A Generalized Algorithm for producing Integer Power Reduction Formulas of Cosine

March 29, 2018

The idea behind this algorithm was inspired by a video entitled *cos(1) + ... + cos(n)* by Peyam R. Tabrizian https://www.youtube.com/watch?v=7LBQTpiK-Xg

Prior to seeing this video, I understood that complex numbers stemmed from the idea that $\sqrt{-1} = i$, and that complex numbers had some properties and could be dealt with in particular ways, but I did not believe they had any utility.

After watching the video however, I became a believer. So I present a way a way I've discovered, for generating the integer power reduction formulas for cosine which hinges on the imaginary representation of cosine.

This analysis begins with (1)Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
Plugging in $-\theta$, we see that $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta)$

We can then add these two equations to get the following identity(2):
$e^{i\theta} + e^{-i\theta} = 2\cos(\theta) \rightarrow \cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

This will be very important, because it means that whenever we can turn, for instance, $e^{7i} + e^{-7i} \rightarrow 2\cos(7x)$

So lets start applying what we now know.

$$\begin{aligned}
\cos^2\theta &= \cos(\theta)\cdot\cos(\theta) && (1)\\
&= \frac{1}{4}(e^{i\theta} + e^{-i\theta})^2 && (2)\\
&= \frac{1}{4}(e^{2i\theta} + 2 + e^{-2i\theta}) && (3)\\
&= \frac{1}{4}(2cos(2\theta) + 2) && (4)\\
&= \frac{cos(2\theta) + 1}{2} && (5)
\end{aligned}$$

Which is precisely the power reduction formula! (You can verify this with a search online) Lets apply this to some more powers of cosine. We'll be reusing the expanded complex form for each subsequent power.
I will not prove the formulas are correct rigorously. But viewing them on desmos they certainly appear to match up.
https://www.desmos.com/calculator/1a6havywh1

$$
\begin{align}
\cos^3\theta &= cos^2\theta \cdot cos\theta \tag{6}\\
&= \frac{1}{8}(e^{2i\theta} + 2 + e^{-2i\theta})(e^{i\theta} + e^{-i\theta}) \tag{7}\\
&= \frac{1}{8}(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}) \tag{8}\\
&= \frac{1}{8}(2\cos(3\theta) + 6cos(\theta)) \tag{9}\\
&= \frac{1}{4}\cos(3\theta) + \frac{3}{4}\cos(\theta) \tag{10}
\end{align}
$$

$$
\begin{align}
\cos^4\theta &= cos^3\theta \cdot cos\theta \tag{11}\\
&= \frac{1}{16}(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta})(e^{i\theta} + e^{-i\theta}) \tag{12}\\
&= \frac{1}{16}(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}) \tag{13}\\
&= \frac{1}{16}(2\cos(4\theta) + 8\cos(2\theta) + 6) \tag{14}\\
&= \frac{1}{8}\cos(4\theta) + \frac{1}{2}\cos(2\theta) + \frac{3}{8} \tag{15}
\end{align}
$$

$$
\begin{align}
\cos^5\theta &= cos^4\theta \cdot cos\theta \tag{16}\\
&= \frac{1}{32}(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta})(e^{i\theta} + e^{-i\theta}) \tag{17}\\
&= \frac{1}{32}(e^{5i\theta} + 5e^{3i\theta} + 10e^{i\theta} + 10e^{-i\theta} + 5e^{-3i\theta} + e^{-5i\theta}) \tag{18}\\
&= \frac{1}{32}(2\cos(5\theta) + 10\cos(3\theta) + 20\cos(\theta)) \tag{19}\\
&= \frac{1}{16}\cos(5\theta) + \frac{5}{16}\cos(3\theta) + \frac{5}{8}\cos(\theta) \tag{20}
\end{align}
$$

If you were paying attention to the coefficients at (3), (8), (13), (18), you might have noticed a relationship to pascal's triangle...

Let's draw something like pascal's triangle, where each column signifies a coefficient in the expanded complex form of $cos^n x$. Each row is the previous row multiplied by $(e^{i\theta} + e^{-i\theta})$.

cos^n	$e^{-5i\theta}$	$e^{-4i\theta}$	$e^{-3i\theta}$	$e^{-2i\theta}$	$e^{-i\theta}$	e^0	$e^{i\theta}$	$e^{i2\theta}$	$e^{i3\theta}$	$e^{i4\theta}$	$e^{i5\theta}$
1						1					
$\cos(\theta)$					1	0	1				
$\cos^2(\theta)$				1	0	2	0	1			
$\cos^3(\theta)$			1	0	3	0	3	0	1		
$\cos^4(\theta)$		1	0	4	0	6	0	4	0	1	
$\cos^5(\theta)$	1	0	5	0	10	0	10	0	5	0	1