

Unit 1. Set Theory

* Set of Natural Number

$$N = \{1, 2, 3, 4, \dots\}$$

* Set of integers

$$Z = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$$

$$Z^+ = \{1, 2, 3, 4, \dots\}$$

$$Z^- = \{-2, -1\}$$

* Non Negative ~~negative~~ integer = $\{0, 1, 2, 3, \dots\}$

* Set of rational number.

$$Q = \left\{ \frac{p}{q} \mid p \in Z, \text{ and } q \neq 0 \right\}$$

$$\text{eg: } p=2 \quad q=3 = \frac{2}{3}$$

* Set of real number

$$R = \text{Set of all rational and irrational numbers}$$

* Set of complex number

$$C = x+iy = a+ib$$

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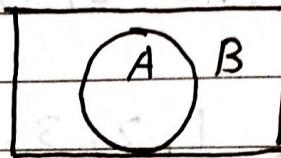
* Types of Sets.

* Empty set = ϕ = null set

* Subsets:

The set A is said to be subset of B if and only if every element of A is present in B.

$$A \subseteq B$$



$$A \subseteq B$$

B = super set of
 A

i) Every set is subset of itself

$$A \subseteq A$$

ii) Empty set is subset of any set.

* Size of set

Numbering of distinct elements present in the set.

Suppose S is the set

$$|S| = 2$$

Eg: S denotes the five int less than or equal to 10

$$|S| = 10$$

for empty set $|\emptyset| = 1$

* Power Set

Given set S , then power set of S is the set of all subsets of S .

$$\text{Ex: } S = \{0, 1, 2\}$$

subset of $S = \emptyset, S, \{0\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{1\}, \{2\}$.

$$P(S) = \left\{ \emptyset, \{1\}, \{2\}, \{0\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{0, 1, 2\} \right\}$$

$$|P(S)| = 8 = 2^3$$

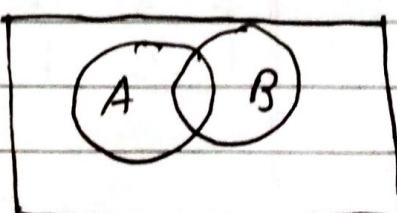
$$\text{General form} = 2^n$$

* Union

Let A & B be two sets then A union B is set of those elements that are either in A or in B or in Both.

$$\therefore \Rightarrow A \cup B$$

$$\text{ex: } A = \{1, 3, 4, 5\}\\ B = \{2, 4, 6, 7\}$$

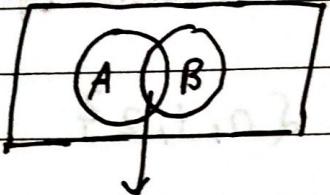


$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$

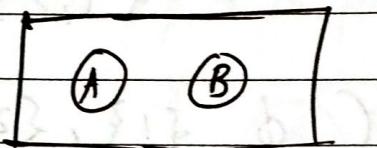
* Intersection

Let A, B be sets

$$A \cap B =$$



$$A \cap B =$$



$$= \emptyset$$

* Difference of sets

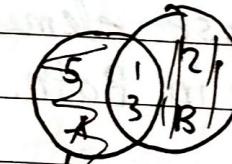
Let A, B be sets

$A - B$ = elements that are in A but not in B

$$A = \{1, 3, 5\} \Rightarrow A - B = \{5\}$$

$$B = \{1, 2, 3\}$$

$$A - B$$



$$B - A$$

$$A - B$$

Computer representation of sets.

Ex:- let $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

- i) What bit string represent the subset of all odd integer in V ?

$$A = \{1, 3, 5, 7, 9\}$$

1	2	3	4	5	6	7	8	9	10
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
(1)	0	1	0	1	0	1	0	1	0

Group of 4 from right

$$10 \quad 1010 \quad 1010$$

- ii) What bit string represent the subset of all even integer in V ?

$$B = \{2, 4, 6, 8, 10\}$$

1	2	3	4	5	6	7	8	9	10
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
0	1	0	1	0	1	0	1	0	1

$$01 \quad 0101 \quad 0101$$

Wise

iii) What bit string represent the subset of even integers in \mathbb{N}^2 not exceeding 5 in V^2 ?

$$C = \{1, 2, 3, 4, 5\}$$

1	2	3	4	5	6	7	8	9	10
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
1	1	1	1	1	0	0	0	0	0

$$\text{11 } 1110 \text{ 0000}$$

$$\begin{aligned} A \cup B &= (10 \ 1010 \ 1010) \cup (01 \ 0101 \ 0101) \\ &= (11 \ 1111 \ 1111) \end{aligned}$$

$$\begin{aligned} A \cap B &= (10 \ 1010 \ 1010) \cap (01 \ 0101 \ 0101) \\ &= (00 \ 0000 \ 0000) = \emptyset \end{aligned}$$

Product of set / cartesian Product

let A & B be the sets

$$A = \{a_1, a_2, \dots, a_n\} \quad |A| = n$$

$$B = \{b_1, b_2, \dots, b_m\} \quad |B| = m$$

Then cartesian product of two sets A and B is denoted by $A \times B$ and is defined as

$$A \times B = \{ (a_i, b_j) \mid a_i \in A, b_j \in B \}_{i=1,2,3 \dots n, j=1,2,3 \dots m}$$

$$\text{Ex: } A = \{1, 2, 3\}, B = \{4, 5, 6\}$$

$$A \times B = \{ (1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6) \}.$$

$$|A \times B| = 9$$

$$B \times A = \{ (4, 1), (5, 1), (6, 1), (4, 2), (5, 2), (6, 2), (4, 3), (5, 3), (6, 3) \}$$

$$|B \times A| = 9$$

$$A \times B \neq B \times A \text{ (elements not equal)}$$

* If $|A|=n$ & $|B|=m$ then $|A \times B| = |B \times A| = m \times n$

Covering of Sets.

Ex:- 1)

$$A = \{1, 2, 3, 4\}$$

$$A_1 = \{1, 2\} \quad A_2 = \{2, 3\} \quad A_3 = \{3, 4\}$$

$$A_4 = \{1, 2, 3, 4\}$$

$$A_1 \cup A_2 \cup A_3 \cup A_4 = A$$

\therefore The collection $\{A_1, A_2, A_3, A_4\}$ is called partition of A .

2)

$$A = \{1, 2, 3, 4\}$$

$$A_1 = \{1, 2\} \quad A_2 = \{3, 4\}$$

$$\Rightarrow A_1 \cap A_2 = \emptyset$$

$$A_1 \cup A_2 = A$$

Every partition is covering but not every covering is partition.

Relation

$$\text{Let } A = \{1, 2\} \quad B = \{3, 4, 5\}$$

$$A \times B = \{(1,3) (1,4) (1,5) (2,3) (2,4) (2,5)\}$$

$$P(A \times B) = 2^6$$

let A & B be sets A binary relation from A to B is a subset of $A \times B$.

Ex:-

$$A = \{1\} \quad B = \{2, 3\}$$

$$A \times B = \{(1, 2) (1, 3)\}$$

$$|A| = 1 \quad |B| = 2 \quad |A \times B| = 2$$

- i) $R_1 = \emptyset$
- ii) $R_2 = A \times B = \{(1, 2) (1, 3)\}$
- iii) $R_3 = \{(1, 2)\}$
- iv) $R_4 = \{(1, 3)\}$

$$\therefore P(A \times B) = 2^2 = 4.$$

Number of elements in the power set = $P(A \times B)$
 $= 2^{nm}$

\therefore Number of relation from A to $B = 2^{nm}$

let A & B non empty sets

Relation is subset of its cartesian product

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$$R \subseteq A \times B$$

$$|A|=m \quad |B|=n$$

if $A=B$ i.e $|A|=n$

$$A = \{1, 2, 3\} \quad B = \{1, 2, 3\}$$

$$R \subseteq A \times A$$

No. of relation from A to $A = 2^{n \times n} = 2^n$

Type of Relation

i) Reflexive Relation

A Relation R on set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

$$\text{Ex :- } A = \{1, 2, 3, 4\}$$

$$A = \{1, 2, 3, 4\}$$

$$A \times A = \left\{ \begin{array}{l} (1,1) \dots \dots (1,4) \\ (2,1) \dots \dots (2,4) \\ (3,1) \dots \dots (3,4) \\ (4,1) \dots \dots (4,4) \end{array} \right\}$$

* Same ordered pair should include them only reflexive.

$$1) R_1 = \{(1,1) (2,2) (3,3) (4,4)\}$$

$$(1,1) (2,2) (3,3) (4,4) \in R,$$

i.e. $(a,a) \in R_1$ for every element $a \in A$
 $\therefore R_1$ is Reflexive relation.

$$2) R_2 = \{(1,1) (2,2) (3,3) (4,4) (3,1) (4,1)\}$$

R_2 is reflexive.

$$3) R_3 = \{(1,1) (2,2) (3,3)\}$$

$$(4,4) \notin R_3$$

R_3 is not reflexive relation

$$4) R_4 = \{(1,1) (2,2) (3,3) (4,4) (5,5)\}$$

$$(5,5) \notin A \times A$$

It is not in cartesian product

$\therefore R_4$ is not a relation.

$$5) R_5 = \{A \times A\}$$

$\therefore R_5$ is a reflexive relation.

2) Symmetric relation

A relation R on set A is called symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$.

$$\text{Ex: } A = \{1, 2, 3, 4\}$$

$$R_1 = \{(1,2), (2,1), (3,1), (1,3)\}$$

R_1 is symmetric relation

$$\text{Ex: } R_2 = \{(1,1), (2,1), (1,2), (3,3), (1,3)\}$$

$(1,3) \in R_2$ But $(3,1) \notin R_2$

$\therefore R_2$ is not symmetric relation

3) Anti symmetric relation

A Relation R on set A is called antisymmetric Reln If $(a,b) \in R$ then $(b,a) \notin R$ for $b \neq a$.

$$\text{Ex: } A = \{1, 2, 3, 4\}$$

$$R_1 = \{(1,2), (1,3), (1,4)\}$$

But $(2,1) \notin R_1$

$(3,1) \notin R_1$

$(4,1) \notin R_1$

R_1 is antisymmetric relation

4) Transitive relation

A relation R on set A is called transitive If whenever is $(a,b) \in R$ and $(b,c) \in R$ Then $(a,c) \in R$ for all $a,b,c \in A$.

Ex: i) $A = \{1, 2, 3, 4\}$

$$\rightarrow R_1 = \{(1,2) (2,3) (1,3)\}$$

If $(1,2)$ and $(2,3) \in R_1$ Then $(1,3) \in R_1$,

$\therefore R_1$ is transitive relation

Ex: ii) $R_2 = \{(1,3) (3,1) (1,1) (3,2)\}$

\rightarrow

If $(1,3) \notin R_2$ & $(3,2) \in R_2$ Then $(1,2) \notin R_2$

$\therefore R_2$ is not transitive Relation.

Ex: iii) $R_3 = \{(1,4) (4,1) (1,1) (4,3) (1,3)\}$

\rightarrow

i) $(1,4) (4,1) \in R_3 \Rightarrow (1,1) \in R_3$

ii) $(1,4) (4,3) \in R_3 \Rightarrow (1,3) \in R_3$

iii) $(1,1) (1,3) \in R_3 \Rightarrow (1,3) \in R_3$

iv) $(4,1) (1,3) \in R_3 \Rightarrow (4,3) \in R_3$

v) $(4,1) (1,1) \in R_3 \Rightarrow (4,1) \in R_3$

$\therefore R_3$ is transitive relation.

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Ex: iv) $R_4 = A \times A$
 $\rightarrow R_4$ is transitive

Ex: v) $R_5 = \{(1,1) (2,2) (3,3) (4,4)\}$
 \rightarrow yes it is anti-sym.

* $A = \{1, 2, 3, 4\}$

i) $R_1 = \{(1,1) (2,2) (3,3) (4,4)\}$

Ref \rightarrow Yes it is reflexive

Sym \rightarrow Yes it is symmetric

Anti-Sym \rightarrow Yes it is antisymmetric

Transitive \rightarrow Yes it is transitive

ii) $R_2 = A \times A$

Ref \rightarrow ✓

Sym \rightarrow ✓

Anti \rightarrow ✗

Trans \rightarrow ✓

iii) $R_3 = \{(1,2) (1,3) (1,4) (3,3) (4,1)\}$

Ref \rightarrow ✗

Sym \rightarrow ✗

Anti \rightarrow ✓

Trans \rightarrow ✓

iv) $R_4 = \{(1,1) (1,2) (1,3) (2,1) (2,4)\}$

Ref \rightarrow ✗

Sym \rightarrow ✗

Anti \rightarrow ✗

Trans \rightarrow ✗

5) Equivalence Relation

A relation R on set A is called an equivalent relation If it is reflexive, symmetrical and transitive.

$$\text{Ex: } A = \{1, 2, 3, 4\}$$

$$R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$R_2 = A \times A$$

6) Partial order Relation

A relation R on set A is called partial order If it is reflexive, antisymmetric, transitive.

$$\text{Ex: } A = \{1, 2, 3, 4\}$$

$$R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$R_2 = A \times A$$

It is not partial order set.

7) Representing Relation:

i) Representing relation using matrices:

Suppose that R is relation from $A = \{a_1, a_2, \dots, a_n\}$ to $B = \{b_1, b_2, \dots, b_m\}$. Then Relation R can be represented by the matrix $M_R = [M_{ij}]_{m \times n}$

where M_{ij}

$$M_R = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & & & \\ m_{m1} & m_{m2} & \dots & m_{mn} \end{bmatrix}_{m \times n} \quad \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Ex: i)

$$A = \{1, 2, 3\} \quad B = \{4, 5\}$$

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$$

$$R = \{(1, 4), (2, 5), (3, 5)\}$$

$$M_R = \begin{bmatrix} M_{14} & M_{15} \\ M_{24} & M_{25} \\ M_{34} & M_{35} \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

2)

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

→

$$M_R = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix}_{4 \times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

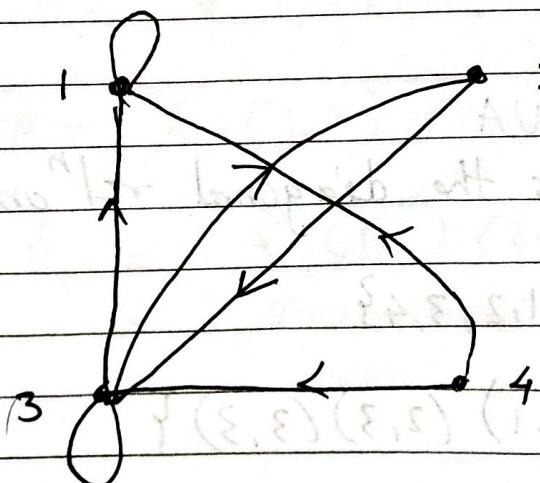
2) Representing Relation Using Diagrams

A directed graph or diagram, consists of set V of vertices together with set E of ordered pairs of elements of V called edges. The vertex a is called initial vertex of the edge (a,b) and the vertex b is called the terminal vertex of this edge an edge of the form (a,a) .

Ex: $A = \{1, 2, 3, 4\} \Rightarrow$ (v) Vertices set loop vertex

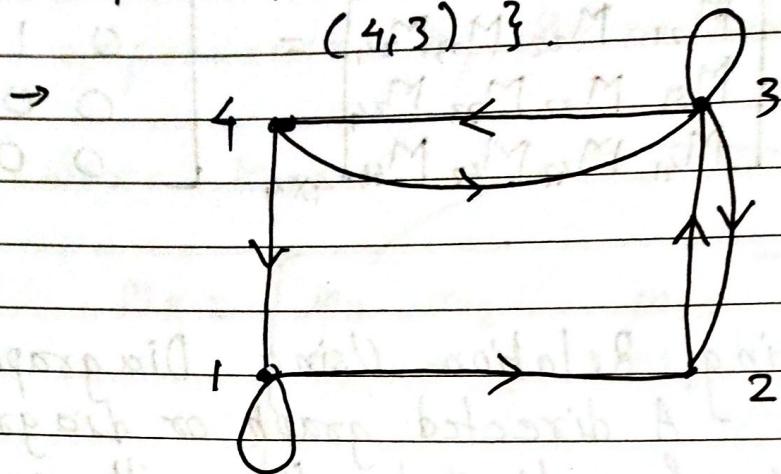
$R = \{(1,1), (2,3), (3,1), (3,2), (4,1), (4,3), (3,3)\}$
 $(E) \Rightarrow$ set of edges.

initial vertex terminal vertex



Ans

2) $A = \{1, 2, 3, 4\}$
 $R = \{(1,1), (2,3), (3,3), (4,1), (3,2), (3,4), (4,3)\}$



* Closure of relation.

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1,1), (2,2), (1,2), (3,2), (4,4)\}$$

$(3,3) \notin R$

$$\bar{R}_r = \{(1,1), (2,2), (1,2), (3,2), (4,4), (3,3)\}$$

↳ reflexive closure of relation R

$$R_\Delta = R \cup \Delta$$

where Δ is the diagonal reln on set A

$$\text{Ex:- } A = \{1, 2, 3, 4\}$$

$$R = \{(1,1), (2,3), (3,3)\}$$

$$\bar{R} = R \cup \Delta$$

where $\Delta = \{(1,1) (2,2) (3,3) (4,4)\}$

2) $A = \{1, 2, 3, 4\}$

$$R = \{(1,1) (1,2) (2,1) (3,2)\}$$

\rightarrow

$$(2,3) \notin R$$

$$\bar{R}_S = \{(1,1) (1,2) (2,1) (3,2) (2,3)\}$$

\hookrightarrow symmetric closure of R .

$$\bar{R}_S = R \cup R^{-1}$$

\hookrightarrow Inverse of $\text{rel}^n R$.

Ex: $A = \{1, 2, 3, 4\}$

$$R = \{(1,1) (2,3) (2,1) (2,4) (3,3) (3,2) (4,4)\}$$

$$\rightarrow R^{-1} = \{(1,1) (3,2) (1,2) (4,2) (3,3) (2,3) (4,4)\}$$

$$R_S = R \cup R^{-1}$$

$$= \{(1,1) (2,3) (3,2) (2,1) (1,2) (4,2) (2,4) (3,3) (4,4)\}$$

3) $A = \{1, 2, 3, 4\}$.

$$R = \{(1,2) (2,1)\}$$

$$\bar{R}_T = \{(1,2) (2,1) (1,1)\}$$

\hookrightarrow Transitive closure of R .

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Warshall Algorithm.

Find the transitive closure of relation R where

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$$

→

$$M_R = \begin{bmatrix} 0 & I_{(1,2)} & 0 & 0 \\ I_{(4,1)} & 0 & I_{(2,3)} & 0 \\ 0 & 0 & 0 & I_{(3,4)} \\ I_{(4,1)} & 0 & 0 & 0 \end{bmatrix}_{4 \times 4} \quad \therefore \text{Express given in matrix form.}$$

$$\omega_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \therefore \text{Write } M_R \text{ as it is.}$$

$$C_1 = \text{column}(1) = \{2, 4\} \quad R_1 = \text{Row}(1) = \{2\}$$

$$C_1 \times R_1 = \{(2, 2), (4, 2)\}$$

$$\omega_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1_{(2,2)} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1_{(4,2)} & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \hookrightarrow \text{Replace 1 on this place in } \omega_0 \\ \text{for } \omega_1, \text{ If already 1 then keep it.} \end{array}$$

$$C_2 = \{1, 2, 4\} \quad R_2 = \{1, 2, 3\}$$

$$C_2 \times R_2 = \{ (1,1) (1,2) (1,3) (2,1) (2,2) (2,3) \\ (4,1) (4,2) (4,3) \}.$$

$$\omega_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$C_3 = \{ 1, 2, 4 \} \quad R_3 = \{ 4 \}$$

$$C_3 \times R_3 = \{ (1,4) (2,4) (4,4) \}$$

$$\omega_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$C_4 = \{ 1, 2, 3, 4 \} \quad R_4 = \{ 1, 2, 3, 4 \}$$

$$C_4 \times R_4 = \left\{ \begin{array}{l} (1,1) (1,2) (1,3) (1,4) \\ (2,1) (2,2) (2,3) (2,4) \\ (3,1) (3,2) (3,3) (3,4) \\ (4,1) (4,2) (4,3) (4,4) \end{array} \right\}$$

$$= A \times A$$

$$\omega_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Wise

$$\bar{R}_1 = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{matrix} \right\} = A \times A$$

2) $A = \{1, 2, 3, 4\}$
 $R = \{(2,1) (2,3) (3,1) (3,4) (4,1) (4,3)\}$

$\Rightarrow M_R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4 \times 4} \rightarrow \text{Depend on } A$

$$\omega_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$C_1 = \{2, 3, 4\} \quad R_1 = \{\} = \emptyset$$

$$C_1 \times R_1 = \emptyset$$

$$\omega_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$C_2 = \{\emptyset\} = \emptyset \quad R_2 = \{1, 3\}$$

$$C_2 \times R_2 = \emptyset$$

$$\omega_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$C_3 = \{2, 4\}$$

$$R_3 = \{1, 4\}$$

$$C_3 \times R_3 = \{(2, 1) (2, 4) (4, 1) (4, 4)\}$$

$$\omega_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$C_4 = \{2, 3, 4\} \quad R_3 = \{1, 3, 4\}$$

$$C_4 \times R_3 = \{(2, 1) (2, 3) (2, 4) (3, 1) (3, 3) (3, 4) (4, 1) (4, 3) (4, 4)\}$$

$$\omega_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\bar{R}_x = \left\{ \begin{array}{l} \{(2, 1) (2, 3) (2, 4)\} \\ \{(3, 1) (3, 3) (3, 4)\} \\ \{(4, 1) (4, 3) (4, 4)\} \end{array} \right\}$$

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\omega_0 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_1 = \{1, 3\} \quad R_1 = \{1, 3\}$$

$$C_1 \times R_1 = \{(1,1), (1,3), (3,1), (3,3)\}$$

$$\omega_1 = \begin{bmatrix} 1_{(1,1)} & 0 & 1_{(1,3)} & 0 \\ 0 & 1 & 1 & 0 \\ 1_{(3,1)} & 0 & 1_{(3,3)} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$