Q1. What are Eigenvalues and Eigenvectors? How are they related to Eigen-Decomposition? Explain with an example.

- **Eigenvector:** A non-zero vector **v** that, when a square matrix **A** acts on it, changes only in magnitude, not direction.
- **Eigenvalue:** The scalar **λ** representing how much the eigenvector is scaled by **A**.

Mathematically:

Av=λvA v = \lambda vAv=λv

Example:

A=[2003],v1=[10],v2=[01]A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad v_1 = \begin{bmatrix}1 \\ 0\end{bmatrix}, \quad v_2 = \begin{bmatrix}0 \\ 1\end{bmatrix}A=[2003],v1=[10],v2=[01]

- Av1=2v1A v_1 = 2 v_1Av1=2v1 → λ = 2
- Av2=3v2A v_2 = 3 v_2Av2=3v2 → λ = 3

Eigen-Decomposition:
If **A** is diagonalizable, it can be written as:

A=VΛV−1A = V \Lambda V^{-1}A=VΛV−1

- **V:** matrix of eigenvectors
- **Λ:** diagonal matrix of eigenvalues

This decomposition simplifies computations like matrix powers and exponentials.

Q2. What is eigen decomposition and its significance in linear algebra?

Eigen-Decomposition: The process of decomposing a square matrix **A** into its eigenvectors and eigenvalues:

A=VΛV−1A = V \Lambda V^{-1}A=VΛV−1

Significance:

1. Simplifies matrix operations: An=VΛnV−1A^n = V \Lambda^n V^{-1}An=VΛnV−1
2. Helps understand matrix properties like invertibility, rank, and stability
3. Fundamental for PCA, SVD, and many ML techniques

Q3. Conditions for diagonalizability using Eigen-Decomposition

A square matrix **A** is **diagonalizable** if:

1. **A has n linearly independent eigenvectors** (for an n×n matrix)
2. This implies **geometric multiplicity = algebraic multiplicity** for each eigenvalue

Proof sketch:

- Suppose A has n independent eigenvectors v1,v2,...,vnv_1, v_2, ..., v_nv1,v2,...,vn
- Form matrix V=[v1,v2,...,vn]V = [v_1, v_2, ..., v_n]V=[v1,v2,...,vn]
- Then AV=VΛ ⟹ A=VΛV−1A V = V \Lambda \implies A = V \Lambda V^{-1}AV=VΛ⟹A=VΛV−1

  If eigenvectors are linearly independent, **V** is invertible → diagonalizable.

Q4. Significance of the spectral theorem

Spectral Theorem:

- Any **symmetric matrix** A=ATA = A^TA=AT can be diagonalized by an **orthogonal matrix** VVV:

A=VΛVTA = V \Lambda V^TA=VΛVT

- Eigenvalues are **real**, eigenvectors are **orthogonal**.

Example:

A=[2112],eigenvalues 1,3,orthogonal eigenvectors [1,−1]T,[1,1]TA = \begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}, \quad \text{eigenvalues } 1, 3, \text{orthogonal eigenvectors } [1,-1]^T, [1,1]^TA=[2112],eigenvalues 1,3,orthogonal eigenvectors [1,−1]T,[1,1]T

Significance:

- Guarantees diagonalizability for symmetric matrices
- Makes PCA and covariance matrix decomposition possible, because covariance matrices are symmetric.

Q5. How do you find eigenvalues and what do they represent?

1. Solve the **characteristic equation**:

det(A−λI)=0\det(A - \lambda I) = 0det(A−λI)=0

- The solutions λ are the eigenvalues.

Meaning:

- Each eigenvalue represents the **scaling factor** along its corresponding eigenvector direction.

Q6. What are eigenvectors and how are they related to eigenvalues?

- Eigenvectors are directions in which a matrix **acts by scaling only**.
- Eigenvalues indicate **how much scaling** occurs along each eigenvector.

Q7. Geometric interpretation of eigenvectors and eigenvalues

- **Eigenvectors:** Directions in which a transformation (matrix) stretches or compresses the space.
- **Eigenvalues:** The factor by which the transformation stretches or compresses along that direction.

Example:

- A 2D scaling matrix A=[[3,0],[0,2]]A = [[3,0],[0,2]]A=[[3,0],[0,2]]
- x-axis vector ([1,0]) → scaled by 3
- y-axis vector ([0,1]) → scaled by 2

Q8. Real-world applications of eigen decomposition

1. **Principal Component Analysis (PCA):** Eigenvectors of covariance matrix → principal directions
2. **Vibration analysis:** Eigenvalues of stiffness/mass matrices → natural frequencies
3. **Graph theory:** Eigenvectors of adjacency matrices → clustering and centrality
4. **Markov chains:** Eigenvectors of transition matrices → steady states
5. **Image compression:** Eigenfaces in face recognition

Q9. Can a matrix have more than one set of eigenvectors and eigenvalues?

- **Eigenvalues are unique up to multiplicity** (algebraic multiplicity > 1 can exist)
- **Eigenvectors are not unique:** any scalar multiple of an eigenvector is also valid
- **Example:** $\lambda = 2$, eigenvector $v = [1,0]$ → $2v = [2,0]$ is also an eigenvector

---

Q10. Eigen-Decomposition in data analysis and ML

1. **PCA (Principal Component Analysis):**
   - Eigenvectors of covariance matrix → principal components
   - Eigenvalues → variance explained by each component
2. **Spectral clustering:**
   - Eigenvectors of Laplacian matrix → low-dimensional embedding for clustering
3. **Dimensionality reduction / feature extraction:**
   - Reduce high-dimensional data while preserving variance or structure
4. **Recommendation systems:**
   - Eigenvectors in matrix factorization → latent factors

Eigen-decomposition underpins **many algorithms that require understanding the directions and magnitudes of data variation**.
