarly,  $\angle AUV = \angle AZV$ , so AZVU is a parallelogram. This means that  $AZ = UV$ .  $AZ = \frac{1}{2} AB$  from symmetry, so  $YZ = \frac{1}{2} AB$ , so  $UV = YZ$ . Repeat to prove that the two triangles are congruent. (Note that there are many ways to use symmetry to solve this part)

If UVW and ZYX are congruent then  $ZY = VW$ ,  $VU = YX$ , and  $UW = ZX$ . Both triangles are concyclic.  $\angle UWV = \angle UIV/2 = 180 - \angle UIW/2 - \angle VIW/2$ .  $\angle XZY = \frac{1}{2} * \angle YIX = \frac{1}{2} * (180 - \angle C) = 90 - \angle C/2$ .  $\angle UWV = \angle XZY$ , so  $180 - \angle UIW/2 - \angle VIW/2 = 90 - \angle C/2$ .  $90 = \angle UIW/2 + \angle VIW/2 - \angle C/2 = 180 - \angle AIB/2 - \angle C/2$ .  $90 = 3\angle C/2$ , so  $\angle C = 60$  deg. Repeat to prove that ABC is equilateral. **Q.E.D**

50.

Set the center of the incircle to be I.  $\angle DBI = \frac{1}{2}\angle B$ .  $\angle DIB + \angle DBI = \angle D = \angle B$ , so  $\angle DIB = \angle DBI$  and  $DI = DB$ . Therefore, the perimeter of ADE is 43, so  $DE = \frac{43}{63} * 20 = \frac{860}{63}$ . **923**