

Vadique Myself

# PHYSICS *of* ELASTIC CONTINUA

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## WHAT IS THIS BOOK ABOUT?

In this book, all models of an elastic continuum are presented: non-linear and linear, micropolar and classical momentless; three-dimensional, two-dimensional (shells and plates), one-dimensional (rods, including thin-walled ones). Fundamentals of dynamics — oscillations, waves and stability — are explained. For thermoelasticity and magnetoelasticity, the summary of theories of thermodynamics and electrodynamics is given. Theories of defects and fractures are described. Approaches to modeling inhomogeneous materials — composites — are shown.

The book is written using compact and elegant direct indexless tensor notation, operating with invariant objects — tensors. The mathematical apparatus for interpreting direct tensor relations is contained in the first chapter.

*[github.com/VadiqueMe/PhysicsOfElasticContinua](https://github.com/VadiqueMe/PhysicsOfElasticContinua)*



# CONTENTS

<i>Chapter 1</i>	<b>Mathematical apparatus</b>	<b>1</b>
§ 1.	Vector	2
§ 2.	Tensor and its components	7
§ 3.	Tensor algebra, or operations with tensors	9
§ 4.	Polyadic representation (decomposition)	12
§ 5.	Matrices, permutations and determinants	14
§ 6.	Cross product and the Levi-Civita tensor	18
§ 7.	Symmetric and antisymmetric tensors	24
§ 8.	Diagonalization of tensor. Eigenvectors and eigenvalues. Invariants	26
§ 9.	Rotation tensor	29
§ 10.	Variations	35
§ 11.	Polar decomposition	36
§ 12.	Tensors in oblique basis	37
§ 13.	Tensor functions	43
§ 14.	Spatial differentiation	44
§ 15.	Integral theorems	49
§ 16.	Curvature tensors	52
<i>Chapter 2</i>	<b>Fundamentals of classical generic mechanics</b>	<b>56</b>
§ 1.	Discrete collection of particles	56
§ 2.	Absolutely rigid undeformable solid body	59
§ 3.	Principle of virtual work	64
§ 4.	Balance of momentum, rotational momentum, and energy	67
§ 5.	Hamilton's principle and Lagrange's equations	68

§ 6. Statics .....	70
§ 7. Mechanics of relative motion .....	73
§ 8. Small oscillations (vibrations) .....	74
<b>Chapter 3 Nonlinear elastic momentless continuum</b> .....	<b>77</b>
§ 1. Continuum. Two descriptions of processes .....	77
§ 2. Motion gradient .....	81
§ 3. Measures (tensors) of deformation .....	82
§ 4. Velocity field .....	87
§ 5. Area vector. Surface change .....	88
§ 6. Forces in continuum. Existence of the Cauchy stress tensor .....	89
§ 7. Balance of momentum and angular momentum .....	91
§ 8. Eigenvalues of the Cauchy stress tensor. Mohr's circles .....	92
§ 9. Principle of virtual work (without Lagrange multipliers) .....	93
§ 10. Constitutive relations of elasticity .....	94
§ 11. Piola–Kirchhoff stress tensors and other measures of stress ...	96
§ 12. Variation of present configuration .....	101
§ 13. Internal constraints .....	101
§ 14. Hollow sphere under pressure .....	102
§ 15. Stresses as Lagrange multipliers .....	103
<b>Chapter 4 Classical linear elasticity</b> .....	<b>105</b>
§ 1. Complete set of equations .....	105
§ 2. Uniqueness of the solution in dynamics .....	108
§ 3. Hooke's law .....	110
§ 4. Theorems of statics .....	113
§ 5. Equations in displacements .....	117
§ 6. Concentrated force in an infinite medium .....	117
§ 7. Finding displacements by deformations .....	117
§ 8. Equations in stresses .....	119
§ 9. Principle of minimum potential energy .....	119
§ 10. Principle of minimum complementary energy .....	121
§ 11. Mixed principles of stationarity .....	122

§ 12. Antiplane shear .....	123
§ 13. Torsion of rods .....	124
§ 14. Plane deformation .....	125
<b>Chapter 5 Micropolar three-dimensional continuum</b> .....	<b>126</b>
§ 1. Introduction to linear micropolar theory .....	126
§ 2. Relations of elasticity .....	130
§ 3. Compatibility equations .....	132
§ 4. Theorems of statics .....	132
§ 5. Cosserat pseudocontinuum .....	132
§ 6. Plane deformation .....	133
§ 7. Nonlinear theory .....	133
§ 8. Nonlinear model with constrained rotation .....	135
<b>Chapter 6 Thermoelasticity</b> .....	<b>137</b>
§ 1. First law of thermodynamics .....	137
§ 2. Second law .....	139
§ 3. Constitutive equations .....	141
§ 4. Heat equation .....	141
§ 5. Linear thermoelasticity .....	142
§ 6. Equations for displacements .....	142
§ 7. Thermal stress .....	142
§ 8. Variational formulations .....	142
<b>Chapter 7 Magnetoelasticity</b> .....	<b>144</b>
§ 1. Electromagnetic field .....	144
§ 2. Electromagnetic waves .....	147
§ 3. Electrostatics .....	149
§ 4. Dielectrics .....	149
§ 5. Magnetostatics .....	149
§ 6. Magnetism .....	150
§ 7. Magnetic rigidity .....	150

<i>Chapter 8</i>	<b>Perturbation methods (asymptotic methods)</b>	<b>153</b>
§ 1.	Asymptotic decompositions	153
§ 2.	Splitting in a linear algebraic system	155
§ 3.	Poincaré method	156
§ 4.	Van der Pol averaging method	156
§ 5.	Coalescence of asymptotic decompositions	157
§ 6.	Multiple-scale analysis (method of multiple scales)	157
§ 7.	Equations with slowly varying parameters	158
§ 8.	Thin bodies	158
<i>Chapter 9</i>	<b>Rods</b>	<b>159</b>
§ 1.	Initial concepts	159
§ 2.	Kinematics of Cosserat lines	165
§ 3.	Balance of forces and moments	166
§ 4.	Principle of virtual work and consequences	167
§ 5.	Classical Kirchhoff's model	167
§ 6.	Euler's problem about stability of rods	168
§ 7.	Variational equations	168
§ 8.	Non-shear model with (ex)tension	168
§ 9.	Mechanics of flexible thread	168
§ 10.	Linear theory	169
§ 11.	Case of small thickness	169
§ 12.	Saint-Venant's problem	170
§ 13.	Finding stiffness by energy	170
§ 14.	Variational method of building one-dimensional model	171
§ 15.	Asymptotic splitting of three-dimensional problem	172
§ 16.	Thermal deformation and stress	173
<i>Chapter 10</i>	<b>Thin-walled rods</b>	<b>174</b>
§ 1.	Variational approach	174
§ 2.	Equations with small parameter	175
§ 3.	First step of the asymptotic procedure	175
§ 4.	Second step	175



§ 5. Third step .....	176
§ 6. Fourth step .....	176
§ 7. Displacements .....	177
§ 8. Results of asymptotic analysis .....	177
<b>Chapter 11 Shells and plates</b>	<b>178</b>
§ 1. Surface geometry .....	178
§ 2. Model of a shell .....	180
§ 3. Balance of forces and moments for a shell .....	180
§ 4. Shells: Relations of elasticity .....	181
§ 5. Classical theory of shells .....	181
§ 6. Shells: A plate .....	181
§ 7. Shells: Approach with Lagrange multipliers .....	181
§ 8. Cylindrical shell .....	182
§ 9. Shells: Common theorems .....	182
§ 10. Shells: Boundary conditions .....	182
§ 11. Shells of revolution .....	182
§ 12. Momentless theory of shells .....	183
§ 13. Shells: Nonlinear momentless theory .....	183
§ 14. Shells: Other variant of classical theory .....	183
§ 15. Plates: Overall concepts .....	183
§ 16. Timoshenko-like model of a plate (direct approach) .....	184
§ 17. Kirchhoff's classical theory of plates .....	184
§ 18. Plates: Asymptotic matching of two-dimensional models .....	184
§ 19. Plates: Variational transition from three-dimensional model .....	184
§ 20. Plates: Splitting of three-dimensional bending problem .....	185
§ 21. Circular plates .....	185
§ 22. Plates: Plane stress .....	185
<b>Chapter 12 Oscillations and waves</b>	<b>187</b>
§ 1. Vibrations of three-dimensional bodies .....	187
§ 2. Vibrations of rods .....	189
§ 3. Small perturbations of parameters .....	189

§ 4. Vibrations of shells .....	189
§ 5. Waves in an elastic continuum .....	190
§ 6. Waves in a rod .....	190
§ 7. Nonlinear oscillations .....	190
<b>Chapter 13 Stability</b>	<b>191</b>
§ 1. Various approaches to the problem of stability .....	191
§ 2. Classical problems with rods .....	192
§ 3. “Tracking” loads .....	192
§ 4. The role of additional pliabilities .....	192
§ 5. Variational formulations .....	192
§ 6. Nonconservative problems .....	193
§ 7. Case of multiple roots .....	193
<b>Chapter 14 Defects</b>	<b>194</b>
§ 1. Volterra dislocations .....	194
§ 2. Straight-line dislocations .....	194
§ 3. Action of stress field on dislocation .....	194
§ 4. About movement of dislocations .....	195
§ 5. Point defects .....	195
§ 6. Force acting on a point defect .....	195
§ 7. Continuously distributed dislocations .....	195
§ 8. Stress during winding of coil .....	195
<b>Chapter 15 Fractures</b>	<b>197</b>
§ 1. Traditional criteria of durability .....	197
§ 2. Antiplane deformation of continuum with a crack .....	198
§ 3. Crack in plane deformation .....	198
§ 4. Crack-driving force .....	198
§ 5. Criterion of crack growth .....	199
§ 6. J-integral .....	199
§ 7. Stress intensity factors .....	199
§ 8. Barenblatt’s model .....	199
§ 9. Deformational criterion .....	200

§ 10. Growth of cracks .....	200
§ 11. Elastic field ahead of a moving crack .....	200
§ 12. Balance of energy for a moving crack .....	201
<i>Chapter 16</i> <b>Composites</b> .....	<b>202</b>
§ 1. Introductory thoughts .....	202
§ 2. Effective fields .....	203
§ 3. Boundary value problems for representative volume .....	203
§ 4. Hill's fork .....	204
§ 5. Eshelby formulas .....	204
§ 6. Effective moduli for material with spherical inclusions .....	204
§ 7. Self-consistent method .....	205
§ 8. Hashin–Shtrikman principle .....	205
<i>Chapter 17</i> <b>Periodic composites</b> .....	<b>207</b>
§ 1. One-dimensional problem .....	207
§ 2. Three-dimensional continuum .....	207
§ 3. Fibrous structure .....	207
§ 4. Statics of a periodic rod .....	207
<b>List of referenced publications</b> .....	<b>208</b>



*chapter 1*

## MATHEMATICAL APPARATUS

### *0.1. Lines, curved and straight (linear)*

The hypothesis of the existence of straight lines.

### *0.2. Straight line and vector*

A (geometric) vector may be like a straight line with an arrow at one of its ends. **Then** it is fully described (characterized) by the magnitude and the direction.

Why are vectors always straight (linear)?

Vectors are linear (straight), they cannot be curved.

Vectors are neither straight nor curved. A vector has the magnitude and the direction. A vector is not a line or a curve, albeit it can be represented by a straight line.

A vector has the magnitude and the direction. It can't be thought of as a line.

### *0.3. Number line*

### *0.4. Plane and space*

### *0.5. Differentiation into small differential chunks*

Small differential chunks

A mention of tensors may scare away the reader, commonly avoiding needless complications. Don't be afraid: tensors are used just due to their wonderful property of invariance — independence from coordinate systems.

## §1. Vector

I propose to begin familiarizing with tensors via memoirs about such a phenomenon as a vector\*.

- ✓ A *point* has position in space. The only characteristic that distinguishes one point from another is its position.
- ✓ A *vector* has both magnitude and direction, but no specific position in space.

### 1.1. What is a vector?

What is “linear”?

- (1) straight
- (2) relating to, resembling, or having a [graph](#) that is a straight line

All vectors are linear objects.

Examples of vectors:

- ✓ A force acts on an object.
- ✓ The velocity of an object describes what’s happening with this object at an instant.

The sum (combination) of two or more vectors is the new (“resultant”) vector. There are two similar methods to calculate the resultant vector geometrically.

The “*head to tail*” method involves lining up the head of the one vector with the tail of the other. Here the resultant goes from the initial point (“tail”) of the first addend to the end point (“head”) of the second addend when the tail (initial point) of the second one coincides with the head (end point) of the first one.

[ .... figure here .... ]

The “*parallelogram*” method.

\* Within the abstract algebra, the word *vector* is about any object which can be summed with similar objects and scaled (multiplied) by scalars, and vector space is a synonym of linear space. Therefore I clarify that in this book *vector* is nothing else than three-dimensional geometric (Ευκλείδειος, Euclidean) vector.

Vector addition is commutative

$$\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}.$$

Multiplication of a vector by a scalar.

Multiplication by minus one.

The Newton’s action–reaction principle “действие равно противодействию по магнитуде и обратно ему по направлению”.

In every mechanical interaction, there’s a pair of forces acting on the two objects that interact. These forces can be represented as vectors, they are equal in magnitude and reverse in direction.

Multiplying a vector by negative one reverses the direction but doesn’t change the magnitude.

For every action, there’s an equal (in magnitude) and opposite (in direction) reaction force.

A vector may be also represented as the sum (combination) of some trio (called “basis”) of other vectors scaled by numbers (coefficients). Such a representation is called a “linear combination” of basis vectors. A list (array, tuple) of coefficients alone, without basis vectors, is not enough and can’t represent a vector.

У элементов векторного пространства — векторов — компонент нет. Компоненты появляются тогда, когда выбран базис. И в разных базисах компоненты одного и того же вектора будут разные.

Here it is — the vector,  $\boldsymbol{v}$  seems like a suitable name for it.

Like all geometric vectors,  $\boldsymbol{v}$  is pretty well characterized by the two mutually independent properties: its length (magnitude, norm, modulus) and its direction in space. This characterization is complete, so two vectors with the same magnitude and the same direction are considered equal.

Every vector exists objectively by itself, independently of methods and units of measurement of both lengths and directions (including any abstractions of such units and methods).

plane geometry is two-dimensional Euclidean geometry

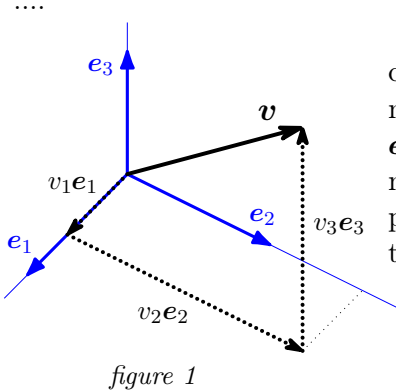
*The modern version of the Euclidean geometry is the theory of multi-dimensional Euclidean (coordinate) spaces, where distance is measured by the Pythagorean theorem.*

Euclidean geometry  $\longleftrightarrow$  “flat” space

Euclidean geometry, sometimes called parabolic geometry, follows a set of propositions that are based on Euclid's five postulates.

The fundamental concepts of Euclidean geometry are a point, a line, and a plane. A point has no dimensions (length or width), it only has a location. A line is straight and extends infinitely **in the opposite directions**. A plane is a flat surface that extends infinitely.

## 1.2. The method of coordinates



Introduce rectangular (“cartesian”) coordinates by picking some three mutually perpendicular unit vectors  $e_1, e_2, e_3$  as a basis for measurements. Within such a system, “ $\cdot$ ”-products of basis vectors are equal to the Kronecker delta:

$$e_i \cdot e_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for any orthonormal basis.

Decomposing vector  $v$  in some orthonormal basis  $e_i$  ( $i = 1, 2, 3$ ), we get coefficients  $v_i$  — components of vector  $v$  in that basis (fig. 1)

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 \equiv \sum_{i=1}^3 v_i e_i \equiv v_i e_i, \quad v_i = v \cdot e_i. \quad (1.1)$$

Here and hereinafter, the Einstein's summation convention is accepted: an index repeated twice (and no more than twice) in a single term implies summation over this index. And a non-repeating index is called “free”, it is identical in all parts of the equation. These are examples:

$$\sigma = \tau_{ii}, \quad p_j = n_i \tau_{ij}, \quad m_i = e_{ijk} x_j f_k, \quad a_i = \lambda b_i + \mu c_i.$$

(But equations  $a = b_{kkk}$ ,  $c = f_i + g_k$ ,  $a_{ij} = k_i \gamma_{ij}$  are incorrect.)

Having components of a vector in an orthonormal basis, the length of this vector is retrieved by the “Πυθαγόρας’ equation”

$$v \cdot v = v_i e_i \cdot v_j e_j = v_i \delta_{ij} v_j = v_i v_i, \quad \|v\| = \sqrt{v \cdot v} = \sqrt{v_i v_i}. \quad (1.2)$$



The direction of a vector in space is measured by <sup>the</sup> three angles (cosines of angles) between this vector and each of the basis ones:

$$\cos \angle(\mathbf{v}, \mathbf{e}_i) = \frac{\mathbf{v} \cdot \mathbf{e}_i}{\|\mathbf{v}\|} = \frac{v_i}{\sqrt{v_j v_j}} \Leftrightarrow v_i = \|\mathbf{v}\| \cos \angle(\mathbf{v}, \mathbf{e}_i). \quad (1.3)$$

*Measurement of angles.* The cosine of the angle between two vectors is the same as the dot product of these vectors when they are normalized to both have the magnitude equal to the one unit of length

$$\cos \angle(\mathbf{v}, \mathbf{w}) = \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|}.$$

To accompany the magnitude, which represents the length independent of direction, there's a way to represent the direction of a vector independent of its length. For this purpose, we use unit vectors, which are vectors with a magnitude of 1.

A rotation matrix is just a transform that expresses the basis vectors of the input space in a different orientation. The length of the basis vectors will be the same, and the origin will not change. Also, the angle between the basis vectors will not change. All that changes is the relative direction of all of the basis vectors.

Therefore, a rotation matrix is not really a “rotation” matrix; it is an orientation matrix.

*The angle between two random vectors.* According to (1.3)

$$\begin{aligned} \cos \angle(\mathbf{v}, \mathbf{e}_m) &= \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \mathbf{e}_m = \frac{v_m}{\sqrt{v_j v_j}}, \\ \cos \angle(\mathbf{w}, \mathbf{e}_n) &= \frac{\mathbf{w}}{\|\mathbf{w}\|} \cdot \mathbf{e}_n = \frac{w_n}{\sqrt{w_k w_k}}. \end{aligned}$$

The length (1.2) and the direction in space (1.3) — measurable by means of the trio of basic vectors — describe a vector, and every vector possesses these properties\*. However, this is not enough (“not sufficient” in jargon of math books) for vector.

\* And what is the direction of the null vector  $\mathbf{0}$  with zero length  $\|\mathbf{0}\| = 0$ ? (The zero vector with no magnitude ends exactly where it begins and is not directed anywhere, thus its direction is *undefined*.)

A vector is not just a collection of components in some basis. Vector is *invariant*, it doesn't depend on any basis of any coordinate system.

A triple of pairwise perpendicular unit vectors can only rotate and thereby characterize the angular orientation.

Decomposition of the same vector  $\mathbf{v}$  in two cartesian systems with basis unit vectors  $\mathbf{e}_i$  and  $\mathbf{e}'_i$  (fig. 2) gives

$$\mathbf{v} = v_i \mathbf{e}_i = v'_i \mathbf{e}'_i,$$

where

$$\begin{aligned} v_i &= \mathbf{v} \cdot \mathbf{e}_i = v'_k \mathbf{e}'_k \cdot \mathbf{e}_i, \\ v'_i &= \mathbf{v} \cdot \mathbf{e}'_i = v_k \mathbf{e}_k \cdot \mathbf{e}'_i. \end{aligned}$$

Appeared here two-index objects (two-dimensional arrays)  $o_{k'i} \equiv \mathbf{e}'_k \cdot \mathbf{e}_i$  and  $o_{ki'} \equiv \mathbf{e}_k \cdot \mathbf{e}'_i$  are used to shorten formulas.

The “ $\cdot$ ”-product (dot product) of two vectors is commutative — that is, swapping multipliers doesn't change the result. Thus

$$o_{k'i} = \mathbf{e}'_k \cdot \mathbf{e}_i = \cos \angle(\mathbf{e}'_k, \mathbf{e}_i) = \cos \angle(\mathbf{e}_i, \mathbf{e}'_k) = \mathbf{e}_i \cdot \mathbf{e}'_k = o_{ik'}, \quad (1.3a)$$

$$o_{ki'} = \mathbf{e}_k \cdot \mathbf{e}'_i = \cos \angle(\mathbf{e}_k, \mathbf{e}'_i) = \cos \angle(\mathbf{e}'_i, \mathbf{e}_k) = \mathbf{e}'_i \cdot \mathbf{e}_k = o_{i'k}. \quad (1.3b)$$

Lines of equalities (1.3a) and (1.3b) are mutually reciprocal by multiplication

$$o_{k'i} o_{ki'} = o_{ki'} o_{k'i} = 1, \quad o_{k'i} o_{i'k} = o_{i'k} o_{k'i} = 1.$$

.....

Components of the same vector  $\mathbf{v}$  in an “old” and a “new” — rotated — orthonormal bases, transformed into each other:

$$v_i = v'_k o_{k'i}, \quad v'_i = v_k o_{ki'}. \quad (1.4)$$

At the same time, orthonormal basis vectors of “old” and “new” — rotated — rectangular coordinates are connected via the matrix of cosines as

$$\mathbf{e}'_i = o_{i'k} \mathbf{e}_k, \quad \mathbf{e}_k = \mathbf{e}'_i o_{i'k} = o_{ki'} \mathbf{e}'_i. \quad (1.5)$$

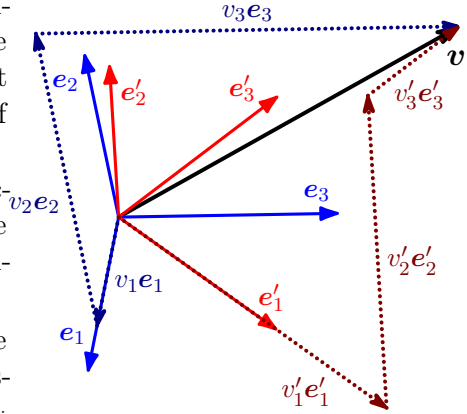


figure 2

...

The distributive property of the “ $\bullet$ ”-product...

“ $\bullet$ ”-multiplication distributes over addition of vectors:

$$\mathbf{e}_k \bullet \mathbf{v} = \mathbf{e}_k \bullet (v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3) = \mathbf{e}_k \bullet v_1 \mathbf{e}_1 + \mathbf{e}_k \bullet v_2 \mathbf{e}_2 + \mathbf{e}_k \bullet v_3 \mathbf{e}_3$$

Therefore, I don’t need brackets when using the summation convention

$$\mathbf{e}_1 \bullet \mathbf{v} = \mathbf{e}_1 \bullet (v_j \mathbf{e}_j) = \mathbf{e}_1 \bullet v_j \mathbf{e}_j$$

....

$$(1.5) \Rightarrow \delta_{kj} = \mathbf{e}_k \bullet \mathbf{e}_j = o_{ki'} \mathbf{e}'_i \bullet \mathbf{e}_j = o_{ki'} o_{i'j}$$

The matrix of cosines is orthogonal, meaning that it inverses with “transposing” (swapping letters of indices)\*:

$$o_{i'k} o_{j'k} = o_{k'i} o_{k'j} = \delta_{ij} \quad (1.6)$$

(proof:  $\delta_{ij} = \mathbf{e}'_i \bullet \mathbf{e}'_j = o_{i'k} \mathbf{e}_k \bullet o_{j'n} \mathbf{e}_n = o_{i'k} o_{j'n} \delta_{kn} = o_{i'k} o_{j'k}$ ).

Multiplying an orthogonal matrix by components of any vector retains the length of this vector:

$$\|\mathbf{v}\|^2 = \mathbf{v} \bullet \mathbf{v} = v'_i v'_i = o_{i'k} v_k o_{i'n} v_n = v_n v_n$$

— this conclusion **leans on** (1.6).

Orthogonal transformation of vector components

$$\mathbf{v} \bullet \mathbf{e}'_i = v_k \mathbf{e}_k \bullet \mathbf{e}'_i = \mathbf{e}'_i \bullet \mathbf{e}_k v_k = o_{i'k} v_k = v'_i \quad (1.7)$$

is sometimes used for defining a vector itself. If in each orthonormal basis  $\mathbf{e}_i$  a triplet of numbers  $v_i$  is known, and with a rotation of basis as a whole (with a transition to a new orthonormal basis) it is transformed according to (1.7), then this triplet of components represents an invariant object — vector  $\mathbf{v}$ .

## § 2. Tensor and its components

When in each orthonormal basis  $\mathbf{e}_i$  we have a set of nine ( $3^2 = 9$ ) numbers  $B_{ij}$  ( $i, j = 1, 2, 3$ ), and this set is transformed during a transition to a new (rotated) orthonormal basis  $\mathbf{e}'_i$  as

$$B'_{ij} = \mathbf{e}'_i \bullet \mathbf{e}_m B_{mn} \mathbf{e}_n \bullet \mathbf{e}'_j = \mathbf{e}'_i \bullet \mathbf{e}_m \mathbf{e}'_j \bullet \mathbf{e}_n B_{mn} = o_{i'm} o_{j'n} B_{mn}, \quad (2.1)$$

\* More about matrices — in § 5.

then this set of components represents an invariant object — tensor of second complexity (of second valence, bivalent)  ${}^2\mathbf{B}$ .

In other words, tensor  ${}^2\mathbf{B}$  reveals in every basis as a collection of its components  $B_{ij}$ , changing along with a basis according to (2.1).

The key example of a second complexity tensor is a dyad. Having two vectors  $\mathbf{a} = a_i \mathbf{e}_i$  and  $\mathbf{b} = b_i \mathbf{e}_i$ , in each basis  $\mathbf{e}_i$  assume  $d_{ij} \equiv a_i b_j$ . It's easy to see how components  $d_{ij}$  transform according to (2.1):

$$a'_i = o_{i'm} a_m, \quad b'_j = o_{j'n} b_n \Rightarrow d'_{ij} = a'_i b'_j = o_{i'm} a_m o_{j'n} b_n = o_{i'm} o_{j'n} d_{mn}.$$

Resulting tensor  ${}^2\mathbf{d}$  is called a dyadic product or just dyad and is written as  $\mathbf{a} \otimes \mathbf{b}$  or  $\mathbf{ab}$ . I choose notation  ${}^2\mathbf{d} = \mathbf{ab}$  without symbol  $\otimes$ .

Essential exemplar of a bivalent tensor is the unit tensor (other names are unit dyad, identity tensor and metric tensor). Let for any orthonormal (cartesian) basis  $E_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ . These are really components of tensor, (2.1) is actual:  $E'_{mn} = o_{m'i} o_{n'j} E_{ij} = o_{m'i} o_{n'i} = \delta_{mn}$ . I write this tensor as  $\mathbf{E}$  (other popular choices are  $\mathbf{I}$  and  ${}^2\mathbf{1}$ ).

Immutability of components upon any rotation makes tensor  $\mathbf{E}$  isotropic. There are no non-null vectors with such property (all components of the null vector  $\mathbf{0}$  are zero in any basis).

The next example is related to a linear transformation (linear mapping) of vectors. If  $\mathbf{b} = b_i \mathbf{e}_i$  is linear (preserving addition and multiplication by number) function of  $\mathbf{a} = a_j \mathbf{e}_j$ , then  $b_i = c_{ij} a_j$  in every basis. Transformation coefficients  $c_{ij}$  alter when a basis rotates:

$$b'_i = c'_{ij} a'_j = o_{i'k} b_k = o_{i'k} c_{kn} a_n, \quad a_n = o_{j'n} a'_j \Rightarrow c'_{ij} = o_{i'k} o_{j'n} c_{kn}.$$

It turns out that a set of two-index objects  $c_{ij}$ ,  $c'_{ij}$ ,  $\dots$ , describing the same linear mapping  $\mathbf{a} \mapsto \mathbf{b}$ , but in various bases, represents a single invariant object — a tensor of second complexity  ${}^2\mathbf{c}$ . And many book authors introduce tensors in that way, by means of linear mappings (linear transformations).

And the last example is a bilinear form  $F(\mathbf{a}, \mathbf{b}) = f_{ij} a_i b_j$ , where  $f_{ij}$  are coefficients,  $a_i$  and  $b_j$  are components of vector arguments  $\mathbf{a} = a_i \mathbf{e}_i$  and  $\mathbf{b} = b_j \mathbf{e}_j$ . The result  $F$  is invariant (independent of basis) with the transformation (2.1) for coefficients  $f_{ij}$ :

$$F' = f'_{ij} a'_i b'_j = f_{mn} \underbrace{a'_m b'_n}_{o_{i'm} a'_i o_{j'n} b'_j} = F \Leftrightarrow f'_{ij} = o_{i'm} o_{j'n} f_{mn}.$$

If  $f_{ij} = \delta_{ij}$ , then  $F = \delta_{ij} a_i b_j = a_i b_i$  — the “ $\cdot$ ”-product (dot product, scalar product) of two vectors. When both arguments are the same, such a homogeneous polynomial of second degree (quadratic) of one vector’s components  $F(\mathbf{a}, \mathbf{a}) = f_{ij} a_i a_j$  is called a quadratic form.

Now about more complex tensors (of valence larger than two). Tensor of third complexity  ${}^3\mathbf{C}$  is represented by a collection of  $3^3 = 27$  numbers  $C_{ijk}$ , changing with a rotation of basis as

$$C'_{ijk} = \mathbf{e}'_i \cdot \mathbf{e}_p \mathbf{e}'_j \cdot \mathbf{e}_q \mathbf{e}'_k \cdot \mathbf{e}_r C_{pqr} = o_{i'p} o_{j'q} o_{k'r} C_{pqr}. \quad (2.2)$$

The primary example is a triad of three vectors  $\mathbf{a} = a_i \mathbf{e}_i$ ,  $\mathbf{b} = b_j \mathbf{e}_j$  and  $\mathbf{c} = c_k \mathbf{e}_k$

$$t_{ijk} \equiv a_i b_j c_k \Leftrightarrow {}^3\mathbf{t} = \mathbf{abc}.$$

It is seen that orthogonal transformations (2.2) and (2.1) are results of “repeating” vector’s (1.7). The reader will easily compose a transformation of components for tensor of any complexity and will write a corresponding polyad as an example.

Vectors with transformation (1.7) are tensors of first complexity.

In the end consider the least complex objects — scalars, they are tensors of zeroth complexity. A scalar is a single ( $3^0 = 1$ ) number, which doesn’t depend on a basis: energy, mass, temperature et al. But what are components, for example, of vector  $\mathbf{v} = v_i \mathbf{e}_i$ ,  $v_i = \mathbf{v} \cdot \mathbf{e}_i$ ? If not scalars, then what? Here could be no simple answer. In each particular basis,  $\mathbf{e}_i$  are vectors and  $v_i$  are scalars.

### § 3. Tensor algebra, or operations with tensors

These are four operations.

The first — **linear combination** — aggregates addition and multiplication by number. Arguments of this operation and the result are of the same complexity. For two tensors it looks like this:

$$\lambda a_{ij\dots} + \mu b_{ij\dots} = c_{ij\dots} \Leftrightarrow \lambda \mathbf{a} + \mu \mathbf{b} = \mathbf{c}. \quad (3.1)$$

Here  $\lambda$  and  $\mu$  are scalar coefficients;  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are tensors of the same complexity. It’s easy to show that components of result  $\mathbf{c}$  satisfy the orthogonal transformation like (2.1).

“Decomposition of a vector in a basis” — representation of a vector as the sum  $\mathbf{v} = v_i \mathbf{e}_i$  — is nothing else but the linear combination of basis vectors  $\mathbf{e}_i$  with coefficients  $v_i$ .

The second operation is **multiplication (tensor product, direct product)**. It takes arguments of any complexities, returning the result of cumulative complexity. Examples:

$$\begin{aligned} v_i a_{jk} &= C_{ijk} \Leftrightarrow \mathbf{v}^2 \mathbf{a} = {}^3\mathbf{C}, \\ a_{ij} B_{abc} &= D_{ijabc} \Leftrightarrow {}^2\mathbf{a} {}^3\mathbf{B} = {}^5\mathbf{D}. \end{aligned} \quad (3.2)$$

Transformation of a collection of result’s components, such as  $C_{ijk} = v_i a_{jk}$ , during a rotation of basis is orthogonal, similar to (2.2), thus here’s no doubt that this collection is a set of tensor components.

Primary and already known (from §2) subtype of multiplication is the dyadic product of two vectors  ${}^2\mathbf{A} = \mathbf{b}\mathbf{c}$ .

The third operation is called **contraction**. It applies to bivalent and more complex tensors. This operation acts upon a single tensor, without other “participants”. Roughly speaking, contracting a tensor is summing of its components over some pair of indices. As a result, tensor’s complexity decreases by two.

For trivalent tensor  ${}^3\mathbf{D}$ , three variants of contraction are possible, giving vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  with components

$$a_i = D_{kki}, \quad b_i = D_{kik}, \quad c_i = D_{ikk}. \quad (3.3)$$

A rotation of basis

$$a'_i = D'_{kki} = \underbrace{o_{k'p} o_{k'q}}_{\delta_{pq}} o_{i'r} D_{pqr} = o_{i'r} D_{ppr} = o_{i'r} a_r$$

shows “the tensorial nature” of the result of contraction.

For a tensor of second complexity, the only one variant of contraction is possible, giving a scalar called the trace or the first invariant

$$\mathbf{B}_\bullet \equiv \text{trace } \mathbf{B} \equiv \text{I}(\mathbf{B}) = B_{kk}.$$

The trace of the unit tensor (“contraction of the Kronecker delta”) is equal to the dimension of space

$$\text{trace } \mathbf{E} = \mathbf{E}_\bullet = \delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

The fourth operation is also applicable to a single tensor of second and bigger complexities. It is named as **index swap, index juggling, transposing**. From components of a tensor, the new collection is

emerged with another sequence of indices, the result's complexity stays the same. For example, trivalent tensor  ${}^3\mathbf{D}$  can give tensors  ${}^3\mathbf{A}$ ,  ${}^3\mathbf{B}$ ,  ${}^3\mathbf{C}$  with components

$$\begin{aligned} {}^3\mathbf{A} = {}^3\mathbf{D}_{1\rightleftharpoons 2} &\Leftrightarrow A_{ijk} = D_{jik}, \\ {}^3\mathbf{B} = {}^3\mathbf{D}_{1\rightleftharpoons 3} &\Leftrightarrow B_{ijk} = D_{kji}, \\ {}^3\mathbf{C} = {}^3\mathbf{D}_{2\rightleftharpoons 3} &\Leftrightarrow C_{ijk} = D_{ikj}. \end{aligned} \quad (3.4)$$

For a bivalent tensor, the only one transposition is possible:  $\mathbf{A}^\top \equiv \mathbf{A}_{1\rightleftharpoons 2} = \mathbf{B} \Leftrightarrow B_{ij} = A_{ji}$ . Obviously,  $(\mathbf{A}^\top)^\top = \mathbf{A}$ .

For the dyadic multiplication of two vectors,  $\mathbf{a}\mathbf{b} = \mathbf{b}\mathbf{a}^\top$ .

### Combining operations

The four presented operations are combined in various sequences.

The combination of multiplication (3.2) and contraction (3.3) — the “ $\bullet$ ”-product (dot product) — is the most frequently used. In the direct indexless notation this is denoted by large dot “ $\bullet$ ”, which shows the contraction by adjacent indices:

$$\mathbf{a} = \mathbf{B} \bullet \mathbf{c} \Leftrightarrow a_i = B_{ij}c_j, \quad \mathbf{A} = \mathbf{B} \bullet \mathbf{C} \Leftrightarrow A_{ij} = B_{ik}C_{kj}. \quad (3.5)$$

The defining property of the unit tensor — it is the neutral element for the dyadic product with the subsequent contraction by adjacent indices (“ $\bullet$ ”-product)

$${}^n\mathbf{a} \bullet \mathbf{E} = \mathbf{E} \bullet {}^n\mathbf{a} = {}^n\mathbf{a} \quad \forall {}^n\mathbf{a} \quad \forall n > 0. \quad (3.6)$$

In the commutative scalar product of two vectors, the dot represents the same: the dyadic product and the subsequent contraction

$$\mathbf{a} \bullet \mathbf{b} = (\mathbf{a}\mathbf{b})_\bullet = a_ib_i = b_ia_i = (\mathbf{b}\mathbf{a})_\bullet = \mathbf{b} \bullet \mathbf{a}. \quad (3.7)$$

The following identity describes how to swap multipliers for the “ $\bullet$ ”-product (dot product) of two second complexity tensors

$$\begin{aligned} \mathbf{B} \bullet \mathbf{Q} &= (\mathbf{Q}^\top \bullet \mathbf{B}^\top)^\top \\ (\mathbf{B} \bullet \mathbf{Q})^\top &= \mathbf{Q}^\top \bullet \mathbf{B}^\top. \end{aligned} \quad (3.8)$$

For two dyads  $\mathbf{B} = \mathbf{b}\mathbf{d}$  and  $\mathbf{Q} = \mathbf{p}\mathbf{q}$

$$\begin{aligned} (\mathbf{b}\mathbf{d} \bullet \mathbf{p}\mathbf{q})^\top &= \mathbf{p}\mathbf{q}^\top \bullet \mathbf{b}\mathbf{d}^\top \\ d_ip_ibq^\top &= qp^\top db \\ d_ip_iqb &= p_id_iqb. \end{aligned}$$

For a vector and a bivalent tensor

$$\mathbf{c} \cdot \mathbf{B} = \mathbf{B}^\top \cdot \mathbf{c}, \quad \mathbf{B} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{B}^\top. \quad (3.9)$$

Contraction can be repeated two times or more:  $(\mathbf{A} \cdot \mathbf{B})_\bullet = \mathbf{A} \bullet \mathbf{B} = A_{ij} B_{ji}$ , and here are useful equations for second complexity tensors

$$\begin{aligned} \mathbf{A} \bullet \mathbf{B} &= \mathbf{B} \bullet \mathbf{A}, \quad \mathbf{d} \cdot \mathbf{A} \cdot \mathbf{b} = \mathbf{A} \bullet \mathbf{b} \mathbf{d} = \mathbf{b} \mathbf{d} \bullet \mathbf{A} = b_j d_i A_{ij}, \\ \mathbf{A} \bullet \mathbf{B} &= \mathbf{A}^\top \bullet \mathbf{B}^\top = A_{ij} B_{ji}, \quad \mathbf{A} \bullet \mathbf{B}^\top = \mathbf{A}^\top \bullet \mathbf{B} = A_{ij} B_{ij}, \\ \mathbf{A} \bullet \mathbf{E} &= \mathbf{E} \bullet \mathbf{A} = \mathbf{A}_\bullet = A_{jj}, \\ \mathbf{A} \cdot \mathbf{B} \bullet \mathbf{E} &= A_{ij} B_{jk} \delta_{ki} = \mathbf{A} \bullet \mathbf{B}, \quad \mathbf{A} \cdot \mathbf{A} \bullet \mathbf{E} = \mathbf{A} \bullet \mathbf{A}, \\ \mathbf{A} \bullet \mathbf{B} \cdot \mathbf{C} &= \mathbf{A} \cdot \mathbf{B} \bullet \mathbf{C} = \mathbf{C} \bullet \mathbf{A} \cdot \mathbf{B} = A_{ij} B_{jk} C_{ki}, \\ \mathbf{A} \bullet \mathbf{B} \cdot \mathbf{C} \cdot \mathbf{D} &= \mathbf{A} \cdot \mathbf{B} \bullet \mathbf{C} \cdot \mathbf{D} = \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} \bullet \mathbf{D} = \\ &= \mathbf{D} \bullet \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = A_{ij} B_{jk} C_{kh} D_{hi}. \end{aligned} \quad (3.10)$$

## § 4. Polyadic representation (decomposition)

Before in § 2, a tensor was presented as some invariant object, showing itself in every basis as a collection of numbers (components). Such a presentation is typical for majority of books about tensors. Index notation can be convenient, especially when only rectangular coordinates are used, but very often it is not. And the relevant case is physics of elastic continua: it needs more elegant, more powerful and perfect apparatus of the direct tensor calculus, operating with indexless invariant objects.

Linear combination  $\mathbf{v} = v_i \mathbf{e}_i$  from decomposition (1.1) connects vector  $\mathbf{v}$  with basis  $\mathbf{e}_i$  and vector's components  $v_i$  in that basis. Soon we will get a similar relation for a tensor of any complexity.

Any bivalent tensor  ${}^2\mathbf{B}$  has nine components  $B_{ij}$  in each basis. The number of various dyads  $\mathbf{e}_i \mathbf{e}_j$  for the same basis is nine ( $3^2$ ) too. Linear combining these dyads with coefficients  $B_{ij}$  gives the sum  $B_{ij} \mathbf{e}_i \mathbf{e}_j$ . This is tensor, — but what are its components, and how this representation changes or doesn't change with a rotation of basis?

Components of the constructed sum

$$(B_{ij} \mathbf{e}_i \mathbf{e}_j)_{pq} = B_{ij} \delta_{ip} \delta_{jq} = B_{pq}$$



are components of tensor  ${}^2\mathbf{B}$ . And with a rotation of basis

$$B'_{ij}e'_ie'_j = o_{i'p}o_{j'q}B_{pq}o_{i'n}e_n o_{j'm}e_m = \delta_{pn}\delta_{qm}B_{pq}e_ne_m = B_{pq}e_pe_q.$$

Doubts are dropped: a tensor of second complexity can be (re)presented as the linear combination

$${}^2\mathbf{B} = B_{ij}e_ie_j \quad (4.1)$$

— the dyadic decomposition of a bivalent tensor.

For the unit tensor

$$\mathbf{E} = E_{ij}e_ie_j = \delta_{ij}e_ie_j = e_ie_i = e_1e_1 + e_2e_2 + e_3e_3,$$

that's why  $\mathbf{E}$  is called the unit dyad.

Polyadic representations like (4.1) help to operate with tensors easier and with more comprehension:

$$\begin{aligned} \mathbf{v} \cdot {}^2\mathbf{B} &= v_ie_i \cdot e_j B_{jk}e_k = v_i\delta_{ij}B_{jk}e_k = v_iB_{ik}e_k, \\ e_i \cdot {}^2\mathbf{B} \cdot e_j &= e_i \cdot B_{pq}e_pe_q \cdot e_j = B_{pq}\delta_{ip}\delta_{qj} = B_{ij} = {}^2\mathbf{B} \cdot e_ie_i. \end{aligned} \quad (4.2)$$

The last line here is quite interesting: tensor components are represented through the tensor itself. An orthogonal transformation of components with a rotation of basis (2.1) turns out to be just a version of (4.2).

And any tensor, of any complexity above zero, may be decomposed into basis polyads. The following equalities are for a trivalent tensor

$$\begin{aligned} {}^3\mathbf{C} &= C_{ijk}e_ie_je_k, \\ C_{ijk} &= {}^3\mathbf{C} \cdots e_ke_je_i = e_i \cdot {}^3\mathbf{C} \cdots e_ke_j = e_je_i \cdots {}^3\mathbf{C} \cdot e_k. \end{aligned} \quad (4.3)$$

Using decomposition into polyads, it's easy to see the actuality of property (3.6) — the “unitness” of tensor  $\mathbf{E}$ :

$$\begin{aligned} {}^n\mathbf{a} &= a_{ij\dots q}e_ie_j\dots e_q, \quad \mathbf{E} = e_e e_e \\ {}^n\mathbf{a} \cdot \mathbf{E} &= a_{ij\dots q}e_ie_j\dots \underbrace{e_q \cdot e_e}_{\delta_{eq}}e_e = a_{ij\dots q}e_ie_j\dots e_q = {}^n\mathbf{a}, \\ \mathbf{E} \cdot {}^n\mathbf{a} &= e_e e_e \cdot a_{ij\dots q}e_ie_j\dots e_q = a_{ij\dots q}\delta_{ei}e_e e_j\dots e_q = {}^n\mathbf{a}. \end{aligned}$$

The polyadic representation connects direct and index notations together. It's not worth contraposing one another. The direct notation is compact, elegant, it much more than others suits for final relations. But sometimes the index notation is very convenient too, as it is for cumbersome manipulations with tensors.

## § 5. Matrices, permutations and determinants

A two-dimensional array can be visualized as a table consisting of rows and columns — a matrix.

Matrices are rectangular arrangement of elements (numbers or expressions) in rows and columns. They are sometimes called arrays. A matrix is a two-dimensional or one-dimensional array of elements. A matrix is a rectangular array (a table) — it is made up of rows and columns. A matrix has the same number of elements in each row and the same number of elements in each column.

### *Matrix dimension*

Matrices come in all sizes or dimensions.

The dimension of a matrix consists of the number of rows, then a multiplication sign (“ $\times$ ” is used the most often), and then the number of columns.

**Examples.**

$$[\mathcal{A}] = \dots_{3 \times 3}$$

Matrix  $[A]$  is a  $3 \times 3$  matrix, because it has 3 rows and 3 columns. Matrix  $[B]$  has 2 rows and 4 columns, so its dimension is  $2 \times 4$ . Matrix  $[C]$  is a column matrix (that is a matrix with just one column), and its dimension is  $3 \times 1$ . And  $[D]$  is a row matrix with dimension  $1 \times 6$ .

The matrix algebra includes linear operations — the addition of matrices and the scalar multiplication (multiplication of a matrix by a scalar).

The dimension of a matrix (described earlier) is essential for binary operations, that is operations involving the two matrices.

- ✓ Adding or subtracting the two matrices is possible only when ..... the same dimension. **Addition of matrices is possible only when** both addends have the same size.
- ✓ For multiplying the two matrices, the number of columns in the first matrix has to match the number of rows in the second matrix.

$$[\mathcal{A}]_{m \times n} = \dots$$

The resulting matrix, known as the matrix product, has the number of rows of the first matrix and the number of columns of the second matrix. The product of matrices ...A... and ...B... is denoted as ...AB....

### *Square matrices*

....

### *Matrices and one-dimensional arrays*

The two indices of a table is more than the single index of a one-dimensional array. Due to this, a one-dimensional array may be represented as both a table of rows and a table of columns. Thus, these arrays have the ambiguity: they are either horizontal tables with dimensions “one row many columns” ( $1 \times n$ )

$$[h_{11} \ h_{12} \ h_{13}],$$

or vertical tables with dimensions “many rows one column” ( $n \times 1$ )

$$\begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}.$$

And such a case is possible, when the elements of both these matrices are equal to each other ( $h_{11} = v_{11}$ ,  $h_{12} = v_{21}$ ,  $h_{13} = v_{31}$ ). Then this case is commonly considered as transposing. The transpose of a matrix is an operation of the matrix algebra which flips a matrix around its main diagonal. The main diagonal consists of elements whose row number and column number coincide, it goes from the top left corner to the bottom right corner. For a “vertical” or a “horizontal” matrix representing a one-dimensional array the main diagonal goes a similar way with just the only one element on it.

.....

### *Matrices and components of bivalent tensors*

It's sometimes pretty convenient to represent the set of components of a bivalent tensor as a matrix.

$$B_{ij} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

Because the matrix multiplication is calculated similar to the tensor multiplication with contraction (the “ $\bullet$ ”-product).

$$C_{ik} = A_{ij}B_{jk}$$

...

Widespread in literature substitution of tensors for matrices of components leads to mistakes — without tracking the bases which these matrices correspond to.

...

$$\delta_{ij} = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

...

The transpose of a matrix is computed/performed by swapping columns for rows in that matrix. For example

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}^T = \begin{bmatrix} M_{11} & M_{21} & M_{31} \\ M_{12} & M_{22} & M_{32} \\ M_{13} & M_{23} & M_{33} \end{bmatrix}$$

(swapping letters of indices)

...

To write permutations, the O.Veblen's permutation symbol  $e_{ijk}$  is introduced

$$\begin{aligned} e_{123} &= e_{231} = e_{312} = 1, \\ e_{213} &= e_{321} = e_{132} = -1, \end{aligned}$$

all others are zeroes.

The number of indices in  $e$ -symbols is equal to the dimension of space.

Symbol  $e_{ijk}$  ( $\pm 1$  or 0) changes its sign with a permutation of any two indices, it does not change with a double (“circular”) permutation, and it nullifies if any index is repeated.

$$e_{ijk} = \begin{cases} 0 & \text{when there is a repeated index in } ijk, \\ +1 & \text{when an even number of permutations of } ijk \text{ is present,} \\ -1 & \text{when an odd number of permutations of } ijk \text{ is present.} \end{cases}$$

...

Using  $e$ -symbols, the determinant of a matrix is expressed as

$$e_{pqr} \det_{m,n} A_{mn} = e_{ijk} A_{pi} A_{qj} A_{rk} \quad (5.1)$$

$$\text{or} \quad (5.2)$$

$$e_{pqr} \det_{m,n} A_{mn} = e_{ijk} A_{ip} A_{jq} A_{kr}. \quad (5.3)$$

For  $p=1, q=2, r=3$

$$\det_{m,n} A_{mn} = e_{ijk} A_{1i} A_{2j} A_{3k} = e_{ijk} A_{i1} A_{j2} A_{k3}$$

$$\det_{m,n} A_{mn} = A_{11} A_{22} A_{33} - A_{11} A_{23} A_{32} - A_{12} A_{21} A_{33} + \\ + A_{12} A_{23} A_{31} + A_{13} A_{21} A_{32} - A_{13} A_{22} A_{31}$$

...

$$e_{pqr} \det_{m,n} A_{mn} = \det \begin{bmatrix} A_{1p} & A_{1q} & A_{1r} \\ A_{2p} & A_{2q} & A_{2r} \\ A_{3p} & A_{3q} & A_{3r} \end{bmatrix} = \det \begin{bmatrix} A_{p1} & A_{p2} & A_{p3} \\ A_{q1} & A_{q2} & A_{q3} \\ A_{r1} & A_{r2} & A_{r3} \end{bmatrix}$$

...

$$\det_{i,j} \delta_{ij} = 1$$

...

А символ перестановки через детерминант — как

$$e_{pqr} = e_{ijk} \delta_{pi} \delta_{qj} \delta_{rk} = e_{ijk} \delta_{ip} \delta_{jq} \delta_{kr}, \\ e_{pqr} = \det \begin{bmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{bmatrix} = \det \begin{bmatrix} \delta_{p1} & \delta_{p2} & \delta_{p3} \\ \delta_{q1} & \delta_{q2} & \delta_{q3} \\ \delta_{r1} & \delta_{r2} & \delta_{r3} \end{bmatrix}. \quad (5.4)$$

...

A determinant is not sensitive to transposing:

$$\det_{i,j} A_{ij} = \det_{i,j} A_{ji} = \det_{j,i} A_{ij}.$$

...

“The determinant of the matrix product of the two matrices is equal to the product of determinants of these matrices”

$$\det_{i,k} B_{ik} \det_{k,j} C_{kj} = \det_{i,j} B_{ik} C_{kj} \quad (5.5)$$

$$e_{fgh} \det_{m,n} B_{ms} C_{sn} = e_{pqr} B_{fi} C_{ip} B_{gj} C_{jq} B_{hk} C_{kr}$$

$$e_{fgh} \det_{m,s} B_{ms} = e_{ijk} B_{fi} B_{gj} B_{hk}$$

$$e_{ijk} \det_{s,n} C_{sn} = e_{pqr} C_{ip} C_{jq} C_{kr}$$

$$e_{fgh} e_{ijk} \det_{m,s} B_{ms} \det_{s,n} C_{sn} = e_{ijk} e_{pqr} B_{fi} B_{gj} B_{hk} C_{ip} C_{jq} C_{kr}$$

...

Определитель компонент of a bivalent tensor is invariant, он не меняется с поворотом базиса

$$A'_{ij} = o_{i'm} o_{j'n} A_{mn}$$

...

## § 6. Cross product and the Levi-Civita tensor

By habitual notions, the “ $\times$ ”-product (“cross product”, “vector product”, sometimes “oriented area product”) of two vectors is a vector, heading perpendicular to the plane of multipliers, whose length is equal to the area of parallelogram, spanned by multipliers

$$\| \mathbf{a} \times \mathbf{b} \| = \| \mathbf{a} \| \| \mathbf{b} \| \sin \angle(\mathbf{a}, \mathbf{b}).$$

However, a “ $\times$ ”-product isn’t quite a vector, since it is not completely invariant.

The multipliers of “ $\times$ ”-product  $\mathbf{c} = \mathbf{a} \times \mathbf{b}$  determine the result’s direction in space, but except the sign (the orientation), fig. 3.

The space becomes oriented and results of “ $\times$ ”-products become completely determined as soon as the one direction from the possible two is chosen as “positive”. The choice is arbitrary (random).

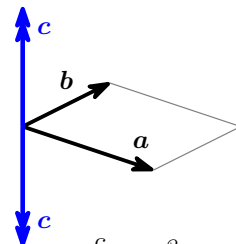


figure 3

The two possible orientations of space are commonly called “right-chiral” or “right-handed” and “left-chiral” or “left-handed”.

Chiral = asymmetric in such a way that the thing and its mirror image are not superimposable, a picture cannot be superposed on its mirror image by any combination of rotations and translations.

An object is chiral if it is distinguishable from its mirror image (cannot be superimposed onto it).

Vectors are usually measured using some basis  $\mathbf{e}_i$ . They are decomposed into linear combinations like  $\mathbf{a} = a_i \mathbf{e}_i$ . So the orientation of space is equivalent to the orientation of the sequential triple of basis vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ . It means that the sequence of basis vectors becomes significant (for linear combinations, the sequence of addends doesn’t affect anything).

If two bases consist of different sequences of the same vectors within an oriented space, then their orientations differ by some permutation.

The orientation of space is the (kind of) asymmetry. This asymmetry makes it impossible to replicate a reflection by means of any rotations\* in this space (without advancing to a higher-dimensional space).

\* Applying only rotations, it’s impossible to replace the left hand of a human figure into the right hand. But it is possible reflecting a figure in a mirror.

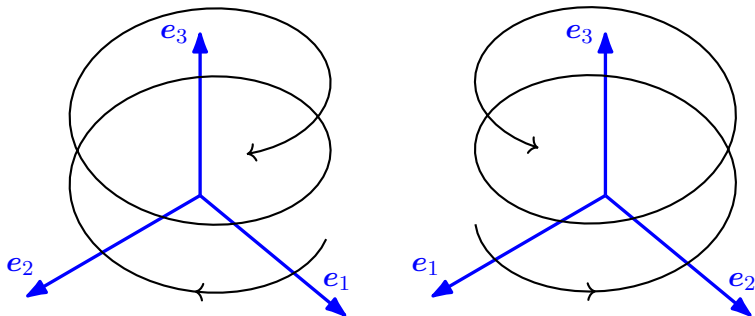


figure 4

A pseudovector (axial vector) is a vector-like object that is invariant under any rotation\*, but on a reflection it additionally flips its direction backwards.

... put the figure here ...

Except in rare cases, the direction of a fully invariant (polar) vector will change on a reflection together with the entire space.

A pseudovector (axial vector), unlike a (polar) vector, when reflected doesn't change the component orthogonal to the plane of reflection, and turns out to be flipped relatively to the polar vectors and the geometry of the entire space. This happens because the sign (and, accordingly, the direction) of each axial vector changes along with changing the sign of the “ $\times$ ”-product — which corresponds to reflection.

The otherness of pseudovectors narrows the variety of formulas: a pseudovector is not additive with a vector. Formula  $\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}$  is correct, because  $\boldsymbol{\omega}$  is pseudovector there, and with the cross product two “pseudo” give  $(-1)^2 = 1$  (“mutually compensate each other”).

Here is the Levi-Civita tensor — the volumetric tensor of third complexity

$${}^3\epsilon = \epsilon_{ijk} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k, \quad \epsilon_{ijk} \equiv \mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k \quad (6.1)$$

\* Rotations cannot change orientation of a triple of basis vectors, only a reflection can.



with components  $\epsilon_{ijk}$  equal to “triple” (“mixed”, “cross-dot”) products of basis vectors.

The absolute value (modulus) of each nonzero component of tensor  ${}^3\epsilon$  is equal to the volume  $\sqrt{g}$  of a parallelepiped, drew upon a basis. For basis  $\mathbf{e}_i$  of pairwise perpendicular one-unit long vectors  $\sqrt{g} = 1$ .

Tensor  ${}^3\epsilon$  is isotropic, its components are constant enand independent of rotations of a basis. But a reflection — изменение ориентации тройки базисных векторов (перемена “направления винта”) — changes the sign of  ${}^3\epsilon$ , so this is pseudotensor (axial tensor).

If  $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ , then  $\mathbf{e}_i$  is ориентированная положительно тройка, произвольно выбираемая из двух вариантов (рис. 3). В этом случае компоненты  ${}^3\epsilon$  равны символу перестановки Veblen:  $\epsilon_{ijk} = e_{ijk}$ . Когда же  $\mathbf{e}_1 \times \mathbf{e}_2 = -\mathbf{e}_3$ , then triple  $\mathbf{e}_i$  is oriented negatively (or “mirrored”). For mirrored triples  $\epsilon_{ijk} = -e_{ijk}$  (and  $e_{ijk} = -\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k$ ).

With the Levi-Civita tensor  ${}^3\epsilon$  it is possible to take a fresh look at the cross product:

$$\begin{aligned}\epsilon_{ijk} &= \mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k \Leftrightarrow \mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k, \\ \mathbf{a} \times \mathbf{b} &= a_i \mathbf{e}_i \times b_j \mathbf{e}_j = a_i b_j \mathbf{e}_i \times \mathbf{e}_j = a_i b_j \epsilon_{ijk} \mathbf{e}_k = \\ &= b_j a_i \mathbf{e}_j \mathbf{e}_i \cdot \epsilon_{mnk} \mathbf{e}_m \mathbf{e}_n \mathbf{e}_k = \mathbf{b} \mathbf{a} \cdot \cdot {}^3\epsilon, \\ &= a_i \epsilon_{ijk} \mathbf{e}_k b_j = -a_i \epsilon_{ikj} \mathbf{e}_k b_j = -\mathbf{a} \cdot {}^3\epsilon \cdot \mathbf{b}. \quad (6.2)\end{aligned}$$

So that, the cross product is not another new, entirely distinct operation. With the Levi-Civita tensor it reduces to the four already described (§ 3) and is applicable to tensors of any complexity.

“The cross product” is just the dot product — the combination of multiplication and contraction (§ 3) — involving tensor  ${}^3\epsilon$ . Such combinations are possible with any tensors:

$$\begin{aligned}\mathbf{a} \times {}^2\mathbf{B} &= a_i \mathbf{e}_i \times B_{jk} \mathbf{e}_j \mathbf{e}_k = \underbrace{a_i B_{jk} \epsilon_{ijn}}_{-a_i \epsilon_{inj} B_{jk}} \mathbf{e}_n \mathbf{e}_k = -\mathbf{a} \cdot {}^3\epsilon \cdot {}^2\mathbf{B}, \\ {}^2\mathbf{C} \times \mathbf{d} \mathbf{b} &= C_{ij} \mathbf{e}_i \mathbf{e}_j \times d_p b_q \mathbf{e}_p \mathbf{e}_q = \mathbf{e}_i C_{ij} d_p \underbrace{\epsilon_{jpk}}_{-\epsilon_{pjk} = -\epsilon_{jpk}} \mathbf{e}_k b_q \mathbf{e}_q = \\ &= -{}^2\mathbf{C} d \cdot \cdot {}^3\epsilon \mathbf{b} = -{}^2\mathbf{C} \cdot {}^3\epsilon \cdot \mathbf{d} \mathbf{b}, \\ \mathbf{E} \times \mathbf{E} &= \mathbf{e}_i \mathbf{e}_i \times \mathbf{e}_j \mathbf{e}_j = -\epsilon_{ijk} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k = -{}^3\epsilon. \quad (6.3)\end{aligned}$$

The last equation connects the isotropic tensors of second and third complexities.

Generalizing to all tensors of nonzero complexity

$${}^n\boldsymbol{\xi} \times {}^m\boldsymbol{\zeta} = -{}^n\boldsymbol{\xi} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^m\boldsymbol{\zeta} \quad \forall {}^n\boldsymbol{\xi}, {}^m\boldsymbol{\zeta} \quad \forall n > 0, m > 0. \quad (6.4)$$

When one of the operands is the unit (metric) tensor, from (6.4) and (3.6)  $\forall {}^n\boldsymbol{\Upsilon} \quad \forall n > 0$

$$\begin{aligned} \boldsymbol{E} \times {}^n\boldsymbol{\Upsilon} &= -\boldsymbol{E} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^n\boldsymbol{\Upsilon} = -{}^3\boldsymbol{\epsilon} \cdot {}^n\boldsymbol{\Upsilon}, \\ {}^n\boldsymbol{\Upsilon} \times \boldsymbol{E} &= -{}^n\boldsymbol{\Upsilon} \cdot {}^3\boldsymbol{\epsilon} \cdot \boldsymbol{E} = -{}^n\boldsymbol{\Upsilon} \cdot {}^3\boldsymbol{\epsilon}. \end{aligned}$$

The cross product of two vectors is not commutative but anti-commutative:

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{E}) = (\boldsymbol{a} \times \boldsymbol{E}) \cdot \boldsymbol{b} = -\boldsymbol{ab} \cdot {}^3\boldsymbol{\epsilon} = -{}^3\boldsymbol{\epsilon} \cdot \boldsymbol{ab}, \\ \boldsymbol{b} \times \boldsymbol{a} &= \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{E}) = (\boldsymbol{b} \times \boldsymbol{E}) \cdot \boldsymbol{a} = -\boldsymbol{ba} \cdot {}^3\boldsymbol{\epsilon} = -{}^3\boldsymbol{\epsilon} \cdot \boldsymbol{ba}, \\ \boldsymbol{a} \times \boldsymbol{b} &= -\boldsymbol{ab} \cdot {}^3\boldsymbol{\epsilon} = \boldsymbol{ba} \cdot {}^3\boldsymbol{\epsilon} \Rightarrow \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}. \end{aligned} \quad (6.5)$$

For any bivalent tensor  ${}^2\boldsymbol{B}$  and a tensor of first complexity (vector)  $\boldsymbol{a}$

$${}^2\boldsymbol{B} \times \boldsymbol{a} = \boldsymbol{e}_i B_{ij} \boldsymbol{e}_j \times a_k \boldsymbol{e}_k = (-a_k \boldsymbol{e}_k \times \boldsymbol{e}_j B_{ij} \boldsymbol{e}_i)^\top = -(\boldsymbol{a} \times {}^2\boldsymbol{B}^\top)^\top.$$

However, in the particular case of the unit tensor  $\boldsymbol{E}$  and a vector

$$\begin{aligned} \boldsymbol{E} \times \boldsymbol{a} &= -(\boldsymbol{a} \times \boldsymbol{E}^\top)^\top = -(\boldsymbol{a} \times \boldsymbol{E})^\top = \boldsymbol{a} \times \boldsymbol{E}, \\ \boldsymbol{E} \times \boldsymbol{a} &= \boldsymbol{a} \times \boldsymbol{E} = -\boldsymbol{a} \cdot {}^3\boldsymbol{\epsilon} = -{}^3\boldsymbol{\epsilon} \cdot \boldsymbol{a}. \end{aligned} \quad (6.6)$$

Справедливо такое соотношение

$$e_{ijk} e_{pqr} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix} \quad (6.7)$$

○ Доказательство начнём с представлений символов Веблена как определителей (5.4).  $e_{ijk} = \pm \boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k$  по строкам,  $e_{pqr} = \pm \boldsymbol{e}_p \times \boldsymbol{e}_q \cdot \boldsymbol{e}_r$  по столбцам, с “−” для “левой” тройки

$$e_{ijk} = \det \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}, \quad e_{pqr} = \det \begin{bmatrix} \delta_{p1} & \delta_{q1} & \delta_{r1} \\ \delta_{p2} & \delta_{q2} & \delta_{r2} \\ \delta_{p3} & \delta_{q3} & \delta_{r3} \end{bmatrix}.$$

Левая часть (6.7) есть произведение  $e_{ijk} e_{pqr}$  этих определителей. Но  $\det(AB) = (\det A)(\det B)$  — определитель произведения матриц равен произведению определителей (5.5). В матрице-произведении

$$\overbrace{+e_{ijk} \boldsymbol{e}_i \boldsymbol{e}_k \boldsymbol{e}_j}$$

элемент  $[\dots]_{11}$  равен  $\delta_{is}\delta_{ps} = \delta_{ip}$ , как и в (6.7); **легко проверить и другие фрагменты.** ●

Contraction of (6.7) приводит к полезным формулам

$$\begin{aligned}
 e_{ijk}e_{pqk} &= \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & \delta_{kk} \end{bmatrix} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & 3 \end{bmatrix} = \\
 &= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jk}\delta_{kp} + \delta_{ik}\delta_{jp}\delta_{kq} - \delta_{ik}\delta_{jq}\delta_{kp} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jk}\delta_{kq} = \\
 &= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} + \delta_{ik}\delta_{jp} - \delta_{ip}\delta_{jq} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} = \\
 &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \\
 e_{ijk}e_{pjk} &= \delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip}, \\
 e_{ijk}e_{ijk} &= 2\delta_{ii} = 6.
 \end{aligned}$$

Or in short

$$e_{ijk}e_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \quad e_{ijk}e_{pjk} = 2\delta_{ip}, \quad e_{ijk}e_{ijk} = 6. \quad (6.8)$$

The first of these formulas даёт представление двойного векторного произведения

$$\begin{aligned}
 \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= a_i e_i \times \epsilon_{pqj} b_p c_q e_j = \epsilon_{kij} \epsilon_{pqj} a_i b_p c_q e_k = \\
 &= (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip}) a_i b_p c_q e_k = a_i b_k c_i e_k - a_i b_i c_k e_k = \\
 &= \mathbf{a} \cdot \mathbf{cb} - \mathbf{a} \cdot \mathbf{bc} = \mathbf{a} \cdot (\mathbf{cb} - \mathbf{bc}) = \mathbf{a} \cdot \mathbf{cb} - \mathbf{cb} \cdot \mathbf{a}. \quad (6.9)
 \end{aligned}$$

By another interpretation, the dot product of a dyad and a vector is not commutative:  $\mathbf{bd} \cdot \mathbf{c} \neq \mathbf{c} \cdot \mathbf{bd}$ , and this difference can be expressed as

$$\mathbf{bd} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{bd} = \mathbf{c} \times (\mathbf{b} \times \mathbf{d}). \quad (6.10)$$

$$\mathbf{a} \cdot \mathbf{bc} = \mathbf{cb} \cdot \mathbf{a} = \mathbf{ca} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{ac}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{c} \times (\mathbf{b} \times \mathbf{a})$$

**The same way it may be derived that**

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{ba} - \mathbf{ab}) \cdot \mathbf{c} = \mathbf{ba} \cdot \mathbf{c} - \mathbf{ab} \cdot \mathbf{c}. \quad (6.11)$$

And following identities for any two vectors  $\mathbf{a}$  and  $\mathbf{b}$

$$\begin{aligned}
 (\mathbf{a} \times \mathbf{b}) \times \mathbf{E} &= \epsilon_{ijk} a_i b_j e_k \times e_n e_n = a_i b_j \epsilon_{ijk} \epsilon_{knq} e_q e_n = \\
 &= a_i b_j (\delta_{in}\delta_{jq} - \delta_{iq}\delta_{jn}) e_q e_n = a_i b_j e_j e_i - a_i b_i e_j e_j = \\
 &= \mathbf{ba} - \mathbf{ab}, \quad (6.12)
 \end{aligned}$$

$$\begin{aligned}
(\mathbf{a} \times \mathbf{E}) \cdot (\mathbf{b} \times \mathbf{E}) &= (\mathbf{a} \cdot {}^3\epsilon) \cdot (\mathbf{b} \cdot {}^3\epsilon) = \\
&= a_i \epsilon_{ipn} e_p e_n \cdot b_j \epsilon_{jsk} e_s e_k = a_i b_j \epsilon_{ipn} \epsilon_{nkj} e_p e_k = \\
&= a_i b_j (\delta_{ik} \delta_{pj} - \delta_{ij} \delta_{pk}) e_p e_k = a_i b_j e_j e_i - a_i b_i e_k e_k = \\
&= \mathbf{b}\mathbf{a} - \mathbf{a} \cdot \mathbf{b} \mathbf{E}. \quad (6.13)
\end{aligned}$$

Finally, one more correlation between isotropic tensors of second and third complexities:

$${}^3\epsilon \cdot {}^3\epsilon = \epsilon_{ijk} e_i \epsilon_{kjn} e_n = -2\delta_{in} e_i e_n = -2\mathbf{E}. \quad (6.14)$$

## § 7. Symmetric and antisymmetric tensors

A tensor that does not change upon permutation of some pair of its indices is called symmetric for this pair of indices. But if, upon permutation of some pair of indices, a tensor alternates its sign  $(+/-)$ , then it is called antisymmetric or skew-symmetric for this pair of indices.

The Levi-Civita tensor  ${}^3\epsilon$  is antisymmetric for any pair of indices, that is, it is completely (absolutely) antisymmetric (skew-symmetric).

Тензор второй сложности  $\mathbf{B}$  симметричен, если  $\mathbf{B} = \mathbf{B}^\top$ . Когда транспонирование меняет знак тензора, то есть  $\mathbf{A}^\top = -\mathbf{A}$ , тогда он антисимметричен (кососимметричен).

Любой тензор второй сложности представим суммой симметричной и антисимметричной частей

$$\begin{aligned}
\mathbf{C} &= \mathbf{C}^S + \mathbf{C}^A, \quad \mathbf{C}^\top = \mathbf{C}^S - \mathbf{C}^A; \\
\mathbf{C}^S &\equiv \frac{1}{2} (\mathbf{C} + \mathbf{C}^\top), \quad \mathbf{C}^A \equiv \frac{1}{2} (\mathbf{C} - \mathbf{C}^\top). \quad (7.1)
\end{aligned}$$

Для диады  $\mathbf{cd} = \mathbf{cd}^S + \mathbf{cd}^A = \frac{1}{2} (\mathbf{cd} + \mathbf{dc}) + \frac{1}{2} (\mathbf{cd} - \mathbf{dc})$ .

Произведение двух симметричных тензоров  $\mathbf{C}^S \cdot \mathbf{D}^S$  симметрично далеко не всегда, но только когда  $\mathbf{D}^S \cdot \mathbf{C}^S = \mathbf{C}^S \cdot \mathbf{D}^S$ , ведь по (??)  $(\mathbf{C}^S \cdot \mathbf{D}^S)^\top = \mathbf{D}^S \cdot \mathbf{C}^S$ .

В нечётномерных пространствах любой антисимметричный тензор второй сложности необратим, определитель матрицы компонент для него — нулевой.

Существует взаимно-однозначное соответствие между антисимметричными тензорами второй сложности и (псевдо)векторами.

Компоненты кососимметричного тензора полностью определяются тройкой чисел (диагональные элементы матрицы компонент — нули, а недиагональные — попарно противоположны). Dot product кососимметричного  $\mathbf{A}$  и какого-либо тензора  ${}^n\boldsymbol{\xi}$  однозначно соответствует cross product'у псевдовектора  $\mathbf{a}$  и того же тензора  ${}^n\boldsymbol{\xi}$

$$\begin{aligned} \mathbf{b} &= \mathbf{A} \cdot {}^n\boldsymbol{\xi} \Leftrightarrow \mathbf{a} \times {}^n\boldsymbol{\xi} = \mathbf{b} \quad \forall \mathbf{A} = \mathbf{A}^A \quad \forall {}^n\boldsymbol{\xi} \quad \forall n > 0, \\ \mathbf{d} &= {}^n\boldsymbol{\xi} \cdot \mathbf{A} \Leftrightarrow {}^n\boldsymbol{\xi} \times \mathbf{a} = \mathbf{d} \quad \forall \mathbf{A} = \mathbf{A}^A \quad \forall {}^n\boldsymbol{\xi} \quad \forall n > 0. \end{aligned} \quad (7.2)$$

Раскроем это соответствие  $\mathbf{A} = \mathbf{A}(\mathbf{a})$ :

$$\begin{aligned} \mathbf{A} \cdot {}^n\boldsymbol{\xi} &= \mathbf{a} \times {}^n\boldsymbol{\xi} \\ A_{hi} \mathbf{e}_h \mathbf{e}_i \cdot \xi_{jk\dots q} \mathbf{e}_j \mathbf{e}_k \dots \mathbf{e}_q &= a_i \mathbf{e}_i \times \xi_{jk\dots q} \mathbf{e}_j \mathbf{e}_k \dots \mathbf{e}_q \\ A_{hj} \xi_{jk\dots q} \mathbf{e}_h \mathbf{e}_k \dots \mathbf{e}_q &= a_i \in_{ijh} \xi_{jk\dots q} \mathbf{e}_h \mathbf{e}_k \dots \mathbf{e}_q \\ A_{hj} &= a_i \in_{ijh} \\ A_{hj} &= -a_i \in_{ihj} \\ \mathbf{A} &= -\mathbf{a} \cdot {}^3\boldsymbol{\epsilon} \end{aligned}$$

Так же из  ${}^n\boldsymbol{\xi} \cdot \mathbf{A} = {}^n\boldsymbol{\xi} \times \mathbf{a}$  получается  $\mathbf{A} = -{}^3\boldsymbol{\epsilon} \cdot \mathbf{a}$ .

Или проще, согласно (6.4)

$$\begin{aligned} \mathbf{A} &= \mathbf{A} \cdot \mathbf{E} = \mathbf{a} \times \mathbf{E} = -\mathbf{a} \cdot {}^3\boldsymbol{\epsilon}, \\ \mathbf{A} &= \mathbf{E} \cdot \mathbf{A} = \mathbf{E} \times \mathbf{a} = -{}^3\boldsymbol{\epsilon} \cdot \mathbf{a}. \end{aligned}$$

(Псевдо)вектор  $\mathbf{a}$  называется сопутствующим для тензора  $\mathbf{A}$ .

В общем, для взаимно-однозначного соответствия между  $\mathbf{A}$  and  $\mathbf{a}$  имеем

$$\begin{aligned} \mathbf{A} &= -\mathbf{a} \cdot {}^3\boldsymbol{\epsilon} = \mathbf{a} \times \mathbf{E} = -{}^3\boldsymbol{\epsilon} \cdot \mathbf{a} = \mathbf{E} \times \mathbf{a}, \\ \mathbf{a} &= \mathbf{a} \cdot \mathbf{E} = \mathbf{a} \cdot \left( -\frac{1}{2} {}^3\boldsymbol{\epsilon} \cdot {}^3\boldsymbol{\epsilon} \right) = \frac{1}{2} \mathbf{A} \cdot {}^3\boldsymbol{\epsilon}. \end{aligned} \quad (7.3)$$

Компоненты кососимметричного  $\mathbf{A}$  через компоненты сопутствующего ему псевдовектора  $\mathbf{a}$

$$\begin{aligned} \mathbf{A} &= -{}^3\boldsymbol{\epsilon} \cdot \mathbf{a} = -\in_{ijk} \mathbf{e}_i \mathbf{e}_j a_k, \\ A_{ij} &= -\in_{ijk} a_k = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \end{aligned}$$

и наоборот

$$\mathbf{a} = \frac{1}{2} \mathbf{A} \cdot \cdot^3 \boldsymbol{\epsilon} = \frac{1}{2} A_{jk} \epsilon_{kji} \mathbf{e}_i,$$

$$a_i = \frac{1}{2} \epsilon_{ikj} A_{jk} = \frac{1}{2} \begin{bmatrix} \epsilon_{123} A_{32} + \epsilon_{132} A_{23} \\ \epsilon_{213} A_{31} + \epsilon_{231} A_{13} \\ \epsilon_{312} A_{21} + \epsilon_{321} A_{12} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}.$$

Легко запоминающийся вспомогательный “псевдовекторный инвариант”  $\mathbf{A}_\times$  получается из тензора  $\mathbf{A}$  заменой диадного произведения на векторное

$$\mathbf{A}_\times \equiv A_{ij} \mathbf{e}_i \times \mathbf{e}_j = -\mathbf{A} \cdot \cdot^3 \boldsymbol{\epsilon},$$

$$\mathbf{A}_\times = (\mathbf{a} \times \mathbf{E})_\times = -2\mathbf{a}, \quad \mathbf{a} = -\frac{1}{2} \mathbf{A}_\times = -\frac{1}{2} (\mathbf{a} \times \mathbf{E})_\times. \quad (7.4)$$

Обоснование (7.4):

$$\begin{aligned} \mathbf{a} \times \mathbf{E} &= -\frac{1}{2} \mathbf{A}_\times \times \mathbf{E} = -\frac{1}{2} A_{ij} \underbrace{(\mathbf{e}_i \times \mathbf{e}_j)}_{\epsilon_{ijn} \mathbf{e}_n} \times \mathbf{e}_k \mathbf{e}_k = \\ &= -\frac{1}{2} A_{ij} \underbrace{\epsilon_{nij} \epsilon_{nkp}}_{\delta_{jp} \delta_{ik} - \delta_{ip} \delta_{jk}} \mathbf{e}_p \mathbf{e}_k = -\frac{1}{2} A_{ij} (\mathbf{e}_j \mathbf{e}_i - \mathbf{e}_i \mathbf{e}_j) = \\ &= -\frac{1}{2} (\mathbf{A}^\top - \mathbf{A}) = \mathbf{A}^\mathbf{A} = \mathbf{A}. \end{aligned}$$

Сопутствующий вектор **можно ввести** для любого бивалентного тензора, но лишь антисимметричная часть при этом даёт вклад:  $\mathbf{C}^\mathbf{A} = -\frac{1}{2} \mathbf{C}_\times \times \mathbf{E}$ . Для симметричного тензора сопутствующий вектор — нулевой:  $\mathbf{B}_\times = \mathbf{0} \Leftrightarrow \mathbf{B} = \mathbf{B}^\top = \mathbf{B}^\mathbf{S}$ .

With (7.4) разложение какого-либо тензора  $\mathbf{C}$  на симметричную и антисимметричную части выглядит как

$$\mathbf{C} = \mathbf{C}^\mathbf{S} - \frac{1}{2} \mathbf{C}_\times \times \mathbf{E}. \quad (7.5)$$

For a dyad

$$(6.12) \Rightarrow (\mathbf{c} \times \mathbf{d}) \times \mathbf{E} = \mathbf{dc} - \mathbf{cd} = -2\mathbf{cd}^\mathbf{A}, \quad (\mathbf{cd})_\times = \mathbf{c} \times \mathbf{d},$$

and its decomposition

$$\mathbf{cd} = \frac{1}{2} (\mathbf{cd} + \mathbf{dc}) - \frac{1}{2} (\mathbf{c} \times \mathbf{d}) \times \mathbf{E}. \quad (7.6)$$

## § 8. Diagonalization of tensor. Eigenvectors and eigenvalues. Invariants

If for some tensor  ${}^2\mathbf{B}$  and nonzero vector  $\mathbf{a}$

$${}^2\mathbf{B} \cdot \mathbf{a} = \eta \mathbf{a}, \quad \mathbf{a} \neq \mathbf{0} \quad (8.1)$$

$${}^2\mathbf{B} \cdot \mathbf{a} = \eta \mathbf{E} \cdot \mathbf{a}, \quad ({}^2\mathbf{B} - \eta \mathbf{E}) \cdot \mathbf{a} = \mathbf{0},$$

then  $\eta$  is called the eigenvalue (or characteristic value) of tensor  ${}^2\mathbf{B}$ , and the axis (direction) of eigenvector  $\mathbf{a}$  is called its characteristic axis (direction).

In components, this is the eigenvalue problem for a matrix. A homogeneous system of linear equations  $(B_{ij} - \eta \delta_{ij})a_j = 0$  has a non-zero solution if the determinant of a matrix of components  $\det(B_{ij} - \eta \delta_{ij})$  is equal to zero:

$$\det \begin{bmatrix} B_{11} - \eta & B_{12} & B_{13} \\ B_{21} & B_{22} - \eta & B_{23} \\ B_{31} & B_{32} & B_{33} - \eta \end{bmatrix} = -\eta^3 + \text{I}\eta^2 - \text{II}\eta + \text{III} = 0; \quad (8.2)$$

$$\text{I} = \text{trace } {}^2\mathbf{B} = B_{kk} = B_{11} + B_{22} + B_{33},$$

$$\text{II} = B_{11}B_{22} - B_{12}B_{21} + B_{11}B_{33} - B_{13}B_{31} + B_{22}B_{33} - B_{23}B_{32}, \quad (8.3)$$

$$\text{III} = \det {}^2\mathbf{B} = \det_{i,j} B_{ij} = e_{ijk} B_{1i} B_{2j} B_{3k} = e_{ijk} B_{i1} B_{j2} B_{k3}.$$

The roots of characteristic equation (8.2) — eigenvalues — don't depend on the basis and therefore are invariants.

Coefficients of (8.3) also don't depend on the basis; they are called the first, the second and the third invariants of a tensor. The first invariant I — the trace — was described earlier in §3. The second invariant II is the trace of the adjugate matrix — the transpose of the cofactor matrix (matrix of algebraic complements):  $\text{II}({}^2\mathbf{B}) \equiv \text{trace}(\text{adj } B_{ij})$ . Or

$$\text{II}({}^2\mathbf{B}) \equiv \frac{1}{2} [({}^2\mathbf{B}_\bullet)^2 - {}^2\mathbf{B} \cdot {}^2\mathbf{B}] = \frac{1}{2} [(B_{kk})^2 - B_{ij}B_{ji}].$$

And the third invariant III is the determinant of a matrix of tensor components:  $\text{III}({}^2\mathbf{B}) \equiv \det {}^2\mathbf{B}$ .

This applies to all second complexity tensors. Besides that, in case of a symmetric tensor, the following is true:

- 1° The eigenvalues of a symmetric bivalent tensor are real numbers.
  - 2° The characteristic axes (directions) for different eigenvalues are orthogonal to each other.
- The first statement is proved by contradiction. If  $\eta$  is a complex root of (8.2) corresponding to eigenvector  $\mathbf{a}$ , then conjugate number  $\bar{\eta}$

will also be the root of (8.2). Eigenvector  $\bar{\mathbf{a}}$  with the conjugate components corresponds to it. And then

$$(8.1) \Rightarrow (\bar{\mathbf{a}} \cdot) {}^2\mathbf{B} \cdot \mathbf{a} = \eta \mathbf{a}, \quad (\mathbf{a} \cdot) {}^2\mathbf{B} \cdot \bar{\mathbf{a}} = \bar{\eta} \bar{\mathbf{a}} \Rightarrow \\ \Rightarrow \bar{\mathbf{a}} \cdot {}^2\mathbf{B} \cdot \mathbf{a} - \mathbf{a} \cdot {}^2\mathbf{B} \cdot \bar{\mathbf{a}} = (\eta - \bar{\eta}) \mathbf{a} \cdot \bar{\mathbf{a}}.$$

Here on the left is zero, because  $\mathbf{a} \cdot {}^2\mathbf{B} \cdot \mathbf{c} = \mathbf{c} \cdot {}^2\mathbf{B}^\top \cdot \mathbf{a}$  and  ${}^2\mathbf{B} = {}^2\mathbf{B}^\top$ . Thence  $\eta = \bar{\eta}$ , that is a real number.

Just as simple looks the proof of 2°:

$$\underbrace{\mathbf{a}_2 \cdot {}^2\mathbf{B} \cdot \mathbf{a}_1 - \mathbf{a}_1 \cdot {}^2\mathbf{B} \cdot \mathbf{a}_2}_{=0} = (\eta_1 - \eta_2) \mathbf{a}_1 \cdot \mathbf{a}_2, \quad \eta_1 \neq \eta_2 \Rightarrow \\ \Rightarrow \mathbf{a}_1 \cdot \mathbf{a}_2 = 0. \quad \bullet$$

If the roots of the characteristic equation (the eigenvalues) are different, then one unit long eigenvectors  $\boldsymbol{\alpha}_i$  compose an orthonormal basis. What are tensor components in such a basis?

$${}^2\mathbf{B} \cdot \boldsymbol{\alpha}_k = \sum_k \eta_k \boldsymbol{\alpha}_k, \quad k = 1, 2, 3 \\ {}^2\mathbf{B} \cdot \underbrace{\boldsymbol{\alpha}_k \boldsymbol{\alpha}_k}_E = \sum_k \eta_k \boldsymbol{\alpha}_k \boldsymbol{\alpha}_k$$

In a common case  $B_{ij} = \mathbf{e}_i \cdot {}^2\mathbf{B} \cdot \mathbf{e}_j$ . In the basis  $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$  of mutually perpendicular one unit long  $\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j = \delta_{ij}$  eigenvectors of a symmetric tensor

$$B_{11} = \boldsymbol{\alpha}_1 \cdot (\eta_1 \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_1 + \eta_2 \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_2 + \eta_3 \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_3) \cdot \boldsymbol{\alpha}_1 = \eta_1, \\ B_{12} = \boldsymbol{\alpha}_1 \cdot (\eta_1 \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_1 + \eta_2 \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_2 + \eta_3 \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_3) \cdot \boldsymbol{\alpha}_2 = 0, \\ \dots$$

The matrix of components is diagonal and  ${}^2\mathbf{B} = \sum \eta_i \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i$ . Here goes a summation over three repeating indices, because the special basis is used.

The case of multiplicity of eigenvalues is considered in the limit. If  $\eta_2 \rightarrow \eta_1$ , then any linear combination of vectors  $\mathbf{a}_1$  and  $\mathbf{a}_2$  in the limit satisfies equation (8.1). This means that any axis in plane  $\mathbf{a}_1, \mathbf{a}_2$  becomes characteristic. When all three eigenvalues coincide, any axis in space is characteristic. Then  ${}^2\mathbf{B} = \eta \mathbf{E}$ , such tensors are called isotropic or “spherical”.



## §9. Rotation tensor

The relation between two “right” (or two “left”) orthonormal bases  $\mathbf{e}_i$  and  $\mathring{\mathbf{e}}_i$  can be described by a two-index array represented as a matrix (§1, §5)

$$\mathbf{e}_i = \mathbf{e}_i \cdot \underbrace{\mathring{\mathbf{e}}_j \mathring{\mathbf{e}}_j}_{\mathbf{E}} = o_{ij} \mathring{\mathbf{e}}_j, \quad o_{ij} \equiv \mathbf{e}_i \cdot \mathring{\mathbf{e}}_j$$

(“a matrix of cosines”).

Also, a rotation of a tensor can be described by another tensor, called rotation tensor  $\mathbf{O}$

$$\mathbf{e}_i = \mathbf{e}_j \underbrace{\mathring{\mathbf{e}}_j \cdot \mathring{\mathbf{e}}_i}_{\delta_{ji}} = \mathbf{O} \cdot \mathring{\mathbf{e}}_i, \quad \mathbf{O} \equiv \mathbf{e}_j \mathring{\mathbf{e}}_j = e_1 \mathring{\mathbf{e}}_1 + e_2 \mathring{\mathbf{e}}_2 + e_3 \mathring{\mathbf{e}}_3. \quad (9.1)$$

Components of  $\mathbf{O}$  both in an initial  $\mathring{\mathbf{e}}_i$  and in a rotated  $\mathbf{e}_i$  bases are the same

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{O} \cdot \mathbf{e}_j &= \underbrace{\mathbf{e}_i \cdot \mathbf{e}_k}_{\delta_{ik}} \mathring{\mathbf{e}}_k \cdot \mathbf{e}_j = \mathring{\mathbf{e}}_i \cdot \mathbf{e}_j, \\ \mathring{\mathbf{e}}_i \cdot \mathbf{O} \cdot \mathring{\mathbf{e}}_j &= \mathring{\mathbf{e}}_i \cdot \mathbf{e}_k \underbrace{\mathring{\mathbf{e}}_k \cdot \mathring{\mathbf{e}}_j}_{\delta_{kj}} = \mathring{\mathbf{e}}_i \cdot \mathbf{e}_j. \end{aligned} \quad (9.2)$$

In matrix notation, these components  $\overset{\delta_{kj}}{\text{present the } \text{???transposed????}}$  matrix of cosines  $o_{ji} \equiv \mathring{\mathbf{e}}_i \cdot \mathbf{e}_j$ :

$$\mathbf{O} = o_{ji} \mathbf{e}_i \mathbf{e}_j = o_{ji} \mathring{\mathbf{e}}_i \mathring{\mathbf{e}}_j.$$

Spatial transformations in the 3-dimensional Euclidean space  $\mathbb{R}^3$  are distinguished into active or alibi transformations, and passive or alias transformations. An active transformation is a transformation which actually changes the physical position (alibi, elsewhere) of objects, which can be defined in the absence of a coordinate system; whereas a passive transformation is merely a change in the coordinate system in which the object is described (alias, other name) (change of coordinates, or change of basis). By transformation, math texts usually refer to active transformations.

Tensor  $\mathbf{O}$  relates the two vectors — “before rotation”  $\mathring{\mathbf{r}} = \rho_i \mathring{\mathbf{e}}_i$  and “after rotation”  $\mathbf{r} = \rho_i \mathbf{e}_i$ . Components  $\rho_i = \text{constant}$  of  $\mathbf{r}$  in rotated basis  $\mathbf{e}_i$  are the same as of  $\mathring{\mathbf{r}}$  in immobile basis  $\mathring{\mathbf{e}}_i$ . So that the rotation tensor describes the rotation of the vector together with the basis. And since  $\mathbf{e}_i = \mathbf{e}_j \mathring{\mathbf{e}}_j \cdot \mathring{\mathbf{e}}_i \Leftrightarrow \rho_i \mathbf{e}_i = \mathbf{e}_j \mathring{\mathbf{e}}_j \cdot \rho_i \mathring{\mathbf{e}}_i$ , then

$$\mathbf{r} = \mathbf{O} \cdot \mathring{\mathbf{r}} \quad (9.3)$$

(this is the Rodrigues rotation formula).

**Olinde Rodrigues.** Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendants des causes qui peuvent les produire. *Journal de mathématiques pures et appliquées*, tome 5 (1840), pages 380–440.

For a second complexity tensor  $\overset{\circ}{\mathbf{C}} = C_{ij} \overset{\circ}{\mathbf{e}}_i \overset{\circ}{\mathbf{e}}_j$ , a rotation into the current position  $\mathbf{C} = C_{ij} \mathbf{e}_i \mathbf{e}_j$  looks like

$$\mathbf{e}_i C_{ij} \mathbf{e}_j = \mathbf{e}_i \overset{\circ}{\mathbf{e}}_i \cdot \overset{\circ}{\mathbf{e}}_p C_{pq} \overset{\circ}{\mathbf{e}}_q \cdot \overset{\circ}{\mathbf{e}}_j \mathbf{e}_j \Leftrightarrow \mathbf{C} = \mathbf{O} \cdot \overset{\circ}{\mathbf{C}} \cdot \mathbf{O}^\top. \quad (9.4)$$

Essential property of a rotation tensor — orthogonality — is expressed as

$$\underset{\mathbf{e}_i \overset{\circ}{\mathbf{e}}_i}{\mathcal{Q}} \cdot \underset{\overset{\circ}{\mathbf{e}}_j \mathbf{e}_j}{\mathcal{Q}^\top} = \underset{\overset{\circ}{\mathbf{e}}_i \mathbf{e}_i}{\mathcal{Q}^\top} \cdot \underset{\mathbf{e}_j \overset{\circ}{\mathbf{e}}_j}{\mathcal{Q}} = \underset{\overset{\circ}{\mathbf{e}}_i \overset{\circ}{\mathbf{e}}_i}{\mathbf{E}}, \quad (9.5)$$

that is the transposed tensor coincides with the reciprocal tensor:  $\mathbf{O}^\top = \mathbf{O}^{-1} \Leftrightarrow \mathbf{O} = \mathbf{O}^{-\top}$ .

An orthogonal tensor retains lengths and angles (the metric) because it does not change the “ $\cdot$ ”-product of vectors

$$(\mathbf{O} \cdot \mathbf{a}) \cdot (\mathbf{O} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{O}^\top \cdot \mathbf{O} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{E} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}. \quad (9.6)$$

For all orthogonal tensors  $(\det \mathbf{Q})^2 = 1$ :

$$1 = \det \mathbf{E} = \det (\mathbf{Q} \cdot \mathbf{Q}^\top) = (\det \mathbf{Q}) (\det \mathbf{Q}^\top) = (\det \mathbf{Q})^2.$$

A rotation tensor is an orthogonal tensor with  $\det \mathbf{O} = 1$ .

But not only rotation tensors possess the property of orthogonality. When in (9.1) the first basis is “left”, and the second one is “right”, then there’s a combination of a rotation and a reflection (“rotoreflection”)  $\mathbf{O} = -\mathbf{E} \cdot \mathbf{O}$  with  $\det (-\mathbf{E} \cdot \mathbf{O}) = -1$ .

У любого бивалентного тензора в трёхмерном пространстве как минимум одно собственное число — the root of (8.2) is non-complex (real). For a rotation tensor, it is equal to one

$$\mathbf{O} \cdot \mathbf{a} = \eta \mathbf{a} \Rightarrow \mathbf{a} \cdot \overbrace{\mathbf{O}^\top \cdot \mathbf{O}}^{\mathbf{E}} \cdot \mathbf{a} = \eta \mathbf{a} \cdot \eta \mathbf{a} \Rightarrow \eta^2 = 1.$$

Соответствующая собственная ось называется осью поворота. Теорема Euler’a о конечном повороте в том и состоит, что такая

ось существует. Если  $\mathbf{k}$  — орт этой оси, а  $\vartheta$  — конечная величина угла поворота, то тензор поворота представим как

$$\mathbf{O}(\mathbf{k}, \vartheta) = \mathbf{E} \cos \vartheta + \mathbf{k} \times \mathbf{E} \sin \vartheta + \mathbf{k}\mathbf{k} (1 - \cos \vartheta). \quad (9.7)$$

Доказывается эта формула так. Направление  $\mathbf{k}$  при повороте не меняется ( $\mathbf{O} \cdot \mathbf{k} = \mathbf{k}$ ), поэтому на оси поворота  $\hat{\mathbf{e}}_3 = \mathbf{e}_3 = \mathbf{k}$ . В перпендикулярной плоскости (рис. 5)  $\hat{\mathbf{e}}_1 = \mathbf{e}_1 \cos \vartheta - \mathbf{e}_2 \sin \vartheta$ ,  $\hat{\mathbf{e}}_2 = \mathbf{e}_1 \sin \vartheta + \mathbf{e}_2 \cos \vartheta$ ,  $\mathbf{O} = \mathbf{e}_i \hat{\mathbf{e}}_i \Rightarrow (9.7)$ .

Из (9.7) и (9.3) получаем формулу поворота Родрига в параметрах  $\mathbf{k}$  и  $\vartheta$ :

$$\mathbf{r} = \hat{\mathbf{r}} \cos \vartheta + \mathbf{k} \times \hat{\mathbf{r}} \sin \vartheta + \mathbf{k}\mathbf{k} \cdot \hat{\mathbf{r}} (1 - \cos \vartheta).$$

В параметрах конечного поворота транспонирование, оно же обращение, тензора  $\mathbf{O}$  эквивалентно перемене направления поворота — знака угла  $\vartheta$

$$\mathbf{O}^\top = \mathbf{O}|_{\vartheta=-\vartheta} = \mathbf{E} \cos \vartheta - \mathbf{k} \times \mathbf{E} \sin \vartheta + \mathbf{k}\mathbf{k} (1 - \cos \vartheta).$$

Пусть теперь тензор поворота меняется со временем:  $\mathbf{O} = \mathbf{O}(t)$ . Псевдовектор угловой скорости  $\boldsymbol{\omega}$  вводится через тензор поворота  $\mathbf{O}$  таким путём. Дифференцируем тождество ортогональности (9.5) по времени\*

$$\dot{\mathbf{O}} \cdot \mathbf{O}^\top + \mathbf{O} \cdot \dot{\mathbf{O}}^\top = 2\mathbf{0}.$$

Тензор  $\dot{\mathbf{O}} \cdot \mathbf{O}^\top$  (по (??)  $(\dot{\mathbf{O}} \cdot \mathbf{O}^\top)^\top = \mathbf{O} \cdot \dot{\mathbf{O}}^\top$ ) оказался антисимметричным. Поэтому согласно (7.3) он представим сопутствующим вектором как  $\dot{\mathbf{O}} \cdot \mathbf{O}^\top = \boldsymbol{\omega} \times \mathbf{E} = \boldsymbol{\omega} \times \mathbf{O} \cdot \mathbf{O}^\top$ . То есть

$$\dot{\mathbf{O}} = \boldsymbol{\omega} \times \mathbf{O}, \quad \boldsymbol{\omega} \equiv -\frac{1}{2} \left( \dot{\mathbf{O}} \cdot \mathbf{O}^\top \right)_\times \quad (9.8)$$

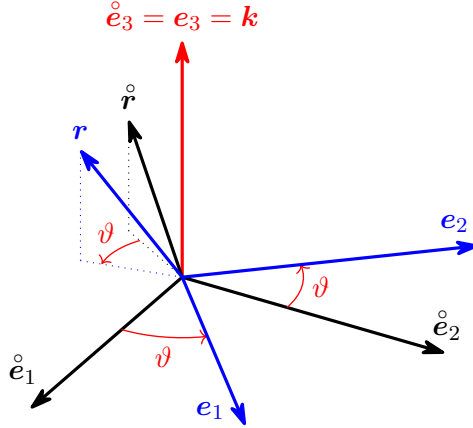
Помимо этого общего представления вектора  $\boldsymbol{\omega}$ , для него есть и другие. Например, через параметры конечного поворота.

Производная  $\dot{\mathbf{O}}$  в параметрах конечного поворота в общем случае (оба параметра — и единичный вектор  $\mathbf{k}$ , и угол  $\vartheta$  — переменны во времени):

\* Various notations are used to designate the time derivative. In addition to the Leibniz's notation  $dx/dt$ , the very popular one is the “dot-above” Newton's notation  $\dot{x}$ .

$$\mathring{e}_i = \mathring{e}_i \cdot e_j e_j$$

$$\begin{bmatrix} \mathring{e}_1 \\ \mathring{e}_2 \\ \mathring{e}_3 \end{bmatrix} = \begin{bmatrix} \mathring{e}_1 \cdot e_1 & \mathring{e}_1 \cdot e_2 & \mathring{e}_1 \cdot e_3 \\ \mathring{e}_2 \cdot e_1 & \mathring{e}_2 \cdot e_2 & \mathring{e}_2 \cdot e_3 \\ \mathring{e}_3 \cdot e_1 & \mathring{e}_3 \cdot e_2 & \mathring{e}_3 \cdot e_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$



$$\begin{bmatrix} \mathring{e}_1 \cdot e_1 & \mathring{e}_1 \cdot e_2 & \mathring{e}_1 \cdot e_3 \\ \mathring{e}_2 \cdot e_1 & \mathring{e}_2 \cdot e_2 & \mathring{e}_2 \cdot e_3 \\ \mathring{e}_3 \cdot e_1 & \mathring{e}_3 \cdot e_2 & \mathring{e}_3 \cdot e_3 \end{bmatrix} = \begin{bmatrix} \cos \vartheta & \cos(90^\circ + \vartheta) & \cos 90^\circ \\ \cos(90^\circ - \vartheta) & \cos \vartheta & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} = \begin{bmatrix} \cos \vartheta & -\sin \vartheta & 0 \\ \sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathring{e}_1 = e_1 \cos \vartheta - e_2 \sin \vartheta$$

$$\mathring{e}_2 = e_1 \sin \vartheta + e_2 \cos \vartheta$$

$$\mathring{e}_3 = e_3 = \mathbf{k}$$

$$\mathbf{O} = e_1 \mathring{e}_1 + e_2 \mathring{e}_2 + e_3 \mathring{e}_3 =$$

$$\begin{aligned} &= \overbrace{e_1 \mathring{e}_1}^{e_1 e_1 \cos \vartheta - e_1 e_2 \sin \vartheta} + \overbrace{e_2 \mathring{e}_2}^{e_2 e_1 \sin \vartheta + e_2 e_2 \cos \vartheta} + \overbrace{e_3 \mathring{e}_3}^{\mathbf{k} \mathbf{k}} = \\ &= \mathbf{E} \cos \vartheta - \underbrace{e_3 e_3}_{\mathbf{k} \mathbf{k}} \cos \vartheta + \underbrace{(e_2 e_1 - e_1 e_2)}_{e_3 \times e_i e_i = \epsilon_{3ij} e_j e_i} \sin \vartheta + \mathbf{k} \mathbf{k} = \\ &= \mathbf{E} \cos \vartheta + \mathbf{k} \times \mathbf{E} \sin \vartheta + \mathbf{k} \mathbf{k} (1 - \cos \vartheta) \end{aligned}$$

рисунок 5  
“Finite rotation”

$$\begin{aligned}
\dot{\mathbf{O}} &= (\mathbf{O}^S + \mathbf{O}^A)^\bullet = \left( \overbrace{\mathbf{E} \cos \vartheta + \mathbf{k} \mathbf{k} (1 - \cos \vartheta)}^{\mathbf{O}^S} + \overbrace{\mathbf{k} \times \mathbf{E} \sin \vartheta}^{\mathbf{O}^A} \right)^\bullet = \\
&= \underbrace{(\mathbf{k} \mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta + (\mathbf{k} \dot{\mathbf{k}} + \dot{\mathbf{k}} \mathbf{k})(1 - \cos \vartheta)}_{\dot{\mathbf{O}}^S} + \underbrace{\mathbf{k} \times \mathbf{E} \dot{\vartheta} \cos \vartheta + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta}_{\dot{\mathbf{O}}^A}.
\end{aligned}$$

Находим

$$\begin{aligned}
\dot{\mathbf{O}} \cdot \mathbf{O}^\top &= (\dot{\mathbf{O}}^S + \dot{\mathbf{O}}^A) \cdot (\mathbf{O}^S - \mathbf{O}^A) = \\
&= \dot{\mathbf{O}}^S \cdot \mathbf{O}^S + \dot{\mathbf{O}}^A \cdot \mathbf{O}^S - \dot{\mathbf{O}}^S \cdot \mathbf{O}^A - \dot{\mathbf{O}}^A \cdot \mathbf{O}^A,
\end{aligned}$$

using

$$\begin{aligned}
\mathbf{k} \cdot \mathbf{k} &= 1 = \text{constant} \Rightarrow \mathbf{k} \cdot \dot{\mathbf{k}} + \dot{\mathbf{k}} \cdot \mathbf{k} = 0 \Leftrightarrow \dot{\mathbf{k}} \cdot \mathbf{k} = \mathbf{k} \cdot \dot{\mathbf{k}} = 0, \\
\mathbf{k} \mathbf{k} \cdot \mathbf{k} \mathbf{k} &= \mathbf{k} \mathbf{k}, \quad \dot{\mathbf{k}} \mathbf{k} \cdot \mathbf{k} \mathbf{k} = \dot{\mathbf{k}} \mathbf{k}, \quad \mathbf{k} \dot{\mathbf{k}} \cdot \mathbf{k} \mathbf{k} = {}^2\mathbf{0}, \\
(\mathbf{k} \mathbf{k} - \mathbf{E}) \cdot \mathbf{k} &= \mathbf{k} - \mathbf{k} = \mathbf{0}, \quad (\mathbf{k} \mathbf{k} - \mathbf{E}) \cdot \mathbf{k} \mathbf{k} = \mathbf{k} \mathbf{k} - \mathbf{k} \mathbf{k} = {}^2\mathbf{0}, \\
\mathbf{k} \cdot (\mathbf{k} \times \mathbf{E}) &= (\mathbf{k} \times \mathbf{E}) \cdot \mathbf{k} = \mathbf{k} \times \mathbf{k} = \mathbf{0}, \quad \mathbf{k} \mathbf{k} \cdot (\mathbf{k} \times \mathbf{E}) = (\mathbf{k} \times \mathbf{E}) \cdot \mathbf{k} \mathbf{k} = {}^2\mathbf{0}, \\
(\mathbf{k} \mathbf{k} - \mathbf{E}) \cdot (\mathbf{k} \times \mathbf{E}) &= -\mathbf{k} \times \mathbf{E}, \\
(\mathbf{a} \times \mathbf{E}) \cdot \mathbf{b} &= \mathbf{a} \times (\mathbf{E} \cdot \mathbf{b}) = \mathbf{a} \times \mathbf{b} \Rightarrow (\dot{\mathbf{k}} \times \mathbf{E}) \cdot \mathbf{k} \mathbf{k} = \dot{\mathbf{k}} \times \mathbf{k} \mathbf{k}, \\
(6.13) \Rightarrow (\mathbf{k} \times \mathbf{E}) \cdot (\mathbf{k} \times \mathbf{E}) &= \mathbf{k} \mathbf{k} - \mathbf{E}, \quad (\dot{\mathbf{k}} \times \mathbf{E}) \cdot (\mathbf{k} \times \mathbf{E}) = \mathbf{k} \dot{\mathbf{k}} - \dot{\mathbf{k}} \cdot \mathbf{k} \mathbf{E}, \\
(6.12) \Rightarrow \dot{\mathbf{k}} \mathbf{k} - \mathbf{k} \dot{\mathbf{k}} &= (\mathbf{k} \times \dot{\mathbf{k}}) \times \mathbf{E}, \quad (\dot{\mathbf{k}} \times \mathbf{k}) \mathbf{k} - \mathbf{k} (\dot{\mathbf{k}} \times \mathbf{k}) = \mathbf{k} \times (\dot{\mathbf{k}} \times \mathbf{k}) \times \mathbf{E}
\end{aligned}$$

$$\dot{\mathbf{P}}^S \cdot \mathbf{P}^S =$$

$$\begin{aligned}
&= (\mathbf{k} \mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cdot \mathbf{E} \cos \vartheta + (\mathbf{k} \dot{\mathbf{k}} + \dot{\mathbf{k}} \mathbf{k})(1 - \cos \vartheta) \cdot \mathbf{E} \cos \vartheta + \\
&\quad + (\mathbf{k} \mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cdot \mathbf{k} \mathbf{k} (1 - \cos \vartheta) + (\mathbf{k} \dot{\mathbf{k}} + \dot{\mathbf{k}} \mathbf{k})(1 - \cos \vartheta) \cdot \mathbf{k} \mathbf{k} (1 - \cos \vartheta) = \\
&= (\mathbf{k} \mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta + (\mathbf{k} \dot{\mathbf{k}} + \dot{\mathbf{k}} \mathbf{k}) \cos \vartheta (1 - \cos \vartheta) + (\mathbf{k} \dot{\mathbf{k}} \cdot \mathbf{k} \mathbf{k} + \dot{\mathbf{k}} \mathbf{k} \cdot \mathbf{k} \mathbf{k}) (1 - \cos \vartheta)^2 = \\
&= (\mathbf{k} \mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta + \mathbf{k} \dot{\mathbf{k}} \cos \vartheta (1 - \cos \vartheta) + \\
&\quad + \dot{\mathbf{k}} \mathbf{k} \cos \vartheta - \dot{\mathbf{k}} \mathbf{k} \cos^2 \vartheta + \dot{\mathbf{k}} \mathbf{k} - 2 \dot{\mathbf{k}} \mathbf{k} \cos \vartheta + \dot{\mathbf{k}} \mathbf{k} \cos^2 \vartheta = \\
&= (\mathbf{k} \mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta + \mathbf{k} \dot{\mathbf{k}} \cos \vartheta - \dot{\mathbf{k}} \mathbf{k} \cos^2 \vartheta + \dot{\mathbf{k}} \mathbf{k} (1 - \cos \vartheta),
\end{aligned}$$

$$\dot{\mathbf{P}}^A \cdot \mathbf{P}^S =$$

$$\begin{aligned}
&= (\mathbf{k} \times \mathbf{E}) \cdot \mathbf{E} \dot{\vartheta} \cos^2 \vartheta + (\dot{\mathbf{k}} \times \mathbf{E}) \cdot \mathbf{E} \sin \vartheta \cos \vartheta + \\
&\quad + (\mathbf{k} \times \mathbf{E}) \cdot \mathbf{k} \dot{\mathbf{k}} \dot{\vartheta} \cos \vartheta (1 - \cos \vartheta) + (\dot{\mathbf{k}} \times \mathbf{E}) \cdot \mathbf{k} \mathbf{k} \sin \vartheta (1 - \cos \vartheta) = \\
&= \mathbf{k} \times \mathbf{E} \dot{\vartheta} \cos^2 \vartheta + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta \cos \vartheta + \dot{\mathbf{k}} \times \mathbf{k} \mathbf{k} \sin \vartheta (1 - \cos \vartheta),
\end{aligned}$$

$$\dot{\mathbf{P}}^S \cdot \mathbf{P}^A =$$

$$\begin{aligned}
&= (\mathbf{k} \mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cdot (\mathbf{k} \times \mathbf{E}) \sin \vartheta + (\mathbf{k} \dot{\mathbf{k}} + \dot{\mathbf{k}} \mathbf{k})(1 - \cos \vartheta) \cdot (\mathbf{k} \times \mathbf{E}) \sin \vartheta = \\
&= \mathbf{k} \mathbf{k} \cdot (\mathbf{k} \times \mathbf{E}) \dot{\vartheta} \sin^2 \vartheta - \mathbf{E} \cdot (\mathbf{k} \times \mathbf{E}) \dot{\vartheta} \sin^2 \vartheta + \left( \mathbf{k} \dot{\mathbf{k}} \cdot (\mathbf{k} \times \mathbf{E}) + \dot{\mathbf{k}} \mathbf{k} \cdot (\mathbf{k} \times \mathbf{E}) \right) \sin \vartheta (1 - \cos \vartheta) = \\
&= -\mathbf{k} \times \mathbf{E} \dot{\vartheta} \sin^2 \vartheta + \mathbf{k} \dot{\mathbf{k}} \times \mathbf{k} \sin \vartheta (1 - \cos \vartheta),
\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{P}}^A \cdot \mathbf{P}^A &= (\mathbf{k} \times \mathbf{E}) \dot{\vartheta} \cos \vartheta \cdot (\mathbf{k} \times \mathbf{E}) \sin \vartheta + (\dot{\mathbf{k}} \times \mathbf{E}) \cdot (\mathbf{k} \times \mathbf{E}) \sin^2 \vartheta = \\ &= (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta + \mathbf{k}\dot{\mathbf{k}} \sin^2 \vartheta;\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{P}} \cdot \mathbf{P}^\Gamma &= \dot{\mathbf{P}}^S \cdot \mathbf{P}^S + \dot{\mathbf{P}}^A \cdot \mathbf{P}^S - \dot{\mathbf{P}}^S \cdot \mathbf{P}^A - \dot{\mathbf{P}}^A \cdot \mathbf{P}^A = \\ &= (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta + \mathbf{k}\dot{\mathbf{k}} \cos \vartheta - \mathbf{k}\dot{\mathbf{k}} \cos^2 \vartheta + \dot{\mathbf{k}}\mathbf{k} (1 - \cos \vartheta) + \\ &\quad + \mathbf{k} \times \mathbf{E} \dot{\vartheta} \cos^2 \vartheta + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta \cos \vartheta + \dot{\mathbf{k}} \times \mathbf{k}\mathbf{k} \sin \vartheta (1 - \cos \vartheta) + \\ &\quad + \mathbf{k} \times \mathbf{E} \dot{\vartheta} \sin^2 \vartheta - \mathbf{k}\dot{\mathbf{k}} \times \mathbf{k} \sin \vartheta (1 - \cos \vartheta) - (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta - \mathbf{k}\dot{\mathbf{k}} \sin^2 \vartheta = \\ &= \mathbf{k} \times \mathbf{E} \dot{\vartheta} + (\mathbf{k}\mathbf{k} - \mathbf{k}\dot{\mathbf{k}})(1 - \cos \vartheta) + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta \cos \vartheta + (\dot{\mathbf{k}} \times \mathbf{k}\mathbf{k} - \mathbf{k}\dot{\mathbf{k}} \times \mathbf{k}) \sin \vartheta (1 - \cos \vartheta) = \\ &= \mathbf{k} \times \mathbf{E} \dot{\vartheta} + \mathbf{k} \times \dot{\mathbf{k}} \times \mathbf{E} (1 - \cos \vartheta) + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta \cos \vartheta + \mathbf{k} \times (\dot{\mathbf{k}} \times \mathbf{k}) \times \mathbf{E} \sin \vartheta (1 - \cos \vartheta) = \\ &= \mathbf{k} \times \mathbf{E} \dot{\vartheta} + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta \cos \vartheta + (\mathbf{k}\mathbf{k} \cdot \mathbf{k} - \mathbf{k}\dot{\mathbf{k}} \cdot \mathbf{k}) \times \mathbf{E} \sin \vartheta (1 - \cos \vartheta) + \mathbf{k} \times \dot{\mathbf{k}} \times \mathbf{E} (1 - \cos \vartheta) = \\ &= \mathbf{k} \times \mathbf{E} \dot{\vartheta} + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta + \mathbf{k} \times \dot{\mathbf{k}} \times \mathbf{E} (1 - \cos \vartheta).\end{aligned}$$

Этот результат, подставленный в определение (9.8) псевдовектора  $\boldsymbol{\omega}$ , даёт

$$\boldsymbol{\omega} = \mathbf{k}\dot{\vartheta} + \dot{\mathbf{k}} \sin \vartheta + \mathbf{k} \times \dot{\mathbf{k}} (1 - \cos \vartheta). \quad (9.9)$$

Вектор  $\boldsymbol{\omega}$  получился разложенным по трём взаимно ортогональным направлениям —  $\mathbf{k}$ ,  $\dot{\mathbf{k}}$  и  $\mathbf{k} \times \dot{\mathbf{k}}$ . При неподвижной оси поворота  $\dot{\mathbf{k}} = \mathbf{0} \Rightarrow \boldsymbol{\omega} = \mathbf{k}\dot{\vartheta}$ .

Ещё одно представление  $\boldsymbol{\omega}$  связано с компонентами тензора поворота (9.2). Поскольку  $\mathbf{P} = o_{ji} \circ \dot{\mathbf{e}}_i \dot{\mathbf{e}}_j$ ,  $\mathbf{P}^\Gamma = o_{ij} \circ \dot{\mathbf{e}}_i \dot{\mathbf{e}}_j$ , а векторы начального базиса  $\dot{\mathbf{e}}_i$  неподвижны (со временем не меняются), то

$$\begin{aligned}\dot{\mathbf{P}} &= \dot{o}_{ji} \circ \dot{\mathbf{e}}_i \dot{\mathbf{e}}_j, \quad \dot{\mathbf{P}} \cdot \mathbf{P}^\Gamma = \dot{o}_{ni} \circ o_{nj} \circ \dot{\mathbf{e}}_i \dot{\mathbf{e}}_j, \\ \boldsymbol{\omega} &= -\frac{1}{2} \dot{o}_{ni} \circ o_{nj} \circ \dot{\mathbf{e}}_i \times \dot{\mathbf{e}}_j = \frac{1}{2} \in_{jik} o_{nj} \circ \dot{o}_{ni} \circ \dot{\mathbf{e}}_k.\end{aligned} \quad (9.10)$$

Отметим и формулы

$$\begin{aligned}(9.8) \Rightarrow \dot{\mathbf{e}}_i \dot{\mathbf{e}}_i &= \boldsymbol{\omega} \times \mathbf{e}_i \dot{\mathbf{e}}_i \Rightarrow \dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i, \\ (9.8) \Rightarrow \boldsymbol{\omega} &= -\frac{1}{2} (\dot{\mathbf{e}}_i \dot{\mathbf{e}}_i \cdot \dot{\mathbf{e}}_j \mathbf{e}_j)_{\times} = -\frac{1}{2} (\dot{\mathbf{e}}_i \mathbf{e}_i)_{\times} = \frac{1}{2} \mathbf{e}_i \times \dot{\mathbf{e}}_i.\end{aligned} \quad (9.11)$$

Не всё то вектор, что имеет величину и направление. Поворот тела вокруг оси представляет, казалось бы, вектор, имеющий численное значение, равное углу поворота, и направление, совпадающее с направлением оси вращения. Однако, два таких поворота не складываются как векторы, когда углы поворота не бесконечно-малые. На самом же деле последовательные повороты не складываются, а умножаются.

Можно ли складывать угловые скорости? — Да, ведь угол поворота в  $\hat{y}$  бесконечно малый. — Но только при вращении вокруг неподвижной оси?

...

## § 10. Variations

Далее часто будет использоваться сходная с дифференцированием операция варьирования. Не отсылая читателя к книгам о вариационном исчислении (calculus of variations), ограничимся представлениями о вариации  $\delta x$  величины  $x$  как о задаваемом нами бесконечно малом приращении, совместимом с ограничениями (связями, constraints). Если ограничений для  $x$  нет, то  $\delta x$  произвольна (случайна). But when  $x = x(y)$  — функция независимого аргумента  $y$ , then  $\delta x = x'(y)\delta y$ .

Writings with variations have the same features as writings with differentials. Если, например,  $\delta x$  and  $\delta y$  are variations of  $x$  and  $y$ ,  $u$  and  $v$  are конечные величины, то пишем  $u\delta x + v\delta y = \delta w$ , а не  $w$  — даже когда  $\delta w$  не является вариацией величины  $w$ ; в этом случае  $\delta w$  это единое обозначение. Surely, if  $u = u(x, y)$ ,  $v = v(x, y)$  and  $\partial_x v = \partial_y u$  ( $\frac{\partial}{\partial x} v = \frac{\partial}{\partial y} u$ ), then the sum  $\delta w$  will be a variation of some  $w$ .

Варьируя тождество (9.5), получим  $\delta \mathbf{O} \cdot \mathbf{O}^\top = -\mathbf{O} \cdot \delta \mathbf{O}^\top$ . Этот тензор антисимметричен, и потому выражается через свой сопутствующий вектор  $\delta \mathbf{o}$  как  $\delta \mathbf{O} \cdot \mathbf{O}^\top = \delta \mathbf{o} \times \mathbf{E}$ . Приходим к соотношениям

$$\delta \mathbf{O} = \delta \mathbf{o} \times \mathbf{O}, \quad \delta \mathbf{o} = -\frac{1}{2} \left( \delta \mathbf{O} \cdot \mathbf{O}^\top \right)_\times, \quad (10.1)$$

аналогичным (9.8). Вектор бесконечно малого поворота  $\delta \mathbf{o}$  это не “вариация  $\mathbf{o}$ ”, но единый символ (в отличие от  $\delta \mathbf{O}$ ).

Малый поворот определяется вектором  $\delta \mathbf{o}$ , но конечный поворот тоже возможно представить как вектор

...

## § 11. Polar decomposition

Any tensor of second complexity  $\mathbf{F}$  with  $\det F_{ij} \neq 0$  (not singular tensor) can be decomposed as

...

*Example.* Polar decompose tensor  $\mathbf{C} = C_{ij}\mathbf{e}_i\mathbf{e}_j$ , where  $\mathbf{e}_k$  are mutually perpendicular unit vectors of basis, and  $C_{ij}$  are tensor's components

$$\begin{aligned}
 C_{ij} &= \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix} \\
 \mathbf{O} &= O_{ij}\mathbf{e}_i\mathbf{e}_j = \mathbf{O}_1 \cdot \mathbf{O}_2 \\
 O_{ij} &= \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix} \\
 \mathbf{C} &= \mathbf{O} \cdot \mathbf{S}_R, \quad \mathbf{O}^T \cdot \mathbf{C} = \mathbf{S}_R \\
 \mathbf{C} &= \mathbf{S}_L \cdot \mathbf{O}, \quad \mathbf{C} \cdot \mathbf{O}^T = \mathbf{S}_L \\
 S_{Rij} &= \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix} \\
 S_{Lij} &= \begin{bmatrix} 104/5 & 47/5 & 5 \\ 47/5 & -129/5 & -10 \\ 5 & -10 & 3 \end{bmatrix}
 \end{aligned}$$

...



## § 12. Tensors in oblique basis

Until now a basis of three mutually perpendicular unit vectors  $\mathbf{e}_i$  was used. Presently take a basis of any three linearly independent (non-coplanar) vectors  $\mathbf{a}_i$ .

Decomposition of vector  $\mathbf{v}$  in basis  $\mathbf{a}_i$  (fig. 6) is linear combination

$$\mathbf{v} = v^i \mathbf{a}_i. \quad (12.1)$$

The summation convention gains the new conditions: a summation index is repeated at different levels of the same monomial, and a free index stays at the equal height in every part of the expression ( $a_i = b_{ij}c^j$  is correct,  $a_i = b_{kk}^i$  is wrong twice).

В таком базисе уже  $\mathbf{v} \cdot \mathbf{a}_i = v^k \mathbf{a}_k \cdot \mathbf{a}_i \neq v^i$ , ведь тут  $\mathbf{a}_i \cdot \mathbf{a}_k \neq \delta_{ik}$ .

Дополним же базис  $\mathbf{a}_i$  ещё другой тройкой векторов  $\mathbf{a}^i$ , называемых кобазисом или взаимным базисом, чтобы

$$\begin{aligned} \mathbf{a}_i \cdot \mathbf{a}^j &= \delta_i^j, \quad \mathbf{a}^i \cdot \mathbf{a}_j = \delta_j^i, \\ \mathbf{E} &= \mathbf{a}^i \mathbf{a}_i = \mathbf{a}_i \mathbf{a}^i. \end{aligned} \quad (12.2)$$

Это — основное свойство кобазиса. Ортонормированный (ортонормальный) базис может быть определён как совпадающий со своим кобазисом:  $\mathbf{e}^i = \mathbf{e}_i$ .

Для, к примеру, первого вектора кобазиса  $\mathbf{a}^1$

$$\begin{cases} \mathbf{a}^1 \cdot \mathbf{a}_1 = 1 \\ \mathbf{a}^1 \cdot \mathbf{a}_2 = 0 \\ \mathbf{a}^1 \cdot \mathbf{a}_3 = 0 \end{cases} \Rightarrow \begin{cases} \mathbf{a}^1 \cdot \mathbf{a}_1 = 1 \\ \gamma \mathbf{a}^1 = \mathbf{a}_2 \times \mathbf{a}_3 \end{cases} \Rightarrow \begin{cases} \mathbf{a}^1 = 1/\gamma \mathbf{a}_2 \times \mathbf{a}_3 \\ \gamma = \mathbf{a}_2 \times \mathbf{a}_3 \cdot \mathbf{a}_1 \end{cases}$$

Коэффициент  $\gamma$  получился равным (с точностью до знака для “левой” тройки  $\mathbf{a}_i$ ) объёму параллелепипеда, построенного на векторах  $\mathbf{a}_i$ . In § 6 the same volume was presented as  $\sqrt{g}$ , and this is not without reason, because it coincides with the square root

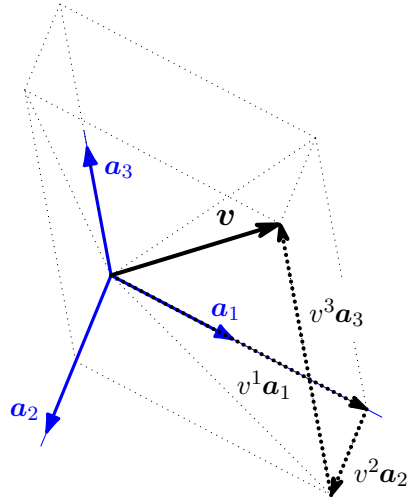


figure 6

of gramian  $g \equiv \det g_{ij}$  — determinant of the symmetric J. P. Gram matrix  $g_{ij} \equiv \mathbf{a}_i \cdot \mathbf{a}_j$ .

○ The proof resembles the derivation of (6.7). The “triple product”  $\mathbf{a}_i \times \mathbf{a}_j \cdot \mathbf{a}_k$  in some orthonormal basis  $\mathbf{e}_i$  вычисли́мо как детерминант (с “—” для “левой” тройки  $\mathbf{a}_i$ ) по строкам

$$\in_{ijk} \equiv \mathbf{a}_i \times \mathbf{a}_j \cdot \mathbf{a}_k = \pm \det \begin{bmatrix} \mathbf{a}_i \cdot \mathbf{e}_1 & \mathbf{a}_i \cdot \mathbf{e}_2 & \mathbf{a}_i \cdot \mathbf{e}_3 \\ \mathbf{a}_j \cdot \mathbf{e}_1 & \mathbf{a}_j \cdot \mathbf{e}_2 & \mathbf{a}_j \cdot \mathbf{e}_3 \\ \mathbf{a}_k \cdot \mathbf{e}_1 & \mathbf{a}_k \cdot \mathbf{e}_2 & \mathbf{a}_k \cdot \mathbf{e}_3 \end{bmatrix}$$

или по столбцам

$$\in_{pqr} \equiv \mathbf{a}_p \times \mathbf{a}_q \cdot \mathbf{a}_r = \pm \det \begin{bmatrix} \mathbf{a}_p \cdot \mathbf{e}_1 & \mathbf{a}_q \cdot \mathbf{e}_1 & \mathbf{a}_r \cdot \mathbf{e}_1 \\ \mathbf{a}_p \cdot \mathbf{e}_2 & \mathbf{a}_q \cdot \mathbf{e}_2 & \mathbf{a}_r \cdot \mathbf{e}_2 \\ \mathbf{a}_p \cdot \mathbf{e}_3 & \mathbf{a}_q \cdot \mathbf{e}_3 & \mathbf{a}_r \cdot \mathbf{e}_3 \end{bmatrix}.$$

Произведение определителей  $\in_{ijk} \in_{pqr}$  равно определителю произведения матриц, and elements of the latter are sums like  $\mathbf{a}_i \cdot \mathbf{e}_s \mathbf{a}_p \cdot \mathbf{e}_s = \mathbf{a}_i \cdot \mathbf{e}_s \mathbf{e}_s \cdot \mathbf{a}_p = \mathbf{a}_i \cdot \mathbf{E} \cdot \mathbf{a}_p = \mathbf{a}_i \cdot \mathbf{a}_p$ , в результате

$$\in_{ijk} \in_{pqr} = \det \begin{bmatrix} \mathbf{a}_i \cdot \mathbf{a}_p & \mathbf{a}_i \cdot \mathbf{a}_q & \mathbf{a}_i \cdot \mathbf{a}_r \\ \mathbf{a}_j \cdot \mathbf{a}_p & \mathbf{a}_j \cdot \mathbf{a}_q & \mathbf{a}_j \cdot \mathbf{a}_r \\ \mathbf{a}_k \cdot \mathbf{a}_p & \mathbf{a}_k \cdot \mathbf{a}_q & \mathbf{a}_k \cdot \mathbf{a}_r \end{bmatrix};$$

$$i=p=1, j=q=2, k=r=3 \Rightarrow \in_{123} \in_{123} = \det_{i,j} (\mathbf{a}_i \cdot \mathbf{a}_j) = \det_{i,j} g_{ij}. \quad \bullet$$

Representing  $\mathbf{a}^1$  and other cobasis vectors as the sum

$$\pm 2\sqrt{g} \mathbf{a}^1 = \mathbf{a}_2 \times \mathbf{a}_3 - \overbrace{\mathbf{a}_3 \times \mathbf{a}_2}^{+\mathbf{a}_2 \times \mathbf{a}_3},$$

приходим к общей формуле (с “—” для “левой” тройки  $\mathbf{a}_i$ )

$$\mathbf{a}^i = \pm \frac{1}{2\sqrt{g}} e^{ijk} \mathbf{a}_j \times \mathbf{a}_k, \quad \sqrt{g} \equiv \pm \mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 > 0. \quad (12.3)$$

Здесь  $e^{ijk}$  по-прежнему символ перестановки Veblen’a ( $\pm 1$  или 0):  $e^{ijk} \equiv e_{ijk}$ . Произведение  $\mathbf{a}_j \times \mathbf{a}_k = \epsilon_{jkn} \mathbf{a}^n$ , компоненты тензора Лёви-Чивиты  $\epsilon_{jkn} = \pm e_{jkn} \sqrt{g}$ , and by (6.8)  $e^{ijk} e_{jkn} = 2\delta_n^i$ . Thus

$$\mathbf{a}^1 = \pm 1/\sqrt{g} (\mathbf{a}_2 \times \mathbf{a}_3), \quad \mathbf{a}^2 = \pm 1/\sqrt{g} (\mathbf{a}_3 \times \mathbf{a}_1), \quad \mathbf{a}^3 = \pm 1/\sqrt{g} (\mathbf{a}_1 \times \mathbf{a}_2).$$

*Example.* Get cobasis for basis  $\mathbf{a}_i$  when

$$\mathbf{a}_1 = \mathbf{e}_1 + \mathbf{e}_2,$$

$$\mathbf{a}_2 = \mathbf{e}_1 + \mathbf{e}_3,$$

$$\mathbf{a}_3 = \mathbf{e}_2 + \mathbf{e}_3.$$

$$\sqrt{g} = -\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 = -\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 2;$$

$$-\mathbf{a}_2 \times \mathbf{a}_3 = \det \begin{bmatrix} 1 & \mathbf{e}_1 & 0 \\ 0 & \mathbf{e}_2 & 1 \\ 1 & \mathbf{e}_3 & 1 \end{bmatrix} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3,$$

$$-\mathbf{a}_3 \times \mathbf{a}_1 = \det \begin{bmatrix} 0 & \mathbf{e}_1 & 1 \\ 1 & \mathbf{e}_2 & 1 \\ 1 & \mathbf{e}_3 & 0 \end{bmatrix} = \mathbf{e}_1 + \mathbf{e}_3 - \mathbf{e}_2,$$

$$-\mathbf{a}_1 \times \mathbf{a}_2 = \det \begin{bmatrix} 1 & \mathbf{e}_1 & 1 \\ 1 & \mathbf{e}_2 & 0 \\ 0 & \mathbf{e}_3 & 1 \end{bmatrix} = \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_1$$

and finally

$$\mathbf{a}^1 = \frac{1}{2} (\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3),$$

$$\mathbf{a}^2 = \frac{1}{2} (\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3),$$

$$\mathbf{a}^3 = \frac{1}{2} (-\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3).$$

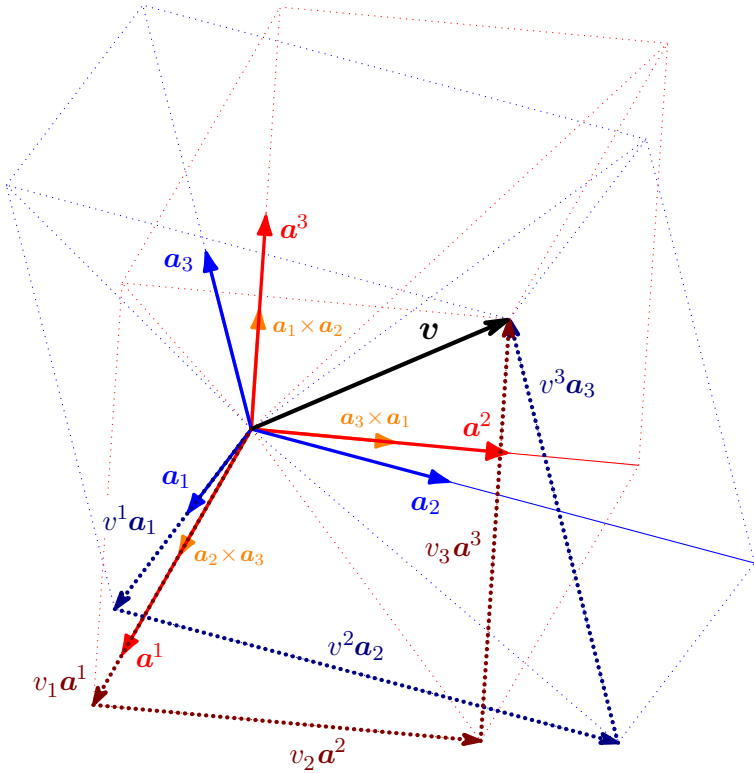
Имея кобазис, возможно не только разложить по нему любой вектор (рис. 7), но и найти коэффициенты разложения (12.1):

$$\begin{aligned} \mathbf{v} &= v^i \mathbf{a}_i = v_i \mathbf{a}^i, \\ \mathbf{v} \cdot \mathbf{a}^i &= v^k \mathbf{a}_k \cdot \mathbf{a}^i = v^i, \quad v_i = \mathbf{v} \cdot \mathbf{a}_i. \end{aligned} \tag{12.4}$$

Коэффициенты  $v_i$  называются ковариантными компонентами вектора  $\mathbf{v}$ , а  $v^i$  — его контравариантными\* компонентами.

Есть литература о тензорах, где introducing existence and различают ковариантные и контравариантные... векторы (and “covectors”, “dual vectors”). Не сто́ит вводить читателя в заблуждение: вектор-то один и тот же, просто разложение по двум разным базисам даёт два набора компонент.

\* Потому что они меняются обратно (contra) изменению длин базисных векторов  $\mathbf{a}_i$ .



$$\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 = \sqrt{g} = 0.56274$$

$$1/\sqrt{g} = 1.77703$$

$$\mathbf{a}_i \cdot \mathbf{a}^j = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}^1 & \mathbf{a}_1 \cdot \mathbf{a}^2 & \mathbf{a}_1 \cdot \mathbf{a}^3 \\ \mathbf{a}_2 \cdot \mathbf{a}^1 & \mathbf{a}_2 \cdot \mathbf{a}^2 & \mathbf{a}_2 \cdot \mathbf{a}^3 \\ \mathbf{a}_3 \cdot \mathbf{a}^1 & \mathbf{a}_3 \cdot \mathbf{a}^2 & \mathbf{a}_3 \cdot \mathbf{a}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_i^j$$

рисунок 7  
 “Decomposition of vector in oblique basis”

От векторов перейдём к тензорам второй сложности. Имеем четыре комплекта диад:  $\mathbf{a}_i \mathbf{a}_j$ ,  $\mathbf{a}^i \mathbf{a}^j$ ,  $\mathbf{a}_i \mathbf{a}^j$ ,  $\mathbf{a}^i \mathbf{a}_j$ . Сопасаующиеся коэффициенты в декомпозиции тензора называются его контравариантными, ковариантными и смешанными компонентами:

$$\begin{aligned} {}^2\mathbf{B} &= B^{ij} \mathbf{a}_i \mathbf{a}_j = B_{ij} \mathbf{a}^i \mathbf{a}^j = B_j^i \mathbf{a}_i \mathbf{a}^j = B_i^j \mathbf{a}^i \mathbf{a}_j, \\ B^{ij} &= \mathbf{a}^i \cdot {}^2\mathbf{B} \cdot \mathbf{a}^j, \quad B_{ij} = \mathbf{a}_i \cdot {}^2\mathbf{B} \cdot \mathbf{a}_j, \\ B_j^i &= \mathbf{a}^i \cdot {}^2\mathbf{B} \cdot \mathbf{a}_j, \quad B_i^j = \mathbf{a}_i \cdot {}^2\mathbf{B} \cdot \mathbf{a}^j. \end{aligned} \quad (12.5)$$

Для двух видов смешанных компонент точка в индексе это просто свободное место: у  $B_j^i$  верхний индекс “ $i$ ” — первый, а нижний “ $j$ ” — второй.

Компоненты единичного (“метрического”) тензора  $\mathbf{E}$

$$\begin{aligned} \mathbf{E} &= \mathbf{a}^k \mathbf{a}_k = \mathbf{a}_k \mathbf{a}^k = g_{jk} \mathbf{a}^j \mathbf{a}^k = g^{jk} \mathbf{a}_j \mathbf{a}_k: \\ \mathbf{a}_i \cdot \mathbf{E} \cdot \mathbf{a}^j &= \mathbf{a}_i \cdot \mathbf{a}^j = \delta_i^j, \quad \mathbf{a}^i \cdot \mathbf{E} \cdot \mathbf{a}_j = \mathbf{a}^i \cdot \mathbf{a}_j = \delta_j^i, \\ \mathbf{a}_i \cdot \mathbf{E} \cdot \mathbf{a}_j &= \mathbf{a}_i \cdot \mathbf{a}_j \equiv g_{ij}, \quad \mathbf{a}^i \cdot \mathbf{E} \cdot \mathbf{a}^j = \mathbf{a}^i \cdot \mathbf{a}^j \equiv g^{ij}; \\ \mathbf{E} \cdot \mathbf{E} &= g_{ij} \mathbf{a}^i \mathbf{a}^j \cdot g^{nk} \mathbf{a}_n \mathbf{a}_k = g_{ij} g^{jk} \mathbf{a}^i \mathbf{a}_k = \mathbf{E} \Rightarrow g_{ij} g^{jk} = \delta_i^k. \end{aligned} \quad (12.6)$$

Вдобавок к (12.2) и (12.3) открылся ещё один способ найти векторы кобазиса — через матрицу  $g^{ij}$ , обратную матрице Грама  $g_{ij}$ . И наоборот:

$$\begin{aligned} \mathbf{a}^i &= \mathbf{E} \cdot \mathbf{a}^i = g^{jk} \mathbf{a}_j \mathbf{a}_k \cdot \mathbf{a}^i = g^{jk} \mathbf{a}_j \delta_k^i = g^{ji} \mathbf{a}_j, \\ \mathbf{a}_i &= \mathbf{E} \cdot \mathbf{a}_i = g_{jk} \mathbf{a}^j \mathbf{a}^k \cdot \mathbf{a}_i = g_{jk} \mathbf{a}^j \delta_i^k = g_{ji} \mathbf{a}^j. \end{aligned} \quad (12.7)$$

*Example.* Using reversed Gram matrix, get cobasis for basis  $\mathbf{a}_i$  when

$$\mathbf{a}_1 = \mathbf{e}_1 + \mathbf{e}_2,$$

$$\mathbf{a}_2 = \mathbf{e}_1 + \mathbf{e}_3,$$

$$\mathbf{a}_3 = \mathbf{e}_2 + \mathbf{e}_3.$$

$$g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \det g_{ij} = 4,$$

$$\text{adj } g_{ij} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}^T,$$

$$g^{ij} = g_{ij}^{-1} = \frac{\text{adj } g_{ij}}{\det g_{ij}} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

Using  $\mathbf{a}^i = g^{ij} \mathbf{a}_j$

$$\mathbf{a}^1 = g^{11} \mathbf{a}_1 + g^{12} \mathbf{a}_2 + g^{13} \mathbf{a}_3 = \frac{1}{2} \mathbf{e}_1 + \frac{1}{2} \mathbf{e}_2 - \frac{1}{2} \mathbf{e}_3,$$

$$\mathbf{a}^2 = g^{21} \mathbf{a}_1 + g^{22} \mathbf{a}_2 + g^{23} \mathbf{a}_3 = \frac{1}{2} \mathbf{e}_1 - \frac{1}{2} \mathbf{e}_2 + \frac{1}{2} \mathbf{e}_3,$$

$$\mathbf{a}^3 = g^{31} \mathbf{a}_1 + g^{32} \mathbf{a}_2 + g^{33} \mathbf{a}_3 = -\frac{1}{2} \mathbf{e}_1 + \frac{1}{2} \mathbf{e}_2 + \frac{1}{2} \mathbf{e}_3.$$

...

Едини́чный тензор (unit tensor, identity tensor, metric tensor)

$$\mathbf{E} \cdot \boldsymbol{\xi} = \boldsymbol{\xi} \cdot \mathbf{E} = \boldsymbol{\xi} \quad \forall \boldsymbol{\xi}$$

$$\mathbf{E} \cdot \mathbf{a} \mathbf{b} = \mathbf{a} \mathbf{b} \cdot \mathbf{E} = \mathbf{a} \cdot \mathbf{E} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$$

$$\mathbf{E} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{E} = \text{trace } \mathbf{A}$$

$$\mathbf{E} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{E} = \text{trace } \mathbf{A} \neq \text{not anymore } A_{jj}$$

Thus for, say, trace of some tensor  $\mathbf{A} = A_{ij} \mathbf{r}^i \mathbf{r}^j$ :  $\mathbf{A} \cdot \mathbf{E} = \text{trace } \mathbf{A}$ , you have

$$\mathbf{A} \cdot \mathbf{E} = A_{ij} \mathbf{r}^i \mathbf{r}^j \cdot \mathbf{r}_{\partial k} \mathbf{r}^k = A_{ij} \mathbf{r}^i \cdot \mathbf{r}^j = A_{ij} g^{ij}$$

...

Тензор поворо́та (rotation tensor)

$$\mathbf{P} = \mathbf{a}_i \hat{\mathbf{a}}^i = \hat{\mathbf{a}}^i \mathbf{a}_i = \mathbf{P}^{-\top}$$

$$\mathbf{P}^{-1} = \hat{\mathbf{a}}_i \mathbf{a}^i = \hat{\mathbf{a}}^i \mathbf{a}_i = \mathbf{P}^{\top}$$

$$\mathbf{P}^{\top} = \hat{\mathbf{a}}^i \mathbf{a}_i = \hat{\mathbf{a}}_i \mathbf{a}^i = \mathbf{P}^{-1}$$

...

... Характеристическое уравнение (8.2) быстро приводит к тождеству Кэли–Гамильтона (Cayley–Hamilton)

$$\begin{aligned} -\mathbf{B} \cdot \mathbf{B} \cdot \mathbf{B} + \mathbf{I} \mathbf{B} \cdot \mathbf{B} - \mathbf{II} \mathbf{B} + \mathbf{III} \mathbf{E} &= {}^2\mathbf{0}, \\ -\mathbf{B}^3 + \mathbf{I} \mathbf{B}^2 - \mathbf{II} \mathbf{B} + \mathbf{III} \mathbf{E} &= {}^2\mathbf{0}. \end{aligned} \quad (12.8)$$

### §13. Tensor functions

In the concept of function  $y=f(x)$  as of mapping (morphism)  $f: x \mapsto y$ , an input (argument)  $x$  and an output (result)  $y$  may be tensors of any complexities.

Consider at least a scalar function of a bivalent tensor  $\varphi=\varphi(\mathbf{B})$ . Examples are  $\mathbf{B} \cdot \cdot \mathbf{\Phi}$  (or  $\mathbf{p} \cdot \mathbf{B} \cdot \mathbf{q}$ ) and  $\mathbf{B} \cdot \cdot \mathbf{B}$ . Then in each basis  $\mathbf{a}_i$  paired with cobasis  $\mathbf{a}^i$  we have function  $\varphi(B_{ij})$  of nine numeric arguments — components  $B_{ij}$  of tensor  $\mathbf{B}$ . For example

$$\varphi(\mathbf{B}) = \mathbf{B} \cdot \cdot \mathbf{\Phi} = B_{ij} \mathbf{a}^i \mathbf{a}^j \cdot \cdot \mathbf{a}_m \mathbf{a}_n \mathbf{\Phi}^{mn} = B_{ij} \mathbf{\Phi}^{ji} = \varphi(B_{ij}).$$

With any transition to a new basis, the result doesn't change:  $\varphi(B_{ij}) = \varphi(B'_{ij}) = \varphi(\mathbf{B})$ .

Differentiation of  $\varphi(\mathbf{B})$  looks like

$$d\varphi = \frac{\partial \varphi}{\partial B_{ij}} dB_{ij} = \frac{\partial \varphi}{\partial \mathbf{B}} \cdot \cdot d\mathbf{B}^\top. \quad (13.1)$$

Tensor  $\partial\varphi/\partial\mathbf{B}$  is called the derivative of function  $\varphi$  by argument  $\mathbf{B}$ ;  $d\mathbf{B}$  is the differential of tensor  $\mathbf{B}$ ,  $d\mathbf{B} = dB_{ij} \mathbf{a}^i \mathbf{a}^j$ ;  $\partial\varphi/\partial B_{ij}$  are components (contravariant ones) of  $\partial\varphi/\partial\mathbf{B}$

$$\mathbf{a}^i \cdot \frac{\partial \varphi}{\partial \mathbf{B}} \cdot \mathbf{a}^j = \frac{\partial \varphi}{\partial B} \cdot \cdot \mathbf{a}^j \mathbf{a}^i = \frac{\partial \varphi}{\partial B_{ij}} \Leftrightarrow \frac{\partial \varphi}{\partial \mathbf{B}} = \frac{\partial \varphi}{\partial B_{ij}} \mathbf{a}_i \mathbf{a}_j.$$

...

$$\varphi(\mathbf{B}) = \mathbf{B} \cdot \cdot \mathbf{\Phi}$$

$$d\varphi = d(\mathbf{B} \cdot \cdot \mathbf{\Phi}) = d\mathbf{B} \cdot \cdot \mathbf{\Phi} = \mathbf{\Phi} \cdot \cdot d\mathbf{B} = \mathbf{\Phi}^\top \cdot \cdot d\mathbf{B}^\top$$

$$d\varphi = \frac{\partial \varphi}{\partial \mathbf{B}} \cdot \cdot d\mathbf{B}^\top, \quad \frac{\partial(\mathbf{B} \cdot \cdot \mathbf{\Phi})}{\partial \mathbf{B}} = \mathbf{\Phi}^\top$$

$$\mathbf{p} \cdot \mathbf{B} \cdot \mathbf{q} = \mathbf{B} \cdot \cdot \mathbf{qp}$$

$$\frac{\partial(\mathbf{p} \cdot \mathbf{B} \cdot \mathbf{q})}{\partial \mathbf{B}} = \mathbf{pq}$$

...

$$\varphi(\mathbf{B}) = \mathbf{B} \cdot \mathbf{B}$$

$$d\varphi = d(\mathbf{B} \cdot \mathbf{B}) = d...$$

...

Но согласно опять-таки (12.8)  $-\mathbf{B}^2 + \mathbf{I}\mathbf{B} - \mathbf{II}\mathbf{E} + \mathbf{III}\mathbf{B}^{-1} = {}^2\mathbf{0}$ , поэтому

...

Скалярная функция  $\varphi(\mathbf{B})$  называется изотропной, если она не чувствительна к повороту аргумента:

$$\varphi(\mathbf{B}) = \varphi(\mathbf{O} \cdot \overset{\circ}{\mathbf{B}} \cdot \mathbf{O}^\top) = \varphi(\overset{\circ}{\mathbf{B}}) \quad \forall \mathbf{O} = \mathbf{a}_i \overset{\circ}{\mathbf{a}}^i = \mathbf{a}^i \overset{\circ}{\mathbf{a}}_i = \mathbf{O}^{-\top}$$

для любого ортогонального тензора  $\mathbf{O}$  (тензора поворота, § 9).

Симметричный тензор  $\mathbf{B}^S$  полностью определяется тройкой инвариантов и угловой ориентацией собственных осей (они же взаимно ортогональны, § 8). Ясно, что изотропная функция  $\varphi(\mathbf{B}^S)$  симметричного аргумента является функцией лишь инвариантов  $\mathbf{I}(\mathbf{B}^S)$ ,  $\mathbf{II}(\mathbf{B}^S)$ ,  $\mathbf{III}(\mathbf{B}^S)$ ; она дифференцируется согласно (?), где транспонирование излишне.

## § 14. Spatial differentiation

««« rename: remove fields

*Tensor field* is a tensor varying from point to point (variable in space, coordinate dependent).

«««&lt;

Пусть at each point of some region of a three-dimensional space определена величина  $\varsigma$ . Тогда говорят, что есть тензорное поле  $\varsigma = \varsigma(\mathbf{r})$ , where  $\mathbf{r}$  is location vector (radius vector) of a point in space.

Величина  $\varsigma$  может быть тензором любой сложности. Пример скалярного поля — поле температуры в среде, векторного поля — скорости частиц жидкости.

Концепт тензорного поля никак не связан с концептом поля с операциями  $+$  и  $*$  с 11 свойствами этих операций.

Не только для решения прикладных задач, но нередко и в “чистой теории” вместо аргумента  $\mathbf{r}$  используется набор (какая-



либо тройка) криволинейных координат  $q^i$ . Если непрерывно менять лишь одну координату из трёх, получается координатная линия. Каждая точка трёхмерного пространства лежит на пересечении трёх координатных линий (рис. 8). Вектор положения точки выражается через набор координат as relation  $\mathbf{r} = \mathbf{r}(q^i)$ .

Commonly used sets of coordinates «Rectangular (“cartesian”), spherical and cylindrical coordinates are

Curvilinear coordinates may be derived from a set of rectangular (“cartesian”) coordinates by using a transformation that is locally invertible (a one-to-one map) at each point. Therefore rectangular coordinates of any point of space can be converted to some curvilinear coordinates and vice versa.

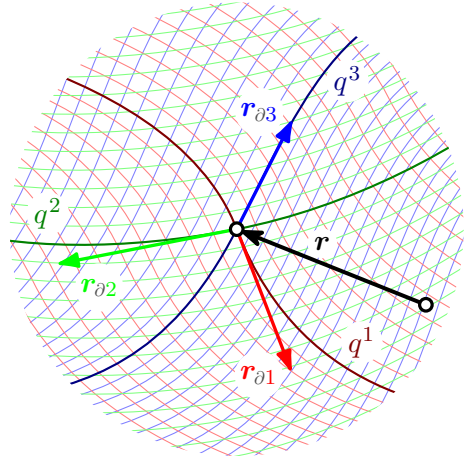


figure 8

...

The differential of a function presents a change in the linearization of this function.

...

partial derivative

$$\partial_i \equiv \frac{\partial}{\partial q^i}$$

...

differential of  $\varsigma(q^i)$

$$d\varsigma = \frac{\partial \varsigma}{\partial q^i} dq^i = \partial_i \varsigma dq^i \quad (14.1)$$

...

Linearity

$$\partial_i (\lambda p + \mu q) = \lambda (\partial_i p) + \mu (\partial_i q) \quad (14.2)$$

“Product rule”

$$\partial_i(p \circ q) = (\partial_i p) \circ q + p \circ (\partial_i q) \quad (14.3)$$

...

Local basis  $\mathbf{r}_{\partial i}$

The differential of location vector  $\mathbf{r}(q^i)$  is

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^i} dq^i = dq^i \mathbf{r}_{\partial i}, \quad \mathbf{r}_{\partial i} \equiv \frac{\partial \mathbf{r}}{\partial q^i} \equiv \partial_i \mathbf{r} \quad (14.4)$$

...

Local cobasis  $\mathbf{r}^i$ ,  $\mathbf{r}^i \cdot \mathbf{r}_{\partial j} = \delta_j^i$

...

$$\begin{aligned} \frac{\partial \zeta}{\partial \mathbf{r}} &= \frac{\partial \zeta}{\partial q^i} \mathbf{r}^i = \partial_i \zeta \mathbf{r}^i \\ d\zeta &= \frac{\partial \zeta}{\partial \mathbf{r}} \cdot d\mathbf{r} = \partial_i \zeta \mathbf{r}^i \cdot dq^j \mathbf{r}_{\partial j} = \partial_i \zeta dq^i \end{aligned} \quad (14.5)$$

...

The bivalent unit tensor (metric tensor)  $\mathbf{E}$ , which is neutral (3.6) to the “ $\cdot$ ”-product (dot product), can be represented as

$$\mathbf{E} = \mathbf{r}^i \mathbf{r}_{\partial i} = \underbrace{\mathbf{r}^i \partial_i}_{\nabla} \mathbf{r}, \quad (14.6)$$

where appears the differential “nabla” operator

$$\nabla \equiv \mathbf{r}^i \partial_i. \quad (14.7)$$

...

$$d\zeta = \frac{\partial \zeta}{\partial \mathbf{r}} \cdot d\mathbf{r} = d\mathbf{r} \cdot \nabla \zeta = \partial_i \zeta dq^i \quad (14.8)$$

$$d\mathbf{r} = d\mathbf{r} \cdot \underbrace{\mathbf{E}}_{\nabla} \mathbf{r}$$

...

Divergence of the dyadic product of two vectors

$$\begin{aligned} \nabla \cdot (\mathbf{a}\mathbf{b}) &= \mathbf{r}^i \partial_i \cdot (\mathbf{a}\mathbf{b}) = \mathbf{r}^i \cdot \partial_i (\mathbf{a}\mathbf{b}) = \mathbf{r}^i \cdot (\partial_i \mathbf{a}) \mathbf{b} + \mathbf{r}^i \cdot \mathbf{a} (\partial_i \mathbf{b}) = \\ &= (\mathbf{r}^i \cdot \partial_i \mathbf{a}) \mathbf{b} + \mathbf{a} \cdot \mathbf{r}^i (\partial_i \mathbf{b}) = (\mathbf{r}^i \partial_i \cdot \mathbf{a}) \mathbf{b} + \mathbf{a} \cdot (\mathbf{r}^i \partial_i \mathbf{b}) = \\ &= (\nabla \cdot \mathbf{a}) \mathbf{b} + \mathbf{a} \cdot (\nabla \mathbf{b}) \end{aligned} \quad (14.9)$$

— here’s no need to expand vectors  $\mathbf{a}$  and  $\mathbf{b}$ , expanding just differential operator  $\nabla$ .

...

Gradient of cross product of two vectors, applying “product rule” (14.3) and relation (6.5) for any two vectors (partial derivative  $\partial_i$  of some vector by scalar coordinate  $q^i$  is a vector too)

$$\begin{aligned}\nabla(\mathbf{a} \times \mathbf{b}) &= \mathbf{r}^i \partial_i (\mathbf{a} \times \mathbf{b}) = \mathbf{r}^i (\partial_i \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \partial_i \mathbf{b}) = \\ &= \mathbf{r}^i (\partial_i \mathbf{a} \times \mathbf{b} - \partial_i \mathbf{b} \times \mathbf{a}) = \mathbf{r}^i \partial_i \mathbf{a} \times \mathbf{b} - \mathbf{r}^i \partial_i \mathbf{b} \times \mathbf{a} = \\ &= \nabla \mathbf{a} \times \mathbf{b} - \nabla \mathbf{b} \times \mathbf{a}. \quad (14.10)\end{aligned}$$

...

Gradient of dot product of two vectors

$$\begin{aligned}\nabla(\mathbf{a} \cdot \mathbf{b}) &= \mathbf{r}^i \partial_i (\mathbf{a} \cdot \mathbf{b}) = \mathbf{r}^i (\partial_i \mathbf{a}) \cdot \mathbf{b} + \mathbf{r}^i \mathbf{a} \cdot (\partial_i \mathbf{b}) = \\ &= (\mathbf{r}^i \partial_i \mathbf{a}) \cdot \mathbf{b} + \mathbf{r}^i (\partial_i \mathbf{b}) \cdot \mathbf{a} = (\nabla \mathbf{a}) \cdot \mathbf{b} + (\nabla \mathbf{b}) \cdot \mathbf{a}. \quad (14.11)\end{aligned}$$

...

...

## § 15. Integral theorems

Для векторных полей известны интегральные теоремы Gauss'а и Stokes'а.

Gauss' theorem (divergence theorem) enables an integral taken over a volume to be replaced by one taken over the closed surface bounding that volume, and vice versa.

Stokes' theorem enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve. Stokes' theorem relates a line integral around a closed path to a surface integral over what is called a *capping surface* of the path.

Теорема Гаусса о дивергенции — про то, как заменить объёмный интеграл поверхностным (and vice versa). В этой теореме рассматривается поток (ef)flux вектора через ограничивающую объём  $V$  замкнутую поверхность  $\mathcal{O}(\partial V)$  с единичным вектором внешней нормали  $\mathbf{n}$

$$\oint_{\mathcal{O}(\partial V)} \mathbf{n} \cdot \mathbf{a} d\mathcal{O} = \int_V \nabla \cdot \mathbf{a} dV. \quad (15.1)$$

Объём  $V$  нарезается тремя семействами координатных поверхностей на множество бесконечно малых элементов. Поток через поверхность  $\mathcal{O}(\partial V)$  равен сумме потоков через края получившихся элементов. В бесконечной малости каждый такой элемент — маленький локальный дифференциальный кубик (параллелепипед). ... Поток вектора  $\mathbf{a}$  через грани малого кубика объёма  $dV$  есть  $\sum_{i=1}^6 \mathbf{n}_i \cdot \mathbf{a} \mathcal{O}_i$ , а через сам этот объём поток равен  $\nabla \cdot \mathbf{a} dV$ .

Похожая трактовка этой теоремы есть, к примеру, в курсе Richard'a Feynman'a [85].

( рисунок с кубиками )

to dice — нарезать кубиками

small cube, little cube

локально ортонормальные координаты  $\boldsymbol{\xi} = \xi_i \mathbf{n}_i$ ,  $d\boldsymbol{\xi} = d\xi_i \mathbf{n}_i$ ,  
 $\nabla = \mathbf{n}_i \partial_i$

разложение вектора  $\mathbf{a} = a_i \mathbf{n}_i$

Теорема Стокса о циркуляции выражается равенством

...

...

## § 16. Curvature tensors

The *Riemann curvature tensor* or *Riemann–Christoffel tensor* (after **Bernhard Riemann** and **Elwin Bruno Christoffel**) is the most common method used to express the curvature of Riemannian manifolds. It's a tensor field, it assigns a tensor to each point of a Riemannian manifold, that measures the extent to which the metric tensor is not locally isometric to that of “flat” space. The curvature tensor measures noncommutativity of the covariant derivative, and as such is the integrability obstruction for the existence of an isometry with “flat” space.

Рассматривая тензорные поля в криволинейных координатах (§ 14), мы исходили из представления вектора-радиуса (вектора положения) точки функцией этих координат:  $\mathbf{r} = \mathbf{r}(q^i)$ . Этим отношением порождаются выражения

- ✓ векторов локального касательного базиса  $\mathbf{r}_{\partial i} \equiv \partial \mathbf{r} / \partial q^i \equiv \partial_i \mathbf{r}$ ,
- ✓ компонент  $g_{ij} \equiv \mathbf{r}_{\partial i} \cdot \mathbf{r}_{\partial j}$  и  $g^{ij} \equiv \mathbf{r}^i \cdot \mathbf{r}^j = g_{ij}^{-1}$  единичного “метрического” тензора  $\mathbf{E} = \mathbf{r}_{\partial i} \mathbf{r}^i = \mathbf{r}^i \mathbf{r}_{\partial i} = g_{jk} \mathbf{r}^j \mathbf{r}^k = g^{jk} \mathbf{r}_{\partial j} \mathbf{r}_{\partial k}$ ,
- ✓ векторов локального взаимного кокасательного базиса  $\mathbf{r}^i \cdot \mathbf{r}_{\partial j} = \delta_j^i$ ,  $\mathbf{r}^i = g^{ij} \mathbf{r}_{\partial j}$ ,
- ✓ дифференциального набла-оператора Hamilton’а  $\nabla \equiv \mathbf{r}^i \partial_i$ ,  $\mathbf{E} = \nabla \mathbf{r}$ ,
- ✓ полного дифференциала  $d\xi = d\mathbf{r} \cdot \nabla \xi$ ,
- ✓ частных производных касательного базиса (вторых частных производных  $\mathbf{r}$ )  $\mathbf{r}_{\partial i \partial j} \equiv \partial_i \partial_j \mathbf{r} = \partial_i \mathbf{r}_{\partial j}$ ,
- ✓ символов “связности” Христоффеля (Christoffel symbols)  $\Gamma_{ij}^k \equiv \mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}^k$  и  $\Gamma_{ijk} \equiv \mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}_{\partial k}$ .

Представим теперь, что функция  $\mathbf{r}(q^k)$  не известна, но зато в каждой точке пространства известны шесть независимых компонент положительно определённой (all Gram matrices are non-negative definite) симметричной метрической матрицы  $\text{Gram } g_{ij}(q^k)$ .

the Gram matrix (or Gramian)

Билинейная форма ...

...



Поскольку шесть функций  $g_{ij}(q^k)$  происходят от векторной функции  $\mathbf{r}(q^k)$ , то между элементами  $g_{ij}$  существуют некие соотношения.

Differential  $d\mathbf{r}$  (14.4) is exact. This is true if and only if second partial derivatives commute:

$$d\mathbf{r} = \mathbf{r}_{\partial k} dq^k \Leftrightarrow \partial_i \mathbf{r}_{\partial j} = \partial_j \mathbf{r}_{\partial i} \text{ or } \mathbf{r}_{\partial i \partial j} = \mathbf{r}_{\partial j \partial i}.$$

Но это условие уже обеспечено симметрией  $g_{ij}$

...

metric (“affine”) connection  $\nabla_i$ , её же называют “covariant derivative”

$$\mathbf{r}_{\partial i \partial j} = \underbrace{\mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}^k}_{\Gamma_{ij}^k} \mathbf{r}_{\partial k} = \underbrace{\mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}_{\partial k}}_{\Gamma_{ij\cdot}^k} \mathbf{r}^k$$

$$\Gamma_{ij}^k \mathbf{r}_{\partial k} = \mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}^k \mathbf{r}_{\partial k} = \mathbf{r}_{\partial i \partial j}$$

covariant derivative (affine connection) is only defined for vector fields

$$\nabla \mathbf{v} = \mathbf{r}^i \partial_i (v^j \mathbf{r}_{\partial j}) = \mathbf{r}^i (\partial_i v^j \mathbf{r}_{\partial j} + v^j \mathbf{r}_{\partial i \partial j})$$

$$\nabla \mathbf{v} = \mathbf{r}^i \mathbf{r}_{\partial j} \nabla_i v^j, \quad \nabla_i v^j \equiv \partial_i v^j + \Gamma_{in}^j v^n$$

$$\nabla \mathbf{r}_{\partial i} = \mathbf{r}^k \partial_k \mathbf{r}_{\partial i} = \mathbf{r}^k \mathbf{r}_{\partial k \partial i} = \mathbf{r}^k \mathbf{r}_{\partial n} \Gamma_{ki}^n, \quad \nabla_i \mathbf{r}_{\partial n} = \Gamma_{in\cdot}^k \mathbf{r}_{\partial k}$$

Christoffel symbols describe a metric (“affine”) connection, that is how the basis changes from point to point.

символы Christoffel’я это “components of connection” in local coordinates

...

torsion tensor  ${}^3\mathfrak{T}$  with components

$$\mathfrak{T}_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$$

determines the antisymmetric part of a connection

...

симметрия  $\Gamma_{ij\cdot}^k = \Gamma_{ji\cdot}^k$ , поэтому  $3^3 - 3 \cdot 3 = 18$  разных (независимых)  $\Gamma_{ij\cdot}^k$

$$\begin{aligned}
\Gamma_{ij}^n g_{nk} &= \Gamma_{ijk} = \mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}_{\partial k} = \\
&= \frac{1}{2}(\mathbf{r}_{\partial i \partial j} + \mathbf{r}_{\partial j \partial i}) \cdot \mathbf{r}_{\partial k} + \frac{1}{2}(\mathbf{r}_{\partial j \partial k} - \mathbf{r}_{\partial k \partial j}) \cdot \mathbf{r}_{\partial i} + \frac{1}{2}(\mathbf{r}_{\partial i \partial k} - \mathbf{r}_{\partial k \partial i}) \cdot \mathbf{r}_{\partial j} = \\
&= \frac{1}{2}(\mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}_{\partial k} + \mathbf{r}_{\partial i \partial k} \cdot \mathbf{r}_{\partial j}) + \frac{1}{2}(\mathbf{r}_{\partial j \partial i} \cdot \mathbf{r}_{\partial k} + \mathbf{r}_{\partial j \partial k} \cdot \mathbf{r}_{\partial i}) - \frac{1}{2}(\mathbf{r}_{\partial k \partial i} \cdot \mathbf{r}_{\partial j} + \mathbf{r}_{\partial k \partial j} \cdot \mathbf{r}_{\partial i}) = \\
&= \frac{1}{2}(\partial_i(\mathbf{r}_{\partial j} \cdot \mathbf{r}_{\partial k}) + \partial_j(\mathbf{r}_{\partial i} \cdot \mathbf{r}_{\partial k}) - \partial_k(\mathbf{r}_{\partial i} \cdot \mathbf{r}_{\partial j})) = \\
&= \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}). \quad (16.1)
\end{aligned}$$

Все символы Christoffel'я тождественно равны нулю лишь в ортонормальной (декартовой) системе. (А какие они для ко-соугольной?)

Дальше:  $d\mathbf{r}_{\partial i} = d\mathbf{r} \cdot \nabla \mathbf{r}_{\partial i} = dq^k \partial_k \mathbf{r}_{\partial i} = \mathbf{r}_{\partial k \partial i} dq^k$  — тоже полные дифференциалы.

$$d\mathbf{r}_{\partial k} = \partial_i \mathbf{r}_{\partial k} dq^i = \frac{\partial \mathbf{r}_{\partial k}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}_{\partial k}}{\partial q^2} dq^2 + \frac{\partial \mathbf{r}_{\partial k}}{\partial q^3} dq^3$$

Поэтому  $\partial_i \partial_j \mathbf{r}_{\partial k} = \partial_j \partial_i \mathbf{r}_{\partial k}$ ,  $\partial_i \mathbf{r}_{\partial j \partial k} = \partial_j \mathbf{r}_{\partial i \partial k}$ , и трёхиндексный объект из векторов третьих частных производных

$$\mathbf{r}_{\partial i \partial j \partial k} \equiv \partial_i \partial_j \partial_k \mathbf{r} = \partial_i \mathbf{r}_{\partial j \partial k} \quad (16.2)$$

симметричен по первому и второму индексам (а не только по второму и третьему). И тогда равен нулю <sup>4</sup> $\mathbf{0}$  следующий тензор четвёртой сложности — *Riemann curvature tensor* (or *Riemann–Christoffel tensor*)

$${}^4\mathfrak{R} = \mathfrak{R}_{hijk} \mathbf{r}^h \mathbf{r}^i \mathbf{r}^j \mathbf{r}^k, \quad \mathfrak{R}_{hijk} \equiv \mathbf{r}_{\partial h} \cdot (\mathbf{r}_{\partial j \partial i \partial k} - \mathbf{r}_{\partial i \partial j \partial k}). \quad (16.3)$$

Выразим компоненты  $\mathfrak{R}_{ijkn}$  через метрическую матрицу  $g_{ij}$ . Начнём с дифференцирования локального кобазиса:

$$\mathbf{r}^i \cdot \mathbf{r}_{\partial k} = \delta_k^i \Rightarrow \partial_j \mathbf{r}^i \cdot \mathbf{r}_{\partial k} + \mathbf{r}^i \cdot \mathbf{r}_{\partial j \partial k} = 0 \Rightarrow \partial_j \mathbf{r}^i = -\Gamma_{jk}^i \mathbf{r}^k.$$

...

Шесть независимых компонент:  $\mathfrak{R}_{1212}$ ,  $\mathfrak{R}_{1213}$ ,  $\mathfrak{R}_{1223}$ ,  $\mathfrak{R}_{1313}$ ,  $\mathfrak{R}_{1323}$ ,  $\mathfrak{R}_{2323}$ .

...

Symmetric bivalent *Ricci curvature tensor*

$$\mathcal{R} \equiv \frac{1}{4} \mathfrak{R}_{abij} \mathbf{r}^a \times \mathbf{r}^b \mathbf{r}^i \times \mathbf{r}^j = \frac{1}{4} \epsilon^{abp} \epsilon^{ijq} \mathfrak{R}_{abij} \mathbf{r}_{\partial p} \mathbf{r}_{\partial q} = \mathcal{R}^{pq} \mathbf{r}_{\partial p} \mathbf{r}_{\partial q}$$

(coefficient  $\frac{1}{4}$  is used here for convenience) with components

$$\begin{aligned}\mathcal{R}^{11} &= \frac{1}{g} \mathfrak{R}_{2323}, \\ \mathcal{R}^{21} &= \frac{1}{g} \mathfrak{R}_{1323}, \quad \mathcal{R}^{22} = \frac{1}{g} \mathfrak{R}_{1313}, \\ \mathcal{R}^{31} &= \frac{1}{g} \mathfrak{R}_{1223}, \quad \mathcal{R}^{32} = \frac{1}{g} \mathfrak{R}_{1213}, \quad \mathcal{R}^{33} = \frac{1}{g} \mathfrak{R}_{1212}.\end{aligned}$$

Равенство тензора Риччи нулю  $\mathcal{R} = {}^2\mathbf{0}$  (в компонентах это шесть уравнений  $\mathcal{R}^{ij} = \mathcal{R}^{ji} = 0$ ) is the **necessary** condition of integrability (“compatibility”) для нахождения вектора-радиуса  $\mathbf{r}(q^k)$  по полю  $g_{ij}(q^k)$ .

## Bibliography

Many books exist which describe only apparatus of tensor calculus [97, 98, 99, 100, 101]. However, an index notation — it’s when tensors are considered as matrices of components — is still more popular than the direct indexless notation. The direct notation is used, for instance, in the appendices of books by А. И. Лурье (A. I. Lurie) [22, 23]. R. Feynman’s lectures [85] contain a vivid description of vector fields. Also, information about tensor calculus is part of original and profound book by C. Truesdell [58].

*chapter 2*

# FUNDAMENTALS OF CLASSICAL GENERIC MECHANICS

## §1. Discrete collection of particles

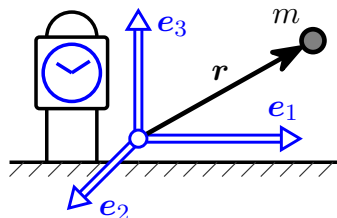
Classical generic mechanics models physical objects by discretizing them into a collection of particles (“pointlike masses”, “material points”<sup>\*</sup>).

In a collection of  $N$  particles, each  $k$ -th particle has its nonzero mass  $m_k = \text{constant} > 0$  and motion function  $\mathbf{r}_k(t)$ . Function  $\mathbf{r}_k(t)$  is measured relative to the chosen reference system.

A reference system (also “reference frame”) consists of (fig. 9)

- ✓ some “null” reference point,
- ✓ a set of coordinates,
- ✓ any clock.

“Any” clock — because (in classical generic mechanics) time tick-tocks, flows and passes identically in any clock in any place, and all clocks are perfectly synchronized.



*figure 9*

Long time ago, the reference system was some absolute space: empty at first, and then filled with the continuous elastic medium — the æther. Later it became clear that for classical mechanics any reference systems can be used, but the preference is given to the so called “inertial” systems, where a point moves with a constant

<sup>\*</sup> Point mass (pointlike mass, material point) is the concept of an object, typically matter, that has nonzero mass and is (or is being thought of as) infinitesimal in its volume (dimensions).

velocity without acceleration ( $\ddot{\mathbf{r}} = \mathbf{0}$ ,  $\dot{\mathbf{r}} = \text{constant}$ ) in the absence of external interactions.

The measure of interaction in mechanics is the vector of force  $\mathbf{F}$ . In the widely known\* Newton's equation

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) \quad (1.1)$$

the right-hand side can depend only on position, velocity and explicitly presented time, whereas acceleration  $\ddot{\mathbf{r}}$  is directly proportional to force  $\mathbf{F}$  with coefficient  $1/m$ .

Here're theses of the dynamics of a collection of particles.

The force acting on the  $k$ -th particle (fig. 10)

$$m_k \ddot{\mathbf{r}}_k = \mathbf{F}_k, \quad \mathbf{F}_k = \mathbf{F}_k^{(e)} + \sum_j \mathbf{F}_{kj}^{(i)}. \quad (1.2)$$

$\mathbf{F}_k^{(e)}$  is the external force — such forces emanate from objects outside the considered system. The second addend is the sum of internal forces (force  $\mathbf{F}_{kj}^{(i)}$  is the interaction induced by the  $j$ -th particle on the  $k$ -th particle). Internal interactions happen only between elements of the observed system and don't affect (mechanically) anything other. Neither particle interacts with itself,  $\mathbf{F}_{kk}^{(i)} = \mathbf{0} \forall k$ .

From (1.2) together with the action–reaction principle

$$\mathbf{F}_{kj}^{(i)} = -\mathbf{F}_{jk}^{(i)} \forall k, j \Rightarrow \sum_k \sum_j \mathbf{F}_{kj}^{(i)} = \mathbf{0},$$

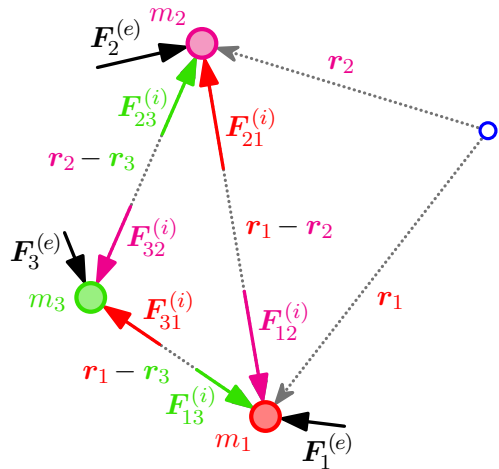


figure 10

\*“Axiomata sive Leges Motus” (“Axioms or Laws of Motion”) were written by Isaac Newton in his *Philosophiæ Naturalis Principia Mathematica*, first published in 1687. Reprint (en Latin), 1871. Translated into English by Andrew Motte, 1846.

ensues the balance of momentum

$$\left( \sum_k m_k \dot{\mathbf{r}}_k \right)^{\bullet} = \sum_k m_k \ddot{\mathbf{r}}_k = \sum_k \mathbf{F}_k^{(e)}. \quad (1.3)$$

The moment acting on the  $k$ -th particle

$$\mathbf{r}_k \times m_k \ddot{\mathbf{r}}_k = \mathbf{r}_k \times \mathbf{F}_k = \mathbf{r}_k \times \mathbf{F}_k^{(e)} + \mathbf{r}_k \times \sum_j \mathbf{F}_{kj}^{(i)}. \quad (1.4)$$

It is relative to the reference point.

When in addition to the action–reaction principle, internal interactions between particles are central, that is

$$\mathbf{F}_{kj}^{(i)} \parallel (\mathbf{r}_k - \mathbf{r}_j) \Leftrightarrow (\mathbf{r}_k - \mathbf{r}_j) \times \mathbf{F}_{kj}^{(i)} = \mathbf{0},$$

the balance of angular (rotational) momentum comes out\*

$$\left( \sum_k \mathbf{r}_k \times m_k \dot{\mathbf{r}}_k \right)^{\bullet} = \sum_k \mathbf{r}_k \times \mathbf{F}_k^{(e)}. \quad (1.5)$$

Changes in momentum and angular momentum are determined only by external forces  $\mathbf{F}_k^{(e)}$ .

...

$$\begin{aligned} * \left( \sum_k \mathbf{r}_k \times m_k \dot{\mathbf{r}}_k \right)^{\bullet} &= \sum_k \dot{\mathbf{r}}_k \times m_k \dot{\mathbf{r}}_k + \sum_k \mathbf{r}_k \times m_k \ddot{\mathbf{r}}_k = \sum_k \mathbf{r}_k \times m_k \ddot{\mathbf{r}}_k, \\ \mathbf{F}_{kj}^{(i)} &= -\mathbf{F}_{jk}^{(i)} \text{ and } (\mathbf{r}_k - \mathbf{r}_j) \times \mathbf{F}_{kj}^{(i)} = \mathbf{0} \Rightarrow \\ &\Rightarrow \sum_{k=1}^N \mathbf{r}_k \times \sum_{j=1}^N \mathbf{F}_{kj}^{(i)} = \sum_{k=1}^N \sum_{j=k+1}^N (\mathbf{r}_k - \mathbf{r}_j) \times \mathbf{F}_{kj}^{(i)} = \mathbf{0} \end{aligned}$$

## §2. Absolutely rigid undeformable solid body

An absolutely rigid undeformable body is a solid\* body, in which deformation is zero (or negligibly small — so small that it can be neglected). The distance between any two points of a non-deformable rigid body remains constant regardless of external forces exerted on it.

A non-deformable rigid body is modeled using the “continual approach” — as a continuous distribution of mass (material continuum), rather than using the “discrete approach” — modeling as a discrete collection of body’s particles.

The mass of a material continuum is distributed in its volume continuously

$$dm \equiv \rho dV \quad (2.1)$$

( $\rho$  is volume(tric) mass density and  $dV$  is infinitesimal volume).

A formula with summation over discrete points turns into a formula for a continuous body by replacing particle masses with mass (2.1) of volume element  $dV$  and integrating over the whole volume of a body. In particular, here are formulas for the (linear) momentum

$$\sum_k m_k \dot{\mathbf{r}}_k \text{ becomes } \int_V \dot{\mathbf{r}} dm \quad (2.2)$$

and the angular (rotational) momentum

$$\sum_k \mathbf{r}_k \times m_k \dot{\mathbf{r}}_k \text{ becomes } \int_V \mathbf{r} \times \dot{\mathbf{r}} dm. \quad (2.3)$$

To fully describe the location (position, place) of any non-deformable body with all its points, it’s enough to choose some unique point as the “pole”, to find or to set the location  $\mathbf{p} = \mathbf{p}(t)$  of the chosen point, as well as the angular orientation of a body relative to the pole (fig. 11). As a result, any motion of an undeformable rigid body is either a rotation around the chosen pole, or an equal

\*“Rigid” is inelastic and not flexible, and “solid” is not fluid. A solid substance retains its size and shape without a container (as opposed to a fluid substance — liquid or gas).

displacement of the pole and all body's points — translation (linear motion)\*, or a combination of them both.

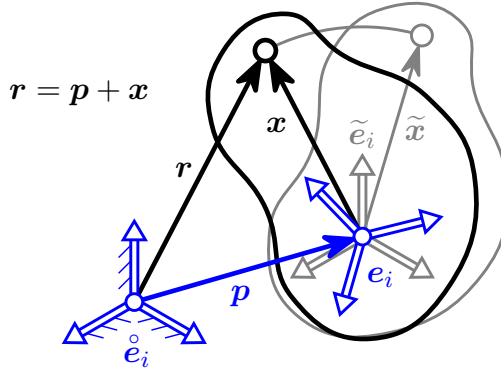


figure 11

$\dot{\mathbf{e}}_i$  — triplet of orthonormal basis vectors, immovable relatively to the absolute (or any inertial) reference system

Имея неподвижный базис  $\dot{\mathbf{e}}_i$  and движущийся вместе с телом базис  $\mathbf{e}_i$ , ...

Если добавить базис  $\mathbf{e}_i$  (этот базис движется вместе с телом), то угловая ориентация тела может быть определена тензором поворота  $\mathbf{O} \equiv \mathbf{e}_i \tilde{\mathbf{e}}_i$ .

Then any motion of a body is completely described by two functions  $\mathbf{p}(t)$  and  $\mathbf{O}(t)$ .

Вектор положения некоторой точки тела

$$\mathbf{r} = \mathbf{p} + \mathbf{x} \quad (2.4)$$

$$\tilde{\mathbf{x}} = x_i \tilde{\mathbf{e}}_i, \mathbf{x} = x_i \mathbf{e}_i$$

(9.3), § 1.9

$$\mathbf{x} = \mathbf{O} \cdot \tilde{\mathbf{x}}$$

$$\dot{\mathbf{r}} = \dot{\mathbf{p}} + \dot{\mathbf{x}},$$

For a non-deformable rigid body, components  $x_i$  don't depend on time:  $x_i = \text{constant}(t)$  and  $\dot{\mathbf{x}} = x_i \dot{\mathbf{e}}_i$

\* A translation can also be thought of as a rotation with the revolution center at infinity.



$$\dot{\mathbf{x}} = \dot{\mathbf{O}} \cdot \overset{\circ}{\mathbf{x}}$$

$$x_i \dot{\mathbf{e}}_i = \dot{\mathbf{O}} \cdot x_i \overset{\circ}{\mathbf{e}}_i \Leftrightarrow \dot{\mathbf{e}}_i = \dot{\mathbf{O}} \cdot \overset{\circ}{\mathbf{e}}_i$$

...

The linear momentum and the rotational (angular) momentum of a non-deformable continuous body are described by the following integrals

...

...

$$\int_{\mathcal{V}} \mathbf{p} dm = \mathbf{p} \int_{\mathcal{V}} dm = \mathbf{p} m$$

$$\int_{\mathcal{V}} \mathbf{x} dm = \mathbf{\Xi} m, \quad \mathbf{\Xi} \equiv m^{-1} \int_{\mathcal{V}} \mathbf{x} dm$$

Three inertial characteristics of the body:

- ✓ integral mass  $m = \int_{\mathcal{V}} dm = \int_{\mathcal{V}} \rho d\mathcal{V}$  — the mass of the whole body,
- ✓ eccentricity vector  $\mathbf{\Xi}$  — measures the offset of the chosen pole from the body's “center of mass”,
- ✓ inertia tensor  ${}^2\mathfrak{J}$ .

The eccentricity vector is equal to the null vector only when the chosen pole coincides with the “center of mass” — the unique point within a body with location vector  $\mathbf{n}$ , in short

$$\mathbf{\Xi} = \mathbf{0} \Leftrightarrow \mathbf{p} = \mathbf{n}.$$

$$\mathbf{x} = \mathbf{r} - \mathbf{p}, \quad \mathbf{\Xi} m = \int_{\mathcal{V}} (\mathbf{r} - \mathbf{n}) dm = \mathbf{0},$$

$$\int_{\mathcal{V}} \mathbf{r} dm - \mathbf{n} \int_{\mathcal{V}} dm = \mathbf{0} \Rightarrow \mathbf{n} = m^{-1} \int_{\mathcal{V}} \mathbf{r} dm$$

...

Introducing (pseudo)vector of angular velocity  $\boldsymbol{\omega}$ , ...

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i$$

...

inertia tensor  ${}^2\mathfrak{J}$

$${}^2\mathfrak{J} \equiv - \int_{\mathcal{V}} (\mathbf{x} \times \mathbf{E}) \cdot (\mathbf{x} \times \mathbf{E}) dm = \int_{\mathcal{V}} (\mathbf{x} \cdot \mathbf{x} \mathbf{E} - \mathbf{x} \mathbf{x}) dm$$

It is assumed (can be proven?) that the inertia tensor changes only due to a rotation

$${}^2\mathfrak{J} = \mathbf{O} \cdot {}^2\mathfrak{J}^{\circ} \cdot \mathbf{O}^{\top}$$

and its components in basis  $\mathbf{e}_i$  (moving together with a body) don't change over time

$${}^2\mathfrak{J} = \mathfrak{J}_{ab} \mathbf{e}_a \mathbf{e}_b, \quad \mathfrak{J}_{ab} = \text{constant}(t)$$

thus the time derivative is

$$\begin{aligned} {}^2\dot{\mathfrak{J}} &= \mathfrak{J}_{ab} (\dot{\mathbf{e}}_a \mathbf{e}_b + \mathbf{e}_a \dot{\mathbf{e}}_b) = \mathfrak{J}_{ab} (\boldsymbol{\omega} \times \mathbf{e}_a \mathbf{e}_b + \mathbf{e}_a \boldsymbol{\omega} \times \mathbf{e}_b) = \\ &= \mathfrak{J}_{ab} (\boldsymbol{\omega} \times \mathbf{e}_a \mathbf{e}_b - \mathbf{e}_a \mathbf{e}_b \times \boldsymbol{\omega}) = \boldsymbol{\omega} \times {}^2\mathfrak{J} - {}^2\mathfrak{J} \times \boldsymbol{\omega} \end{aligned}$$

Substitution of (...) into (1.3) and (1.5) gives equations of balance — the balance of linear momentum and the balance of rotational momentum — for a model of continuous non-deformable rigid body

...

here  $\mathbf{f}$  is the external force per mass unit,  $\mathbf{F}$  is the resultant of external forces (also called the “equally acting force” or the “main vector”),  $\mathbf{M}$  is the resultant of external couples (the “main couple”, the “main moment”).

...

— Are there any scenarios for which the center of mass is not almost exactly equivalent to the center of gravity?

— Non-uniform gravity field. In a uniform gravitational field, the center of mass is equal to the center of gravity.

...

*Work*

$$W(\mathbf{F}, \mathbf{u}) = \mathbf{F} \cdot \mathbf{u}$$

as the exact (full) differential

$$dW = \frac{\partial W}{\partial \mathbf{F}} \cdot d\mathbf{F} + \frac{\partial W}{\partial \mathbf{u}} \cdot d\mathbf{u}$$

by “product rule”

$$dW = d(\mathbf{F} \cdot \mathbf{u}) = d\mathbf{F} \cdot \mathbf{u} + \mathbf{F} \cdot d\mathbf{u}$$

$$\frac{\partial W}{\partial \mathbf{F}} = \mathbf{u}, \quad \frac{\partial W}{\partial \mathbf{u}} = \mathbf{F}$$

...

### *Constraints*

Imposed on the positions and velocities of particles, there are restrictions of a geometrical or kinematical nature, called constraints.

Holonomic constraints are relations between position variables (and possibly time) which can be expressed as equality like

$$f(q^1, q^2, q^3, \dots, q^n, t) = 0,$$

where  $q^1, q^2, q^3, \dots, q^n$  are  $n$  parameters (coordinates) that fully describe the system.

A constraint that cannot be expressed as such is nonholonomic.

Holonomic constraint depends only on coordinates and time. It does not depend on velocities or any higher time derivatives.

Velocity-dependent constraints like

$$f(q^1, q^2, \dots, q^n, \dot{q}^1, \dot{q}^2, \dots, \dot{q}^n, t) = 0$$

are mostly not holonomic.

For example, the motion of a particle constrained to lie on a sphere's surface is subject to a holonomic constraint, but if the particle is able to fall off a sphere under the influence of gravity, the constraint becomes non-holonomic. For the first case the holonomic constraint may be given by the equation:  $r^2 - a^2 = 0$ , where  $r$  is the distance from the centre of a sphere of radius  $a$ . Whereas the second non-holonomic case may be given by:  $r^2 - a^2 \geq 0$ .

Three examples of nonholonomic constraints are: when the constraint equations are nonintegrable, when the constraints have inequalities, or with complicated non-conservative forces like friction.

$$\mathbf{r}_i = \mathbf{r}_i(q^1, q^2, \dots, q^n, t)$$

(assuming  $n$  independent parameters/coordinates)

### §3. Principle of virtual work

*Mécanique analytique* (1788–89) is a two volume French treatise on analytical mechanics, written by Joseph Louis Lagrange, and published 101 years following Isaac Newton's *Philosophiæ Naturalis Principia Mathematica*.

**Joseph Louis Lagrange.** *Mécanique analytique*. Nouvelle édition, revue et augmentée par l'auteur. Tome premier. Mme Ve Courcier, Paris, 1811. 490 pages.

**Joseph Louis Lagrange.** *Mécanique analytique*. Troisième édition, revue, corrigée et annotée par M. J. Bertrand. Tome second. Mallet-Bachelier, Paris, 1855. 416 pages.

The historical transition from geometrical methods, as presented in Newton's *Principia*, to methods of mathematical analysis.

Consider the exact differential of any set of location vectors  $\mathbf{r}_i$ , that are functions of other variable parameters (coordinates)  $q^1, q^2, \dots, q^n$  and time  $t$ .

The actual displacement is the differential

$$d\mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial t} dt + \sum_{j=1}^n \frac{\partial \mathbf{r}_i}{\partial q^j} dq^j$$

Now, imagine an arbitrary path through the configuration space/manifold. This means it has to satisfy the constraints of the system but not the actual applied forces

$$\delta \mathbf{r}_i = \sum_{j=1}^n \frac{\partial \mathbf{r}_i}{\partial q^j} \delta q^j$$

A virtual infinitesimal displacement of a system of particles refers to a change in the configuration of a system as the result of any arbitrary infinitesimal change of location vectors (or coordinates)  $\delta \mathbf{r}_k$ , consistent with the forces and constraints imposed on the system at the current/given instant  $t$ . This displacement is called “virtual” to distinguish it from an actual displacement of the system occurring in a time interval  $dt$ , during which the forces and constraints may be changing.

Assume the system is in equilibrium, that is the full force on each particle vanishes,  $\mathbf{F}_i = \mathbf{0} \forall i$ . Then clearly the term  $\mathbf{F}_i \cdot \delta \mathbf{r}_i$ , which is the virtual work of force  $\mathbf{F}_i$  in displacement  $\delta \mathbf{r}_i$ , also vanishes for each

particle,  $\mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \ \forall i$ . The sum of these vanishing products over all particles is likewise equal to zero:

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0.$$

Decompose the full force  $\mathbf{F}_i$  into the applied (active) force  $\mathbf{F}_i^{(a)}$  and the force of constraint  $\Phi_i$ ,

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \Phi_i$$

We now restrict ourselves to systems for which the net virtual work of the force of every constraint is zero:

$$\sum_i \Phi_i \cdot \delta \mathbf{r}_i = 0.$$

We therefore have as the condition for equilibrium of a system that the virtual work of all applied forces vanishes:

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = 0.$$

— the principle of virtual work.

Note that coefficients  $\mathbf{F}_i^{(a)}$  can no longer be thought equal to zero: in common  $\mathbf{F}_i^{(a)} \neq 0$ , since  $\delta \mathbf{r}_i$  are not independent but are bound by constraints.

Виртуальным перемещением частицы с вектором-радиусом  $\mathbf{r}_k$  называется вариация  $\delta \mathbf{r}_k$  — любое бесконечно малое приращение вектора  $\mathbf{r}_k$ , происходящее мгновенно и совместимое со связями (constraints). Если связей нет, то есть система свободна, тогда виртуальные перемещения  $\delta \mathbf{r}_k$  совершенно любые.

Связи бывают голономные (holonomic, или геометрические), связывающие только положения (перемещения) — это функции лишь координат и, возможно, времени

$$c(\mathbf{r}, t) = 0 \tag{3.1}$$

— и неголономные (или дифференциальные), содержащие производные координат по времени:  $c(\mathbf{r}, \dot{\mathbf{r}}, t) = 0$  и не интегрируемые до геометрических связей.

Далее рассматриваем системы, все связи в которых — голономные. В системе с голономными связями виртуальные перемещения частицы “ $k$ ” удовлетворяют уравнению

$$\sum_{j=1}^m \frac{\partial c_j}{\partial \mathbf{r}_k} \cdot \delta \mathbf{r}_k = 0. \quad (3.2)$$

В несвободных системах все силы делятся на две группы: активные и реакции связей. Реакция  $\Phi_k$  действует со стороны всех материальных ограничителей на частицу “ $k$ ” и меняется согласно уравнению (3.1) для каждой связи. Связи предполагаются идеальными:

$$\sum_k \Phi_k \cdot \delta \mathbf{r}_k = 0 \quad — \quad (3.3)$$

работа реакций на любых виртуальных перемещениях равна нулю.

Принцип виртуальной работы is

$$\sum_k \left( \mathbf{F}_k^{(a)} - m_k \ddot{\mathbf{r}}_k \right) \cdot \delta \mathbf{r}_k = 0, \quad (3.4)$$

где  $\mathbf{F}_k^{(a)}$  — лишь активные силы, без реакций связей.

Дифференциальное вариационное уравнение (3.4) может показаться тривиальным следствием закона Ньютона (1.1) и условия идеальности связей (3.3). Однако содержание (3.4) несравненно обширнее. Читатель вскоре увидит, что принцип (3.4) может быть положен в основу механики [91]. Разные модели упругих тел, описываемые в этой книге, построены с опорой на этот принцип.

Для примера рассмотрим совершенно жёсткое (недеформируемое) твёрдое тело.

$$\dots (2.4) \Rightarrow \delta \mathbf{r} = \delta \mathbf{p} + \delta \mathbf{x}$$

(begin copied from § 1.10)

Варьируя тождество (9.5), получим  $\delta \mathbf{O} \cdot \mathbf{O}^\top = -\mathbf{O} \cdot \delta \mathbf{O}^\top$ . Этот тензор антисимметричен, и потому выражается через свой сопутствующий вектор  $\delta \mathbf{o}$  как  $\delta \mathbf{O} \cdot \mathbf{O}^\top = \delta \mathbf{o} \times \mathbf{E}$ . Приходим к соотношениям

$$\delta \mathbf{O} = \delta \mathbf{o} \times \mathbf{O}, \quad \delta \mathbf{o} = -\frac{1}{2} \left( \delta \mathbf{O} \cdot \mathbf{O}^\top \right)_{\times}, \quad (3.5)$$

(end of copied from § 1.10)

...

Проявилась замечательная особенность (3.4): это уравнение эквивалентно системе такого порядка, каково число степеней свободы системы, то есть сколько независимых вариаций  $\delta \mathbf{r}_k$  мы имеем. Если в системе  $N$  точек есть  $m$  связей, то число степеней свободы  $n = 3N - m$ .

...

#### § 4. Balance of momentum, rotational momentum, and energy

Эти уравнения баланса можно связать со свойствами пространства и времени [92]. Сохранение импульса (количества движения) в изолированной\* системе выводится из однородности пространства (*при любом параллельном переносе — трансляции — замкнутой системы как целого свойства этой системы не меняются*). Сохранение момента импульса — следствие изотропии пространства (*свойства замкнутой системы не меняются при любом повороте этой системы как целого*). Энергия же изолированной системы сохраняется, так как время однородно (энергия  $E \equiv K(q, \dot{q}) + \Pi(q)$  такой системы не зависит явно от времени).

Уравнения баланса могут быть выведены из принципа виртуальной работы (3.4). Перепишем его в виде

$$\sum_k \left( \mathbf{F}_k^{(e)} - m_k \ddot{\mathbf{r}}_k \right) \cdot \delta \mathbf{r}_k + \delta W^{(i)} = 0, \quad (4.1)$$

где выделены внешние силы  $\mathbf{F}_k^{(e)}$  и виртуальная работа внутренних сил  $\delta W^{(i)} = \sum_k \sum_j \mathbf{F}_{kj}^{(i)} \cdot \delta \mathbf{r}_k$ .

Предполагается, что внутренние силы не совершают работы на виртуальных перемещениях тела как жёсткого целого ( $\delta \mathbf{r}$  и  $\delta \mathbf{o}$  —

\* Изолированная (замкнутая) система — это система частиц, взаимодействующих друг с другом, но ни с какими другими телами.

произвольные постоянные векторы, определяющие трансляцию и поворот)

$$\begin{aligned} \delta \mathbf{r}_k &= \delta \mathbf{p} + \delta \mathbf{o} \times \mathbf{r}_k, \\ \delta \mathbf{p} &= \text{constant}, \delta \mathbf{o} = \text{constant} \end{aligned} \Rightarrow \delta W^{(i)} = 0. \quad (4.2)$$

Предпосылки-соображения для этого предположения таковы.

Первое — для случая упругих (потенциальных) внутренних сил. Тогда  $\delta W^{(i)} = -\delta \Pi$  — вариация потенциала с противоположным знаком. Достаточно очевидно, что только лишь деформирование меняет  $\Pi$ .

Второе соображение — в том, что суммарный вектор и суммарный момент внутренних сил равен нулю

$$\sum \dots$$

...

Принимая (4.2) и подставляя в (4.1) сначала  $\delta \mathbf{r}_k = \delta \mathbf{p}$  (трансляция), а затем  $\delta \mathbf{r}_k = \delta \mathbf{o} \times \mathbf{r}_k$  (поворот), получаем баланс импульса (...) и баланс момента импульса (...).

...

## § 5. Hamilton's principle and Lagrange's equations

Two branches of analytical mechanics are Lagrangian mechanics (using generalized coordinates and corresponding generalized velocities in configuration space) and Hamiltonian mechanics (using coordinates and corresponding momenta in phase space). Both formulations are equivalent by a Legendre transformation on the generalized coordinates, velocities and momenta, therefore both contain the same information for describing the dynamics of a system.



Вариационное уравнение (3.4) удовлетворяется в любой момент времени. Проинтегрируем его\* по какому-либо промежутку  $[t_1, t_2]$

$$\int_{t_1}^{t_2} \left( \delta K + \sum_k \mathbf{F}_k \cdot \delta \mathbf{r}_k \right) dt - \left[ \sum_k m_k \dot{\mathbf{r}}_k \cdot \delta \mathbf{r}_k \right]_{t_1}^{t_2} = 0. \quad (5.1)$$

Без ущерба для общности можно принять  $\delta \mathbf{r}_k(t_1) = \delta \mathbf{r}_k(t_2) = \mathbf{0}$ , тогда внеинтегральный член исчезает.

Вводятся обобщённые координаты  $q^i$  ( $i = 1, \dots, n$  — число степеней свободы). Векторы-радиусы становятся функциями вида  $\mathbf{r}_k(q^i, t)$ , тождественно удовлетворяющими уравнениям связей (3.1). Если связи стационарны, то есть (3.1) не содержат  $t$ , то остаётся  $\mathbf{r}_k(q^i)$ . Кинетическая энергия превращается в функцию  $K(q^i, \dot{q}^i, t)$ , где явно входящее  $t$  характерно лишь для нестационарных связей.

Весьма существенно понятие обобщённых сил  $Q_i$ . Они вводятся через выражение виртуальной работы

$$\sum_k \mathbf{F}_k \cdot \delta \mathbf{r}_k = \sum_i Q_i \delta q^i, \quad Q_i \equiv \sum_k \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q^i}. \quad (5.2)$$

Стойит акцентировать происхождение обобщённых сил через работу. Установив набор обобщённых координат системы, следует сгруппировать приложенные силы  $\mathbf{F}_k$  в комплексы  $Q_i$ .

Если силы потенциальны с энергией  $\Pi = \Pi(q^i, t)$ , то

$$\sum_i Q_i \delta q^i = -\delta \Pi, \quad Q_i = -\frac{\partial \Pi}{\partial q^i}. \quad (5.3)$$

Явное присутствие  $t$  может быть при нестационарности связей или зависимости физических полей от времени.

...

$$\begin{aligned} * \delta K &= \sum_k m_k \dot{\mathbf{r}}_k \cdot \delta \dot{\mathbf{r}}_k, \quad \left( \sum_k m_k \dot{\mathbf{r}}_k \cdot \delta \mathbf{r}_k \right) \cdot = \sum_k m_k \ddot{\mathbf{r}}_k \cdot \delta \mathbf{r}_k + \underbrace{\sum_k m_k \dot{\mathbf{r}}_k \cdot \delta \dot{\mathbf{r}}_k}_{\delta K} \\ &- \int_{t_1}^{t_2} \sum_k m_k \ddot{\mathbf{r}}_k \cdot \delta \mathbf{r}_k dt = \int_{t_1}^{t_2} \delta K dt - \left[ \sum_k m_k \dot{\mathbf{r}}_k \cdot \delta \mathbf{r}_k \right]_{t_1}^{t_2} \end{aligned}$$

Известны уравнения Lagrange'а не только второго, но и первого рода. Рассмотрим их ради методики вывода, много раз применяемой в этой книге.

При наличии связей (3.1) равенство  $\mathbf{F}_k = m_k \ddot{\mathbf{r}}_k$  не следует из вариационного уравнения (3.4), ведь тогда виртуальные перемещения  $\delta \mathbf{r}_k$  не независимы. Каждое из  $m$  ( $m$  — число связей) условий для вариаций (3.2) умножим на некий скаляр  $\lambda_\alpha$  ( $\alpha = 1, \dots, m$ ) и добавим к (3.4):

$$\sum_{k=1}^N \left( \mathbf{F}_k + \sum_{\alpha=1}^m \lambda_\alpha \frac{\partial c_\alpha}{\partial \mathbf{r}_k} - m_k \ddot{\mathbf{r}}_k \right) \cdot \delta \mathbf{r}_k = 0. \quad (5.4)$$

Среди  $3N$  компонент вариаций  $\delta \mathbf{r}_k$  зависимых  $m$ . Но столько же и множителей Лагранжа: подберём  $\lambda_\alpha$  так, чтобы коэффициенты (??какие?) при зависимых вариациях обратились в нуль. Но при остальных вариациях коэффициенты (??) также должны быть нулями из-за независимости. Следовательно, все выражения в скобках  $(\dots)$  равны нулю — это и есть уравнения Lagrange'а первого рода.

Поскольку для каждой частицы

...

## § 6. Statics

Рассмотрим систему со стационарными (постоянными во времени) связями при статических (не меняющихся со временем) активных силах  $\mathbf{F}_k$ . В равновесии  $\mathbf{r}_k = \text{constant}$ , и формулировка принципа виртуальной работы следующая:

$$\sum_k \mathbf{F}_k \cdot \delta \mathbf{r}_k = 0 \Leftrightarrow \sum_k \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q^i} = Q_i = 0. \quad (6.1)$$

Существенны обе стороны этого положения: и вариационное уравнение, и равенство нулю обобщённых сил.

Соотношения (6.1) — это самые общие уравнения статики. В литературе распространено узкое представление об уравнениях равновесия как балансе сил и моментов. Но при этом нужно понимать, что набор уравнений равновесия точно соответствует обоб-

щённым координатам. The resultant force (также называемая “равнодействующей силой” или “главным вектором”) and the resultant couple (“главный момент”) в уравнениях равновесия фигурируют\*, поскольку у системы есть степени свободы трансляции и поворота. Огромная популярность сил и моментов связана не столько с известностью статики совершенно недеформируемого твёрдого тела, но с тем, что виртуальная работа внутренних сил на перемещениях системы как жёсткого целого равна нулю в любой среде.

Пусть в системе действуют два вида сил: потенциальные с энергией от обобщённых координат  $\Pi(q^i)$  и дополнительные внешние  $\dot{Q}_i$ . Из (6.1) следуют уравнения равновесия

$$\frac{\partial \Pi}{\partial q^i} = \dot{Q}_i, \quad (6.2)$$

$$d\Pi = \sum_i \frac{\partial \Pi}{\partial q^i} dq^i = \sum_i \dot{Q}_i dq^i.$$

Здесь содержится нелинейная в общем случае задача статики о связи положения равновесия  $q^i$  с нагрузками  $\dot{Q}_i$ .

В линейной системе с квадратичным потенциалом вида  $\Pi = \frac{1}{2} C_{ik} q^k q^i$

$$\sum_k C_{ik} q^k = \dot{Q}_i. \quad (6.3)$$

Тут фигурируют “матрица жёсткости”  $C_{ik}$ , координаты  $q^k$  и нагрузки  $\dot{Q}_i$ .

Сказанное возможно обобщить и на континуальные линейные упругие среды.

Матрица жёсткости  $C_{ik}$  обычно бывает положительной (таково свойство конструкций и в природе, и в технике). Тогда  $\det C_{ik} > 0$ ,

\* Since describing a composition of any system of forces, acting on the same absolutely rigid body, into a single force and a single couple (about a chosen point) in the book “Éléments de statique” by Louis Poinso.

линейная алгебраическая система (6.3) однозначно разрешима, а решение её можно заменить минимизацией квадратичной формы

$$\mathcal{E}(q^j) \equiv \Pi - \sum_i \mathring{Q}_i q^i = \frac{1}{2} \sum_{i,k} q^i C_{ik} q^k - \sum_i \mathring{Q}_i q^i \rightarrow \min. \quad (6.4)$$

Бывает однако, что конструкция неудачно спроектирована, тогда матрица жёсткости сингулярна (необратима) и  $\det C_{ik} = 0$  (или же весьма близок к нулю — nearly singular матрица с  $\det C_{ik} \approx 0$ ). Тогда решение линейной проблемы статики (6.3) существует лишь при ортогональности столбца нагрузок  $\mathring{Q}_i$  всем линейно независимым решениям однородной сопряжённой системы

...

Известные теоремы статики линейно **упругих** систем легко доказываются в случае конечного числа степеней свободы. Теорема Клапейрона выражается равенством

...

Reciprocal work theorem (“работа  $W_{12}$  сил первого варианта на перемещениях от сил второго равна работе  $W_{21}$  сил второго варианта на перемещениях от сил первого”) мгновенно выводится из (6.3):

(...)

Тут существенна симметрия матрицы жёсткости  $C_{ij}$ , то есть консервативность системы.

...

Но вернёмся к проблеме (6.2), иногда называемой теоремой Lagrange’a. Её можно обратить преобразованием Лежандра Legendre (involution) transform(ation):

$$\begin{aligned} d\left(\sum_i \mathring{Q}_i q^i\right) &= \sum_i d\left(\mathring{Q}_i q^i\right) = \sum_i \left(q^i d\mathring{Q}_i + \mathring{Q}_i dq^i\right), \\ d\left(\sum_i \mathring{Q}_i q^i\right) - \sum_i \mathring{Q}_i dq^i &= \sum_i q^i d\mathring{Q}_i, \\ d\left(\sum_i \mathring{Q}_i q^i - \Pi\right) &= \sum_i q^i d\mathring{Q}_i = \sum_i \frac{\partial \widehat{\Pi}}{\partial \mathring{Q}_i} d\mathring{Q}_i; \end{aligned}$$

$$q^i = \frac{\partial \hat{\Pi}}{\partial \dot{Q}_i}, \quad \hat{\Pi}(\dot{Q}_i) = \sum_i \dot{Q}_i q^i - \Pi. \quad (6.5)$$

Это теорема Castigliano,  $\hat{\Pi}$  называется дополнительной энергией. В линейной системе (6.3)  $\Rightarrow \hat{\Pi} = \Pi$ . Теорема (6.5) бывает очень полезна — когда легко находится  $\hat{\Pi}(\dot{Q}_i)$ . Встречаются так называемые статически определимые системы, в которых все внутренние силы удаётся найти лишь из баланса сил и моментов. Для них (6.5) эффективна.

В отличие от линейной задачи (6.3), нелинейная задача (6.2) может не иметь решений вовсе или же иметь их несколько.

....

Рассказ о статике в общей механике закончим the d'Alembert's principle: уравнения динамики отличаются от статических лишь наличием дополнительных “сил инерции”  $m_k \ddot{\mathbf{r}}_k$ . Принцип d'Alembert'a достаточно очевиден, но бездумное применение может привести к ошибкам. Например, уравнения вязкой жидкости в статике и в динамике отличаются не только лишь инерционными добавками. Но для твёрдых упругих тел принцип d'Alembert'a полностью справедлив.

## § 7. Mechanics of relative motion

До этого не ставился вопрос о системе отсчёта, всё рассматривалось в некой “абсолютной” системе или одной из инерциальных систем (§ 1). Теперь представим себе две системы: “абсолютную” и “подвижную”

...

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{v} + \mathbf{x} \\ \mathbf{r} &= \rho_i \mathbf{e}_i, \quad \mathbf{x} = x_i \mathbf{e}_i \\ \dot{\mathbf{r}} &= \dot{\mathbf{r}} + \dot{\mathbf{x}} \\ \dot{\mathbf{r}} &= \dot{\rho}_i \mathbf{e}_i, \quad \dot{\mathbf{x}} = (\dot{x}_i \mathbf{e}_i)^\bullet = \dot{x}_i \mathbf{e}_i + x_i \dot{\mathbf{e}}_i \end{aligned}$$

$$x_i \neq \text{constant} \Rightarrow \dot{x}_i \neq 0$$

By (9.11, § 1.9)

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i \Rightarrow x_i \dot{\mathbf{e}}_i = \boldsymbol{\omega} \times x_i \mathbf{e}_i = \boldsymbol{\omega} \times \mathbf{x}$$

$$\dot{\mathbf{x}} = \dot{x}_i \mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{x}$$

$$\mathbf{v} \equiv \dot{\mathbf{x}} = \dot{\mathbf{r}} + \dot{\mathbf{x}} = \underbrace{\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{x}}_{\mathbf{v}_e} - \underbrace{\boldsymbol{\omega} \times \mathbf{x} + \dot{\mathbf{x}}}_{\mathbf{v}_{rel}}$$

$$\dot{\mathbf{x}} - \boldsymbol{\omega} \times \mathbf{x} = \dot{x}_i \mathbf{e}_i \equiv \mathbf{v}_{rel} - \text{relative velocity, } \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{x} \equiv \mathbf{v}_e$$

$$\mathbf{v} = \mathbf{v}_e + \mathbf{v}_{rel} \quad (7.1)$$

...

$$\dot{\dot{\mathbf{r}}} = \dot{\mathbf{r}} + \dot{\mathbf{x}}$$

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}} + \ddot{\mathbf{x}}$$

$$\mathbf{w} \equiv \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \ddot{\mathbf{r}} + \ddot{\mathbf{x}}$$

$$\ddot{\mathbf{r}} = \ddot{\rho}_i \dot{\mathbf{e}}_i, \quad \ddot{\mathbf{x}} = (x_i \mathbf{e}_i)^{\bullet\bullet} = (\dot{x}_i \mathbf{e}_i + x_i \dot{\mathbf{e}}_i)^{\bullet} = \ddot{x}_i \mathbf{e}_i + \dot{x}_i \dot{\mathbf{e}}_i + \dot{x}_i \dot{\mathbf{e}}_i + x_i \ddot{\mathbf{e}}_i$$

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i \Rightarrow \ddot{\mathbf{e}}_i = (\boldsymbol{\omega} \times \mathbf{e}_i)^{\bullet} = \dot{\boldsymbol{\omega}} \times \mathbf{e}_i + \boldsymbol{\omega} \times \dot{\mathbf{e}}_i = \dot{\boldsymbol{\omega}} \times \mathbf{e}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{e}_i)$$

$$x_i \ddot{\mathbf{e}}_i = x_i (\boldsymbol{\omega} \times \mathbf{e}_i)^{\bullet} = \dot{\boldsymbol{\omega}} \times x_i \mathbf{e}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times x_i \mathbf{e}_i) = \dot{\boldsymbol{\omega}} \times \mathbf{x} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})$$

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i \Rightarrow \dot{x}_i \dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \dot{x}_i \mathbf{e}_i = \boldsymbol{\omega} \times \mathbf{v}_{rel}$$

$$\ddot{x}_i \mathbf{e}_i \equiv \mathbf{w}_{rel} - \text{relative acceleration}$$

$$2\dot{x}_i \dot{\mathbf{e}}_i = 2\boldsymbol{\omega} \times \mathbf{v}_{rel} \equiv \mathbf{w}_{Cor} - \text{Coriolis acceleration}$$

$$\ddot{\mathbf{x}} = \mathbf{w}_{rel} + \mathbf{w}_{Cor} + x_i \ddot{\mathbf{e}}_i$$

$$(x_i \dot{\mathbf{e}}_i)^{\bullet} = \dot{x}_i \dot{\mathbf{e}}_i + x_i \ddot{\mathbf{e}}_i = \frac{1}{2} \mathbf{w}_{Cor} + x_i \ddot{\mathbf{e}}_i$$

$$(x_i \dot{\mathbf{e}}_i)^{\bullet} = (\boldsymbol{\omega} \times \mathbf{x})^{\bullet} = \dot{\boldsymbol{\omega}} \times \mathbf{x} + \boldsymbol{\omega} \times \dot{\mathbf{x}}$$

$$\boldsymbol{\omega} \times \dot{\mathbf{x}} = \boldsymbol{\omega} \times (\dot{x}_i \mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{x}) = \underbrace{\boldsymbol{\omega} \times \dot{x}_i \mathbf{e}_i}_{\dot{x}_i \dot{\mathbf{e}}_i = \frac{1}{2} \mathbf{w}_{Cor}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})$$

...

## § 8. Small oscillations (vibrations)

If the statics of a linear system is described by equation (6.3), then in the dynamics we have

$$\sum_k \left( A_{ik} \ddot{q}_k + C_{ik} q^k \right) = \dot{Q}_i(t), \quad (8.1)$$

where  $A_{ik}$  is the symmetric and positive “matrix of kinetic energy”.

Any description of oscillations almost always includes the term “mode”. A mode of vibration can be defined as a way of vibrating or a pattern of vibration. A normal mode is a pattern of periodic motion, when all parts of an oscillating system move sinusoidally with the same frequency and with a fixed phase relation. The free motion described by the normal modes takes place at fixed frequencies — the natural resonant frequencies of an oscillating system.

The most generic motion of an oscillating system is some superposition of normal modes of this system.\*

A research of an oscillating system most often begins with orthogonal (normal) “modes”— harmonics, free (without any driving or damping force) sinusoidal oscillations

$$q^k(t) = \overset{*}{q}_k \sin \omega_k t.$$

Multipliers  $\overset{*}{q}_k = \text{constant}$  are orthogonal (normal) “modes” of oscillation,  $\omega_k$  are natural (resonant, eigen-) frequencies. This set, dependent on the structure of an oscillating object, the materials and the boundary conditions, is found from the eigenvalue problem

$$\begin{aligned} \overset{\circ}{Q}_i &= 0, \quad \ddot{\overset{*}{q}}_k = -\omega_k^2 \overset{*}{q}_k \sin \omega_k t, \quad (8.1) \Rightarrow \\ &\Rightarrow \sum_k \left( C_{ik} - A_{ik} \omega_k^2 \right) \overset{*}{q}_k \sin \omega_k t = 0 \end{aligned} \quad (8.2)$$

...

Duhamel’s integral is a way of calculating the response of linear systems to arbitrary time-varying external perturbation.

...

\* The modes are “normal” in the sense that they move independently, and an excitation of one mode will never cause a motion of another mode. In mathematical terms, normal modes are orthogonal to each other. In music, normal modes of vibrating instruments (strings, air pipes, percussion and others) are called “harmonics” or “overtones”.

## Bibliography

In a long list of books on generic mechanics one can find works of both specialists in mechanics [86, 87, 93, 94, 95] and broadly oriented theoretical physicists [92, 88]. The book by F. R. Gantmacher (Ф. Р. ГАНТМАХЕР) [91] with compact but complete narration of fundamentals of the theory is pretty interesting.



*chapter 3*

## NONLINEAR ELASTIC MOMENTLESS CONTINUUM

### §1. Continuum. Two descriptions of processes

According to the atomic theory, a substance is composed of discrete particles — atoms. Therefore a model of a system of particles with masses  $m_k$  and location vectors  $\mathbf{r}_k(t)$  may seem suitable yet despite an unimaginable number of degrees of freedom, because amounts of memory and speed of computers are characterized also by astronomical numbers.

But anyway, it's worth choosing a fundamentally and qualitatively different model — a model of material continuum, where the mass is continuously distributed in a volume: volume  $\mathcal{V}$  contains mass

$$m = \int_{\mathcal{V}} \rho d\mathcal{V}, \quad dm = \rho d\mathcal{V}, \quad (1.1)$$

here  $\rho$  is the volume(tric) mass density and  $d\mathcal{V}$  is the infinitesimal volume element.

Real matter is modelled as a continuum, which can be thought of as an infinite set of vanishingly small particles, joined together.

A space of material points is only the first and simple idea of a continuous distribution of mass. More complex models are possible too, where particles have more degrees of freedom: not only of translation, but also of rotation, of internal deformation, and others. Knowing that such models are attracting more and more interest, in this chapter we will consider the classical concept of a continuous medium as “made of simple points”.

At every moment of time  $t$  a deformable continuum occupies a certain volume  $\mathcal{V}$  of space. This volume moves and deforms, but

the set of particles in this volume is constant — the balance of mass (“matter is neither created nor annihilated”)

$$dm = \rho d\mathcal{V} = \rho' d\mathcal{V}' = \overset{\circ}{\rho} d\overset{\circ}{\mathcal{V}}, \quad m = \int_{\mathcal{V}} \rho d\mathcal{V} = \int_{\mathcal{V}'} \rho' d\mathcal{V}' = \int_{\overset{\circ}{\mathcal{V}}} \overset{\circ}{\rho} d\overset{\circ}{\mathcal{V}}. \quad (1.2)$$

Introducing some variable parameters  $q^i$  — the curvilinear coordinates, we have a relation for locations of particles

$$\mathbf{r} = \mathbf{r}(q^i, t). \quad (1.3)$$

...

*Material description*

at the initial moment, in the so-called initial (original, reference, “material”) configuration

at some initial moment  $t=0$

“запоминается” начальная (“материальная”) конфигурация — locations in space of particles at some arbitrarily chosen “initial” moment  $t=0$

$$\overset{\circ}{\mathbf{r}}(q^i) \equiv \mathbf{r}(q^i, 0)$$

Morphism (function)  $\overset{\circ}{\mathbf{r}} = \overset{\circ}{\mathbf{r}}(q^i)$

isomorphism (bijective mapping) (invertible one-to-one relation) (взаимно однозначное)

Subsequent locations in space of particles are then dependent variables — functions of time and of the initial (material, “Lagrangian”) coordinates/location  $\overset{\circ}{\mathbf{r}}$

$$\mathbf{r} = \mathbf{r}(\overset{\circ}{\mathbf{r}}, t).$$

Для пространственного дифференцирования (постоянных во времени) отношений like  $\varphi = \varphi(\overset{\circ}{\mathbf{r}})$

вводится локальный касательный базис  $\overset{\circ}{\mathbf{r}}_{\partial i}$  и взаимный базис  $\overset{\circ}{\mathbf{r}}^i$

$$\overset{\circ}{\mathbf{r}}_{\partial i} \equiv \partial_i \overset{\circ}{\mathbf{r}} \left( \partial_i \equiv \frac{\partial}{\partial q^i} \right), \quad \overset{\circ}{\mathbf{r}}^i \cdot \overset{\circ}{\mathbf{r}}_{\partial j} = \delta_j^i$$

“материальный” оператор Hamilton’a  $\overset{\circ}{\nabla}$

$$\mathbf{E} = \overset{\circ}{\mathbf{r}}^i \overset{\circ}{\mathbf{r}}_{\partial i} = \overset{\circ}{\mathbf{r}}^i \partial_i \overset{\circ}{\mathbf{r}} = \overset{\circ}{\nabla} \overset{\circ}{\mathbf{r}}, \quad \overset{\circ}{\nabla} \equiv \overset{\circ}{\mathbf{r}}^i \partial_i, \quad (1.4)$$

тогда  $d\varphi = d\overset{\circ}{\mathbf{r}} \cdot \overset{\circ}{\nabla} \varphi$ .

...

But yet another approach may be effective — *the spatial (or “Eulerian”) description*, when instead of focusing on how particles of a continuum move from the initial configuration through space and time, processes are considered at fixed points in space as time progresses. With relations like  $\rho = \rho(\mathbf{r}, t)$ , we track what’s happening exactly in this place. Various particles, continuously leaving and coming here, do not confuse us.

...

the balance of mass in spatial description (the continuity equation for mass)

...

Jaumann derivative (“**corotational** time derivative”) was first introduced by Gustav Jaumann\*

Es sei  $\frac{\partial}{\partial t}$  der Operator der lokalen Fluxion, d. i. der partiellen Fluxion in einem gegen das Koordinatensystem ruhenden Punkte des Raumes. Ferner sei  $\frac{d}{dt}$  der Operator der totalen Fluxion, welcher definiert wird durch

$$\frac{da}{dt} = \frac{\partial a}{\partial t} + \mathbf{v} \cdot \nabla a,$$

$$\frac{d\mathbf{a}}{dt} = \frac{\partial \mathbf{a}}{\partial t} + \mathbf{v} \cdot \nabla; \mathbf{a} - \frac{1}{2}(\text{rot } \mathbf{v}) \times \mathbf{a},$$

$$\frac{d\boldsymbol{\alpha}}{dt} = \frac{\partial \boldsymbol{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla; \boldsymbol{\alpha} - \frac{1}{2}(\text{rot } \mathbf{v} \times \boldsymbol{\alpha} - \boldsymbol{\alpha} \times \text{rot } \mathbf{v}).$$

Endlich verwenden wir die körperliche Fluxion eines Skalars:

$$\frac{\delta}{\delta t} a = \frac{\partial}{\partial t} a + \text{div } a\mathbf{v} = \frac{d}{dt} a + a \text{ div } \mathbf{v}.$$

körperliche — bodily/телесная, material/вещественная(материальная),  
physical/физическая

$$\nabla \cdot (a\mathbf{v}) = a \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla a$$

\* **Gustav Jaumann.** Geschlossenes System physikalischer und chemischer Differentialgesetze (I. Mitteilung) // Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Klasse, Abteilung IIa, Band CXX, 1911. Seiten 385–530.

...

Пусть  $\mathbf{v}(\overset{\circ}{\mathbf{r}}, t)$  — какое-либо поле (?? только в материальном описании от  $\overset{\circ}{\mathbf{r}}$  ??). Найдём скорость изменения интеграла по объёму

$$\Upsilon \equiv \int_{\mathcal{V}} \rho \mathbf{v} d\mathcal{V}$$

(“ $\mathbf{v}$  is  $\Upsilon$  per mass unit”). Seemingly difficult calculation of  $\dot{\Upsilon}$  (since  $\mathcal{V}$  is deforming) is actually quite simple together with (1.2):

$$\Upsilon = \int_{\overset{\circ}{\mathcal{V}}} \overset{\circ}{\rho} \mathbf{v} d\overset{\circ}{\mathcal{V}} \Rightarrow \dot{\Upsilon} = \int_{\overset{\circ}{\mathcal{V}}} \overset{\circ}{\rho} \dot{\mathbf{v}} d\overset{\circ}{\mathcal{V}} = \int_{\mathcal{V}} \rho \dot{\mathbf{v}} d\mathcal{V}. \quad (1.5)$$

$$\Psi = \int_{\mathcal{V}} \rho \psi d\mathcal{V} = \int_{\mathcal{V}'} \rho' \psi d\mathcal{V}' \Rightarrow \dot{\Psi} = \int_{\mathcal{V}} \rho \dot{\psi} d\mathcal{V} = \int_{\mathcal{V}'} \rho' \dot{\psi} d\mathcal{V}'$$

...

It is not worth it to contrapose the material and the spatial descriptions. In this book both are used, depending on the situation.

## § 2. Motion gradient

Having motion function  $\mathbf{r} = \mathbf{r}(q^i, t)$ ,  $\mathring{\mathbf{r}}(q^i) \equiv \mathbf{r}(q^i, 0)$ , “nabla” operators  $\nabla \equiv \mathbf{r}^i \partial_i$ ,  $\mathring{\nabla} \equiv \mathring{\mathbf{r}}^i \partial_i$  and looking at differential relations for a certain infinitesimal vector in two configurations, current with  $d\mathbf{r}$  and initial with  $d\mathring{\mathbf{r}}$

$$\begin{aligned} d\mathbf{r} &= d\mathring{\mathbf{r}} \cdot \overbrace{\mathring{\nabla}}^{\mathbf{F}^\top} = \overbrace{\nabla \mathbf{r}^\top}^{\mathbf{F}} \cdot d\mathring{\mathbf{r}} \\ d\mathring{\mathbf{r}} &= d\mathbf{r} \cdot \overbrace{\nabla \mathring{\mathbf{r}}}^{\mathbf{r}^i \mathring{\mathbf{r}}_{\partial_i}} = \overbrace{\nabla \mathring{\mathbf{r}}^\top}^{\mathring{\mathbf{r}}_{\partial_i} \mathbf{r}^i} \cdot d\mathbf{r} \\ &\quad \mathbf{F}^{-\top} \quad \mathbf{F}^{-1} \end{aligned} \quad (2.1)$$

here comes to mind to introduce the “motion gradient”<sup>\*</sup>, picking one of these tensor multipliers for it:  $\mathbf{F} \equiv \mathring{\nabla} \mathbf{r}^\top = \mathbf{r}_{\partial_i} \mathring{\mathbf{r}}^i$ .

Why this one? The reason to choose  $\mathring{\nabla} \mathbf{r}^\top$  is another expression for the differential

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial \mathring{\mathbf{r}}} \cdot d\mathring{\mathbf{r}} \\ \mathbf{F} &= \frac{\partial \mathbf{r}}{\partial \mathring{\mathbf{r}}} \\ d\mathring{\mathbf{r}} &= \frac{\partial \mathring{\mathbf{r}}}{\partial \mathbf{r}} \cdot d\mathbf{r} \\ \mathbf{F}^{-1} &= \frac{\partial \mathring{\mathbf{r}}}{\partial \mathbf{r}} \\ \frac{\partial \boldsymbol{\zeta}}{\partial \mathring{\mathbf{r}}} &= \partial_i \boldsymbol{\zeta} \mathring{\mathbf{r}}^i & \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{r}} &= \partial_i \boldsymbol{\zeta} \mathbf{r}^i \\ \dots & \end{aligned}$$

$$\begin{aligned} \mathbf{E} &= \overbrace{\mathring{\nabla} \mathring{\mathbf{r}}}^{\frac{\partial \mathring{\mathbf{r}}}{\partial \mathring{\mathbf{r}}}} = \overbrace{\nabla \mathbf{r}}^{\frac{\partial \mathbf{r}}{\partial \mathbf{r}}} \\ \dots & \end{aligned}$$

For cartesian coordinates with orthonormal basis  $\mathbf{e}_i = \text{constant}$

$$\mathbf{r} = x_i(t) \mathbf{e}_i, \quad \mathring{\mathbf{r}} = x_i(0) \mathbf{e}_i = \mathring{x}_i \mathbf{e}_i, \quad \mathring{x}_i \equiv x_i(0),$$

<sup>\*</sup> Tensor  $\mathbf{F}$  doesn’t well suit its more popular name “deformation gradient”, because this tensor describes not only the deformation itself, but also the rotation of a body as a whole without deformation.

$$\begin{aligned}\overset{\circ}{\nabla} &= \mathbf{e}_i \frac{\partial}{\partial \overset{\circ}{x}_i} = \mathbf{e}_i \overset{\circ}{\partial}_i, \quad \nabla = \mathbf{e}_i \frac{\partial}{\partial x_i} = \mathbf{e}_i \partial_i, \\ \overset{\circ}{\nabla} \mathbf{r} &= \mathbf{e}_i \frac{\partial \mathbf{r}}{\partial \overset{\circ}{x}_i} = \mathbf{e}_i \frac{\partial(x_j \mathbf{e}_j)}{\partial \overset{\circ}{x}_i} = \frac{\partial x_j}{\partial \overset{\circ}{x}_i} \mathbf{e}_i \mathbf{e}_j = \overset{\circ}{\partial}_i x_j \mathbf{e}_i \mathbf{e}_j, \\ \nabla \overset{\circ}{\mathbf{r}} &= \mathbf{e}_i \frac{\partial \overset{\circ}{\mathbf{r}}}{\partial x_i} = \frac{\partial \overset{\circ}{x}_j}{\partial x_i} \mathbf{e}_i \mathbf{e}_j = \partial_i \overset{\circ}{x}_j \mathbf{e}_i \mathbf{e}_j\end{aligned}$$

...

By the polar decomposition theorem (§ 1.11), the motion gradient decomposes into the rotation tensor  $\mathbf{O}$  and the symmetric positive stretch tensors  $\mathbf{U}$  and  $\mathbf{V}$ :

$$\mathbf{F} = \mathbf{O} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{O}$$

...

Когда нет поворота ( $\mathbf{O} = \mathbf{E}$ ), тогда  $\mathbf{F} = \mathbf{U} = \mathbf{V}$ .

...

### § 3. Measures (tensors) of deformation

Motion gradient  $\mathbf{F}$  characterizes both the deformation of a body and the rotation of a body as a whole. “Pure” deformation tensors are the stretch tensors  $\mathbf{U}$  and  $\mathbf{V}$  from polar decomposition  $\mathbf{F} = \mathbf{O} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{O}$ , as well as other tensors, originating from  $\mathbf{U}$  or (and)  $\mathbf{V}$ .

Widely used ones are “squares” of  $\mathbf{U}$  and  $\mathbf{V}$

$$\begin{aligned}(\mathbf{U}^2) &= \mathbf{U} \cdot \mathbf{U} = \mathbf{F}^\top \cdot \mathbf{F} \equiv \mathbf{G}, \\ (\mathbf{V}^2) &= \mathbf{V} \cdot \mathbf{V} = \mathbf{F} \cdot \mathbf{F}^\top \equiv \mathbf{\Phi}.\end{aligned}\tag{3.1}$$

These are the Green’s deformation tensor (or the right Cauchy–Green tensor)  $\mathbf{G}$  and the Finger’s deformation tensor (or the left Cauchy–Green tensor)  $\mathbf{\Phi}$ . The convenient relation to  $\mathbf{F}$  — without extracting roots as with  $\mathbf{U}$  and  $\mathbf{V}$  — explains the big popularity of tensors  $\mathbf{G}$  and  $\mathbf{\Phi}$ . Tensor  $\mathbf{G}$  was first used by George Green\*.

\* **Green, George.** (1839) On the propagation of light in crystallized media. *Transactions of the Cambridge Philosophical Society.* 1842, vol. 7, part II, pages 121–140.

Inversion of  $\Phi$  and  $G$  gives two more deformation tensors

$$\begin{aligned} V^{-2} &= \Phi^{-1} = (F \cdot F^\top)^{-1} = F^{-\top} \cdot F^{-1} \equiv {}^2c, \\ U^{-2} &= G^{-1} = (F^\top \cdot F)^{-1} = F^{-1} \cdot F^{-\top} \equiv {}^2f, \end{aligned} \quad (3.2)$$

each of which is sometimes called the Piola tensor or the Finger tensor. The inverse of the left Cauchy–Green tensor  $\Phi$  is the Cauchy deformation tensor  ${}^2c$ .

Components of the deformation tensors

$$\begin{aligned} G &= \mathring{r}^i r_{\partial i} \cdot r_{\partial j} \mathring{r}^j = G_{ij} \mathring{r}^i \mathring{r}^j, \quad G_{ij} \equiv r_{\partial i} \cdot r_{\partial j}, \\ {}^2f &= \mathring{r}_{\partial i} r^i \cdot r^j \mathring{r}_{\partial j} = G^{ij} \mathring{r}_{\partial i} \mathring{r}_{\partial j}, \quad G^{ij} \equiv r^i \cdot r^j, \\ {}^2c &= r^i \mathring{r}_{\partial i} \cdot \mathring{r}_{\partial j} r^j = g_{ij} r^i r^j, \quad g_{ij} \equiv \mathring{r}_{\partial i} \cdot \mathring{r}_{\partial j}, \\ \Phi &= r_{\partial i} \mathring{r}^i \cdot \mathring{r}^j r_{\partial j} = g^{ij} r_{\partial i} r_{\partial j}, \quad g^{ij} \equiv \mathring{r}^i \cdot \mathring{r}^j \end{aligned}$$

coincide with components of the unit (“metric”) tensor

$$\begin{aligned} E &= r_{\partial i} r^i = G_{ij} r^i r^j = r^i r_{\partial i} = G^{ij} r_{\partial i} r_{\partial j} = \\ &= \mathring{r}^i \mathring{r}_{\partial i} = g^{ij} \mathring{r}_{\partial i} \mathring{r}_{\partial j} = \mathring{r}_{\partial i} \mathring{r}^i = g_{ij} \mathring{r}^i \mathring{r}^j, \end{aligned}$$

but these components are in different bases. Using the index notation, it’s easy to get confused about differences between the unit (“metric”) tensor  $E$  and tensors  $G$ ,  $\Phi$ ,  ${}^2f$ ,  ${}^2c$  of deformation. The direct indexless notation has the obvious advantage here.

As was mentioned in § 1.11, the stretch tensors  $U$  and  $V$  have the same invariants. If  $U_k$  ( $k = 1, 2, 3$ ) are eigenvalues of  $U$  and  $V$ , then the invariants will be

$$\begin{aligned} \text{I}(U) &= \text{I}(V) = \sum U_k, \\ \text{II}(U) &= \text{II}(V) = U_1 U_2 + U_1 U_3 + U_2 U_3, \\ \text{III}(U) &= \text{III}(V) = U_1 U_2 U_3. \end{aligned}$$

Invariants of  $G$  and  $\Phi$  coincide too:  $\text{I}(G) = \text{I}(\Phi) = \sum U_k^2, \dots$

Without deformation

$$F = U = V = G = \Phi = {}^2f = {}^2c = E,$$

thus as characteristics of deformation it’s worth taking the differences like  $U - E$ ,  $U \cdot U - E$ ,  $\dots$

...

*The right Cauchy–Green deformation tensor*

George Green discovered a deformation tensor known as the right Cauchy–Green deformation tensor or Green’s deformation tensor

$$\mathbf{G} = \mathbf{F}^\top \cdot \mathbf{F} = \mathbf{U}^2 \quad \text{or} \quad G_{ij} = F_{k'i} F_{k'j} = \frac{\partial x_{k'}}{\partial \hat{x}_i} \frac{\partial x_{k'}}{\partial \hat{x}_j}.$$

This tensor gives the “square” of local change in distances due to deformation:  $d\mathbf{r} \cdot d\mathbf{r} = d\hat{\mathbf{r}} \cdot \mathbf{G} \cdot d\hat{\mathbf{r}}$

Invariants of  $\mathbf{G}$  are used in expressions for (density of) isotropic body’s potential energy of deformation. The most popular invariants are

$$\begin{aligned} \text{I}(\mathbf{G}) &\equiv \text{trace } \mathbf{G} = G_{ii} = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \\ \text{II}(\mathbf{G}) &\equiv \frac{1}{2}(G_{jj}^2 - G_{ik}G_{ki}) = \gamma_1^2\gamma_2^2 + \gamma_2^2\gamma_3^2 + \gamma_3^2\gamma_1^2 \\ \text{III}(\mathbf{G}) &\equiv \det \mathbf{G} = \gamma_1^2\gamma_2^2\gamma_3^2 \end{aligned}$$

where  $\gamma_i$  are stretch ratios for unit fibers that are initially oriented along directions of eigenvectors of the right stretch tensor  $\mathbf{U}$ .

*The inverse of Green’s deformation tensor*

Sometimes called the Finger tensor or the Piola tensor, the inverse of the right Cauchy–Green deformation tensor

$${}^2\mathbf{f} = \mathbf{G}^{-1} = \mathbf{F}^{-1} \cdot \mathbf{F}^{-\top} \quad \text{or} \quad f_{ij} = \frac{\partial \hat{x}_i}{\partial x_{k'}} \frac{\partial \hat{x}_j}{\partial x_{k'}}$$

*The left Cauchy–Green or Finger deformation tensor*

Swapping multipliers in the formula for the right Green–Cauchy deformation tensor leads to the left Cauchy–Green deformation tensor, defined as

$$\Phi = \mathbf{F} \cdot \mathbf{F}^\top = \mathbf{V}^2 \quad \text{or} \quad \Phi_{ij} = \frac{\partial x_i}{\partial \hat{x}_k} \frac{\partial x_j}{\partial \hat{x}_k}$$

The left Cauchy–Green deformation tensor is often called the Finger’s deformation tensor, named after Josef Finger (1894).



Invariants of  $\boldsymbol{\Phi}$  are also used in expressions for strain energy density functions. The conventional invariants are defined as

$$\begin{aligned} I_1 &\equiv \Phi_{ii} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &\equiv \frac{1}{2}(\Phi_{ii}^2 - \Phi_{jk}\Phi_{kj}) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\ I_3 &\equiv \det \boldsymbol{\Phi} = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned}$$

( $J \equiv \det \mathbf{F}$  is the Jacobian, determinant of the motion gradient)

### *The Cauchy deformation tensor*

The Cauchy deformation tensor is defined as the inverse of the left Cauchy–Green deformation tensor

$${}^2\mathbf{c} = \boldsymbol{\Phi}^{-1} = \mathbf{F}^{-\top} \cdot \mathbf{F}^{-1} \quad \text{or} \quad c_{ij} = \frac{\partial \hat{x}_k}{\partial x_i} \frac{\partial \hat{x}_k}{\partial x_j}$$

$$d\hat{\mathbf{r}} \cdot d\hat{\mathbf{r}} = d\mathbf{r} \cdot {}^2\mathbf{c} \cdot d\mathbf{r}$$

This tensor is also called the Piola tensor or the Finger tensor in rheology and fluid dynamics literature.

### *Finite strain tensors*

The concept of *strain* is used to evaluate how much a given displacement differs locally from a body displacement as a whole (a “rigid body displacement”). One of such strains for large deformations is the *Green strain tensor* (*Green–Lagrangian strain tensor*, *Green–Saint-Venant strain tensor*), defined as

$$\mathbf{C} = \frac{1}{2}(\mathbf{G} - \mathbf{E}) \quad \text{or} \quad C_{ij} = \frac{1}{2} \left( \frac{\partial x_{k'}}{\partial \hat{x}_i} \frac{\partial x_{k'}}{\partial \hat{x}_j} - \delta_{ij} \right)$$

or as the function of the displacement gradient tensor

$$\mathbf{C} = \frac{1}{2} \left( \hat{\nabla} \mathbf{u} + \hat{\nabla} \mathbf{u}^\top + \hat{\nabla} \mathbf{u} \cdot \hat{\nabla} \mathbf{u}^\top \right)$$

in cartesian coordinates

$$C_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial \hat{x}_i} + \frac{\partial u_i}{\partial \hat{x}_j} + \frac{\partial u_k}{\partial \hat{x}_i} \frac{\partial u_k}{\partial \hat{x}_j} \right)$$

The Green strain tensor measures how much  $\mathbf{G}$  differs from  $\mathbf{E}$ .

The *Almansi–Hamel strain tensor*, referenced to the deformed configuration (“Eulerian description”), is defined as

$${}^2\mathbf{a} = \frac{1}{2}(\mathbf{E} - {}^2\mathbf{c}) = \frac{1}{2}(\mathbf{E} - \mathbf{\Phi}^{-1}) \quad \text{or} \quad a_{ij} = \frac{1}{2} \left( \delta_{ij} - \frac{\partial \overset{\circ}{x}_k}{\partial x_i} \frac{\partial \overset{\circ}{x}_k}{\partial x_j} \right)$$

or as function of the displacement gradient

$$\begin{aligned} {}^2\mathbf{a} &= \frac{1}{2}(\nabla \mathbf{u}^\top + \nabla \mathbf{u} - \nabla \mathbf{u} \cdot \nabla \mathbf{u}^\top) \\ a_{ij} &= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) \end{aligned}$$

### *Seth–Hill family of abstract strain tensors*

B. R. Seth was the first to show that the Green and Almansi strain tensors are special cases of a more abstract measure of deformation. The idea was further expanded upon by Rodney Hill in 1968 (publication??). The Seth–Hill family of strain measures (also called Doyle–Ericksen tensors) is expressed as

$$\mathbf{D}_{(m)} = \frac{1}{2m} (\mathbf{U}^{2m} - \mathbf{E}) = \frac{1}{2m} (\mathbf{G}^m - \mathbf{E})$$

For various  $m$  it gives

$$\begin{aligned} \mathbf{D}_{(1)} &= \frac{1}{2}(\mathbf{U}^2 - \mathbf{E}) = \frac{1}{2}(\mathbf{G} - \mathbf{E}) && \text{Green strain tensor} \\ \mathbf{D}_{(1/2)} &= \mathbf{U} - \mathbf{E} = \mathbf{G}^{1/2} - \mathbf{E} && \text{Biot strain tensor} \\ \mathbf{D}_{(0)} &= \ln \mathbf{U} = \frac{1}{2} \ln \mathbf{G} && \text{logarithmic strain, Hencky strain} \\ \mathbf{D}_{(-1)} &= \frac{1}{2}(\mathbf{E} - \mathbf{U}^{-2}) && \text{Almansi strain} \end{aligned}$$

The second-order approximation of these tensors is

$$\mathbf{D}_{(m)} = \boldsymbol{\varepsilon} + \frac{1}{2} \nabla \mathbf{u} \cdot \nabla \mathbf{u}^\top - (1 - m) \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}$$

where  $\boldsymbol{\varepsilon} \equiv \nabla \mathbf{u}^S$  is the infinitesimal deformation tensor.

Many other different definitions of measures  $\mathbf{D}$  are possible, provided that they satisfy these conditions:

- ✓  $\mathbf{D}$  vanishes for any movement of a body as a rigid whole
- ✓ dependence of  $\mathbf{D}$  on displacement gradient tensor  $\nabla \mathbf{u}$  is continuous, continuously differentiable and monotonic
- ✓ it's desired that  $\mathbf{D}$  reduces to the infinitesimal linear deformation tensor  $\boldsymbol{\varepsilon}$  when  $\nabla \mathbf{u} \rightarrow 0$

For example, tensors from the set

$$\mathbf{D}^{(n)} = (\mathbf{U}^n - \mathbf{U}^{-n}) / 2n$$

aren't from the Seth–Hill family, but for any  $n$  they have the same 2nd-order approximation as Seth–Hill measures with  $m = 0$ .

*Wikipedia, the free encyclopedia* — Finite strain theory

...

## § 4. Velocity field

This topic is discussed in nearly any book about continuum mechanics, however for solid elastic continua it's not very vital. Among various models of a material continuum, an elastic solid body is distinguished by interesting possibility of deriving the complete set (system) of equations for it via the single logically flawless procedure. But now we follow the way, usual for fluid continuum mechanics.

So, there's velocity field in spatial description  $\mathbf{v} \equiv \dot{\mathbf{r}} = \mathbf{v}(\mathbf{r}, t)$ . Decomposition of tensor  $\nabla \mathbf{v} = \nabla \dot{\mathbf{r}} = \mathbf{r}^i \partial_i \dot{\mathbf{r}} = \mathbf{r}^i \dot{\mathbf{r}}_{\partial i}^*$  into symmetric and skew-symmetric parts (§ 1.7)

$$\nabla \dot{\mathbf{r}} = \nabla \dot{\mathbf{r}}^S - \frac{1}{2} (\nabla \times \dot{\mathbf{r}}) \times \mathbf{E}$$

or, introducing the rate of deformation tensor (rate of stretching tensor, strain rate tensor)  $\mathcal{D}$  and the vorticity tensor (rate of rotation tensor, spin tensor)  $\mathcal{W}$

$$\begin{aligned} \nabla \mathbf{v} &= \mathcal{D} - \mathcal{W}, \\ \mathcal{D} \equiv \nabla \mathbf{v}^S &= \nabla \dot{\mathbf{r}}^S = \frac{1}{2} (\mathbf{r}^i \dot{\mathbf{r}}_{\partial i} + \dot{\mathbf{r}}_{\partial i} \mathbf{r}^i), \\ -\mathcal{W} \equiv \nabla \mathbf{v}^A &= -\mathbf{w} \times \mathbf{E}, \quad \mathbf{w} \equiv \frac{1}{2} \nabla \times \mathbf{v} = \frac{1}{2} \mathbf{r}^i \times \dot{\mathbf{r}}_{\partial i}, \end{aligned} \quad (4.1)$$

where also figures vorticity (pseudo)vector  $\mathbf{w}$ , the companion of vorticity tensor  $\mathcal{W}$ .

\* For sufficiently smooth functions, partial derivatives always commute, space and time ones too. Thus

$$\frac{\partial}{\partial q^i} \frac{\partial \mathbf{r}}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \mathbf{r}}{\partial q^i} \quad \text{or} \quad \partial_i \dot{\mathbf{r}} = \dot{\mathbf{r}}_{\partial i}$$

Components of the rate of deformation tensor in the current configuration's basis

$$\begin{aligned}\mathcal{D} &= \mathcal{D}_{ij} \mathbf{r}^i \mathbf{r}^j, \quad \mathcal{D}_{ij} = \mathbf{r}_{\partial i} \cdot \mathcal{D} \cdot \mathbf{r}_{\partial j} = \frac{1}{2} \mathbf{r}_{\partial i} \cdot (\mathbf{r}^k \dot{\mathbf{r}}_k + \dot{\mathbf{r}}_k \mathbf{r}^k) \cdot \mathbf{r}_{\partial j} = \\ &= \frac{1}{2} (\dot{\mathbf{r}}_{\partial i} \cdot \mathbf{r}_{\partial j} + \mathbf{r}_{\partial i} \cdot \dot{\mathbf{r}}_{\partial j}) = \frac{1}{2} (\mathbf{r}_{\partial i} \cdot \mathbf{r}_{\partial j})^\bullet\end{aligned}$$

...

$$\dot{G}_{ij}$$

$$G_{ij} \equiv \mathbf{r}_{\partial i} \cdot \mathbf{r}_{\partial j}$$

...

For elastic solid media, there's no need for discussion about rotations: the true representation appears along the way of logically harmonious conclusions without additional hypotheses.

## § 5. Area vector. Surface change

Take an infinitesimal surface. The area vector by length is equal to the surface's area and is directed along the normal to this surface.

In the initial (original, undeformed, “material”, reference) configuration, the area vector can be represented as  $\mathring{\mathbf{n}} do$ . Surface's area  $do$  is infinitely small, and  $\mathring{\mathbf{n}}$  is unit normal vector.

In the present (current, actual, deformed, “spatial”) configuration, the same surface has area vector  $\mathbf{n} d\mathcal{O}$ .

With differential precision, these infinitesimal surfaces are parallelograms, thus

$$\begin{aligned}\mathring{\mathbf{n}} do &= d\mathring{\mathbf{r}}' \times d\mathring{\mathbf{r}}'' = \frac{\partial \mathring{\mathbf{r}}}{\partial q^i} dq^i \times \frac{\partial \mathring{\mathbf{r}}}{\partial q^j} dq^j = \mathring{\mathbf{r}}_{\partial i} \times \mathring{\mathbf{r}}_{\partial j} dq^i dq^j, \\ \mathbf{n} d\mathcal{O} &= d\mathbf{r}' \times d\mathbf{r}'' = \frac{\partial \mathbf{r}}{\partial q^i} dq^i \times \frac{\partial \mathbf{r}}{\partial q^j} dq^j = \mathbf{r}_{\partial i} \times \mathbf{r}_{\partial j} dq^i dq^j.\end{aligned}\tag{5.1}$$

Applying the transformation of volume (??), we have

$$\begin{aligned}d\mathcal{V} &= J d\mathcal{V}^\circ \Rightarrow \mathbf{r}_{\partial i} \times \mathbf{r}_{\partial j} \cdot \mathbf{r}_{\partial k} = J \mathring{\mathbf{r}}_{\partial i} \times \mathring{\mathbf{r}}_{\partial j} \cdot \mathring{\mathbf{r}}_{\partial k} \Rightarrow \\ &\Rightarrow \mathbf{r}_{\partial i} \times \mathbf{r}_{\partial j} \cdot \mathbf{r}_{\partial k} \mathbf{r}^k = J \mathring{\mathbf{r}}_{\partial i} \times \mathring{\mathbf{r}}_{\partial j} \cdot \mathring{\mathbf{r}}_{\partial k} \mathbf{r}^k \Rightarrow \\ &\Rightarrow \mathbf{r}_{\partial i} \times \mathbf{r}_{\partial j} = J \mathring{\mathbf{r}}_{\partial i} \times \mathring{\mathbf{r}}_{\partial j} \cdot \mathbf{F}^{-1}.\end{aligned}$$

Hence with (5.1) we come to the relation

$$\mathbf{n}d\mathcal{O} = J\mathbf{n}^{\circ}d\mathbf{o} \cdot \mathbf{F}^{-1}, \quad (5.2)$$

called the Nanson's formula.

## § 6. Forces in continuum. Existence of the Cauchy stress tensor

Augustin-Louis Cauchy founded the *continuum mechanics* by assuming that two parts of a continuum interact by means of surface densities of the contact forces concentrated on the dividing surface.

Assuming that these contact forces depend only on the perpendicular (normal) to the dividing surface and that the contact forces are balanced by some volume density of force (including inertia), Cauchy played with tetrahedrons and proved the existence of the stress tensor.

De la pression ou tension dans un corps solide. *Exercices de mathématiques*, par M. **Augustin-Louis Cauchy**. Seconde année: 1827. Paris, Chez de Bure frères. Pages 42 à 59.

Since particles of the momentless model of continuum are points with only translational degrees of freedom\*, then there're no moments among generalized forces, there're no applied external couples neither in a volume nor on a surface.

Force  $\rho \mathbf{f} d\mathcal{V}$  acts on infinitesimal volume  $d\mathcal{V}$ ; if  $\mathbf{f}$  is a mass force (acting per unit of mass), then  $\rho \mathbf{f}$  is a volume one. Such forces originate from force fields, for example: gravitational forces ("weight"), forces of inertia in non-inertial reference systems, electromagnetic forces in a medium with charges and currents.

Surface force  $\mathbf{p} d\mathcal{O}$  acts on infinitesimal surface  $d\mathcal{O}$ . It may be a contact pressure or/and a friction, an electrostatic force with charges concentrated on the surface.

In a material continuum, as in any mechanical system, external and internal forces are distinguished. Internal forces balance the action

\* Translational degrees of freedom arise from a particle's ability to move freely in space.

of external forces, and inside a medium they are transmitted continuously from point to point. Since times of Euler and Cauchy, the internal forces in a continuum are assumed to be the surface short-range contact forces: on infinitesimal surface  $\mathbf{n}d\mathcal{O}$  inside a medium acts force  $\mathbf{t}(\mathbf{n})d\mathcal{O}$ . Clarifying(?): acts from?? the side where the unit normal  $\mathbf{n}$  is directed.

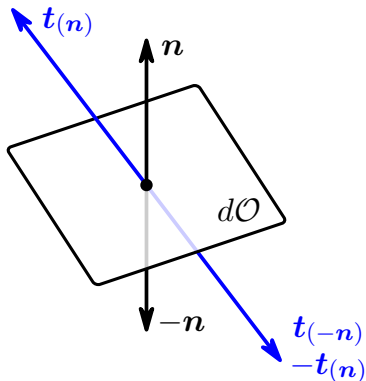


figure 12

By the action–reaction principle, traction vector  $\mathbf{t}(\mathbf{n})$  changes its sign with the reversal of the direction of normal  $\mathbf{n}$ :  $\mathbf{t}(-\mathbf{n}) = -\mathbf{t}(\mathbf{n})$ . (Sometimes this thesis is proved via the balance of momentum for an infinitely short cylinder with bases  $\mathbf{n}d\mathcal{O}$  and  $-\mathbf{n}d\mathcal{O}$ .)

Traction vector  $\mathbf{t}(\mathbf{n})$  on the surface with unit normal  $\mathbf{n}$  is called the surface traction vector or force-stress vector. However,  $\mathbf{t}(\mathbf{n})$  is **not a vector field**: traction

$\mathbf{t} = \mathbf{t}(\mathbf{n}, \mathbf{r}, t)$  depends not only on location  $\mathbf{r}$  of the point, but also on the local direction (defined by  $\mathbf{n}$ ) of the surface element. An infinite number of surfaces of any direction contain the same point, and infinitely many traction vectors  $\mathbf{t}(\mathbf{n})$  exist at each point.

Thereby, the stress at the point of a continuum is **not a vector field**, but an infinite collection (set) of all traction vectors for any and in every way directed infinitesimal surfaces, containing this point in self.

Actually, the infinite set of all traction vectors  $\mathbf{t}(\mathbf{n})$  at the point is completely determined by the only one single second complexity tensor — the Cauchy stress tensor  $\boldsymbol{\tau}$ .

The derivation of this thesis is described in many books. It is known as the theorem about existence of the Cauchy stress tensor — the one with the impressive tetrahedron argument.

*The Cauchy tetrahedron argument and the proof of the existence of the Cauchy stress tensor.*

На поверхности бесконечно малого материального тетраэдра ...

...

## § 7. Balance of momentum and angular momentum

Consider some random finite volume  $\mathcal{V}$  of an elastic medium, contained within surface  $\mathcal{O}(\partial\mathcal{V})$ . It is loaded with external forces, surface contact ones  $\mathbf{p}d\mathcal{O}$  and body (mass or volume) ones  $\mathbf{f}dm = \rho\mathbf{f}d\mathcal{V}$ .

The integral formulation of the balance of momentum is as follows

$$\left( \int_{\mathcal{V}} \rho \mathbf{v} d\mathcal{V} \right)^{\bullet} = \int_{\mathcal{V}} \rho \mathbf{f} d\mathcal{V} + \oint_{\mathcal{O}(\partial\mathcal{V})} \mathbf{p} d\mathcal{O}. \quad (7.1)$$

...  $\mathbf{p} = \mathbf{t}_{(n)} = \mathbf{n} \cdot \boldsymbol{\tau}$  ...

Импульс слева найдём по (1.5), а поверхностный интеграл превратим в объёмный по теореме о дивергенции. Получим

$$\int_{\mathcal{V}} \left( \nabla \cdot \boldsymbol{\tau} + \rho(\mathbf{f} - \dot{\mathbf{v}}) \right) d\mathcal{V} = \mathbf{0}.$$

Но объём  $\mathcal{V}$  случаен, поэтому равно нулю подынтегральное выражение. Приходим к уравнению баланса импульса в локальной (дифференциальной) форме

$$\nabla \cdot \boldsymbol{\tau} + \rho(\mathbf{f} - \dot{\mathbf{v}}) = \mathbf{0}. \quad (7.2)$$

...

Now about the balance of the angular (rotational) momentum. Here is the integral formulation:

$$\left( \int_{\mathcal{V}} \mathbf{r} \times \rho \mathbf{v} d\mathcal{V} \right)^{\bullet} = \int_{\mathcal{V}} \mathbf{r} \times \rho \mathbf{f} d\mathcal{V} + \oint_{\mathcal{O}(\partial\mathcal{V})} \mathbf{r} \times \mathbf{p} d\mathcal{O}. \quad (7.3)$$

Дифференцируя левую часть ( $\mathbf{v} \equiv \dot{\mathbf{r}}$ )

$$\left( \int_{\mathcal{V}} \mathbf{r} \times \rho \dot{\mathbf{r}} d\mathcal{V} \right)^{\bullet} = \int_{\mathcal{V}} \mathbf{r} \times \rho \ddot{\mathbf{r}} d\mathcal{V} + \int_{\mathcal{V}} \underbrace{\dot{\mathbf{r}} \times \rho \dot{\mathbf{r}}}_{\mathbf{0}} d\mathcal{V},$$

применяя теорему о дивергенции к поверхностному интегралу ....

$$(\dots \mathbf{p} = \mathbf{t}_{(n)} = \mathbf{n} \cdot \boldsymbol{\tau} \dots)$$

$$\mathbf{r} \times (\mathbf{n} \cdot \boldsymbol{\tau}) = -(\mathbf{n} \cdot \boldsymbol{\tau}) \times \mathbf{r} = -\mathbf{n} \cdot (\boldsymbol{\tau} \times \mathbf{r}) \Rightarrow$$

$$\Rightarrow \oint_{\mathcal{O}(\partial\mathcal{V})} \mathbf{r} \times (\mathbf{n} \cdot \boldsymbol{\tau}) d\mathcal{O} = - \int_{\mathcal{V}} \nabla \cdot (\boldsymbol{\tau} \times \mathbf{r}) d\mathcal{V},$$

...

$$\int_{\mathcal{V}} \mathbf{r} \times \rho \ddot{\mathbf{r}} d\mathcal{V} = \int_{\mathcal{V}} \mathbf{r} \times \rho \mathbf{f} d\mathcal{V} - \int_{\mathcal{V}} \nabla \cdot (\boldsymbol{\tau} \times \mathbf{r}) d\mathcal{V},$$

$$\int_{\mathcal{V}} \mathbf{r} \times \rho (\mathbf{f} - \ddot{\mathbf{r}}) d\mathcal{V} - \int_{\mathcal{V}} \nabla \cdot (\boldsymbol{\tau} \times \mathbf{r}) d\mathcal{V} = \mathbf{0},$$

...

$$\underbrace{\nabla \cdot (\boldsymbol{\tau} \times \mathbf{r})}_{\mathbf{r}^i \cdot \partial_i (\boldsymbol{\tau} \times \mathbf{r})} = \underbrace{(\nabla \cdot \boldsymbol{\tau}) \times \mathbf{r} + \mathbf{r}^i \cdot (\boldsymbol{\tau} \times \partial_i \mathbf{r})}_{\mathbf{r}^i \cdot (\partial_i \boldsymbol{\tau}) \times \mathbf{r}}$$

$$\boldsymbol{\tau} = e_i \mathbf{t}_{(i)}, e_i = \text{constant}$$

$$\begin{aligned} \mathbf{r}^i \cdot (\boldsymbol{\tau} \times \partial_i \mathbf{r}) &= \mathbf{r}^i \cdot (e_j \mathbf{t}_{(j)} \times \mathbf{r}_i) = \mathbf{r}^i \cdot e_j \mathbf{t}_{(j)} \times \mathbf{r}_i = \\ &= -e_j \cdot \mathbf{r}^i \mathbf{r}_i \times \mathbf{t}_{(j)} = -e_j \cdot \mathbf{E} \times \mathbf{t}_{(j)} = -e_j \times \mathbf{t}_{(j)} = -\boldsymbol{\tau}_{\times} \end{aligned}$$

...

## § 8. Eigenvalues of the Cauchy stress tensor. Mohr's circles

Like any symmetric bivalent tensor, the Cauchy stress tensor  $\boldsymbol{\tau}$  has три вещественных собственных числá  $\sigma_i$ , а также тройку взаимно перпендикулярных собственных векторов единичной длины (§ 1.8). Собственные числа тензора  $\boldsymbol{\tau}$  называются главными напряжениями (principal stresses).

In representation  $\boldsymbol{\tau} = \sum \sigma_i e_i e_i$  most often indices are sorted descending  $\sigma_1 \geq \sigma_2 \geq \sigma_3$ , а тройка  $e_i$  ориентирована как “правая”.

Известна теорема о кругах Мора (Mohr's circles)\*

\* Mohr's circles, named after Christian Otto Mohr, is a two-dimensional graphical representation of transformation for the Cauchy stress tensor.



...

Чтобы замкнуть набор (систему) уравнений модели сплошной среды, нужно добавить определяющие отношения (constitutive relations) — уравнения, связывающие напряжение с деформацией (и другие необходимые связи). Однако, for a solid elastic continuum такой длинный путь построения модели излишен, что читатель и увидит ниже.

## § 9. Principle of virtual work (without Lagrange multipliers)

According to the principle of virtual work for some finite volume of continuum

$$\int_V (\rho \mathbf{f} \cdot \delta \mathbf{r} + \delta W^{(i)}) dV + \oint_{\mathcal{O}(\partial V)} \mathbf{n} \cdot \boldsymbol{\tau} \cdot \delta \mathbf{r} d\mathcal{O} = 0. \quad (9.1)$$

Here  $\delta W^{(i)}$  is work of internal forces per volume unit in the current configuration;  $\mathbf{f}$  is mass force, with dynamics  $(\mathbf{f} - \ddot{\mathbf{r}})$ ;  $\mathbf{p} = \mathbf{t}_{(n)} = \mathbf{n} \cdot \boldsymbol{\tau}$  is surface force.

Applying the divergence theorem to the surface integral, using\*

$$\nabla \cdot (\boldsymbol{\tau} \cdot \delta \mathbf{r}) = \nabla \cdot \boldsymbol{\tau} \cdot \delta \mathbf{r} + \boldsymbol{\tau} \cdot \nabla \delta \mathbf{r}^\top$$

and randomness of  $\mathcal{V}$ , we get the local differential version of (9.1)

$$(\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}) \cdot \delta \mathbf{r} + \boldsymbol{\tau} \cdot \nabla \delta \mathbf{r}^\top + \delta W^{(i)} = 0. \quad (9.2)$$

When a body virtually moves as a rigid whole, the work of internal forces nullifies

$$\begin{aligned} \delta \mathbf{r} &= \delta \mathbf{o} + \delta \mathbf{o} \times \mathbf{r} \Rightarrow \delta W^{(i)} = 0, \\ (\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}) \cdot (\delta \mathbf{o} + \delta \mathbf{o} \times \mathbf{r}) + \boldsymbol{\tau}^\top \cdot \nabla (\delta \mathbf{o} + \delta \mathbf{o} \times \mathbf{r}) &= 0, \\ \delta \mathbf{o} = \text{constant} \Rightarrow \nabla \delta \mathbf{o} &= \mathbf{0}, \quad \delta \mathbf{o} = \text{constant} \Rightarrow \nabla \delta \mathbf{o} = \mathbf{0}, \\ \nabla (\delta \mathbf{o} + \delta \mathbf{o} \times \mathbf{r}) &= \nabla (\delta \mathbf{o} \times \mathbf{r}) = \nabla \delta \mathbf{o} \times \mathbf{r} - \nabla \mathbf{r} \times \delta \mathbf{o} = \\ &= -\nabla \mathbf{r} \times \delta \mathbf{o} = -\mathbf{E} \times \delta \mathbf{o} \end{aligned}$$

...

$$\begin{aligned} * \mathbf{r}^i \cdot \partial_i (\boldsymbol{\tau} \cdot \delta \mathbf{r}) &= \mathbf{r}^i \cdot (\partial_i \boldsymbol{\tau}) \cdot \delta \mathbf{r} + \mathbf{r}^i \cdot \boldsymbol{\tau} \cdot \partial_i (\delta \mathbf{r}), \\ \mathbf{r}^i \cdot \boldsymbol{\tau} \cdot \partial_i (\delta \mathbf{r}) &= \boldsymbol{\tau} \cdot \partial_i (\delta \mathbf{r}) \mathbf{r}^i = \boldsymbol{\tau} \cdot (\mathbf{r}^i \partial_i \delta \mathbf{r})^\top \end{aligned}$$

Assuming  $\delta \mathbf{o} = \mathbf{0}$  (just translation)  $\Rightarrow \nabla \delta \mathbf{r} = \nabla \delta \rho = {}^2\mathbf{0}$ , it turns into the balance of forces (of momentum)

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f} = \mathbf{0}.$$

If  $\delta \mathbf{r} = \delta \mathbf{o} \times \mathbf{r}$  (just rotation) with  $\delta \mathbf{o} = \text{constant}$ , then

$$(14.10, \S 1.14) \Rightarrow \nabla \delta \mathbf{r} = \nabla \delta \mathbf{o} \times \mathbf{r} - \nabla \mathbf{r} \times \delta \mathbf{o} = -\mathbf{E} \times \delta \mathbf{o},$$

$$\nabla \delta \mathbf{r}^\top = \mathbf{E} \times \delta \mathbf{o}$$

With

$$(7.4, \S 1.7) \Rightarrow \boldsymbol{\tau}_\times = -\boldsymbol{\tau} \cdot \cdot {}^3\boldsymbol{\epsilon},$$

$$\boldsymbol{\tau} \cdot \cdot (\mathbf{E} \times \delta \mathbf{o}) = \boldsymbol{\tau} \cdot \cdot (-{}^3\boldsymbol{\epsilon} \cdot \delta \mathbf{o}) = (-\boldsymbol{\tau} \cdot \cdot {}^3\boldsymbol{\epsilon}) \cdot \delta \mathbf{o} = \boldsymbol{\tau}_\times \cdot \delta \mathbf{o}$$

...

In an elastic continuum, internal forces are potential

$$\delta W^{(i)} = -\rho \delta \tilde{\Pi}$$

...

$$\boldsymbol{\tau} \cdot \cdot \nabla \delta \mathbf{r}^S = -\delta W^{(i)} = \rho \delta \tilde{\Pi} \quad (9.3)$$

...

Вид потенциала  $\tilde{\Pi}$  per mass unit пока неизвестен, но очевидно что  $\tilde{\Pi}$  определяется деформацией.

With the balance of mass  $\rho J = \overset{\circ}{\rho} \Leftrightarrow \rho = J^{-1} \overset{\circ}{\rho}$  ( $J \equiv \det \mathbf{F}$  is the Jacobian, determinant of the motion gradient), потенциал на единицу объёма в недеформированной конфигурации  $\overset{\circ}{\Pi}$  имеет вид

$$\overset{\circ}{\Pi} \equiv \overset{\circ}{\rho} \tilde{\Pi} \Rightarrow \delta \overset{\circ}{\Pi} = \overset{\circ}{\rho} \delta \tilde{\Pi},$$

$$\rho \delta \tilde{\Pi} = J^{-1} \delta \overset{\circ}{\Pi}. \quad (9.4)$$

Полным аналогом (...) является равенство

...

## § 10. Constitutive relations of elasticity

Fundamental relation of elasticity (??)

...

$$\Pi(C) = \int_0^C \boldsymbol{\tau} \cdot \cdot dC$$

If the strain energy density is path independent, then it acts as a potential for stress, that is

$$\boldsymbol{\tau} = \frac{\partial \Pi(\boldsymbol{C})}{\partial \boldsymbol{C}}$$

For adiabatic processes,  $\Pi$  is equal to the change in internal energy per unit volume.

For isothermal processes,  $\Pi$  is equal to the Helmholtz free energy per unit volume.

The natural configuration of a body is defined as the configuration in which the body is in stable thermal equilibrium with no external loads and zero stress and strain.

When we apply energy methods in elasticity, we implicitly assume that a body returns to its natural configuration after loads are removed. This implies that the Gibbs' condition is satisfied:

$$\Pi(\boldsymbol{C}) \geq 0 \quad \text{with} \quad \Pi(\boldsymbol{C}) = 0 \quad \text{iff} \quad \boldsymbol{C} = \mathbf{0}$$

...

Начальная конфигурация считается естественной (natural configuration) — недеформированной ненапряжённой:  $\boldsymbol{C} = {}^2\mathbf{0} \Leftrightarrow \boldsymbol{\tau} = {}^2\mathbf{0}$ , поэтому в  $\Pi$  нет линейных членов.

Тензор жёсткости  ${}^4\mathbf{A}$

...

Rubber-like material (elastomer)

Материалу типа резины (эластомеру) характерны большие деформации, и функция  $\Pi(\text{I}, \text{II}, \text{III})$  для него бывает весьма сложной\*.

При больших деформациях исчезают преимущества использования  $\boldsymbol{u}$  и  $\boldsymbol{C}$  — проще остаться с вектором-радиусом  $\boldsymbol{r}$  ...

...

\* **Harold Alexander.** A constitutive relation for rubber-like materials // International Journal of Engineering Science, volume 6 (September 1968), pages 549–563.

## § 11. Piola–Kirchhoff stress tensors and other measures of stress

Соотношение Nanson’a  $\mathbf{n}d\mathcal{O} = J\mathring{\mathbf{n}}do \cdot \mathbf{F}^{-1}$  между векторами бесконечно малой площадки в начальной ( $\mathring{\mathbf{n}}do$ ) и в текущей ( $\mathbf{n}d\mathcal{O}$ ) конфигурациях\*

$$(5.2) \Rightarrow \mathbf{n}d\mathcal{O} \cdot \boldsymbol{\tau} = J\mathring{\mathbf{n}}do \cdot \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \Rightarrow \mathbf{n} \cdot \boldsymbol{\tau}d\mathcal{O} = \mathring{\mathbf{n}} \cdot J\mathbf{F}^{-1} \cdot \boldsymbol{\tau}do$$

gives the dual expression of a surface force

$$\mathbf{n} \cdot \boldsymbol{\tau}d\mathcal{O} = \mathring{\mathbf{n}} \cdot \mathbf{T}do, \quad \mathbf{T} \equiv J\mathbf{F}^{-1} \cdot \boldsymbol{\tau}. \quad (11.1)$$

Тензор  $\mathbf{T}$  называется первым (несимметричным) тензором напряжения Piola–Kirchhoff, иногда — “номинальным напряжением” (“nominal stress”) или “инженерным напряжением” (“engineering stress”). Бывает и когда какое-либо из этих (на)именований даётся транспонированному тензору

$$\mathbf{T}^\top = J\boldsymbol{\tau}^\top \cdot \mathbf{F}^{-\top} = J\boldsymbol{\tau} \cdot \mathbf{F}^{-\top}.$$

Обращение (11.1)

$$J^{-1}\mathbf{F} \cdot \mathbf{T} = J^{-1}\mathbf{F} \cdot J\mathbf{F}^{-1} \cdot \boldsymbol{\tau} \Rightarrow \boldsymbol{\tau} = J^{-1}\mathbf{F} \cdot \mathbf{T}$$

...

$$\delta\Pi = \mathbf{T} \cdot \delta\mathring{\nabla}\mathbf{r}^\top \Rightarrow \Pi = \Pi(\mathring{\nabla}\mathbf{r}) \quad (11.2)$$

— этот немного неожиданный результат получился благодаря коммутативности  $\delta$  и  $\mathring{\nabla}$ :  $\mathring{\nabla}\delta\mathbf{r}^\top = \delta\mathring{\nabla}\mathbf{r}^\top$  ( $\nabla$  and  $\delta$  don’t commute).

Тензор  $\mathbf{T}$  оказался энергетически сопряжённым с  $\mathbf{F} \equiv \mathring{\nabla}\mathbf{r}^\top$

$$\mathbf{T} = \frac{\partial\Pi}{\partial\mathring{\nabla}\mathbf{r}^\top} = \frac{\partial\Pi}{\partial\mathbf{F}}. \quad (11.3)$$

Второй (симметричный) тензор напряжения Piola–Kirchhoff  $\mathbf{S}$  энергетически сопряжён с  $\mathbf{G} \equiv \mathbf{F}^\top \cdot \mathbf{F}$  и  $\mathbf{C} \equiv \frac{1}{2}(\mathbf{G} - \mathbf{E})$

$$\begin{aligned} \delta\Pi(\mathbf{C}) = \mathbf{S} \cdot \delta\mathbf{C} &\Rightarrow \mathbf{S} = \frac{\partial\Pi}{\partial\mathbf{C}}, \\ d\mathbf{G} = 2d\mathbf{C} &\Rightarrow \delta\Pi(\mathbf{G}) = \frac{1}{2}\mathbf{S} \cdot \delta\mathbf{G}, \quad \mathbf{S} = 2\frac{\partial\Pi}{\partial\mathbf{G}}. \end{aligned} \quad (11.4)$$

\* Like before,  $\mathbf{F} = \frac{\partial\mathbf{r}}{\partial\mathbf{r}^i} = \mathbf{r}_{\partial i}\mathring{\mathbf{r}}^i = \mathring{\nabla}\mathbf{r}^\top$  is motion gradient,  $J \equiv \det \mathbf{F}$  is the Jacobian (the Jacobian determinant).

Связь между первым и вторым тензорами

$$\mathbf{S} = \mathbf{T} \cdot \mathbf{F}^{-\top} = \mathbf{F}^{-1} \cdot \mathbf{T}^{\top} \Leftrightarrow \mathbf{T} = \mathbf{S} \cdot \mathbf{F}^{\top}, \quad \mathbf{T}^{\top} = \mathbf{F} \cdot \mathbf{S}$$

и между тензором  $\mathbf{S}$  и тензором напряжения Cauchy  $\boldsymbol{\tau}$

$$\mathbf{S} = J \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{-\top} \Leftrightarrow J^{-1} \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^{\top} = \boldsymbol{\tau}.$$

...

$$\begin{aligned} \mathbf{T} &= \frac{\partial \Pi}{\partial \mathbf{C}} \cdot \mathbf{F}^{\top} = 2 \frac{\partial \Pi}{\partial \mathbf{G}} \cdot \mathbf{F}^{\top} \\ \delta \mathbf{S} &= \frac{\partial \mathbf{S}}{\partial \mathbf{C}} \cdot \delta \mathbf{C} = \frac{\partial^2 \Pi}{\partial \mathbf{C} \partial \mathbf{C}} \cdot \delta \mathbf{C} \\ \delta \mathbf{T} &= \delta \mathbf{S} \cdot \mathbf{F}^{\top} + \mathbf{S} \cdot \delta \mathbf{F}^{\top} \end{aligned}$$

...

The quantity  $\boldsymbol{\kappa} = J \boldsymbol{\tau}$  is called the *Kirchhoff stress tensor* and is used widely in numerical algorithms in metal plasticity (where there's no change in volume during plastic deformation). Another name for it is *weighted Cauchy stress tensor*.

...

Here's balance of forces (of momentum) with tensor  $\mathbf{T}$  for any undeformed volume  $\overset{\circ}{\mathcal{V}}$

$$\int_{\mathcal{V}} \rho \mathbf{f} d\mathcal{V} + \int_{\mathcal{O}(\partial \mathcal{V})} \mathbf{n} \cdot \boldsymbol{\tau} d\mathcal{O} = \int_{\overset{\circ}{\mathcal{V}}} \overset{\circ}{\rho} \mathbf{f} d\overset{\circ}{\mathcal{V}} + \int_{o(\partial \overset{\circ}{\mathcal{V}})} \overset{\circ}{\mathbf{n}} \cdot \mathbf{T} d\mathcal{O} = \int_{\overset{\circ}{\mathcal{V}}} \left( \overset{\circ}{\rho} \mathbf{f} + \overset{\circ}{\nabla} \cdot \mathbf{T} \right) d\overset{\circ}{\mathcal{V}} = \mathbf{0}$$

or in the local (differential) version

$$\overset{\circ}{\nabla} \cdot \mathbf{T} + \overset{\circ}{\rho} \mathbf{f} = \mathbf{0}. \quad (11.5)$$

Advantages of this equation in comparison with (7.2) are: here figures the known mass density  $\overset{\circ}{\rho}$  of undeformed volume  $\overset{\circ}{\mathcal{V}}$ , and the operator  $\overset{\circ}{\nabla} \equiv \overset{\circ}{\mathbf{r}}^i \partial_i$  is defined through known vectors  $\overset{\circ}{\mathbf{r}}^i$ . Appearance of  $\mathbf{T}$  **reflects** the specific property of an elastic solid body — “to remember” its initial configuration. Tensor  $\mathbf{T}$  is unlikely useful in fluid mechanics.

Principle of virtual work for an arbitrary volume  $\overset{\circ}{\mathcal{V}}$  of the elastic ( $\delta W^{(i)} = -\delta\Pi$ ) continuum:

$$\int_{\overset{\circ}{\mathcal{V}}} \left( \overset{\circ}{\rho} \mathbf{f} \cdot \delta \mathbf{r} - \delta\Pi \right) d\overset{\circ}{\mathcal{V}} + \int_{o(\partial\overset{\circ}{\mathcal{V}})} \overset{\circ}{\mathbf{n}} \cdot \mathbf{T} \cdot \delta \mathbf{r} \, do = 0,$$

$$\overset{\circ}{\nabla} \cdot (\mathbf{T} \cdot \delta \mathbf{r}) = \overset{\circ}{\nabla} \cdot \mathbf{T} \cdot \delta \mathbf{r} + \mathbf{T}^\top \cdot \overset{\circ}{\nabla} \delta \mathbf{r}, \quad \mathbf{T}^\top \cdot \overset{\circ}{\nabla} \delta \mathbf{r} = \mathbf{T} \cdot \overset{\circ}{\nabla} \delta \mathbf{r}^\top$$

$$\delta\Pi = \left( \overset{\circ}{\rho} \mathbf{f} + \overset{\circ}{\nabla} \cdot \mathbf{T} \right) \cdot \delta \mathbf{r} + \mathbf{T} \cdot \overset{\circ}{\nabla} \delta \mathbf{r}^\top$$

....

The first one is non-symmetric, it links forces in the deformed stressed configuration to the underformed geometry and mass (volumes, areas, densities as they were initially), and it is energetically conjugate to the motion gradient (often mistakenly called the “deformation gradient”, forgetting about rigid rotations). The first (or sometimes its transpose) is also known as “nominal stress” and “engineering stress”.

The second one is symmetric, it links loads in the initial undeformed configuration to the initial mass and geometry, and it is conjugate to the right Cauchy–Green deformation tensor (and thus to the Cauchy–Green–Venant measure of deformation).

The first is simpler when you use just the motion gradient and is more universal, but the second is simpler when you prefer right Cauchy–Green deformation and its offsprings.

There’s also popular Cauchy stress, which relates forces in the deformed configuration to the deformed geometry and mass.

“energetically conjugate” means that their product is kind of energy, here: elastic potential energy per unit of volume

.....

In the case of finite deformations, the Piola–Kirchhoff tensors  $\mathbf{T}$  and  $\mathbf{S}$  describe the stress relative to the initial configuration. In contrast with them, the Cauchy stress tensor  $\boldsymbol{\tau}$  describes the stress relative to the current configuration. For infinitesimal deformations, the Cauchy and Piola–Kirchhoff stress tensors are identical.

### *1st Piola–Kirchhoff stress tensor*

The 1st Piola–Kirchhoff stress tensor  $\mathbf{T}$  relates forces in the current (present, “spatial”) configuration with areas in the initial (“material”) configuration

$$\mathbf{T} = J \boldsymbol{\tau} \cdot \mathbf{F}^{-\top}$$

where  $\mathbf{F}$  is the motion gradient and  $J \equiv \det \mathbf{F}$  is the Jacobi determinant, Jacobian.

Because it relates different coordinate systems, the 1st Piola–Kirchhoff stress is a two-point tensor. Commonly, it’s not symmetric.

The 1st Piola–Kirchhoff stress is the 3D generalization of the 1D concept of engineering stress.

If the material rotates without a change in stress (rigid rotation), the components of the 1st Piola–Kirchhoff stress tensor will vary with material orientation.

The 1st Piola–Kirchhoff stress is energy conjugate to the motion gradient.

### *2nd Piola–Kirchhoff stress tensor*

The 2nd Piola–Kirchhoff stress tensor  $\mathbf{S}$  relates forces in the initial configuration to areas in the initial configuration. The force in the initial configuration is obtained via mapping that preserves the relative relationship between the force direction and the area normal in the initial configuration.

$$\mathbf{S} = J \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{-\top}$$

This tensor is a one-point tensor and it is symmetric.

If the material rotates without a change in stress (rigid rotation), the 2nd Piola–Kirchhoff stress tensor remain constant, irrespective of material orientation.

The 2nd Piola–Kirchhoff stress tensor is energy conjugate to the Green–Lagrange finite strain tensor.

...



## § 12. Variation of present configuration

Usually the two configurations of a nonlinear elastic medium are considered: the initial one with location vectors  $\overset{\circ}{\mathbf{r}}$  and the current (present) one with  $\mathbf{r}$ .

And the following equations describe a small change of the current configuration with infinitesimal changes to the location vector  $\delta\mathbf{r}$ , to the vector of mass forces  $\delta\mathbf{f}$ , to the first Piola–Kirchhoff stress tensor  $\delta\mathbf{T}$  and to the deformation tensor  $\delta\mathbf{C}$ .

By varying (11.5), (.....)\* and (.....), we get

$$\begin{aligned}\overset{\circ}{\rho}\delta\mathbf{f} + \overset{\circ}{\nabla} \cdot \delta\mathbf{T} &= \mathbf{0}, \quad \delta\mathbf{T} = \left( \frac{\partial^2 \Pi}{\partial \mathbf{C} \partial \mathbf{C}} \cdot \delta\mathbf{C} \right) \cdot \mathbf{F}^\top + \frac{\partial \Pi}{\partial \mathbf{C}} \cdot \delta\mathbf{F}^\top, \\ \delta\mathbf{F}^\top &= \delta\overset{\circ}{\nabla}\mathbf{r} = \overset{\circ}{\nabla}\delta\mathbf{r} = \mathbf{F}^\top \cdot \nabla\delta\mathbf{r}, \quad \delta\mathbf{F} = \delta\overset{\circ}{\nabla}\mathbf{r}^\top = \nabla\delta\mathbf{r}^\top \cdot \mathbf{F}, \\ \delta\mathbf{C} &= \mathbf{F}^\top \cdot \delta\boldsymbol{\varepsilon} \cdot \mathbf{F}, \quad \delta\boldsymbol{\varepsilon} \equiv \nabla\delta\mathbf{r}^S.\end{aligned}\tag{12.1}$$

...

$$\begin{aligned}(5.2) \Rightarrow \overset{\circ}{\mathbf{n}}d\mathcal{O} &= J^{-1}\mathbf{n}d\mathcal{O} \cdot \mathbf{F} \Rightarrow \overset{\circ}{\mathbf{n}} \cdot \delta\mathbf{T}d\mathcal{O} = J^{-1}\mathbf{n} \cdot \mathbf{F} \cdot \delta\mathbf{T}d\mathcal{O} \Rightarrow \\ &\Rightarrow \overset{\circ}{\mathbf{n}} \cdot \delta\mathbf{T}d\mathcal{O} = \mathbf{n} \cdot \delta\boldsymbol{\tau}d\mathcal{O}, \quad \delta\boldsymbol{\tau} \equiv J^{-1}\mathbf{F} \cdot \delta\mathbf{T}\end{aligned}$$

— введённый так tensor  $\delta\boldsymbol{\tau}$  связан с вариацией  $\delta\mathbf{T}$  как  $\boldsymbol{\tau}$  связан с  $\mathbf{T}$  ( $\boldsymbol{\tau} = J^{-1}\mathbf{F} \cdot \mathbf{T}$ ). Из (12.1) и ...

...

... корректируя коэффициенты линейной функции  $\delta\boldsymbol{\tau}(\nabla\delta\mathbf{r})$ .

## § 13. Internal constraints

До сих пор деформация считалась свободной, мера деформации  $\mathbf{C}$  могла быть любой. Однако, существуют материалы со значительным сопротивлением некоторым видам деформации. Резина, например, изменению формы сопротивляется намного меньше, чем изменению объёма — некоторые виды резины можно считать несжимаемым материалом.

Понятие геометрической связи, развитое в общей механике ...

$$\begin{aligned}^* \nabla &= \nabla \cdot \overset{\circ}{\nabla}\overset{\circ}{\mathbf{r}} = \overset{\circ}{r}^i \partial_i \cdot \overset{\circ}{r}^j \partial_j \overset{\circ}{r} \stackrel{?}{=} \overset{\circ}{r}^i \partial_i \overset{\circ}{r} \cdot \overset{\circ}{r}^j \partial_j = \overset{\circ}{\nabla}\overset{\circ}{r} \cdot \overset{\circ}{\nabla} = \mathbf{F}^\top \cdot \overset{\circ}{\nabla} \\ \overset{\circ}{\nabla} &= \overset{\circ}{\nabla} \cdot \nabla\mathbf{r} = \overset{\circ}{r}^i \partial_i \cdot \mathbf{r}^j \partial_j \mathbf{r} \stackrel{?}{=} \overset{\circ}{r}^i \partial_i \mathbf{r} \cdot \mathbf{r}^j \partial_j = \overset{\circ}{\nabla}\mathbf{r} \cdot \nabla = \mathbf{F}^\top \cdot \nabla\end{aligned}$$

...

for incompressible materials  $\Pi = \Pi(I, II)$

Mooney–Rivlin model of incompressible material

$$\Pi = c_1(I - 3) + c_2(II - 3)$$

incompressible Treloar (neo-Hookean) material

$$c_2 = 0 \Rightarrow \Pi = c_1(I - 3)$$

...

## § 14. Hollow sphere under pressure

Решение этой относительно простой задачи описано во многих книгах. В начальной (ненапряжённой) конфигурации имеем сферу с внутренним радиусом  $r_0$  и наружным  $r_1$ . Давление равно  $p_0$  внутри и  $p_1$  снаружи.

Введём удобную для этой задачи сферическую систему координат в отсчётной конфигурации  $q^1 = \theta$ ,  $q^2 = \phi$ ,  $q^3 = r$  (рис. ??). Эти же координаты будут и материальными. Имеем

...

## § 15. Stresses as Lagrange multipliers

Описанному ранее в § 9 использованию принципа виртуальной работы предшествовало введение тензора напряжения Cauchy через баланс сил для бесконечно малого тетраэдра (§ 6). Но теперь мы увидим, что сей принцип применим и без рассуждений с тетраэдром.

Рассмотрим тело — не только лишь упругое, с любой виртуальной работой внутренних сил  $\delta W^{(i)}$  per mass unit — нагруженное массовыми  $\mathbf{f} dm$  (для краткости пишем  $\mathbf{f}$  вместо  $\mathbf{f} - \ddot{\mathbf{r}}$ , так что динамика присутствует) и поверхностными  $\mathbf{p} d\mathcal{O}$  внешними силами. Имеем вариационное уравнение

$$\int_V \rho (\mathbf{f} \cdot \delta \mathbf{r} + \delta W^{(i)}) dV + \int_{\mathcal{O}(\partial V)} \mathbf{p} \cdot \delta \mathbf{r} d\mathcal{O} = 0. \quad (15.1)$$

Полагаем, что внутренние силы не совершают работу при виртуальном движении тела как целого — когда от бесконечно малых виртуальных перемещений  $\delta \mathbf{r}$  частиц тела нет деформации  $\delta \boldsymbol{\varepsilon}$

$$\delta \boldsymbol{\varepsilon} = \nabla \delta \mathbf{r}^S = \mathbf{0} \Rightarrow \delta W^{(i)} = 0. \quad (15.2)$$

Отбросив  $\delta W^{(i)}$  в (15.1) при условии (15.2), получим вариационное уравнение со связью. Приём с множителями Лагранжа даёт возможность считать вариации  $\delta \mathbf{r}$  независимыми. Поскольку в каждой точке связь представлена симметричным тензором второй сложности, то таким же тензором будут и множители Лагранжа  ${}^2\lambda$ . Приходим к уравнению

$$\int_V \left( \rho \mathbf{f} \cdot \delta \mathbf{r} - {}^2\lambda \cdot \nabla \delta \mathbf{r}^S \right) dV + \int_{\mathcal{O}(\partial V)} \mathbf{p} \cdot \delta \mathbf{r} d\mathcal{O} = 0. \quad (15.3)$$

Благодаря симметрии  ${}^2\lambda$  имеем\*

$$\begin{aligned} {}^2\lambda &= {}^2\lambda^\top \Rightarrow {}^2\lambda \cdot \nabla \delta \mathbf{r}^S = {}^2\lambda \cdot \nabla \delta \mathbf{r}^\top, \\ {}^2\lambda \cdot \nabla \delta \mathbf{r}^S &= \nabla \cdot ({}^2\lambda \cdot \delta \mathbf{r}) - \nabla \cdot {}^2\lambda \cdot \delta \mathbf{r}. \end{aligned}$$

\*  $\Lambda^S \cdot X = \Lambda^S \cdot X^\top = \Lambda^S \cdot X^S, \quad \nabla \cdot (B \cdot a) = (\nabla \cdot B) \cdot a + B^\top \cdot \nabla a$

Подставив это в (15.3) и применив теорему о дивергенции, получаем

$$\int_{\mathcal{V}} (\rho \mathbf{f} + \nabla \cdot {}^2\boldsymbol{\lambda}) \cdot \delta \mathbf{r} d\mathcal{V} + \int_{\mathcal{O}(\partial\mathcal{V})} (\mathbf{p} - \mathbf{n} \cdot {}^2\boldsymbol{\lambda}) \cdot \delta \mathbf{r} d\mathcal{O} = 0.$$

But  $\delta \mathbf{r}$  is random on a surface and inside a volume, thus

$$\mathbf{p} = \mathbf{n} \cdot {}^2\boldsymbol{\lambda}, \quad \nabla \cdot {}^2\boldsymbol{\lambda} + \rho \mathbf{f} = \mathbf{0}$$

— formally introduced symmetric multiplier  ${}^2\boldsymbol{\lambda}$  happened to be the Cauchy stress tensor.

Similar introduction of stresses was presented in book [50]. Here are no new results, но интересна сама возможность одновременного вывода тех уравнений механики сплошной среды, которые раньше считались независимыми. В следующих главах эта техника используется для построения новых континуальных моделей.

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# CLASSICAL LINEAR ELASTICITY

Geometrically linear model: displacements are small. Operators  $\overset{\circ}{\nabla}$  and  $\nabla$  are indistinguishable,  $\mathcal{V} = \overset{\circ}{\mathcal{V}}$ ,  $\rho = \overset{\circ}{\rho}$  — “equations can be written in the initial configuration”, operators  $\delta$  and  $\nabla$  commute ( $\delta \nabla \mathbf{u} = \nabla \delta \mathbf{u}$ ).

## §1. Complete set of equations

**E**quations of nonlinear elasticity, even in simplest cases, bring to mathematically complicated problems. Therefore the linear theory of small displacements is applied everywhere. Equations of this theory were derived in the first half of the XIX<sup>th</sup> century by Cauchy, Navier, Lamé, Clapeyron, Poisson, Saint-Venant, George Green and other scientists.

The complete closed set (system) of equations of the classical linear theory in the direct invariant tensor notation is

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}, \quad \boldsymbol{\sigma} = \frac{\partial \Pi}{\partial \boldsymbol{\varepsilon}} = {}^4\mathbf{A} \cdot \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \nabla \mathbf{u}^S. \quad (1.1)$$

Here  $\boldsymbol{\sigma}$  is linear stress tensor,  $\mathbf{f}$  is vector of volume loads,  $\boldsymbol{\varepsilon}$  is infinitesimal linear deformation tensor,  $\Pi(\boldsymbol{\varepsilon})$  is potential energy of deformation per volume unit,  ${}^4\mathbf{A}$  is stiffness tensor (tetravalent, with symmetry  ${}^4\mathbf{A}_{12\rightleftharpoons 34} = {}^4\mathbf{A}$ ,  ${}^4\mathbf{A}_{1\rightleftharpoons 2} = {}^4\mathbf{A}$ ,  ${}^4\mathbf{A}_{3\rightleftharpoons 4} = {}^4\mathbf{A}$ ).

Equations (1.1) are exact, they can be derived via variation of equations of the nonlinear theory. Variation from an arbitrary configuration is described in §3.12. The linear theory is the result of variation from the initial unstressed configuration, where

$$\begin{aligned} \mathbf{F} = \mathbf{E}, \quad \mathbf{C} = {}^2\mathbf{0}, \quad \delta \mathbf{C} = \nabla \delta \mathbf{r}^S \equiv \delta \boldsymbol{\varepsilon}, \\ \boldsymbol{\tau} = {}^2\mathbf{0}, \quad \delta \boldsymbol{\tau} = \delta \mathbf{T} = \frac{\partial^2 \Pi}{\partial \mathbf{C} \partial \mathbf{C}} \cdot \delta \mathbf{C}, \quad \nabla \cdot \delta \boldsymbol{\tau} + \rho \delta \mathbf{f} = \mathbf{0}. \end{aligned} \quad (1.2)$$

It remains to change  $\delta \mathbf{r}$  to  $\mathbf{u}$ ,  $\delta \boldsymbol{\varepsilon}$  to  $\boldsymbol{\varepsilon}$ ,  $\delta \boldsymbol{\tau}$  to  $\boldsymbol{\sigma}$ ,  $\partial^2 \Pi / \partial \mathbf{C} \partial \mathbf{C}$  to  ${}^4\mathbf{A}$ , and  $\rho \delta \mathbf{f}$  to  $\mathbf{f}$ .

If derivation (1.2) seems abstruse to the reader, it's possible to proceed from equations

$$\begin{aligned}\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f} &= \mathbf{0}, \quad \nabla = \mathbf{F}^{-\top} \cdot \overset{\circ}{\nabla}, \quad \mathbf{F} = \mathbf{E} + \overset{\circ}{\nabla} \mathbf{u}^\top, \\ \boldsymbol{\tau} &= J^{-1} \mathbf{F} \cdot \frac{\partial \Pi}{\partial \mathbf{C}} \cdot \mathbf{F}^\top, \quad \mathbf{C} = \overset{\circ}{\nabla} \mathbf{u}^S + \frac{1}{2} \overset{\circ}{\nabla} \mathbf{u} \cdot \overset{\circ}{\nabla} \mathbf{u}^\top.\end{aligned}\quad (1.3)$$

Assuming displacement  $\mathbf{u}$  is small, we'll move from (1.3) to (1.1).

Or so. Instead of  $\mathbf{u}$  take some small enough parameter  $\chi \mathbf{u}$ ,  $\chi \rightarrow 0$ . And to represent thereafter the unknowns by series in integer exponents of the parameter  $\chi$

$$\begin{aligned}\boldsymbol{\tau} &= \boldsymbol{\tau}^{(0)} + \chi \boldsymbol{\tau}^{(1)} + \dots, \quad \mathbf{C} = \mathbf{C}^{(0)} + \chi \mathbf{C}^{(1)} + \dots, \\ \nabla &= \overset{\circ}{\nabla} + \chi \nabla^{(1)} + \dots, \quad \mathbf{F} = \mathbf{E} + \chi \overset{\circ}{\nabla} \mathbf{u}^\top, \quad J = 1 + \chi J^{(1)} + \dots\end{aligned}$$

The complete set of equations (1.1) comes from the first (zeroth) terms of these series. In book [58] this is called the “formal approximation”.

It is impossible to tell unambiguously how small the parameter  $\chi$  should be — the answer depends on the situation and is determined by whether the linear model describes the effect of interest or not. When, as example, I'm interested in a relation between the frequency of a freely vibrating motion after initial offset with the amplitude of vibrations, then a nonlinear model is needed.

A linear problem is posed in the initial volume  $\mathcal{V} = \overset{\circ}{\mathcal{V}}$ , bounded by surface  $o$  with area vector  $\mathbf{n} do$  (“the principle of initial dimensions”).

Boundary conditions most often are: on part  $o_1$  of the surface displacements are known (geometrical/kinematical condition), and on another part  $o_2$  forces are known (mechanical condition)

$$\mathbf{u}|_{o_1} = \mathbf{u}_0, \quad \mathbf{n} \cdot \boldsymbol{\sigma}|_{o_2} = \mathbf{p}. \quad (1.4)$$

More complex combinations happen too, if we know certain components of both  $\mathbf{u}$  and  $\mathbf{t}_{(n)} = \mathbf{n} \cdot \boldsymbol{\sigma}$  simultaneously. For example, on a flat face  $x = \text{constant}$  when pressing a stamp with a smooth surface  $u_x = \nu(y, z)$ ,  $\tau_{xy} = \tau_{xz} = 0$  (function  $\nu$  is determined by the stamp's shape).

Initial conditions for dynamic problems, when instead of  $\mathbf{f}$  we have  $\mathbf{f} - \rho \ddot{\mathbf{u}}$ , are set as is common in mechanics — on positions and on velocities: at the given moment of time  $t=0$   $\mathbf{u}$  and  $\dot{\mathbf{u}}$  are known.

Due to the linearity of problems the principle of superposition (or independence) of the action of loads is fulfilled. When there are several loads, the problem can be solved for each load separately, and the complete solution is obtained by summing. For statics this means, for example, the following: if external loads  $\mathbf{f}$  and  $\mathbf{p}$  increase by  $m$  times (body is fixed on  $o_1$ ), then  $\mathbf{u}$ ,  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\sigma}$  will increase by  $m$  times too. Potential energy density  $\Pi$  will increase by  $m^2$  times. In reality such is observed only when loads are small.

Density of the potential energy of elastic deformation  $\Pi$

$$\Pi(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon} \cdot \cdot \mathbf{A} \cdot \cdot \boldsymbol{\varepsilon}$$

and its variation

$$\begin{aligned} \delta\Pi &= \frac{1}{2} \delta(\boldsymbol{\varepsilon} \cdot \cdot \mathbf{A} \cdot \cdot \boldsymbol{\varepsilon}) = \frac{1}{2} (\delta\boldsymbol{\varepsilon} \cdot \cdot \mathbf{A} \cdot \cdot \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot \cdot \mathbf{A} \cdot \cdot \delta\boldsymbol{\varepsilon}) = \underbrace{\boldsymbol{\varepsilon} \cdot \cdot \mathbf{A} \cdot \cdot \delta\boldsymbol{\varepsilon}}_{\boldsymbol{\sigma}} \\ \delta\Pi(\boldsymbol{\varepsilon}) &= \frac{\partial\Pi}{\partial\boldsymbol{\varepsilon}} \cdot \cdot \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot \cdot \delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot \cdot \mathbf{A} \cdot \cdot \delta\boldsymbol{\varepsilon} \\ \delta^2\Pi(\boldsymbol{\varepsilon}) &= \delta\boldsymbol{\varepsilon} \cdot \cdot \frac{\partial^2\Pi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}} \cdot \cdot \delta\boldsymbol{\varepsilon} = \delta\boldsymbol{\varepsilon} \cdot \cdot \mathbf{A} \cdot \cdot \delta\boldsymbol{\varepsilon} = 2\Pi(\delta\boldsymbol{\varepsilon}) \end{aligned}$$

As was noted in [chapter 2](#), the principle of virtual work (d'Alembert–Lagrange principle) can be put into the basis of mechanics. This principle is true for the linear theory too (internal forces in an elastic medium are potential:  $\delta W^{(i)} = -\delta\Pi$ )

$$\int_{\mathcal{V}} [(\mathbf{f} - \rho\ddot{\mathbf{u}}) \cdot \delta\mathbf{u} - \delta\Pi] d\mathcal{V} + \int_{o_2} \mathbf{p} \cdot \delta\mathbf{u} do = 0, \quad \mathbf{u}|_{o_1} = \mathbf{0}, \quad (1.5)$$

because

$$\begin{aligned} \delta\Pi &= \boldsymbol{\sigma} \cdot \cdot \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot \cdot \nabla\delta\mathbf{u}^S = \nabla \cdot (\boldsymbol{\sigma} \cdot \delta\mathbf{u}) - \nabla \cdot \boldsymbol{\sigma} \cdot \delta\mathbf{u}, \\ \int_{\mathcal{V}} \delta\Pi d\mathcal{V} &= \oint_{o(\partial\mathcal{V})} \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \delta\mathbf{u} do - \int_{\mathcal{V}} \nabla \cdot \boldsymbol{\sigma} \cdot \delta\mathbf{u} d\mathcal{V} \end{aligned}$$

and the left part of (1.5) becomes

$$\int_{\mathcal{V}} (\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} - \rho\ddot{\mathbf{u}}) \cdot \delta\mathbf{u} d\mathcal{V} + \int_{o_2} (\mathbf{p} - \mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \delta\mathbf{u} do,$$

which, of course, is equal to zero. Notice the boundary condition  $\mathbf{u}|_{o_1} = \mathbf{0}$ : virtual displacements are compatible with this constraint —  $\delta\mathbf{u}|_{o_1} = \mathbf{0}$ .

## § 2. Uniqueness of the solution in dynamics

As is typical for linear mathematical physics, the uniqueness theorem is proven “by contradiction”. Assume there are some two solutions:  $\mathbf{u}_1(\mathbf{r}, t)$  and  $\mathbf{u}_2(\mathbf{r}, t)$ . If difference  $\mathbf{u}^* \equiv \mathbf{u}_1 - \mathbf{u}_2$  will be equal to  $\mathbf{0}$ , then these solutions coincide, that is the solution is unique.

But first we'll make sure the existence of the energy integral — by deriving the balance of energy equation for the linear model of the small displacement theory

$$\begin{aligned} \int_{\mathcal{V}} (\mathbf{K} + \Pi)^\bullet d\mathcal{V} &= \int_{\mathcal{V}} \mathbf{f} \cdot \dot{\mathbf{u}} d\mathcal{V} + \int_{o_2} \mathbf{p} \cdot \dot{\mathbf{u}} d\sigma, \\ \mathbf{u}|_{o_1} &= \mathbf{0}, \quad \mathbf{n} \cdot \boldsymbol{\sigma}|_{o_2} = \mathbf{p}, \\ \mathbf{u}|_{t=0} &= \mathbf{u}^o, \quad \dot{\mathbf{u}}|_{t=0} = \dot{\mathbf{u}}^o. \end{aligned} \quad (2.1)$$

For the left-hand side we have

$$\begin{aligned} \dot{\mathbf{K}} &= \frac{1}{2} (\rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}})^\bullet = \frac{1}{2} \rho (\dot{\mathbf{u}} \cdot \ddot{\mathbf{u}} + \ddot{\mathbf{u}} \cdot \dot{\mathbf{u}}) = \rho \ddot{\mathbf{u}} \cdot \dot{\mathbf{u}}, \\ \dot{\Pi} &= \frac{1}{2} \underbrace{(\boldsymbol{\varepsilon} \cdot \cdot \cdot \mathcal{A} \cdot \cdot \cdot \boldsymbol{\varepsilon})^\bullet}_{2\boldsymbol{\varepsilon} \cdot \cdot \cdot \mathcal{A} \cdot \cdot \cdot \dot{\boldsymbol{\varepsilon}}} = \boldsymbol{\sigma} \cdot \cdot \cdot \dot{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma} \cdot \cdot \cdot \nabla \dot{\mathbf{u}}^S = \nabla \cdot (\boldsymbol{\sigma} \cdot \dot{\mathbf{u}}) - \underbrace{\nabla \cdot \boldsymbol{\sigma} \cdot \cdot \cdot \dot{\mathbf{u}}}_{-(\mathbf{f} - \rho \ddot{\mathbf{u}})} = \\ &= \nabla \cdot (\boldsymbol{\sigma} \cdot \dot{\mathbf{u}}) + (\mathbf{f} - \rho \ddot{\mathbf{u}}) \cdot \dot{\mathbf{u}} \end{aligned}$$

(balance of momentum  $\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} - \rho \ddot{\mathbf{u}} = \mathbf{0}$  is used),

$$\dot{\mathbf{K}} + \dot{\Pi} = \nabla \cdot (\boldsymbol{\sigma} \cdot \dot{\mathbf{u}}) + \mathbf{f} \cdot \dot{\mathbf{u}}.$$

Applying the divergence theorem

$$\int_{\mathcal{V}} \nabla \cdot (\boldsymbol{\sigma} \cdot \dot{\mathbf{u}}) d\mathcal{V} = \oint_{o(\partial\mathcal{V})} \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \dot{\mathbf{u}} d\sigma$$

and boundary condition  $\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{p}$  on  $o_2$ , we get (2.1).

From (2.1) it follows that without loads (when there're no external forces, neither volume nor surface), and the full mechanical energy doesn't change:

$$\mathbf{f} = \mathbf{0} \text{ and } \mathbf{p} = \mathbf{0} \Rightarrow \int_{\mathcal{V}} (\mathbf{K} + \Pi) d\mathcal{V} = \text{constant}(t). \quad (2.2)$$

If at the moment  $t=0$  there was unstressed ( $\Pi=0$ ) rest ( $\mathbf{K}=0$ ), then

$$\int_{\mathcal{V}} (\mathbf{K} + \Pi) d\mathcal{V} = 0. \quad (2.2')$$

The kinetic energy is positive:  $\mathbf{K} > 0$  if  $\dot{\mathbf{u}} \neq \mathbf{0}$  and nullifies only when  $\dot{\mathbf{u}} = \mathbf{0}$  — this ensues from its definition  $\mathbf{K} \equiv \frac{1}{2} \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}$ . The



potential energy, being a quadratic form  $\Pi(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \boldsymbol{\varepsilon}$ , is positive too:  $\Pi > 0$  if  $\boldsymbol{\varepsilon} \neq {}^2\mathbf{0}$ . Such is a priori requirement of the positive definiteness for stiffness tensor  ${}^4\mathcal{A}$ . This is one of “additional inequalities in the theory of elasticity” [22, 58].

Since  $K$  and  $\Pi$  are positive-definite, (2.2') gives

$K = 0, \Pi = 0 \Rightarrow \dot{\mathbf{u}} = \mathbf{0}, \boldsymbol{\varepsilon} = \nabla \mathbf{u}^S = {}^2\mathbf{0} \Rightarrow \mathbf{u} = \mathbf{u}^\circ + \boldsymbol{\omega}^\circ \times \mathbf{r}$  ( $\mathbf{u}^\circ$  and  $\boldsymbol{\omega}^\circ$  are some constants of translation and rotation). With an immobile part of the surface

$$\mathbf{u}|_{o_1} = \mathbf{0} \Rightarrow \mathbf{u}^\circ = \mathbf{0} \text{ and } \boldsymbol{\omega}^\circ = \mathbf{0} \Rightarrow \mathbf{u} = \mathbf{0} \text{ everywhere.}$$

Now remember two solutions  $\mathbf{u}_1$  and  $\mathbf{u}_2$ . Their difference  $\mathbf{u}^* \equiv \mathbf{u}_1 - \mathbf{u}_2$  is a solution of an entirely “homogeneous” (with no constant terms at all) linear problem: in a volume  $\mathbf{f} = \mathbf{0}$ , in boundary and in initial conditions — zeroes. Therefore  $\mathbf{u}^* = \mathbf{0}$ , and the uniqueness is proven.

As for the existence of a solution — it cannot be proven in the generic case by simple conclusions. I could only tell that a dynamic problem is evolutionary, it describes the progress of a process in time.

The balance (conservation) of momentum gives acceleration  $\ddot{\mathbf{u}}$ . Then, moving to the “next time layer”  $t + dt$ :

$$\begin{aligned} \dot{\mathbf{u}}(\mathbf{r}, t + dt) &= \dot{\mathbf{u}}(\mathbf{r}, t) + \ddot{\mathbf{u}} dt, \\ \mathbf{u}(\mathbf{r}, t + dt) &= \mathbf{u}(\mathbf{r}, t) + \dot{\mathbf{u}} dt, \\ \boldsymbol{\varepsilon}(\mathbf{r}, t + dt) &= (\nabla \mathbf{u}(\mathbf{r}, t + dt))^S \Rightarrow \boldsymbol{\sigma}, \\ \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} &= \rho \ddot{\mathbf{u}}(\mathbf{r}, t + dt) \end{aligned}$$

and so forth.

Surely, these considerations lack the mathematical scrupulosity, inherent, for example, for the Philippe Ciarlet’s monograph [53].

$$\sigma = \frac{\partial \Pi}{\partial \varepsilon} = {}^4\mathcal{A} \cdots \varepsilon = \varepsilon \cdots {}^4\mathcal{A}$$

### § 3. Hooke's law

That relation between stress and deformation (strain), which in the XVII<sup>th</sup> century Robert Hooke could only phrase pretty vaguely\*, is written in the direct invariant notation in (1.1) and is carried out by the stiffness tensor

$${}^4\mathcal{A} = \frac{\partial^2 \Pi}{\partial \varepsilon \partial \varepsilon} = A^{ijkl} \mathbf{r}_{\partial i} \mathbf{r}_{\partial j} \mathbf{r}_{\partial k} \mathbf{r}_{\partial l}, \quad A^{ijkl} = \frac{\partial^2 \Pi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}. \quad (3.1)$$

The stiffness tensor, as the partial derivative of the scalar elastic potential energy density  $\Pi$  twice by the same bivalent tensor  $\varepsilon$ , is symmetric in pairs of indices:  ${}^4\mathcal{A}_{12\equiv 34} = {}^4\mathcal{A} \Leftrightarrow A^{ijkl} = A^{klij}$ . Therefrom 36 components out of  $3^4 = 81$  “have a twin” and only 45 are independent. Furthermore, due to the symmetry of infinitesimal deformation tensor  $\varepsilon$ , tensor  ${}^4\mathcal{A}$  is symmetric inside each pair

\* “*ceiinnosssttuu, id est, Ut tensio sic vis*”—**Robert Hooke**. Lectures de Potentia Restitutiva, Or of Spring Explaining the Power of Springing Bodies. London, 1678. 56 pages.

of indices too:  $A^{ijkl} = A^{jikl} = A^{ijlk} (= A^{jilk})$ . This reduces the number of independent components to 21:

$$\begin{aligned}
 A^{abcd} &= A^{cdab} = A^{bacd} = A^{abdc} \\
 A^{1111} \\
 A^{1112} &= A^{1121} = A^{1211} = A^{2111} \\
 A^{1113} &= A^{1131} = A^{1311} = A^{3111} \\
 A^{1122} &= A^{2211} \\
 A^{1123} &= A^{1132} = A^{2311} = A^{3211} \\
 A^{1133} &= A^{3311} \\
 A^{1212} &= A^{1221} = A^{2112} = A^{2121} \\
 A^{1213} &= A^{1231} = A^{1312} = A^{1321} = A^{2113} = A^{2131} = A^{3112} = A^{3121} \\
 A^{1222} &= A^{2122} = A^{2212} = A^{2221} \\
 A^{1223} &= A^{1232} = A^{2123} = A^{2132} = A^{2312} = A^{2321} = A^{3212} = A^{3221} \\
 A^{1233} &= A^{2133} = A^{3312} = A^{3321} \\
 A^{1313} &= A^{1331} = A^{3113} = A^{3131} \\
 A^{1322} &= A^{2213} = A^{2231} = A^{3122} \\
 A^{1323} &= A^{1332} = A^{2313} = A^{2331} = A^{3123} = A^{3132} = A^{3213} = A^{3231} \\
 A^{1333} &= A^{3133} = A^{3313} = A^{3331} \\
 A^{2222} \\
 A^{2223} &= A^{2232} = A^{2322} = A^{3222} \\
 A^{2233} &= A^{3322} \\
 A^{2323} &= A^{2332} = A^{3223} = A^{3232} \\
 A^{2333} &= A^{3233} = A^{3323} = A^{3332} \\
 A^{3333}
 \end{aligned}$$

Components of the stiffness tensor are often written as a symmetric  $6 \times 6$  matrix

$$[\mathcal{A}]_{6 \times 6} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix} \equiv \begin{bmatrix} A^{1111} & A^{1122} & A^{1133} & A^{1112} & A^{1113} & A^{1123} \\ A^{2211} & A^{2222} & A^{2233} & A^{1222} & A^{1322} & A^{2223} \\ A^{3311} & A^{3322} & A^{3333} & A^{1233} & A^{1333} & A^{2333} \\ A^{1211} & A^{2212} & A^{3312} & A^{1212} & A^{1213} & A^{1223} \\ A^{1311} & A^{2213} & A^{3313} & A^{1312} & A^{1313} & A^{1323} \\ A^{2311} & A^{2322} & A^{3323} & A^{2312} & A^{2313} & A^{2323} \end{bmatrix}$$

Даже в декартовых координатах  $x, y, z$  квадратичная форма упругой энергии  $\Pi(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \boldsymbol{\varepsilon}$  довольно-таки громоздкая:

$$\begin{aligned}
 2\Pi &= a_1 \varepsilon_x^2 + a_2 \varepsilon_y^2 + a_3 \varepsilon_z^2 + a_4 \varepsilon_{xy}^2 + a_5 \varepsilon_{xz}^2 + a_6 \varepsilon_{yz}^2 + \\
 &+ 2 \left[ \varepsilon_x (a_{12} \varepsilon_y + a_{13} \varepsilon_z + a_{14} \varepsilon_{xy} + a_{15} \varepsilon_{xz} + a_{16} \varepsilon_{yz}) + \right. \\
 &\quad + \varepsilon_y (a_{23} \varepsilon_z + a_{24} \varepsilon_{xy} + a_{25} \varepsilon_{xz} + a_{26} \varepsilon_{yz}) + \\
 &\quad + \varepsilon_z (a_{34} \varepsilon_{xy} + a_{35} \varepsilon_{xz} + a_{36} \varepsilon_{yz}) + \\
 &\quad \left. + \varepsilon_{xy} (a_{45} \varepsilon_{xz} + a_{46} \varepsilon_{yz}) + a_{56} \varepsilon_{xz} \varepsilon_{yz} \right]. \tag{3.2}
 \end{aligned}$$

Когда добавляется материальная симметрия, число независимых компонент тензора <sup>4</sup>**A** ещё уменьшается.

Пусть материал имеет плоскость симметрии упругих свойств  $z = \text{constant}$ . Тогда энергия  $\Pi$  не меняется при перемене знаков у  $\varepsilon_{zx}$  и  $\varepsilon_{yz}$ . А это возможно лишь если

$$\Pi \Big|_{\substack{\varepsilon_{xz} = -\varepsilon_{xz} \\ \varepsilon_{yz} = -\varepsilon_{yz}}} = \Pi \Leftrightarrow \begin{aligned} 0 = a_{15} = a_{16} = a_{25} = a_{26} = \\ = a_{35} = a_{36} = a_{45} = a_{46} \end{aligned} \quad (3.3)$$

— число независимых констант падает до 13.

Пусть далее плоскостей симметрии две:  $z = \text{constant}$  и  $y = \text{constant}$ . Поскольку  $\Pi$  в таком случае не чувствительна к знакам  $\varepsilon_{yx}$  и  $\varepsilon_{yz}$ , вдобавок к (3.3) имеем

$$a_{14} = a_{24} = a_{34} = a_{56} = 0 \quad (3.4)$$

— осталось 9 констант.

Ортотропным (ортогонально анизотропным) называется материал с тремя ортогональными плоскостями симметрии — пусть это координатные плоскости  $x, y, z$ . Легко увидеть, что (3.3) и (3.4) — это весь набор нулевых констант и в этом случае. Итак, ортотропный материал характеризуется девятью константами, и “для ортотропности” достаточно двух перпендикулярных плоскостей симметрии. Вид упругой энергии упрощается до

$$\begin{aligned} \Pi = \frac{1}{2}a_1\varepsilon_x^2 + \frac{1}{2}a_2\varepsilon_y^2 + \frac{1}{2}a_3\varepsilon_z^2 + \frac{1}{2}a_4\varepsilon_{xy}^2 + \frac{1}{2}a_5\varepsilon_{xz}^2 + \frac{1}{2}a_6\varepsilon_{yz}^2 + \\ + a_{12}\varepsilon_x\varepsilon_y + a_{13}\varepsilon_x\varepsilon_z + a_{23}\varepsilon_y\varepsilon_z. \end{aligned}$$

В ортотропном материале сдвиговые (угловые) деформации  $\varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}$  никак не влияют на нормальные напряжения  $\sigma_x = \partial\Pi/\partial\varepsilon_x$ ,  $\sigma_y = \partial\Pi/\partial\varepsilon_y$ ,  $\sigma_z = \partial\Pi/\partial\varepsilon_z$  (и наоборот). Популярный ортотропный материал — древесина; её упругие свойства различны по трём взаимно перпендикулярным направлениям: по радиусу, вдоль окружности и по высоте ствола.

Ещё один случай анизотропии — трансверсально изотропный (transversely isotropic) материал. Он характеризуется

...

æolotropic (anisotropic)

...

$$2\Pi = \boldsymbol{\varepsilon} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \boldsymbol{\varepsilon}, \quad \boldsymbol{\sigma} = \frac{\partial \Pi}{\partial \boldsymbol{\varepsilon}} = {}^4\mathcal{A} \cdot \cdot \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot \cdot {}^4\mathcal{A} \Rightarrow$$

$$2\Pi = \boldsymbol{\sigma} \cdot \cdot \boldsymbol{\varepsilon} \quad (3.5)$$

...

преобразование Лежандра

Legendre (involution) transform(ation)

плотность потенциальной энергии внутренних сил (напряжений)

complementary energy

$$\boldsymbol{\varepsilon} = \frac{\partial \hat{\Pi}}{\partial \boldsymbol{\sigma}} = {}^4\mathcal{B} \cdot \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \cdot {}^4\mathcal{B},$$

$$\hat{\Pi}(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot \cdot \boldsymbol{\varepsilon} - \Pi(\boldsymbol{\varepsilon}) \quad (3.6)$$

...

В линейной теории дополнительная энергия численно равна энергии деформации (“упругому потенциалу”).

## § 4. Theorems of statics

### *Clapeyron's theorem*

In equilibrium with external forces, volume ones  $\mathbf{f}$  and surface ones  $\mathbf{p}$ , the work of these forces (“statically frozen” — constant along time) through actual displacements is equal to the doubled\* energy of deformation

$$2 \int_V \Pi dV = \int_V \mathbf{f} \cdot \mathbf{u} dV + \int_{o_2} \mathbf{p} \cdot \mathbf{u} do. \quad (4.1)$$

\*“Ce produit représentait d’ailleurs le double de la force vive que le ressort pouvait absorber par l’effet de sa flexion et qui était la mesure naturelle de sa puissance.” —

**Benoît Paul Émile Clapeyron.** Mémoire sur le travail des forces élastiques dans un corps solide élastique déformé par l’action de forces extérieures. *Comptes rendus*, Tome XLVI, Janvier–Juin 1858. Page 208–212.

$$\begin{aligned}
\bigcirc \quad 2\Pi &= \boldsymbol{\sigma} \cdots \boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdots \nabla \mathbf{u}^S = \nabla \cdot (\boldsymbol{\sigma} \cdot \mathbf{u}) - \underbrace{\nabla \cdot \boldsymbol{\sigma} \cdot \mathbf{u}}_{-f} \Rightarrow \\
&\Rightarrow 2 \int_V \Pi dV = \int_{o_2} \underbrace{\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{u}}_p d\mathbf{o} + \int_V \mathbf{f} \cdot \mathbf{u} dV \quad \bullet
\end{aligned}$$

Из (4.1) следует также, что без нагрузки  $\int_V \Pi dV = 0$ . Поскольку  $\Pi$  положительна, то и напряжение  $\boldsymbol{\sigma}$ , и деформация  $\boldsymbol{\varepsilon}$  без нагрузки — нулевые.

$$\begin{aligned}
2\Pi &= \boldsymbol{\sigma} \cdots \boldsymbol{\varepsilon} \\
\dot{\Pi} &= \boldsymbol{\sigma} \cdots \dot{\boldsymbol{\varepsilon}} \\
\delta\Pi &= \boldsymbol{\sigma} \cdots \delta\boldsymbol{\varepsilon}
\end{aligned}$$

**Paradox of elastostatics:**  $\Pi$  is equal to only the half of the work of external forces.

The accumulated potential energy of deformation  $\Pi$  is equal to only the half of the work done by external forces, acting through displacements from the unstressed configuration to the equilibrium.

Clapeyron's theorem implies that the accumulated elastic energy accounts for only the half of energy spent on deformation. The remaining half of the work done by external forces is lost somewhere before reaching the equilibrium.

This apparent paradox is reached within the framework of purely conservative linear elasticity. Alternatively, however, within elastostatics the common characterization of the work done to reach equilibrium is conceptually ambiguous, and a novel interpretation may be needed.

**Roger Fosdick and Lev Truskinovsky.** About Clapeyron's Theorem in Linear Elasticity. *Journal of Elasticity*, Volume 72, July 2003. Pages 145–172.

There is always heating due to energy dissipation.

Для решения парадокса в теории распространена концепция of “static loading” — infinitely slow gradual application of the load.

Статика рассматривает “замороженное” равновесие вне времени. Динамика нагружения до равновесия — предыстория. In the linear small displacement theory в равновесии затраченная на деформацию работа внешних сил на актуальных перемещениях равна удвоенной потенциальной энергии деформации. “Запасается” всего половина потраченной энергии. Вторая половина есть дополнительная энергия, она теряется до обретения равновесия на динамику — на внутреннюю энергию

частиц (“диссипацию”), на колебания и волны. Так в теории. Однако, в реальности не бывает ни моментального “мёртвого” нагружения, ни бесконечно медленного “следящего”. Это две крайности. Реальная динамика нагружения всегда где-то между ними.

В области же бесконечно малых вариаций и виртуальных работ, работа реальных внешних сил на виртуальных перемещениях точно равна вариации упругого потенциала. А упругая среда есть такая, в которой вариация работы сил внутренних (напряжений) на виртуальных деформациях это минус вариация потенциала.

$-\delta W^{(i)} = \delta \Pi = \delta W^{(e)}$ , когда варьируются только перемещения (нагрузки не варьируются). Потому в принципе виртуальной работы и варьируются лишь перемещения, чтобы виртуальная работа внешних неварьируемых реальных сил на вариациях перемещений была равна минус вариации внутренней энергии (в случае упругой среды — вариации упругого потенциала).

### *Uniqueness of solution theorem*

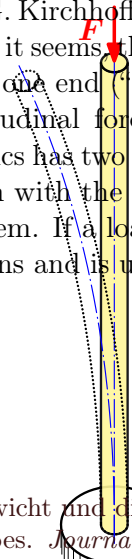
Как и в динамике (§ 2), допускаем существование двух решений и ищем их разность

...  
The uniqueness of solution, discovered by G. Kirchhoff for bodies with simply connected contour\*, is contrary to, as it seems, the everyday experience. Imagine a straight rod, clamped at one end (“cantilever”) and compressed at second end with a longitudinal force (fig. 13). When a load is large enough, the problem of statics has two solutions — “straight” and “bent”. But such a contradiction with the uniqueness theorem comes from nonlinearity of this problem. If a load is small, then the solution is described by linear equations and is unique.

...  
...  
...  
...

\* **Gustav Robert Kirchhoff.** Über das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes. *Journal für die reine und angewandte Mathematik (Crelle’s journal)*, 36. Band (1859). Seiten 285–313. (Seite 291)

figure 13



Proposed by Enrico Betti\*.

For a body with a fixed part  $o_1$  of the surface two options are considered: the first with loads  $\mathbf{f}_1$ ,  $\mathbf{p}_1$  and the second with loads  $\mathbf{f}_2$ ,  $\mathbf{p}_2$ . The verbal formulation of the theorem is the same as in § 2.6. Mathematical notation

$$\overbrace{\int_{\mathcal{V}} \mathbf{f}_1 \cdot \mathbf{u}_2 d\mathcal{V} + \int_{o_2} \mathbf{p}_1 \cdot \mathbf{u}_2 d\mathbf{o}}^{W_{12}} = \overbrace{\int_{\mathcal{V}} \mathbf{f}_2 \cdot \mathbf{u}_1 d\mathcal{V} + \int_{o_2} \mathbf{p}_2 \cdot \mathbf{u}_1 d\mathbf{o}}^{W_{21}}. \quad (4.2)$$

...

Reciprocal work theorem, also known as Betti's theorem, claims that for a linear elastic structure subject to two sets of forces  $P$  and  $Q$ , the work done by set  $P$  through displacements produced by set  $Q$  is equal to the work done by set  $Q$  through displacements produced by set  $P$ . This theorem has applications in structural engineering where it is used to define influence lines and derive the boundary element method.

...

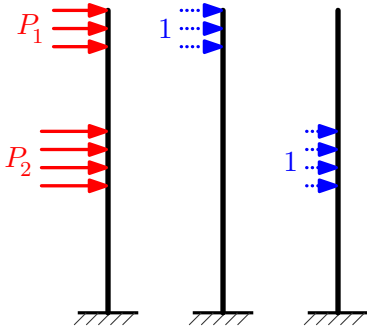


figure 14

The reciprocal work theorem finds unexpected and effective applications. For example, consider a rod-beam clamped at one end ("cantilever") and bent by two forces with integral values  $P_1$  and  $P_2$  (fig. 14). While the linear theory is applied, displacements-deflections can be represented as

$$\begin{aligned} u_1 &= \alpha_{11}P_1 + \alpha_{12}P_2, \\ u_2 &= \alpha_{21}P_1 + \alpha_{22}P_2. \end{aligned}$$

...

\* **Enrico Betti.** Teoria della elasticità. *Il Nuovo Cimento* (1869–1876), VII e VIII (1872). Pagina 69.



## § 5. Equations in displacements

The complete set of equations (1.1) contains unknowns  $\sigma$ ,  $\varepsilon$  and  $u$ . Excluding  $\sigma$  and  $\varepsilon$ , we come to the formulation in displacements (symmetrization of  $\nabla u$  is redundant here, because  ${}^4\mathcal{A}_{3\rightleftharpoons 4} = {}^4\mathcal{A}$ )

$$\begin{aligned} \nabla \cdot ({}^4\mathcal{A} \cdot \nabla u) + f &= 0, \\ u|_{o_1} &= u_0, \quad n \cdot {}^4\mathcal{A} \cdot \nabla u|_{o_2} = p. \end{aligned} \quad (5.1)$$

In an isotropic medium (5.1) takes the form

...

Общее решение однородного уравнения (...) нашёл Heinz Neuber  
П. Ф. Папкович

...

## § 6. Concentrated force in an infinite medium

Concentrated force is useful mathematical idealization, but cannot be found in the real world, where all forces are either body forces acting over a volume or surface forces acting over an area.

Начнём с риторического вопроса: почему упругое тело сопротивляется приложенной нагрузке, выдерживает её? Удачный ответ можно найти ...

...

## § 7. Finding displacements by deformations

Like any bivalent tensor, displacement gradient  $\nabla u$  can be decomposed into the sum of the symmetric and antisymmetric parts

$$\nabla u = \overbrace{\varepsilon}^{\nabla u} - \overbrace{\omega \times E}^{\nabla u}, \quad \omega \equiv \frac{1}{2} \nabla \times u, \quad (7.1)$$

The symmetric part  $\nabla u^S$  is the linear deformation tensor  $\varepsilon$ . The antisymmetric part  $\nabla u^A$  we will denote as  $\Omega$  and will call it the tensor of small rotations. Any antisymmetric bivalent tensor can

be represented by a vector (§ 1.7). So, to find displacements  $\mathbf{u}$  by deformations  $\boldsymbol{\varepsilon}$ , one more field is needed — the field of rotations  $\boldsymbol{\omega}(\mathbf{r})$ .

....

*The compatibility conditions in the linear elasticity*

The Saint-Venant’s compatibility conditions represent the integrability conditions for a symmetric bivalent tensor field. When such a tensor field is compatible, then it describes some deformation (strain).

In the displacement  $\mapsto$  deformation relation  $\boldsymbol{\varepsilon} = \nabla \mathbf{u}^S$ , the six components  $\varepsilon_{ij}$  of deformation  $\boldsymbol{\varepsilon}$  originate from only three components  $u_k$  of the displacement vector  $\mathbf{u}$ .

The compatibility conditions determine whether this deformation does not cause any gaps and/or overlaps (... and add picture here .....).

...

$$\text{inc } \boldsymbol{\varepsilon} \equiv \nabla \times (\nabla \times \boldsymbol{\varepsilon})^\top$$

A contour is arbitrary here, so we have the relation

$$\text{inc } \boldsymbol{\varepsilon} = {}^2\mathbf{0}, \tag{7.2}$$

called the equation of compatibility of deformations.

...

Expression (7.2) provides constraints on possible variants of a deformation (strain) field. ( ... the figure with cut squares ... )

...

Tensor  $\text{inc } \boldsymbol{\varepsilon}$  is symmetric together with  $\boldsymbol{\varepsilon}$

...

All equations of the linear theory have an analogue (primary source) in the nonlinear theory. To find it for (7.2), remember the Cauchy–Greendeformation tensor (§ 3.3) and curvature tensors (§ 1.16)

...

## § 8. Equations in stresses

Balance of forces (of momentum)

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \quad (8.1)$$

does not quite yet determine stresses. It's necessary as well that deformations (strains)  $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$  corresponding to stresses (3.6)

$$\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \frac{\partial \hat{\Pi}}{\partial \boldsymbol{\sigma}} = {}^4\mathcal{B} \cdot \boldsymbol{\sigma} \quad (8.2)$$

were compatible (§ 7)

$$\text{inc } \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) \equiv \nabla \times (\nabla \times \boldsymbol{\varepsilon}(\boldsymbol{\sigma}))^\top = {}^2\mathbf{0}. \quad (8.3)$$

Gathered together, (8.1), (8.2) and (8.3) present the complete closed set (system) of equations in stresses.

...

## § 9. Principle of minimum potential energy

When the existence of a deformation energy function is assured, and external forces are assumed constant during variation of displacements, the principle of virtual work leads to the principle of minimum potential energy.

Formulation of the principle:

$$\mathcal{E}(\mathbf{u}) \equiv \int_{\mathcal{V}} (\Pi(\mathbf{u}) - \mathbf{f} \cdot \mathbf{u}) d\mathcal{V} - \int_{o_2} \mathbf{p} \cdot \mathbf{u} do \rightarrow \min, \quad \mathbf{u}|_{o_1} = \mathbf{u}_0. \quad (9.1)$$

Functional  $\mathcal{E}(\mathbf{u})$ , called the (full) potential energy of a linear-elastic body, is minimal when displacements  $\mathbf{u}$  are true — that is for the solution of problem (5.1). Input functions  $\mathbf{u}$  must satisfy the geometrical condition on  $o_1$  (so they don't break existing constraints) and be continuous (or else  $\Pi(\mathbf{u})$  will not be integrable).

For the true field of displacements  $\mathbf{u}$ , quadratic function

$$\Pi(\mathbf{u}) = \frac{1}{2} \nabla \mathbf{u} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u}$$

becomes equal to the true deformation energy. Then  $\mathcal{E} = \mathcal{E}_{\min}$ , which according to the Clapeyron's theorem (4.1) is

$$\mathcal{E}_{\min} = \int_{\mathcal{V}} \Pi(\mathbf{u}) d\mathcal{V} - \left( \int_{\mathcal{V}} \mathbf{f} \cdot \mathbf{u} d\mathcal{V} + \int_{o_2} \mathbf{p} \cdot \mathbf{u} do \right) = - \int_{\mathcal{V}} \Pi(\mathbf{u}) d\mathcal{V}.$$

Taking some other satisfactory (“admissible”) field of displacements  $\mathbf{u}'$ , have a look at finite difference

$$\mathcal{E}(\mathbf{u}') - \mathcal{E}(\mathbf{u}) = \int_{\mathcal{V}} \left( \Pi(\mathbf{u}') - \Pi(\mathbf{u}) - \mathbf{f} \cdot (\mathbf{u}' - \mathbf{u}) \right) d\mathcal{V} - \int_{o_2} \mathbf{p} \cdot (\mathbf{u}' - \mathbf{u}) d\sigma,$$

seeking  $\mathcal{E}(\mathbf{u}') - \mathcal{E}(\mathbf{u}) \geq 0$  or (ditto)  $\mathcal{E}(\mathbf{u}') \geq \mathcal{E}(\mathbf{u})$ .

$\mathbf{f} = \text{constant}$  and  $\mathbf{p} = \text{constant}$

$\Pi(\mathbf{a}) = \frac{1}{2} \nabla \mathbf{a} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{a}$  (but *not* linear  $\frac{1}{2} \nabla \mathbf{u} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{a}$  — this means  $\Pi(\mathbf{a}) \neq \frac{1}{2} \boldsymbol{\sigma} \cdot \cdot \nabla \mathbf{a}$ )

Constraints don't change:  $(\mathbf{u}' - \mathbf{u})|_{o_1} = \mathbf{u}_0 - \mathbf{u}_0 = \mathbf{0}$ . External surface force  $\mathbf{p}|_{o_2} = \mathbf{t}_{(n)} = \mathbf{n} \cdot \boldsymbol{\sigma}$  on  $o_2$  and  $= \mathbf{0}$  elsewhere on  $o(\partial\mathcal{V})$ .  $\boldsymbol{\sigma} = \nabla \mathbf{u} \cdot \cdot {}^4\mathcal{A} = {}^2\text{constant}$  along with constant  $\mathbf{p}$  and  $\mathbf{f}$ . Therefore

$$\begin{aligned} \int_{o_2} \mathbf{p} \cdot (\mathbf{u}' - \mathbf{u}) d\sigma &= \oint_{o(\partial\mathcal{V})} \mathbf{n} \cdot \boldsymbol{\sigma} \cdot (\mathbf{u}' - \mathbf{u}) d\sigma = \int_{\mathcal{V}} \nabla \cdot (\boldsymbol{\sigma} \cdot (\mathbf{u}' - \mathbf{u})) d\mathcal{V} = \\ &= \int_{\mathcal{V}} (\nabla \cdot \boldsymbol{\sigma}) \cdot (\mathbf{u}' - \mathbf{u}) d\mathcal{V} + \int_{\mathcal{V}} \boldsymbol{\sigma}^T \cdot \cdot \nabla (\mathbf{u}' - \mathbf{u}) d\mathcal{V}. \end{aligned}$$

Due to symmetry  $\boldsymbol{\sigma}^T = \boldsymbol{\sigma} \Rightarrow \boldsymbol{\sigma}^T \cdot \cdot \nabla \mathbf{a} = \boldsymbol{\sigma} \cdot \cdot \nabla \mathbf{a} = \boldsymbol{\sigma} \cdot \cdot \nabla \mathbf{a}^S \forall \mathbf{a}$ . **Разность преобразуется до**

$$\begin{aligned} \mathcal{E}(\mathbf{u}') - \mathcal{E}(\mathbf{u}) &= \\ &= \int_{\mathcal{V}} \left( \Pi(\mathbf{u}') - \Pi(\mathbf{u}) - (\nabla \cdot \boldsymbol{\sigma} + \mathbf{f}) \cdot (\mathbf{u}' - \mathbf{u}) - \boldsymbol{\sigma} \cdot \cdot \nabla (\mathbf{u}' - \mathbf{u}) \right) d\mathcal{V}. \end{aligned}$$

And with the balance of momentum  $\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}$

$$\mathcal{E}(\mathbf{u}') - \mathcal{E}(\mathbf{u}) = \int_{\mathcal{V}} \left( \Pi(\mathbf{u}') - \Pi(\mathbf{u}) - \boldsymbol{\sigma} \cdot \cdot \nabla (\mathbf{u}' - \mathbf{u}) \right) d\mathcal{V}.$$

Here

$$\Pi(\mathbf{u}') = \frac{1}{2} \nabla \mathbf{u}' \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u}', \quad \Pi(\mathbf{u}) = \frac{1}{2} \nabla \mathbf{u} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u},$$

$$\Pi(\mathbf{u}') - \Pi(\mathbf{u}) = \frac{1}{2} \left( \nabla \mathbf{u}' \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u}' - \nabla \mathbf{u} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u} \right)$$

$${}^4\mathcal{A}_{12 \rightleftharpoons 34} = {}^4\mathcal{A} \Rightarrow \nabla \mathbf{u} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u}' = \nabla \mathbf{u}' \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u}$$

$$\frac{1}{2} \left( \nabla \mathbf{u}' \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u}' - \nabla \mathbf{u} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u} + \nabla \mathbf{u} \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u}' - \nabla \mathbf{u}' \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla \mathbf{u} \right)$$

dot product is distributive ...

differentiation is linear

$$(\nabla u' - \nabla u) = \nabla(u' - u)$$

... для конечной разности потенциалов

$$\frac{1}{2} \nabla(u' + u) \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla(u' - u) = \Pi(u') - \Pi(u),$$

добавляя к которой

$$-\nabla u \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla(u' - u) = -\sigma \cdot \cdot \nabla(u' - u)$$

получаем

$$\frac{1}{2} \nabla(u' - u) \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla(u' - u) = \Pi(u' - u)$$

и в концё-концёв\*

$$\mathcal{E}(u') - \mathcal{E}(u) = \int_V \Pi(u' - u) dV.$$

Since  ${}^4\mathcal{A}$  is positive definite (§ 2),  $\Pi(w) = \frac{1}{2} \nabla w \cdot \cdot {}^4\mathcal{A} \cdot \cdot \nabla w \geq 0 \ \forall w$  (and = 0 only if  $\nabla w = \mathbf{0} \Leftrightarrow w = \text{constant}$ : for a case of translation as a whole without deformation).

...

$$\delta \nabla u = \nabla \delta u$$

...

Ritz method

Задача о минимуме функционала  $\mathcal{E}(u)$  приближённо решается как

...

метод конечных элементов, finite element method

...

## § 10. Principle of minimum complementary energy

When the constitutive stress–strain relations assure the existence of a complementary energy function and the geometrical boundary conditions are assumed constant during variation of stresses, then the principle of minimum complementary energy emerges.

$$* \ b^2 - a^2 - 2a(b - a) = (b + a)(b - a) - 2a(b - a) = (b - a)^2$$

The complementary energy of a linear-elastic body is the following functional over the field of stresses:

$$\mathcal{D}(\sigma) \equiv \int_{\mathcal{V}} \hat{\Pi}(\sigma) d\mathcal{V} - \int_{o_1} \mathbf{n} \cdot \sigma \cdot \mathbf{u}_0 do, \quad \mathbf{u}_0 \equiv \mathbf{u}|_{o_1}, \quad (10.1)$$

$$\nabla \cdot \sigma + \mathbf{f} = \mathbf{0}, \quad \mathbf{n} \cdot \sigma|_{o_2} = \mathbf{p}.$$

...

$$\delta(\nabla \cdot \sigma + \mathbf{f}) = \nabla \cdot \delta\sigma = \mathbf{0} \text{ inside volume } \mathcal{V}$$

...

The principle of minimum complementary energy is very useful for estimating inexact (approximate) solutions. But for computations it isn't so essential as the (Lagrange) principle of minimum potential energy (9.1).

## § 11. Mixed principles of stationarity

### **Prange–Hellinger–Reissner Variational Principle,**

named after *Ernst Hellinger*, *Georg Prange* and *Eric Reissner*.

Working independently of Hellinger and Prange, Eric Reissner published his famous six-page paper “On a variational theorem in elasticity” in 1950. In this paper he develops — without, however, considering Hamilton–Jacobi theory — a variational principle same to that of Prange and Hellinger.

### **Hu–Washizu Variational Principle,**

named after *Hu Haichang* and *Kyuichiro Washizu*.

The following functional over displacements and stresses

$$\mathcal{R}(\mathbf{u}, \sigma) = \int_{\mathcal{V}} \left[ \sigma \cdot \nabla \mathbf{u}^S - \hat{\Pi}(\sigma) - \mathbf{f} \cdot \mathbf{u} \right] d\mathcal{V} -$$

$$- \int_{o_1} \mathbf{n} \cdot \sigma \cdot (\mathbf{u} - \mathbf{u}_0) do - \int_{o_2} \mathbf{p} \cdot \mathbf{u} do \quad (11.1)$$

carries names of Reissner, Prange and Hellinger.

...

The advantage of the Reissner–Hellinger principle — freedom of variation. But it also has a drawback: on the true solution the functional has no extremum, but only stationarity.

Принцип можно использовать для построения приближённых решений методом Ritz (Ritz method). Задавая аппроксимации

...

Принцип Hu–Washizu [103] формулируется так:

$$\begin{aligned} \delta \mathcal{W}(\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}) &= 0, \\ \mathcal{W} \equiv \int_V & \left[ \boldsymbol{\sigma} \cdot \cdot (\nabla \mathbf{u}^S - \boldsymbol{\varepsilon}) + \Pi(\boldsymbol{\varepsilon}) - \mathbf{f} \cdot \mathbf{u} \right] dV - \\ & - \int_{o_1} \mathbf{n} \cdot \boldsymbol{\sigma} \cdot (\mathbf{u} - \mathbf{u}_0) d\mathbf{o} - \int_{o_2} \mathbf{p} \cdot \mathbf{u} d\mathbf{o}. \quad (11.2) \end{aligned}$$

Как и в принципе Рейсснера–Хеллингера, здесь нет ограничений ни в объёме, ни на поверхности, но добавляется третий независимый аргумент  $\boldsymbol{\varepsilon}$ . Поскольку  $\hat{\Pi} = \boldsymbol{\sigma} \cdot \cdot \boldsymbol{\varepsilon} - \Pi$ , то (11.1) and (11.2) кажутся почти одним и тем же.

From the Hu–Washizu principle ensues the complete system of equations with boundary conditions, так как

...

Об истории открытия вариационных принципов и соотношении их можно прочитать, например, у Ю. Н. Работнова [50].

## § 12. Antiplane shear

This is such a problem of the linear theory of elasticity, where non-trivial results are obtained by simple conclusions\*.

Рассматривается изотропная среда в декартовых координатах  $x_i$  ( $x_1$  и  $x_2$  в плоскости,  $x_3$  перпендикулярна плоскости) с базисными ортами  $\mathbf{e}_i = \partial_i \mathbf{r}$ ,  $\mathbf{r} = x_i \mathbf{e}_i$ ,  $\mathbf{e}_i \mathbf{e}_i = \mathbf{E} \Leftrightarrow \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ . In case of antiplane deformation/strain (antiplane shear) поле перемещений  $\mathbf{u}$  параллельно координате  $x_3$ :  $\mathbf{u} = v \mathbf{e}_3$ , и  $v$  не зависит от  $x_3$ :  $v = v(x_1, x_2)$ ,  $\partial_3 v = 0$ .

\* Non-trivial in the theory of elasticity is, for example, when the division of a force by an area gives an infinitely large error in stress calculation.

Деформация

$$\varepsilon \equiv \nabla \mathbf{u}^S = \nabla (\mathbf{v} \mathbf{e}_3)^S = \mathbf{e}_3 \nabla \mathbf{v}^S + \underbrace{\mathbf{v} \nabla \mathbf{e}_3}_{{}^2\mathbf{0}}^S = \frac{1}{2} (\nabla \mathbf{v} \mathbf{e}_3 + \mathbf{e}_3 \nabla \mathbf{v})$$

...

Возможна неоднородность среды в плоскости  $x_1, x_2$ :  
 $\mu = \mu(x_1, x_2)$ ,  $\partial_3 \mu = 0$ .

...

### § 13. Torsion of rods

**M. de Saint-Venant.** Memoire sur la torsion des prismes (1853)

Adhémar-Jean-Claude Barré de Saint-Venant. Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exerçant simultanément. 1856. 327 pages.

1. Memoire sur la torsion des prismes, avec des considerations sur leur flexion, etc. Memoires presentes par divers savants a l'Academie des sciences, t. 14, 1856.

2. Memoire sur la flexion des prismes, etc. Journal de mathematiques pures et appliquees, publie par J. Liouville, 2me serie, t. 1, 1856.

Перевод на русский язык: **Сен-Венан Б.** Мемуар о кручении призм. Мемуар об изгибе призм. М.: Физматгиз, 1961. 518 страниц. стр. 379–494

This problem, studied in detail by Saint-Venant, is contained in almost every book about classical elasticity. It considers a cylinder of some section, loaded only by surface forces at the ends (... figure ...)

$$\begin{aligned} z = \ell : \mathbf{k} \cdot \boldsymbol{\sigma} &= \mathbf{p}(x_\alpha), \\ z = 0 : -\mathbf{k} \cdot \boldsymbol{\sigma} &= \mathbf{p}_0(x_\alpha), \end{aligned}$$

where  $\mathbf{k} \equiv \mathbf{e}_3$ ,  $\alpha = 1, 2$ ,  $\mathbf{x} \equiv x_\alpha \mathbf{e}_\alpha$ . Coordinates are  $x_1, x_2, z$ .

The resultant (the sum) of external forces is equal to  $\mathbf{0}$ , and the resultant couple is directed along the  $z$  axis:

$$\int_o \mathbf{p} do = \mathbf{0}, \quad \int_o \mathbf{x} \times \mathbf{p} do = M \mathbf{k}.$$



It is known that the torsion gives tangential components of stress  $\tau_{z1} \equiv \mathbf{k} \cdot \boldsymbol{\sigma} \cdot \mathbf{e}_1$  and  $\tau_{z2} \equiv \mathbf{k} \cdot \boldsymbol{\sigma} \cdot \mathbf{e}_2$ . Assuming that only these components of tensor  $\boldsymbol{\sigma}$  are non-zero

$$\boldsymbol{\sigma} = \mathbf{s}\mathbf{k} + \mathbf{k}\mathbf{s}, \quad \mathbf{s} \equiv \tau_{z\alpha}\mathbf{e}_\alpha.$$

The solution of this problem simplifies if the equations in stresses are used.

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0} \Rightarrow \nabla_\perp \cdot \mathbf{s} = \mathbf{0} (\nabla_\perp \equiv \mathbf{e}_\alpha \partial_\alpha), \partial_z \mathbf{s} = \mathbf{0}, \quad (13.1)$$

$$\nabla \cdot \nabla \boldsymbol{\sigma} + \frac{1}{1+\nu} \nabla \nabla \boldsymbol{\sigma} = {}^2\mathbf{0} \Rightarrow \Delta_\perp \mathbf{s} = \mathbf{0} (\Delta_\perp \equiv \partial_\alpha \partial_\alpha). \quad (13.2)$$

Independence of  $\mathbf{s}$  from  $z$  makes it possible to replace three-dimensional operators with two-dimensional ones.

...

## § 14. Plane deformation

Тут вектор перемещения  $\mathbf{u}$  параллелен плоскости  $x_1, x_2$  и не зависит от третьей координаты  $z$

...

Для примера рассмотрим полуплоскость с сосредоточенной нормальной силой  $Q$  на краю (?? рисунок ??)

...

## Bibliography

Можно назвать несколько десятков книг по классической теории упругости, представляющих несомненный интерес несмотря на возрастающую отдалённость во времени. Подробные литературные указания содержатся в фундаментальной монографии ...

chapter 5

## MICROPOLAR THREE-DIMENSIONAL CONTINUUM

### §1. Introduction to linear micropolar theory

The characteristic distinctive feature of classical elastic media (chapter 3 and 4) — they consist of “simple material points”. A particle of a classical continuum has only translational degrees of freedom, and only single vector  $\mathbf{r}(q^i, t)$  determines its movement. Therefore loads (“force factors”) in such a model are only forces, volume and surface ones. No moments.

But it is not so hard to build more complex models of a continuous medium, where the particles have not only degrees of translation freedom, but also some additional ones. New degrees of freedom are associated with new loads.

The most natural of non-classical models of a three-dimensional medium was proposed by Cosserat brothers in 1909 [19]. Every particle of the Cosserat continuum is infinitesimal absolutely rigid body with six degrees of freedom — three translational and three rotational. Loads in such a medium are forces and moments. The work of the Cosserat brothers remained unnoticed for half a century, but then interest in this topic arose [31, 37].

*from Nowacki W. The Linear Theory of Micropolar Elasticity. In: Micropolar Elasticity. International Centre for Mechanical Sciences (Courses and Lectures), vol. 151, 1974, pp. 1–43*

Woldemar Voigt tried to remove the shortcomings of the classical theory of elasticity [W. Voigt. Theoretische Studien über die Elasticitätsverhältnisse der Krystalle. Abhandlungen der Königlichen Gesellschaft

der Wissenschaften in Göttingen, 34: 3–51, 1887] by the assumption that the interaction of two parts of the body is transmitted through an area element  $do$  by means not only of the force vector  $\mathbf{p}do$  but also by the moment vector  $\mathbf{m}do$ . Thus, besides the force stresses  $\sigma_{ji}$  also the moment stresses have been defined.

However, the complete theory of asymmetric elasticity was developed by the brothers **François et Eugène Cosserat** who published it in 1909 in the work “*Théorie des corps déformables*”.

They assumed that the body consists of interconnected particles in the form of small rigid bodies. During the deformation each particle is displaced by  $\mathbf{u}(\mathbf{x}, t)$  and rotated by  $\boldsymbol{\varphi}(\mathbf{x}, t)$ , the functions of the position  $\mathbf{x}$  and time  $t$ .

Thus an elastic continuum has been described such that its points possess the orientation (polar media) and for which we can speak of the rotation of a point. The vectors  $\mathbf{u}$  and  $\boldsymbol{\varphi}$  are mutually independent and determine the deformation of the body. The introduction of the vectors  $\mathbf{u}$  and  $\boldsymbol{\varphi}$  and the assumption that the transmission of forces through an area element  $do$  is carried out by means of the force vector  $\mathbf{p}$  and the moment vector  $\mathbf{m}$  leads in the consequence to asymmetric stress tensors  $\sigma_{ji}$  and  $\mu_{ji}$ .

The theory of the brothers E. and F. Cosserat remained unnoticed and was not duly appreciated during their lifetime. This was so because the presentation was very general (the theory was non-linear, including large deformations) and because its frames exceeded the frames of the theory of elasticity. They attempted to construct the unified field theory, containing mechanics, optics and electrodynamics and combined by a general principle of the least action.

The research in the field of the general theories of continuous media conducted in the last fifteen years, drew the attention of the scientists to Cosserats' work. Looking for the new models, describing more precisely the behaviour of the real elastic media, the models similar to, or identical with that of Cosserats' have been encountered. Here, we mention, first of all, the papers by C. Truesdell and R. A. Toupin [**C. Truesdell and R. A. Toupin**. *The classical field theories*. *Encyclopædia of Physics*, Chapter 1, Springer-Verlag, Berlin, 1960], G. Grioli [**Grioli G.** *Elasticité asymétrique*. *Ann. di Mat. Pura et Appl. Ser. IV*, 50 (1960)], R. D. Mindlin and H. F. Tiersten [**Mindlin, R. D.; Tiersten, H. F.** *Effects of couple-stresses in linear elasticity*. *Arch. Rational Mech. Anal.* 11. 1962. 415–448].

In the truly micropolar continuum, vector fields of displacements  $\mathbf{u}(\mathbf{r}, t)$  and rotations  $\boldsymbol{\varphi}(\mathbf{r}, t)$  are mutually independent. It is also called the model with free rotation.

Consider at first the geometrically linear model, that's the case of small displacements and small rotations. Here operators  $\overset{\circ}{\nabla}$  and  $\nabla$  are indistinguishable,  $\mathcal{V} = \overset{\circ}{\mathcal{V}}$ ,  $\rho = \overset{\circ}{\rho}$  — equations “can be written in the initial configuration”, operators  $\delta$  and  $\nabla$  commute ( $\delta \nabla \mathbf{u} = \nabla \delta \mathbf{u}$ ,  $\delta \nabla \boldsymbol{\varphi} = \nabla \delta \boldsymbol{\varphi}$ ).

As the foundation for building this model we put the principle of virtual work (“the variation of work of real external forces on virtual displacements is equal with negative sign to the variation of work of internal forces — real stresses on virtual deformations”)

$$\int_{\mathcal{V}} (\mathbf{f} \cdot \delta \mathbf{u} + \mathbf{m} \cdot \delta \boldsymbol{\varphi}) d\mathcal{V} + \int_o (\mathbf{p} \cdot \delta \mathbf{u} + \mathbf{M} \cdot \delta \boldsymbol{\varphi}) do = - \int_{\mathcal{V}} \delta W^{(i)} d\mathcal{V}.$$

Here  $\mathbf{f}$  and  $\mathbf{m}$  are external forces and moments per volume unit;  $\mathbf{p}$  and  $\mathbf{M}$  — they too, but per surface unit (surface loads act only on a certain part  $o$  of surface  $\partial \mathcal{V}$ , bounding volume  $\mathcal{V}$ );  $\delta W^{(i)}$  — the work of internal forces per volume unit.

As before, suppose that  $\delta W^{(i)}$  nullifies when a body moves as a rigid whole without deformation:

$$\delta \mathbf{u} = \delta \boldsymbol{\varphi} \times \mathbf{r} + \text{constant}, \quad \delta \boldsymbol{\varphi} = \text{constant} \Rightarrow \delta W^{(i)} = 0,$$

$$\nabla \delta \mathbf{u} = \nabla \delta \boldsymbol{\varphi} \times \mathbf{r} - \nabla \mathbf{r} \times \delta \boldsymbol{\varphi} = -\mathbf{E} \times \delta \boldsymbol{\varphi} = -\delta \boldsymbol{\varphi} \times \mathbf{E}, \quad \nabla \delta \boldsymbol{\varphi} = {}^2\mathbf{0}.$$

Introducing deformation tensors — tensor of relative displacement between particles (distortion tensor, strain tensor)  $\boldsymbol{\gamma}$  and curvature-twist tensor (torsion-flexure tensor, wryness tensor)  $\boldsymbol{\kappa}$

$$\boldsymbol{\gamma} \equiv \nabla \mathbf{u} + \boldsymbol{\varphi} \times \mathbf{E}, \quad \boldsymbol{\kappa} \equiv \nabla \boldsymbol{\varphi}, \quad (1.1)$$

$$\boldsymbol{\gamma}_{\times} = \nabla \times \mathbf{u} - 2\boldsymbol{\varphi}, \quad \boldsymbol{\kappa}_{\times} = \nabla \times \boldsymbol{\varphi},$$

$$\delta \boldsymbol{\gamma} = \nabla \delta \mathbf{u} + \delta \boldsymbol{\varphi} \times \mathbf{E}, \quad \delta \boldsymbol{\kappa} = \nabla \delta \boldsymbol{\varphi},$$

with the needed absence of virtual deformations  $\delta \boldsymbol{\gamma} = {}^2\mathbf{0}$  and  $\delta \boldsymbol{\kappa} = {}^2\mathbf{0}$ .

In § 3.15 for the momentless continuum, stresses appear as Lagrange multipliers in the principle of virtual work, when  $\delta W^{(i)} = 0$ . The same here:

$$\int_V \left( \mathbf{f} \cdot \delta \mathbf{u} + \mathbf{m} \cdot \delta \boldsymbol{\varphi} - \boldsymbol{\tau} \cdot \delta \boldsymbol{\gamma}^T - \boldsymbol{\mu} \cdot \delta \boldsymbol{\kappa}^T \right) dV + \int_0 \left( \mathbf{p} \cdot \delta \mathbf{u} + \mathbf{M} \cdot \delta \boldsymbol{\varphi} \right) d\sigma = 0. \quad (1.2)$$

Множители Лагранжа в каждой точке — это несимметричные тензоры второй сложности  $\boldsymbol{\tau}$  и  $\boldsymbol{\mu}$ .

Преобразуем  $-\boldsymbol{\tau} \cdot \delta \boldsymbol{\gamma}^T$  и  $-\boldsymbol{\mu} \cdot \delta \boldsymbol{\kappa}^T$

$$\delta \boldsymbol{\gamma}^T = \nabla \delta \mathbf{u}^T - \delta \boldsymbol{\varphi} \times \mathbf{E}, \quad \delta \boldsymbol{\kappa}^T = \nabla \delta \boldsymbol{\varphi}^T,$$

$$-\boldsymbol{\tau} \cdot \delta \boldsymbol{\gamma}^T = -\boldsymbol{\tau} \cdot \nabla \delta \mathbf{u}^T + \boldsymbol{\tau} \cdot (\delta \boldsymbol{\varphi} \times \mathbf{E}), \quad -\boldsymbol{\mu} \cdot \delta \boldsymbol{\kappa}^T = -\boldsymbol{\mu} \cdot \nabla \delta \boldsymbol{\varphi}^T.$$

Using

$$(7.4, \S 1.7) \Rightarrow \mathbf{A}_\times = -\mathbf{A} \cdot \cdot {}^3\boldsymbol{\epsilon},$$

$$\begin{aligned} \mathbf{A} \cdot \cdot (\mathbf{b} \times \mathbf{E}) &= \mathbf{A} \cdot \cdot (\mathbf{E} \times \mathbf{b}) = \mathbf{A} \cdot \cdot (-{}^3\boldsymbol{\epsilon} \cdot \mathbf{b}) = \\ &= (-\mathbf{A} \cdot \cdot {}^3\boldsymbol{\epsilon}) \cdot \mathbf{b} = \mathbf{A}_\times \cdot \mathbf{b} \Rightarrow \end{aligned}$$

and “product rule”  $\Rightarrow \boldsymbol{\tau} \cdot (\delta \boldsymbol{\varphi} \times \mathbf{E}) = \boldsymbol{\tau}_\times \cdot \delta \boldsymbol{\varphi}$

$$\nabla \cdot (\boldsymbol{\tau} \cdot \delta \mathbf{u}) = (\nabla \cdot \boldsymbol{\tau}) \cdot \delta \mathbf{u} + \boldsymbol{\tau} \cdot \nabla \delta \mathbf{u}^T,$$

$$\nabla \cdot (\boldsymbol{\mu} \cdot \delta \boldsymbol{\varphi}) = (\nabla \cdot \boldsymbol{\mu}) \cdot \delta \boldsymbol{\varphi} + \boldsymbol{\mu} \cdot \nabla \delta \boldsymbol{\varphi}^T,$$

получаем

$$-\boldsymbol{\tau} \cdot \delta \boldsymbol{\gamma}^T = (\nabla \cdot \boldsymbol{\tau}) \cdot \delta \mathbf{u} - \nabla \cdot (\boldsymbol{\tau} \cdot \delta \mathbf{u}) + \boldsymbol{\tau}_\times \cdot \delta \boldsymbol{\varphi},$$

$$-\boldsymbol{\mu} \cdot \delta \boldsymbol{\kappa}^T = (\nabla \cdot \boldsymbol{\mu}) \cdot \delta \boldsymbol{\varphi} - \nabla \cdot (\boldsymbol{\mu} \cdot \delta \boldsymbol{\varphi}).$$

После интегрирования с применением теоремы о дивергенции\*

$$\int_V \nabla \cdot (\boldsymbol{\tau} \cdot \delta \mathbf{u}) dV = \oint_{\partial(\partial V)} \mathbf{n} \cdot \boldsymbol{\tau} \cdot \delta \mathbf{u} d\sigma, \quad \int_V \nabla \cdot (\boldsymbol{\mu} \cdot \delta \boldsymbol{\varphi}) dV = \oint_{\partial(\partial V)} \mathbf{n} \cdot \boldsymbol{\mu} \cdot \delta \boldsymbol{\varphi} d\sigma$$

(1.2) приобретает вид

$$\begin{aligned} \int_V \left( (\nabla \cdot \boldsymbol{\tau} + \mathbf{f}) \cdot \delta \mathbf{u} + (\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \mathbf{m}) \cdot \delta \boldsymbol{\varphi} \right) dV + \\ + \int_0 \left( (\mathbf{p} - \mathbf{n} \cdot \boldsymbol{\tau}) \cdot \delta \mathbf{u} + (\mathbf{M} - \mathbf{n} \cdot \boldsymbol{\mu}) \cdot \delta \boldsymbol{\varphi} \right) d\sigma = 0. \end{aligned}$$

\*  $\mathbf{a} \cdot ({}^2\mathbf{B} \cdot \mathbf{c}) = (\mathbf{a} \cdot {}^2\mathbf{B}) \cdot \mathbf{c} = \mathbf{a} \cdot {}^2\mathbf{B} \cdot \mathbf{c}$

From randomness of variations  $\delta \mathbf{u}$  and  $\delta \varphi$  inside a volume ensues the balance of forces and moments

$$\nabla \cdot \boldsymbol{\tau} + \mathbf{f} = \mathbf{0}, \quad \nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \mathbf{m} = \mathbf{0}, \quad (1.3)$$

and from randomness on a surface — boundary conditions

$$\mathbf{n} \cdot \boldsymbol{\tau} = \mathbf{p}, \quad \mathbf{n} \cdot \boldsymbol{\mu} = \mathbf{M}. \quad (1.4)$$

Force stress tensor  $\boldsymbol{\tau}$  удовлетворяет тем же дифференциальным “уравнениям равновесия”<sup>\*</sup> и краевым условиям, что и в безмоментной среде. But tensor  $\boldsymbol{\tau}$  is asymmetric: instead of  $\boldsymbol{\tau}_{\times} = \mathbf{0}$  here is  $\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \mathbf{m} = \mathbf{0}$  — couple stresses  $\boldsymbol{\mu}$  appear, and volume moment load  $\mathbf{m}$  is non-zero.

Meaning of components of the couple stress tensor  $\boldsymbol{\mu}$  is revealed similarly as for  $\boldsymbol{\tau}$ . For an orthonormal basis, moment  $\mathbf{M}_i = \mathbf{e}_i \cdot \boldsymbol{\mu} = \mu_{ik} \mathbf{e}_k$  acts on an area with normal  $\mathbf{e}_i$ . Diagonal components  $\mu_{11}, \mu_{22}, \mu_{33}$  are twisting moments, nondiagonal are bending ones (?? figure??).

...

eccentricity vector  $\boldsymbol{\alpha}$  and inertia tensor  ${}^2\mathfrak{J}$

For an isotropic medium  $\boldsymbol{\alpha} = \mathbf{0}$ ,  ${}^2\mathfrak{J} = \mathfrak{J}\mathbf{E}$ .

...

## § 2. Relations of elasticity

В этой книге упругой называется среда с потенциальными внутренними силами:  $\delta W^{(i)} = -\delta \Pi$ , где  $\Pi$  — энергия деформации на единицу объёма (continuing to model the geometrically linear material,  $\mathcal{V} = \overset{\circ}{\mathcal{V}}$ ).

Имея соотношения (...)

...

$$\begin{aligned} \delta \Pi = -\delta W^{(i)} &= \boldsymbol{\tau} \bullet \bullet \delta \boldsymbol{\gamma}^{\top} + \boldsymbol{\mu} \bullet \bullet \delta \boldsymbol{\kappa}^{\top} \Rightarrow \\ &\Rightarrow \boldsymbol{\tau} = \frac{\partial \Pi}{\partial \boldsymbol{\gamma}}, \quad \boldsymbol{\mu} = \frac{\partial \Pi}{\partial \boldsymbol{\kappa}}. \end{aligned} \quad (2.1)$$

<sup>\*</sup> Кавычки здесь оттого, что *уравнения равновесия* это вообще всё, что вытекает из принципа виртуальной работы в статике.

Последние равенства — соотношения упругости (определяющие уравнения, constitutive equations).

Разлагая тензоры деформации и напряжения на симметричные и антисимметричные части

$$\begin{aligned}\boldsymbol{\gamma} &= \boldsymbol{\gamma}^S - \frac{1}{2} \boldsymbol{\gamma}_\times \times \boldsymbol{E}, \quad \boldsymbol{\kappa} = \boldsymbol{\kappa}^S - \frac{1}{2} \boldsymbol{\kappa}_\times \times \boldsymbol{E}, \\ \delta \boldsymbol{\gamma}^\top &= \delta \boldsymbol{\gamma}^S + \frac{1}{2} \delta \boldsymbol{\gamma}_\times \times \boldsymbol{E}, \quad \delta \boldsymbol{\kappa}^\top = \delta \boldsymbol{\kappa}^S + \frac{1}{2} \delta \boldsymbol{\kappa}_\times \times \boldsymbol{E}, \\ \boldsymbol{\tau} &= \boldsymbol{\tau}^S - \frac{1}{2} \boldsymbol{\tau}_\times \times \boldsymbol{E}, \quad \boldsymbol{\mu} = \boldsymbol{\mu}^S - \frac{1}{2} \boldsymbol{\mu}_\times \times \boldsymbol{E},\end{aligned}$$

преобразуем выражение  $\delta \Pi = \boldsymbol{\tau} \bullet \bullet \delta \boldsymbol{\gamma}^\top + \boldsymbol{\mu} \bullet \bullet \delta \boldsymbol{\kappa}^\top$  как

$$\delta \Pi = \dots \quad (2.2)$$

...

$$\begin{aligned}\boldsymbol{\gamma}_\times &= \nabla \times \boldsymbol{u} - 2\boldsymbol{\varphi}, \\ \boldsymbol{\kappa}_\times &= \nabla \times \boldsymbol{\varphi}, \\ \boldsymbol{\varepsilon} \equiv \boldsymbol{\gamma}^S &= \nabla \boldsymbol{u}^S,\end{aligned}$$

...

The classical isotropic linear elastic material behavior is described by two material parameters, for example, the Young's modulus and the Poisson's ratio, while the isotropic Cosserat continuum needs six material parameters

even when assumed to be linear, homogeneous and isotropic, it requires six independent material constants, in contrast to only two such constants for the classical continuum

...

Соотношения (2.1) обращаются преобразованием Лежандра

$$\begin{aligned}\boldsymbol{\gamma} &= \frac{\partial \hat{\Pi}}{\partial \boldsymbol{\tau}}, \quad \boldsymbol{\kappa} = \frac{\partial \hat{\Pi}}{\partial \boldsymbol{\mu}}, \\ \hat{\Pi}(\boldsymbol{\tau}, \boldsymbol{\mu}) &= \boldsymbol{\tau} \bullet \bullet \boldsymbol{\gamma}^\top + \boldsymbol{\mu} \bullet \bullet \boldsymbol{\kappa}^\top - \Pi(\boldsymbol{\gamma}, \boldsymbol{\kappa}).\end{aligned} \quad (2.3)$$

...

material's intrinsic (internal) length scale  $\ell$

Если устремить  $\ell$  к нулю, то исчезает вклад  $\boldsymbol{\kappa}$  в  $\Pi$ , а с ним и моментные напряжения  $\boldsymbol{\mu}$ . Когда вдобавок нет объёмной моментной нагрузки  $\boldsymbol{m}$ , тогда тензор  $\boldsymbol{\tau}$  становится симметричным:  $\nabla \bullet \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \boldsymbol{m} = \mathbf{0}$ ,  $\boldsymbol{\mu} = {}^2\mathbf{0}$ ,  $\boldsymbol{m} = \mathbf{0} \Rightarrow \boldsymbol{\tau}_\times = \mathbf{0}$ , и модель превращается в классическую безмоментную.

Yet using of the micropolar model is natural in a case when the real material has a certain smallest volume “which is impossible to enter into”. And such a situation occurs quite often: composites with “representative” volume, polycrystalline materials, polymers with large molecules (macromolecules).

### § 3. Compatibility equations

Having identity  $\nabla \times \nabla \mathbf{a} = {}^2\mathbf{0} \ \forall \mathbf{a}$  and definitions of deformation tensors (1.1),

$$\begin{aligned} \boldsymbol{\kappa} &\equiv \nabla \boldsymbol{\varphi} \Rightarrow \nabla \times \boldsymbol{\kappa} = {}^2\mathbf{0}, \\ \boldsymbol{\gamma} - \boldsymbol{\varphi} \times \mathbf{E} &= \nabla \mathbf{u} \Rightarrow \nabla \times (\boldsymbol{\gamma} - \boldsymbol{\varphi} \times \mathbf{E}) = {}^2\mathbf{0} \dots \end{aligned} \quad (3.1)$$

### § 4. Theorems of statics

Теоремы статики линейных консервативных систем, легко выводимые для конечного числа степеней свободы (минимальность энергии, теорема Клапейрона, reciprocal work theorem et al. — § 2.6), справедливы и для континуальных линейных упругих сред (гл. 4), including the micropolar model of continuum (a medium with force couples, moments).

...

### § 5. Cosserat pseudocontinuum

Besides the model with free rotation (“the truly micropolar continuum”), there is the simplified model of a medium with force couples, in which rotations are expressed via displacements as in the classical momentless continuum:

$$\boldsymbol{\varphi} = \frac{1}{2} \nabla \times \mathbf{u} \Leftrightarrow \boldsymbol{\gamma}_\times = \mathbf{0} \Leftrightarrow \boldsymbol{\gamma} = \boldsymbol{\varepsilon} = \nabla \mathbf{u}^S \quad (5.1)$$

— the model with constrained rotation\*.

\* Cosserat brothers called it cas de trièdre caché (case of latent trihedron).



Равенство  $\gamma_x = 0$  (симметрию  $\gamma$ ) возможно понимать как внутреннюю связь (§ 3.13). Аргумент  $\gamma_x$  исчезает из энергии  $\Pi$ , соотношение упругости для  $\tau_x$  не может быть написано. Его место в полной системе занимает уравнение связи.

В классической (линейной безмоментной) теории упругости полная система сводится к одному уравнению для вектора  $u$  (§ 4.??). В моментной теории

...

## § 6. Plane deformation

Все переменные в этой постановке не зависят от декартовой координаты  $z \equiv x_3$  (орт оси —  $k$ ). Перемещения и силы перпендикулярны оси  $z$ , а повороты и моменты — параллельны ей:

...

Это краткое изложение плоской задачи относится к модели с независимыми поворотами. Псевдоконтинуум Коссера (модель со стеснённым вращением) получается либо при наложении внутренней связи  $\gamma_x = 0$ , либо при предельном переходе ...

Подробнее о плоской моментной задаче написано в книгах Н. Ф. Морозова [32, 33].

## § 7. Nonlinear theory

Кажущееся на первый взгляд чрезвычайно трудным, построение теории конечных деформаций континуума Коссера становится прозрачным, если опираться на общую механику, тензорное исчисление и нелинейную теорию безмоментной среды.

При построении модели упругого континуума обычно проходят четыре этапа:

- ✓ определение степеней свободы частиц,
- ✓ выявление нагрузок (“силовых факторов”, напряжений) и условий их баланса,

- ✓ подбор соответствующих мер деформации
- и, наконец,
- ✓ вывод соотношений упругости между напряжением и деформацией.

Этот традиционный путь очень сокращается, если опираться на принцип виртуальной работы.

Как и в гл. 3, среда состоит из частиц с материальными координатами  $q^i$  и вектором-радиусом  $\mathbf{r}(q^i, t)$ . В начальной (исходной, отсчётной) конфигурации  $\mathbf{r}(q^i, 0) \equiv \mathring{\mathbf{r}}(q^i)$ . Но кроме трансляции, частицы имеют независимые степени свободы поворота, описываемого ортогональным тензором

$$\mathbf{O}(q^i, t) \equiv \mathbf{a}_j \mathring{\mathbf{a}}^j = \mathbf{a}^j \mathring{\mathbf{a}}_j = \mathbf{O}^{-\top},$$

где тройка векторов  $\mathbf{a}_j(q^i, t)$  жёстко связана с каждой частицей, показывая угловую ориентацию относительно как-либо выбираемых\* векторов  $\mathring{\mathbf{a}}_j(q^i) \equiv \mathbf{a}_j(q^i, 0)$ ,  $\mathbf{a}_j = \mathbf{O} \cdot \mathring{\mathbf{a}}_j$ ;  $\mathbf{a}^j$  — тройка взаимных векторов:  $\mathbf{a}_j \mathbf{a}^j = \mathbf{a}^j \mathbf{a}_j = \mathbf{E}$  ( $t=0$ ,  $\mathring{\mathbf{a}}^j$ :  $\mathring{\mathbf{a}}_j \mathring{\mathbf{a}}^j = \mathring{\mathbf{a}}^j \mathring{\mathbf{a}}_j = \mathbf{E}$ ). Движение среды полностью определяется функциями  $\mathbf{r}(q^i, t)$  and  $\mathbf{O}(q^i, t)$ .

Имея представления  $\mathring{\mathbf{r}}(q^i)$  и  $\mathbf{r}(q^i, t)$ , вводим базис  $\mathbf{r}_i \equiv \partial_i \mathbf{r}$ , взаимный базис  $\mathbf{r}^i$ :  $\mathbf{r}_j \cdot \mathbf{r}^i = \delta_j^i$ , differential operators  $\mathring{\nabla}$  and  $\nabla$ , а также motion gradient  $\mathbf{F}$

$$\mathring{\nabla} \equiv \mathring{\mathbf{r}}^i \partial_i, \quad \nabla \equiv \mathbf{r}^i \partial_i, \quad \nabla = \mathbf{F}^{-\top} \cdot \mathring{\nabla}, \quad \mathbf{F} \equiv \mathring{\nabla} \mathbf{r}^\top = \mathbf{r}_i \mathring{\mathbf{r}}^i. \quad (7.1)$$

Вариационное уравнение принципа виртуальной работы для континуума с нагрузками в объёме и на поверхности:

$$\int_{\mathcal{V}} \left( \rho (\mathbf{f} \cdot \delta \mathbf{r} + \mathbf{m} \cdot \delta \boldsymbol{\varphi}) + \delta W^{(i)} \right) d\mathcal{V} + \int_{\mathcal{O}} \left( \mathbf{p} \cdot \delta \mathbf{r} + \mathbf{M} \cdot \delta \boldsymbol{\varphi} \right) d\mathcal{O} = 0. \quad (7.2)$$

Here  $\rho$  is mass density;  $\mathbf{f}$  and  $\mathbf{m}$  are external force and moment per mass unit;  $\mathbf{p}$  and  $\mathbf{M}$  — они же per surface unit;  $\delta W^{(i)}$  — работа

\* Один из вариантов:  $\mathring{\mathbf{a}}_j = \mathring{\mathbf{r}}_j \equiv \partial_j \mathbf{r}$ . Другое предложение:  $\mathring{\mathbf{a}}_j$  это ортонормальная тройка собственных векторов тензора инерции частицы. Вообще,  $\mathring{\mathbf{a}}_j$  могут быть любой тройкой линейно-независимых векторов.

внутренних сил per volume unit в текущей конфигурации. Вектор мального поворота  $\delta\varphi$

$$\begin{aligned} O \cdot O^T &= E \Rightarrow \delta O \cdot O^T = -O \cdot \delta O^T \Rightarrow \\ &\Rightarrow \delta O \cdot O^T = \delta\varphi \times E = \delta\varphi \times O \cdot O^T \Rightarrow \delta O = \delta\varphi \times O, \\ \delta\varphi &= -\frac{1}{2}(\delta O \cdot O^T)_{\times} \end{aligned}$$

При движении среды как жёсткого целого нет деформаций, и работа  $\delta W^{(i)}$  внутренних сил равна нулю:

$$\begin{aligned} \delta r &= \text{constant} + \delta\varphi \times r, \quad \delta\varphi = \text{constant} \Rightarrow \delta W^{(i)} = 0, \\ \nabla \delta\varphi &= {}^20, \quad \nabla \delta r = \nabla \delta\varphi \times r - \nabla r \times \delta\varphi = -E \times \delta\varphi = -\delta\varphi \times E, \\ \nabla \delta r + \delta\varphi \times E &= {}^20. \end{aligned}$$

К нагрузкам. Несимметричные тензоры напряжения, силового  $\tau$  и моментного  $\mu$ , введём как множители Lagrange'a:

$$\begin{aligned} \int_V \left( \rho(f \cdot \delta r + m \cdot \delta\varphi) - \tau \cdot (\nabla \delta r + \delta\varphi \times E)^T - \mu \cdot \nabla \delta\varphi^T \right) dV + \\ + \int_O \left( p \cdot \delta r + M \cdot \delta\varphi \right) dO = 0. \quad (7.3) \end{aligned}$$

После тех же преобразований, что и в § 1, получаем

...

Отсюда вытекают уравнения баланса сил и моментов в объёме и краевые условия в виде формул типа Cauchy. Они по существу те же, что и в линейной теории.

Найдём теперь тензоры деформации. Их можно вводить по-разному, если требовать лишь одного — нечувствительности к движению среды как жёсткого целого. Читатель найдёт не один вариант таких тензоров. Однако, вид тензоров деформации “подсказывает” принцип виртуальной работы.

...

## § 8. Nonlinear model with constrained rotation

Вспомним переход к модели со стеснённым вращением в линейной теории (§ 5). Разделились соотношения упругости для симметрич-

ной части тензора силового напряжения  $\boldsymbol{\tau}^S$  и кососимметричной его части  $\boldsymbol{\tau}_\chi$ . Возникла внутренняя связь  $\boldsymbol{\gamma}_\chi = \mathbf{0}$

...

## Bibliography

Все работы по моментной теории упругости упоминают книгу братьев Eugène et François Cosserat [19], где трёхмерной среде посвящена одна глава из шести. Переведённая монография W. Nowacki [37] была одной из первых книг на русском языке с изложением линейной моментной теории. Ранее эта область представлялась статьями — например, R. D. Mindlin'a и H. F. Tiersten'a [31]. Краткое изложение моментной теории, но с подробным рассмотрением задач содержится в книгах Н. Ф. Морозова [32, 33].

## chapter 6

# THERMOELASTICITY

### §1. First law of thermodynamics

**H**itherto modeling was limited to mechanics only. It is widely known, however, that a change in temperature causes a deformation. Temperature deformation and stress often play the primary role and can lead to a breakage.

So effective in mechanics, the principle of virtual work is not applicable to thermomechanics\*. Considering thermal effects, it's possible to lean on the two laws of thermodynamics.

The first, discovered by Joule, Mayer, and Helmholtz — this is the adapted version of the balance of energy: rate of internal energy change  $\dot{E}$  is equal to the sum of power of external forces  $P^{(e)}$  and rate of heat supply  $\dot{Q}$

$$\dot{E} = P^{(e)} + \dot{Q}. \quad (1.1)$$

Internal energy  $E$  is the sum of the kinetic and potential energies of particles. For any finite volume of a material continuum

$$E = \int_{\mathcal{V}} \rho \left( \frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + e \right) d\mathcal{V}. \quad (1.2)$$

With the balance of mass  $dm = \rho d\mathcal{V} = \rho' d\mathcal{V}'$ ,  $m = \int_{\mathcal{V}} \rho d\mathcal{V} = \int_{\mathcal{V}'} \rho' d\mathcal{V}'$  and

$$\Psi = \int_{\mathcal{V}} \rho \psi d\mathcal{V} = \int_{\mathcal{V}'} \rho' \psi d\mathcal{V}' \Rightarrow \dot{\Psi} = \int_{\mathcal{V}} \rho \dot{\psi} d\mathcal{V} = \int_{\mathcal{V}'} \rho' \dot{\psi} d\mathcal{V}',$$

it's easy to get the time derivative of internal energy

$$\dot{E} = \int_{\mathcal{V}} \rho \left( \ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \dot{e} \right) d\mathcal{V}. \quad (1.3)$$

\* Analogue of the principle of virtual work will be presented below in §8.

The power of external forces for some finite volume of continuum (momentless — in this chapter only the momentless model is considered)

$$P^{(e)} = \int_V \rho \mathbf{f} \cdot \dot{\mathbf{r}} dV + \oint_{O(\partial V)} \mathbf{n} \cdot \boldsymbol{\tau} \cdot \dot{\mathbf{r}} dO = \int_V \left( \rho \mathbf{f} \cdot \dot{\mathbf{r}} + \nabla \cdot (\boldsymbol{\tau} \cdot \dot{\mathbf{r}}) \right) dV = \int_V \left( (\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}) \cdot \dot{\mathbf{r}} + \boldsymbol{\tau} \cdot \nabla \dot{\mathbf{r}}^S \right) dV. \quad (1.4)$$

As before (chapter 3),  $\boldsymbol{\tau}$  is Cauchy stress tensor,  $\mathbf{f}$  is mass force (without inertial part  $-\ddot{\mathbf{r}}$ , which is included in  $\dot{\mathbf{E}}$ ),  $\mathbf{n} \cdot \boldsymbol{\tau}$  is surface force. The symmetry of  $\boldsymbol{\tau}$  is used for expanding  $\nabla \cdot (\boldsymbol{\tau} \cdot \dot{\mathbf{r}})$ :

$$\boldsymbol{\tau}^T = \boldsymbol{\tau} = \boldsymbol{\tau}^S \Rightarrow \boldsymbol{\tau} \cdot \nabla \dot{\mathbf{r}}^T = \boldsymbol{\tau} \cdot \nabla \dot{\mathbf{r}}^S, \\ \nabla \cdot (\boldsymbol{\tau} \cdot \dot{\mathbf{r}}) = (\nabla \cdot \boldsymbol{\tau}) \cdot \dot{\mathbf{r}} + \boldsymbol{\tau} \cdot \nabla \dot{\mathbf{r}}^T = \nabla \cdot \boldsymbol{\tau} \cdot \dot{\mathbf{r}} + \boldsymbol{\tau} \cdot \nabla \dot{\mathbf{r}}^S.$$

Denominating the rate of deformation tensor as  $\mathcal{D} \equiv \nabla \dot{\mathbf{r}}^S$

$$P^{(e)} = \int_V \left( (\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}) \cdot \dot{\mathbf{r}} + \boldsymbol{\tau} \cdot \mathcal{D} \right) dV. \quad (1.4')$$

Heat arrives in a volume of continuum by two ways. The first is a surface heat transfer (heat conduction, thermal conductivity, convection, diffusion), occurring via matter, upon a contact of two media. This can be described by heat flux vector  $\mathbf{q}$ . Through an infinitesimal area in the current configuration towards normal vector  $\mathbf{n}$  per unit of time passes heat flux  $\mathbf{q} \cdot \mathbf{n} dO$ . For a surface with finite dimensions this expression needs to be integrated. It's usually assumed

$$\mathbf{q} = -^2\mathbf{k} \cdot \nabla \Theta, \quad (1.5)$$

where  $\Theta$  is temperature (temperature field);  $^2\mathbf{k}$  is thermal conductivity tensor as property of the material, for isotropic material  $^2\mathbf{k} = k\mathbf{E}$  and  $\mathbf{q} = -k \nabla \Theta$ .

The second way is a volume heat transfer (thermal radiation). Solar energy, flame of a campfire, a microwave oven are familiar examples of pervasive heating by radiation. Thermal radiation occurs via electromagnetic waves and doesn't need an interjacent medium. Heat is radiated (emitted) by any matter (with temperature above the absolute zero 0 K). The rate of heat transfer by radiation per mass unit  $b$  or per volume unit  $B = \rho b$  is assumed as known.

Therefore, the rate of heat supply for a finite volume is

$$\dot{Q} = - \oint_{O(\partial\mathcal{V})} \mathbf{n} \cdot \mathbf{q} dO + \int_{\mathcal{V}} \rho b d\mathcal{V} = \int_{\mathcal{V}} (-\nabla \cdot \mathbf{q} + \rho b) d\mathcal{V}. \quad (1.6)$$

Applying (1.3), (1.4') and (1.6) to formulation (1.1) gives the equality of integrals over a volume

$$\int_{\mathcal{V}} \rho (\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \dot{\mathbf{e}}) d\mathcal{V} = \int_{\mathcal{V}} ((\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}) \cdot \dot{\mathbf{r}} + \boldsymbol{\tau} \cdot \cdot \mathcal{D} - \nabla \cdot \mathbf{q} + \rho b) d\mathcal{V}.$$

And since volume  $\mathcal{V}$  is random, integrands are equal too

$$\rho \ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \rho \dot{\mathbf{e}} = (\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}) \cdot \dot{\mathbf{r}} + \boldsymbol{\tau} \cdot \cdot \mathcal{D} - \nabla \cdot \mathbf{q} + \rho b.$$

With the balance of momentum

$$\nabla \cdot \boldsymbol{\tau} + \rho (\mathbf{f} - \ddot{\mathbf{r}}) = \mathbf{0} \quad (7.2, \S 3.7)$$

this simplifies to

$$\rho \dot{\mathbf{e}} = \boldsymbol{\tau} \cdot \cdot \mathcal{D} - \nabla \cdot \mathbf{q} + \rho b \quad (1.7)$$

— the local (differential) version of the balance of energy.

...

## § 2. Second law

The following concept of laws of thermodynamics is widespread: change in internal energy  $dE$  is equal to the sum of work of external forces  $\partial W^{(e)}$  and supplied heat  $\partial Q$

$$dE = \partial W^{(e)} + \partial Q.$$

Work  $\partial W^{(e)}$  and heat  $\partial Q$  are inexact differentials\*, but quotient  $\partial Q/\theta$  becomes the exact differential — differential  $dS$  of the entropy.

Further, processes are divided into reversible ones, for which  $dS = \partial Q/\theta$ , and irreversible ones with characteristic Clausius inequality  $dS \geq \partial Q/\theta$ .

But how to adapt this for a continuum with an inhomogeneous temperature field?

\* Because work and heat depend on the path of the process (are path functions), they can't be full (exact) differentials, contrasting with the idea of the exact differential, expressed via the gradient of another function and therefore path independent.

Sometimes a process in an infinitesimal volume is thought of as reversible, then the equality like

$$\rho\theta\dot{s} = -\nabla\cdot\mathbf{q} + \rho b \quad (2.1)$$

is proposed ( $s$  is entropy per mass unit and  $\dot{s}$  is the time derivative of it, that is the rate of entropy change).

However, there's always heat dissipation — an irreversible process, and therefore (2.1) looks disputable.

The most appropriate expression of the second law of thermodynamics for material continuum seems to be the Clausius–Duhem inequality

$$\left( \int_{\mathcal{V}} \rho s d\mathcal{V} \right) \geq - \oint_{O(\partial\mathcal{V})} \frac{\mathbf{q}}{\theta} \cdot \mathbf{n} dO + \int_{\mathcal{V}} \frac{\rho b}{\theta} d\mathcal{V}. \quad (2.2)$$

This inequality as the imbalance of entropy defines the rate of entropy production.

$$\begin{aligned} - \oint_{O(\partial\mathcal{V})} \mathbf{n} \cdot \mathbf{q} \theta^{-1} dO &= - \int_{\mathcal{V}} \nabla \cdot (\mathbf{q} \theta^{-1}) d\mathcal{V} \\ - \nabla \cdot (\mathbf{q} \theta^{-1}) &= - (\nabla \cdot \mathbf{q}) \theta^{-1} - \mathbf{q} \cdot \left( \nabla \frac{1}{\theta} \right) \\ - \nabla \frac{1}{\theta} &= \frac{1}{\theta^2} \nabla \theta \\ - \nabla \cdot (\mathbf{q} \theta^{-1}) &= - (\nabla \cdot \mathbf{q}) \theta^{-1} + (\mathbf{q} \cdot \nabla \theta) \theta^{-2} \\ \rho \dot{s} &\geq (-\nabla \cdot \mathbf{q} + \rho b) \theta^{-1} + (\mathbf{q} \cdot \nabla \theta) \theta^{-2} \end{aligned} \quad (2.3)$$

The Clausius–Duhem inequality is also called the dissipation inequality. For a real matter, the dissipation is always greater than zero, it can never be negative and can't be zero whenever irreversible processes are present.

...

Helmholtz free energy per mass unit

$$a \equiv e - \theta s, \quad (2.4)$$

$$\dot{a} = \dot{e} - \theta \dot{s} - \dot{\theta} s$$



### § 3. Constitutive equations

К балансу импульса, балансу момента импульса и законам термодинамики нужно добавить определяющие уравнения, выражающие свойства среды. Эти уравнения

...

Термоупругим называется материал, в котором свободная энергия  $a$  и энтропия  $s$  — функции деформации  $\mathbf{C}$  и температуры  $\theta$

$$a = a(\mathbf{C}, \theta)$$

$$\dot{a} = \frac{\partial a}{\partial \mathbf{C}} \cdot \dot{\mathbf{C}} + \frac{\partial a}{\partial \theta} \dot{\theta}$$

...

### § 4. Heat equation

In mathematical physics, a parabolic differential equation, similar to

$$k \Delta \theta + B = c \dot{\theta}, \quad (4.1)$$

is declared as a “heat equation”. Here  $k$  is thermal conductivity,  $B = \rho b$  is rate of heat transfer by radiation per volume unit,  $c$  is thermal capacity per volume unit. Boundary conditions most often are external temperature  $\theta_1^{(e)}$  on part  $O_1$  of the surface and heat flux  $q^{(e)}$  from the outside of part  $O_2$  of the surface:

$$\theta|_{O_1} = \theta_1^{(e)}, \quad k \partial_n \theta|_{O_2} = q^{(e)}.$$

Sometimes, flux  $q^{(e)}$  is thought to be proportional to the difference between temperature  $\theta^{(e)}$  of the ambient and body temperature  $\theta$

$$k \partial_n \theta + h (\theta - \theta^{(e)}) = 0.$$

If heat transfer coefficient  $h$  is infinitely large, it turns into the first condition  $\theta = \theta^{(e)}$ , and when  $h \rightarrow 0$  — into condition  $\partial_n \theta = 0$  of thermal insulation.

But how is equation (4.1) related to fundamental principles of balance? Since there’s no special “thermal energy”, but there is internal energy, changing according to the first law of thermodynamics

...

...

$$e = a + \theta s \Rightarrow \dot{e} = \dot{a} + \dot{\theta} s + \theta \dot{s}$$

$$\rho \dot{e} = \rho (\dot{a} + \dot{\theta} s + \theta \dot{s}) = \rho \left( \overbrace{\left( \frac{\partial a}{\partial \mathbf{C}} \cdot \dot{\mathbf{C}} + \frac{\partial a}{\partial \theta} \dot{\theta} + \dot{\theta} s + \theta \dot{s} \right)}_{=0 \Leftarrow s = -\frac{\partial a}{\partial \theta}} \right)$$

...

## § 5. Linear thermoelasticity

Квадратичная аппроксимация свободной энергии наиболее естественна в линейной теории

...

## § 6. Equations for displacements

Полагая поле температуры известным

...

## § 7. Thermal stress

Это напряжение сто́ит рассмотреть детально, хотя оно и определяется очевидным образом полями перемещений и температуры. При равновесии свободного тела без внешних нагрузок

...

## § 8. Variational formulations

Когда температура постоянна, уравнения термоупругости выглядят как в механике.

...

Для переноса вариационного метода на термоупругость достаточно заменить in the (Lagrange) principle of minimum potential energy  $\Pi(\mathbf{C})$  на Helmholtz free energy  $A(\mathbf{C}, \theta)$ , а в принципе Reissner'a–Hellinger'a заменить  $\hat{\Pi}(\boldsymbol{\tau})$  на Gibbs free enthalpy (Gibbs function)  $G(\boldsymbol{\tau}, \theta)$ .

The *Gibbs free energy* (*Gibbs energy* or *Gibbs function* or *free enthalpy* to distinguish it from Helmholtz free energy) is a thermodynamic potential that measures the maximum of reversible work by a thermodynamic system at a constant temperature and pressure.

...

...

Более сложные вариационные постановки для нестационарных задач можно найти, например, в книге [89].

## Bibliography

Шириной и глубиной описания термоупругости выделяются книги W. Nowacki [36, 37], книга E. Melan'a и H. Parkus'a [28] и монография H. Parkus'a [43]. С. Truesdell [58] внёс большой вклад в создание и распространение новых взглядов на термодинамику сплошной среды. Чёткое изложение основных законов есть у С. Teodosiu [54]. Методы расчёта температурных полей представлены у Н. М. Беляева и А. А. Рядно [89].

## chapter 7

# MAGNETOELASTICITY

Problems, solved using models of elastic continua, usually include already known external loads (how to find them is a distinct problem). When electric currents flow in an elastic body, the magnetic field produces a load. With deforming of a body, the magnetic field changes too. If the field's sensitivity to deformation is high, then solving of the integrated problem of elasticity and magnetism is needed.

### §1. Electromagnetic field

**H**ere is the summary of the theory of electromagnetism. This theory describes the two vector fields: electric  $\mathbb{E}(\mathbf{r}, t)$  and magnetic  $\mathbb{B}(\mathbf{r}, t)$ . What are vector  $\mathbb{E}$  and pseudovector  $\mathbb{B}$  is seen from the expression of electromagnetic force (Lorentz force)  $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t, q)$ , acting on a point charge — a particle of infinitesimal size that contains electric charge  $q$  and moves with velocity  $\mathbf{v} \equiv \dot{\mathbf{r}}$

$$\mathbf{F} = q(\mathbb{E} + \mathbf{v} \times \mathbb{B}). \quad (1.1)$$

In essence, the part of the electromagnetic force arising from interaction with a moving charge — magnetic force  $q\mathbf{v} \times \mathbb{B}$  — reveals magnetic field  $\mathbb{B}$ , while the other part — electric force  $q\mathbb{E}$  — reveals electric field  $\mathbb{E}$ .

Acute question: in which exactly frame of reference is velocity  $\mathbf{v} \equiv \dot{\mathbf{r}}$  of a charged particle measured, leads to the special theory of relativity\*

the classical concept about the existence of absolute space and time as the preferred frame of reference

a vacuum is a space without matter, “free space”

\* **Albert Einstein.** Zur Elektrodynamik bewegter Körper. // Annalen der Physik, IV. Folge, Band 17, 1905. Seiten 891–921.

...

continuous charge distribution

charge density (electric charge per volume unit)  $\varrho(\mathbf{r}, t)$ 

$$dq = \varrho d\mathcal{V}, \quad q = \int_{\mathcal{V}} \varrho d\mathcal{V}$$

the density of electric current is the flux of electric charge  
 $\mathbf{j}(\mathbf{r}, t) \equiv \varrho \dot{\mathbf{r}} \equiv \varrho \mathbf{v}$

в вакууме нет  $\varrho$  and  $\mathbf{j}$ 

in a region with no charges ( $\varrho = 0$ ) and no currents ( $\mathbf{j} = \mathbf{0}$ ), such  
 as in a vacuum, ...

...

ponderomotive force is electromagnetic force (Lorentz force) per  
 volume unit

In a material continuum with charges and currents acts the volume  
 “ponderomotive” force

$$\mathbf{f} = \varrho \mathbf{E} + \mathbf{j} \times \mathbf{B} \quad (1.2)$$

— it is the differential (local, microscopic, continual) version of (1.1).

...

Electromagnetic phenomena are usually described by Maxwell’s  
 equations. Differential versions of Maxwell’s equations are

$$\nabla \cdot \mathbf{E} = \frac{\varrho}{\varepsilon_0} \quad \text{Gauss’s theorem for electricity} \quad (1.3^\alpha)$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \quad \begin{array}{l} \text{Maxwell–Faraday equation} \\ \text{(Faraday’s law of induction)} \end{array} \quad (1.3^\beta)$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{Gauss’s theorem for magnetism} \quad (1.3^\gamma)$$

$$c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\varepsilon_0} + \dot{\mathbf{E}} \quad \begin{array}{l} \text{Ampère’s circuital law with} \\ \text{Maxwell’s term } \dot{\mathbf{E}} \text{ for the balance} \\ \text{of electric charge} \end{array} \quad (1.3^\delta)$$

...

speed of light in vacuum  $c = 299\,792\,458 \text{ m/s}$ “electric constant”, vacuum permittivity  $\varepsilon_0$  $\varepsilon_0 \approx 8.8541878 \cdot 10^{-12} \text{ F} \cdot \text{m}^{-1}$  (farads per metre)

“magnetic constant”, vacuum permeability  $\mu_0 = \frac{1}{\varepsilon_0 c^2}$

...

With  $\mu_0$  equation (1.3<sup>δ</sup>) is sometimes written as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \dot{\mathbf{E}} \quad \text{or} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \dot{\mathbf{E}}.$$

...

The balance of electric charge — the continuity equation for electric charge — mathematically follows from Maxwell's equations

$$\left. \begin{aligned} \nabla \cdot (1.3^\delta) &\Rightarrow c^2 \nabla \cdot (\nabla \times \mathbf{B}) = \frac{\nabla \cdot \mathbf{j}}{\varepsilon_0} + \nabla \cdot \dot{\mathbf{E}} \\ (1.3^\alpha)^\bullet &\Rightarrow \nabla \cdot \dot{\mathbf{E}} = \frac{\dot{\varrho}}{\varepsilon_0} \\ \nabla \cdot (\nabla \times \mathbf{a}) &= 0 \quad \forall \mathbf{a}, \quad \mathbf{j} \equiv \varrho \dot{\mathbf{r}} \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow \nabla \cdot (\varrho \dot{\mathbf{r}}) + \dot{\varrho} = 0. \quad (1.4)$$

Besides the ponderomotive force vector (1.2), with which the electromagnetic field acts on the continuum, another expression of interaction exists — the bivalent “Maxwell” stress tensor

$${}^2\mathbf{M} \equiv \varepsilon_0 \left( \mathbf{E}\mathbf{E} + c^2 \mathbf{B}\mathbf{B} - \frac{1}{2} (\mathbf{E} \cdot \mathbf{E} + c^2 \mathbf{B} \cdot \mathbf{B}) \mathbf{E} \right). \quad (1.5)$$

It derives from (1.2) and Maxwell's equations

$$\begin{aligned} (1.3^\alpha) &\Rightarrow \varrho = \varepsilon_0 \nabla \cdot \mathbf{E} \\ (1.3^\delta) &\Rightarrow \mathbf{j} = \varepsilon_0 c^2 \nabla \times \mathbf{B} - \varepsilon_0 \dot{\mathbf{E}} \\ (1.2) &\Rightarrow \mathbf{f} = \varepsilon_0 \nabla \cdot \mathbf{E}\mathbf{E} + (\varepsilon_0 c^2 \nabla \times \mathbf{B} - \varepsilon_0 \dot{\mathbf{E}}) \times \mathbf{B} = \\ &= \varepsilon_0 \nabla \cdot \mathbf{E}\mathbf{E} + \varepsilon_0 c^2 (\nabla \times \mathbf{B}) \times \mathbf{B} - \varepsilon_0 \dot{\mathbf{E}} \times \mathbf{B} \\ (1.3^\beta) &\Rightarrow \dot{\mathbf{B}} = -\nabla \times \mathbf{E} \end{aligned} \left\} \Rightarrow \dot{\mathbf{E}} \times \mathbf{B} = (\mathbf{E} \times \mathbf{B})^\bullet + \mathbf{E} \times (\nabla \times \mathbf{E})$$

Then

$$\mathbf{f} = \varepsilon_0 \nabla \cdot \mathbf{E}\mathbf{E} - \varepsilon_0 c^2 \mathbf{B} \times (\nabla \times \mathbf{B}) - \varepsilon_0 \mathbf{E} \times (\nabla \times \mathbf{E}) - \varepsilon_0 (\mathbf{E} \times \mathbf{B})^\bullet$$

For the symmetry with  $\nabla \cdot \mathbf{E}\mathbf{E}$ , the null vector

$$(1.3^\gamma) \Rightarrow \nabla \cdot \mathbf{B}\mathbf{B} = \mathbf{0}, \quad c^2 \nabla \cdot \mathbf{B}\mathbf{B} = \mathbf{0}$$

is added to  $\mathbf{f}$ .

...

$$\begin{aligned} (14.9, \S 1.14) &\Rightarrow \nabla \cdot (\mathbf{a}\mathbf{a}) = (\nabla \cdot \mathbf{a})\mathbf{a} + \mathbf{a} \cdot \nabla \mathbf{a} \\ (14.11, \S 1.14) &\Rightarrow \nabla (\mathbf{a} \cdot \mathbf{a}) = 2 \nabla \mathbf{a} \cdot \mathbf{a} \\ \nabla \cdot (\mathbf{a} \cdot \mathbf{a} \mathbf{E}) &= \nabla \cdot \mathbf{E} (\mathbf{a} \cdot \mathbf{a}) = \nabla (\mathbf{a} \cdot \mathbf{a}) = 2 \nabla \mathbf{a} \cdot \mathbf{a} \end{aligned}$$

$$\begin{aligned}
\nabla \cdot {}^2M &= \varepsilon_0 \left( \nabla \cdot (\mathbb{E}\mathbb{E}) + c^2 \nabla \cdot (\mathbb{B}\mathbb{B}) - \frac{1}{2} \nabla \cdot (\mathbb{E} \cdot \mathbb{E} \mathbf{E} + c^2 \mathbb{B} \cdot \mathbb{B} \mathbf{E}) \right) = \\
&= \varepsilon_0 \left( \nabla \cdot \mathbb{E}\mathbb{E} + \mathbb{E} \cdot \nabla \mathbb{E} + c^2 \nabla \cdot \mathbb{B}\mathbb{B} + c^2 \mathbb{B} \cdot \nabla \mathbb{B} - \right. \\
&\quad \left. - \nabla \mathbb{E} \cdot \mathbb{E} - c^2 \nabla \mathbb{B} \cdot \mathbb{B} \right) \\
&\dots
\end{aligned}$$

## § 2. Electromagnetic waves

To derive wave equations

$$\begin{aligned}
\nabla \times (1.3^\beta) &\Rightarrow \nabla \times (\nabla \times \mathbb{E}) = -\nabla \times \dot{\mathbb{B}} \\
\nabla \times (1.3^\delta) &\Rightarrow \nabla \times (\nabla \times \mathbb{B}) = \nabla \times \left( \frac{\dot{\mathbf{j}}}{\varepsilon_0 c^2} + \frac{\dot{\mathbb{E}}}{c^2} \right) \\
\nabla \times \dot{\mathbb{E}} &= (\nabla \times \mathbb{E})^\bullet, (1.3^\beta) \Rightarrow \nabla \times \dot{\mathbb{E}} = -\ddot{\mathbb{B}} \\
\nabla \times \dot{\mathbb{B}} &= (\nabla \times \mathbb{B})^\bullet, (1.3^\delta) \Rightarrow \nabla \times \dot{\mathbb{B}} = \frac{\dot{\mathbf{j}}^\bullet}{\varepsilon_0 c^2} + \frac{\ddot{\mathbb{E}}}{c^2}
\end{aligned}$$

$$\nabla \times (\nabla \times \mathbb{E}) = \nabla \nabla \cdot \mathbb{E} - \Delta \mathbb{E}$$

$$\nabla \times (\nabla \times \mathbb{B}) = \nabla \nabla \cdot \mathbb{B} - \Delta \mathbb{B}$$

$$\begin{aligned}
\Delta \mathbb{E} - \underbrace{\nabla \nabla \cdot \mathbb{E}}_{\varrho/\varepsilon_0 (1.3^\alpha)} &= \frac{\ddot{\mathbb{E}}}{c^2} + \frac{\dot{\mathbf{j}}^\bullet}{\varepsilon_0 c^2} \\
\Delta \mathbb{B} - \underbrace{\nabla \nabla \cdot \mathbb{B}}_{0 (1.3^\gamma)} &= \frac{\ddot{\mathbb{B}}}{c^2} - \frac{\nabla \times \dot{\mathbf{j}}}{\varepsilon_0 c^2}
\end{aligned}$$

...

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0 \quad \forall \mathbf{a}$$

$$\nabla \times \nabla = \mathbf{0}, \quad \nabla \times \nabla \alpha = \mathbf{0} \quad \forall \alpha$$

vector potential  $\mathbb{A}$

$$\nabla \cdot \mathbb{B} = 0 \quad (1.3^\gamma) \Leftrightarrow \mathbb{B} = \nabla \times \mathbb{A}$$

potential  $\mathbb{A}$  is not unique and has gauge freedom  $\mathbb{A} + \nabla a$

$$\mathbb{B} = \nabla \times (\mathbb{A} + \nabla a) \Leftrightarrow \nabla \cdot \mathbb{B} = 0 \quad (1.3^\gamma)$$

scalar potential  $\phi$

$$\begin{aligned}\nabla \times \mathbf{E} &= -\dot{\mathbf{B}} \quad (1.3^\beta) \Rightarrow \nabla \times \mathbf{E} = -\nabla \times (\dot{\mathbf{A}} + \nabla \dot{a}) \Rightarrow \\ &\Rightarrow \nabla \times (\mathbf{E} + \dot{\mathbf{A}} + \nabla \dot{a}) = \mathbf{0} \Rightarrow -\nabla \phi = \mathbf{E} + \dot{\mathbf{A}} + \nabla \dot{a} \Rightarrow \\ &\Rightarrow \mathbf{E} = -\nabla(\phi + \dot{a}) - \dot{\mathbf{A}}. \quad (2.1)\end{aligned}$$

And

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \quad (1.3^\alpha) \Rightarrow \begin{aligned} -\Delta(\phi + \dot{a}) - \nabla \cdot \dot{\mathbf{A}} &= \frac{\rho}{\varepsilon_0} \\ -\Delta\phi - \nabla \cdot (\dot{\mathbf{A}} + \nabla \dot{a}) &= \frac{\rho}{\varepsilon_0} \end{aligned} \quad (2.2)\end{aligned}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \nabla \cdot \mathbf{A} - \Delta \mathbf{A}$$

$\nabla \nabla \cdot \nabla a - \nabla \cdot \nabla \nabla a = \mathbf{0}$  (partial derivatives of a smooth function commute)

$$\begin{aligned}c^2 \nabla \times \mathbf{B} &= \frac{\mathbf{j}}{\varepsilon_0} + \dot{\mathbf{E}} \quad (1.3^\delta) \Rightarrow \\ &\Rightarrow c^2 \nabla \times (\nabla \times \mathbf{A}) = \frac{\mathbf{j}}{\varepsilon_0} - \nabla(\dot{\phi} + \ddot{a}) - \ddot{\mathbf{A}} \Rightarrow \\ &\Rightarrow c^2(\nabla \nabla \cdot \mathbf{A} - \Delta \mathbf{A}) = \frac{\mathbf{j}}{\varepsilon_0} - \nabla \dot{\phi} - \nabla \ddot{a} - \ddot{\mathbf{A}}. \quad (2.3)\end{aligned}$$

With a gauge freedom it's possible to simplify wave equations (2.3) and (2.2), assuming that

$$\left. \begin{aligned} -\nabla \ddot{a} &= \nabla \dot{\phi} + c^2 \nabla \nabla \cdot \mathbf{A} \Rightarrow \ddot{a} = -\dot{\phi} - c^2 \nabla \cdot \mathbf{A} \\ -\nabla \cdot (\dot{\mathbf{A}} + \nabla \dot{a}) &= \frac{1}{c^2} \ddot{\phi} \Rightarrow \dot{\phi} = -c^2 \nabla \cdot (\mathbf{A} + \nabla a) \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow c^2 \nabla \cdot \mathbf{A} + c^2 \Delta a - c^2 \nabla \cdot \mathbf{A} = \ddot{a},$$

finally presenting as homogeneous wave equation for  $a$

$$\ddot{a} = c^2 \Delta a. \quad (2.4)$$

A more popular option is even more tight condition

$$\Delta a = 0 \Rightarrow \ddot{a} = 0, \quad \dot{\phi} + c^2 \nabla \cdot \mathbf{A} = 0$$

— Lorenz gauge condition, which gives the same effect, being just a particular — harmonic — case of (2.4).



Following from (2.2) and (2.3) with condition (2.4), equations of electromagnetic waves in the potential formulation are

$$\begin{aligned} -\Delta\phi + \frac{1}{c^2}\ddot{\Phi} &= \frac{\varrho}{\varepsilon_0}, \\ -c^2\Delta\mathbb{A} &= \frac{\mathbf{j}}{\varepsilon_0} - \ddot{\mathbb{A}}. \end{aligned} \quad (2.5)$$

...

### § 3. Electrostatics

Рассмотрение этого вопроса полезно и для последующего описания магнетизма. В статике

$$\mathbf{v} \equiv \dot{\mathbf{r}} = \mathbf{0} \Rightarrow \mathbb{B} = \mathbf{0}$$

...

The volume “ponderomotive” force, с которой электростатическое поле действует на среду ...

...

Maxwell stress tensor (1.5) in electrostatics is

$${}^2\mathbf{M} = \varepsilon_0 \left( \mathbb{E}\mathbb{E} - \frac{1}{2} \mathbb{E} \cdot \mathbb{E} \mathbf{E} \right)$$

...

### § 4. Dielectrics

Начнём с рассмотрения электростатического поля

...

В диэлектриках нет свободных зарядов: charge density  $\varrho = 0$ . Здесь вводится плотность дипольного момента

...

### § 5. Magnetostatics

Если поле (а с ним ...)

...

## § 6. Magnetics

Выяснив законы магнитостатики в общем случае, обратимся к веществу — некий опыт у нас уже есть в электростатике диэлектриков.

Начнём с рассмотрения

...

...

Насколько соответствует поведение реальных материалов представленным здесь формальным построениям — сей вопрос is out of scope этой книги.

## § 7. Magnetic rigidity

В электротехнике распространены обмотки всевозможной формы, в которых провод намотан так, что образуется некое массивное тело. Такие обмотки есть в статоре генератора автомобиля (да и в роторе), в больших промышленных электромагнитах и в магнитных системах установок “токама́к” (**тороидальная камера с магнитными катушками**) для управляемого термоядерного синтеза — примеров много. Сочетание токопровода и изоляции образует периодический композит, и одной из главных нагрузок для него является пондеромоторная магнитная сила. Рассчитывая деформации и механические напряжения в обмотке, начинают с определения магнитных сил. Поскольку распределение токов задано известной геометрией проводов, достаточно интегрирования по формуле Био-Савара (??). Термин “магнитоупругость” при этом неуместен, так как задачи магнитостатики и упругости решаются отдельно.

Однако при деформации обмотки меняются и поле  $\mathbf{j}$ , и вызываемое им поле  $\mathbf{B}$ . Объёмная сила становится равной

$$\mathbf{f} = (\mathbf{j} \times \mathbf{B})_0 + \dots \quad (7.1)$$

**Подчёркнутое слагаемое** соответствует недеформированному состоянию. Обусловленное деформацией изменение объёмной силы линейно связано с малым перемещением  $\mathbf{u}$ , поэтому матричное (по-

сле дискретизации) уравнение в перемещениях можно представить в виде

$$(C + C_m)u = \overset{\circ}{F}. \quad (7.2)$$

К “обычному” оператору линейной упругости  $C$  добавилась магнитная жёсткость  $C_m$ ;  $\overset{\circ}{F}$  — силы в недеформированном состоянии.

Добавка  $C_m$  пропорциональна квадрату тока и может стать весьма существенной в магнитных системах с сильным полем. Учёт её необходим и при недостаточной величине  $C$ ; в номинальном режиме конструкция может держать нагрузку, но дополнительная нагрузка неблагоприятного направления может оказаться “невыносимой”.

Но особенно важна роль магнитной жёсткости в задачах устойчивости. Поскольку магнитные силы потенциальны, матрица  $C_m$  симметрична, и критические параметры могут быть найдены статическим методом Euler’a.

Как иллюстрацию рассмотрим простую задачу о балке в продольном магнитном поле. Балка располагается на декартовой оси  $z$ , концы  $z=0$  и  $z=l$  закреплены, магнитная индукция  $\mathbb{B} = B\mathbf{k} = \text{constant}$ , по балке течёт постоянный (по величине) ток  $I$ . В классической модели балки при равных жёсткостях на изгиб для прогиба  $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$  легко получить следующую постановку:

...

Вводя комплексную комбинацию  $u \equiv u_1 + iu_2$ , будем иметь

...

с общим решением

...

Подстановка в граничные условия приводит к однородной системе для постоянных  $A_k$ ; приравняв нулю определитель, придём к характеристическому уравнению

...

Наименьший положительный корень  $x = 3.666$ , так что критическая комбинация параметров такова:

$$(IBl^3/a)_* = 394.2.$$

В этом решении поле  $\mathbb{B}$  считалось внешним и не варьировалось. Но если собственное поле тока в стержне сравнимо с  $\mathbb{B}$ , решение изменится и усложнится.

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# PERTURBATION METHODS (ASYMPTOTIC METHODS)

Of approximate approaches to the analysis of nonlinear systems, perturbation methods are applied the most often.

## §1. Asymptotic decompositions

Until now, function arguments were only coordinates and time. Dimensions of a body, elastic moduli, ranges of action changes in space and time — all these parameters were considered known. All asymptotic methods are based on studying how the solution depends on parameters.

Decomposition of a perturbation usually consists of the first two terms. If a problem with unknown  $u$  contains parameter  $a$ , then, assuming

$$a = a_0 + \chi a_1,$$

a solution is sought as series

$$u = u_0 + \chi u_1 + \chi^2 u_2 + \dots \quad (1.1)$$

Additional argument  $\chi$  is called a formal small parameter.

A perturbation decomposition may be divergent, but at the same time it can be more useful description of the solution than a representation that is uniformly and absolutely convergent.

Decomposition (1.1) looks like a common power series. However, perturbation methods use a different approach: series are considered as asymptotic with convergence not  $n \rightarrow \infty$  ( $n$  — number of withheld terms), but  $\chi \rightarrow 0$

$$\phi = \sum_{k=0}^{\infty} \phi_k(\chi), \quad \lim_{\chi \rightarrow 0} \frac{\phi - \sum_{k=0}^n \phi_k}{\phi_n} = 0. \quad (1.2)$$

In other words, the remainder of series is a higher order infinitesimal in comparison with the last withheld term. Obviously, a power series (1.1) also converges as an asymptotic series.

But the decomposable unknowns usually also depend on the “main” arguments — on coordinates and time. The convergence for  $\chi \rightarrow 0$  must be uniform across these “main” arguments — this is a requirement for effective use of asymptotic decompositions. For example

$$\sin(1 + \chi)t = \sin t + \chi t \cos \chi t - \frac{1}{2}\chi^2 t^2 \sin \chi t + \dots$$

doesn't satisfy the requirement of uniformity, because with  $t \rightarrow \infty$  subsequent terms prevail over previous ones.

Why asymptotic methods are so attractive? As example take a solution of the equation

$$f(u, \chi) = 0.$$

Substituting decomposition (1.1) into (1.2) and equating coefficients at the same degrees of  $\chi$ , we get

$$\begin{aligned} f(u_0, 0) &= 0, \\ (\partial_u f)_0 + u_1 (\partial_\chi f)_0 &= 0, \\ (\partial_u f)_0 u_2 + \frac{1}{2} (\partial_u^2 f)_0 u_1^2 + \frac{1}{2} (\partial_\chi^2 f)_0 + (\partial_u \partial_\chi f)_0 u_1 &= 0, \\ &\dots \end{aligned}$$

If the problem for  $u_0$  is uniquely solvable at the first step, then subsequent steps will give corrections  $u_1, u_2, \dots$ . Small corrections are barely important and  $u_0$  is enough, but then the asymptotic analysis disappears, since formally small terms in (1.2) are just discarded. However, it happens that these corrections contain in self some important information, absent in  $u_0$  — then they play the main role. It's worth mentioning that all corrections follow from the linear problem with the same operator  $(\partial_u f)_0$ .

But all non-trivial and effective solutions are obtained in other ways, non-uniqueness of the solution at the first step characterizes them. About this — in the next paragraphs.

And yes, asymptotic methods change the initially complex formulation of the problem (1.2) to a simpler one. Essential is that this does not happen by “simple” discarding of terms, but quite correctly —

equalities remain equalities. However, the convergence is not proven, so there is no complete mathematical accuracy.

In previous chapters, asymptotic problems have already arisen. The linear theory follows from the nonlinear theory via asymptotic decomposition by the value of load (§ 4.1). The momentless theory derives (accurate to edge effects) from the moment (micropolar) theory (chapter 5) when the “moment” rigidities are approaching infinity. In thermoelasticity (chapter 6), use of the heat equation (4.1, § 6.4) instead of the full balance of energy must be supported by asymptotic methods.

To a certain degree the introduction of a small parameter  $\chi \rightarrow 0$  is the weakness of all asymptotic approaches. A protestation like “infinitesimal parameters do not exist, all quantities are finite” is barely constructive here. The more relevant question is: what is a small parameter? Usually a problem is reformulated in “dimensionless” quantities, then that “dimensionless” parameter is taken as a small parameter  $\chi$ , which turns out to be small. But another way is possible too: if it’s known that some parameter  $\omega$  influences the solution only a little, then, redesignating it as  $\chi\omega$ , do asymptotic analysis for  $\chi \rightarrow 0$ .

Surely, these are not “laws” of asymptotics, not recommendations, but only considerations. There’s no common theory of asymptotic methods, their application is to some extent an art.

A deeper description of asymptotic methods is written in the books by Ali Hasan Nayfeh [107, 108].

## § 2. Splitting in a linear algebraic system

This simple case illustrates asymptotic methods well.

A linear system is considered

$$C_{ij}u_j = f_i, \quad C_{ij} = C_{ij}^{(0)} + \chi C_{ij}^{(1)} \quad (2.1)$$

с матрицей  $C_{ij}$  и столбцами неизвестных  $u_j$  и нагрузок  $f_i$ . Процесс построения асимптотического решения определяется тем, вырождена матрица  $C_{ij}^{(0)}$  или нет. Three cases are possible.

1°:  $\det C_{ij}^{(0)} \neq 0$ . Однородная задача

$$C_{ij}^{(0)} u_j = 0 \quad (2.2)$$

имеет лишь тривиальное (нулевое) решение. Матрица  $C_{ij}^{(0)}$  обратима, неоднородная задача всегда однозначно решима. Решение строится так:

$$u_j = \dots \quad (2.3)$$

...

$$2^\circ \det C_{ij}^{(0)} = 0$$

...

$$3^\circ \det C_{ij}^{(0)} = 0$$

...

### § 3. Poincaré method

This method associated with the name Jules Henri Poincaré is widely known in the theory of nonlinear oscillations. It is intended, in particular, to determine periodic solutions of equation

$$\ddot{u} + u = \chi f(u, \dot{u}) \quad (3.1)$$

....

### § 4. Van der Pol averaging method

Опять рассматривается уравнение (3.1), но теперь ищутся не только периодические решения. Вводится фазовая плоскость

...

Процедура осреднения применяется в разных разделах теории упругости (тонкие тела, композиты и др.). Осреднение вне асимптотических методов приводит обычно к незамкнутым системам уравнений; для замыкания приходится принимать некие гипотезы, что умаляет убедительность теории. Иная ситуация в асимптотике: условия разрешимости для поправочных членов с необходимостью приводят к соответствующим интегральным соотношениям.



## § 5. Coalescence of asymptotic decompositions

Основоположник метода сращивания внешних и внутренних асимптотических разложений — Ludwig Prandtl. Рассматривая течение вязкой жидкости, он заметил, что влияние малой вязкости локализовано у края — в тонком слое на краю. Вдали от края жидкость ведёт себя как идеальная. Одни и те же уравнения Navier–Stokes по-разному упрощаются вдали от края и около него [...]

Метод сращивания состоит из трёх процедур: построения внешнего разложения, построения внутренних разложений и сращивания внешнего разложения с внутренними. Метод предназначен для дифференциальных уравнений с малым параметром при старших производных. Вдали от края решение меняется плавно, формально малые члены можно отбросить, уравнение имеет пониженный порядок — всё это характерно для внешнего разложения. У края наоборот: решение меняется быстро, первостепенную роль играют старшие производные, хотя имеют малые коэффициенты. Но внешнее и внутреннее разложения — это разные формы одного решения, они должны быть состыкованы процедурой сращивания. Рассмотрим пример.

Задача о прогибе  $u(x)$  натянутой струны с закреплёнными концами под действием равномерно распределённой нагрузки может быть поставлена так:

...

## § 6. Multiple-scale analysis (method of multiple scales)

Этот метод привлекателен, естественен и — как написано у Ali Hasan Nayfeh

...

## § 7. Equations with slowly varying parameters

Рассмотрим гармонический осциллятор, собственная частота которого медленно меняется во времени

...

## § 8. Thin bodies

Задачи теории упругости часто ставятся для тонких тел — стержней, пластин и оболочек. Таковы многие элементы конструкций, но и в природе вне человека тонкие тела встречаются довольно часто.

Решение задач упругости для тонких тел многие десятилетия основывалось на неких гипотезах о распределении решения по толщине и о порядках одних неизвестных относительно других. Построенные так теории сыграли большую роль в практике инженерных расчётов. Однако, им не хватало логической стройности и убедительности, их хотелось обосновать, уточнить — а в последнее время и уничтожить (в связи с появлением великолепных компьютеров). Но открытие асимптотического расщепления прояснило картину: в тонком теле трёхмерная задача расщепляется на задачи меньшей размерности. Классические теории тонких тел получили и подтверждение, и развитие.

Рассмотрим задачу о кручении из ...

...

## Bibliography

Ali Hasan Nayfeh's book [108] is an excellent introduction to perturbation methods (asymptotic methods).

Всё разнообразие асимптотических методов представлено в монографиях ...

## §1. Initial concepts

**R**od is a thin long body. It is thought of (and modeled) as a spatial curve — the axis of rod, coated with a material ( [a figure](#) ).

The axis of rod is described as a curve by parameterizing the location vector of points of a curve. This is a morphism (a function) of one variable coordinate  $s$ ,

$$\mathbf{r} = \mathbf{r}(s). \quad (1.1)$$

Material coating gives at each rod's point a plane figure, perpendicular to the axis — normal section  $\Omega(s)$ .

...

$\mathbf{c} = \mathbf{c}(t)$  is a parametric curve parameterized by parameter  $t$ . If  $dt \neq d\ell$ , then a parameterization is not natural. For the natural parameterization  $dt = d\ell$ , where

$$d\ell = \sqrt{(dq^1)^2 + (dq^2)^2 + (dq^3)^2}.$$

Many different functions draw the same curve. But among various parametrizations of a curve, the parametrization by the arc length is special, it is also called the *natural parametrization*.

The length of an infinitesimal piece of a curve is described by the  $\Pi\nu\partial\alpha\gamma\acute{o}\rho\alpha\varsigma$ -formula

$$d\ell = \sqrt{dx^2 + dy^2 + dz^2}$$

where  $dx \equiv dq^1$ ,  $dy \equiv dq^2$ ,  $dz \equiv dq^3$  are infinitesimal changes of coordinates.  $d\ell$  is called the differential length, that is the length of an almost straight very small piece of a curve.

$\mathbf{c}(s)$  is a parametric curve parameterized by the arc length (the natural parametrization), its derivative by the arc length parameter is denoted as  $\mathbf{c}' \equiv \frac{d\mathbf{c}}{ds}$ .

If the arc length (natural) parametrization  $\mathbf{r} = \mathbf{r}(s)$  is used, then the length of derivative  $\mathbf{r}'(s) \equiv \frac{d\mathbf{r}(s)}{ds}$  (of the tangent vector) is always equal to the one unit of length:

$$\begin{aligned}\mathbf{r}(s) &= q_i(s) \mathbf{e}_i(s) = q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + q_3 \mathbf{e}_3, \\ \mathbf{r}'(s) &= \frac{d\mathbf{r}(s)}{ds} = \frac{dq_i(s)}{ds} \mathbf{e}_i(s) = q'_i(s) \mathbf{e}_i(s), \\ \|\mathbf{r}'(s)\|^2 &\equiv \mathbf{r}'(s) \cdot \mathbf{r}'(s) = q'_i(s) \mathbf{e}_i \cdot q'_j(s) \mathbf{e}_j = q'_i(s) q'_j(s) \delta_{ij} = \sum_{i=1}^3 (q'_i(s))^2, \\ ds &= \sqrt{(dq_1)^2 + (dq_2)^2 + (dq_3)^2} \Rightarrow ds^2 = \sum_{i=1}^3 (dq_i(s))^2, \\ \|\mathbf{r}'(s)\|^2 &= \frac{\sum_{i=1}^3 (dq_i(s))^2}{(ds)^2} = 1 \Rightarrow \|\mathbf{r}'(s)\| = 1. \\ &\dots\end{aligned}$$

In each section we select two perpendicular axes  $x_\alpha$  with co-directed unit vectors  $\mathbf{e}_\alpha$  ( $\alpha=1,2$ ). The reason of selection for all rod sections is the same, for example, the main axes of inertia of the section are chosen everywhere.

*The actual axis and the initial axis are different*

When vector  $\mathring{\mathbf{e}}_3$  is directed along the tangent to the initial axis with location  $\mathring{\mathbf{r}}$ , it is written as  $\mathring{\mathbf{r}}' \equiv \frac{\partial \mathring{\mathbf{r}}}{\partial s} \equiv \mathring{\mathbf{r}}_{\partial s} \equiv \mathring{\mathbf{e}}_3$ .

Vector  $\mathbf{e}_3 \equiv \mathbf{k}$  is directed along the tangent (tangentially) to the actual axis.

Together with the unit vector, tangent to the actual axis

$$\mathbf{r}' \equiv \frac{d\mathbf{r}}{ds} \equiv \mathbf{r}_{\partial s} \equiv \mathbf{e}_3 \equiv \mathbf{k},$$

we'll get for each  $s$  a triple of mutually perpendicular unit vectors.

The curvature and the torsion of the rod's axis [can be described](#) by vector  $\boldsymbol{\psi} = \psi_j \mathbf{e}_j$ :

$$\mathbf{e}'_j = \boldsymbol{\psi} \times \mathbf{e}_j, \quad \boldsymbol{\psi} = \frac{1}{2} \mathbf{e}_j \times \mathbf{e}'_j. \quad (1.2)$$

For a cylindrical (prismatic) rod  $\boldsymbol{\psi} = \mathbf{0}$ .

However, (1.2) is only an initial concept of vector  $\psi$  as of geometric features. Further in §2, after adopting the material structure of a rod, a concept of  $\psi$  will change.

Moreover, at each point of the rod's axis, thought of as a curve, there's also another triple of mutually perpendicular unit vectors, the one with the normal and the binormal vectors.

Tangent  $\mathbf{T}$ , normal  $\mathbf{N}$  and binormal  $\mathbf{B}$  vectors, together called the *Frenet–Serret frame*, are defined as:

- ✓  $\mathbf{T}$  is the unit vector tangent to a curve. The length of the tangent vector is always one unit, if the natural (arc length) parametrization of a curve is used. The tangent vector points to where a curve continues further.
- ✓  $\mathbf{N}$  is the normal unit vector, the derivative of  $\mathbf{T}$  by the curve's parameter (for instance, the arc length of a curve). The normal vector is always perpendicular to the tangent vector and points towards the center of curvature. It is divided by its length  $\|\mathbf{N}\|$  to be the one unit long.
- ✓  $\mathbf{B}$  is the binormal unit vector, the “ $\times$ ”-product (“cross product”) of  $\mathbf{T}$  and  $\mathbf{N}$ ,  $\mathbf{B} \equiv \mathbf{T} \times \mathbf{N}$ .

The Frenet–Serret formulas describe the derivatives of tangent, normal and binormal unit vectors through relations with each other.

$$\begin{aligned}\frac{d\mathbf{T}}{ds} &= \kappa\mathbf{N}, \\ \frac{d\mathbf{N}}{ds} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \frac{d\mathbf{B}}{ds} &= -\tau\mathbf{N},\end{aligned}$$

where  $d/ds$  denotes the derivative by the arc length,  $\kappa$  is the curvature and  $\tau$  is the curve's torsion. The associated collection —  $\mathbf{T}$ ,  $\mathbf{N}$ ,  $\mathbf{B}$ ,  $\kappa$ ,  $\tau$  — is called the Frenet–Serret apparatus.

Two scalars  $\kappa$  and  $\tau$  effectively define the curvature and the torsion of a space curve. Intuitively, the curvature measures the deviation of a curve from a straight line, while the torsion measures the deviation of a curve from being planar.

Two functions  $\kappa(s)$  and  $\tau(s)$  completely define the geometry of a curve, because they are coefficients of a system of ordinary differential

equations for  $\mathbf{T}$ ,  $\mathbf{N}$  and  $\mathbf{B}$ . Knowing  $\mathbf{T}(s)$ , by integration we'll obtain  $\mathbf{r}(s)$  with accuracy up to a constant rigid displacement without deformations.

...

$$\mathbf{T} = \frac{d\mathbf{r}}{ds} \quad \text{or} \quad \mathbf{T} = \mathbf{r}'$$

The derivative of  $\mathbf{T}$  consists of two multipliers — the curvature  $\kappa$  and the unit normal vector  $\mathbf{N}$

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N} \quad \text{or} \quad \mathbf{T}' = \kappa \mathbf{N}$$

Curvature  $\kappa$  is equal to the magnitude (length) of vector  $\mathbf{N}$  (the derivative of vector  $\mathbf{T}$ , the second derivative of location vector  $\mathbf{r}$ )

$$\kappa = \|\mathbf{N}\| = \|\mathbf{T}'\| = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{r}''\| = \left\| \frac{d^2\mathbf{r}}{ds^2} \right\|$$

Vector  $\mathbf{N}$  itself is divided by its length thus its length is equal to the one unit.

$$\mathbf{N} = \frac{\frac{d\mathbf{T}}{ds}}{\left\| \frac{d\mathbf{T}}{ds} \right\|} \quad \text{or} \quad \mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

The radius of curvature is the reciprocal of curvature.

$$\frac{1}{\kappa} \frac{d\mathbf{T}}{ds} = \mathbf{N} \quad \text{or} \quad \frac{1}{\kappa} \mathbf{T}' = \mathbf{N}$$

...

$$\kappa \geq 0$$

The Frenet–Serret frame is defined only if the curvature is nonzero ( $\kappa > 0$ ), it is not defined if  $\kappa = 0$ .

A line with the nonzero curvature  $\kappa \neq 0$  is considered a curve.

The zero curvature implies that a line is straight, and it lies in a plane, making the torsion equal to zero too ( $\tau = 0$ ).

...

$\mathbf{T}$  always has the unit magnitude (length). Since the length of  $\mathbf{T}$  is constant, then  $\mathbf{N}$  — the derivative of  $\mathbf{T}$  and the second derivative of location vector  $\mathbf{r}$  — is always perpendicular to  $\mathbf{T}$

$$\mathbf{T} \cdot \mathbf{T} = 1 \Rightarrow \mathbf{T}' \cdot \mathbf{T} = 0 \Rightarrow \mathbf{N} \cdot \mathbf{T} = 0.$$

Vectors of the Frenet–Serret frame make an orthonormal basis  $\mathbf{f}_i$ :

$$\mathbf{f}_1 = \mathbf{T}, \quad \mathbf{f}_2 = \mathbf{N}, \quad \mathbf{f}_3 = \mathbf{B}.$$

The location vector in the Frenet–Serret basis

$$\mathbf{r}(s) = q_j(s) \mathbf{f}_j(s) = q_1(s) \underbrace{\mathbf{f}_1(s)}_{\mathbf{T}} + q_2(s) \underbrace{\mathbf{f}_2(s)}_{\mathbf{N}} + q_3(s) \underbrace{\mathbf{f}_3(s)}_{\mathbf{B}}.$$

The tensor version of the Frenet–Serret formulas

$$\mathbf{f}_i' = {}^2\mathbf{d} \cdot \mathbf{f}_i. \quad (1.3)$$

The Frenet–Serret formulas written using the matrix notation

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

Tensor  ${}^2\mathbf{d}$  is skew-symmetric, therefore it can be represented via the companion (pseudo)vector (§ 1.7). This pseudovector is known as the Darboux vector.

$$\mathbf{D} = \tau \mathbf{T} + \kappa \mathbf{B}$$

$$\mathbf{D} = \tau \mathbf{T} + 0 \mathbf{N} + \kappa \mathbf{B}$$

With the Darboux vector, the Frenet–Serret formulas turn into the following:

$$\mathbf{T}' = \mathbf{D} \times \mathbf{T},$$

$$\mathbf{N}' = \mathbf{D} \times \mathbf{N},$$

$$\mathbf{B}' = \mathbf{D} \times \mathbf{B}$$

or as the vector version of (1.3)

$$\mathbf{f}_i' = \mathbf{D} \times \mathbf{f}_i. \quad (1.4)$$

The Darboux vector is the angular velocity vector of the Frenet–Serret frame.

...

Approximate applied theories of rods like the “strength of materials” use such concepts as internal force  $\mathbf{Q}$  and internal moment  $\mathbf{M}$ . The following relations connect them with the stress tensor

$$\mathbf{Q}(s) = \int_{\Omega} \mathbf{t}_{(\mathbf{k})} d\Omega = \int_{\Omega} \mathbf{k} \cdot \boldsymbol{\tau} d\Omega, \quad (1.5)$$

$$\mathbf{M}(s) = \int_{\Omega} \mathbf{x} \times \mathbf{t}_{(\mathbf{k})} d\Omega = \int_{\Omega} \mathbf{x} \times \mathbf{k} \cdot \boldsymbol{\tau} d\Omega. \quad (1.6)$$

$$\mathbf{k} \equiv \mathbf{e}_3, \quad \mathbf{x} = x_{\alpha} \mathbf{e}_{\alpha}, \quad \alpha = 1, 2$$

...

These thoughts about geometry and about mechanics — in particular, about internal force (1.5) and internal moment (1.6) — concern only some unique single configuration of a rod. It’s meaningless to continue these thoughts, because in reality plane and normal sections do not remain plane and normal after deforming.

*synonyms for “deplanations”*

warp, warping = деформация (deform, deformation, strain), искривление, искажение (distort, distortion), перекоc (skew), перекоcить, перекашиваться, коробиться, покоробить, коробление, out of shape, become twisted or bent

And the addition of some assumptions-hypotheses to the model, like “there are no warping (deplanations)” and initially plane sections remain plane\* introduces essential contradictions with reality. Enough to recall just one fact that without deplanations it’s impossible to acceptably describe the torsion of a rod (and not only torsion).

The very reasonable approach to modeling deformations of an elastic rod consists in asymptotic splitting of the three-dimensional

\* The two very popular beam models exist which postulate the hypothesis about the absence of deplanations. In the Euler–Bernoulli beam theory, shear deformations are neglected, plane sections remain plane and perpendicular to the axis. In the Timoshenko beam theory there’s a constant transverse shear along the section, so plane sections still remain plane, but they are no longer perpendicular to the axis.



problem with a small thickness. But for a complex asymptotic procedure it would be much simpler to have whatever solution version beforehand. And the direct approach, when the one-dimensional model of a rod is a material line, gives such a version.

The primary question for building the one-dimensional model: what degrees of freedom — besides translation — do the particles of a material line possess?

It is known that rods are sensitive to the moment loads. And the presence of moments among generalized forces indicates the presence of rotational degrees of freedom. Therefore as the one-dimensional model of a rod it is reasonable to take the Cosserat line — it consists of infinitesimal absolutely rigid bodies. However, another new degrees of freedom may also appear — as for thin-walled rods, described in the dedicated chapter ([chapter 10](#)).

In the mechanics of continuous elastic media, the place of rods is specific. First, moments play the main role here, and not the role of small additions as in a three-dimensional Cosserat continuum. Second, rods can be used to test models with additional degrees of freedom — and before the presence of these degrees will be researched on a three-dimensional model.

The following section presents and describes the simple one-dimensional Cosserat-like moment model.

## § 2. Kinematics of Cosserat lines

Model described further is a simplified version of [chapter 5](#).

There's no more a triple of material coordinates  $q_i$ , but the only one —  $s$ . It may be the arc length parameter in the initial configuration. The motion of a particle over time is described by the location vector  $\mathbf{r}(s, t)$  and the rotation tensor  $\mathbf{O}(s, t)$ . Linear and angular ([9.8](#), [§ 1.9](#)) velocities of a rod's particle are

$$\dot{\mathbf{r}} = \mathbf{v}, \quad (2.1)$$

$$\dot{\mathbf{O}} = \boldsymbol{\omega} \times \mathbf{O} \Leftrightarrow \boldsymbol{\omega} = -\frac{1}{2}(\dot{\mathbf{O}} \cdot \mathbf{O}^T)_{\times}. \quad (2.2)$$

Deformation of a rod as a Cosserat line is defined by two vectors

$$\mathbf{\Gamma} \equiv \mathbf{r}' - \mathbf{O} \cdot \mathring{\mathbf{r}}', \quad (2.3)$$

$$\boldsymbol{\kappa} \equiv -\frac{1}{2}(\mathbf{O}' \cdot \mathbf{O}'^\top)_\times \Leftrightarrow \mathbf{O}' = \boldsymbol{\kappa} \times \mathbf{O}. \quad (2.4)$$

$$\left( \mathring{\mathbf{r}}(s) \equiv \mathbf{r}(s, 0) \right)$$

$$\|\mathbf{e}_3\| = \|\mathring{\mathbf{e}}_3\| = 1 = \text{constant}$$

(2.3) and (2.4) are really the deformation vectors, it follows from their equality to zero on displacements of a body as a rigid whole (.... add some equation(s) here describing displacements as a rigid whole .....).

Next we will clarify the idea of the first deformation vector  $\mathbf{\Gamma}$ . Without loss of universality, the parameter  $s$  is the initial arc length and the initial basis vector  $\mathring{\mathbf{e}}_3$  is directed along the tangent in the initial configuration:  $\mathring{\mathbf{e}}_3 = \mathring{\mathbf{r}}'$ . Then

$$\begin{aligned} \mathbf{\Gamma} \equiv \mathbf{r}' - \mathbf{O} \cdot \mathring{\mathbf{r}}', \quad \mathring{\mathbf{r}}' = \mathring{\mathbf{e}}_3, \quad \mathbf{O} \cdot \mathring{\mathbf{r}}' = \mathbf{O} \cdot \mathring{\mathbf{e}}_3 = \mathbf{e}_3 \\ \Rightarrow \mathbf{\Gamma} = \mathbf{r}' - \mathbf{e}_3, \end{aligned} \quad (2.5)$$

$$\begin{aligned} \mathbf{r}' = \mathbf{\Gamma} + \mathbf{e}_3, \quad \|\mathbf{r}'\|^2 = \mathbf{r}' \cdot \mathbf{r}' = (\mathbf{\Gamma} + \mathbf{e}_3) \cdot (\mathbf{\Gamma} + \mathbf{e}_3) \\ = \mathbf{\Gamma} \cdot \mathbf{\Gamma} + 2\mathbf{\Gamma} \cdot \mathbf{e}_3 + \mathbf{e}_3 \cdot \mathbf{e}_3, \end{aligned}$$

$$\|\mathbf{r}'\| = \sqrt{\|\mathbf{\Gamma}\|^2 + 2\Gamma_3 + 1},$$

$$\|\mathbf{r}'\| - 1 = \sqrt{\|\mathbf{\Gamma}\|^2 + 2\Gamma_3 + 1} - 1 = \Gamma_3 + \infty^{-1}(\|\mathbf{\Gamma}\|^2). \quad (2.6)$$

Equality (2.6) describes a relative elongation. Roughly speaking, component  $\Gamma_3 \equiv \mathbf{\Gamma} \cdot \mathbf{e}_3$  can be considered an elongation, and components  $\Gamma_1 \equiv \mathbf{\Gamma} \cdot \mathbf{e}_1$ ,  $\Gamma_2 \equiv \mathbf{\Gamma} \cdot \mathbf{e}_2$  present a transverse shear. It's more accurate to rely on formulas (2.5) and (2.6).

.....

The Cosserat-like model of a rod doesn't have a section as a plane figure.

....

### § 3. Balance of forces and moments

Possible loads acting on a rod as the Cosserat line are forces and moments: on infinitesimal element  $ds$  of a rod act external

force  $qds$  and external moment  $mds$ . The internal interactions will be force  $Q(s)$  and moment  $M(s)$  — this is the action of the particle with coordinate  $s+0$  on the particle with  $s-0$ . The action–reaction principle gives that a reverse (a change of direction) of coordinate  $s$  changes signs of  $Q$  and  $M$ .

...

#### § 4. Principle of virtual work and consequences

For a piece of rod  $s_0 \leq s \leq s_1$  formulation of the principle is as follows

.....

Conventionally  $\mathbf{a}$  is the tensor of stiffness for bending and twisting,  $\mathbf{b}$  is the tensor of stiffness for (ex)tension and shear, and  $\mathbf{c}$  is the tensor of crosslinks.

Stiffness tensors rotate along with a particle:

...

#### § 5. Classical Kirchhoff's model

It is also called the *Kirchhoff's rod theory*.

Until now functions  $\mathbf{r}(s, t)$  and  $\mathbf{O}(s, t)$  were independent. The Kirchhoff's classical theory postulates the internal constraint

$$\mathbf{\Gamma} = \mathbf{0} \Leftrightarrow \mathbf{r}_{\partial s} = \mathbf{O} \cdot \overset{\circ}{\mathbf{r}}_{\partial s} \text{ or } \mathbf{r}' = \mathbf{O} \cdot \overset{\circ}{\mathbf{r}}'. \quad (5.1)$$

Having the idea of vector  $\mathbf{\Gamma}$  (2.5), here we can tell that: (1) a rod is non-extensible, (2) there are no transverse shears.

If basis vector  $\overset{\circ}{\mathbf{e}}_3$  was directed along the tangent to the axis in the initial configuration, then it will remain on the tangent also after deforming. The rod particles rotate only together with the tangent to the axis and around it.

Уравнения баланса сил и моментов (импульса и момента импульса) не меняются от введения связи (5.1). Но локальное вариационное соотношение (...) становится короче:

...

## § 6. Euler's problem about stability of rods

Рассматривается прямой стержень, закреплённый на одном конце и нагруженный силой  $\mathbf{P}$  на другом (рисунок ?? 123 ??). Сила “мёртвая” (не меняется в процессе деформирования)

...

## § 7. Variational equations

In the nonlinear mechanics of elastic media variational equations are useful, which describe a small change in the current configuration (§ 3.12).

Variating equations of the complete system of the Cosserat model, we get

$$\begin{aligned}\delta Q' + \delta \mathbf{q} &= \rho (\mathbf{u} + \delta \boldsymbol{\tau} \times \boldsymbol{\varepsilon})^{\bullet\bullet}, \\ \delta \mathbf{M}' + \mathbf{u}' \times \mathbf{Q} + \mathbf{r}' \times \delta \mathbf{Q} + \delta \mathbf{m} &= \dots\end{aligned}$$

...

## § 8. Non-shear model with (ex)tension

Kirchhoff's model with internal constraint  $\boldsymbol{\Gamma} = \mathbf{0}$  (5.1) doesn't describe the simplest case of extension/compression of a straight rod. This nuisance disappears with “softening” of the constraint, for example adding the possibility of (ex)tension and inhibiting only the transverse shear

$$\boldsymbol{\Gamma} = \Gamma \mathbf{e}_3 \Leftrightarrow \Gamma_\alpha = 0 \quad (8.1)$$

...

## § 9. Mechanics of flexible thread

A thread is a momentless rod.

A flexible thread (chain) is simpler than a rod, because its particles are “simple” material points with only translational degrees of freedom. Therefore среди нагрузок нет моментов, только “ли-

нейные” силы — внешние распределённые  $\mathbf{q}$  и внутренние сосредоточенные  $\mathbf{Q}$ . Движение нити полностью определяется одним вектором-радиусом  $\mathbf{r}(s, t)$ , а инерционные свойства — линейной плотностью  $\rho(s)$ .

Вот принцип виртуальной работы для куска нити  $s_0 \leq s \leq s_1$

$$\int_{s_0}^{s_1} \left( (\mathbf{q} - \rho \ddot{\mathbf{r}}) \cdot \delta \mathbf{r} - \delta \Pi \right) ds + \left[ \mathbf{Q} \cdot \delta \mathbf{r} \right]_{s_0}^{s_1} = 0. \quad (9.1)$$

...

Механика нити детально описана в книге [29].

## § 10. Linear theory

В линейной теории внешние воздействия считаются малыми, а отсчётная конфигурация — ненапряжённым состоянием покоя. Уравнения в вариациях в этом случае дают

...

## § 11. Case of small thickness

Когда относительная толщина стержня мала, тогда модель типа Cosserat уступает место классической. Понятие “толщина” определяется соотношением жёсткостей:  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  — разных единиц измерения; полагая  $\mathbf{a} = h^2 \hat{\mathbf{a}}$  and  $\mathbf{c} = h \hat{\mathbf{c}}$ , where  $h$  is some scope of length, получим тензоры  $\hat{\mathbf{a}}$ ,  $\hat{\mathbf{b}}$  and  $\hat{\mathbf{c}}$  с одной и той же единицей. Подбирая  $h$  так, чтобы сблизилсь характерные значения тензоров  $\hat{\mathbf{a}}$ ,  $\hat{\mathbf{b}}$  and  $\hat{\mathbf{c}}$ , найдём equivalent thickness  $h$  стержня (для реальных трёхмерных стержней  $h$  где-то на уровне диаметра сечения).

Представив  $\mathbf{Q}$  and  $\mathbf{M}$  через векторы бесконечномалой линейной деформации

...

Переход модели типа Cosserat в классическую кажется более очевидным, если непосредственно интегрировать уравнения (...)

...

## § 12. Saint-Venant's problem

Трудно переоценить ту роль, которую играет в механике стержней классическое решение Saint-Venant. О нём уже шла речь в § 4.13.

Вместо условий ...

...

## § 13. Finding stiffness by energy

Для определения тензоров жёсткости  $\mathbf{a}$ ,  $\mathbf{b}$  и  $\mathbf{c}$  одномерной модели достаточно решений трёхмерных задач для стержня. Но тут возникают два вопроса: какие именно задачи рассматривать и что нужно взять из решений?

Saint-Venant's problem выделяется среди прочих, ведь оттуда берётся жёсткость на кручение.

Вдобавок есть много точных решений, получаемых таким путём: задаётся поле  $\mathbf{u}(\mathbf{r})$ , определяется  $\check{\boldsymbol{\tau}} = {}^4\mathbf{C} \cdot \nabla \mathbf{u}$ , затем находятся объёмные  $\mathbf{f} = -\nabla \cdot \check{\boldsymbol{\tau}}$  и поверхностные  $\mathbf{p} = \mathbf{n} \cdot \check{\boldsymbol{\tau}}$  нагрузки.

Но что делать с решением? Ясно, что  $\mathbf{Q}$  and  $\mathbf{M}$  в стержне — это интегралы по сечению (...). И совсем не ясно, что считать перемещением и поворотом в одномерной модели. Если предложить, например, такой вариант (индекс  $u$  — размерность модели)

$$\mathbf{u}_1(z) = \Omega^{-1} \int_{\Omega} \mathbf{u}_3(\mathbf{x}, z) d\Omega, \quad \boldsymbol{\theta}(z) = \frac{1}{2} \Omega^{-1} \int_{\Omega} \nabla \times \mathbf{u}_3 d\Omega,$$

то чем другие возможные представления хуже?

Помимо  $\mathbf{Q}$  and  $\mathbf{M}$ , есть ещё величина, не вызывающая сомнений — упругая энергия. Естественно потребовать, чтобы в одномерной и в трёхмерной моделях энергии на единицу длины совпали. При этом, чтобы уйти от различий в трактовках  $\mathbf{u}_1$  и  $\boldsymbol{\theta}$ , будем исходить из complementary energy  $\hat{\Pi}(\mathbf{M}, \mathbf{Q})$ :

$$\hat{\Pi}(\mathbf{M}, \mathbf{Q}) = \int_{\Omega} \Pi_3 d\Omega$$

...

## § 14. Variational method of building one-dimensional model

Мы только что определили жёсткости стержня, полагая, что одномерная модель линии типа Cosserat адекватно описывает поведение трёхмерной модели. “Одномерные” представления ассоциируются со следующей картиной перемещений в сечении:

$$\mathbf{u}(s, \mathbf{x}) = \mathbf{U}(s) + \boldsymbol{\theta}(s) \times \mathbf{x}. \quad (14.1)$$

Однако, такое поле  $\mathbf{u}$  не удовлетворяет уравнениям трёхмерной теории(??добавить, каким именно). Невозможно пренебречь возникающими невязками в дифференциальных уравнениях и краевых условиях.

Формально “чистым” является вариационный метод сведения трёхмерной проблемы к одномерной, называемый иногда методом внутренних связей. Аппроксимация (14.1) подставляется в трёхмерную формулировку вариационного принципа минимума потенциальной энергии (9.1, § 4.9)

$$\mathcal{E}(\mathbf{u}) = \int_V (\Pi(\mathbf{u}) - \mathbf{f} \cdot \mathbf{u}) dV - \int_{\sigma_2} \mathbf{p} \cdot \mathbf{u} d\sigma \rightarrow \min,$$

которая после интегрирования по сечению становится одномерной. Если  $\mathbf{U}$  и  $\boldsymbol{\theta}$  варьируются независимо, получаем модель типа Коссера. В случае  $\mathbf{U}' = \boldsymbol{\theta} \times \mathbf{t}$  приходим к классической модели.

Метод внутренних связей привлекателен, его продолжают “переоткрывать”. С его помощью возможно моделировать тела с неоднородностью и анизотропией, он легко обобщается на динамику, если  $\mathbf{f}$  дополнить неварьируемой динамической добавкой до  $\mathbf{f} - \rho \ddot{\mathbf{u}}$ . Можно рассматривать и стержни переменного сечения, и даже нелинейно упругие, ведь вариационная постановка есть (гл. 3).

Аппроксимацию (14.1) можно дополнить слагаемыми с внутренними степенями свободы. Понимая необходимость учёта депланаций, некоторые авторы

...

...

For variational construction of one-dimensional models it's convenient to use the Reissner–Hellinger principle (§ 4.11) with independent approximation of stresses [13]. In this case, some consistency between  $\mathbf{u}$  and  $\boldsymbol{\tau}$  is needed.

Many advantages of the variational method are opposed by the one, but significant disadvantage. Introducing approximations within cross-sections, we impose our unreal simplifications on reality. The variational method is more suitable for applied calculations.

## § 15. Asymptotic splitting of three-dimensional problem

Asymptotic splitting can be considered fundamental for describing the mechanics of rods. One-dimensional models paint only the part of a picture, two-dimensional problems in cross-sections paint the other part, and together they present the solution of a three-dimensional problem for a small thickness.

How to introduce a small parameter  $\chi$  into a three-dimensional problem? The easiest way to do it is through representation of the location vector (§ 1):

$$\mathbf{R}(x_\alpha, s) = \chi^{-1} \mathbf{r}(s) + \mathbf{x}, \quad \mathbf{x} \equiv x_\alpha \mathbf{e}_\alpha(s), \quad \alpha = 1, 2.$$

For an orthonormal basis, upper and lower indices do not differ

$$q_i = q^i, \quad \mathbf{R}^i = \mathbf{R}_i = \mathbf{R}_{\partial i} = \frac{\partial \mathbf{R}}{\partial q^i}.$$

Three coordinates are

$$q^1 = x_1, \quad q^2 = x_2, \quad q^3 = s.$$

The basis vectors are

$$\mathbf{e}_1, \quad \mathbf{e}_2, \quad \mathbf{t} \equiv \mathbf{e}_3.$$

Representation of the Hamilton operator  $\nabla$

$$\nabla = \mathbf{R}^i \frac{\partial}{\partial q^i} = \nabla_\perp + v^{-1} \mathbf{t} (\partial_s - \psi_t D), \quad \nabla_\perp \equiv \mathbf{e}_\alpha \frac{\partial}{\partial x_\alpha},$$

$$v \equiv \mathbf{R}_{\partial 1} \times \mathbf{R}_{\partial 2} \cdot \mathbf{R}_{\partial 3} = \chi^{-1} + \mathbf{t} \cdot \boldsymbol{\psi}_\perp \times \mathbf{x}, \quad D \equiv \mathbf{t} \cdot \mathbf{x} \times \nabla_\perp,$$

$$\boldsymbol{\psi}_\perp + \psi_t \mathbf{t} = \boldsymbol{\psi}$$

(the meaning of vector  $\boldsymbol{\psi}$  is the same as in § 1).



The Cauchy stress tensor

$$\boldsymbol{\tau} = \boldsymbol{\tau}_\perp + \sigma_t \boldsymbol{t} \boldsymbol{t} + 2 \boldsymbol{t} \boldsymbol{\tau}_\parallel^S, \quad \boldsymbol{\tau}_\perp \equiv \tau_{\alpha\beta} \boldsymbol{e}_\alpha \boldsymbol{e}_\beta, \quad \boldsymbol{\tau}_\parallel \equiv \tau_{3\alpha} \boldsymbol{e}_\alpha.$$

...

## § 16. Thermal deformation and stress

The direct approach, so efficient for making one-dimensional Cosserat and Kirchhoff models, isn't applicable for problems of thermoelasticity. The transition from the three-dimensional model to the one-dimensional can be realized either by variational or asymptotic way.

Описанный в § 14 вариационный метод целиком переносится на термоупругость — включая задачи с неоднородностью и анизотропией, переменным сечением, динамические и даже нелинейные. Для этого нужно (§ 6.8) in the Lagrange principle of minimum potential energy заменить потенциал  $\Pi(\boldsymbol{\varepsilon})$  свободной энергией

...

## Bibliography

Unlike other topics of the theory of elasticity, rods are presented in books very modestly. Narration style of “strength of materials” prevails, more exact and perfect approaches seem impossible or unnecessary to many authors.

But many interesting articles have been published, their reviews are presented, for example, by S. Antman [1], В.В. Елисеев [13] and А.А. Илюхин [16].

*chapter 10*

## THIN-WALLED RODS

### §1. Variational approach

In chapter 9, rods with massive cross-sections were described. But other rods are also widely used — thin-walled ones, with sections of narrow bands of various shapes: corner, Z-beam, channel (C-beam), I-beam, ... If rods are like lines, then in a thin-walled rod the section itself also looks like line. The three dimensions — the thickness and the length of a section, as well as the length of a rod — have different decimal orders.

The applied theories of thin-walled rods are known, they are described, for example, in books ...

There is also the exact theory, based on asymptotic splitting of the three-dimensional problem [13]. In sum, the complex asymptotic analysis confirmed most of the hypotheses of applied theories.

As introduction to the mechanics of thin-walled rods, here is the following variational procedure with warping (deplanation) of cross-sections.

Consider the simplest case of a cylindrical rod with a thin, simply connected section (... рисунок ...). Площадь сечения *do*, боковая поверхность свободна, нагрузка в объёме —  $\mathbf{f}$ , на торце  $z=z_1$  известны поверхностные силы  $\mathbf{p}(\mathbf{x})$  ( $\mathbf{x}=x_\alpha \mathbf{e}_\alpha$ ), торец  $z=z_0$  закреплён.

...

## § 2. Equations with small parameter

Рассмотрим призматический стержень с односвязным сечением в виде тонкой криволинейной полосы постоянной толщины  $h$ . Радиус-вектор в объёме представим следующим образом:

...

## § 3. First step of the asymptotic procedure

*Outer decomposition*

Из системы

...

*Inner decomposition near  $s_0$*

Выпишем уравнения для

...

*Coalescence*

Стыковка внутреннего и внешнего разложений

...

## § 4. Second step

*Outer decomposition*

Из системы

...

*Inner decomposition near  $s=s_0$*

Из общей системы

...

Поскольку рассматриваются поправочные члены асимптотических разложений

...

## § 5. Third step

### *Outer decomposition*

Из системы

....

### *Inner decomposition around $s = s_0$*

Как уже отмечалось, внутренние разложения нужны для постановки краевых условий на концах

...

### *Coalescence*

В плоской задаче имеем следующее двучленное внешнее разложение:

....

## § 6. Fourth step

Здесь понадобится лишь внешнее разложение. Более того: в этом приближении мы не будем искать решения уравнений — будет достаточно лишь условий разрешимости. Напомним, что философия наша такова: разыскиваются лишь главные члены асимптотических разложений, но для полного их определения могут понадобиться

...

## § 7. Displacements

Расписывая тензорное соотношение

...

## § 8. Results of asymptotic analysis

Определение главных членов асимптотики напряжений и перемещений для тонкостенных стержней оказалось намного сложнее, чем в случае массивного сечения. Дадим сводку полученных выше итоговых результатов.

Перемещение:

...

## Bibliography

Помимо известных книг ... Материал главы содержится в [13], где можно найти и обширный список статей.

# chapter 11

## SHELLS AND PLATES

### §1. Surface geometry

Surface is described by function (morphism)

$$\mathbf{r} = \mathbf{r}(q^\alpha), \quad \alpha = 1, 2 \quad (1.1)$$

of two mutually independent variable parameters (coordinates)  $q^\alpha$ , then  $\mathbf{r}$  is location vector (radius vector) of surface's points.

#### Examples

- ✓ a linear mapping is a plane  $\mathbf{r}(a, b) = a\mathbf{e}_1 + b(\mathbf{e}_2 + \mathbf{e}_3)$
- ✓ a helicoid  $\mathbf{r}(u, v) = u \sin v \mathbf{e}_1 + u \cos v \mathbf{e}_2 + v \mathbf{e}_3$
- ✓ a cone  $\mathbf{r}(u, v) = u \sin v \mathbf{e}_1 + u \cos v \mathbf{e}_2 + u \mathbf{e}_3$
- ✓ a cylinder of radius  $r = \text{constant}$ 

$$\mathbf{r}(u, v) = r(\cos u \mathbf{e}_1 + \sin u \mathbf{e}_2) + v \mathbf{e}_3$$
- ✓ a torus of revolution with radii  $r$  and  $R$ 

$$\mathbf{r}(p, q) = \mathbf{e}_1(r \cos p + R) \cos q + \mathbf{e}_2(r \cos p + R) \sin q + \mathbf{e}_3 r \sin p$$
- ✓ a 2-sphere — a torus with  $R = 0$ 

$$\mathbf{r}(p, q) = r(\cos p \cos q \mathbf{e}_1 + \cos p \sin q \mathbf{e}_2 + \sin p \mathbf{e}_3)$$
- ✓ a paraboloid  $\mathbf{r}(u, v) = u\mathbf{e}_1 + v\mathbf{e}_2 + (u^2 + v^2)\mathbf{e}_3$  or cylindrical parameterization  $\mathbf{r}(\rho, \vartheta) = \rho \cos \vartheta \mathbf{e}_1 + \rho \sin \vartheta \mathbf{e}_2 + \rho^2 \mathbf{e}_3$  for paraboloid of revolution.

Continuous change of one coordinate  $q^1$ , while second coordinate  $q^2 = u^* = \text{constant}$  is “frozen”, gives coordinate line  $\overset{(1)}{\mathbf{r}}(q^1) = \mathbf{r}(q^1, u^*)$ . A crossing of two coordinate lines  $q^1 = v^*$  and  $q^2 = w^*$  uniquely identifies a point  $\mathbf{r}(v^*, w^*)$  of the surface.

Vectors

$$\mathbf{r}_{\partial\alpha} \equiv \partial_\alpha \mathbf{r}, \quad \partial_\alpha \equiv \frac{\partial}{\partial q^\alpha} \quad (1.2)$$

are tangent to coordinate lines. If they are linearly independent (that is  $\mathbf{r}_{\partial 1} \times \mathbf{r}_{\partial 2} \neq \mathbf{0}$ )\*, they compose a local basis for representing any vector  $\mathbf{v}^{(2)}$  in the tangent plane as linear combination

$$\begin{aligned} \mathbf{v}^{(2)} &= v^\alpha \mathbf{r}_{\partial \alpha} = v_\alpha \mathbf{r}^\alpha, \\ v^\alpha &= \mathbf{v}^{(2)} \cdot \mathbf{r}^\alpha, \quad v_\alpha = \mathbf{v}^{(2)} \cdot \mathbf{r}_{\partial \alpha}, \quad \mathbf{r}^\alpha \cdot \mathbf{r}_{\partial \beta} = \delta_\beta^\alpha. \end{aligned} \quad (1.3)$$

Here  $\mathbf{r}^\alpha$  is local reciprocal basis in the (co)tangent plane.

Field of unit normal vectors  $\mathbf{n}(q^\alpha)$  adds at every point of the surface ( $\forall \mathbf{r}(q^\alpha) \Leftrightarrow \forall q^\alpha$ ) a unit\*\* normal

$$\mathbf{n} = \frac{\mathbf{r}_{\partial 1} \times \mathbf{r}_{\partial 2}}{\|\mathbf{r}_{\partial 1} \times \mathbf{r}_{\partial 2}\|}. \quad (1.4)$$

At non-singular points, three vectors  $\mathbf{r}_{\partial 1}$ ,  $\mathbf{r}_{\partial 2}$  and  $\mathbf{n}$  can be taken as a basis for the entire three-dimensional space, giving a decomposition of any vector (and any tensor too), for example  $\mathbf{u} = u^\alpha \mathbf{r}_{\partial \alpha} + u^n \mathbf{n}$ .

$$u^n = u_n$$

for a 2-sphere of unit radius

$$\mathbf{r}_{\partial p} \times \mathbf{r}_{\partial q} = -\det \begin{bmatrix} -\sin p \cos q & \mathbf{e}_1 & -\cos p \sin q \\ -\sin p \sin q & \mathbf{e}_2 & \cos p \cos q \\ \cos p & \mathbf{e}_3 & 0 \end{bmatrix} = \dots$$

Единичные (“метрические”) бивалентные тензоры  $\mathbf{E}$  в пространстве and  $\mathbf{I}$  в касательной плоскости

$$\mathbf{E} = \mathbf{I} + \mathbf{n}\mathbf{n}, \quad \mathbf{I} \equiv \mathbf{E} - \mathbf{n}\mathbf{n} = \mathbf{r}_{\partial \alpha} \mathbf{r}^\alpha = \mathbf{r}^\alpha \mathbf{r}_{\partial \alpha}.$$

Представление вектора положения  $\mathbf{h}^{(3)}$  любой точки пространства на расстоянии  $h$  от поверхности ( $\partial_\alpha h = 0$ )

$$\mathbf{h}^{(3)}(q^\alpha, h) = \mathbf{r}(q^\alpha) + h \mathbf{n}(q^\alpha) \quad (1.5)$$

\* Sometimes somewhere — at so-called singular points — it is not so.

As example, for a 2-sphere of unit radius

$$\begin{aligned} \mathbf{r}(p, q) &= \cos p \cos q \mathbf{e}_1 + \cos p \sin q \mathbf{e}_2 + \sin p \mathbf{e}_3, \\ \mathbf{r}_{\partial p} &= \partial_p \mathbf{r} = -\sin p \cos q \mathbf{e}_1 - \sin p \sin q \mathbf{e}_2 + \cos p \mathbf{e}_3, \\ \mathbf{r}_{\partial q} &= \partial_q \mathbf{r} = -\cos p \sin q \mathbf{e}_1 + \cos p \cos q \mathbf{e}_2 + 0 \mathbf{e}_3, \end{aligned}$$

a pole  $p = \pm \frac{\pi}{2}$  is a singular point:  $\mathbf{r}_{\partial q}|_{p=\pm\pi/2} = \mathbf{0}$ .

\*\*  $\|\mathbf{a}\| \equiv \sqrt{\mathbf{a} \cdot \mathbf{a}}$  is the length of vector  $\mathbf{a}$ .

даёт локальный касательный базис

$$\begin{aligned} \mathbf{n}_{\partial n} &\equiv \mathbf{n} = \mathbf{n}^{(3)}, \\ \mathbf{n}_{\partial \alpha} &\equiv \partial_\alpha \mathbf{n} = \partial_\alpha \mathbf{r} + h \partial_\alpha \mathbf{n} = \mathbf{r}_{\partial \alpha} + h \mathbf{r}_{\partial \alpha} \cdot \mathbf{r}^\beta \partial_\beta \mathbf{n}. \end{aligned}$$

...

дифференциальный оператор “набла”

$$\text{в пространстве } \nabla \equiv \mathbf{n}^i \partial_i$$

$$\text{в касательной плоскости } \nabla \equiv \mathbf{r}^\alpha \partial_\alpha$$

...

$$\mathbf{n}_{\partial \alpha} = \mathbf{r}_{\partial \alpha} \cdot \left( \mathbf{r}^\beta \mathbf{r}_{\partial \beta} + h \mathbf{r}^\beta \partial_\beta \mathbf{n} \right) = \mathbf{r}_{\partial \alpha} \cdot \left( \mathbf{I} + h \nabla \mathbf{n} \right) = \mathbf{r}_{\partial \alpha} \cdot \left( \mathbf{I} - {}^2\mathbf{c}h \right)$$

Two-coordinate bivalent tensor

$${}^2\mathbf{c} \equiv -\nabla \mathbf{n} = -\mathbf{r}^\alpha \partial_\alpha \mathbf{n} \quad (1.6)$$

characterizes the surface’s curvature.

...

$$\text{кобазис } \mathbf{n}^\alpha \cdot \mathbf{n}_{\partial \beta} = \delta_\beta^\alpha, \quad \mathbf{n}^i \cdot \mathbf{n}_{\partial j} = \delta_j^i$$

$$\mathbf{n}^\alpha = \left( \mathbf{I} + h \nabla \mathbf{n} \right)^{-1} \cdot \mathbf{r}^\alpha, \quad \mathbf{n}^n = \mathbf{n}$$

связь между  $\nabla$  и  $\nabla$

$$\nabla = \left( \mathbf{I} + h \nabla \mathbf{n} \right)^{-1} \cdot \nabla + \mathbf{n} \partial_n$$

...

## § 2. Model of a shell

Располагая моделями трёхмерного моментного континуума, *стержней и пластин*, не так уж тяжело разобраться в механике оболочек.

Как геометрический объект, оболочка определяется срединной поверхностью и толщиной  $h$ , так что в представлении (1.5)  $-h/2 \leq h \leq h/2$ .

...

## § 3. Balance of forces and moments for a shell

При  $\delta u = \text{constant}$  и  $\delta \varphi = 0$  (трансляция) ...



...

## § 4. Shells: Relations of elasticity

Локальное соотношение (??) после вывода уравнений баланса ...

...

## § 5. Classical theory of shells

Вышеизложенная теория (напоминающая балку Тимошенко и континуумы Cosserat) рассматривает rotations  $\varphi$  независимо от displacements  $\mathbf{u}$ . Но обыденный опыт подсказывает: материальный элемент, нормальный к срединной поверхности до деформации, остаётся таковым и после (кинематическая гипотеза Kirchhoff'a). В классической теории Kirchhoff'a, Арона и Лява поле  $\varphi$  выражается через  $\mathbf{u}$ , что в конце концов даёт свести всё к одному векторному уравнению для  $\mathbf{u}$ .

Предположим, что в основе классической теории лежит внутренняя связь

...

## § 6. Shells: A plate

Plate is the simplest kind of shell. Единичный перпендикуляр  $\mathbf{n} = \mathbf{k}$  направлен по декартовой оси  $z$ , в качестве координат ...

...

## § 7. Shells: Approach with Lagrange multipliers

Уязвимым местом этого изложения теории оболочек являются формулы

...

## § 8. Cylindrical shell

Существуют разные уравнения цилиндрической оболочки. Приводятся громоздкие выкладки с отбрасыванием некоторых малых членов, и не всегда ясно, какие именно члены действительно можно отбросить.

Предлагаемая читателю теория оболочек иного свойства: лишних членов нет, все уравнения записаны в компактной тензорной форме — остаётся лишь **грамотно** действовать с компонентами тензоров. В качестве иллюстрации рассмотрим цилиндрическую оболочку.

В декартовой системе

...

## § 9. Shells: Common theorems

Пусть край закреплён

...

## § 10. Shells: Boundary conditions

В рамках рассматриваемого прямого подхода к оболочкам как материальным поверхностям наиболее надёжным способом вывода граничных условий представляется вариационный. Исходим из вариационного уравнения:

...

## § 11. Shells of revolution

*Surface of revolution (reference surface of shell of revolution) is created by rotating a plane curve (the meridian, the generatrix) about a straight line in the plane of curve (an axis of rotation).*

Разберёмся в геометрии поверхности вращения ( **рисунок** ). Меридиан можно задать зависимостью декартовых координат

...

## § 12. Momentless theory of shells

В отличие от пластины, оболочка способна выдерживать нормальную распределённую нагрузку без появления внутренних моментов. В безмоментном состоянии напряжения равномерно распределены по толщине оболочки, безмоментные оболочечные конструкции можно считать оптимально спроектированными.

Уравнения безмоментной теории

...

## § 13. Shells: Nonlinear momentless theory

Вышеизложенную безмоментную теорию оболочек возможно просто и корректно обобщить на нелинейную постановку. Материальная поверхность состоит из частиц

...

## § 14. Shells: Other variant of classical theory

Выше при изложении моментной теории оболочек частицы материальной поверхности считались твёрдыми телами с шестью степенями свободы

...

## § 15. Plates: Overall concepts

**П**ластиной называется тонкое трёхмерное тело, ограниченное двумя параллельными плоскостями и боковой цилиндрической поверхностью (?? [рисунок](#) ??). В декартовых координатах  $x_1, x_2, z$  поперечная координата ...

...

В теории пластин рассматриваются двумерные задачи. Переход от трёхмерной задачи наиболее достоверен на пути асимптотики. Но логическая стройность и эффективность присуща и вариационному подходу, основанному на аппроксимации по толщине решения трёхмерной вариационной задачи. Самое же простое корректное изложение теории пластин характерно для прямого подхода к ним как материальным плоскостям.

...

## § 16. Timoshenko-like model of a plate (direct approach)

Пластина рассматривается как материальная плоскость, частицы которой

...

## § 17. Kirchhoff's classical theory of plates

Принимается внутренняя связь

...

## § 18. Plates: Asymptotic matching of two-dimensional models

При малой толщине из теории типа Тимошенко вытекает классическая. Толщина  $h$  определяется отношением жёсткостей. Перепишем

...

## § 19. Plates: Variational transition from three-dimensional model

Используя вариационные принципы Lagrange'a или Рейсснера–Хеллингера с аппроксимацией решения по толщине, можно по-

лучить двумерные вариационные постановки. Из них вытекают и соотношения внутри области, и естественные краевые условия.

Для примера построим модель типа Тимошенко с аппроксимацией перемещений

...

...

Рассмотренные вариационные переходы легко обобщаются на случаи неоднородности и анизотропии материала, температурных деформаций, динамики. Достоинство принципа Рейсснера–Хеллингера — в явном представлении напряжений. Зато принцип Лагранжа применим к нелинейным задачам ([глава 3](#) содержит трёхмерную постановку).

## § 20. Plates: Splitting of three-dimensional bending problem

Двумерная классическая теория изгиба пластин легко выводится из трёхмерной постановки с малым параметром. Представив радиус-вектор в объёме

...

## § 21. Circular plates

В качестве иллюстрации рассмотрим широко представленный в литературе вопрос об уравнениях теории Kirchhoff'a в полярных координатах.

...

## § 22. Plates: Plane stress

Это вторая из двух задач, о которых говорилось в [§ 15](#). Силы

...

Теория оболочек изложена в монографиях А. Л. Гольденвейзера [7], В. В. Новожилова [39], А. И. Лурье [25], В. С. Черниной [63] и ряде других. Достоинства этих книг перекрывают неразвитость формального аппарата. Переход от трёхмерной модели оболочки к двумерной рассмотрен у ...

...

Техническая теория изгиба пластин изложена ...

...

chapter 12

## OSCILLATIONS AND WAVES

### §1. Vibrations of three-dimensional bodies

**D**ynamic problem of the classical linear elasticity is

$$\begin{aligned}\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} &= \rho \ddot{\mathbf{u}}, \quad \boldsymbol{\sigma} = {}^4\mathcal{A} \cdot \nabla \mathbf{u}, \\ \mathbf{u}|_{o_1} &= \mathbf{0}, \quad \mathbf{n} \cdot \boldsymbol{\sigma}|_{o_2} = \mathbf{p}, \\ \mathbf{u}|_{t=0} &= \mathbf{u}^o, \quad \dot{\mathbf{u}}|_{t=0} = \dot{\mathbf{u}}^o.\end{aligned}\tag{1.1}$$

According to the common theory (§2.8), we begin with the analysis of harmonics (orthogonal oscillations):

$$\begin{aligned}\mathbf{f} &= \mathbf{0}, \quad \mathbf{p} = \mathbf{0}, \quad \mathbf{u}(\mathbf{r}, t) = \mathbf{U}(\mathbf{r}) \sin \omega t, \\ \nabla \cdot ({}^4\mathcal{A} \cdot \nabla \mathbf{U}) + \rho \omega^2 \mathbf{U} &= \mathbf{0}.\end{aligned}\tag{1.2}$$

If a homogeneous problem has a nontrivial solution, then values of  $\omega$  are natural resonant frequencies, and  $\mathbf{U}(\mathbf{r})$  are orthogonal (normal) “modes”.

Time independent equation (1.2) looks like equation of linear elastostatics (§4.??) when the volume load is equal to  $\omega^2 \rho \mathbf{U}$ . The surface load on  $o_2$  is equal to zero. Clapeyron’s identity (4.1, §4.4) gives

$$\omega^2 \int_{\mathcal{V}} \rho \mathbf{U} \cdot \mathbf{U} d\mathcal{V} = 2 \int_{\mathcal{V}} \Pi(\nabla \mathbf{U}^S) d\mathcal{V}.\tag{1.3}$$

It also means that  $\omega^2 \geq 0$ , with zero for motion  $\mathbf{U}$  of a continuum as a rigid whole. If at least a small part of the surface is fixed, then all  $\omega_i > 0$ .

And here is assumed that  $\omega^2$  and  $\mathbf{U}$  are real. This can be proven “by contradiction”. If  $\Im \omega^2 \neq 0$ , then conjugate frequency  $\bar{\omega}^2$  is part of the oscillation spectrum too, and “mode”  $\bar{\mathbf{U}}$  for this

frequency has conjugate components. Using then the reciprocal work theorem (4.2, § 4.4) for  $\mathbf{U}$  and  $\bar{\mathbf{U}}$  as two static variants,

$$\omega^2 \int_V \rho \mathbf{U} \cdot \bar{\mathbf{U}} dV = \bar{\omega}^2 \int_V \rho \bar{\mathbf{U}} \cdot \mathbf{U} dV \Rightarrow \omega^2 = \bar{\omega}^2 \Rightarrow \Im \omega^2 = 0.$$

...

However, a bright picture with decomposition by modes is of little use for practical calculations of oscillations (vibrations) of a three-dimensional elastic body. The reason is density of the spectrum, driven oscillations excite many modes. When the natural frequency density is high, even a small friction qualitatively changes the resonance curve. Damping (decrease in amplitudes) in real bodies is also important. In addition, wave nature of non-stationary processes hinders to just transfer the theory of oscillations of discrete systems to the continuum: in case of sudden local excitation it's more correct to consider waves instead of superposing modes.

The way from a continuous dynamic model to a discrete one goes through the variational approach.

$$\int_V \left( (\mathbf{f} - \rho \ddot{\mathbf{u}}) \cdot \delta \mathbf{u} - \boldsymbol{\sigma} \cdot \delta \boldsymbol{\varepsilon} \right) dV + \int_{o_2} \mathbf{p} \cdot \delta \mathbf{u} d\sigma = 0. \quad (1.4)$$

This is the principle of virtual work with forces of inertia. Looking for an approximate solution in series

$$\mathbf{u}(\mathbf{r}, t) = \sum_{k=1}^N \alpha_k(t) \varphi_k(\mathbf{r}),$$

where  $\varphi_k$  are given  $(\varphi_k|_{o_1} = \mathbf{0})$ , and  $\alpha_k(t)$  are varying. The solution???? is a/?????the system of ordinary equations

...

In place of the principle of virtual work (1.4) one can use the Hellinger-Reissner mixed formulation with an independent approximation of stresses.

In the dynamic theory of elasticity, the Laplace integral transform is often applied. For simple shaped bodies it is sometimes possible to find an analytical solution in transforms. The original can be found by the numerical inversion, but sometimes it's possible to take



the Riemann–Mellin integral<sup>\*</sup> using the “saddle-point method” (or “method of steepest descents”) with a deformation of contour in the complex plane [49, 75].

## § 2. Vibrations of rods

In the linear dynamics of rods we have the following system for forces  $\mathbf{Q}$ , force couples (moments)  $\mathbf{M}$ , displacements  $\mathbf{u}$  and rotations  $\boldsymbol{\theta}$  (§ 9.10):

$$\mathbf{Q}' + \mathbf{q} = \rho(\ddot{\mathbf{u}} + \ddot{\boldsymbol{\theta}} \times \boldsymbol{\varepsilon}), \quad \mathbf{M}' + \mathbf{r}' \times \mathbf{Q} + \mathbf{m} = \mathbf{J} \cdot \ddot{\boldsymbol{\theta}} + \rho \boldsymbol{\varepsilon} \times \ddot{\mathbf{u}}, \quad (2.1)$$

$$\dots \quad (2.2)$$

...

## § 3. Small perturbations of parameters

Рассмотрим задачу об определении собственных частот и форм with small perturbations масс и жёсткостей:

$$\begin{aligned} (C_{ij} - \omega^2 A_{ij}) U_j &= 0, \\ C_{ij} &= C_{ij}^{(0)} + \chi C_{ij}^{(1)}, \quad A_{ij} = A_{ij}^{(0)} + \chi A_{ij}^{(1)}, \quad \chi \rightarrow 0. \end{aligned} \quad (3.1)$$

Находя решение в виде

$$\omega = \omega^{(0)} + \chi \omega^{(1)} + \dots, \quad U_j = U_j^{(0)} + \chi U_j^{(1)} + \dots,$$

получаем последовательность задач

...

## § 4. Vibrations of shells

Динамика оболочек рассматривалась многими

...

<sup>\*</sup> The Riemann–Mellin integral determines the inverse Laplace transform  $F(s) \mapsto f(t)$ .

## § 5. Waves in an elastic continuum

Рассмотрим линейные уравнения динамики однородной изотропной среды без объёмных сил

...

## § 6. Waves in a rod

Рассмотрим прямой стержень. Продольная деформация описывается уравнениями

...

## § 7. Nonlinear oscillations

Рассмотрим простой пример: продольные колебания прямого стержня с малой нелинейной добавкой в соотношениях упругости

...

## Bibliography

Методы решения динамических задач упругости представлены в книгах Л. И. Слепяна [75] и В. Б. Поручикова [49]. О малых линейных колебаниях (вибрациях) написано у С. П. Тимошенко, D. Н. Young'a и W. Weaver'a [67], И. М. Бабакова [68], В. Л. Бидермана [69], В. Т. Гринченко и В. В. Мелешко [71]. Асимптотические проблемы колебаний оболочек освещены у ...

*chapter 13*  
**STABILITY**

## §1. Various approaches to the problem of stability

There's the well developed theory of stability by Ляпунов (Lyapunov). It says that if initial deviations are “close enough”, they don't increase in the future and stay “close enough” forever, then the process is stable. This applies to the condition of equilibrium too. Considering the dynamics of small deviations from an equilibrium configuration, we make sure that these deviations do not increase. This is the **dynamic approach** to the problem of stability, recognized as the most reasonable.

However, for problems about the stability of equilibrium of elastic systems, the different approach has gained popularity, called the **static approach**. It associates with the Leonhard Euler's name. When the equations of statics give a nontrivial solution for small deviations, then the values of parameters are assumed to be “critical”. In other words, if an equilibrium condition ceases to be isolated, then it's considered “critical”, and many adjacent forms of the equilibrium appear then (“indifferent equilibrium”). Using this approach, it's enough to solve the eigenvalue problem.

And moreover, there are also other approaches. For example, the **imperfection method**: if small random changes in the initial shape, stiffnesses, loads and so on cause only a small change of a deformed configuration in equilibrium, then this equilibrium is stable. Or the **energy approach**: a loss of stability occurs when it becomes energetically beneficial, that is it leads to a decrease in energy.

The mentioned approaches draw a motley picture. Yet it is pretty easy to visualize for a discrete model — a model with a finite number of degrees of freedom.

Большой общностью обладают

...

## § 2. Classical problems with rods

Состояние перед варьированием описывается уравнениями нелинейной теории стержней Kirchhoff'a

...

## § 3. "Tracking" loads

В проблемах устойчивости весьма весомо поведение нагрузки в процессе деформирования. Ведь в уравнения входит вариация (of what?)  $\delta q$ , она равна нулю лишь для "dead" loads. **Распространены** "tracking" loads, which change in a definite way along with displacements of the particles of an elastic continuum.

Euler's static approach to the problem of stability

...

## § 4. The role of additional pliabilities

Для прямого **консольного** стержня, сжатого постоянной силой  $F$  на свободном конце, критическая нагрузка определяется формулой Euler'a

...

## § 5. Variational formulations

Во всех разделах линейной теории упругости большую роль играют вариационные постановки. Среди прочего, они составляют основу метода конечных элементов как варианта метода Ritz'a.

Менее развиты вариационные постановки для проблем устойчивости. Здесь получил популярность метод

...

## § 6. Nonconservative problems

В уравнении динамики (...) матрица позиционных сил

...

## § 7. Case of multiple roots

Вернёмся к проблеме устойчивости (...) в случае циркуляционных сил. Как уже отмечалось (где??), критическая ситуация характеризуется

...

## Bibliography

Увлекательные вопросы устойчивости упругих систем освещены в книгах ...

## § 1. Volterra dislocations

Consider the classical linear three-dimensional elastic medium (chapter 4). As shown in § 4.7, the equation of compatibility of deformations

...

## § 2. Straight-line dislocations

A dislocation line may be any curve, closed inside a body or with ends on the surface. For a dislocation of a random shape in a limitless medium, it's not so difficult to obtain an appropriate solution [14]. Мы же ограничимся простейшим случаем прямолинейной дислокации. Ищется решение

...

## § 3. Action of stress field on dislocation

Рассмотрим тело, содержащее внутри дислокацию с замкнутой линией  $C$ . Тело нагружено объёмными  $\mathbf{f}$  и поверхностными  $\mathbf{p}$  силами. Обозначим

...

## § 4. About movement of dislocations

Рассмотрим это явление, следуя [20]. Ограничимся случаем прямолинейной винтовой дислокации, движущейся с постоянной скоростью

...

## § 5. Point defects

Речь пойдёт о континуальной модели таких явлений как вакансии, примесные частицы или междоузельные атомы в кристаллической решётке. В случае дислокации рассматривались

...

## § 6. Force acting on a point defect

Дефект находится в теле, нагруженном объёмными  $f$  и поверхностными  $p$  силами. Суперпозиция

...

## § 7. Continuously distributed dislocations

Начнём со сложения векторов Бюргерса. При обходе сразу двух дислокаций (рис. ?? 40 ??) по контуру

...

## § 8. Stress during winding of coil

Не только дислокации и точечные дефекты, но и макроскопические факторы могут быть источниками собственных напряжений. При намотке катушки (рис. ?? 42 ??) в ней возникают напряжения от натяжения ленты. Расчёт этих напряжений очень сложен, если рассматривать детально процесс укладки ленты.

Но существует чёткий алгоритм Southwell'a [51] расчёта напряжений в катушке: укладка каждого нового витка вызывает внутри катушки приращение напряжений, определяемые соотношениями линейной упругости. There are two stages, and the first one consists in solving the Lamé problem about deformation of a hollow cylinder under external pressure (рис. ?? 43 ??)

...

## Bibliography

Dislocations and point defects in linear elastic bodies have been considered by many authors: John Eshelby [14], Roland deWit [11], Cristian Teodosiu [54], Alan Cottrell [20]. The theory of self-stresses (eigenspannungen) is explained by Ekkehart Kröner in [21]. Calculation of stresses when winding a coil is described by Richard Southwell in his book [51].



## §1. Traditional criteria of durability

**К**ак судить о прочности тела после определения напряжения в нём? При одноосном растяжении напряжением  $\sigma$  есть, очевидно, некий “предел прочности”  $\sigma_*$ , выше которого материал разрушается. Прочность считают достаточной при  $\sigma \leq \sigma_*/k$ , где  $k$  есть так называемый коэффициент запаса. Но такой подход не вполне удовлетворяет, поскольку получаемые из опытов значения  $\sigma_*$  have a big scatter, а выбор коэффициента запаса sometimes becomes a bureaucratic act.

Тем не менее, подобные взгляды на анализ прочности широко распространены. Воздерживаясь от критики, стоит упомянуть самые популярные из них.

**Критерий максимального нормального напряжения:** разрушение происходит при  $\sigma_1 = \sigma_*$  (наибольшее из главных напряжений достигает предельного значения). Но это положение несправедливо для одноосного сжатия, при котором  $\sigma_1 = 0$ .

**Критерий максимального касательного напряжения (Tresca criterion):** разрушение наступает при  $\sigma_1 - \sigma_3 = \sigma_*$  ( $\sigma_3$  — наименьшее из главных напряжений). Это более соответствует началу пластического течения.

**Критерий максимального удлинения:** наибольшее из собственных значений тензора деформации  $\varepsilon_1 = \varepsilon_*$ . Это приемлемо и при сжатии с  $\varepsilon_1 > 0$ .

**Критерий энергии деформации:**  $\Pi = \Pi_*$ . Здесь учитывается, что разрушение требует энергии, а источником её может быть лишь само деформированное тело. Однако достаточный запас энергии — необходимое, но не единственное условие разрушения; должен

включиться некий механизм преобразования упругой энергии в работу разрушения.

**Критерий энергии формоизменения (von Mises yield criterion):**  $s \bullet s = 2\tau_*^2$ ,  $s \equiv \tau - \frac{1}{3} E \text{ trace } \tau$  (“девиатор напряжений”). Здесь не играет роли энергия объёмной деформации. Richard von Mises предложил\* этот критерий как гладкую аппроксимацию условия Henri Tresca.

**Критерий Mohr’a.** Представим себе множество предельных состояний ...

...

## § 2. Antiplane deformation of continuum with a crack

Любая регулярная функция комплексного переменного  $z = x + iy$  содержит в себе решение какой-либо антиплоской задачи статики без ...

...

## § 3. Crack in plane deformation

Рассмотрим плоскую область произвольного очертания с трещиной внутри; нагрузка приложена и “в объёме”, и на внешнем крае. Как и при антиплоской деформации, решение строится в два этапа

...

## § 4. Crack-driving force

Это едва ли не основное понятие механики трещин. Рассмотрим его, следуя

\* **R. von Mises.** Mechanik der festen Körper im plastisch-deformablen Zustand. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse.* 1913, Seiten 582–592.

...

## § 5. Criterion of crack growth

Связанная с энергией  $\mathcal{E}$  трещинодвижущая сила  $F$  — не единственное воздействие на передний край трещины. Должна быть ещё некая сила сопротивления  $F_*$ ; рост трещины начинается при условии

...

## § 6. J-integral

Одно из самых известных понятий в механике трещин выражается интегралом

$$J = \dots \quad (6.1)$$

...

## § 7. Stress intensity factors

Расчёт прочности тела с трещиной сводится к определению коэффициентов интенсивности напряжений. Методы расчёта таких коэффициентов — как аналитические, так и численные — хорошо освещены в литературе.

Рассмотрим ещё один подход к задачам механики трещин, разработанный

...

## § 8. Barenblatt's model

An unlimited increase of stress at the edge of a crack seems indeed dubious. Singular solutions desire support by some additional reasonings or by using of another model. And such a support дала работа

...

## § 9. Deformational criterion

D. S. Dugdale\*, а также М. Я. Леонов и В. В. Панасюк\*\* предложили модель, напоминающую построения Баренблатта. Также есть силы сцепления  $q$  и равен нулю итоговый коэффициент интенсивности напряжений. Но, во-первых,  $q$  имеет иной вид:

...

Второе отличие рассматриваемой модели — в формулировке критерия прочности: трещина начинает расти, когда расхождение берегов в конце свободного участка достигает критического значения  $\delta_*$  (этот параметр — константа материала), то есть при

...

## § 10. Growth of cracks

Пусть нагрузка на тело с трещиной выросла настолько, что выполняется условие

...

## § 11. Elastic field ahead of a moving crack

Рассмотрим этот вопрос

...

\* **Dugdale, D. S.** Yielding of steel sheets containing slits. *Journal of the Mechanics and Physics of Solids*. 1960, Volume 8, Issue 2, pages 100–104.

\*\* **Леонов М. Я., Панасюк В. В.** Развитие мельчайших трещин в твёрдом теле. *Прикладная механика*. 1959, Т. 5, № 4, с. 391–401.

## § 12. Balance of energy for a moving crack

Уравнение баланса энергии в линейной теории ( $\Pi = \frac{1}{2} \boldsymbol{\varepsilon} \cdot \boldsymbol{\mathcal{A}} \cdot \boldsymbol{\varepsilon}$ ,  $K = \frac{1}{2} \rho \dot{\mathbf{u}} \cdot \dot{\mathbf{u}}$ ):

$$\int_{\mathcal{V}} \left( K + \Pi \right) \dot{\phantom{x}} d\mathcal{V} = \int_{\mathcal{V}} \mathbf{f} \cdot \dot{\mathbf{u}} d\mathcal{V} + \int_{\mathcal{O}} \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \dot{\mathbf{u}} d\mathcal{O}. \quad (12.1)$$

...

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Список книг по механике трещин уже велик. В нём сто́ит отметить работы Л. М. Качанова [17], Н. Ф. Морозова [33], В. З. Партона и Е. М. Морозова [46], Г. П. Черепанова [62]. Обзор статей есть у ... Экспериментальные данные представлены, например, в [18].

## chapter 16

# COMPOSITES

### §1. Introductory thoughts

When clay is used as a building material, a shredded straw is added to it. Working with epoxy gum, it's useful before solidifying to blend with a filler: powder, fibers, pieces of fabric. These are examples of composites (composite materials, composite mixtures). New types of composites are used more widely, displacing steel, aluminum alloys and other popular homogeneous materials.

Composites can be defined as (micro-)inhomogeneous materials, where happens some averaging with emerging of new properties. “Usual” mechanics of continua is applicable, surely, to composites too. But it's barely possible to model each aspect of a composite material — and rather absurd. Some new approach is needed that will deal with the complexity of material structure. As for a gas, when instead of describing dynamics of individual molecules we introduce parameters “pressure”, “temperature” and others.

Mechanics of a homogeneous non-composite continuum features only single — macroscopic — length  $\ell$ , the characteristic dimension of a body (volume divided by surface area), assuming that volumes of any smallness have the same properties as finite ones.

For a composite it's different, there is the three scopes of length:  $\ell \gg \ell' \gg \ell_0$ . The largest  $\ell$  presents dimensions of a body. The smallest  $\ell_0$  is around the size of elements of material structure (for example, particulates of a filler powder). The intermediate scope  $\ell'$  exposes the size of a so-called “representative” volume, “unit cell”: that small volume, in which the specific properties of this composite are noticeable.

In composites, the complex problem for an inhomogeneous body is split into two: for a body as a whole (macrolevel) and for a “representative” volume (microlevel). At the macrolevel (scope  $\ell$ ) a composite is modeled as a homogeneous medium with “effective” properties, where “representative” volumes play the role of particles. At the microlevel (scope  $\ell_0$ ) fields are very inhomogeneous, but some averaging by the “representative” volume concludes to the macrolevel. The complex problem for a composite as an inhomogeneous body splits into two: for a representative volume (microlevel) and for a body as a whole (macrolevel).

These thoughts, however, are not perfectly convincing. To convince more, one can use the random field theory

...

The mechanics of composites appeared not very long ago, but is intensively developing. Due to high fracture toughness, the fracture mechanics of composites is especially popular.

## § 2. Effective fields

Any field in a composite is usually represented by the sum  $u = u * + u'$ , where  $u *$  is some smoothed “effective” field, and  $u'$  is fast oscillating fluctuation. It's often assumed

$$u * (A) = \langle u \rangle \equiv \mathcal{V}^{-1} \int_{\mathcal{V}} u d\mathcal{V}, \quad (2.1)$$

where  $\langle u \rangle$  is an average within a representative volume centered at point  $A$ . Averaging (2.1)

...

## § 3. Boundary value problems for representative volume

How the elastic moduli are determined for a homogeneous medium? Without the real possibility to get a relation between  $\sigma$  and  $\varepsilon$  for a point of infinitesimal size, experiments are carried out with bodies of finite volumes under a so-called “homogeneous stress” — when stress

is the same at all points of a body. In composites, the role of a point plays a “representative” volume.

...

## § 4. Hill's fork

Using Voigt and Reuss theories, Hill derived upper and lower bounds on the effective properties of a composite material [Hill, R. W. The elastic behaviour of a crystalline aggregate. *Proceedings of the Physical Society, Section A*, Volume 65, Issue 5 (May 1952). Pages 349–354.]

The scale separation is motivated by the material properties, at both scales continuum mechanics models the underlying system. Such an approach uses energy equivalence at both scales as proposed in Hill (1972).

Hill R (1972) On constitutive macro-variables for heterogeneous solids at finite strain (pages 131–147)

For a composite material, at least two different materials with known material models and parameters, generate a homogenized material modeled with a predetermined constitutive equation. Determination of material parameters of the homogenized material is a challenging task.

Отметив, что

...

## § 5. Eshelby formulas

Итак, эффективные модули определяются энергией представительного объёма в первой или второй задачах:

...

## § 6. Effective moduli for material with spherical inclusions

В однородной матрице случайным образом, но достаточно равномерно, распределены сферические включения радиусом  $a$ . Получившийся композит на макроуровне будет изотропным, его упругие свойства полностью определяются ...



...

## § 7. Self-consistent method

Выше мы опирались на две задачи for a representative volume и определяли effective modules from the equality of energies. В основе метода самосогласования лежит новая идея: представительный объём помещается в безграничную среду с эффективными свойствами, на бесконечности состояние считается однородным, эффективные модули находятся из некоторых дополнительных условий самосогласования.

Обратимся снова к вопросу об объёмном модуле среды со сферическими включениями. Задача сферически симметрична; для включения по-прежнему

...

## § 8. Hashin–Shtrikman principle

Hashin and Shtrikman derived upper and lower bounds for the effective elastic properties of quasi-isotropic and quasi-homogeneous multiphase materials using a variational approach [**Hashin, Z.; Shtrikman, S.** A variational approach to the theory of the elastic behaviour of multiphase materials. // Journal of the Mechanics and Physics of Solids. Volume 11, Issue 2 (March–April 1963). Pages 127–140.]

Hashin Z., Shtrikman S. (1962) On some variational principles in anisotropic and nonhomogeneous elasticity. Journal of the Mechanics and Physics of Solids 10(4): pages 335–342

Вилка Hill’a основана на обычных экстремальных принципах теории упругости. Специально для механики композитов Hashin и Shtrikman построили очень своеобразный функционал, который на некотором точном решении может иметь как максимум, так и минимум, давая возможность с двух сторон оценивать эффективные модули [66].

Рассмотрим первую из двух задач для представительного объёма

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Книги R. Christensen'a [77] and Б. Е. Победри [79] содержат и основы механики композитов, и постановку не теряющих актуальности проблем. Для самого требовательного читателя представляет интерес монография Т. Д. Шермергора [66]. Немало книг посвящено механике разрушения композитов, здесь стоит отметить труд Г. П. Черепанова [80].

*chapter 17*

## PERIODIC COMPOSITES

Periodic composite structure is made of a finite or infinite repetition of a unit cell.

### §1. One-dimensional problem

**В** одномерной задаче статики имеем уравнение  
...

### §2. Three-dimensional continuum

Исходим из уравнений в перемещениях

...

### §3. Fibrous structure

Тензор  $\mathcal{A}$  в этом случае постоянен вдоль оси

...

### §4. Statics of a periodic rod

В уравнениях линейной статики стержня

...

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Лежащий в основе этой главы асимптотический метод представлен (с разной степенью математической скрупулёзности) в книгах [81, 82, 78, 79].

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