Representation Power of Feedforward Neural Networks

Based on work by Barron (1993), Cybenko (1989), Kolmogorov (1957)

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Feedforward Neural Networks

- Two node types:
  - Linear combinations:
$$x \quad \mapsto \quad \sum_i w_i x_i + w_0.$$
  - Sigmoid thresholded linear combinations:
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- What can a network of these nodes represent?

$$
\begin{aligned}
&\sum_{i=1}^{n} w_i x_i && \text{one layer,}\\
&\sum_{i=1}^{n} w_i \sigma\left(\sum_{j=1}^{n_i} w_{ji} x_j + w_{j0}\right) && \text{two layers,}\\
&\quad\vdots && \quad\vdots
\end{aligned}
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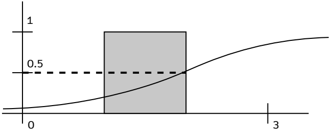

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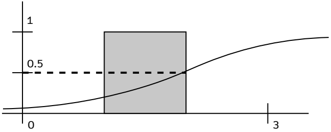
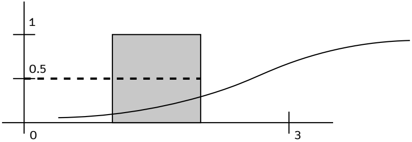


  - $w \geq 0$ and $\sigma(2w + w_0) < 1/2$: mess up on middle bump.



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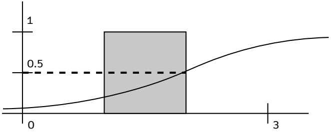

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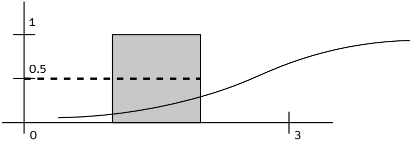

- Can symmetrize $(w < 0)$; no matter what, error $\geq 1/2$.

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Goal: 2-NNs approximate continuous functions over $[0,1]^n$.

Outline

- 2-nn via functional analysis (Cybenko, 1989).
- 2-nn via greedy approx (Barron, 1993).
- 3-nn via histograms (Folklore).
- 3-nn via wizardry (Kolmogorov, 1957).

Overview of Functional Analysis proof (Cybenko, 1989)

- Hidden layer as a basis:

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- Want to show $\mathrm{cl}(\mathrm{span}(B)) = \mathcal{C}([0,1]^n)$.
- Work via contradiction: if $f \in \mathcal{C}([0,1]^n)$ far from $\mathrm{cl}(\mathrm{span}(B))$, can bridge the gap with a sigmoid.

Abstracting σ

- Cybenko needs σ *discriminates*:

$$\mu = 0 \quad \Longleftrightarrow \quad \forall w, w_0 \,.\, \int \sigma(\langle w, x\rangle + w_0) d\mu(x) = 0.$$

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- Most results today only need σ approximates $\mathbb{1}[x \geq 0]$:

$$\sigma(x) \to \begin{cases} 1 & \text{as } x \to +\infty, \\ 0 & \text{as } x \to -\infty. \end{cases}$$

  Combined with σ bounded&measurable gives discriminatory (Cybenko, 1989, Lemma 1).

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- Consider the subspace

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    - Don't need inner products: Hahn-Banach Theorem.

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    - A form of the Riesz Representation Theorem.

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    - $L_{|S} = 0$ implies $\forall w, w_0 \,.\, \int \sigma(\langle w, x\rangle + w_0) d\mu(x) = 0$.
    - But "discriminatory" means $\mu = 0 \iff \forall w, w_0 \,.\, \int \ldots$.

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- Can this be turned into an algorithm?

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- Can rescale B so $f \in \mathrm{cl}(\mathrm{conv}(B))$. (or tweak alg.)
- Is this familiar? Oh let's see.. gradient descent, coordinate descent, Frank-Wolfe, projection pursuit, basis pursuit, boosting.. as usual, cf. Zhang (2003).

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   2.1 Choose (α_t, g_t) apx minimizing (tolerance $\mathcal{O}(1/t^2)$)
   $$\inf_{\alpha \in [0,1], g \in B} \|f - (\alpha \hat{f}_{t-1} + (1-\alpha)g)\|_2^2.$$
   2.2 Update $\hat{f}_t := \alpha_t \hat{f}_{t-1} + (1-\alpha_t) g_t$.

- Can rescale B so $f \in \mathrm{cl}(\mathrm{conv}(B))$. (or tweak alg.)
- Is this familiar? **YES.**
- Barron carefully shows rate $\|f - \hat{f}_t\|_2^2 = \mathcal{O}(1/t)$.

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- (Standard rate proofs use curvature of $\|\cdot\|_2^2$.)

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- From an algorithmic perspective, much remains to be done.

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  the curse of dimension!
- Can defeat this with more layers (cf. Kolmogorov (1957)).

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  - Easiest basis: histograms!

Histogram basis for $\mathcal{C}([0,1]^n)$

- Grid $[0,1]^n$ into $\{R_i\}_{i=1}^m$; consider

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- Now just write individual histogram bars as a NNs.

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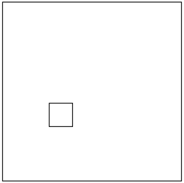

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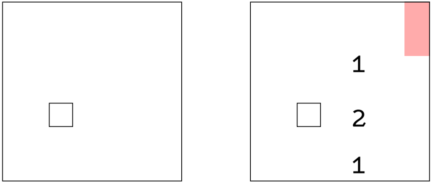

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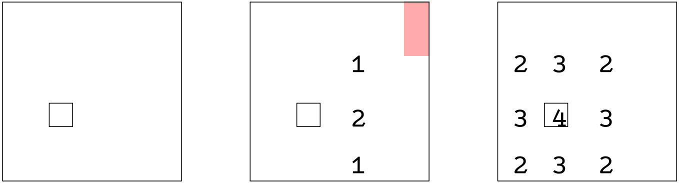

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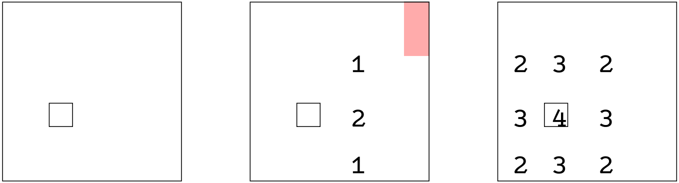
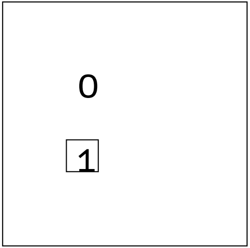


- Now pass through a second layer $\mathbb{1}[\text{input} \geq 3.5]$.



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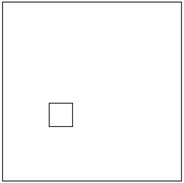

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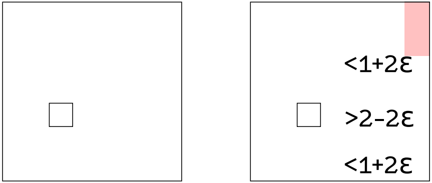

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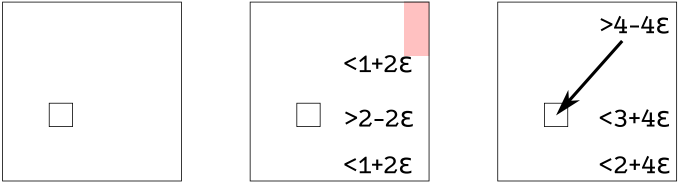

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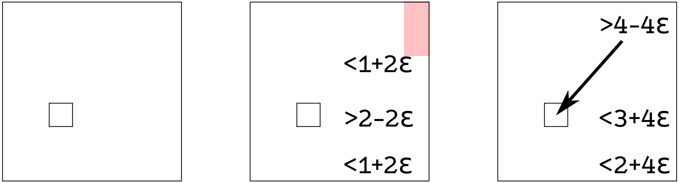

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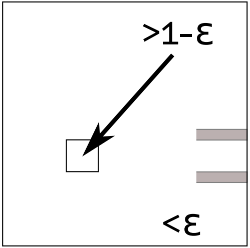

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- Have fuzz $S \subseteq [0,1]^n$ with $m(S) \geq 1 - \epsilon$,

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x \in S \quad \Longrightarrow \quad |f(x) - \hat{f}(x)| < \epsilon.
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Folklore proof discussion

- Curse of dimension!
- Still, high level features useful.

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- The "transfer functions" (i.e., sigmoids), are not fixed across nodes.

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Theorem. (Kolmogorov, 1957)

for

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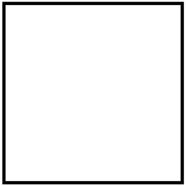

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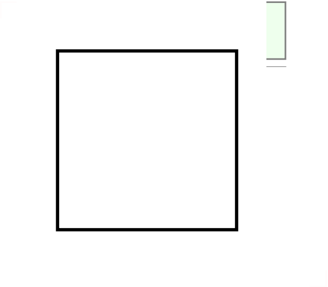

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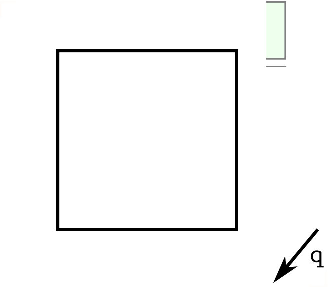

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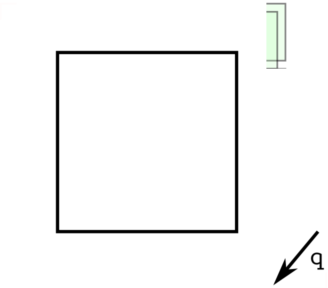

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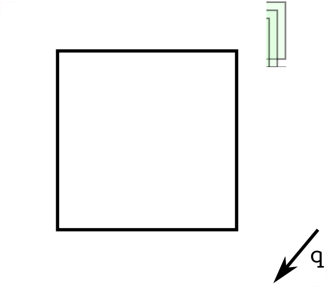
- Given resolution $k \in \mathbb{Z}_{++}$, build a staggered gridding.



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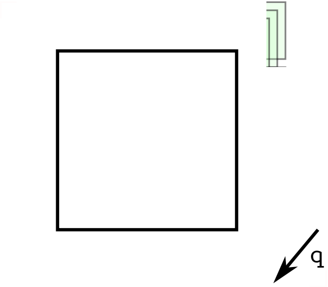

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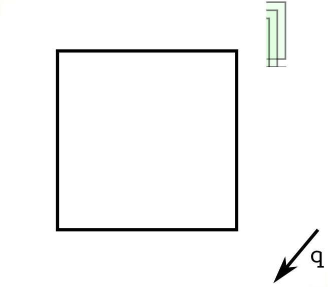

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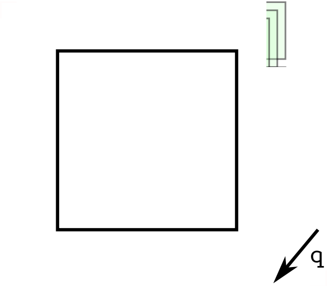

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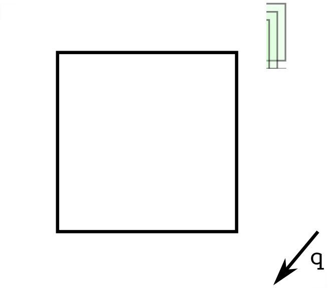

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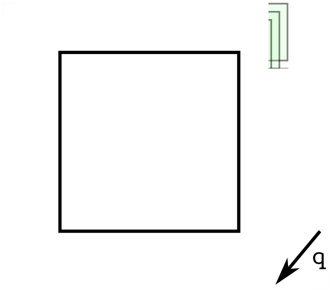

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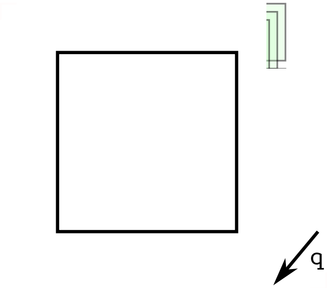

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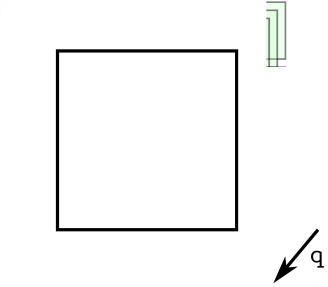

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- *The functions $\psi^{p,q}$ are fractals.*

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- (I'm leaving a lot out =( )

Discussion 1

NOTE — **Communicated by Halbert White**

Representation Properties of Networks: Kolmogorov's Theorem Is Irrelevant

Federico Girosi
Tomaso Poggio
Massachusetts Institute of Technology, Artificial Intelligence Laboratory, Cambridge, MA 02142 USA
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Center for Biological Information Processing, Whitaker College, Cambridge, MA 02142 USA

Many neural networks can be regarded as attempting to approximate a multivariate function in terms of one-input one-output units. This note considers the problem of an exact representation of nonlinear mappings in terms of simpler functions of fewer variables. We review Kolmogorov's theorem on the representation of functions of several variables in terms of functions of one variable and show that it is irrelevant in the context of networks for learning.

1 Kolmogorov's Theorem: An Exact Representation Is Hopeless

A crucial point in approximation theory is the choice of the representation

Discussion 2

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- It shows the value of powerful nodes, whereas the standard apx results just suggest a very wide hidden layer.

Conclusion

- Some fancy mechanisms give 2-NNs $\hat{f}_i \to f \in \mathcal{C}([0,1]^n)$.
- Histogram constructions hint to the power of deeper networks.

Thanks!

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