

A Generalized Algorithm for producing Integer Power Reduction Formulas of Cosine

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The idea behind this algorithm was inspired by a video entitled *cos(1) + ... + cos(n)* by Peyam R. Tabrizian
<https://www.youtube.com/watch?v=7LBQTpiK-Xg>

Prior to seeing this video, I understood that complex numbers stemmed from the idea that $\sqrt{-1} = i$, and that complex numbers had some properties and could be dealt with in particular ways, but I did not believe they had any utility.

After watching the video however, I became a believer. So I present a way a way I've discovered, for generating the integer power reduction formulas for cosine which hinges on the imaginary representation of cosine.

This analysis begins with (1) Euler's Formula: $e^{i\theta} = \cos(\theta) + i \sin(\theta)$
 Plugging in $-\theta$, we see that $e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos(\theta) - i \sin(\theta)$

We can then add these two equations to get the following identity(2):
 $e^{i\theta} + e^{-i\theta} = 2 \cos(\theta) \rightarrow \cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

This will be very important, because it means that whenever we can turn,
 for instance, $e^{7i} + e^{-7i} \rightarrow 2 \cos(7x)$

So lets start applying what we now know.

$$\cos^2 \theta = \cos(\theta) \cdot \cos(\theta) \tag{1}$$

$$= \frac{1}{4}(e^{i\theta} + e^{-i\theta})^2 \tag{2}$$

$$= \frac{1}{4}(e^{2i\theta} + 2 + e^{-2i\theta}) \tag{3}$$

$$= \frac{1}{4}(2\cos(2\theta) + 2) \tag{4}$$

$$= \frac{\cos(2\theta) + 1}{2} \tag{5}$$

Which is precisely the power reduction formula! (You can verify this with a search online) Lets apply this to some more powers of cosine. We'll be reusing the expanded complex form for each subsequent power.

I will not prove the formulas are correct rigorously. But viewing them on desmos they certainly appear to match up.

<https://www.desmos.com/calculator/1a6havywh1>

$$\cos^3 \theta = \cos^2 \theta \cdot \cos \theta \quad (6)$$

$$= \frac{1}{8}(e^{2i\theta} + 2 + e^{-2i\theta})(e^{i\theta} + e^{-i\theta}) \quad (7)$$

$$= \frac{1}{8}(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}) \quad (8)$$

$$= \frac{1}{8}(2 \cos(3\theta) + 6 \cos(\theta)) \quad (9)$$

$$= \frac{1}{4} \cos(3\theta) + \frac{3}{4} \cos(\theta) \quad (10)$$

$$\cos^4 \theta = \cos^3 \theta \cdot \cos \theta \quad (11)$$

$$= \frac{1}{16}(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta})(e^{i\theta} + e^{-i\theta}) \quad (12)$$

$$= \frac{1}{16}(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}) \quad (13)$$

$$= \frac{1}{16}(2 \cos(4\theta) + 8 \cos(2\theta) + 6) \quad (14)$$

$$= \frac{1}{8} \cos(4\theta) + \frac{1}{2} \cos(2\theta) + \frac{3}{8} \quad (15)$$

$$\cos^5 \theta = \cos^4 \theta \cdot \cos \theta \quad (16)$$

$$= \frac{1}{32}(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta})(e^{i\theta} + e^{-i\theta}) \quad (17)$$

$$= \frac{1}{32}(e^{5i\theta} + 5e^{3i\theta} + 10e^{i\theta} + 10e^{-i\theta} + 5e^{-3i\theta} + e^{-5i\theta}) \quad (18)$$

$$= \frac{1}{32}(2 \cos(5\theta) + 10 \cos(3\theta) + 20 \cos(\theta)) \quad (19)$$

$$= \frac{1}{16} \cos(5\theta) + \frac{5}{16} \cos(3\theta) + \frac{5}{8} \cos(\theta) \quad (20)$$

If you were paying attention to the coefficients at step 3, 8, 13, 18, you might have noticed a relationship to pascal's triangle...