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Discrete-Time Fourier Series

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j\frac{2\pi kn}{N}}$$

$$n = \{0, 1, \ldots, N-1\}$$

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}}$$

$$k = \{0, 1, \ldots, N-1\}$$

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1. Consider this signal :

$$x(n) = 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}$$

a) Determine the Fourier coefficients.

b) Find the signal power.

$$x(n) = 2 + \underbrace{2\cos\frac{\pi n}{4}}_{(1)} + \underbrace{\cos\frac{\pi n}{2}}_{(2)} + \underbrace{\frac{1}{2}\cos\frac{3\pi n}{4}}_{(3)}$$

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$$x(n) = \underbrace{2 + 2\cos\frac{\pi n}{4}}_{(1)} + \underbrace{\cos\frac{\pi n}{2}}_{(2)} + \underbrace{\frac{1}{2}\cos\frac{3\pi n}{4}}_{(3)}$$

$$f_1 = \frac{\pi}{2\pi * 4} = \frac{1}{8} \qquad f_2 = \frac{\pi}{2\pi * 2} = \frac{1}{4} \qquad f_3 = \frac{3\pi}{2\pi * 4} = \frac{3}{8}$$

$$N = MCM(4,8) = 8 \qquad C_k = \frac{1}{8}\sum_{n=0}^{7} x(n)e^{-j\frac{\pi k n}{4}}$$

a) Fourier coefficients:

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(1) $$2\cos\left(\frac{2\pi n}{8}\right) = 2\left[\frac{e^{\frac{i2\pi n}{8}} + e^{\frac{-i2\pi n}{8}}}{2}\right]$$ To find the values of K

$$\pm\frac{2\pi n}{8} = \frac{2\pi nk}{N} \longrightarrow k = \pm\frac{8}{8} = \pm 1 \longrightarrow C_1 = 1 \quad \text{y} \quad C_{-1+N} = C_7 = 1$$

Demonstration:

$$e^{\frac{i2\pi(1)n}{8}} + \underbrace{e^{\frac{i2\pi(-1)n}{8}}}_{e^{\frac{i2\pi(7-8)n}{8}}}$$

$$e^{-i2\pi n} = \cos 2\pi n - i\sin 2\pi n = 1$$

$$e^{\frac{i2\pi(7-8)n}{8}} \longrightarrow e^{\frac{i2\pi(7)n}{8}} * \overbrace{e^{\frac{-i2\pi(8)n}{8}}}^{1}$$

$$e^{\frac{i2\pi(1)n}{8}} + e^{\frac{i2\pi(7)n}{8}} \longrightarrow C_1 = 1 \quad \text{y} \quad C_7 = 1$$

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a) Fourier coefficients:

(2) $\cos\left(\frac{2\pi n}{4}\right) = \left[\frac{e^{\frac{i2\pi n}{4}} + e^{\frac{-i2\pi n}{4}}}{2}\right]$ To find the values of K

$$\pm\frac{2\pi n}{4} = \frac{2\pi n k}{N} \longrightarrow k = \pm\frac{8}{4} = \pm 2 \longrightarrow C_2 = \frac{1}{2} \quad y \quad C_{-2+N} = C_6 = \frac{1}{2}$$

Demonstration:

$$e^{\frac{i2\pi(2)n}{8}} + \underbrace{e^{\frac{i2\pi(-2)n}{8}}}_{e^{\frac{i2\pi(6-8)n}{8}}} \qquad e^{-i2\pi n} = \cos 2\pi n - i\sin 2\pi n = 1$$

$$e^{\frac{i2\pi(6-8)n}{8}} \longrightarrow e^{\frac{i2\pi(6)n}{8}} * \overbrace{e^{\frac{-i2\pi(8)n}{8}}}^{e^{-i2\pi n} = 1}$$

$$\frac{1}{2}\left[e^{\frac{i2\pi(2)n}{8}} + e^{\frac{i2\pi(6)n}{8}}\right] \longrightarrow C_2 = \frac{1}{2} \quad y \quad C_6 = \frac{1}{2}$$

a) Fourier coefficients:

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(3) $\frac{1}{2}\cos\left(\frac{3\pi n}{4}\right) = \frac{1}{2}\left[\frac{e^{\frac{i3\pi n}{4}} + e^{\frac{-i3\pi n}{4}}}{2}\right]$ To find the values of K

$$\pm\frac{3\pi n}{4} = \frac{2\pi n k}{N} \longrightarrow k = \pm\frac{24}{8} = \pm 3 \longrightarrow C_3 = \frac{1}{4} \quad \text{y} \quad C_{-3+N} = C_5 = \frac{1}{4}$$

Demonstration:

$$e^{\frac{i2\pi(3)n}{8}} + \underbrace{e^{\frac{i2\pi(-3)n}{8}}}$$

$$e^{-i2\pi n} = \cos 2\pi n - i\sin 2\pi n = 1$$

$$e^{\frac{i2\pi(5-8)n}{8}} \longrightarrow e^{\frac{i2\pi(5)n}{8}} * \overbrace{e^{\frac{-i2\pi(8)n}{8}}}$$

$$\frac{1}{4}\left[e^{\frac{i2\pi(3)n}{8}} + e^{\frac{i2\pi(5)n}{8}}\right] \longrightarrow C_3 = \frac{1}{4} \quad \text{y} \quad C_5 = \frac{1}{4}$$

a) Fourier coefficients:

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$$C_0 = 2 \qquad C_4 = 0$$

$$C_1 = 1 \qquad C_5 = \frac{1}{4}$$

$$C_2 = \frac{1}{2} \qquad C_6 = \frac{1}{2}$$

$$C_3 = \frac{1}{4} \qquad C_7 = 1$$

$$x(n) = 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}$$

$$P = \sum_{k=0}^{N-1} |C_k|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$$

$C_0 = 2$	$C_4 = 0$
$C_1 = 1$	$C_5 = \frac{1}{4}$
$C_2 = \frac{1}{2}$	$C_6 = \frac{1}{2}$
$C_3 = \frac{1}{4}$	$C_7 = 1$

$$\longrightarrow \quad P = 4 + 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{4} + 1 = \frac{53}{8}$$

2. Determine the magnitude spectrum of the signal

$$x(n) = \{\ldots, -1,2,\boxed{1,2,-1,0,-1,2,}1,2,\ldots\} \quad N = 6$$

$$C_k = \frac{1}{6}\sum_{n=0}^{5} x(n)e^{-j\frac{2\pi k n}{6}}$$

The summation is resolved

$$C_k = \frac{1}{6}\left[1e^{-j\frac{2\pi k(0)}{6}} + 2e^{-j\frac{2\pi k(1)}{6}} - 1e^{-j\frac{2\pi k(2)}{6}} + 0e^{-j\frac{2\pi k(3)}{6}} - 1e^{-j\frac{2\pi k(4)}{6}} + 2e^{-j\frac{2\pi k(5)}{6}}\right]$$

$$C_k = \frac{1}{6}\left[1 + 2e^{-j\frac{\pi k}{3}} - 1e^{-j\frac{2\pi k}{3}} - 1e^{-j\frac{4\pi k}{3}} + 2e^{-j\frac{5\pi k}{3}}\right]$$

To find the C_k

$$C_k = \frac{1}{6}\left[1 + 2e^{-j\frac{\pi k}{3}} - 1e^{-j\frac{2\pi k}{3}} - 1e^{-j\frac{4\pi k}{3}} + 2e^{-j\frac{5\pi k}{3}}\right]$$

$$\boxed{\begin{gathered} C_k = \frac{1}{6}\left[1 + 2e^{-j\frac{\pi k}{3}} + 2e^{-(-j\frac{\pi k}{3})} - 1e^{-j\frac{2\pi k}{3}} - 1e^{-(-j\frac{2\pi k}{3})}\right] \\ C_k = \frac{1}{6}\left[1 + 2\left(2\cos\frac{\pi k}{3}\right) - \left(2\cos\frac{2\pi k}{3}\right)\right] \end{gathered}}$$

$C_0 = \frac{1}{2}$ $\quad C_1 = \frac{2}{3}$ $\quad C_2 = 0$

$C_3 = -\frac{5}{6}$ $\quad C_4 = 0$ $\quad C_5 = \frac{2}{3}$

Power

$$P = \sum_{k=0}^{N-1} |C_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$\underbrace{P = \frac{1}{4} + \frac{4}{9} + 0 + \frac{25}{36} + 0 + \frac{4}{9} = \frac{11}{6}}_{\text{Power with } C_k}$$

$$\underbrace{P = \frac{1}{6}[1 + 4 + 1 + 0 + 1 + 4] = \frac{11}{6}}_{\text{Power with } x(n)}$$

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3. Find the Fourier transform of the signal

$$x(n) = [2,0,3,0,\underset{\uparrow}{2},0,3,0,2]$$

Fourier transform of aperiodic signals

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = [2,0,3,0,\underset{\uparrow}{2},0,3,0,2]$$

$$X(\omega) = \sum_{n=-4}^{4} x(n)e^{-j\omega n}$$

$$X(\omega) = 2e^{j4\omega} + 0e^{j3\omega} + 3e^{j2\omega} + 0e^{j\omega} + 2 + 0e^{-j\omega} + 3e^{-j2\omega} + 0e^{-j3\omega} + 2e^{-j4\omega}$$

$$X(\omega) = 2e^{j4\omega} + 3e^{j2\omega} + 2 + 3e^{-j2\omega} + 2e^{-j4\omega}$$

$$X(\omega) = 2e^{j4\omega} + 2e^{-j4\omega} + 2 + 3e^{j2\omega} + 3e^{-j2\omega}$$

$$X(\omega) = 2 + 4\cos 4\omega + 6\cos 2\omega$$

Solution with Real Par signal property.

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Par	$x(n) = x(-n)$	Real	$X_R(\omega) = X(0) + 2\sum_{n=1}^{\infty} x(n)\cos n\omega$
Impar	$x(-n) = -x(n)$	Imaginaria	$X_I(\omega) = 0$

$$X_R(\omega) = 2 + 2[0\cos\omega + 3\cos 2\omega + 0\cos 3\omega + 2\cos 4\omega]$$

$$X_R(\omega) = 2 + 2[3\cos 2\omega + 2\cos 4\omega]$$

$$X_R(\omega) = 2 + 4\cos 4\omega + 6\cos 2\omega$$

4. Find the discrete time Fourier transform.

$$x(n) = [-2,-1,\underset{\uparrow}{0},1,2]$$

$$\sin\omega = \frac{e^{i\omega} - e^{-i\omega}}{2i}$$

$$X(\omega) = \sum_{n=-2}^{2} x(n)e^{-j\omega n}$$

$$X(\omega) = -2e^{j2\omega} - 1e^{j\omega} + 0 + 1e^{-j\omega} + 2e^{-j2\omega}$$

$$X(\omega) = -2\left[e^{j2\omega} - 2e^{-j2\omega}\right] - \left[e^{j\omega} - e^{-j\omega}\right]$$

$$X(\omega) = -4i\sin 2\omega - 2i\sin\omega$$

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Solution with The Real Odd signal property.

Par	$x(n) = x(-n)$	Real	$X_R(\omega) = 0$
Impar	$x(-n) = -x(n)$	Imaginaria	$X_I(\omega) = -2\sum_{n=1}^{\infty} x(n)\sin\omega n$

$$X_I(\omega) = -2[1\sin\omega + 2\sin 2\omega]$$

$$X_I(\omega) = -4i\sin 2\omega - 2i\sin\omega$$

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PROPOSED EXERCISE 1

- Find the discrete time Fourier transform.

$$x(n) = \{6, 9, 3, \underset{\uparrow}{6}, 3, 9, 6\}$$

Solution: $X(\omega) = 6 + 6\cos(\omega) + 18\cos(2\omega) + 12\cos(3\omega)$

PROPOSED EXERCISE 2

- Find the discrete time Fourier transform.

$$x(n) = \{-6, -4, -2, \underset{\uparrow}{0}, 2\ 4\ 6\}$$

Solution: $\mathrm{X}(\omega) = -4\sin(\omega) - 8\sin(2\omega) - 12\sin(3\omega)$

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5. Determine $X_R(\omega)$, $X_I(\omega)$, $|\mathrm{X}(\omega)|$ y $\sphericalangle \mathrm{X}(\omega)$ for the next discrete time Fourier transform

$$X(\omega) = \frac{1}{1 - \frac{1}{3} e^{-i\omega}}$$

multiply and divide by the conjugate of the denominator

$$X(\omega) = \frac{1}{1-\frac{1}{3}e^{-i\omega}} \cdot \frac{1-\frac{1}{3}e^{i\omega}}{1-\frac{1}{3}e^{i\omega}} = \frac{1-\frac{1}{3}e^{i\omega}}{\left(1-\frac{1}{3}e^{-i\omega}\right)\left(1-\frac{1}{3}e^{i\omega}\right)}$$

$$\boxed{e^{\pm i\omega} = \cos(\omega) \pm isen(\omega)}$$

$$\boxed{\frac{1}{2}\left(e^{-i\omega} + e^{i\omega}\right) = \cos(\omega)}$$

$$X(\omega) = \frac{1-\frac{1}{3}e^{i\omega}}{1-\frac{1}{3}e^{i\omega}-\frac{1}{3}e^{-i\omega}+\frac{1}{9}} = \frac{1-\frac{1}{3}e^{i\omega}}{\frac{10}{9}-\frac{1}{3}\left[e^{i\omega}+e^{-i\omega}\right]} \quad ; \quad X(\omega) = \frac{1-\frac{1}{3}(\cos\omega + i\sin\omega)}{\frac{10}{9}-\frac{2}{3}\cos\omega}$$

$$X_R(\omega) = \frac{1-\frac{1}{3}\cos\omega}{\frac{10}{9}-\frac{2}{3}\cos\omega} \qquad X_I(\omega) = \frac{-\frac{1}{3}\sin\omega}{\frac{10}{9}-\frac{2}{3}\cos\omega}$$

Magnitude $|X(\omega)| = \sqrt{X_R(\omega)^2 + X_I(\omega)^2}$

$$\cos^2(\omega) + \sin^2(\omega) = 1$$

$$|X(\omega)| = \sqrt{\left(\frac{1-\frac{1}{3}\cos\omega}{\frac{10}{9}-\frac{2}{3}\cos\omega}\right)^2 + \left(\frac{-\frac{1}{3}\sin\omega}{\frac{10}{9}-\frac{2}{3}\cos\omega}\right)^2} \quad , \quad |X(\omega)| = \sqrt{\frac{1-\frac{2}{3}\cos\omega+\frac{1}{9}\cos^2\omega+\frac{1}{9}\sin^2\omega}{\left(\frac{10}{9}-\frac{2}{3}\cos\omega\right)^2}}$$

$$|X(\omega)| = \sqrt{\frac{1-\frac{2}{3}\cos\omega+\frac{1}{9}}{\left(\frac{10}{9}-\frac{2}{3}\cos\omega\right)^2}} = \sqrt{\frac{\frac{10}{9}-\frac{2}{3}\cos\omega}{\left(\frac{10}{9}-\frac{2}{3}\cos\omega\right)^2}} \quad , \quad |X(\omega)| = \sqrt{\frac{1}{\frac{10}{9}-\frac{2}{3}\cos\omega}}$$

Magnitude $\quad \measuredangle X(\omega) = \tan^{-1}\dfrac{X_I(\omega)}{X_R(\omega)}$

$$\measuredangle X(\omega) = \tan^{-1}\left[\frac{\dfrac{-\frac{1}{3}\sin\omega}{\frac{10}{9}-\frac{2}{3}\cos\omega}}{\dfrac{1-\frac{1}{3}\cos\omega}{\frac{10}{9}-\frac{2}{3}\cos\omega}}\right] = \frac{\left(-\frac{1}{3}\sin\omega\right)}{\left(1-\frac{1}{3}\cos\omega\right)} = \frac{-\sin\omega}{3-\cos\omega} = \frac{\sin(\omega)}{\cos(\omega)-3}$$

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PROPOSED EXERCISE 3

- Determine $Y_R(\omega)$ and $Y_I(\omega)$

$$h(n) = \left(\frac{1}{3}\right)^n u(n) \qquad\qquad x(n) = \left(\frac{1}{10}\right)^n u(n)$$

Solution:

$$Y_R(\omega) = \frac{\frac{1}{30}\cos(2\omega) - \frac{13}{30}\cos(\omega) + 1}{\frac{2}{15}\cos^2(\omega) - \frac{403}{450}\cos(\omega) + \frac{101}{90}} \qquad Y_I(\omega) = \frac{\frac{1}{30}\sin(2\omega) - \frac{13}{30}\sin(\omega)}{\frac{2}{15}\cos^2(\omega) - \frac{403}{450}\cos(\omega) + \frac{101}{90}}$$

6. Determine the system's stable state response $h(n) = \left(\frac{1}{4}\right)^{n-1} u(n-1)$ and input $x(n) = \frac{2}{3} - \frac{1}{3} sin\left(\frac{n\pi}{6}\right) + \frac{3}{2} e^{j\left(\frac{n\pi}{5}\right)} + \frac{3}{2} e^{j\left(-\frac{n\pi}{5}\right)}$

$$\frac{3}{2} e^{j\left(\frac{\pi}{5}n\right)} + \frac{3}{2} e^{j\left(-\frac{\pi}{5}n\right)} = \frac{3}{2}\left(2\, cos\left(\frac{n\pi}{5}\right)\right) = 3\, cos\left(\frac{n\pi}{5}\right)$$

$$x_0(n) = \frac{2}{3} \quad \rightarrow \quad \omega_0 = 0$$

$$x_1(n) = -\frac{1}{3} sin\left(\frac{n\pi}{6}\right) \rightarrow \omega_1 = \frac{\pi}{6}$$

$$x_2(n) = 3\, cos\left(\frac{n\pi}{5}\right) \rightarrow \omega_2 = \frac{\pi}{5}$$

$$H(\omega) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^{n-1} u(n-1)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^{n} u(n)\, e^{-j\omega(n+1)}$$

$$H(\omega) = \sum_{n=-\infty}^{\infty} e^{-j\omega} \left[\left(\frac{1}{3}\right)^{n} u(n) e^{-j\omega n}\right] = \sum_{n=0}^{\infty} e^{-j\omega} \left[\left(\frac{1}{3}\right)^{n} e^{-j\omega n}\right]$$

$$H(\omega) = \frac{e^{-j\omega}}{1 - \frac{1}{3} e^{-j\omega}}$$

$$\sum_{r=0}^{\infty} a^r = \frac{1}{1-a}, \qquad |a| < 1$$

$$H(\omega) = \frac{e^{-j\omega}}{1 - \frac{1}{3}e^{-j\omega}}$$

$$H(0) = \frac{3}{2} \qquad H\left(\frac{\pi}{6}\right) = 1.37\measuredangle - 0.75 \qquad H\left(\frac{\pi}{5}\right) = 1.32\measuredangle - 0.89$$

$$y(n) = \frac{2}{3}\left(\frac{3}{2}\right) - \frac{1}{3}(1.37)\, sin\left(\frac{n\pi}{6} - 0.75\right) + 3(1.32)\, cos\left(\frac{n\pi}{5} - 0.89\right)$$

7. Determine $|\mathrm{H}(\omega)|^2$ for the system.

$$y(n) = 2.5y(n-1) - y(n-2) + x(n) - 5x(n-1)$$

$$|H(Z)|^2 = H(Z) \cdot H(Z^{-1})$$

The Z transform of the equation is performed in differences

$$y(z) = 2.5z^{-1}Y(z) - z^{-2}Y(z) + X(z) - 5z^{-1}X(z)$$

$$y(z) - 2.5z^{-1}Y(z) + z^{-2}Y(z) = X(z) - 5z^{-1}X(z)$$

$$Y(z)[1 - 2.5z^{-1} + z^{-2}] = X(z)[1 - 5z^{-1}]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 5z^{-1}}{1 - 2.5z^{-1} + z^{-2}} \qquad H(z^{-1}) = \frac{1 - 5z}{1 - 2.5z + z^2}$$

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$$H(z) = \frac{1-5z^{-1}}{1-2.5z^{-1}+z^{-2}} \qquad H(z^{-1}) = \frac{1-5z}{1-2.5z+z^2}$$

$$|H(\omega)|^2 = H(Z) * H(Z^{-1})\Big|_{z=e^{j\omega}} = \frac{1-5z^{-1}}{1-2.5z^{-1}+z^{-2}} \cdot \frac{1-5z}{1-2.5z+z^2}\Bigg|_{z=e^{j\omega}}$$

$$|H(\omega)|^2 = \frac{1-5z-5z^{-1}+25}{1-2.5z+z^2-2.5z^{-1}+6.25-2.5z+z^{-2}-2.5z^{-1}+1}\Bigg|_{z=e^{j\omega}}$$

$$|H(\omega)|^2 = \frac{26-5z-5z^{-1}}{8.25-5z-5z^{-1}+z^2+z^{-2}}\Bigg|_{z=e^{j\omega}}$$

$$|H(\omega)|^2 = \left.\frac{26 - 5(z + z^{-1})}{8.25 - 5(z + z^{-1}) + (z^2 + z^{-2})}\right|_{z = e^{j\omega}}$$

$$|H(\omega)|^2 = \frac{26 - 5(e^{i\omega} + e^{-i\omega})}{8.25 - 5(e^{i\omega} + e^{-i\omega}) + (e^{i2\omega} + e^{-i2\omega})}$$

$$|H(\omega)|^2 = \frac{26 - 10\cos\omega}{8.25 - 10\cos\omega + 2\cos 2\omega}$$

$$\boxed{e^{i\omega} = \cos(\omega) + isen(\omega)}$$

$$\boxed{\frac{1}{2}(e^{-i\omega} + e^{i\omega}) = \cos(\omega)}$$

8. Using DFT and IDFT, determine the response of the FIR filter with impulse response h(n) to an input x(n)

$$h(n) = \{3,\underset{\uparrow}{4},2\}$$

$$\underbrace{\qquad\qquad}_{M\,=\,3}$$

$$x(n) = \{1,\underset{\uparrow}{3}\}$$

$$\underbrace{\qquad\qquad}_{L\,=\,2}$$

$$N \geq L + M - 1$$

$$N \geq 2 + 3 - 1$$

$$N \geq 4$$

$$X(k) = \sum_{n=0}^{3} x(n)e^{-j\frac{2\pi kn}{4}}$$

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DFT for $x(n)$

$$X(k) = \sum_{n=0}^{3} x_1(n) e^{-j\frac{2\pi kn}{4}}$$

$$x(n) = \{1,3\}$$
$$\uparrow$$

$$X(k) = 1e^{-\frac{j2\pi k(0)}{4}} + 3e^{-\frac{j2\pi k(1)}{4}} + 0 + 0$$

$$X(0) = 1 + 3 = 4$$

$$X(1) = 1 + 3e^{-j\frac{2\pi}{4}} = 1 - 3i$$

$$X(2) = 1 + 3e^{-j\frac{4\pi}{4}} = -2$$

$$X(3) = 1 + 3e^{-j\frac{6\pi}{4}} = 1 + 3i$$

$$X(k) = \{4, 1 - 3i, -2, 1 + 3i\}$$

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DFT for $h(n)$

$$H(k) = \sum_{n=0}^{3} h(n) e^{-j\frac{2\pi kn}{4}}$$

$$h(n) = \{\underset{\uparrow}{3},4,2\}$$

$$H(k) = 3e^{-\frac{j2\pi k(0)}{4}} + 4e^{-\frac{j2\pi k(1)}{4}} + 2e^{-\frac{j2\pi k(2)}{4}} + 0$$

$$H(0) = 3 + 4 + 2 = 9$$

$$H(1) = 3 + 4e^{-j\frac{2\pi}{4}} + 2e^{-j\frac{4\pi}{4}} = 1 - 4i$$

$$H(2) = 3 + 4e^{-j\frac{4\pi}{4}} + 2e^{-j\frac{8\pi}{4}} = 1$$

$$H(3) = 3 + 4e^{-j\frac{6\pi}{4}} + 2e^{-j\frac{12\pi}{4}} = 1 + 4i$$

$$X(k) = \{4,1-3i,-2,1+3i\}$$

$$H(k) = \{9,1-4i,1,1+4i\}$$

$$Y(k) = \{36,-11-7i,-2,-11+7i\}$$

Using IDFT find $y(n)$

$$y(n) = \frac{1}{4}\sum_{k=0}^{3} Y(k)e^{\frac{j2\pi nk}{4}}$$

$$Y(k) = \{36, -11-7i, -2, -11+7i\}$$

$$y(n) = \frac{1}{4}\left[36e^{\frac{j2\pi n(0)}{4}} + (-11-7i)e^{\frac{j2\pi n(1)}{4}} - 2e^{\frac{j2\pi n(2)}{4}} + (-11+7i)e^{\frac{j2\pi n(3)}{4}}\right]$$

$$y(0) = \frac{1}{4}[36-11-7i-2-11+7i] = \frac{12}{4} = 3$$

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Using IDFT find $y(n)$

$$y(n) = \frac{1}{4}\left[36e^{\frac{j2\pi n(0)}{4}} + (-11 - 7i)e^{\frac{j2\pi n(1)}{4}} - 2e^{\frac{j2\pi n(2)}{4}} + (-11 + 7i)e^{\frac{j2\pi n(3)}{4}}\right]$$

$$y(1) = \frac{1}{4}\left[36 + (-11 - 7i)e^{-j\frac{2\pi}{4}} - 2e^{-j\frac{4\pi}{4}} + (-11 + 7i)e^{-j\frac{6\pi}{4}}\right]$$

$$y(1) = \frac{1}{4}\left[36 + (-11 - 7i)\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right) - 2\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right) + (-11 + 7i)\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right)\right]$$

$$y(1) = \frac{1}{4}[36 + i(-11 - 7i) + 2 - i(-11 + 7i)\,] = \frac{1}{4}[36 - i11 + 7 + 2 + i11 + 7\,] = 13$$

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Using IDFT find $y(n)$

$$y(n) = \frac{1}{4}\left[36e^{\frac{j2\pi n(0)}{4}} + (-11-7i)e^{\frac{j2\pi n(1)}{4}} - 2e^{\frac{j2\pi n(2)}{4}} + (-11+7i)e^{\frac{j2\pi n(3)}{4}}\right]$$

$$y(2) = \frac{1}{4}\left[36 + (-11-7i)e^{-j\frac{4\pi}{4}} - 2e^{-j\frac{8\pi}{4}} + (-11+7i)e^{-j\frac{12\pi}{4}}\right]$$

$$y(2) = \frac{1}{4}\left[36 + (-11-7i)\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right) - 2\left(\cos\frac{8\pi}{4} + i\sin\frac{8\pi}{4}\right) + (-11+7i)\left(\cos\frac{12\pi}{4} + i\sin\frac{12\pi}{4}\right)\right]$$

$$y(2) = \frac{1}{4}[36 - (-11-7i) - 2 - (-11+7i)\,] = \frac{1}{4}[36 + 11 + 7i - 2 + 11 - 7i\,] = 14$$

Using IDFT find $y(n)$

$$y(n) = \frac{1}{4}\left[36e^{\frac{j2\pi n(0)}{4}} + (-11-7i)e^{\frac{j2\pi n(1)}{4}} - 2e^{\frac{j2\pi n(2)}{4}} + (-11+7i)e^{\frac{j2\pi n(3)}{4}}\right]$$

$$y(3) = \frac{1}{4}\left[36 + (-11-7i)e^{-j\frac{6\pi}{4}} - 2e^{-j\frac{12\pi}{4}} + (-11+7i)e^{-j\frac{18\pi}{4}}\right]$$

$$y(3) = \frac{1}{4}\left[36 + (-11-7i)\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right) - 2\left(\cos\frac{12\pi}{4} + i\sin\frac{12\pi}{4}\right) + (-11+7i)\left(\cos\frac{18\pi}{4} + i\sin\frac{18\pi}{4}\right)\right]$$

$$y(3) = \frac{1}{4}[36 - i(-11-7i) + 2 + i(-11+7i)\,] = \frac{1}{4}[36 + 11i - 7 + 2 - 11i - 7\,] = 6$$

$$y(n) = \{\underset{\uparrow}{3},13,14,6\}$$

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To verify the answer of $y(n)$, we do the convolution in time with $h(n)$ and $x(n)$

$$h(n) = \{\underset{\uparrow}{3},4,2\} \qquad x(n) = \{\underset{\uparrow}{1},3\}$$

$$\begin{array}{rrrr}
 & 3 & 4 & 2 \\
 & & 1 & 3 \\
\hline
 & 9 & 12 & 6 \\
3 & 4 & 2 & \\
\hline
3 & 13 & 14 & 6
\end{array}$$

```
>> x = [1 3];
>> h = [3 4 2];
>> conv(h, x)

ans =

     3    13    14     6

>>
```

$$y(n) = \{\underset{\uparrow}{3},13,14,6\}$$

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PROPOSED EXERCISE 4

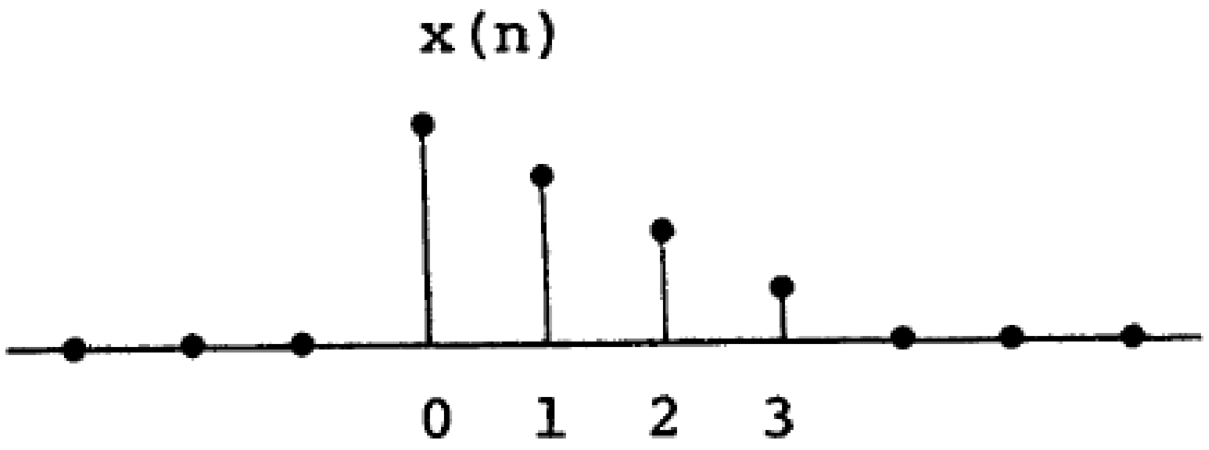
- Determine the output y(n) to an input x(n) and impulse response h(n), using the Discrete Fourier Transform (DFT):

$$x(n) = \{\overrightarrow{2}, 1, 3\} \qquad h(n) = \{\overrightarrow{1}, 3\}$$

Solution:

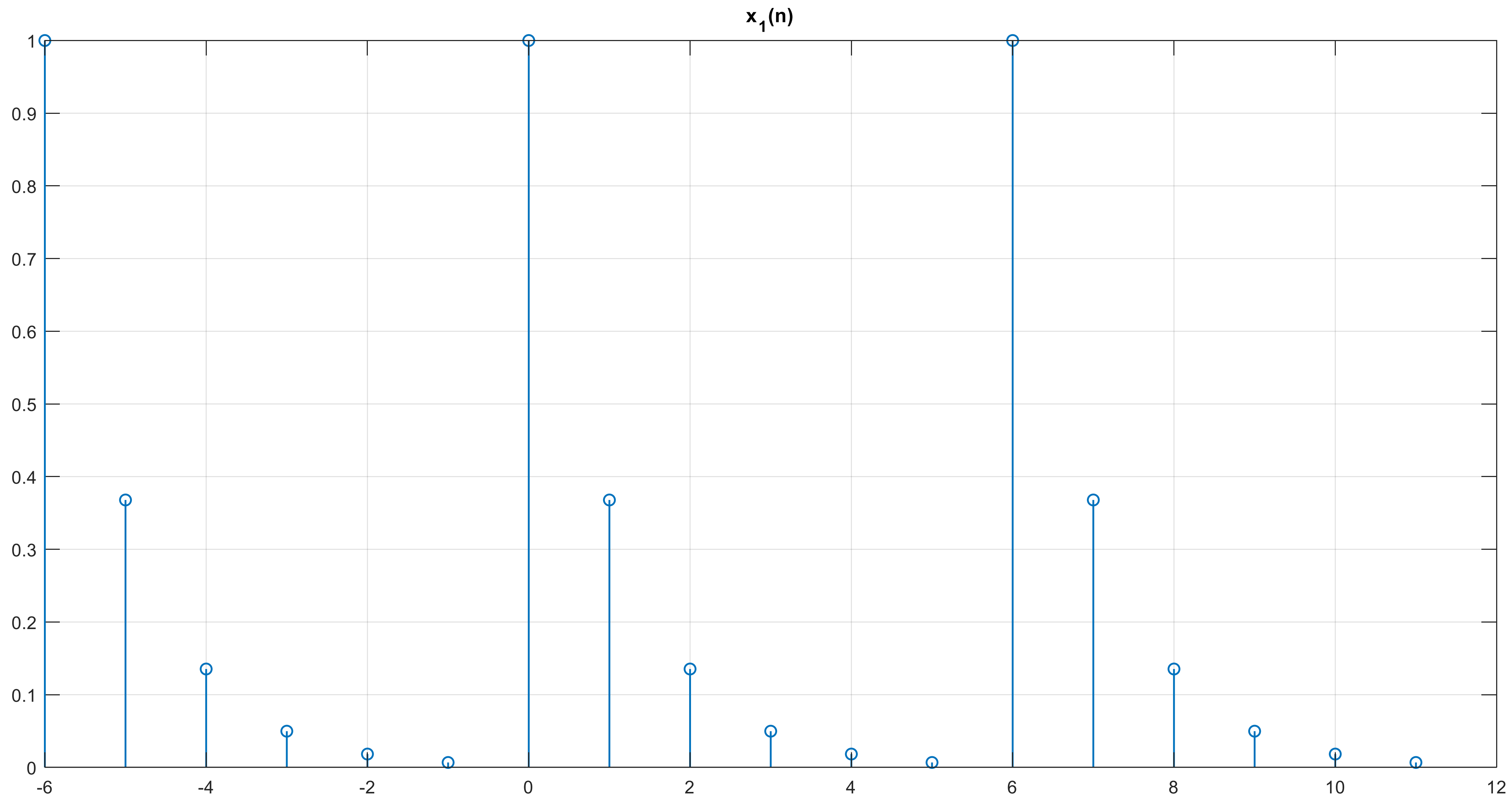
$$y(n) = \{2, 7, 6, 9\}$$

9. We have the following sequence $x(n)$:



Recalling the concept and mathematical notation of circular convolution, DRAW the sequences $x_1(n)$ and $x_2(n)$ (in a window of at least 10 samples) specified as follows:

$$x_1(n) = x\big((n-2)\big)_6 \qquad x_2(n) = x\big((-n)\big)_3$$


$x_1(n)$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-6
-4
-2
0
2
4
6
8
10
12

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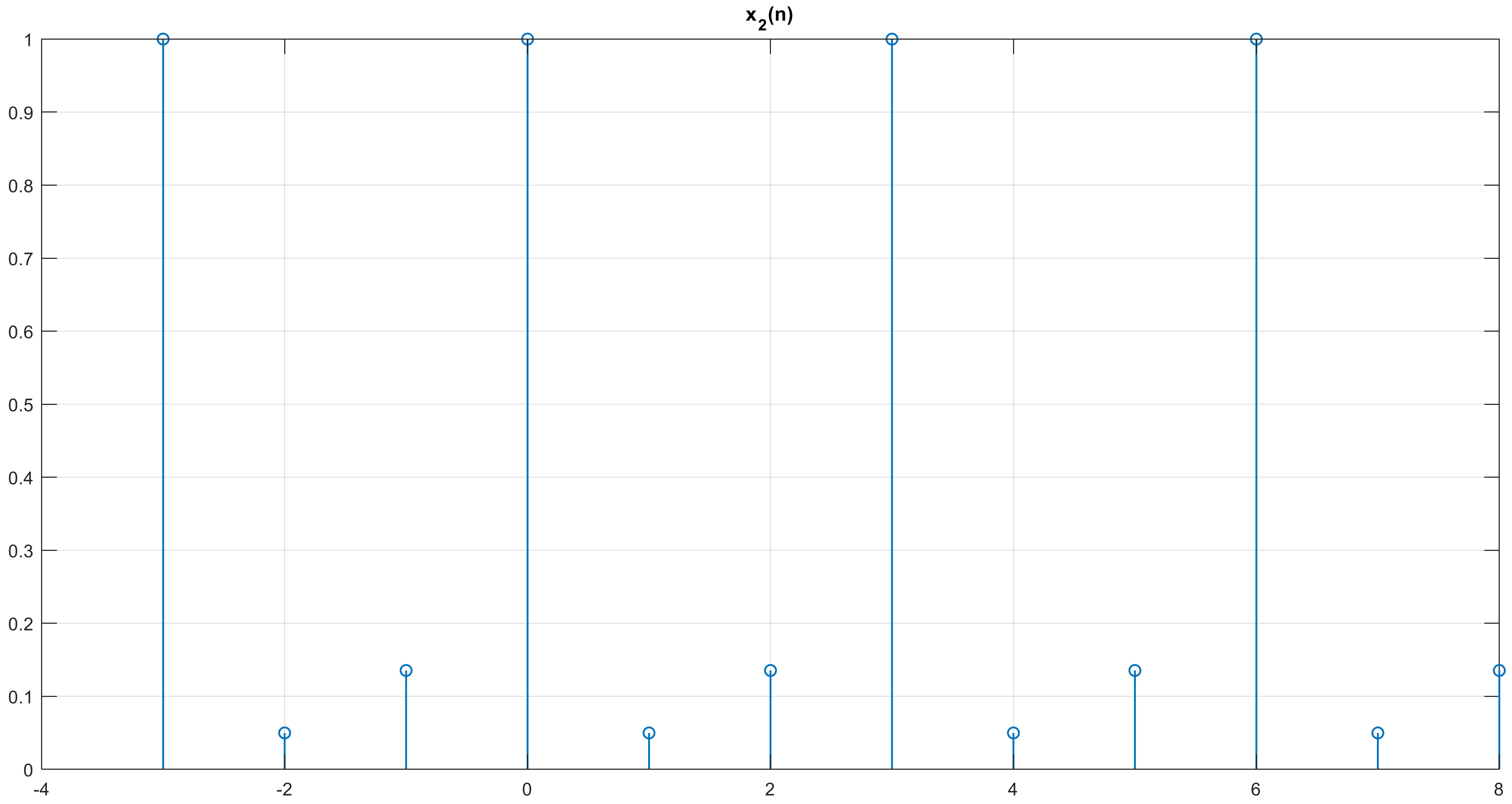

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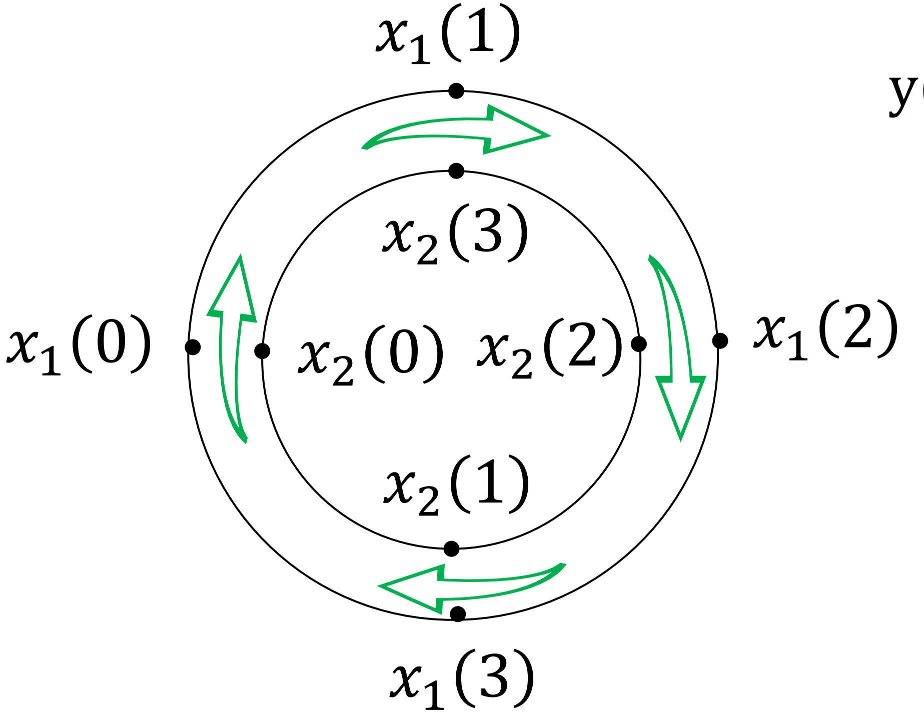
10. Determine the circular convolution of the sequences

$$x_1(n) = \{1,\underset{\uparrow}{2},3,1\} \qquad x_2(n) = \{4,\underset{\uparrow}{3},2,2\} \qquad x_3(n) = x_1(n) \circledast x_2(n)$$

a) In the time domain calculate the circular convolution.

a) In the time domain calculate the circular convolution.

$$x_1(n) = \{\underset{\uparrow}{1},2,3,1\} \qquad x_2(n) = \{\underset{\uparrow}{4},3,2,2\}$$



$$y(0) = x_1(0)x_2(0) + x_1(1)x_2(3) + x_1(2)x_2(2) + x_1(3)x_2(1)$$

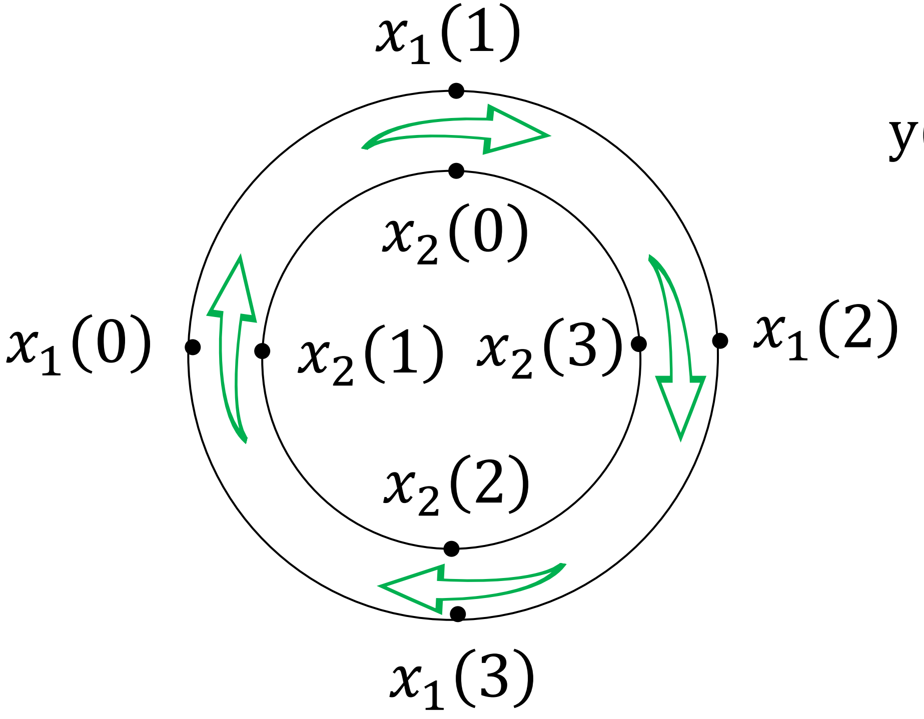
$$y(0) = 1 * 4 + 2 * 2 + 3 * 2 + 1 * 3 = 17$$

The green arrow indicates the direction in which the frequencies of $x_1(n)$.
The red arrow indicates the direction in which the frequencies of $x_2(n)$.

a) In the time domain calculate the circular convolution.

$x_2(n)$ is rotated clockwise

$x_1(n) = \{1,2,3,1\}$ $\quad$ $x_2(n) = \{4,3,2,2\}$



$$y(1) = x_1(0)x_2(1) + x_1(1)x_2(0) + x_1(2)x_2(3) + x_1(3)x_2(2)$$

$$y(1) = 1 * 3 + 2 * 4 + 3 * 2 + 1 * 2 = 19$$

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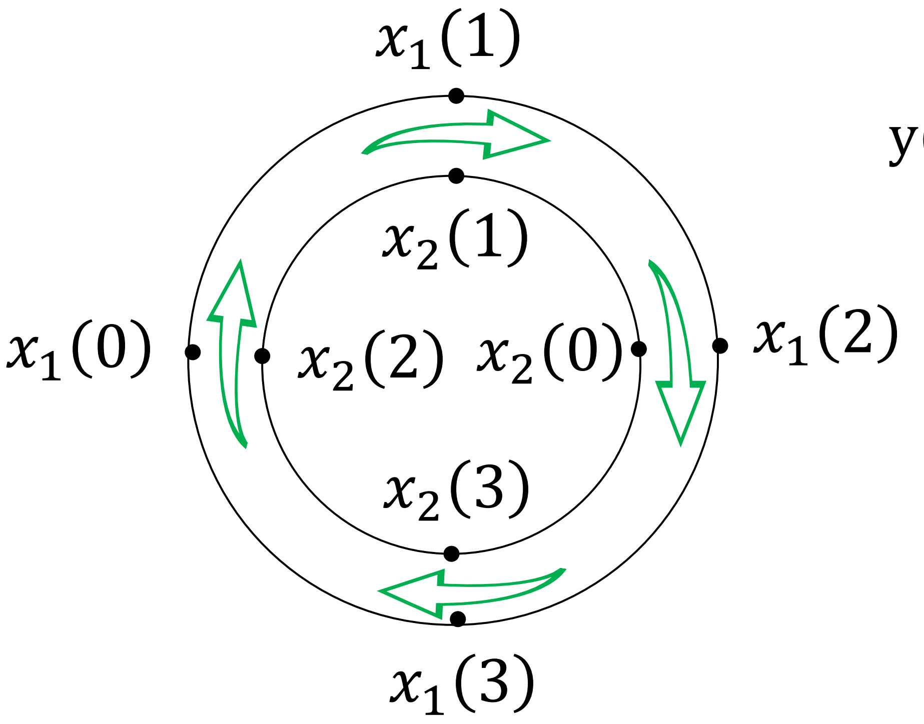
a) In the time domain calculate the circular convolution.

$x_2(n)$ is rotated clockwise

$x_1(n) = \{\underset{\uparrow}{1},2,3,1\}$ $\quad$ $x_2(n) = \{\underset{\uparrow}{4},3,2,2\}$



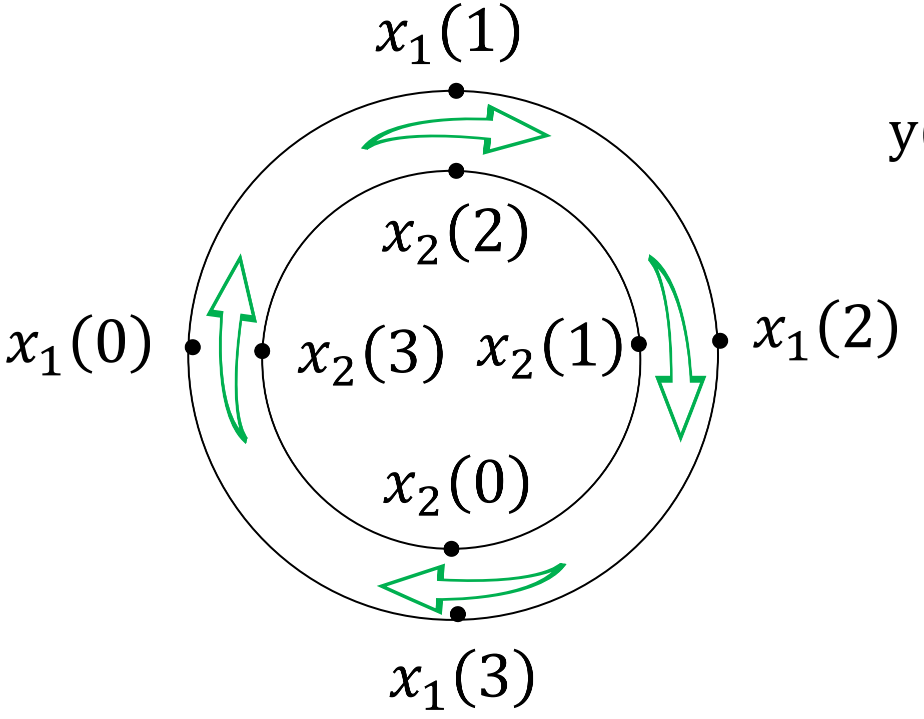
$$y(2) = x_1(0)x_2(2) + x_1(1)x_2(1) + x_1(2)x_2(0) + x_1(3)x_2(3)$$

$$y(2) = 1 * 2 + 2 * 3 + 3 * 4 + 1 * 2 = 22$$

a) In the time domain calculate the circular convolution.

$x_2(n)$ is rotated clockwise

$x_1(n) = \{1,2,3,1\}$ $x_2(n) = \{4,3,2,2\}$



$$y(3) = x_1(0)x_2(3) + x_1(1)x_2(2) + x_1(2)x_2(1) + x_1(3)x_2(0)$$

$$y(3) = 1 * 2 + 2 * 2 + 3 * 3 + 1 * 4 = 19$$

$$y(0) = 17$$

$$y(1) = 19$$

$$y(2) = 22$$

$$y(3) = 19$$

$$y(n) = \{\underset{\uparrow}{17}, 19, 22, 19\}$$

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PROPOSED EXERCISE 5

- Determine the circular convolution betwen x(n) and h(n)

$$x(n) = \{\underset{\uparrow}{3}, 7, 7, 9\}$$

$$h(n) = \{8, \underset{\uparrow}{2}\}$$

Solution: $y(n) = \{78, 38, 70, 74\}$

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PROPOSED EXERCISE 6

- Determine X(k) via the FFT using time decimation

$$x(n) = \{6, 8, \underset{\uparrow}{1}, 0, 7, 1, 10\}$$

Solution:

$$y(n) = \{33, -13.9497 + 4.5355i, -4 - 5i, -4.0503 + 2.5355i, 19, -4.0503 - 2.5355i, -4 + 5i, -13.9497 - 4.5355i\}$$