Bayes AI

Unit 7: Bayesian Regression: Linear and Bayesian Trees

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Temporal Data: Filtering, Event Detection, Pandemics

Example: History of Pandemics

Bill Gates: 12/11/2009: "I'm most worried about a worldwide Pandemic"

Early-period Pandemics	Dates	Size of Mortality
Plague of Athens	430 BC	25% population.
Black Death	1347	30% Europe
London Plague	1666	20% population

Recent Flu Epidemics	Dates 1900-201
Spanish Flu	1918-19
Asian Flu	H2N2, 1957-58
Hong Kong Flu	H3N2, 1968-69

Spanish Flu killed more than WW1

H1N1 Flu 2009: $18,449$ people killed World wide:

SEIR Epidemic Models

Growth *"self-reinforcing"*: More likely if more infectants

- An individual comes into contact with disease at rate β_1
- The susceptible individual contracts the disease with probability β_2
- Each infectant becomes infectious with rate α per unit time
- Each infectant recovers with rate γ per unit time

$S_t + E_t + I_t + R_t = N$

Current Models: SEIR

susceptible-exposed-infectious-recovered model

Dynamic models that extend earlier models to include exposure and recovery.

The coupled SEIR model:
$\dot{S} = -\beta SI$
$\dot{E} = \beta SI - \alpha E$
$\dot{I} = \alpha E - \gamma I$
$\dot{R} = \gamma I$

Infectious disease models

Daniel Bernoulli's (1766) first model of disease transmission in smallpox:

"I wish simply that, in matters which so closely concern the well being of the human race, no decision shall be made without all knowledge which a little analysis and calculation can provide"

- R.A. Ross, (Nobel Medicine winner, 1902) – math model of malaria transmission, which ultimately lead to malaria control.

Ross-McDonald model

- Kermack and McKendrick: susceptible-infectious-recovered (SIR)

London Plague 1665-1666; Cholera: London 1865, Bombay, 1906.

Example: London Plague, 1666: Village Eyam nr. Sheffield

Model of transmission from Infectants, I, to susceptibles, S.

Date 1666	Susceptibles	Infectives
Initial	254	7
July 3	235	15
July 19	201	22
Aug 3	153	29
Aug 19	121	21
Sept 3	108	8
Sept 19	97	8
Oct 3	–	–
Oct 19	83	0

Initial Population $N = 261 = S_0$; Final population $S_\infty = 83$.

Modeling Growth: SI

Coupled Differential eqn $\dot{S} = -\beta SI, \dot{I} = (\beta S - \alpha)I$

- Estimates $\frac{\beta}{\alpha} = 6.54 \times 10^{-3}, \frac{\alpha}{\beta} = 1.53.$

$$\frac{\hat{\beta}}{\alpha} = \frac{\ln(S_0/S_\infty)}{S_0 - S_\infty}$$

Predicted maximum 30.4, very close to observed 29

Key: S and I are observed and α, β are estimated in *hindsight*

Transmission Rates R_0 for 1918 Episode

- 1918-19 influenza pandemic:

Mills et al. 2004:	45 US cities	3 (2-4)
Viboud et al. 2006:	England and Wales	1.8
Massad et al. 2007:	Sao Paulo Brazil	2.7
Nishiura, 2007:	Prussia, Germany	3.41
Chowell et al., 2006:	Geneva, Switzerland	2.7-3.8
Chowell et al., 2007:	San Francisco	2.7-3.5

The larger the R_0 the more severe the epidemic.

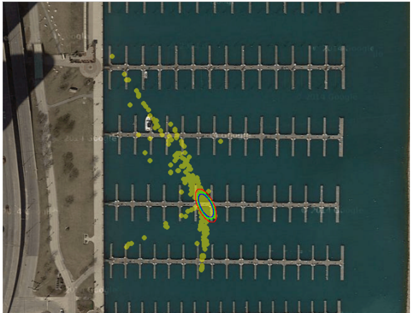
Transmission parameters vary substantially from epidemic to epidemic

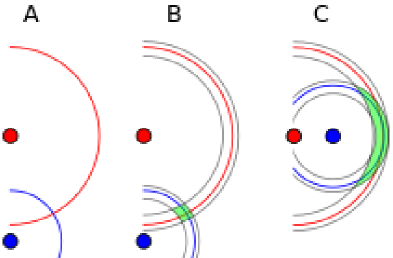
Boat Localization Example

Localization with measurement update

- A boat sails from one island to another
- Boat is trying to identify its location $\theta \sim N(m_0, C_0)$
- Using a sequence of measurements to one of the islands $x_1, \ldots, x_n$

Measurements are noisy due to dilution of precision
http://www.sailingmates.com/your-gps-can-kill-you/





Reckoning

Localization with no measurement updates is called reckoning

Figure 1: source:
http://www.hakaimagazine.com/article-short/traversing-seas

Kalman Filter

$$\theta \sim N(m_0, C_0)$$

$$x_t = \theta + w_t, \quad w_t \sim N(0, \sigma^2)$$

$$x_1, x_2, \ldots \mid \theta \sim N(\theta, \sigma^2)$$

The prior variance C_0 might be quite large if you are very uncertain about your guess m_0

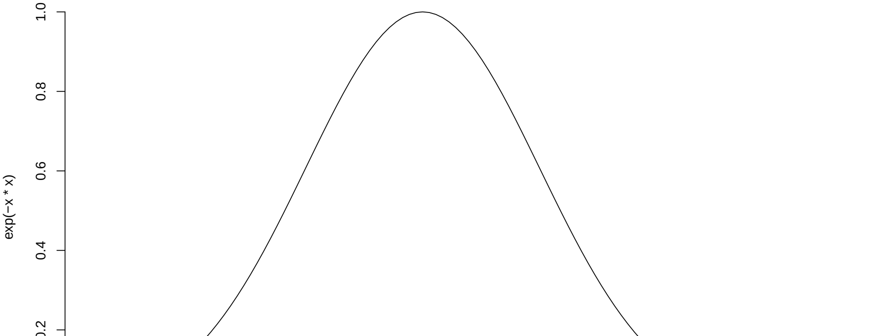
Given the measurements $x^n = (x_1, \ldots, x_n)$, you update your opinion about θ computing its posterior density, using the Bayes formula

Normal Model

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

Or multivariate equivalent

$$f(x) = (2\pi)^{-k/2}|\Sigma|^{-1/2} \exp^{-0.5(x-\mu)^T\Sigma^{-1}(x-\mu)}$$



The Conjugate Prior for the Normal Distribution

We will look at the Gaussian distribution from a Bayesian point of view. In the standard form, the likelihood has two parameters, the mean μ and the variance σ^2

$$p(x^n|\mu, \sigma^2) \propto \frac{1}{\sigma^n} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right)$$

Normal Prior

In case when we know the variance σ^2, but do not know mean μ, we assume μ is random. To have conjugate prior we choose

$$p(\mu|\mu_0, \sigma_0) \propto \frac{1}{\sigma_0} \exp\left(-\frac{1}{2\sigma_0^2}(\mu - \mu_0^2)\right)$$

In practice, when little is known about μ, it is common to set the location hyper-parameter to zero and the scale to some large value.

Normal Model with Unknown Mean, Known Variance

Suppose we wish to estimate a model where the likelihood of the data is normal with an unknown mean μ and a known variance σ^2. Our parameter of interest is μ. We can use a conjugate Normal prior on μ, with mean μ_0 and variance σ_0^2.

$$p(\mu|x^n, \sigma^2) \propto p(x^n|\mu, \sigma^2)p(\mu) \quad \text{(Bayes rule)}$$

$$N(\mu_1, \tau_1) = N(\mu, \sigma^2) \times N(\mu_0, \sigma_0^2)$$

Useful Identity

One of the most useful algebraic tricks for calculating posterior distribution is **completing the square**.

$$\frac{(x-\mu_1)^2}{\sigma_1} + \frac{(x-\mu_2)^2}{\sigma_2} = \frac{(x-\mu_3)^2}{\sigma_3} + \frac{(\mu_1-\mu_2)^2}{\sigma_1+\sigma_2}$$

where

$$\mu_3 = \sigma_3(\mu_1/\sigma_1 + \mu_2/\sigma_2)$$

and

$$\sigma_3 = (1/\sigma_1 + 1/\sigma_2)^{-1}$$

Prior:

$$\theta \sim \frac{e^{-\frac{(\theta-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$$

Likelihood:

$$x \mid \theta \sim \frac{e^{-\frac{(\theta-y)^2}{2r^2}}}{\sqrt{2\pi}r}$$

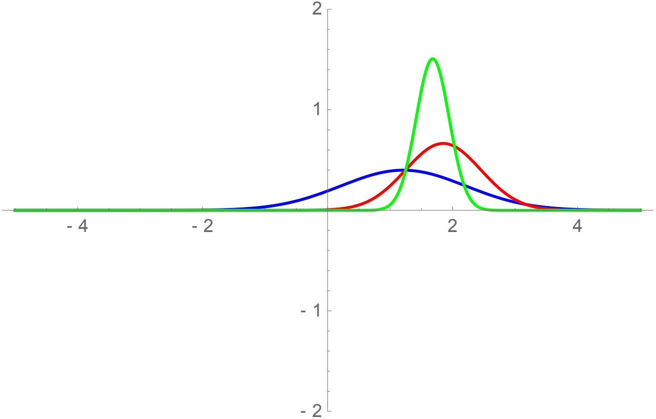
Posterior mean:

$$\frac{x\sigma^2 + \mu r^2}{r^2+\sigma^2}$$

Posterior variance:

1

Prior, Likelihood, Posterior



After n steps

$$
\begin{aligned}
p(\mu|x^n) &\propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) \times \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right) \\
&\propto \exp\left(-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{2\sigma^2} - \frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right) \\
&= \exp\left(-\frac{1}{2}\left[\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma^2} + \frac{(\mu-\mu_0)^2}{\sigma_0^2}\right]\right) \\
&= \exp\left(-\frac{1}{2\sigma^2\sigma_0^2}\left[\sigma_0^2 \sum_{i=1}^{n}(x_i-\mu)^2 + \sigma^2(\mu-\mu_0)^2\right]\right) \\
&= \exp\left(-\frac{1}{2\sigma^2\sigma_0^2}\left[\sigma_0^2 \sum_{i=1}^{n}(x_i^2-2\mu x_i+\mu^2) + \sigma^2(\mu^2-2\mu\mu_0+\mu_0^2)\right.\right.
\end{aligned}
$$

After n steps

We can multiply the $2\mu x_i$ term in the summation by n/n in order to get the equations in terms of the sufficient statistic $\bar{x}^n$

$$p(\mu|x^n) \propto \exp\left(-\frac{1}{2\sigma^2\sigma_0^2}\left[\sigma_0^2\sum_{i=1}^{n}(x_i^2 - \frac{n}{n}2\mu x_i + \mu^2) + \sigma^2(\mu^2 - 2\mu\mu_0 + \mu_0^2\right.\right.$$

$$= \exp\left(-\frac{1}{2\sigma^2\sigma_0^2}\left[\sigma_0^2\sum_{i=1}^{n}x_i^2 - \sigma_0^2 2\mu n\bar{x}^n + \tau_n^0 n\mu^2 + \sigma^2\mu^2 - 2\mu\mu_0\sigma\right.\right.$$

set $k = \sigma_0^2\sum_{i=1}^{n} x_i^2 + \mu_0^2\sigma^2$ (they do not contain μ)

$$p(\mu|x^n) \propto \exp\left(-\frac{1}{2}\left[\mu^2\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right) - 2\mu\left(\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{x}^n}{\sigma^2}\right) + k\right]\right)$$

After n steps

Let's multiply by

$$\frac{1/\sigma_0^2 + n/\sigma^2}{1/\sigma_0^2 + n/\sigma^2}$$

Now

$$p(\mu|x^n) \propto \exp\left(-\frac{1}{2}\left(1/\sigma_0^2 + n/\sigma^2\right)\left(\mu - \frac{\mu_0/\sigma_0^2 + n\bar{x}^n/\sigma^2}{1/\sigma_0^2 + n/\sigma^2}\right)^2\right)$$

$$p(\mu|x^n) \propto \exp\left(-\frac{1}{2}\left(1/\sigma_0^2 + n/\sigma^2\right)\left(\mu - \frac{\mu_0/\sigma_0^2 + n\bar{x}^n/\sigma^2}{1/\sigma_0^2 + n/\sigma^2}\right)^2\right)$$

After n steps

- Posterior mean: $\mu_n = \dfrac{\mu_0/\sigma_0^2 + n\bar{x}^n/\sigma^2}{1/\sigma_0^2 + n/\sigma^2}$
- Posterior variance: $\sigma_n^2 = \left(1/\sigma_0^2 + n/\sigma^2\right)^{-1}$
- Posterior precision:: $\tau_n^2 = 1/\sigma_0^2 + n/\sigma^2$

Posterior Precision is just the sum of the prior precision and the data precision.

Posterior Mean

$$\mu_n = \frac{\mu_0/\sigma_0^2 + n\bar{x}^n/\sigma^2}{1/\sigma_0^2 + n/\sigma^2}$$
$$= \frac{\mu_0\sigma^2}{\sigma^2 + n\sigma_0^2} + \frac{\sigma_0^2 n\bar{x}^n}{\sigma^2 + n\sigma_0^2}$$

- As n increases, data mean dominates prior mean.
- As σ_0^2 decreases (less prior variance, greater prior precision), our prior mean becomes more important.

A state space model

A state space model consists of two equations:

$$Z_t = HS_t + w_t$$
$$S_{t+1} = FS_t + v_t$$

where S_t is a state vector of dimension m, Z_t is the observed time series, F, G, H are matrices of parameters, $\{w_t\}$ and $\{v_t\}$ are *iid* random vectors satisfying

$$\mathsf{E}(w_t) = 0, \quad \mathsf{E}(v_t) = 0, \quad \mathrm{cov}(v_t) = V, \quad \mathrm{cov}(w_t) = W$$

and $\{w_t\}$ and $\{v_t\}$ are independent.

State Space Models

- State space models consider a time series as the output of a dynamic system perturbed by random disturbances.
- Natural interpretation of a time series as the combination of several components, such as trend, seasonal or regressive components.
- Computations can be implemented by recursive algorithms.

Types of Inference

- Model building versus inferring unknown variable. Assume a linear model $Z = HS + \epsilon$
- Model building: know signal S, observe Z, infer H (a.k.a. model identification, learning)
- Estimation: know H, observe Z, estimate S
- Hypothesis testing: unknown takes one of few possible values; aim at small probability of incorrect decision
- Estimation: aim at a small estimation error

Time Series Estimation Tasks

- Filtering: To recover the state vector S_t given Z^t
- Prediction: To predict S_{t+h} or Z_{t+h} for $h > 0$, given Z^t
- Smoothing: To estimate S_t given Z^T , where $T > t$

Property of Multivariate Normal

Under normality, we have

- that normal prior plus normal likelihood results in a normal posterior,
- that if the random vector (X, Y) are jointly normal

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}\right),$$

- then the conditional distribution of X given $Y = y$ is normal

$$X|Y = y \sim N\left[\mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y), \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}\right].$$

From State Space Model

$$\begin{aligned}
S_{t+1}^{t} &= FS_t \\
Z_{t+1}^{t} &= HS_{t+1}^{t} \\
P_{t+1}^{t} &= FP_tF^T + GQG^T \\
V_{t+1}^{t} &= HP_{t+1}^{t}H^T + R \\
C_{t+1}^{t} &= HP_{t+1}^{t}
\end{aligned}$$

- P_{t+j}^{t} = conditional covariance matrix of S_{t+j} given $\{Z_t, Z_{t-1}, \cdots\}$ for $j \geq 0$
- S_{t+j}^{t} = conditional mean of S_{t+j} given $\{Z_t, Z_{t-1}, \cdots\}$
- V_{t+1}^{t} = conditional variance of Z_{t+1} given $Z^t = \{Z_t, Z_{t-1}, \cdots\}$
- C_{t+1}^{t} = conditional covariance between Z_{t+1} and S_{t+1}

Joint conditional distribution $P(S_{t+1}, Z_{t+1}|Z^t)$

$$\begin{bmatrix} S_{t+1} \\ Z_{t+1} \end{bmatrix}_t \sim N\left(\begin{bmatrix} S_{t+1}^t \\ Z_{t+1}^t \end{bmatrix}, \begin{bmatrix} P_{t+1}^t & P_{t+1}^t H' \\ HP_{t+1}^t & HP_{t+1}^t H' + R \end{bmatrix} \right)$$

$P(S_{t+1}|Z_{t+1})$

Finally, when Z_{t+1} becomes available, we may use the property of nromality to update the distribution of S_{t+1} . More specifically,

$$S_{t+1} = S_{t+1}^t + P_{t+1}^t H^T [HP_{t+1}^t H^T + R]^{-1} (Z_{t+1} - Z_{t+1}^t)$$

and

$$P_{t+1} = P_{t+1}^t - P_{t+1}^t H^T [HP_{t+1}^t H' + R]^{-1} HP_{t+1}^t.$$

Predictive residual:

$$R_{t+1}^t = Z_{t+1} - Z_{t+1}^t = Z_{t+1} - HS_{t+1}^t \neq 0$$

means there is new information about the system so that the state vector should be modified. The contribution of r_{t+1}^t to the state vector, of course, needs to be weighted by the variance of r_{t+1}^t and the conditional covariance matrix of S_{t+1}.

Kalman filter

- Predict:

$$\begin{aligned} S_{t+1}^{t} &= FS_t \\ Z_{t+1}^{t} &= HS_{t+1}^{t} \\ P_{t+1}^{t} &= FP_tF^T + GQG^T \\ V_{t+1}^{t} &= HP_{t+1}^{t}H^T + R \end{aligned}$$

- Update:

$$\begin{aligned} S_{t+1|t+1} =& S_{t+1}^{t} + P_{t+1}^{t}H^T[HP_{t+1}^{t}H^T + R]^{-1}(Z_{t+1} - Z_{t+1}^{t}) \\ P_{t+1|t+1} =& P_{t+1}^{t} - P_{t+1}^{t}H^T[HP_{t+1}^{t}H^T + R]^{-1}HP_{t+1}^{t} \end{aligned}$$

Kalman filter

- starts with initial prior information S_0 and P_0
- predicts Z_1^0 and V_1^0
- Once the observation Z_1 is available, uses the updating equations to compute S_1 and P_1

$S_{1|1}$ and $P_{1|1}$ is the prior for the next observation.

This is the Kalman recusion.

Kalman filter

- effect of the initial values S_0 and P_0 is decresing as t increases
- for a stationary time series, all eigenvalues of the coefficient matrix F are less than one in modulus
- Kalman filter recursion ensures that the effect of the initial values indeed vanishes as t increases
- uncertainty about the state is always normal

Local Trend Model

$$
\begin{aligned}
y_t =& \mu_t + e_t, \ e_t \sim N(0, \sigma_e^2) \\
\mu_{t+1} =& \mu_t + \eta_t, \ \eta_t \sim N(0, \sigma_\eta^2)
\end{aligned}
$$

- $\{e_t\}$ and $\{\eta_t\}$ are iid Gaussian white noise
- μ_0 is given (possible as a distributed value)
- trend μ_t is not observable
- we observe some noisy version of the trend y_t
- such a model can be used to analyze realized volatility: μ_t is the log volatility and y_t is constructed from high frequency transactions data

Local Trend Model

$$\begin{aligned} y_t =& \mu_t + e_t, \ e_t \sim N(0, \sigma_e^2) \\ \mu_{t+1} =& \mu_t + \eta_t, \ \eta_t \sim N(0, \sigma_\eta^2) \end{aligned}$$

- if $\sigma_e = 0$, then we have ARIMA(0,1,0) model
- if $\sigma_e > 0$, then we have ARIMA(0,1,1) model, satisfying

$$(1-B)y_t = (1-\theta B)a_t, \ a_t \sim N(0, \sigma_a^2)$$

σ_a and θ are determined by σ_e and σ_η

$$(1-B)y_t = \eta_{t-1} + e_t - e_{t-1}$$

Liner Regression (time dependent parameters)

$$
\begin{aligned}
y_t &= \alpha_t + \beta_t x_t + \epsilon_t & \epsilon_t &\sim N(0, \sigma^2) \\
\alpha_t &= \quad \alpha_{t-1} + \epsilon_t^{\alpha} & \epsilon_t^{\alpha} &\sim N(0, \sigma_\alpha^2) \\
\beta_t &= \quad \beta_{t-1} + \epsilon_t^{\beta} & \epsilon_t^{\beta} &\sim N(0, \sigma_\beta^2)
\end{aligned}
$$

dlm Package

- `dlmModARMA`: for an ARMA process, potentially multivariate
- `dlmModPoly`: for an n^{th} order polynomial
- `dlmModReg` : for Linear regression
- `dlmModSeas`: for periodic – Seasonal factors
- `dlmModTrig`: for periodic – Trigonometric form

Local Linear Trend

$$\begin{aligned} y_t &= \mu_t + \upsilon_t & \upsilon_t &\sim N(0, V) \\ \mu_t &= \mu_{t-1} + \delta_{t-1} + \omega_t^{\mu} & \omega_t^{\mu} &\sim N(0, W^{\mu}) \\ \delta_t &= \delta_{t-1} + \omega_t^{\delta} & \omega_t^{\delta} &\sim N(0, W^{\delta}) \end{aligned}$$

Simple exponential smoothing with additive errors

$$x_t = \ell_{t-1} + \varepsilon_t$$
$$\ell_t = \ell_{t-1} + \alpha\varepsilon_t.$$

Holt's linear method with additive errors

$$
\begin{aligned}
y_t &= \ell_{t-1} + b_{t-1} + \varepsilon_t \\
\ell_t &= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
b_t &= b_{t-1} + \beta\varepsilon_t,
\end{aligned}
$$

Relation to ARMA models

Consider relation with ARMA models. The basic relations are

- an ARMA model can be put into a state space form in "infinite" many ways;
- for a given state space model in, there is an ARMA model.

State space model to ARMA model

The second possibility is that there is an observational noise. Then, the same argument gives

$$(1+\alpha_1 B+\cdots+\alpha_m B^m)(Z_{t+m}-\epsilon_{t+m}) = (1-\theta_1 B-\cdots-\theta_{m-1}B^{m-1})a_{t+m}$$

By combining ϵ_t with a_t , the above equation is an ARMA(m, m) model.

ARMA model to state space model: AR(2)

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$$

For such an AR(2) process, to compute the forecasts, we need Z_{t-1} and Z_{t-2} . Therefore, it is easily seen that

$$\begin{bmatrix} Z_{t+1} \\ Z_t \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Z_t \\ Z_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e_t,$$

where $e_t = a_{t+1}$ and

$$Z_t = [1, 0] S_t$$

where $S_t = (Z_t, Z_{t-1})^T$ and there is no observational noise.

ARMA model to state space model: MA(2)

$$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

Method 1:

$$\begin{bmatrix} a_t \\ a_{t-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{t-1} \\ a_{t-2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} a_t$$

$$Z_t = [-\theta_1, -\theta_2] S_{t-1} + a_t$$

Here the innovation a_t shows up in both the state transition equation and the observation equation. The state vector is of dimension 2.

ARMA model to state space model: MA(2)

Method 2: For an MA(2) model, we have

$$\begin{aligned} Z_t^t &= Z_t \\ Z_{t+1}^t &= -\theta_1 a_t - \theta_2 a_{t-1} \\ Z_{t+2}^t &= -\theta_2 a_t \end{aligned}$$

Let $S_t = (Z_t, -\theta_1 a_t - \theta_2 a_{t-1}, -\theta_2 a_t)^T$. Then,

$$S_{t+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} S_t + \begin{bmatrix} 1 \\ -\theta_1 \\ -\theta_2 \end{bmatrix} a_{t+1}$$

and

$$Z_t = [1, 0, 0] S_t$$

Here the state vector is of dimension 3, but there is no observational noise.

ARMA model to state space model: Akaike's approach

Consider ARMA(p, q) process, let $m = max\{p, q+1\}$, $\phi_i = 0$ for $i > p$ and $\theta_j = 0$ for $j > q$.

$$S_t = (Z_t, Z^t_{t+1}, Z^t_{t+2}, \cdots, Z^t_{t+m-1})^T$$

where $Z^t_{t+\ell}$ is the conditional expectation of $Z_{t+\ell}$ given $\Psi_t = \{Z_t, Z_{t-1}, \cdots\}$. By using the updating equation f forecasts (recall what we discussed before)

$$Z_{t+1}(\ell - 1) = Z_t(\ell) + \psi_{\ell-1} a_{t+1},$$

ARMA model to state space model: Akaike's approach

$$S_t = (Z_t, Z_{t+1}^t, Z_{t+2}^t, \cdots, Z_{t+m-1}^t)^T$$

$$S_{t+1} = FS_t + Ga_{t+1}$$

$$Z_t = [1, 0, \cdots, 0]S_t$$

where

$$F = \left[\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \\ \phi_m & \phi_{m-1} & \cdots & \phi_2 & \phi_1 \end{array}\right], G = \begin{bmatrix} 1 \\ \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{m-1} \end{bmatrix}$$

The matrix F is call a companion matrix of the polynomial $1 - \phi_1 B - \cdots - \phi_m B^m$.

ARMA model to state space model: Aoki's Method

Two-step procedure: First, consider the MA(q) part:

$$W_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

$$\begin{bmatrix} a_t \\ a_{t-1} \\ \vdots \\ a_{t-q+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{t-1} \\ a_{t-2} \\ \vdots \\ a_{t-q} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} a_t$$

$$W_t = [-\theta_1, -\theta_2, \cdots, -\theta_q] S_t + a_t$$

ARMA model to state space model: Aoki's Method

First, consider the AR(p) part:

$$Z_t = \phi_1 Z_{t-1} + ... + \phi_p Z_{t-p} + W_t$$

Define state-space vector as

$$S_t = (Z_{t-1}, Z_{t-2}, \cdots, Z_{t-p}, a_{t-1}, \cdots, a_{t-q})'$$

Then, we have

$$\begin{bmatrix} Z-t \\ Z_{t-1} \\ \vdots \\ Z_{t-p+1} \\ a_t \\ a_{t-1} \\ \vdots \\ a_{t-q+1} \end{bmatrix} = \left[\begin{array}{cccc|cccc} \phi_1 & \phi_2 & \cdots & \phi_p & -\theta_1 & -\theta_2 & \cdots & -\theta_q \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & & & \\ 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & 0 & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \end{array}\right] \begin{bmatrix} Z_{t-1} \\ Z_{t-2} \\ \vdots \\ Z_{t-p} \\ a_{t-1} \\ a_{t-2} \\ \vdots \\ a_{t-q} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$Z_t = [\phi_1, \cdots, \phi_p, -\theta_1, \cdots, -\theta_q] S_t + a_t$$

MLE Estimation

Innovations are given by

$$\epsilon_t = Z_t - HS_t^{t-1}$$

can be shown that $\mathrm{var}(\epsilon_t) = \Sigma_t$, where

$$\Sigma_t = HP_t^{t-1}H^T + R$$

Incomplete Data Likelihood:

$$-\ln L(\Theta) = \frac{1}{2}\sum_{t=1}^{n} \log|\Sigma_t(\Theta)| + \frac{1}{2}\sum_{t=1}^{n} \epsilon_t(\Theta)^T \Sigma(\Theta)^{-1} \epsilon_t(\Theta)$$

Here $\Theta = (F, Q, R)$. Use BFGS to find a sequence of Θ's and stop when stagnation happens.

Kalman Smoother

- Input: initial distribution X_0 and data $Z_1, ..., Z_T$
- Algorithm: forward-backward pass
- Forward pass: Kalman filter: compute S_{t+1}^{t} and S_{t+1}^{t+1} for $0 \leq t < T$
- Backward pass: Compute S_t^T for $0 \leq t < T$

Backward Pass

- Compute X_t^T given $S_{t+1}^T \sim N(m_{t+1}^T, C_{t+1}^T)$
- Reverse arrow: $S_t^t \leftarrow X_{t+1}^t$
- Same as incorporating measurement in filter
- Compute joint (S_t^t, S_{t+1}^t)
- Compute conditional $(S_t^t \mid S_{t+1}^t)$
- New: S_{t+1} is not "known", we only know its distribution: $S_{t+1} \sim S_{t+1}^T$
- "Uncondition" on S_{t+1} to compute S_t^T using laws of total expectation and variance

Kalman Smoother

A smoothed version of data (an estimate, based on the entire data set) If S_n and P_n obtained via Kalman recursions, then for $t = n, .., 1$

$$
\begin{aligned}
S_{t-1}^{t} &= S_{t-1} + J_{t-1}(S_t^n - S_t^{t-1}) \\
P_{t-1}^{n} &= P^{t-1} + J_{t-1}(P_t^n - P_t^{t-1})J_{t-1}^T \\
J_{t-1} &= P_{t-1}F^T[P_t^{t-1}]^{-1}
\end{aligned}
$$

Kalman and Histogran Filter Shortciomings

Kalman:

- linear dynamics
- linear measurement model
- normal errors
- unimodal uncertainty

Histogram:

- discrete states
- approximation
- inefficient in memory

MCMC Financial Econometrics

Set of tools for inference and pricing in continuous-time models.

- Simulation-based and provides a unified approach to state and parameter inference. Can also be applied sequentially.
- Can handle Estimation and Model risk. Important implications for financial decision making
- Bayesian inference. Uses conditional probability to solve an inverse problem and estimates expectations using Monte Carlo.

Filtering, Smoothing, Learning and Prediction

Data y_t depends on a , x_t.

$$\text{Observation equation: } y_t = f\left(x_t, \varepsilon_t^y\right)$$
$$\text{State evolution: } x_{t+1} = g\left(x_t, \varepsilon_{t+1}^x\right),$$

- Posterior distribution of $p\left(x_t|y^t\right)$ where $y^t = \left(y_1, ..., y_t\right)$
- Prediction and Bayesian updating.

$$p\left(x_{t+1}|y^t\right) = \int p\left(x_{t+1}|x_t\right) p\left(x_t|y^t\right) dx_t,$$

updated by Bayes rule

$$\underbrace{p\left(x_{t+1}|y^{t+1}\right)}_{\text{Posterior}} \propto \underbrace{p\left(y_{t+1}|x_{t+1}\right)}_{\text{Likelihood}} \underbrace{p\left(x_{t+1}|y^t\right)}_{\text{Prior}}.$$

Nonlinear Model

- The observation and evolution dynamics are

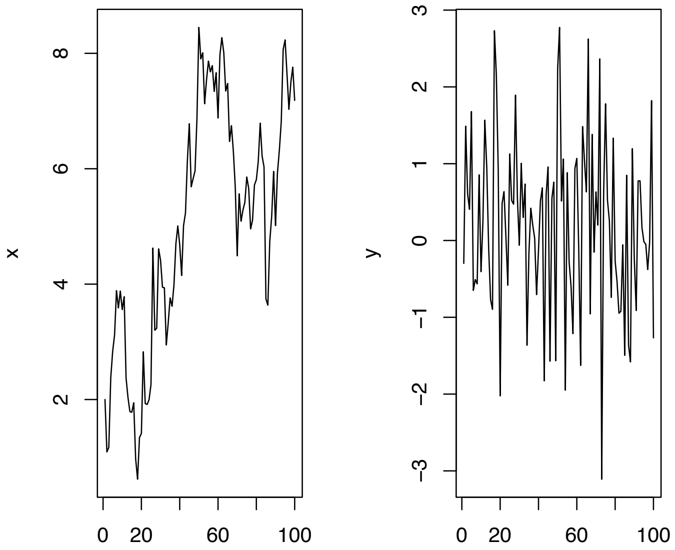
$$y_t = \frac{x_t}{1 + x_t^2} + v_t \text{ , where } v_t \sim N(0, 1)$$
$$x_t = x_{t-1} + w_t \text{ , where } w_t \sim N(0, 0.5)$$

- Initial condition $x_0 \sim N(1, 10)$

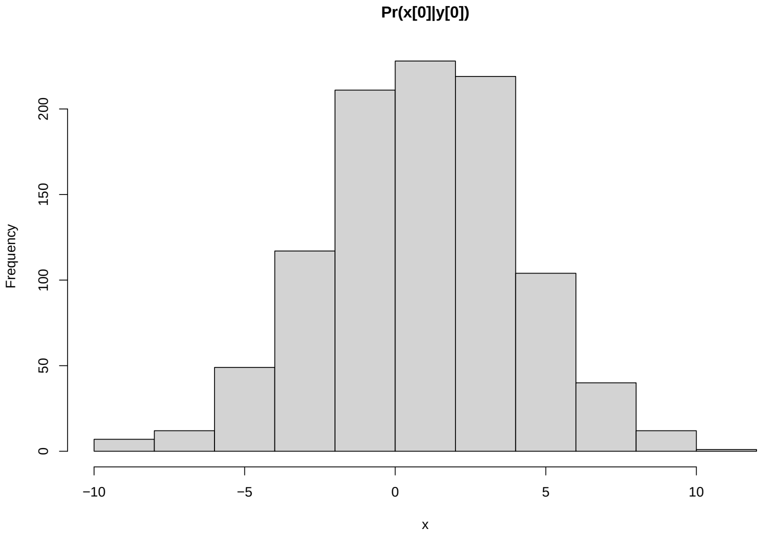
Fundamental question:

How do the filtering distributions $p(x_t|y^t)$ propagate in time?

Nonlinear: $y_t = x_t/(1 + x_t^2) + v_t$

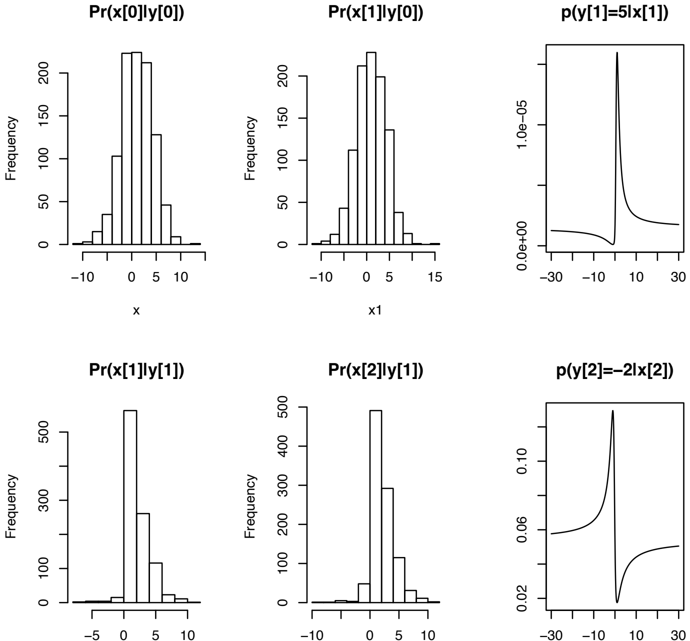


Simulate Data



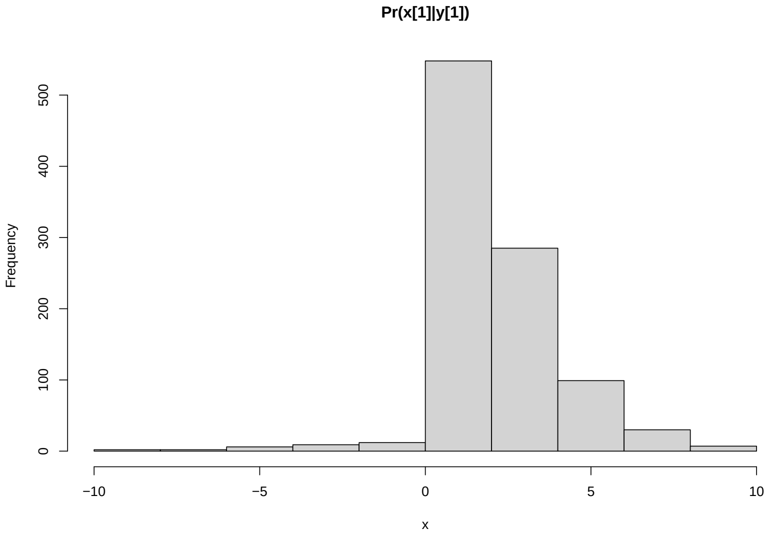
Pr(x[1]|y[0])

Nonlinear Filtering

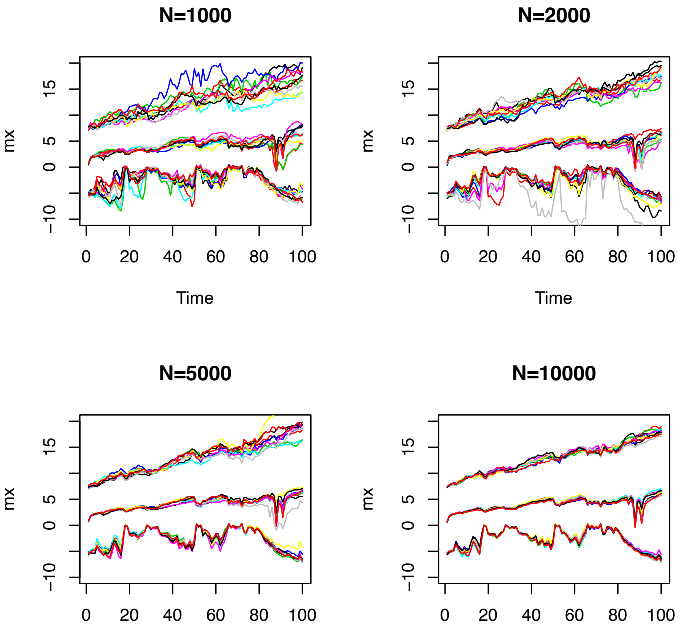


Resampling

Key: resample and propagate particles



Propagation of MC error



Dynamic Linear Model (DLM): Kalman Filter

Kalman filter for linear Gaussian systems

- FFBS (Filter Forward Backwards Sample)

This determines the posterior distribution of the states

$$p(x_t|y^t) \text{ and } p(x_t|y^T)$$

Also the joint distribution $p(x^T|y^T)$ of the hidden states.

- Discrete Hidden Markov Model HMM (Baum-Welch, Viterbi)
- With parameters *known* the Kalman filter gives the exact recursions.

Simulate DLM

Dynamic Linear Models

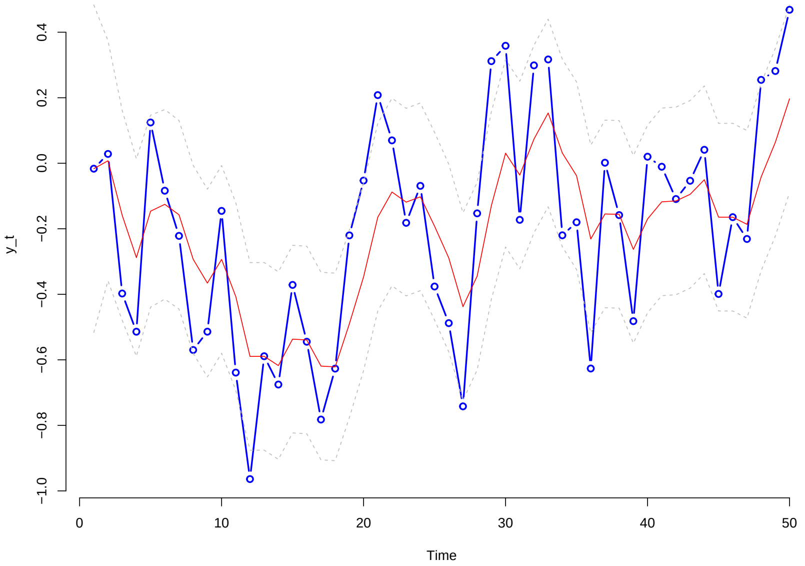
$$y_t = x_t + v_t \text{ and } x_t = \alpha + \beta x_{t-1} + w_t$$

Simulate Data

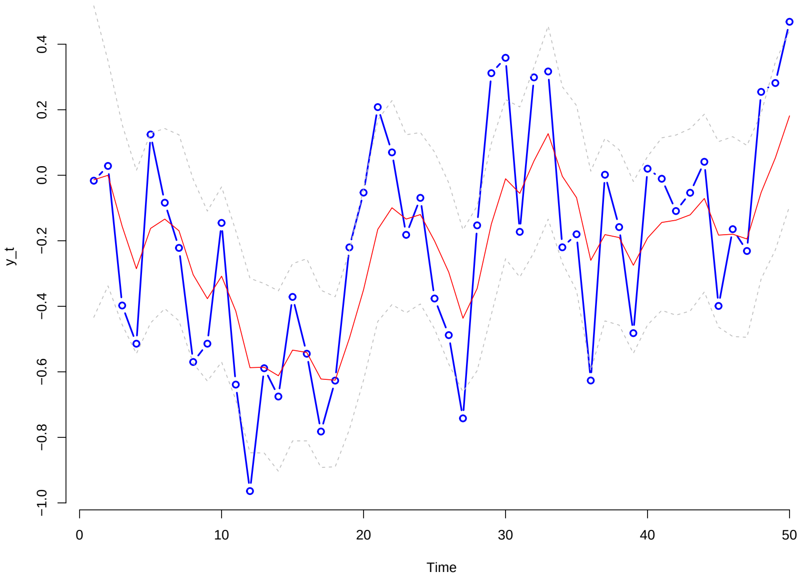
Exact calculations

Kalman Filter recursions

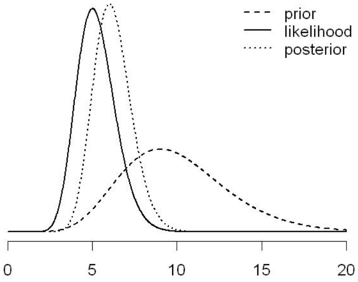
DLM Data



Bootstrap Filter



Streaming Data: How do Parameter Distributions change in Time?



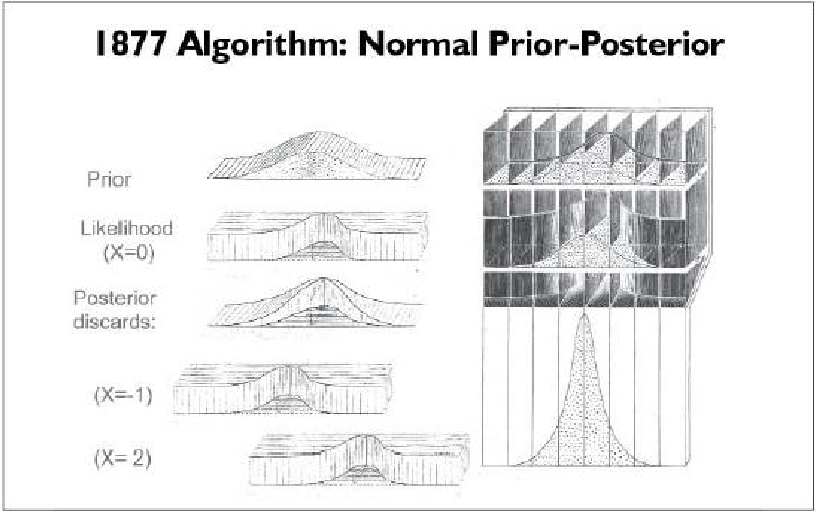
Bayes theorem:

$$p(\theta \mid y^t) \propto p(y_t \mid \theta)\, p(\theta \mid y^{t-1})$$

Online Dynamic Learning

- Real-time surveillance
- Bayes means sequential updating of information
- Update posterior density $p(\theta \mid y_t)$ with every new observation $(t = 1, \ldots, T)$ - "sequential learning"

Galton 1877: First Particle Filter



Streaming Data: Online Learning

Construct an essential state vector Z_{t+1}.

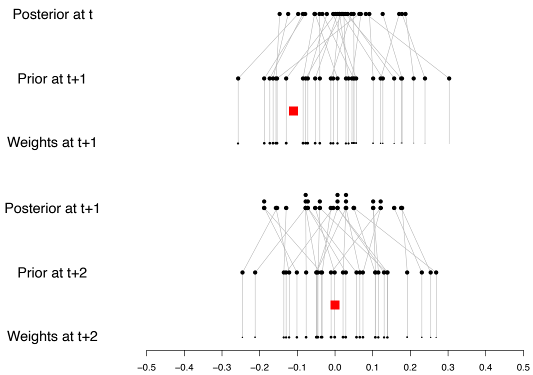
$$p(Z_{t+1}|y^{t+1}) = \int p(Z_{t+1}|Z_t, y_{t+1})\, d\mathbb{P}(Z_t|y^{t+1})$$

$$\propto \int \underbrace{p(Z_{t+1}|Z_t, y_{t+1})}_{propagate} \overbrace{\underbrace{p(y_{t+1}|Z_t)}_{resample}\ d\mathbb{P}(Z_t|y^t)}$$

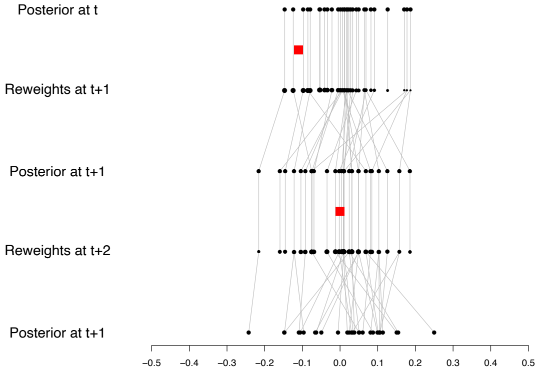
1. *Re-sample* with weights proportional to $p(y_{t+1}|Z_t^{(i)})$ and generate $\{Z_t^{\zeta(i)}\}_{i=1}^N$
2. *Propagate* with $Z_{t+1}^{(i)} \sim p(Z_{t+1}|Z_t^{\zeta(i)}, y_{t+1})$ to obtain $\{Z_{t+1}^{(i)}\}_{i=1}^N$

Parameters: $p(\theta|Z_{t+1})$ drawn "offline"

Sample – Resample



Resample – Sample



Particle Methods: Blind Propagation

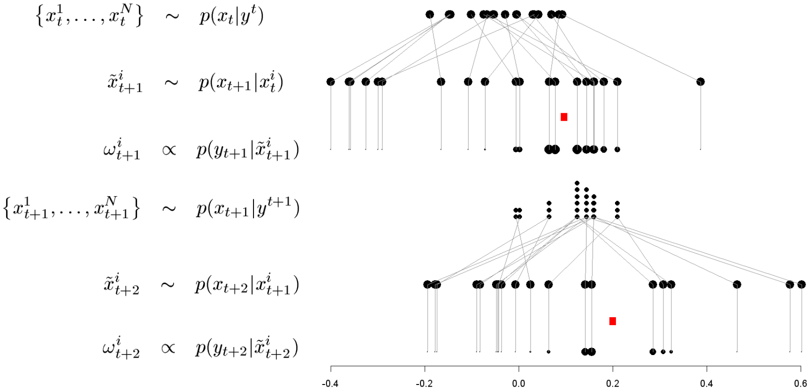


Figure 4: Propagate-Resample is replaced by Resample-Propagate

Traffic Problem

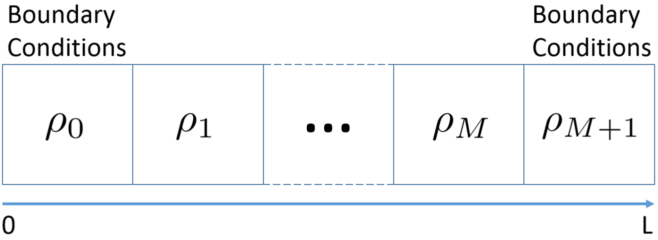


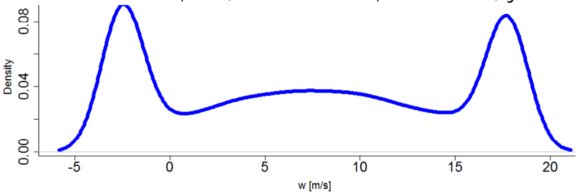
Figure 5: State-Space

Wave Speed Propagation is a Mixture Distribution

Shock wave propagation speed is a mixture, when calculated using Godunov scheme

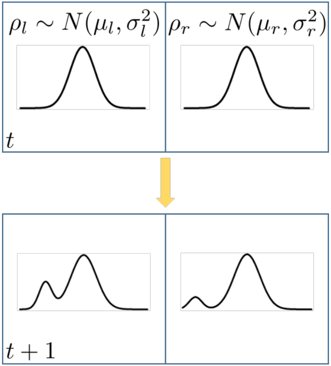
$$w = \frac{q(\rho_l) - q(\rho_r)}{\rho_l - \rho_r} \ \left[\frac{mi}{h}\right] = \left[\frac{veh}{h}\right]\left[\frac{mi}{veh}\right].$$

Assume $\rho_l \sim TN(32, 16, 0, 320)$ and $\rho_r \sim TN(48, 16, 0, 320)$
$q_c = 1600\ veh/h$, $\rho_c = 40\ veh/mi$, and $\rho_{jam} = 320\ veh/mi$



Traffic Flow Speed Forecast is a Mixtrue Dsitribution

Theorem: The solution (including numerical) to the LWR model with stochastic initial conditions is a mixture distribution.



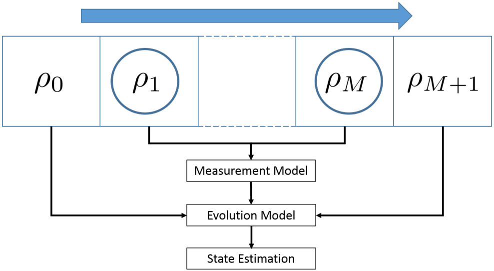
A moment based filters such as Kalman Filter or Extended Kalman Filter would not capture the mixture.

Problem at Hand

The Parameter Learning and State Estimation Problem

- Goal: given sparse sensor measurements, find the distribution over traffic state and underlying traffic flow parameters $p(\theta_t, \phi | y_1, y_2, ..., y_t)$; $\phi = (q_c, \rho_c)$
- Parameters of the evolution equation (LWR) are stochastic
- Distribution over state is a mixture
- Can't use moment based filters (KF, EKF,...)

Data Assimilation: State Space Representation



State space formulation allows to combine knowledge from analytical model with the one from field measurements, while taking model and measurement errors into account

State Space Representation

- State vector $\theta_t = (\rho_{1t}, \ldots, \rho_{nt})$
- Boundary conditionals ρ_{0t} and $\rho_{(n+1)t}$
- Underlying parameters $\phi = (q_c, \rho_c)$ are stochastic

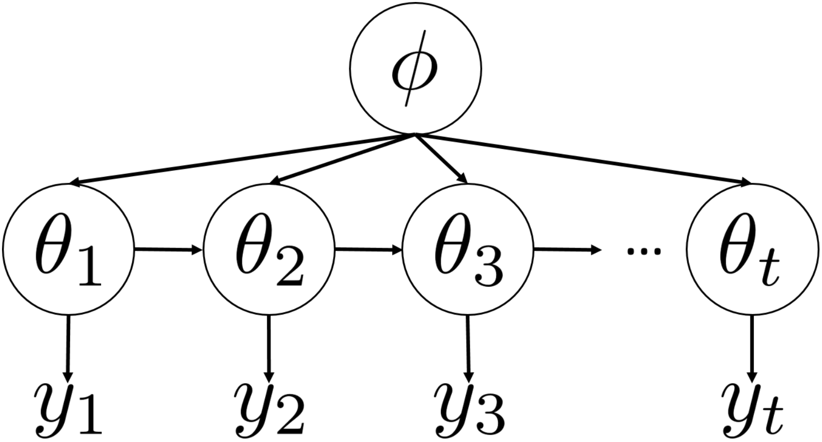
$$\text{Observation: } y_{t+1} = H\theta_{t+1} + v; \;\; v \sim N(0, V) \tag{1}$$

$$\text{Evolution: } \theta_{t+1} = f_\phi(\theta_t) + w; \;\; w \sim N(0, W) \tag{2}$$

$H : \mathbb{R}^M \to \mathbb{R}^k$ in the measurement model. $\phi = (q_c, \rho_c, \rho_{max})$.

Parameter priors: $q_c \sim N(\mu_q, \sigma_c^2)$, $\rho_c = Uniform(\rho_{min}, \rho_{max})$

Particle Parameter Learning



Sample-based PDF Representation

- Regions of high density: Many particles and Large weight of particles
- Uneven partitioning
- Discrete approximation for continuous pdf

$$p^N\left(\theta_{t+1}|y^{t+1}\right) \propto \sum_{i=1}^{N} w_t^{(i)} p\left(\theta_{t+1}|\theta_t^{(i)}, y_{t+1}\right)$$

Particle Filter

Bayes Rule:

$$p(y_{t+1}, \theta_{t+1}|\theta_t) = p(y_{t+1}|\theta_t)\, p(\theta_{t+1}|\theta_t, y_{t+1}).$$

- Given a particle approximation to $p^N\left(\theta_t|y^t\right)$

$$\begin{aligned} p^N\left(\theta_{t+1}|y^{t+1}\right) &\propto \sum_{i=1}^{N} p\left(y_{t+1}|\theta_t^{(i)}\right) p\left(\theta_{t+1}|\theta_t^{(i)}, y_{t+1}\right) \\ &= \sum_{i=1}^{N} w_t^{(i)} p\left(\theta_{t+1}|\theta_t^{(i)}, y_{t+1}\right), \end{aligned}$$

where

$$w_t^{(i)} = \frac{p\left(y_{t+1}|\theta_t^{(i)}\right)}{\sum_{i=1}^{N} p\left(y_{t+1}|\theta_t^{(i)}\right)}.$$

- Essentially a mixture Kalman filter

Particle Parameter Learning

Given particles (a.k.a. random draws) $(\theta_t^{(i)}, \phi^{(i)}, s_t^{(i)})$, $i = 1, \ldots, N$

$$p(\theta_t | y_{1:t}) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\theta^{(i)}} \ .$$

- First resample $(\theta_t^{k(i)}, \phi^{k(i)}, s_t^{k(i)})$ with weights proportional to $p(y_{t+1} | \theta_t^{k(i)}, \phi^{k(i)})$ and $s_t^{k(i)} = S(s_t^{(i)}, \theta_t^{k(i)}, y_{t+1})$ and then propogate to $p(\theta_{t+1} | y_{1:t+1})$ by drawing $\theta_{t+1}^{(i)}$ from $p(\theta_{t+1} | \theta_t^{k(i)}, \phi^{k(i)}, y_{t+1})$, $i = 1, \ldots, N$.
- Next we update the sufficient statistic as

$$s_{t+1} = S(s_t^{k(i)}, \theta_{t+1}^{(i)}, y_{t+1}),$$

for $i = 1, \ldots, N$, which represents a deterministic propogation.

- Finally, parameter learning is completed by drawing $\phi^{(i)}$ using $p(\phi | s_{t+1}^{(i)})$ for $i = 1, \ldots, N$.

Streaming Data

Online Learning

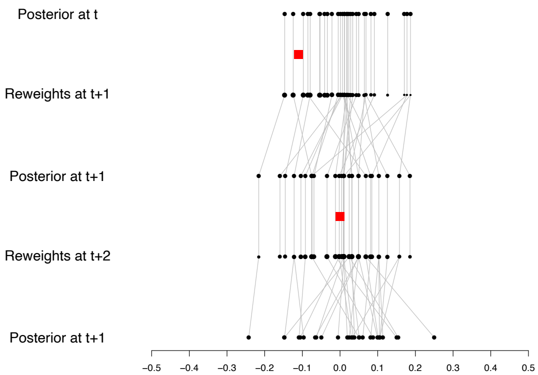
Construct an essential state vector Z_{t+1}.

$$
\begin{aligned}
p(Z_{t+1}|y^{t+1}) &= \int p(Z_{t+1}|Z_t, y_{t+1})\, d\mathbb{P}(Z_t|y^{t+1}) \\
&\propto \int \underbrace{p(Z_{t+1}|Z_t, y_{t+1})}_{\textit{propagate}} \overbrace{\underbrace{p(y_{t+1}|Z_t)}_{\textit{resample}}\; d\mathbb{P}(Z_t|y^t)}
\end{aligned}
$$

1. *Re-sample* with weights proportional to $p(y_{t+1}|Z_t^{(i)})$ and generate $\{Z_t^{\zeta(i)}\}_{i=1}^N$
2. *Propagate* with $Z_{t+1}^{(i)} \sim p(Z_{t+1}|Z_t^{\zeta(i)}, y_{t+1})$ to obtain $\{Z_{t+1}^{(i)}\}_{i=1}^N$

Parameters: $p(\theta|Z_{t+1})$ drawn "offline"

Resample – Propagate



Algorithm

These ingredients then define a particle filtering and learning algorithm for the sequence of joint posterior distributions $p(\theta_t, \phi | y_{1:t})$:

Step 1. (Resample) Draw an index $k_t\,(i) \sim Mult_N\left(w_t^{(1)}, ..., w_t^{(N)}\right)$,

where the weights are given by $w_t^{(i)} \propto p(y_{t+1}|(\theta_t, \phi)^{(i)})$, for $i = 1, ..., N$

Step 2. (Propagate) Draw $\theta_{t+1}^{(i)} \sim p\left(\theta_{t+1}|\theta_t^{k_t(i)}, y_{t+1}\right)$ for $i = 1, ..., N$.

Step 3. (Update) $s_{t+1}^{(i)} = S(s_t^{k_t(i)}, \theta_{t+1}^{(i)}, y_{t+1})$

Step 4. (Replenish) $\phi^{(i)} \sim p(\phi|s_{t+1}^{(i)})$

There are a number of efficiency gains from such an approach, e.g. it does not suffer from degeneracy problems associated with traditional propagate-resample algorithms when y_{t+1} is an outliers.

Obtaining state estimates from particles

- Any estimate of a function $f(\theta_t)$ can be calculated by discrete-approximation

$$\mathrm{E}(f(\theta_t)) = \frac{1}{N}\sum_{j=1}^{N} w_t^{(j)} f(\theta_t^{(j)})$$

- Mean:

$$\mathrm{E}(\theta_t) = \frac{1}{N}\sum_{j=1}^{N} w_t^{(j)} \theta_t^{(j)}$$

- MAP-estimate: particle with largest weight
- Robust mean: mean within window around MAP-estimate

Particle Filters: Pluses

- Estimation of full PDFs
- Non-Gaussian distributions (multi-modal)
- Non-linear state and observation model
- Parallelizable

Particle Filters: Minuses

- Degeneracy problem
- High number of particles needed
- Computationally expensive
- Linear-Gaussian assumption is often sufficient

Applications: Localization

- Track car position in given road map
- Track car position from radio frequency measurements
- Track aircraft position from estimated terrain elevation
- Collision Avoidance (Prediction)

Applications: Model Estimation

- Tracking with multiple motion-models
- Recovery of signal from noisy measurements
- Neural Network model selection (on-line classification)

Applications: Other

- Visual Tracking
- Prediction of (financial) time series
- Quality control in semiconductor industry
- Military applications: Target recognition from single or multiple images, Guidance of missiles
- Reinforcement Learning

Mixture Kalman Filter For Traffic

$$
\begin{aligned}
\text{Observation: } & y_{t+1} = Hx_{t+1} + \gamma^T z_{t+1} + v_{t+1},\ v_{t+1} \sim N(0, V_{t+1}) \\
\text{Evolution: } & x_{t+1} = F_{\alpha_{t+1}} x_t + (1 - F_{\alpha_{t+1}})\mu + \alpha_t \beta_t + \omega_1 \\
& \beta_{t+1} = \max(0, \beta_t + \omega_2) \\
\text{Switching Evolution: } & \alpha_{t+1} \sim p(\alpha_{t+1} | \alpha_t, Z_t)
\end{aligned}
$$

where z_t is an exogenous variable that effects the sensor model, μ is an average free flow speed

$$\alpha_t \in \{0, 1, -1\}$$

$$\omega = (\omega_1, \omega_2)^T \sim N(0, W),\ v \sim N(0, V)$$

$$F_{\alpha_t} = \begin{cases} 1, \ \alpha_t \in \{1, -1\} \\ F, \ \alpha_t = 0 \end{cases}$$

No boundary conditions estimation is needed. No capacity/critical density is needed.