PARTIAL DIFFERENTIAL EQUATIONS PROJECT

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This report is submitted as the fulfillment of the project of Electronics and Circuit Designing
In Semester III of B.Tech (IT & Mathematical Innovation)
at Cluster Innovation Center, University of Delhi

Abstract

Finding solutions for **Second Order Nonlinear Differential Equations** has been a tough challenge as traditional methods do not apply here. This paper concerns with the problems of solving this type of equations (e.g. second order nonlinear differential equation) using numerical methods. We study Hartree's two methods in order to develop a satisfactory method for evaluating the solutions to second order nonlinear equations which is the main goal of this paper.

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1 Introduction

1.1 Second-order linear equations

Partial differential equations arise frequently in formulating fundamental laws of nature and in the study of various fields of physics, chemistry and biology. Second order linear differential equations are of great significance in the study of various phenomena. The reason behind their importance is -

First, second-order linear equations arise in almost all variety of applications.

Second, the study of second-order linear equations is easy and simpler to understand than that of a first order equation.

The above reasons show us how important are second-order linear equations. Generally, a physical phenomena of real life is represented as u(x,y,z,t) which is a function of space variables x,y,z and time t.

1.2 Our Aim: Second-order nonlinear equations

In this paper, we are dealing with second-order nonlinear equations in the internal heat conduction. An example of such kind of equation which we will consider for the explaining the methods of evaluation is as follows -

$$\partial\Theta/\partial t = \partial^2\Theta/\partial x^2 - q^* \, \partial w/\partial t,$$

Where $\partial w/\partial t = -k^*w^*e\hat{}(-A/\Theta)$

The problem with these types of equations is that the presence of the nonlinear term renders the use of formal methods for solving these equations. These equations arise in problems of heat flow when there is an internal generation heat within the medium. Our aim in this paper is to study the 3 methods as described in [1] to find the numerical solution of nonlinear partial differential equations of second-order.

2 Methodology

Now we will study the 3 methods used to evaluate the numerical solutions of second-order nonlinear equations as discussed in 1.2 and apply them to the solve the below second order nonlinear equation -

$$\partial \Theta/\partial t = \partial^2\Theta/\partial x^2 - q^* \ \partial w/\partial t \qquad - \quad (1)$$

Where $\partial w/\partial t = -k*w*e\hat{}(-A/\Theta)$

Subject to boundary conditions -

Θ = constant at t=0 for 0x1
w = constant at t=0 for 0x1
$\partial\Theta/\partial x = H1()$ for t > 0, x=0
$\partial\Theta/\partial x = 0$ for t > 0, x=1 - (2)

2.1 Method 1: Replacing the time derivative

In this method the time derivative is replaced by a finite difference ratio and the resulting into a ordinary differential equation with x as independent variable is integrated numerically or mechanically. This process is repeated for each finite step in time, and a trial and error process of the solution is carried out to ensure that conditions at the two ends of range of x is satisfied.

So, according to the first method the time derivative is replaced by finite ratio by writing Θ(t) for temperature at time t, regarded as a function of x, and considering time interval δ (t), the derivative with respect to time may be written as

$$\partial\Theta/\partial t*(t+1/2\delta t) = \Theta\ (t+\delta t)- \Theta\ (t)/\ \partial t + O(\delta t)^2 \qquad - (3)$$

Now, we apply this method to equation (1) above which approximately becomes -

$$(1/\partial t)*[\ \Theta\ (t+\delta t) - \Theta\ (t)] = 1/2 * \partial\hat{}2[\Theta(t+\delta t)+\Theta(t)]\ /\ \partial x\hat{}2 - (q/\delta t)*[w(t+\delta t) - w(t)] \qquad - (4)$$

2.2 Method 2: Replacing the space derivative

In this method the range of x is divided into a finite number of intervals and space derivative of **theta** at each point is expressed as the **theta** at that point and the neighbouring points on each side. This replaces the differential equation approximately by a set of first-order equations in time whose solutions are obtained by using differential analyser, the integration proceeding in time.

In this method we are replacing the space derivative instead of the time derivative as a finite difference ratio which will reduce the equation (1) to a set of ordinary differential equations of first order. If Θ_{m-1}, Θ_m and Θ_{m+1} are temperatures at time t, at the points $x=(m-1)*\delta x$, $m*\delta x$ and $(m+1)*\delta x$ respectively, then

$$\Theta_{m+1} - 2\Theta_m + \Theta_{m-1} = (\delta x)^2 * (\partial^2 \Theta / \partial x^2)_m + 1/12 * (\delta x)^4 * (\partial^4 \Theta / \partial x^4) + \ldots \quad - (5)$$

Now, by applying this method to the equation (1) above will approximately reduce the equation to -

$$\partial \Theta_m / \partial t = (\Theta_{m-1} - 2 * \Theta_m + \Theta_{m+1}) / (\delta x)^2 - q * \partial w_m / \partial t \quad - (6)$$

2.3 Method 3: Replacing both derivatives by finite difference ratios

This method is the main focus of the paper [1] as mentioned above which discusses the numerical methods to solve this type of equations. In this method, both derivatives i.e., the space and time derivative are replaced by finite difference ratios and the solution proceeds by finite steps in time.

Now, we replace both the time and space derivatives by finite difference ratios which is the main objective of this project to solve second order nonlinear equations which has been developed by evaluating the solutions of equations in the above two methods. On replacing the derivatives with respect to both x and t with respect to a particular way of finite difference ratios, the solution will have to carry out numerically. The particular finite difference form of (1) used by the authors is given below [equation (7)] and is obtained by replacing space and time derivatives at the point $[m * \delta x, (n + 1/2) * \delta t]$ by the usual finite difference ratios.

$$\Theta_m(n+1) - \Theta_m(n) = (\delta t / 2*(\delta x)^2) * [\Theta_{m-1}(n+1) + \Theta_{m+1}(n+1) + \Theta_{m-1}(n) + \Theta_{m+1}(n) - 2*\{\Theta_m(n+1) + \Theta_m(n)\}] - q*[w_m(n+1) - w_m(n)] \quad - (7)$$

Here $\Theta_m(n)$ and $w_m(n)$ are the values at the point $(m*\delta x, n*\delta t)$, which will be referred as (m,n) in future.

Also,

$$w_m(n+1)/w_m(n) = e^{(2*E_m(n+1))} \quad - (8)$$

Where[e^2*E] is a function of mean temperature at $(m\delta t)$.

Now, let the range $0 \leq x \leq 1$ be divided into eight intervals so that $\delta x = 1/8$ and choose interval δt such that

$$\delta t/(\delta x)\hat{}2 = 1 \qquad - \quad (9)$$

On combining equations (7), (8), (9) and the initial boundary conditions (2), we get particularly simpler form of (7) as

$$\Theta_m(n+1) = 1/2*[\Theta_{m-1}(n+1) + \Theta_{m-1}(n) + \Theta_{m-1}(n+1) + \Theta_{m+1}(n)] - 1/2*q*[w_m(n+1) - w_m(n)] \qquad - \quad (10)$$

$$\Theta_0(n+1) = 1/2*[\Theta_1(n+1) + \Theta_1(n) + 1/8*H_1*[\Theta_0(n+1) + \Theta_0(n)] - 1/2*q*[w_0(n+1) - w_0(n)] \qquad - \quad (11)$$

$$\Theta_8(n+1) = 1/2*[\Theta_7(n+1) + \Theta_7(n)] - 1/2*q*[w_8(n+1) - w_8(n)] \qquad - \quad (12)$$

There are seven equations of type (10) for $1 \leq m \leq 7$.
Each equation of subsidiary equation of the form $[w_m(n+1)/w_m(n) = e\hat{}(2*E)]$ with it.

3. Results

GENERAL HEAT EQUATION

The heat equation is a parabolic partial differential equation that describes the distribution of heat or variation in temperature in a given region over time. For a function u(x,y,z,t) of three spatial variables x,y,z see cartesian coordinates and the time variable t,
the heat equation is:

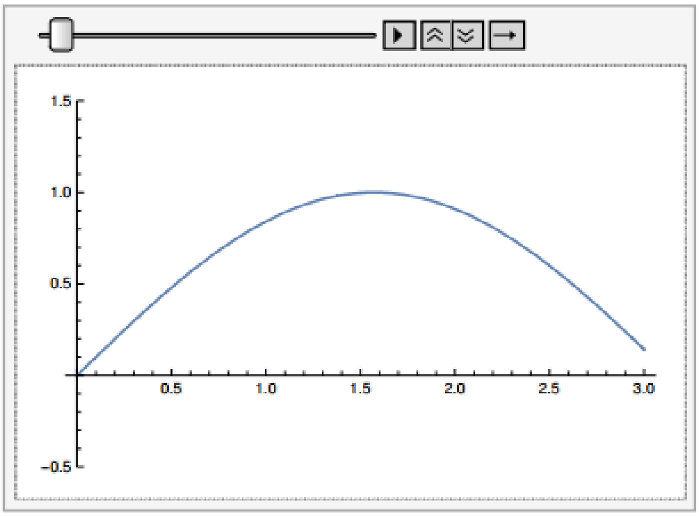
$$u^{(0,0,0,1)}[x,y,z,t] == u^{(2,0,0,0)}[x,y,z,t] + u^{(0,2,0,0)}[x,,y,z,t] + u^{(0,0,2,0)}[x,y,z,t]$$

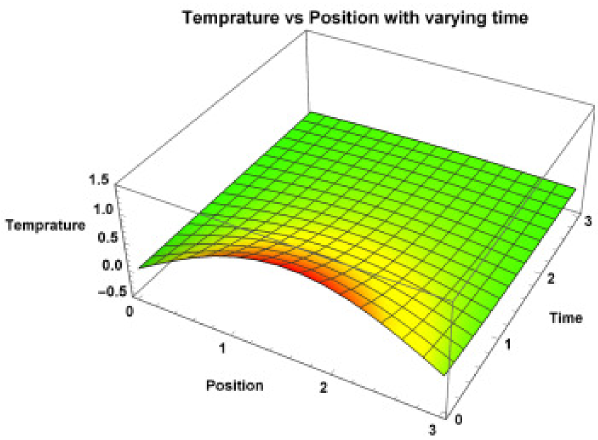
For the sake of simplicity we consider heat equation in only one dimension in this work. Therefore, the heat equation resuces to:

```
u(0,1)[x,t]==u(2,0)[x,t];

heat = D[u[x, t], t] == D[u[x, t], x, x];
cond1 = u[x, 0] == Sin[x];
cond2 = u[0, t] == 0;
cond3 = u[3, t] == 0;
```

On solving the above equation using **NDSOlve** and plotting it on mathematica it produces the given graph.

```
ss = u /. First@NDSolve[{heat, cond1, cond2, cond3}, u, {x, 0, 3}, {t, 0, 3}]

list = Table[Plot[ss[x, t], {x, 0, 3}, PlotRange → {-0.5, 1.5}], {t, 0, 1, 0.1}];
```





INTERNAL HEAT EQUATION

In addition to general heat equation, internal heat factor must also be included into consideration. Suppose that a body obeys the heat equation and, in addition, generates its own heat per unit volume e.g.,(in watts/litre-W/L) at a rate given by a known function q varying in space and time. Then the heat per unit volume u satisfies an equation:

$$u^{(0,1)}[x,t] == \frac{\rho\ Q[x,t]}{c} + u^{(2,0)}[x,t]$$

here,

c specific heat of the material;

P Density of the material;

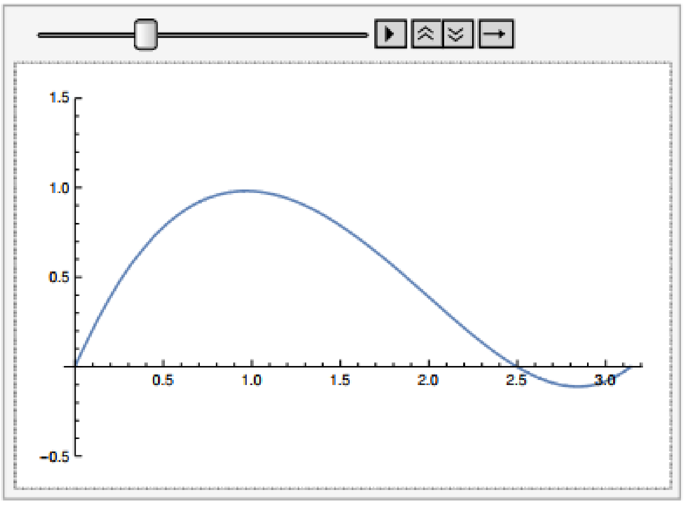
For simplification, in this work we assumed that specific heat and density of a material are constant throughout the experiment.

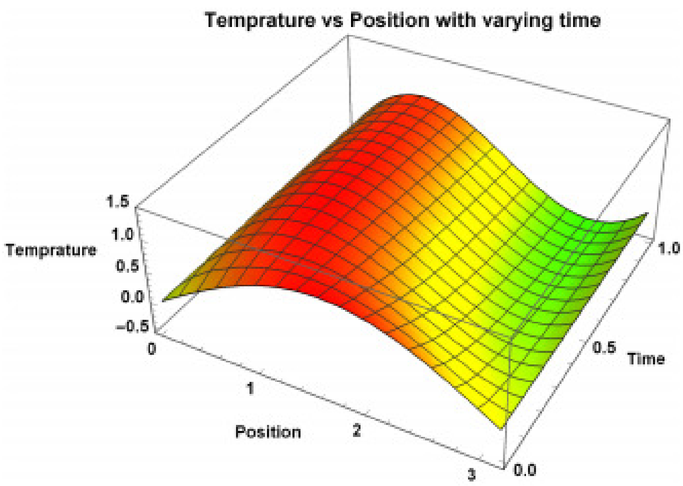
c = 1;

P = 3;

Case 1: Assuming Q[x] as a Trigonometric Function

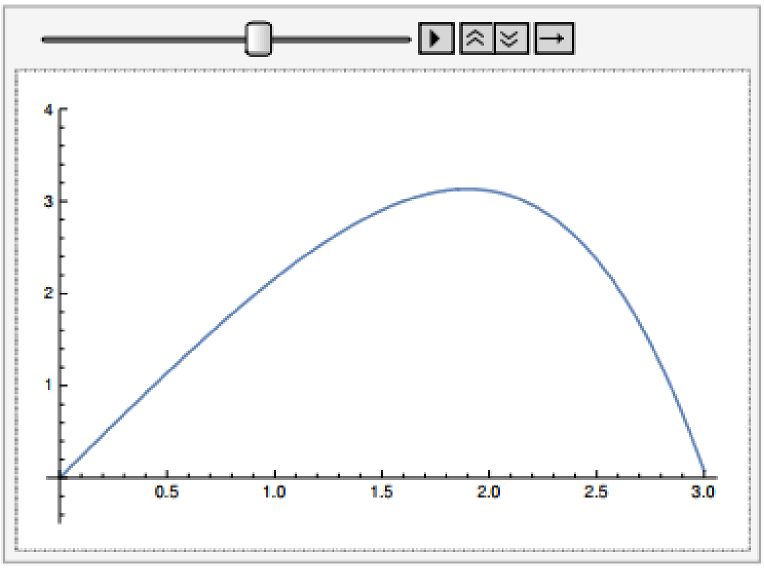
Now assuming Q[x] to be a Trigonometric Function (e.g. Cos[x]) and solving the above internal heat equation using NDSolve it produces these graphs:

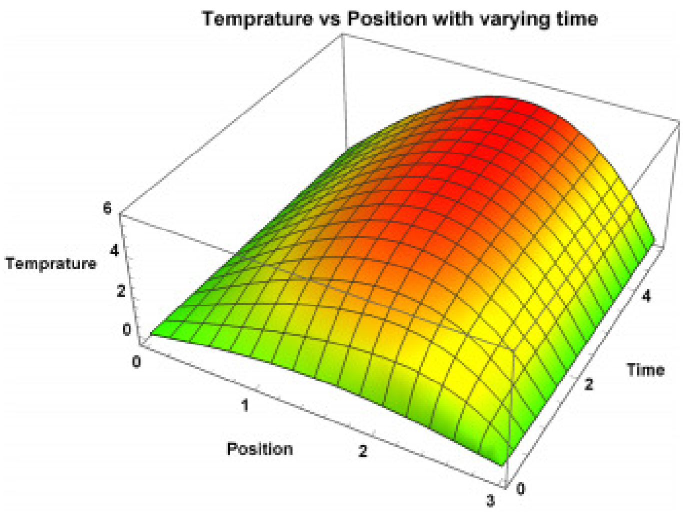




Case 2: Assuming Q[x] as a Linear function

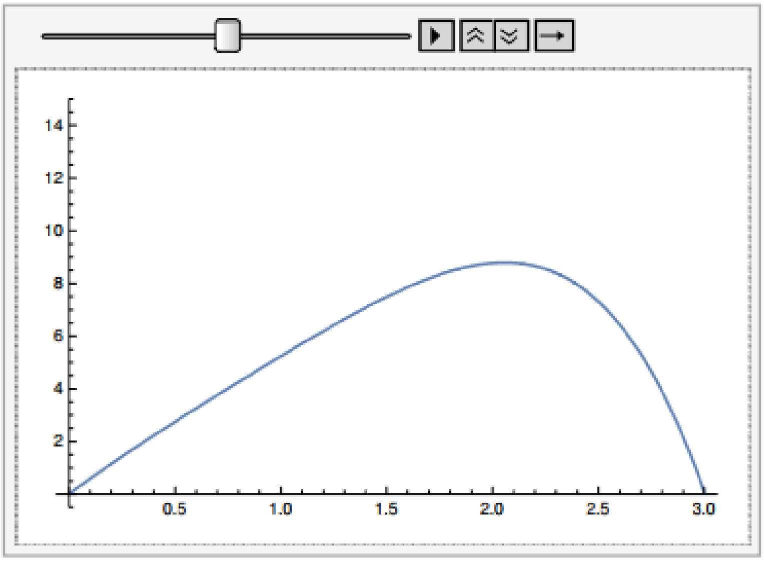
Now assuming Q[x] to be a Linear function (e.g. x) and solving the above internal heat equation using NDSolve it produces these graphs:

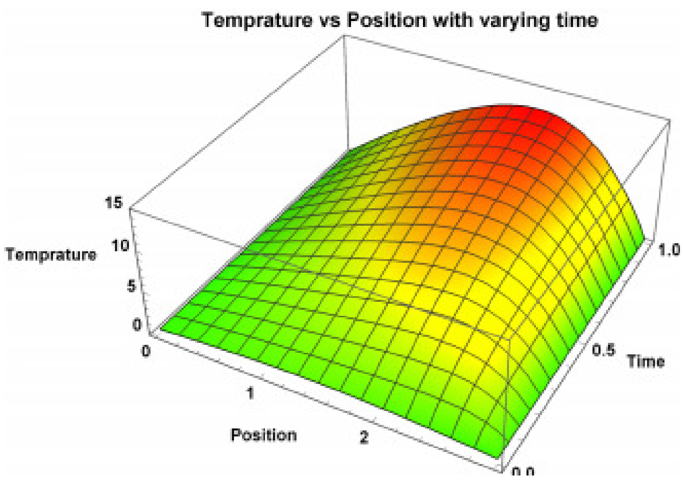




Case 3: Assuming Q[x] as a Polynomial function

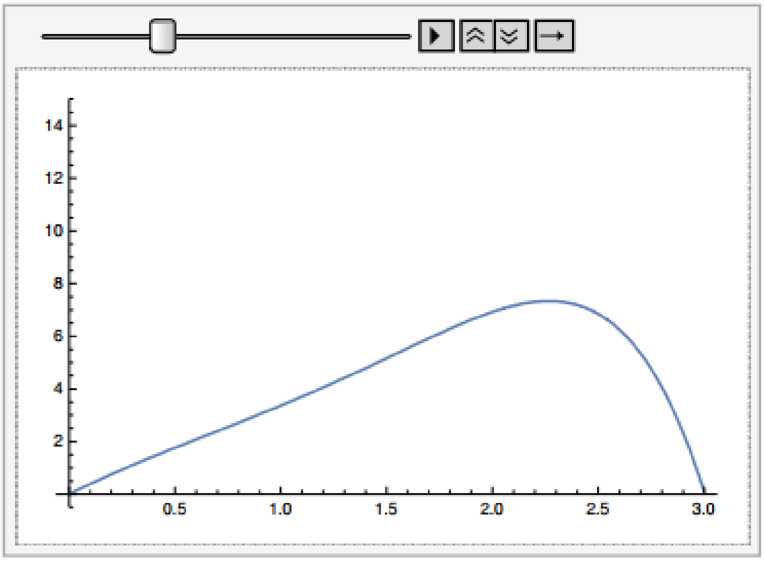
Now assuming Q[x] to be a Polynomial function (e.g. $\square^2 + \square + 1$) and solving the above internal heat equation using NDSolve it produces these graphs:

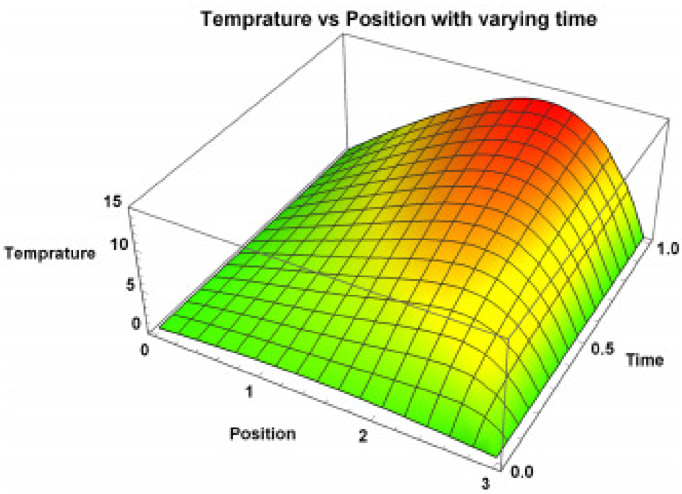




Case 4: Assuming Q[x] as a Exponential function

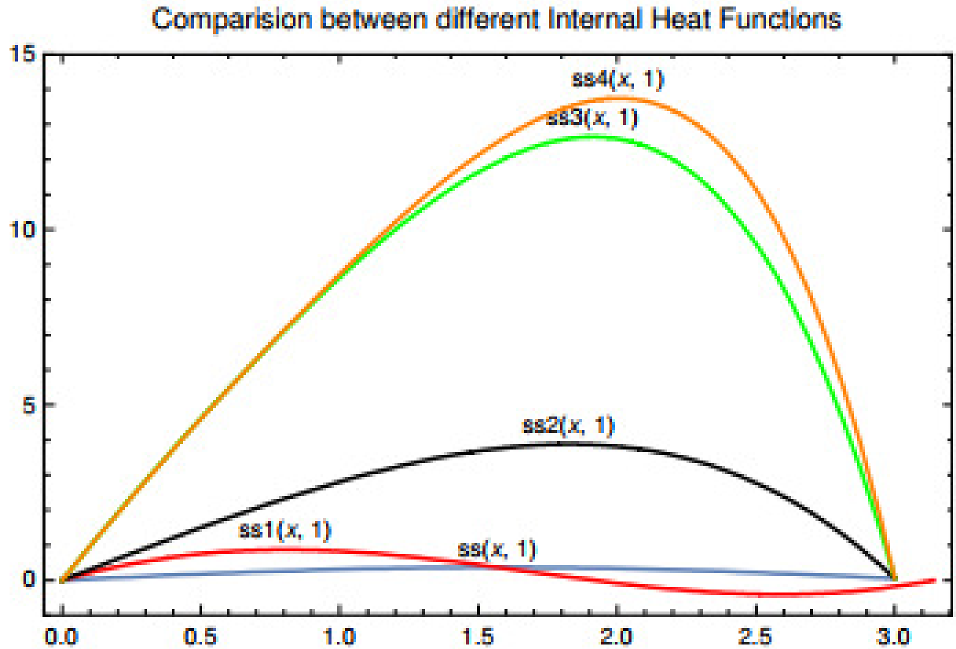
Now assuming Q[x] to be a Exponential function (e.g. □□) and solving the above internal heat equation using NDSolve it produces these graphs:





Comparison between different function as Q[x]

Now comparing all the possible functions of Q[x] by plotting them simultaneously.



As we can see, temperature of the body rises quickly for functions with larger gradient value. It implies that if the internal heat of a body is governed by functions which are exponential or polynomial in nature, the temperature of the body rises more quickly than if they were functions of trigonometric nature.

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Note: For the source code of above graphs kindly refer to the appendix section.

4.References

[1] J. Crank and P. Nicolson (1947). A practical method for numerical evaluation of solutions of partial differential equations of the heat- conduction type. Mathematical Proceedings of the Cambridge Philosophical Society,43, pp 50-67 doi:10.1017/S0305004100023197