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PHYSICS *of* ELASTIC CONTINUA



FOREWORD

In this book I am trying to guide the reader* through the ways of constructing the models of an elastic continuum. These models are: the nonlinear and the linear ones, the micropolar and the classical momentless; the three-dimensional, the two-dimensional (shells and plates) and the one-dimensional (rods, including thin-walled ones). I also explained the fundamentals of dynamics — oscillations, waves and stability. For the thermoelasticity and the magnetoelasticity, I gave the summary of the classical theories of the thermodynamics and the electrodynamics. The dynamics of destruction is described via the theories of defects and fractures. The approaches to modeling of human-made inhomogeneous materials, “composites”, are also shown.

The word “continua” in the title says that an object (a body, a medium) is modeled here not as a discrete collection of particles, but as a continuous space of location vectors, a continuous matter. It gives a large convenience, because the apparatus of calculus of infinitesimals can be used for such models.

When I just began writing this book, I thought of a reader who is pretty acquainted with “higher” mathematics. But later I decided to conduct such an acquaintance by myself, and yet, as a side effect, every reader with any knowledge of math can comprehend the content of the book.

* and myself as well

The book is written using the compact and elegant direct indexless tensor notation. The mathematical apparatus for interpreting the direct tensor relations is located in the first chapter.

I am writing this book simultaneously in the two languages, English and Russian. The reader is free to pick any language of the two.

github.com/VadiqueMe/PhysicsOfElasticContinua

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chapter 1

MATHEMATICAL APPARATUS

Mathematics, or math for short, is abstract. Abstract is the adjective of math, math is the noun of abstract. “Abstract”, “theoretical” and “mathematical” are synonyms. When someone is doing math, he’s playing a game in the far-far-away magical world of imagination.

For example, numbers are not real entities at all. They are purely imaginary concepts. We cannot experience, sensate numbers, can’t see, touch or smell them. Yep, one can compose stories about them, such as $1 + 1 = 2$ — mathematical relations between imaginary entities. Nevertheless, no one can ever feel, perceive it, since there are no such things as *one* and *two**.

And “synthesized by imagination” — it’s not only about numbers. Geometric objects, be it a point, a line, a triangle or a plane, and all kinds of adventures with them are mind derived as well.

§1. The ancient but intuitive geometry

For nearly two thousand years, the freedom of human’s thought was limited by the fairy tale about imaginary perfectly straight one-dimensional lines between some two absolutely dimensionless points and beyond, on and on to the very *ἀπειρο* (infinity) down and up the both sides, about imaginary and completely flat planes-trigons (“triangles”, sometimes “tripoints”), with shortest distances between points in straight lines, with always equal to each other “straight” or “~~straight~~” angles, as well as about many other pretty funny mythical characters and piquant relations between them. Over two thousand years people were in captivity, in slavery to the idea about the existænce of the only one *εὐκλείδειος γεωμετρίας* (the *εὐκλείδ’ean* geometry), and

* I’m not about *two apples* or *two similar bananas* for a couple of days, but about the number “two” itself.

the magic world, described by it, was equated in the past with the real space around themselves.

Εὐκλείδης Eὐkleídēs Euclid, εὐκλείδειος euclidean

εὐκλείδεια γεωμετρία

the plane geometry, or the two-dimensional euclidean geometry

Στοιχεῖα Stoikheía Elements, Principles

(1.1) Points

Στοιχεῖα Εὐκλείδου
Βιβλίον Ι

Euclid's Elements
Book I

Ὅρος α' (1)

Term α' (1)

Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.

A point is that which has no part.

This description shows that Euclid imagines a point as an indivisible location, without width, length or breadth.

(1.2) Lines, curved and straight

Στοιχεῖα Εὐκλείδου
Βιβλίον Ι

Euclid's Elements
Book I

Ὅρος β' (2)

Term β' (2)

Γραμμή δὲ μῆκος ἀπλατές.

A line is breadthless length.

“Line” is the second primitive term in the Elements. “Breadthless length” says that a line will have one dimension, length, but it won't have breadth. The terms “length” and “breadth” are not defined in the Elements.

Linear lines

(1.3) A relation between lines and points

Στοιχεῖα Εὐκλείδου
Βιβλίον Ι

Euclid's Elements
Book I

Ορος γ' (3)

Term γ' (3)

Γραμμῆς δὲ πέρατα σημεῖα.

The ends of a line are points.

This statement doesn't mention how many ends a line can have.

(1.4) *Do straight lines exists?*

The hypothesis on the existænce of straight lines.

The existænce of Euclidean straight lines in space.

Στοιχεῖα Εὐκλείδου
Βιβλίον Ι

Euclid's Elements
Book I

Ορος δ' (4)

Term δ' (4)

Εὐθεῖα γραμμὴ ἐστίν, ἥτις ἐξ ἴσου τοῖς
ἐφ' ἑαυτῆς σημείοις κεῖται.

A straight line is a line which lies
evenly with the points on itself.

To draw a straight line by hand is absolutely impossible.

(1.5) *Vectors. Lines and vectors*

(1.6) *The existænce of vectors. Do vectors exist?*

(1.7) *Continuity of line*

(1.8) *A point of reference*

(1.9) *Translation as the easiest kind of motion. Translations
and vectors*

(1.10) *Straight line and vector*

A (geometric) vector may be like a straight line with an arrow at one of its ends. **Then** it is fully described (characterized) by the magnitude and the direction.

Within the abstract algebra, the word *vector* is about any object which can be summed with similar objects and scaled (multiplied) by scalars, and vector space is a synonym of linear space. Therefore I clarify that in this book *vector* is nothing else than three-dimensional geometric (Εὐκλείδειος, Euclidean) vector.

Why are vectors always straight (linear)?

(a) Vectors are linear (straight), they cannot be curved.

(b) Vectors are neither straight nor curved. A vector has the magnitude and the direction. A vector is not a line or a curve, albeit it can be represented by a straight line.

Vector can't be thought of as a line.

(1.11) The line which figures real numbers

often just “number line”

(1.12) What is a distance?

(1.13) Plane and more dimensional space

(1.14) Distance on plane or more dimensional space

(1.15) What is an angle?

angle \equiv inclination /slope, slant/ of two lines

two lines sharing a common point are usually called intersecting lines

angle \equiv the amount of rotation of line or plane within space

angle \equiv the result of the dot product of two unit vectors gives angle's cosine

(1.16) Differentiation of continuous into small differential chunks

small differential chunks

infinitesimal (infinitely small)

A mention of tensors may scare away the reader, commonly avoiding needless complications. Don't be afraid: tensors are used just due to their wonderful property of the invariance — the independence from a coordinate system.

§2. Vector

I propose to begin familiarizing with tensors via memoirs about such a phenomenon as a vector.

- ✓ A *point* has position in space. The only characteristic that distinguishes one point from another is its position.
- ✓ A *vector* has both magnitude and direction, but no specific position in space.

(2.1) What is a vector?

What is “linear”?

- (1) straight
- (2) relating to, resembling, or having a [graph](#) that is a straight line

All vectors are linear objects.

Examples of vectors:

- ✓ A force acts on an object.
- ✓ The velocity of an object describes what’s happening with this object at an instant.

Multiplication of a vector by a scalar

Multiplication by the minus one

The Newton’s action–reaction principle “действие равно противодействию по магнитуде и обратно ему по направлению”

Each mechanical interaction of two objects is characterized by two forces that act on both interacting objects. These forces can be represented as two vectors that are equal in magnitude and reverse in direction.

Multiplying a vector by the negative one -1 reverses the vector’s direction but doesn’t change its magnitude.

(2.2) The addition and subtraction

The sum (combination) of two or more vectors is the new “resultant” vector. There are two similar methods to calculate the resultant vector geometrically.

The “*head to tail method*” involves lining up the head of one vector with the tail of another. Here the resultant goes from the initial point (the “tail”) of the first addend to the end point (the “head”) of the second addend when the tail (the initial point) of the second one coincides with the head (the end point) of the first one.

[... figure here ...]

The “*parallelogram method*” ...

[... figure here ...]

The vector addition is commutative

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}.$$

....

$$\mathbf{p} + \mathbf{q},$$

$$\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q}) = \mathbf{p} + (-1)\mathbf{q}.$$

For every action, there’s an equal (in magnitude) and opposite (in direction) reaction force.

A vector may be also represented as the sum (combination) of some trio of other vectors, called “basis”, when the each of the three is scaled by a number (coefficient). Such a representation is called a “linear combination” of basis vectors. A list (array, tuple) of coefficients alone, without basis vectors, is not enough and can’t represent a vector.

....

To get the numerical relations from the vector ones, a coordinate system is introduced, and on its axes the vector relations are projected.

....

У самих по себе векторов как элементов векторного пространства компонент нет. Vector components appear only when a certain basis is chosen, then any vector can be decomposed into components. В разных базисах компоненты одного и того же вектора отличаются друг от друга.

Here it is — a vector, \mathbf{v} looks like a suitable name for it.

Like all geometric vectors, \mathbf{v} is pretty well characterized by the two mutually independent properties: its length (magnitude, norm,

modulus) and its direction in space. This characterization is complete, so some two vectors with the same magnitude and the same direction are considered equal.

Every vector exists objectively by itself, independently of methods and units of measurement of both lengths and directions, including any abstractions of such units and methods.

(2.3) The method of coordinates

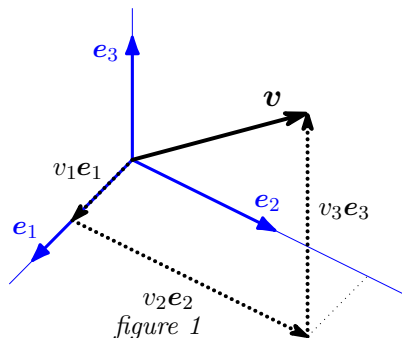
.....

By choosing some mutually perpendicular unit vectors \mathbf{e}_i as the basis for measurements, I introduce the rectangular (“cartesian”) coordinates.

Three ($i = 1, 2, 3$) basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are needed for a three-dimensional — 3D — space.

Within such a system, “ \cdot ”-products of the basis vectors are equal to the Kronecker delta

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$



for any **orthonormal** basis.

Decomposing vector \mathbf{v} in some **orthonormal** basis \mathbf{e}_i ($i = 1, 2, 3$), we get coefficients v_i — the components of vector \mathbf{v} in that basis (figure 1)

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \equiv \sum_{i=1}^3 v_i\mathbf{e}_i \equiv v_i\mathbf{e}_i, \quad v_i = \mathbf{v} \cdot \mathbf{e}_i. \quad (2.1)$$

Here and hereinafter, the Einstein’s summation convention is accepted: an index repeated twice (and no more than twice) in a single term implies a summation over this index. And a non-repeating index is called “free”, and it is identical in the both parts of the equation. These are examples:

$$\sigma = \tau_{ii}, \quad p_j = n_i \tau_{ij}, \quad m_i = e_{ijk} x_j f_k, \quad a_i = \lambda b_i + \mu c_i.$$

(But equations $a = b_{kkk}$, $c = f_i + g_k$, $a_{ij} = k_i \gamma_{ij}$ are incorrect.)

Having components of a vector in an orthonormal basis, the length of this vector is retrieved by the “Πυθαγόρας’ equation”

$$\mathbf{v} \cdot \mathbf{v} = v_i \mathbf{e}_i \cdot v_j \mathbf{e}_j = v_i \delta_{ij} v_j = v_i v_i, \quad \|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_i v_i}. \quad (2.2)$$

The magnitude represents the length independent of direction.

The direction of a vector in space is measured by the three angles (co-sines of angles) between this vector and each of the basis ones:

$$\cos \angle(\mathbf{v}, \mathbf{e}_i) = \frac{\mathbf{v} \cdot \mathbf{e}_i}{\|\mathbf{v}\|} = \frac{v_i}{\sqrt{v_j v_j}} \Leftrightarrow \underbrace{v_i}_{\mathbf{v} \cdot \mathbf{e}_i} = \|\mathbf{v}\| \cos \angle(\mathbf{v}, \mathbf{e}_i). \quad (2.3)$$

Measurement of angles. The cosine of an angle between two vectors is the same as the dot product of these vectors if their magnitudes are equal to the one unit of length

When the magnitudes of two vectors are equal to the one unit of length, then the cosine of the least angle between them is the same as the dot product of these vectors. Any vector with the non-unit magnitude (but the null vector) can be “normalized” via dividing a vector by its magnitude.

$$\cos \angle(\mathbf{v}, \mathbf{w}) = \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|}.$$

To accompany the magnitude, which represents the length independent of direction, there’s a way to represent the direction of a vector independent of its length. For this purpose, the unit vectors (the vectors with the magnitude of 1) are used.

A rotation matrix is just a transform that expresses the basis vectors of the input space in a different orientation. The length of the basis vectors will be the same, and the origin will not change. Also, the angle between the basis vectors will not change. All that changes is the relative direction of all of the basis vectors.

Therefore, a rotation matrix is not really just a “rotation” matrix; it is an orientation matrix.

There are also pseudovectors, waiting for the reader below in §7.

The angle between two random vectors. According to (2.3)

$$\cos \angle(\mathbf{v}, \mathbf{e}_m) = \frac{\mathbf{v} \cdot \mathbf{e}_m}{\|\mathbf{v}\|} = \frac{v_m}{\sqrt{v_j v_j}},$$

$$\cos \angle(\mathbf{w}, \mathbf{e}_n) = \frac{\mathbf{w} \cdot \mathbf{e}_n}{\|\mathbf{w}\|} = \frac{w_n}{\sqrt{w_k w_k}}.$$

The length (2.2) and the direction in space (2.3), that can be measured by the means of the trio of basic vectors, describe a vector. And every vector possesses these properties*. However, this is not enough (“not sufficient” in jargon of the math books).

A vector is not just a collection of components in some basis.

A triple of pairwise perpendicular unit vectors can only rotate and thereby it can characterize the angular orientation of other vectors.

The decomposition of the same vector \mathbf{v} in the two cartesian systems with basis unit vectors \mathbf{e}_i and \mathbf{e}'_i (figure 2) gives

$$\mathbf{v} = v_i \mathbf{e}_i = v'_i \mathbf{e}'_i,$$

where

$$v_i = \mathbf{v} \cdot \mathbf{e}_i = v'_k \mathbf{e}'_k \cdot \mathbf{e}_i,$$

$$v'_i = \mathbf{v} \cdot \mathbf{e}'_i = v_k \mathbf{e}_k \cdot \mathbf{e}'_i.$$

Appeared here two-index objects (the two-dimensional arrays) $o_{ki} \equiv \mathbf{e}'_k \cdot \mathbf{e}_i$ and $o_{ki'} \equiv \mathbf{e}_k \cdot \mathbf{e}'_{i'}$ are used to shorten the formulas.

Написать о пассивном повороте, описанном ниже, и об активном повороте из § 12

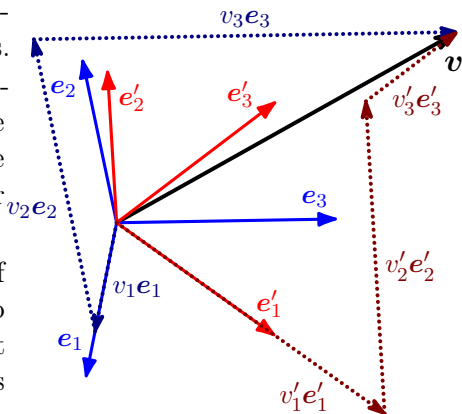


figure 2

* And what is the direction of the null vector (“(vanishing) vector”) $\mathbf{0}$ with the zero length $\|\mathbf{0}\| = 0$? (The zero vector without a magnitude ends exactly where it begins and thus it is not directed anywhere, its direction is *undefined*.)

The “ \bullet ”-product (dot product) of two vectors is commutative — that is, the swapping of multipliers doesn’t change the result. Thus

$$o_{k'i} = \mathbf{e}'_k \bullet \mathbf{e}_i = \cos \angle(\mathbf{e}'_k, \mathbf{e}_i) = \cos \angle(\mathbf{e}_i, \mathbf{e}'_k) = \mathbf{e}_i \bullet \mathbf{e}'_k = o_{ik'}, \quad (2.3a)$$

$$o_{ki'} = \mathbf{e}_k \bullet \mathbf{e}'_i = \cos \angle(\mathbf{e}_k, \mathbf{e}'_i) = \cos \angle(\mathbf{e}'_i, \mathbf{e}_k) = \mathbf{e}'_i \bullet \mathbf{e}_k = o_{i'k}. \quad (2.3b)$$

Lines (2.3a) and (2.3b) are mutually reciprocal by multiplication

$$o_{k'i} o_{ki'} = o_{ki'} o_{k'i} = 1, \quad o_{k'i} o_{i'k} = o_{i'k} o_{k'i} = 1.$$

Multiplying of an orthogonal matrix by the components of any vector retains the length of this vector:

$$\|\mathbf{v}\|^2 = \mathbf{v} \bullet \mathbf{v} = v'_i v_i = o_{i'k} v_k o_{i'n} v_n = v_n v_n$$

— this conclusion **leans on** (??).

Orthogonal transformation of the vector components

$$\mathbf{v} \bullet \mathbf{e}'_i = v_k \mathbf{e}_k \bullet \mathbf{e}'_i = \mathbf{e}'_i \bullet \mathbf{e}_k v_k = o_{i'k} v_k = v'_i \quad (2.4)$$

is sometimes used for defining a vector itself. If in each orthonormal basis \mathbf{e}_i a triplet of numbers v_i is known, and with a rotation of the basis as a whole it is transformed according to (2.4). then this triplet of components represents an invariant object — vector \mathbf{v} .

§ 3. Tensor and its components

When in each orthonormal basis \mathbf{e}_i we have a set of nine ($3^2 = 9$) numbers B_{ij} ($i, j = 1, 2, 3$), and this set is transformed during a transition to a new (rotated) orthonormal basis \mathbf{e}'_i as

$$B'_{ij} = \mathbf{e}'_i \bullet \mathbf{e}_m B_{mn} \mathbf{e}_n \bullet \mathbf{e}'_j = \mathbf{e}'_i \bullet \mathbf{e}_m \mathbf{e}'_j \bullet \mathbf{e}_n B_{mn} = o_{i'm} o_{j'n} B_{mn}, \quad (3.1)$$

then this set of components presents an invariant object — a tensor ${}^2\mathbf{B}$ of second complexity (of second valence, bivalent).

In other words, tensor ${}^2\mathbf{B}$ reveals in every basis as a collection of its components B_{ij} , changing along with a basis according to (3.1).

The key example of a second complexity tensor is a dyad. Having two vectors $\mathbf{a} = a_i \mathbf{e}_i$ and $\mathbf{b} = b_i \mathbf{e}_i$, in each basis \mathbf{e}_i assume $d_{ij} \equiv a_i b_j$. It's easy to see how components d_{ij} transform according to (3.1):

$$a'_i = o_{i'm} a_m, \quad b'_j = o_{j'n} b_n \Rightarrow d'_{ij} = a'_i b'_j = o_{i'm} a_m o_{j'n} b_n = o_{i'm} o_{j'n} d_{mn}.$$

A resulting tensor ${}^2\mathbf{d}$ is called a dyadic product or just dyad and is written as $\mathbf{a} \otimes \mathbf{b}$ or \mathbf{ab} . I choose the notation “ ${}^2\mathbf{d} = \mathbf{ab}$ ”, without the \otimes symbol.

When some bivalent tensor ${}^2\mathbf{B}$ is a dyad $\beta\mathbf{b}$, its components $B_{ij} = \beta_i b_j$ satisfy the equality $B_{pq} B_{mn} = B_{mq} B_{pn}$ to get commutativity of multiplication $\beta_p b_q \beta_m b_n = \beta_m b_q \beta_p b_n$. Here $p \neq m$, or else the equality becomes the identity.

The essential exemplar of a bivalent tensor is the unit tensor (other names are unit dyad, identity tensor and metric tensor). Let for any orthonormal (cartesian) basis $E_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. These are really components of tensor, (3.1) is actual: $E'_{mn} = o_{m'i} o_{n'j} E_{ij} = o_{m'i} o_{n'i} = \delta_{mn}$. I write this tensor as \mathbf{E} (other popular choices are \mathbf{I} and ${}^2\mathbf{1}$).

Invariableness of components upon any rotation makes the tensor \mathbf{E} isotropic. There are no non-null (nonvanishing) isotropic vectors (all components of the null, or “vanishing”, vector $\mathbf{0}$ are equal to zero within any basis).

The next example is related to a linear transformation (a linear mapping) of vectors.

If $\mathbf{b} = b_i \mathbf{e}_i$ is linear (preserving addition and multiplication by number) function of $\mathbf{a} = a_j \mathbf{e}_j$, then $b_i = c_{ij} a_j$ in every basis. Transformation coefficients c_{ij} alter when a basis rotates:

$$b'_i = c'_{ij} a'_j = o_{i'k} b_k = o_{i'k} c_{kn} a_n, \quad a_n = o_{j'n} a'_j \Rightarrow c'_{ij} = o_{i'k} o_{j'n} c_{kn}.$$

It turns out that a set of two-index objects c_{ij} , c'_{ij} , \dots , describing the same linear mapping $\mathbf{a} \mapsto \mathbf{b}$, but in various bases, represents a single invariant object — a tensor of second complexity ${}^2\mathbf{c}$. And many book authors introduce tensors in that way, by means of linear mappings (linear transformations).

And the last example is a bilinear form $F(\mathbf{a}, \mathbf{b}) = f_{ij} a_i b_j$, where f_{ij} are coefficients, a_i and b_j are components of vector arguments $\mathbf{a} = a_i \mathbf{e}_i$

and $\mathbf{b} = b_j \mathbf{e}_j$. The result F is invariant (independent of basis) with the transformation (3.1) for coefficients f_{ij} :

$$F' = f'_{ij} a'_i b'_j = f_{mn} \underbrace{a_m b_n}_{o_{i'm} a'_i o_{j'n} b'_j} = F \Leftrightarrow f'_{ij} = o_{i'm} o_{j'n} f_{mn}.$$

If $f_{ij} = \delta_{ij}$, then $F = \delta_{ij} a_i b_j = a_i b_i$ — the “ \bullet ”-product (dot product, scalar product) of two vectors. When both arguments are the same, such a homogeneous polynomial of second degree (quadratic) of one vector’s components $F(\mathbf{a}, \mathbf{a}) = f_{ij} a_i a_j$ is called a quadratic form.

Now about more complex tensors (of valence larger than two). Tensor of third complexity ${}^3\mathbf{C}$ is represented by a collection of $3^3 = 27$ numbers C_{ijk} , changing with a rotation of basis as

$$C'_{ijk} = \mathbf{e}'_i \bullet \mathbf{e}_p \mathbf{e}'_j \bullet \mathbf{e}_q \mathbf{e}'_k \bullet \mathbf{e}_r C_{pqr} = o_{i'p} o_{j'q} o_{k'r} C_{pqr}. \quad (3.2)$$

The primary example is a triad of three vectors $\mathbf{a} = a_i \mathbf{e}_i$, $\mathbf{b} = b_j \mathbf{e}_j$ and $\mathbf{c} = c_k \mathbf{e}_k$

$$t_{ijk} \equiv a_i b_j c_k \Leftrightarrow {}^3\mathbf{t} = \mathbf{a} \mathbf{b} \mathbf{c}.$$

It is seen that orthogonal transformations (3.2) and (3.1) are results of “repeating” vector’s (2.4). The reader will easily compose a transformation of components for tensor of any complexity and will write a corresponding polyad as an example.

Vectors with transformation (2.4) are tensors of the first complexity (monovalent tensors).

The least complex objects are scalars or tensors of the zeroth complexity. A scalar is a single ($3^0 = 1$) number, which doesn’t depend on a basis: the energy, the mass, the temperature et al. But what are components, for example, of vector $\mathbf{v} = v_i \mathbf{e}_i$, $v_i = \mathbf{v} \bullet \mathbf{e}_i$? If not scalars, then what? Here could be no simple answer. In each particular basis, \mathbf{e}_i are vectors and v_i are scalars.

§ 4. Tensor algebra, or operations with tensors

The whole tensor algebra can be built on the only five* operations (or actions). This section is just about them.

Equality

The first (or the zeroth) is **the equality** “=”. This operation shows whether one tensor “on the left” is equal to another tensor “on the right”. Tensors can be equal only when their complexities (valencies) are the same. Tensors of different valencies cannot be equal or not equal.

$$\dots \quad (4.1)$$

....

Linear combination

The next operation is **the linear combination**. It aggregates the addition and the multiplication by a number (by a scalar, or, in another word, scaling). The arguments of this operation and the result are of the same complexity. For a pair of tensors

$$\lambda a_{ij\dots} + \mu b_{ij\dots} = c_{ij\dots} \Leftrightarrow \lambda \mathbf{a} + \mu \mathbf{b} = \mathbf{c}. \quad (4.2)$$

Here λ and μ are scalar coefficients; \mathbf{a} , \mathbf{b} and \mathbf{c} are tensors of the same complexity. It's easy to show that the components of the result \mathbf{c} satisfy an orthogonal transformation like (3.1).

The decomposition of a vector by some basis, that is the representation of a vector as the sum $\mathbf{v} = v_i \mathbf{e}_i$, is nothing else but the linear combination of the basis vectors \mathbf{e}_i with the coefficients v_i .

This operation is *linear* because the only two atomary kinds of motion are possible on a line: the translation (the movement along a straight line) and the reflexion (mirroring) (the backward movement).

* The four without the equality.

Multiplication of tensors

One more operation — **the multiplication (the tensor product, the direct product)**. It takes arguments of any complexities, returning the result of the cumulative complexity. Examples:

$$\begin{aligned} v_i a_{jk} &= C_{ijk} \Leftrightarrow \mathbf{v}^2 \mathbf{a} = {}^3\mathbf{C}, \\ a_{ij} B_{abc} &= D_{ijabc} \Leftrightarrow {}^2\mathbf{a} {}^3\mathbf{B} = {}^5\mathbf{D}. \end{aligned} \quad (4.3)$$

Transformation of a collection of result's components, such as $C_{ijk} = v_i a_{jk}$, during a rotation of basis is orthogonal, similar to (3.2), thus here's no doubt that such a collection is a set of tensor components.

Primary and already known (from §3) subtype of multiplication is the dyadic product of two vectors ${}^2\mathbf{A} = \mathbf{b}\mathbf{c}$.

Contraction

The fourth (or the third) operation is called **the contraction**. It applies to bivalent and more complex tensors. This operation acts upon a single tensor, without other “participants”. Roughly speaking, contracting a tensor is summing of its components over some pair of indices. As a result, tensor's complexity decreases by two.

For a trivalent tensor ${}^3\mathbf{D}$ there are the three possible contractions. They give vectors \mathbf{a} , \mathbf{b} and \mathbf{c} with components

$$a_i = D_{kki}, \quad b_i = D_{kik}, \quad c_i = D_{ikk}. \quad (4.4)$$

A rotation of basis

$$a'_i = D'_{kki} = \underbrace{o_{k'p} o_{k'q} o_{i'r}}_{\delta_{pq}} D_{pqr} = o_{i'r} D_{ppr} = o_{i'r} a_r$$

shows “the tensorial nature” as the result of contraction.

For a tensor of second complexity, the only one variant of contraction is possible. It gives a scalar, known as the trace

$$\mathbf{B}_\bullet \equiv \text{trace } \mathbf{B} \equiv \mathbf{I}(\mathbf{B}) = B_{kk}.$$

The trace of the unit tensor (“contraction of the Kronecker delta”) is equal to the dimension of space

$$\text{trace } \mathbf{E} = \mathbf{E}_{\bullet} = \delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

Index juggling, transposing

The last operation is applicable to a single tensor of the second* and bigger complexities. It is named as **the index swap, index juggling, transposing**. From components of a tensor, the new collection emerges with another sequence of indices, and the result’s complexity stays the same. For example, a trivalent tensor ${}^3\mathbf{D}$ can give tensors ${}^3\mathbf{A}$, ${}^3\mathbf{B}$, ${}^3\mathbf{C}$ with components

$$\begin{aligned} {}^3\mathbf{A} = {}^3\mathbf{D}_{1\rightleftharpoons 2} &\Leftrightarrow A_{ijk} = D_{jik}, \\ {}^3\mathbf{B} = {}^3\mathbf{D}_{1\rightleftharpoons 3} &\Leftrightarrow B_{ijk} = D_{kji}, \\ {}^3\mathbf{C} = {}^3\mathbf{D}_{2\rightleftharpoons 3} &\Leftrightarrow C_{ijk} = D_{ikj}. \end{aligned} \quad (4.5)$$

For a bivalent tensor, the only one transposition is possible: $\mathbf{A}^\top \equiv \mathbf{A}_{1\rightleftharpoons 2} = \mathbf{B} \Leftrightarrow B_{ij} = A_{ji}$. Obviously, $(\mathbf{A}^\top)^\top = \mathbf{A}$.

For the dyadic multiplication of two vectors, $\mathbf{a}\mathbf{b} = \mathbf{b}\mathbf{a}^\top$.

Combining operations

The four presented algebraic operations (actions) can be combined in various sequences.

The combination of multiplication (4.3) and contraction (4.4) — the “ \bullet ”-product (dot product) — is the most frequently used. In the direct indexless notation this is denoted by a large dot “ \bullet ”, which shows the contraction by adjacent indices :

$$\mathbf{a} = \mathbf{B} \bullet \mathbf{c} \Leftrightarrow a_i = B_{ij}c_j, \quad \mathbf{A} = \mathbf{B} \bullet \mathbf{C} \Leftrightarrow A_{ij} = B_{ik}C_{kj}. \quad (4.6)$$

The defining property of the unit tensor — it is neutral (the “identity element”) for the dyadic product with the subsequent contraction by adjacent indices (for the “ \bullet ”-product)

$${}^n\mathbf{a} \bullet \mathbf{E} = \mathbf{E} \bullet {}^n\mathbf{a} = {}^n\mathbf{a} \quad \forall {}^n\mathbf{a} \quad \forall n > 0. \quad (4.7)$$

* Transposing a vector makes no sense.

In the **commutative** scalar product of two vectors, the dot represents the same : the dyadic product and the subsequent contraction

$$\mathbf{a} \cdot \mathbf{b} = (\mathbf{ab})_{\bullet} = a_i b_i = b_i a_i = (\mathbf{ba})_{\bullet} = \mathbf{b} \cdot \mathbf{a}. \quad (4.8)$$

The following identity describes how to swap multipliers for the “ \cdot ”-product (dot product) of two second complexity tensors

$$\begin{aligned} \mathbf{B} \cdot \mathbf{Q} &= (\mathbf{Q}^{\top} \cdot \mathbf{B}^{\top})^{\top} \\ (\mathbf{B} \cdot \mathbf{Q})^{\top} &= \mathbf{Q}^{\top} \cdot \mathbf{B}^{\top}. \end{aligned} \quad (4.9)$$

For two dyads $\mathbf{B} = \mathbf{bd}$ and $\mathbf{Q} = \mathbf{pq}$

$$\begin{aligned} (\mathbf{bd} \cdot \mathbf{pq})^{\top} &= \mathbf{pq}^{\top} \cdot \mathbf{bd}^{\top} \\ d_i p_i \mathbf{bq}^{\top} &= \mathbf{qp} \cdot \mathbf{db} \\ d_i p_i \mathbf{qb} &= p_i d_i \mathbf{qb}. \end{aligned}$$

For a vector and a bivalent tensor

$$\mathbf{c} \cdot \mathbf{B} = \mathbf{B}^{\top} \cdot \mathbf{c}, \quad \mathbf{B} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{B}^{\top}. \quad (4.10)$$

Contraction can be repeated two times or more: $(\mathbf{A} \cdot \mathbf{B})_{\bullet} = \mathbf{A} \cdot \mathbf{B} = A_{ij} B_{ji}$, and here are useful equations for second complexity tensors

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{A}, \quad \mathbf{d} \cdot \mathbf{A} \cdot \mathbf{b} = \mathbf{A} \cdot \mathbf{bd} = \mathbf{bd} \cdot \mathbf{A} = b_j d_i A_{ij}, \\ \mathbf{A} \cdot \mathbf{B} &= \mathbf{A}^{\top} \cdot \mathbf{B}^{\top} = A_{ij} B_{ji}, \quad \mathbf{A} \cdot \mathbf{B}^{\top} = \mathbf{A}^{\top} \cdot \mathbf{B} = A_{ij} B_{ij}, \\ \mathbf{A} \cdot \mathbf{E} &= \mathbf{E} \cdot \mathbf{A} = \mathbf{A}_{\bullet} = A_{jj}, \\ \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{E} &= A_{ij} B_{jk} \delta_{ki} = \mathbf{A} \cdot \mathbf{B}, \quad \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{E} = \mathbf{A} \cdot \mathbf{A}, \\ \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} &= \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \cdot \mathbf{B} = A_{ij} B_{jk} C_{ki}, \\ \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} \cdot \mathbf{D} &= \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} \cdot \mathbf{D} = \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} \cdot \mathbf{D} = \\ &= \mathbf{D} \cdot \mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = A_{ij} B_{jk} C_{kh} D_{hi}. \end{aligned} \quad (4.11)$$

§ 5. Polyadic representation (decomposition)

Before in § 3, a tensor was presented as some invariant object, revealing itself in every basis as a collection of numbers (components). Such

a presentation is typical for the majority of books about tensors. The index notation can be convenient, especially when only rectangular coordinates are used, but quite often it is not. And the relevant case is *physics of continua, elastic and not very elastic* : it needs more elegant, more powerful and perfect apparatus of the direct tensor calculus, operating with indexless invariant objects.

Linear combination $\mathbf{v} = v_i \mathbf{e}_i$ from decomposition (2.1) connects vector \mathbf{v} with basis \mathbf{e}_i and vector's components v_i in that basis. Soon we will get a similar relation for a tensor of any complexity.

Any bivalent tensor ${}^2\mathbf{B}$ has nine components B_{ij} in each basis. The number of various dyads $\mathbf{e}_i \mathbf{e}_j$ for the same basis is nine (3^2) too. Linear combining these dyads with coefficients B_{ij} gives the sum $B_{ij} \mathbf{e}_i \mathbf{e}_j$. This is tensor, but what are its components, and how this representation changes or doesn't change with a rotation of basis?

Components of the constructed sum

$$(B_{ij} \mathbf{e}_i \mathbf{e}_j)_{pq} = B_{ij} \delta_{ip} \delta_{jq} = B_{pq}$$

are components of tensor ${}^2\mathbf{B}$. And with a rotation of basis

$$B'_{ij} \mathbf{e}'_i \mathbf{e}'_j = o_{i'p} o_{j'q} B_{pq} o_{i'n} \mathbf{e}_n o_{j'm} \mathbf{e}_m = \delta_{pn} \delta_{qm} B_{pq} \mathbf{e}_n \mathbf{e}_m = B_{pq} \mathbf{e}_p \mathbf{e}_q.$$

Doubts are dropped: a tensor of second complexity can be (re)presented as the linear combination

$${}^2\mathbf{B} = B_{ij} \mathbf{e}_i \mathbf{e}_j \tag{5.1}$$

— the dyadic decomposition of a bivalent tensor.

For the unit tensor

$$\mathbf{E} = E_{ij} \mathbf{e}_i \mathbf{e}_j = \delta_{ij} \mathbf{e}_i \mathbf{e}_j = \mathbf{e}_i \mathbf{e}_i = \mathbf{e}_1 \mathbf{e}_1 + \mathbf{e}_2 \mathbf{e}_2 + \mathbf{e}_3 \mathbf{e}_3,$$

that's why \mathbf{E} is called the unit dyad.

Polyadic representations like (5.1) help to operate with the tensors much easier:

$$\mathbf{v} \cdot {}^2\mathbf{B} = v_i \mathbf{e}_i \cdot \mathbf{e}_j B_{jk} \mathbf{e}_k = v_i \delta_{ij} B_{jk} \mathbf{e}_k = v_i B_{ik} \mathbf{e}_k,$$

$$\mathbf{e}_i \cdot {}^2\mathbf{B} \cdot \mathbf{e}_j = \mathbf{e}_i \cdot B_{pq} \mathbf{e}_p \mathbf{e}_q \cdot \mathbf{e}_j = B_{pq} \delta_{ip} \delta_{qj} = B_{ij} = {}^2\mathbf{B} \cdot \mathbf{e}_j \mathbf{e}_i. \tag{5.2}$$

§ 6. Matrices, permutations and determinants

Matrices are the convenient tool for arranging of elements and for solving systems of linear equations.

Does the reader know that matrices are sometimes called “arrays”? Does someone need two-dimensional arrays? Matrices can be presented as tables full of rows and columns. Any matrix has the same number of elements in each row and the same number of elements in each column. The rectangular arrangement of elements, anyone? Matrices are full of numbers and expressions in rows and columns. A column arranges data vertically from top to bottom, while a row arranges horizontally from left to right.

Matrix dimensions

Matrices come in all sizes, or “dimensions”.

By convention, rows are listed first, and columns second.

The dimension of a matrix consists of the number of rows, then a multiplication sign (“ \times ” is used the most often), and then the number of columns.

Here are examples

$$\begin{array}{ll}
 \begin{array}{l} \left[\mathcal{A} \right] \\ 3 \times 3 \end{array} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} & \begin{array}{l} 3 \text{ rows and 3 columns} \\ 3 \times 3 \text{ matrix} \\ \text{“square” matrix} \end{array} \quad \begin{array}{l} \textit{number of rows} \\ \textit{is the same} \\ \textit{as columns} \end{array} \\
 \\
 \begin{array}{l} \left[\mathcal{B} \right] \\ 2 \times 4 \end{array} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \end{bmatrix} & \begin{array}{l} 2 \text{ rows and 4 columns} \\ \text{dimension } 2 \times 4 \end{array} \\
 \\
 \begin{array}{l} \left[\mathcal{C} \right] \\ 3 \times 1 \end{array} = \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix} & \begin{array}{l} 3 \text{ rows, 1 column} \\ 3 \times 1 \\ \text{a “column matrix”} \end{array} \quad \begin{array}{l} \textit{a matrix} \\ \textit{with just one} \\ \textit{column} \end{array} \\
 \\
 \begin{array}{l} \left[\mathcal{D} \right] \\ 1 \times 6 \end{array} = \left[D_{11} \ D_{12} \ D_{13} \ D_{14} \ D_{15} \ D_{16} \right] & \begin{array}{l} 1 \text{ row, 6 columns} \\ 1 \times 6 \\ \text{a “row matrix”} \end{array} \quad \begin{array}{l} \textit{a matrix} \\ \textit{with just one} \\ \textit{row} \end{array}
 \end{array}$$

The matrix algebra

The matrix algebra includes the linear operations — the addition of matrices and the multiplication by scalar.

The dimension of a matrix is essential for binary operations, that is for operations involving two matrices.

An addition or subtraction of the two matrices is possible only if they have the same sizes.

The multiplication of matrices

.....

$$[\mathcal{A}]_{m \times n} = \dots$$

The matrix of the result, known as “the matrix product”, has the number of rows of the first multiplier matrix and the number of columns of the second matrix.

.....

Square matrices

....

Matrices and the one-dimensional arrays

The two indices of a table is more than the single index of a one-dimensional array. Due to this, an one-dimensional array could be presented as a table of rows or as a table of columns.

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix},$$

or the vertical tables

$$\begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}.$$

$$\det_{i,j} \delta_{ij} = 1$$

...

the permutation parity symbols via the determinant

$$e_{pqr} = e_{ijk} \delta_{pi} \delta_{qj} \delta_{rk} = e_{ijk} \delta_{ip} \delta_{jq} \delta_{kr},$$

$$e_{pqr} = \det \begin{bmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{bmatrix} = \det \begin{bmatrix} \delta_{p1} & \delta_{p2} & \delta_{p3} \\ \delta_{q1} & \delta_{q2} & \delta_{q3} \\ \delta_{r1} & \delta_{r2} & \delta_{r3} \end{bmatrix}. \quad (6.1)$$

...

A determinant is not sensitive to transposing:

$$\det A_{ij} = \det A_{ji} = \det A_{ij}.$$

...

“The determinant of the matrix product of the two matrices is equal to the product of determinants of these matrices”

$$\det B_{ik} \det C_{kj} = \det B_{ik} C_{kj} \quad (6.2)$$

$$e_{fgh} \det_{m,n} B_{ms} C_{sn} = e_{pqr} B_{fi} C_{ip} B_{gj} C_{jq} B_{hk} C_{kr}$$

$$e_{fgh} \det_{m,s} B_{ms} = e_{ijk} B_{fi} B_{gj} B_{hk}$$

$$e_{ijk} \det_{s,n} C_{sn} = e_{pqr} C_{ip} C_{jq} C_{kr}$$

$$e_{fgh} e_{ijk} \det_{m,s} B_{ms} \det_{s,n} C_{sn} = e_{ijk} e_{pqr} B_{fi} B_{gj} B_{hk} C_{ip} C_{jq} C_{kr}$$

...

Определитель компонент of a bivalent tensor is invariant, он не меняется с поворотом базиса

$$A'_{ij} = o_{i'm} o_{j'n} A_{mn}$$

§ 7. The cross product

By common notions, the “ \times ”-product (the “cross product”, the “vector product”, sometimes the “oriented area product”) of two vectors is the vector, heading perpendicular to the plane of multipliers, whose length is equal to the area of the parallelogram, spanned by the multipliers

$$\| \mathbf{a} \times \mathbf{b} \| = \| \mathbf{a} \| \| \mathbf{b} \| \sin \angle(\mathbf{a}, \mathbf{b}).$$

However, a “ \times ”-product isn’t quite a vector, since it is not completely invariant.

The multipliers of the “ \times ”-product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ determine the result’s direction in space, with an accuracy up to the sign figure 3.

Once you pick as the positive the “right-chiral” (“right-handed”) or the “left-chiral” (“left-handed”) orientation of space, the one direction from the possible two, then the results of the “ \times ”-products become completely determined.

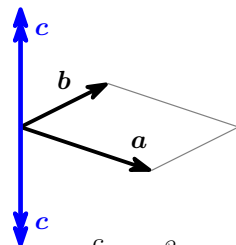


figure 3

“The chiral” means asymmetric in such a way that the thing and its mirror image are not superimposable, a picture cannot be superposed on its mirror image by any combination of rotations and translations.

An object is chiral if it is distinguishable from its mirror image.

Vectors are usually measured via some basis \mathbf{e}_i . They are decomposed into linear combinations like $\mathbf{a} = a_i \mathbf{e}_i$. So the orientation of space is equivalent to the orientation of the sequential triple of basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. It means that the sequence of basis vectors becomes significant (for linear combinations, the sequence of addends doesn’t affect anything).

If two bases consist of different sequences of the same vectors, then orientations of these bases differ by some permutation.

The orientation of the space is a (kind of) asymmetry. This asymmetry makes it impossible to replicate mirroring by the means of any rotations*

A pseudovector is a vector-like object, invariant under any rotation.

**

... put the figure here ...

* Applying only rotations, it’s impossible to replace the left hand with the right hand. But it is possible by mirroring.

** Rotations cannot change the orientation of a triple of basis vectors, it is possible only via mirroring.

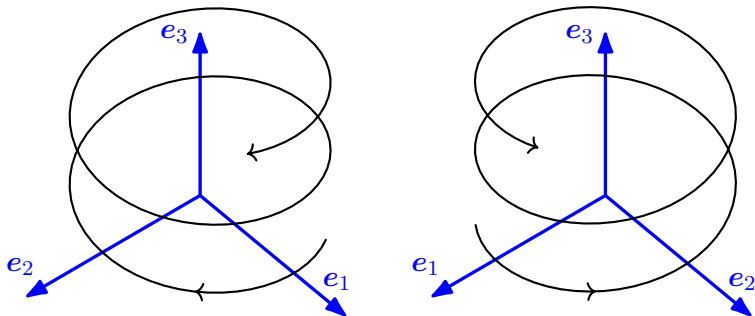


figure 4

Except on rare cases, mirroring changes the direction of a fully invariant (polar) vector.

A pseudovector (an axial vector), unlike a polar vector, doesn't change the component that is perpendicular to the mirroring plane, and turns out to be flipped relatively to the polar vectors and the geometry of the entire space. This happens because the sign (and, accordingly, the direction) of each axial vector changes along with changing the sign of the “ \times ”-product — which corresponds to mirroring.

The otherness of pseudovectors narrows the variety of formulas: a pseudovector is not additive with a vector. The formula $\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}$ from an absolutely rigid undeformable body's kinematics is correct, because $\boldsymbol{\omega}$ is pseudovector there, and with the cross product the two “pseudo” give $(-1)^2 = 1$, mutually compensating each other.

The parity of permutations tensor is the volumetric tensor of third complexity

$${}^3\epsilon = \epsilon_{ijk} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k, \quad \epsilon_{ijk} \equiv \mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k \quad (7.1)$$

with the components ϵ_{ijk} equal to the “triple” (the “mixed”, the “cross-dot”) products of the basis vectors.

The absolute value (the modulus) of each nonzero component of ${}^3\epsilon$ is equal to the volume \sqrt{g} of a parallelopiped drew upon a basis. For a basis \mathbf{e}_i of pairwise perpendicular one unit long vectors $\sqrt{g} = 1$.

The tensor ${}^3\epsilon$ is isotropic, its components are constant and independent of any rotations of a basis. But mirroring — a change in the orientation of a triple of basis vectors (a change in “the direction of screw”) — changes the sign of ${}^3\epsilon$, so this is a pseudotensor (an axial tensor).

If $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$ without the “minus” sign, then the basis triple \mathbf{e}_i is oriented positively. The positive orientation (or “the positive direction”) is chosen for different reasons from the two possible ones (figure 3). For a positively oriented basis triplet, the components of ${}^3\epsilon$ are equal to the parity of permutations symbol $\epsilon_{ijk} = e_{ijk}$. And when $\mathbf{e}_1 \times \mathbf{e}_2 = -\mathbf{e}_3$, then the basis triple \mathbf{e}_i is oriented negatively, or “mirrored”. For mirrored triples $\epsilon_{ijk} = -e_{ijk}$ (and $e_{ijk} = -\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k$).

With the **Levi-Civita tensor** ${}^3\epsilon$ it is possible to take the fresh look at the cross product :

$$\epsilon_{ijk} = \mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k \Leftrightarrow \mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k,$$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= a_i \mathbf{e}_i \times b_j \mathbf{e}_j = a_i b_j \mathbf{e}_i \times \mathbf{e}_j = a_i b_j \epsilon_{ijk} \mathbf{e}_k = \\ &= b_j a_i \mathbf{e}_j \mathbf{e}_i \cdot \epsilon_{mnk} \mathbf{e}_m \mathbf{e}_n \mathbf{e}_k = \mathbf{b} \mathbf{a} \cdot \epsilon \cdot \mathbf{e}, \\ &= a_i \epsilon_{ijk} \mathbf{e}_k b_j = -a_i \epsilon_{ikj} \mathbf{e}_k b_j = -\mathbf{a} \cdot \epsilon \cdot \mathbf{b}. \end{aligned} \quad (7.2)$$

So that, the cross product is not another new, entirely distinct operation. With the **Levi-Civita** tensor it reduces to the four already described (§ 4) and is applicable to tensors of any complexity.

“The cross product” is just the dot product — the combination of multiplication and contraction (§ 4) — involving tensor ${}^3\epsilon$. Such combinations are possible with any tensors :

$$\begin{aligned} \mathbf{a} \times {}^2\mathbf{B} &= a_i \mathbf{e}_i \times B_{jk} \mathbf{e}_j \mathbf{e}_k = \underbrace{a_i B_{jk} \epsilon_{ijn}}_{-a_i \epsilon_{inj} B_{jk}} \mathbf{e}_n \mathbf{e}_k = -\mathbf{a} \cdot \epsilon \cdot {}^2\mathbf{B}, \\ {}^2\mathbf{C} \times \mathbf{d} \mathbf{b} &= C_{ij} \mathbf{e}_i \mathbf{e}_j \times d_p b_q \mathbf{e}_p \mathbf{e}_q = \mathbf{e}_i C_{ij} d_p \underbrace{\epsilon_{jpk}}_{-\epsilon_{pjk} = -\epsilon_{jkp}} \mathbf{e}_k b_q \mathbf{e}_q = \\ &= -{}^2\mathbf{C} \mathbf{d} \cdot \epsilon \mathbf{b} = -{}^2\mathbf{C} \cdot \epsilon \cdot \mathbf{d} \mathbf{b}, \end{aligned}$$

$$\mathbf{E} \times \mathbf{E} = \mathbf{e}_i \mathbf{e}_i \times \mathbf{e}_j \mathbf{e}_j = -\epsilon_{ijk} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k = -{}^3\epsilon. \quad (7.3)$$

The last equation links the isotropic tensor of the second complexity and the isotropic tensor of the third complexity.

Generalizing to all tensors of nonzero complexity

$${}^n\xi \times {}^m\zeta = -{}^n\xi \cdot {}^3\epsilon \cdot {}^m\zeta \quad \forall {}^n\xi, {}^m\zeta \quad \forall n > 0, m > 0. \quad (7.4)$$

When one of the operands is the unit (metric) tensor, from (7.4) and (4.7) for $\forall {}^n\Upsilon \quad \forall n > 0$

$$\begin{aligned} \mathbf{E} \times {}^n\Upsilon &= -\mathbf{E} \cdot {}^3\epsilon \cdot {}^n\Upsilon = -{}^3\epsilon \cdot {}^n\Upsilon, \\ {}^n\Upsilon \times \mathbf{E} &= -{}^n\Upsilon \cdot {}^3\epsilon \cdot \mathbf{E} = -{}^n\Upsilon \cdot {}^3\epsilon. \end{aligned} \quad (7.5)$$

The cross product of two vectors is not commutative, but is anti-commutative :

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{E}) = (\mathbf{a} \times \mathbf{E}) \cdot \mathbf{b} = -\overbrace{\mathbf{a} \mathbf{b} \cdot \cdot {}^3\epsilon}^{\in_{jik} = \in_{kji} \Rightarrow a_i b_j \in_{jik} \mathbf{e}_k = \in_{kji} a_i b_j \mathbf{e}_k} = -{}^3\epsilon \cdot \cdot \mathbf{a} \mathbf{b}, \\ \mathbf{b} \times \mathbf{a} &= \mathbf{b} \cdot (\mathbf{a} \times \mathbf{E}) = (\mathbf{b} \times \mathbf{E}) \cdot \mathbf{a} = -\mathbf{b} \mathbf{a} \cdot \cdot {}^3\epsilon = -{}^3\epsilon \cdot \cdot \mathbf{b} \mathbf{a}, \\ \mathbf{a} \times \mathbf{b} &= -\mathbf{a} \mathbf{b} \cdot \cdot {}^3\epsilon = \mathbf{b} \mathbf{a} \cdot \cdot {}^3\epsilon \Rightarrow \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}. \end{aligned} \quad (7.6)$$

For any bivalent tensor ${}^2\mathbf{B}$ and any vector \mathbf{a}

$$\begin{aligned} {}^2\mathbf{B} \times \mathbf{a} &= \mathbf{e}_i B_{ij} \mathbf{e}_j \times a_k \mathbf{e}_k = -\mathbf{e}_i B_{ij} a_k \mathbf{e}_k \times \mathbf{e}_j \\ &= (-a_k \mathbf{e}_k \times \mathbf{e}_j B_{ij} \mathbf{e}_i)^\top = -(\mathbf{a} \times {}^2\mathbf{B}^\top)^\top, \end{aligned} \quad (7.7)$$

and only for the unit dyad and a vector, the “ \times ”-product is commutative

$$\begin{aligned} \in_{ijk} &= \in_{kij} \Rightarrow \in_{ijk} a_k \mathbf{e}_i \mathbf{e}_j = a_k \in_{kij} \mathbf{e}_i \mathbf{e}_j \\ -\mathbf{E} \times \mathbf{a} &= \overbrace{{}^3\epsilon \cdot \mathbf{a} = \mathbf{a} \cdot {}^3\epsilon}^{\in_{ijk} = \in_{kji} \Rightarrow a_i b_j \in_{jik} \mathbf{e}_k = \in_{kji} a_i b_j \mathbf{e}_k} = -\mathbf{a} \times \mathbf{E}, \end{aligned} \quad (7.8)$$

plus as the particular case of (7.7)

$$\mathbf{E} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{E}^\top)^\top = -(\mathbf{a} \times \mathbf{E})^\top. \quad (7.9)$$

Справедливо такое соотношение

$$\begin{aligned} e_{ijk} e_{pqr} &= \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix} \\ &\quad + \in_{ijk} \mathbf{e}_i \mathbf{e}_k \mathbf{e}_j \end{aligned} \quad (7.10)$$

○ Доказательство начнём с представления символов чётности перестановки как определителей (6.1). $e_{ijk} = \pm \mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k$ по строкам, $e_{pqr} = \pm \mathbf{e}_p \times \mathbf{e}_q \cdot \mathbf{e}_r$ по столбцам, с “−” для “левой” тройки

$$e_{ijk} = \det \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}, \quad e_{pqr} = \det \begin{bmatrix} \delta_{p1} & \delta_{q1} & \delta_{r1} \\ \delta_{p2} & \delta_{q2} & \delta_{r2} \\ \delta_{p3} & \delta_{q3} & \delta_{r3} \end{bmatrix}.$$

Левая часть (7.10) есть произведение $e_{ijk}e_{pqr}$ этих определителей. Но $\det(AB) = (\det A)(\det B)$ — определитель произведения матриц равен произведению определителей (6.2). В матрице-произведении элемент $[\dots]_{11}$ равен $\delta_{is}\delta_{ps} = \delta_{ip}$, как и в (7.10); **легко проверить и другие фрагменты.** ●

The contraction of (7.10) leads to useful formulas

$$\begin{aligned} e_{ijk}e_{pqk} &= \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & \delta_{kk} \end{bmatrix} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & 3 \end{bmatrix} = \\ &= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jk}\delta_{kp} + \delta_{ik}\delta_{jp}\delta_{kq} - \delta_{ik}\delta_{jq}\delta_{kp} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jk}\delta_{kq} = \\ &= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} + \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} = \\ &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \end{aligned}$$

$$e_{ijk}e_{pjk} = \delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip},$$

$$e_{ijk}e_{ijk} = 2\delta_{ii} = 6$$

or in short

$$e_{ijk}e_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \quad e_{ijk}e_{pjk} = 2\delta_{ip}, \quad e_{ijk}e_{ijk} = 6. \quad (7.11)$$

The first of these formulas gives the following representation of the double “ \times ”-product

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= a_i \mathbf{e}_i \times \epsilon_{pqj} b_p c_q \mathbf{e}_j = \epsilon_{kij} \epsilon_{pqj} a_i b_p c_q \mathbf{e}_k = \\ &= (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip}) a_i b_p c_q \mathbf{e}_k = a_i b_k c_i \mathbf{e}_k - a_i b_i c_k \mathbf{e}_k = \\ &= \mathbf{a} \cdot \mathbf{cb} - \mathbf{a} \cdot \mathbf{bc} = \mathbf{a} \cdot (\mathbf{cb} - \mathbf{bc}) = \mathbf{a} \cdot \mathbf{cb} - \mathbf{cb} \cdot \mathbf{a}. \end{aligned} \quad (7.12)$$

By another interpretation, the dot product of a dyad and a vector is not commutative: $\mathbf{bd} \cdot \mathbf{c} \neq \mathbf{c} \cdot \mathbf{bd}$, and the difference can be rendered as

$$\mathbf{bd} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{bd} = \mathbf{c} \times (\mathbf{b} \times \mathbf{d}). \quad (7.13)$$

$$\mathbf{a} \cdot \mathbf{bc} = \mathbf{cb} \cdot \mathbf{a} = \mathbf{ca} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{ac}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{c} \times (\mathbf{b} \times \mathbf{a})$$

The same way it may be derived that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{ba} - \mathbf{ab}) \cdot \mathbf{c} = \mathbf{ba} \cdot \mathbf{c} - \mathbf{ab} \cdot \mathbf{c}. \quad (7.14)$$

And the following identities for any two vectors \mathbf{a} and \mathbf{b}

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \times \mathbf{E} &= \epsilon_{ijk} a_i b_j \mathbf{e}_k \times \mathbf{e}_n \mathbf{e}_n = a_i b_j \epsilon_{ijk} \epsilon_{knq} \mathbf{e}_q \mathbf{e}_n = \\ &= a_i b_j (\delta_{in} \delta_{jq} - \delta_{iq} \delta_{jn}) \mathbf{e}_q \mathbf{e}_n = a_i b_j \mathbf{e}_j \mathbf{e}_i - a_i b_j \mathbf{e}_i \mathbf{e}_j = \\ &= \mathbf{ba} - \mathbf{ab}, \end{aligned} \quad (7.15)$$

$$\begin{aligned} (\mathbf{a} \times \mathbf{E}) \cdot (\mathbf{b} \times \mathbf{E}) &= (\mathbf{a} \cdot {}^3\epsilon) \cdot (\mathbf{b} \cdot {}^3\epsilon) = \\ &= a_i \epsilon_{ipm} \mathbf{e}_p \mathbf{e}_n \cdot b_j \epsilon_{jsk} \mathbf{e}_s \mathbf{e}_k = a_i b_j \epsilon_{ipn} \epsilon_{nkj} \mathbf{e}_p \mathbf{e}_k = \\ &= a_i b_j (\delta_{ik} \delta_{pj} - \delta_{ij} \delta_{pk}) \mathbf{e}_p \mathbf{e}_k = a_i b_j \mathbf{e}_j \mathbf{e}_i - a_i b_i \mathbf{e}_k \mathbf{e}_k = \\ &= \mathbf{ba} - \mathbf{a} \cdot \mathbf{b} \mathbf{E}. \end{aligned} \quad (7.16)$$

Finally, the direct relation between the isotropic tensors of the second and third complexities

$${}^3\epsilon \cdot \cdot {}^3\epsilon = \mathbf{e}_i \epsilon_{ijk} \epsilon_{kjin} \mathbf{e}_n = -2\delta_{in} \mathbf{e}_i \mathbf{e}_n = -2\mathbf{E}. \quad (7.17)$$

§ 8. Symmetric and skewsymmetric tensors

A tensor that does not change upon a permutation of some pair of its indices is called symmetric for that pair of indices. And when a permutation of some pair of indices alternates the sign “+/-” of a tensor, then this tensor is called anti-symmetric or skew-symmetric for that pair of indices.

As example, the tensor of parity of permutations ${}^3\epsilon$ (7.1) is antisymmetric for any & every pair of indices, it is completely (absolutely) skewsymmetric.

Tensor of the second complexity \mathbf{B} is symmetric when $\mathbf{B} = \mathbf{B}^\top$. If transposing changes the tensor's sign, that is $\mathbf{A}^\top = -\mathbf{A}$, then tensor \mathbf{A} is skewsymmetric (antisymmetric).

The sum of a bivalent tensor \mathbf{C} with the transpose \mathbf{C}^\top is always symmetric: $(\mathbf{C} + \mathbf{C}^\top)^\top = \mathbf{C}^\top + \mathbf{C} = \mathbf{C} + \mathbf{C}^\top \quad \forall \mathbf{C}$, while the difference $(\mathbf{C} - \mathbf{C}^\top)^\top = \mathbf{C}^\top - \mathbf{C} = -(\mathbf{C} - \mathbf{C}^\top)$ is always $\forall \mathbf{C}$ antisymmetric.

Denoting

$$\mathbf{C}^S \equiv \frac{1}{2}(\mathbf{C} + \mathbf{C}^\top), \quad \mathbf{C}^A \equiv \frac{1}{2}(\mathbf{C} - \mathbf{C}^\top) \quad (8.1)$$

— the symmetric \mathbf{C}^S and the antisymmetric \mathbf{C}^A parts of some bivalent tensor \mathbf{C} , any bivalent tensor can be presented as the sum of these parts

$$\mathbf{C} = \mathbf{C}^S + \mathbf{C}^A, \quad \mathbf{C}^\top = \mathbf{C}^S - \mathbf{C}^A. \quad (8.2)$$

For a dyad

$$\mathbf{cd} = \overbrace{\frac{1}{2}(\mathbf{cd} + \mathbf{dc})}^{cd^S} + \overbrace{\frac{1}{2}(\mathbf{cd} - \mathbf{dc})}^{cd^A}.$$

The product $\mathbf{C}^S \cdot \mathbf{D}^S$ of two symmetric tensors \mathbf{C}^S and \mathbf{D}^S is symmetric not always, but only when $\mathbf{D}^S \cdot \mathbf{C}^S = \mathbf{C}^S \cdot \mathbf{D}^S$, because by (4.9) $(\mathbf{C}^S \cdot \mathbf{D}^S)^\top = \mathbf{D}^S \cdot \mathbf{C}^S$.

With (7.8) and (7.9), the skew symmetry of the “ \times ”-product for the unit dyad and a vector is obvious

$$\begin{aligned} (\mathbf{E} \times \mathbf{a})^\top &= (\mathbf{e}_j \mathbf{e}_j \times a_i \mathbf{e}_i)^\top = (-\mathbf{e}_j a_i \mathbf{e}_i \times \mathbf{e}_j)^\top = -a_i \mathbf{e}_i \times \mathbf{e}_j \mathbf{e}_j \\ &= -\mathbf{a} \times \mathbf{E} = -\mathbf{E} \times \mathbf{a} = (\mathbf{a} \times \mathbf{E})^\top. \end{aligned} \quad (8.3)$$

In search for a case when a bivalent tensor \mathbf{A} can be represented by just a single vector \mathbf{a} , in such a way that an action of vector \mathbf{a} on other objects is exactly like an action of bivalent \mathbf{A} on the same objects, perhaps there's a chance to find such $\mathbf{A} = \mathbf{A}(\mathbf{a})$ that for $\forall^n \boldsymbol{\xi} \quad \forall n > 0$

$$\begin{aligned} \mathbf{b} &= \mathbf{A} \cdot {}^n \boldsymbol{\xi} \Leftrightarrow \mathbf{a} \times {}^n \boldsymbol{\xi} = \mathbf{b} \quad \forall \mathbf{b}, \\ \mathbf{d} &= {}^n \boldsymbol{\xi} \cdot \mathbf{A} \Leftrightarrow {}^n \boldsymbol{\xi} \times \mathbf{a} = \mathbf{d} \quad \forall \mathbf{d} \end{aligned}$$

or, in words, the “ \cdot ”-product of bivalent \mathbf{A} and some other tensor ${}^n\xi$ is equal to the “ \times ”-product of pseudovector \mathbf{a} and the same tensor ${}^n\xi$.

The relation $\mathbf{a} \mapsto \mathbf{A}$ can be derived from (4.7) and (7.5)

$$\begin{aligned}\mathbf{A} &= \mathbf{A} \cdot \mathbf{E} = \mathbf{a} \times \mathbf{E} = -\mathbf{a} \cdot {}^3\epsilon, \\ \mathbf{A} &= \mathbf{E} \cdot \mathbf{A} = \mathbf{E} \times \mathbf{a} = -{}^3\epsilon \cdot \mathbf{a}.\end{aligned}\tag{8.4}$$

Or, delving into components,

$$\begin{aligned}\mathbf{A} \cdot {}^n\xi &= \mathbf{a} \times {}^n\xi \\ A_{hi} e_h e_i \cdot \xi_{jk\dots q} e_j e_k \dots e_q &= a_i e_i \times \xi_{jk\dots q} e_j e_k \dots e_q \\ A_{hj} \xi_{jk\dots q} e_h e_k \dots e_q &= a_i \in_{ijh} \xi_{jk\dots q} e_h e_k \dots e_q \\ A_{hj} &= a_i \in_{ijh} \\ A_{hj} &= -a_i \in_{ihj} \\ \mathbf{A} &= -\mathbf{a} \cdot {}^3\epsilon\end{aligned}$$

and by similar way from ${}^n\xi \cdot \mathbf{A} = {}^n\xi \times \mathbf{a}$ follows $\mathbf{A} = -{}^3\epsilon \cdot \mathbf{a}$.

(Pseudo)vector \mathbf{a} is sometimes named as “accompanying” or “companion” for tensor \mathbf{A} .

To components

$$\begin{aligned}\mathbf{A} &= -{}^3\epsilon \cdot \mathbf{a} \\ A_{ij} e_i e_j &= -\in_{ijk} e_i e_j a_k \\ A_{ij}(a_k) : A_{ij} &= -\in_{ijk} a_k\end{aligned}$$

or, written as a matrix,

$$\begin{aligned}\begin{bmatrix} A_{ij} \\ 3 \times 3 \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & -\in_{123} a_3 & -\in_{132} a_2 \\ -\in_{213} a_3 & 0 & -\in_{231} a_1 \\ -\in_{312} a_2 & -\in_{321} a_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.\end{aligned}$$

That the bivalent $\mathbf{A} = \mathbf{a} \times \mathbf{E} = \mathbf{E} \times \mathbf{a}$ is skewsymmetric was clear since (8.3). In the three-dimensional space, any antisymmetric tensor of the second complexity has only three independent components out of 9: $A_{ij} = -A_{ji}$ and $A_{jj} = 0$.

The uniqueness of \mathbf{a} for the unique \mathbf{A} , that is if $\mathbf{a}' \times \mathbf{E} = \mathbf{A}$ and $\mathbf{a}'' \times \mathbf{E} = \mathbf{A}$ (or $\mathbf{a}' \times \mathbf{E} - \mathbf{a}'' \times \mathbf{E} = \mathbf{A} - \mathbf{A}$) then $\mathbf{a}' = \mathbf{a}''$ or $\mathbf{a}' - \mathbf{a}'' = \mathbf{0}$

$$\begin{aligned}(\mathbf{a}' - \mathbf{a}'') \times \mathbf{E} &= \mathbf{0} \\ (\mathbf{a}' - \mathbf{a}'') \cdot {}^3\epsilon &= \mathbf{0} \cdot {}^3\epsilon\end{aligned}$$

follows from the equal zeros $\mathbf{0} \cdot {}^3\epsilon = \mathbf{0}$ and the uniqueness of the “ \cdot ”-product’s result ($\mathbf{b} \cdot \mathbf{c} = \mathbf{d} \cdot \mathbf{c}, \mathbf{c} \neq \mathbf{0} \Leftrightarrow \mathbf{b} = \mathbf{d}$, including $\mathbf{b} \cdot \mathbf{c} = \mathbf{0} \cdot \mathbf{c}, \mathbf{c} \neq \mathbf{0} \Leftrightarrow \mathbf{b} = \mathbf{0}$). For $\mathbf{a} = \mathbf{0}$, $\mathbf{A}(\mathbf{0}) = \mathbf{0} \times \mathbf{E} = \mathbf{0}$.

✓ \mathbf{a} is unique for \mathbf{A}

And yet about the reciprocal relation $\mathbf{A} \mapsto \mathbf{a} : \mathbf{a} = \mathbf{a}(\mathbf{A})$. By (7.17), the unit dyad \mathbf{E} via ${}^3\epsilon$

$$\mathbf{E} = -\frac{1}{2} {}^3\epsilon \cdot \cdot {}^3\epsilon,$$

and it is neutral (4.7) for the “ \cdot ”-product

$$\mathbf{a} = \underbrace{\mathbf{a} \cdot \left(-\frac{1}{2} {}^3\epsilon \cdot \cdot {}^3\epsilon\right)}_{-1/2 a_a \in_{abc} \in_{cbm} e_m} = \underbrace{\left(-\frac{1}{2} {}^3\epsilon \cdot \cdot {}^3\epsilon\right) \cdot \mathbf{a}}_{-1/2 e_h \in_{hij} \in_{jik} a_k}$$

or without brackets

$$\mathbf{a} = -\frac{1}{2} {}^3\epsilon \cdot \cdot {}^3\epsilon \cdot \mathbf{a} = -\frac{1}{2} \mathbf{a} \cdot {}^3\epsilon \cdot \cdot {}^3\epsilon.$$

Bivalent \mathbf{A} can be introduced here as in (8.4), $-\mathbf{A} = \mathbf{a} \cdot {}^3\epsilon = {}^3\epsilon \cdot \mathbf{a}$, and then

$$\mathbf{a}(\mathbf{A}) = \frac{1}{2} \mathbf{A} \cdot \cdot {}^3\epsilon = \frac{1}{2} {}^3\epsilon \cdot \cdot \mathbf{A}. \quad (8.5)$$

.....

$$\begin{aligned}a_i e_i &= \frac{1}{2} A_{jk} \in_{kji} e_i = \frac{1}{2} \in_{ikj} A_{jk} e_i, \\ a_i &= \frac{1}{2} \in_{ikj} A_{jk} = \frac{1}{2} \begin{bmatrix} \in_{123} A_{32} + \in_{132} A_{23} \\ \in_{213} A_{31} + \in_{231} A_{13} \\ \in_{312} A_{21} + \in_{321} A_{12} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}.\end{aligned}$$

.....

$$\begin{aligned}
-2a_1 &= \overbrace{\epsilon_{123}}^{(+1)} \overbrace{\epsilon_{321}}^{(-1)} a_1 + \overbrace{\epsilon_{132}}^{(-1)} \overbrace{\epsilon_{231}}^{(+1)} a_1 \\
&\quad \quad \quad -A_{32} \quad \quad \quad -A_{23} \\
-2a_2 &= \overbrace{\epsilon_{213}}^{-1} \overbrace{\epsilon_{312}}^{+1} a_2 + \overbrace{\epsilon_{231}}^{+1} \overbrace{\epsilon_{132}}^{-1} a_2 \\
&\quad \quad \quad -A_{31} \quad \quad \quad -A_{13} \\
-2a_3 &= \overbrace{\epsilon_{312}}^{(+1)} \overbrace{\epsilon_{213}}^{(-1)} a_3 + \overbrace{\epsilon_{321}}^{(-1)} \overbrace{\epsilon_{123}}^{(+1)} a_3 \\
&\quad \quad \quad -A_{21} \quad \quad \quad -A_{12}
\end{aligned}$$

.....

$$\mathbf{a}' \times \mathbf{E} = \mathbf{A}' \quad \text{and} \quad \mathbf{a}'' \times \mathbf{E} = \mathbf{A}''$$

PROVE IT \mathbf{A} is unique for \mathbf{a} **PROVE IT**

.....

There is **PROVE THAT ONLY FOR SKEWSYMMETRIC**
 $\forall \mathbf{A} = \mathbf{A}^A$ **IT IS BIJECTION** a bijection* between antisymmet-
 ric bivalent tensors and (pseudo)vectors. The components of
 a skewsymmetric tensor are fully described by the three numbers
 xxx

.....

All in all, here is the bijection $\mathbf{A} \leftrightarrow \mathbf{a}$

$$\mathbf{A}(\mathbf{a}) = -\mathbf{a} \cdot {}^3\epsilon = \mathbf{a} \times \mathbf{E} = -{}^3\epsilon \cdot \mathbf{a} = \mathbf{E} \times \mathbf{a}, \quad (8.6^{A(a)})$$

$$\mathbf{a}(\mathbf{A}) = \frac{1}{2} \mathbf{A} \cdot \cdot {}^3\epsilon = \frac{1}{2} {}^3\epsilon \cdot \cdot \mathbf{A}. \quad (8.6^{a(A)})$$

Easy to memorize, the “pseudovector invariant” \mathbf{A}_\times comes from
 the original tensor \mathbf{A} by replacing a dyadic product with a cross
 product

$$\begin{aligned}
\mathbf{A}_\times &\equiv A_{ij} \mathbf{e}_i \times \mathbf{e}_j = -\mathbf{A} \cdot \cdot {}^3\epsilon, \\
\mathbf{A}_\times &= (\mathbf{a} \times \mathbf{E})_\times = -2\mathbf{a}, \quad \mathbf{a} = -\frac{1}{2} \mathbf{A}_\times = -\frac{1}{2} (\mathbf{a} \times \mathbf{E})_\times.
\end{aligned} \quad (8.7)$$

*“a bijective relation”, “a reciprocally reversible mapping”, “a one-to-one
 correspondence”

Explanation:

$$\begin{aligned}
 \mathbf{a} \times \mathbf{E} &= -\frac{1}{2} \mathbf{A}_{\times} \times \mathbf{E} = -\frac{1}{2} A_{ij} \underbrace{(\mathbf{e}_i \times \mathbf{e}_j)}_{\in_{ijn} \mathbf{e}_n} \times \mathbf{e}_k \mathbf{e}_k \\
 &= -\frac{1}{2} A_{ij} \underbrace{\in_{nij} \in_{nkp}}_{\delta_{jp} \delta_{ik} - \delta_{ip} \delta_{jk}} \mathbf{e}_p \mathbf{e}_k = -\frac{1}{2} A_{ij} (\mathbf{e}_j \mathbf{e}_i - \mathbf{e}_i \mathbf{e}_j) \\
 &= -\frac{1}{2} (\mathbf{A}^T - \mathbf{A}) = \mathbf{A}^A = \mathbf{A}.
 \end{aligned}$$

The companion vector can be introduced for any bivalent tensor. But only the asymmetric part contributes here: $\mathbf{C}^A = -\frac{1}{2} \mathbf{C}_{\times} \times \mathbf{E}$.

For a symmetric tensor, the companion vector is zero :

$$\mathbf{B}_{\times} = \mathbf{0} \Leftrightarrow \mathbf{B} = \mathbf{B}^T = \mathbf{B}^S.$$

With (8.7) the decomposition of some tensor \mathbf{C} into the symmetric and antisymmetric parts looks like

$$\mathbf{C} = \mathbf{C}^S - \frac{1}{2} \mathbf{C}_{\times} \times \mathbf{E}. \quad (8.8)$$

For a dyad

$$(7.15) \Rightarrow (\mathbf{c} \times \mathbf{d}) \times \mathbf{E} = \mathbf{dc} - \mathbf{cd} = -2\mathbf{cd}^A, \quad (\mathbf{cd})_{\times} = \mathbf{c} \times \mathbf{d},$$

and its decomposition

$$\mathbf{cd} = \frac{1}{2} (\mathbf{cd} + \mathbf{dc}) - \frac{1}{2} (\mathbf{c} \times \mathbf{d}) \times \mathbf{E}. \quad (8.9)$$

§9. Polar decomposition

Any tensor of the second complexity \mathbf{F} with $\det F_{ij} \neq 0$ (not singular) can be decomposed as

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Example. Polar decompose tensor $\mathbf{C} = C_{ij} \mathbf{e}_i \mathbf{e}_j$, where \mathbf{e}_k are pairwise perpendicular unit vectors and C_{ij} are the tensor's components.

$$\begin{aligned}
 C_{ij} &= \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix} \\
 \mathbf{O} &= O_{ij} \mathbf{e}_i \mathbf{e}_j = \mathbf{O}_1 \cdot \mathbf{O}_2 \\
 O_{ij} &= \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix} \\
 \mathbf{C} &= \mathbf{O} \cdot \mathbf{S}_R, \quad \mathbf{O}^\top \cdot \mathbf{C} = \mathbf{S}_R \\
 \mathbf{C} &= \mathbf{S}_L \cdot \mathbf{O}, \quad \mathbf{C} \cdot \mathbf{O}^\top = \mathbf{S}_L \\
 S_{Rij} &= \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix} \\
 S_{Lij} &= \begin{bmatrix} 104/5 & 47/5 & 5 \\ 47/5 & -129/5 & -10 \\ 5 & -10 & 3 \end{bmatrix}
 \end{aligned}$$

...

§ 10. Eigenvectors and eigenvalues

If for some tensor ${}^2\mathbf{B}$ and the nonzero vector \mathbf{a}

$${}^2\mathbf{B} \cdot \mathbf{a} = \eta \mathbf{a}, \quad \mathbf{a} \neq \mathbf{0} \quad (10.1)$$

$${}^2\mathbf{B} \cdot \mathbf{a} = \eta \mathbf{E} \cdot \mathbf{a}, \quad ({}^2\mathbf{B} - \eta \mathbf{E}) \cdot \mathbf{a} = \mathbf{0},$$

then η is called the eigenvalue (or the characteristic value) of tensor ${}^2\mathbf{B}$, and the axis (direction) of eigenvector \mathbf{a} is called its characteristic axis (or direction).

In components, this is the eigenvalue problem for a matrix. A homogeneous system of linear equations $(B_{ij} - \eta \delta_{ij}) a_j = 0$ has a non-zero solution if the determinant of a matrix of components

$$\det_{i,j} (B_{ij} - \eta \delta_{ij})$$

is equal to zero:

$$\det \begin{bmatrix} B_{11} - \eta & B_{12} & B_{13} \\ B_{21} & B_{22} - \eta & B_{23} \\ B_{31} & B_{32} & B_{33} - \eta \end{bmatrix} = -\eta^3 + \text{chaI}\eta^2 - \text{chaII}\eta + \text{chaIII} = 0; \quad (10.2)$$

$$\begin{aligned} \text{chaI} &= \text{trace } {}^2\mathbf{B} = B_{kk} = B_{11} + B_{22} + B_{33}, \\ \text{chaII} &= B_{11}B_{22} - B_{12}B_{21} + B_{11}B_{33} - B_{13}B_{31} + B_{22}B_{33} - B_{23}B_{32}, \\ \text{chaIII} &= \det {}^2\mathbf{B} = \det B_{ij} = e_{ijk} B_{1i} B_{2j} B_{3k} = e_{ijk} B_{i1} B_{j2} B_{k3}. \end{aligned} \quad (10.3)$$

The roots of the characteristic equation (10.2) — the eigenvalues — don't depend on the basis and therefore are invariants.

The coefficients of (10.3) also don't depend on the basis; they are called the first, the second and the third characteristic invariants of a tensor. The first invariant chaI is the trace. It was described earlier in §4. **The second invariant chaII is the trace of the adjugate matrix — the transpose of the cofactor matrix (of the matrix of algebraic complements)**

$$\text{chaII}({}^2\mathbf{B}) \equiv \text{trace}(\text{adj } B_{ij})$$

(it's hard, yeah). Or

$$\text{chaII}({}^2\mathbf{B}) \equiv \frac{1}{2} \left[({}^2\mathbf{B} \cdot) {}^2 - {}^2\mathbf{B} \cdot \cdot {}^2\mathbf{B} \right] = \frac{1}{2} \left[(B_{kk})^2 - B_{ij} B_{ji} \right].$$

And the third invariant chaIII is the determinant of a matrix of tensor components: $\text{chaIII}({}^2\mathbf{B}) \equiv \det {}^2\mathbf{B}$.

This applies to all second complexity tensors. Besides that, in case of a symmetric tensor, the following is true:

- 1° The eigenvalues of a symmetric bivalent tensor are real numbers.
- 2° The characteristic axes (directions) for different eigenvalues are orthogonal to each other.

○ The first statement is proved by contradiction. If η is a complex root of (10.2) corresponding to eigenvector \mathbf{a} , then conjugate number $\bar{\eta}$

will also be the root of (10.2). Eigenvector $\bar{\mathbf{a}}$ with the conjugate components corresponds to it. And then

$$\begin{aligned} (10.1) \Rightarrow (\bar{\mathbf{a}} \cdot) {}^2\mathbf{B} \cdot \mathbf{a} &= \eta \mathbf{a}, \quad (\mathbf{a} \cdot) {}^2\mathbf{B} \cdot \bar{\mathbf{a}} = \bar{\eta} \bar{\mathbf{a}} \Rightarrow \\ &\Rightarrow \bar{\mathbf{a}} \cdot {}^2\mathbf{B} \cdot \mathbf{a} - \mathbf{a} \cdot {}^2\mathbf{B} \cdot \bar{\mathbf{a}} = (\eta - \bar{\eta}) \mathbf{a} \cdot \bar{\mathbf{a}}. \end{aligned}$$

Here on the left is zero, because $\mathbf{a} \cdot {}^2\mathbf{B} \cdot \mathbf{c} = \mathbf{c} \cdot {}^2\mathbf{B}^\top \cdot \mathbf{a}$ and ${}^2\mathbf{B} = {}^2\mathbf{B}^\top$. Thence $\eta = \bar{\eta}$, that is a real number.

Just as simple looks the proof of 2°:

$$\underbrace{\mathbf{a}_2 \cdot {}^2\mathbf{B} \cdot \mathbf{a}_1 - \mathbf{a}_1 \cdot {}^2\mathbf{B} \cdot \mathbf{a}_2}_{=0} = (\eta_1 - \eta_2) \mathbf{a}_1 \cdot \mathbf{a}_2, \quad \eta_1 \neq \eta_2 \Rightarrow \mathbf{a}_1 \cdot \mathbf{a}_2 = 0. \quad \bullet$$

If the roots of the characteristic equation (the eigenvalues) are different, then one unit long eigenvectors $\boldsymbol{\alpha}_i$ compose an orthonormal basis. What are tensor components in such a basis?

$$\begin{aligned} {}^2\mathbf{B} \cdot \boldsymbol{\alpha}_k &= \sum_k \eta_k \boldsymbol{\alpha}_k, \quad k = 1, 2, 3 \\ {}^2\mathbf{B} \cdot \underbrace{\boldsymbol{\alpha}_k \boldsymbol{\alpha}_k}_E &= \sum_k \eta_k \boldsymbol{\alpha}_k \boldsymbol{\alpha}_k \end{aligned}$$

For a common case $B_{ij} = \mathbf{e}_i \cdot {}^2\mathbf{B} \cdot \mathbf{e}_j$. In the basis $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$ of mutually perpendicular one unit long $\boldsymbol{\alpha}_i \cdot \boldsymbol{\alpha}_j = \delta_{ij}$ eigenvectors of a symmetric tensor

$$\begin{aligned} B_{11} &= \boldsymbol{\alpha}_1 \cdot (\eta_1 \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_1 + \eta_2 \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_2 + \eta_3 \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_3) \cdot \boldsymbol{\alpha}_1 = \eta_1, \\ B_{12} &= \boldsymbol{\alpha}_1 \cdot (\eta_1 \boldsymbol{\alpha}_1 \boldsymbol{\alpha}_1 + \eta_2 \boldsymbol{\alpha}_2 \boldsymbol{\alpha}_2 + \eta_3 \boldsymbol{\alpha}_3 \boldsymbol{\alpha}_3) \cdot \boldsymbol{\alpha}_2 = 0, \\ &\dots \end{aligned}$$

The matrix of components is diagonal and ${}^2\mathbf{B} = \sum_i \eta_i \boldsymbol{\alpha}_i \boldsymbol{\alpha}_i$. Yes, here is the summation over the three repeating indices, because the special basis is used.

The transition to a coincidence case of eigenvalues can be acquired via limit calculation. If $\eta_2 \rightarrow \eta_1$, then any linear combination of vectors \mathbf{a}_1 and \mathbf{a}_2 in the limit satisfies the equation (10.1). And then any axis in the plane $(\mathbf{a}_1, \mathbf{a}_2)$ becomes characteristic.

When all the three eigenvalues coincide, then any axis in space is characteristic.

Then ${}^2\mathbf{B} = \eta \mathbf{E}$, such tensors are called isotropic or “spherical”.

§ 11. Collections of invariants of a symmetric bivalent tensor

....

“The algebraic” invariants

...

“The characteristic” invariants

These are coefficients of a characteristic equation (10.1).

...

“The research” invariants

...

“The harmonic” invariants

...

§ 12. Rotations via rotation tensors

The relation between two “right” (or two “left” orthonormal bases \mathbf{e}_i and $\mathring{\mathbf{e}}_i$) can be described by a two-index array represented as a matrix (§ 2, § 6)

$$\mathbf{e}_i = \mathbf{e}_i \cdot \underbrace{\mathring{\mathbf{e}}_j \mathring{\mathbf{e}}_j}_{\mathbf{E}} = o_{ij} \mathring{\mathbf{e}}_j, \quad o_{ij} \mathring{\mathbf{e}}_j \equiv \mathbf{e}_i \cdot \mathring{\mathbf{e}}_j$$

(“a matrix of cosines”).

Also, a rotation of tensor can be described by another tensor, called a rotation tensor \mathbf{O}

$$\mathbf{e}_i = \mathbf{e}_j \underbrace{\mathring{\mathbf{e}}_j \cdot \mathring{\mathbf{e}}_i}_{\delta_{ji}} = \mathbf{O} \cdot \mathring{\mathbf{e}}_i, \quad \mathbf{O} \equiv \mathbf{e}_j \mathring{\mathbf{e}}_j = \mathbf{e}_1 \mathring{\mathbf{e}}_1 + \mathbf{e}_2 \mathring{\mathbf{e}}_2 + \mathbf{e}_3 \mathring{\mathbf{e}}_3. \quad (12.1)$$

Components of \mathbf{O} both in an initial $\mathring{\mathbf{e}}_i$ and in a rotated \mathbf{e}_i bases are the same

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{O} \cdot \mathbf{e}_j &= \underbrace{\mathbf{e}_i \cdot \mathbf{e}_k}_{\delta_{ik}} \mathring{\mathbf{e}}_k \cdot \mathbf{e}_j = \mathring{\mathbf{e}}_i \cdot \mathbf{e}_j, \\ \mathring{\mathbf{e}}_i \cdot \mathbf{O} \cdot \mathring{\mathbf{e}}_j &= \mathring{\mathbf{e}}_i \cdot \mathbf{e}_k \underbrace{\mathring{\mathbf{e}}_k \cdot \mathring{\mathbf{e}}_j}_{\delta_{kj}} = \mathring{\mathbf{e}}_i \cdot \mathbf{e}_j. \end{aligned} \quad (12.2)$$

In the matrix notation, these components present the matrix of cosines $o_{ji}^\circ = \mathring{\mathbf{e}}_i \cdot \mathbf{e}_j$:

$$\mathbf{O} = o_{ji}^\circ \mathbf{e}_i \mathbf{e}_j = o_{ji}^\circ \mathring{\mathbf{e}}_i \mathring{\mathbf{e}}_j.$$

Spatial transformations in the 3-dimensional Euclidean space \mathbb{R}^3 are distinguished into active or alibi transformations, and passive or alias transformations. An active transformation is a transformation which actually changes the physical position (alibi, elsewhere) of objects, which can be defined in the absence of a coordinate system; whereas a passive transformation is merely a change in the coordinate system in which the object is described (alias, other name) (change of coordinates, or change of basis). By transformation, math texts usually refer to active transformations.

Tensor \mathbf{O} relates the two vectors — “before rotation” $\mathring{\mathbf{r}} = \rho_i \mathring{\mathbf{e}}_i$ and “after rotation” $\mathbf{r} = \rho_i \mathbf{e}_i$. Components $\rho_i = \text{constant}$ of \mathbf{r} in rotated basis \mathbf{e}_i are the same as of $\mathring{\mathbf{r}}$ in immobile basis $\mathring{\mathbf{e}}_i$. So that the rotation tensor describes the rotation of the vector together with the basis. And since $\mathbf{e}_i = \mathbf{e}_j \mathring{\mathbf{e}}_j \cdot \mathring{\mathbf{e}}_i \Leftrightarrow \rho_i \mathbf{e}_i = \mathbf{e}_j \mathring{\mathbf{e}}_j \cdot \rho_i \mathring{\mathbf{e}}_i$, then

$$\mathbf{r} = \mathbf{O} \cdot \mathring{\mathbf{r}} \quad (12.3)$$

(this is the Rodrigues rotation formula).

Olinde Rodrigues. Des lois géométriques qui régissent les déplacements d’un système solide dans l’espace, et de la variation des

coordonnées provenant de ces déplacements considérés indépendants des causes qui peuvent les produire. *Journal de mathématiques pures et appliquées*, tome 5 (1840), pages 380–440.

For a second complexity tensor $\overset{\circ}{\mathbf{C}} = C_{ij} \overset{\circ}{\mathbf{e}}_i \overset{\circ}{\mathbf{e}}_j$, a rotation into the current position $\mathbf{C} = C_{ij} \mathbf{e}_i \mathbf{e}_j$ looks like

$$\mathbf{e}_i C_{ij} \mathbf{e}_j = \mathbf{e}_i \overset{\circ}{\mathbf{e}}_i \cdot \overset{\circ}{\mathbf{e}}_p C_{pq} \overset{\circ}{\mathbf{e}}_q \cdot \overset{\circ}{\mathbf{e}}_j \mathbf{e}_j \Leftrightarrow \mathbf{C} = \mathbf{O} \cdot \overset{\circ}{\mathbf{C}} \cdot \mathbf{O}^\top. \quad (12.4)$$

The essential property of a rotation tensor — the orthogonality — is expressed as

$$\underbrace{\mathbf{O} \cdot \mathbf{O}^\top}_{\mathbf{e}_i \overset{\circ}{\mathbf{e}}_i \quad \overset{\circ}{\mathbf{e}}_j \mathbf{e}_j} = \underbrace{\mathbf{O}^\top \cdot \mathbf{O}}_{\overset{\circ}{\mathbf{e}}_i \mathbf{e}_i \quad \mathbf{e}_j \overset{\circ}{\mathbf{e}}_j} = \underbrace{\mathbf{E}}_{\overset{\circ}{\mathbf{e}}_i \overset{\circ}{\mathbf{e}}_i \quad \mathbf{e}_i \mathbf{e}_i}, \quad (12.5)$$

that is the transposed tensor coincides with the reciprocal tensor: $\mathbf{O}^\top = \mathbf{O}^{-1} \Leftrightarrow \mathbf{O} = \mathbf{O}^{-\top}$.

An orthogonal tensor retains lengths and angles (the metric) because it does not change the “ \cdot ”-product of vectors

$$(\mathbf{O} \cdot \mathbf{a}) \cdot (\mathbf{O} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{O}^\top \cdot \mathbf{O} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{E} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}. \quad (12.6)$$

For all orthogonal tensors $(\det \mathbf{Q})^2 = 1$:

$$1 = \det \mathbf{E} = \det (\mathbf{Q} \cdot \mathbf{Q}^\top) = (\det \mathbf{Q}) (\det \mathbf{Q}^\top) = (\det \mathbf{Q})^2.$$

A rotation tensor is an orthogonal tensor with $\det \mathbf{O} = +1$.

But not only rotation tensors possess the property of orthogonality. When in (12.1) the first basis is “left”, and the second one is “right”, then there’s a combination of rotating and mirroring (a “rotoreflexion”) $\mathbf{O} = -\mathbf{E} \cdot \mathbf{O}$ with $\det (-\mathbf{E} \cdot \mathbf{O}) = -1$.

Any bivalent tensor in the three-dimensional (3D) space has at least one eigenvalue — the root of (10.2) — is non-complex, or real. For a rotation tensor, it is equal to one

$$\mathbf{O} \cdot \mathbf{a} = \eta \mathbf{a} \Rightarrow \overbrace{\mathbf{a} \cdot \mathbf{O}^\top \cdot \mathbf{O}}^{\mathbf{O} \cdot \mathbf{a}} \cdot \mathbf{a} = \eta \mathbf{a} \cdot \eta \mathbf{a} \Rightarrow \eta^2 = 1.$$

The corresponding eigenvector is called the axis of rotation. The Euler's theorem on finite rotation is just about that such an axis exists [<http://eulerarchive.maa.org/docs/originals/E478.pdf>]. If \mathbf{k} is the unit vector of that axis, and ϑ is the finite angle of rotation, then the rotation tensor is representable as

$$\mathbf{O}(\mathbf{k}, \vartheta) = \mathbf{E} \cos \vartheta + \mathbf{k} \times \mathbf{E} \sin \vartheta + \mathbf{k}\mathbf{k} (1 - \cos \vartheta). \quad (12.7)$$

Доказывается эта формула так. Направление \mathbf{k} во время поворота не меняется ($\mathbf{O} \cdot \mathbf{k} = \mathbf{k}$), поэтому на оси поворота $\mathring{\mathbf{e}}_3 = \mathbf{e}_3 = \mathbf{k}$. В перпендикулярной плоскости (рисунок 5) $\mathring{\mathbf{e}}_1 = \mathbf{e}_1 \cos \vartheta - \mathbf{e}_2 \sin \vartheta$, $\mathring{\mathbf{e}}_2 = \mathbf{e}_1 \sin \vartheta + \mathbf{e}_2 \cos \vartheta$, $\mathbf{O} = \mathbf{e}_i \mathring{\mathbf{e}}_i \Rightarrow (12.7)$.

Из (12.7) и (12.3) получаем формулу поворота Родрига в параметрах \mathbf{k} и ϑ :

$$\mathbf{r} = \mathring{\mathbf{r}} \cos \vartheta + \mathbf{k} \times \mathring{\mathbf{r}} \sin \vartheta + \mathbf{k}\mathbf{k} \cdot \mathring{\mathbf{r}} (1 - \cos \vartheta).$$

В параметрах конечного поворота транспонирование, оно же обращение, тензора \mathbf{O} эквивалентно перемене направления поворота — знака угла ϑ

$$\mathbf{O}^\top = \mathbf{O}|_{\vartheta=-\vartheta} = \mathbf{E} \cos \vartheta - \mathbf{k} \times \mathbf{E} \sin \vartheta + \mathbf{k}\mathbf{k} (1 - \cos \vartheta).$$

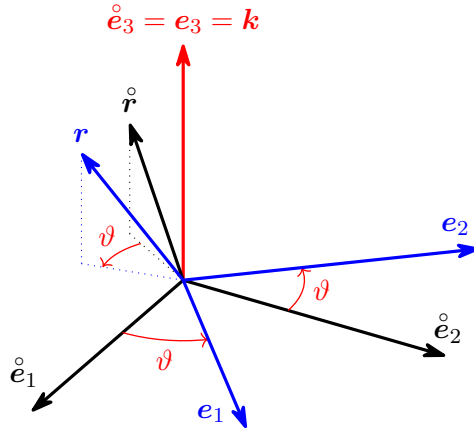
Пусть теперь тензор поворота меняется со временем: $\mathbf{O} = \mathbf{O}(t)$. Псевдовектор угловой скорости $\boldsymbol{\omega}$ вводится через тензор поворота \mathbf{O} таким путём. Дифференцируем тождество ортогональности (12.5) по времени*

$$\dot{\mathbf{O}} \cdot \mathbf{O}^\top + \mathbf{O} \cdot \dot{\mathbf{O}}^\top = \mathbf{0}.$$

* Various notations are used to designate the time derivative. In addition to the Leibniz's notation dx/dt , the very popular one is the “dot above” Newton's notation \dot{x} .

$$\mathring{e}_i = \mathring{e}_i \cdot \mathbf{e}_j \mathbf{e}_j$$

$$\begin{bmatrix} \mathring{e}_1 \\ \mathring{e}_2 \\ \mathring{e}_3 \end{bmatrix} = \begin{bmatrix} \mathring{e}_1 \cdot \mathbf{e}_1 & \mathring{e}_1 \cdot \mathbf{e}_2 & \mathring{e}_1 \cdot \mathbf{e}_3 \\ \mathring{e}_2 \cdot \mathbf{e}_1 & \mathring{e}_2 \cdot \mathbf{e}_2 & \mathring{e}_2 \cdot \mathbf{e}_3 \\ \mathring{e}_3 \cdot \mathbf{e}_1 & \mathring{e}_3 \cdot \mathbf{e}_2 & \mathring{e}_3 \cdot \mathbf{e}_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$



$$\begin{bmatrix} \mathring{e}_1 \cdot \mathbf{e}_1 & \mathring{e}_1 \cdot \mathbf{e}_2 & \mathring{e}_1 \cdot \mathbf{e}_3 \\ \mathring{e}_2 \cdot \mathbf{e}_1 & \mathring{e}_2 \cdot \mathbf{e}_2 & \mathring{e}_2 \cdot \mathbf{e}_3 \\ \mathring{e}_3 \cdot \mathbf{e}_1 & \mathring{e}_3 \cdot \mathbf{e}_2 & \mathring{e}_3 \cdot \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \cos \vartheta & \cos(90^\circ + \vartheta) & \cos 90^\circ \\ \cos(90^\circ - \vartheta) & \cos \vartheta & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} = \begin{bmatrix} \cos \vartheta & -\sin \vartheta & 0 \\ \sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathring{e}_1 = \mathbf{e}_1 \cos \vartheta - \mathbf{e}_2 \sin \vartheta$$

$$\mathring{e}_2 = \mathbf{e}_1 \sin \vartheta + \mathbf{e}_2 \cos \vartheta$$

$$\mathring{e}_3 = \mathbf{e}_3 = \mathbf{k}$$

$$\mathbf{O} = \mathbf{e}_1 \mathring{e}_1 + \mathbf{e}_2 \mathring{e}_2 + \mathbf{e}_3 \mathring{e}_3 =$$

$$\begin{aligned} &= \overbrace{\mathbf{e}_1 \mathring{e}_1}^{\mathbf{e}_1 \mathring{e}_1} + \overbrace{\mathbf{e}_2 \mathring{e}_2}^{\mathbf{e}_2 \mathring{e}_2} + \overbrace{\mathbf{e}_3 \mathring{e}_3}^{\mathbf{e}_3 \mathring{e}_3} = \\ &= \mathbf{E} \cos \vartheta - \underbrace{\mathbf{e}_3 \mathbf{e}_3}_{\mathbf{k}\mathbf{k}} \cos \vartheta + \underbrace{(\mathbf{e}_2 \mathbf{e}_1 - \mathbf{e}_1 \mathbf{e}_2)}_{\mathbf{e}_3 \times \mathbf{e}_i \mathbf{e}_i = \epsilon_{3ij} \mathbf{e}_j \mathbf{e}_i} \sin \vartheta + \mathbf{k}\mathbf{k} = \\ &= \mathbf{E} \cos \vartheta + \mathbf{k} \times \mathbf{E} \sin \vartheta + \mathbf{k}\mathbf{k} (1 - \cos \vartheta) \end{aligned}$$

рисунок 5
“Finite rotation”

По (4.9) $(\dot{\mathbf{O}} \cdot \mathbf{O}^\top)^\top = \mathbf{O} \cdot \dot{\mathbf{O}}^\top$, поэтому тензор $\dot{\mathbf{O}} \cdot \mathbf{O}^\top$ оказывается антисимметричным. Тогда согласно (8.6^{A(a)}) он представим через сопутствующий вектор как $\dot{\mathbf{O}} \cdot \mathbf{O}^\top = \boldsymbol{\omega} \times \mathbf{E} = \boldsymbol{\omega} \times \mathbf{O} \cdot \mathbf{O}^\top$. То есть

$$\dot{\mathbf{O}} = \boldsymbol{\omega} \times \mathbf{O}, \quad \boldsymbol{\omega} \equiv -\frac{1}{2} \left(\dot{\mathbf{O}} \cdot \mathbf{O}^\top \right)_\times \quad (12.8)$$

Besides this generic representation for the (pseudo)vector $\boldsymbol{\omega}$, there are others too. For example, via two parameters of the finite rotation.

Производная $\dot{\mathbf{O}}$ в параметрах конечного поворота в общем случае, когда оба параметра — и единичный вектор \mathbf{k} , и угол ϑ — переменны во времени :

$$\begin{aligned} \dot{\mathbf{O}} &= (\mathbf{O}^S + \mathbf{O}^A)^\bullet = \left(\overbrace{\mathbf{E} \cos \vartheta + \mathbf{k} \mathbf{k} (1 - \cos \vartheta)}^{\mathbf{O}^S} + \overbrace{\mathbf{k} \times \mathbf{E} \sin \vartheta}^{\mathbf{O}^A} \right)^\bullet = \\ &= \underbrace{(\mathbf{k} \mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta + (\mathbf{k} \dot{\mathbf{k}} + \dot{\mathbf{k}} \mathbf{k}) (1 - \cos \vartheta)}_{\dot{\mathbf{O}}^S} + \underbrace{\mathbf{k} \times \mathbf{E} \dot{\vartheta} \cos \vartheta + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta}_{\dot{\mathbf{O}}^A}. \end{aligned}$$

Находим

$$\begin{aligned} \dot{\mathbf{O}} \cdot \mathbf{O}^\top &= (\dot{\mathbf{O}}^S + \dot{\mathbf{O}}^A) \cdot (\mathbf{O}^S - \mathbf{O}^A) = \\ &= \dot{\mathbf{O}}^S \cdot \mathbf{O}^S + \dot{\mathbf{O}}^A \cdot \mathbf{O}^S - \dot{\mathbf{O}}^S \cdot \mathbf{O}^A - \dot{\mathbf{O}}^A \cdot \mathbf{O}^A, \end{aligned}$$

using

$$\begin{aligned} \mathbf{k} \cdot \mathbf{k} &= 1 = \text{constant} \Rightarrow \mathbf{k} \cdot \dot{\mathbf{k}} + \dot{\mathbf{k}} \cdot \mathbf{k} = 0 \Leftrightarrow \dot{\mathbf{k}} \cdot \mathbf{k} = \mathbf{k} \cdot \dot{\mathbf{k}} = 0, \\ \mathbf{k} \mathbf{k} \cdot \mathbf{k} \mathbf{k} &= \mathbf{k} \mathbf{k}, \quad \dot{\mathbf{k}} \mathbf{k} \cdot \mathbf{k} \mathbf{k} = \dot{\mathbf{k}} \mathbf{k}, \quad \mathbf{k} \dot{\mathbf{k}} \cdot \mathbf{k} \mathbf{k} = \dot{\mathbf{k}} \mathbf{k}, \\ (\mathbf{k} \mathbf{k} - \mathbf{E}) \cdot \mathbf{k} &= \mathbf{k} - \mathbf{k} = \mathbf{0}, \quad (\mathbf{k} \mathbf{k} - \mathbf{E}) \cdot \mathbf{k} \mathbf{k} = \mathbf{k} \mathbf{k} - \mathbf{k} \mathbf{k} = \mathbf{0}, \\ \mathbf{k} \cdot (\mathbf{k} \times \mathbf{E}) &= (\mathbf{k} \times \mathbf{E}) \cdot \mathbf{k} = \mathbf{k} \times \mathbf{k} = \mathbf{0}, \quad \mathbf{k} \mathbf{k} \cdot (\mathbf{k} \times \mathbf{E}) = (\mathbf{k} \times \mathbf{E}) \cdot \mathbf{k} \mathbf{k} = \mathbf{0}, \\ (\mathbf{k} \mathbf{k} - \mathbf{E}) \cdot (\mathbf{k} \times \mathbf{E}) &= -\mathbf{k} \times \mathbf{E}, \\ (\mathbf{a} \times \mathbf{E}) \cdot \mathbf{b} &= \mathbf{a} \times (\mathbf{E} \cdot \mathbf{b}) = \mathbf{a} \times \mathbf{b} \Rightarrow (\dot{\mathbf{k}} \times \mathbf{E}) \cdot \mathbf{k} \mathbf{k} = \dot{\mathbf{k}} \times \mathbf{k} \mathbf{k}, \end{aligned}$$

$$(7.16) \Rightarrow (\mathbf{k} \times \mathbf{E}) \cdot (\mathbf{k} \times \mathbf{E}) = \mathbf{k} \mathbf{k} - \mathbf{E}, \quad (\dot{\mathbf{k}} \times \mathbf{E}) \cdot (\mathbf{k} \times \mathbf{E}) = \mathbf{k} \dot{\mathbf{k}} - \dot{\mathbf{k}} \cdot \mathbf{k} \mathbf{E},$$

$$(7.15) \Rightarrow \dot{\mathbf{k}} \mathbf{k} - \mathbf{k} \dot{\mathbf{k}} = (\mathbf{k} \times \dot{\mathbf{k}}) \times \mathbf{E}, \quad (\dot{\mathbf{k}} \times \mathbf{k}) \mathbf{k} - \mathbf{k} (\dot{\mathbf{k}} \times \mathbf{k}) = \mathbf{k} \times (\dot{\mathbf{k}} \times \mathbf{k}) \times \mathbf{E}$$

$$\begin{aligned}
\dot{O}^S \cdot O^S &= \\
&= (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cdot \mathbf{E} \cos \vartheta + (\mathbf{k}\dot{\mathbf{k}} + \dot{\mathbf{k}}\mathbf{k})(1 - \cos \vartheta) \cdot \mathbf{E} \cos \vartheta + \\
&\quad + (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cdot \mathbf{k}\mathbf{k} (1 - \cos \vartheta) + (\mathbf{k}\dot{\mathbf{k}} + \dot{\mathbf{k}}\mathbf{k})(1 - \cos \vartheta) \cdot \mathbf{k}\mathbf{k} (1 - \cos \vartheta) = \\
&= (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta + (\mathbf{k}\dot{\mathbf{k}} + \dot{\mathbf{k}}\mathbf{k}) \cos \vartheta (1 - \cos \vartheta) + (\mathbf{k}\dot{\mathbf{k}} \cdot \mathbf{k}\mathbf{k} + \dot{\mathbf{k}}\mathbf{k} \cdot \mathbf{k}\mathbf{k}) (1 - \cos \vartheta)^2 = \\
&= (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta + \mathbf{k}\dot{\mathbf{k}} \cos \vartheta (1 - \cos \vartheta) + \\
&\quad + \dot{\mathbf{k}}\mathbf{k} \cos \vartheta - \dot{\mathbf{k}}\mathbf{k} \cos^2 \vartheta + \mathbf{k}\dot{\mathbf{k}} - 2 \dot{\mathbf{k}}\mathbf{k} \cos \vartheta + \dot{\mathbf{k}}\mathbf{k} \cos^2 \vartheta = \\
&= (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta + \mathbf{k}\dot{\mathbf{k}} \cos \vartheta - \dot{\mathbf{k}}\mathbf{k} \cos^2 \vartheta + \dot{\mathbf{k}}\mathbf{k} (1 - \cos \vartheta),
\end{aligned}$$

$$\begin{aligned}
\dot{O}^A \cdot O^S &= \\
&= (\mathbf{k} \times \mathbf{E}) \cdot \mathbf{E} \dot{\vartheta} \cos^2 \vartheta + (\dot{\mathbf{k}} \times \mathbf{E}) \cdot \mathbf{E} \sin \vartheta \cos \vartheta + \\
&\quad + (\mathbf{k} \times \mathbf{E}) \cdot \mathbf{k}\dot{\mathbf{k}} \dot{\vartheta} \cos \vartheta (1 - \cos \vartheta) + (\dot{\mathbf{k}} \times \mathbf{E}) \cdot \mathbf{k}\mathbf{k} \sin \vartheta (1 - \cos \vartheta) = \\
&= \mathbf{k} \times \mathbf{E} \dot{\vartheta} \cos^2 \vartheta + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta \cos \vartheta + \dot{\mathbf{k}} \times \mathbf{k}\mathbf{k} \sin \vartheta (1 - \cos \vartheta),
\end{aligned}$$

$$\begin{aligned}
\dot{O}^S \cdot O^A &= \\
&= (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cdot (\mathbf{k} \times \mathbf{E}) \sin \vartheta + (\mathbf{k}\dot{\mathbf{k}} + \dot{\mathbf{k}}\mathbf{k})(1 - \cos \vartheta) \cdot (\mathbf{k} \times \mathbf{E}) \sin \vartheta = \\
&= \mathbf{k}\dot{\mathbf{k}} \cdot (\mathbf{k} \times \mathbf{E}) \dot{\vartheta} \sin^2 \vartheta - \mathbf{E} \cdot (\mathbf{k} \times \mathbf{E}) \dot{\vartheta} \sin^2 \vartheta + \left(\mathbf{k}\dot{\mathbf{k}} \cdot (\mathbf{k} \times \mathbf{E}) + \dot{\mathbf{k}}\mathbf{k} \cdot (\mathbf{k} \times \mathbf{E}) \right) \sin \vartheta (1 - \cos \vartheta) = \\
&= -\mathbf{k} \times \mathbf{E} \dot{\vartheta} \sin^2 \vartheta + \mathbf{k}\dot{\mathbf{k}} \times \mathbf{k} \sin \vartheta (1 - \cos \vartheta),
\end{aligned}$$

$$\begin{aligned}
\dot{O}^A \cdot O^A &= (\mathbf{k} \times \mathbf{E}) \dot{\vartheta} \cos \vartheta \cdot (\mathbf{k} \times \mathbf{E}) \sin \vartheta + (\dot{\mathbf{k}} \times \mathbf{E}) \cdot (\mathbf{k} \times \mathbf{E}) \sin^2 \vartheta = \\
&= (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta + \mathbf{k}\dot{\mathbf{k}} \sin^2 \vartheta;
\end{aligned}$$

$$\begin{aligned}
\dot{O} \cdot O^T &= \dot{O}^S \cdot O^S + \dot{O}^A \cdot O^S - \dot{O}^S \cdot O^A - \dot{O}^A \cdot O^A = \\
&= (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta + \mathbf{k}\dot{\mathbf{k}} \cos \vartheta - \mathbf{k}\dot{\mathbf{k}} \cos^2 \vartheta + \dot{\mathbf{k}}\mathbf{k} (1 - \cos \vartheta) + \\
&\quad + \mathbf{k} \times \mathbf{E} \dot{\vartheta} \cos^2 \vartheta + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta \cos \vartheta + \dot{\mathbf{k}} \times \mathbf{k}\mathbf{k} \sin \vartheta (1 - \cos \vartheta) + \\
&\quad + \mathbf{k} \times \mathbf{E} \dot{\vartheta} \sin^2 \vartheta - \mathbf{k}\dot{\mathbf{k}} \times \mathbf{k} \sin \vartheta (1 - \cos \vartheta) - (\mathbf{k}\mathbf{k} - \mathbf{E}) \dot{\vartheta} \sin \vartheta \cos \vartheta - \mathbf{k}\dot{\mathbf{k}} \sin^2 \vartheta = \\
&= \mathbf{k} \times \mathbf{E} \dot{\vartheta} + (\mathbf{k}\dot{\mathbf{k}} - \mathbf{k}\dot{\mathbf{k}})(1 - \cos \vartheta) + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta \cos \vartheta + (\dot{\mathbf{k}} \times \mathbf{k}\mathbf{k} - \mathbf{k}\dot{\mathbf{k}} \times \mathbf{k}) \sin \vartheta (1 - \cos \vartheta) = \\
&= \mathbf{k} \times \mathbf{E} \dot{\vartheta} + \mathbf{k} \times \dot{\mathbf{k}} \times \mathbf{E} (1 - \cos \vartheta) + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta \cos \vartheta + \mathbf{k} \times (\dot{\mathbf{k}} \times \mathbf{k}) \times \mathbf{E} \sin \vartheta (1 - \cos \vartheta) = \\
&= \mathbf{k} \times \mathbf{E} \dot{\vartheta} + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta \cos \vartheta + (\mathbf{k}\mathbf{k} \cdot \mathbf{k} - \mathbf{k}\dot{\mathbf{k}} \cdot \mathbf{k}) \times \mathbf{E} \sin \vartheta (1 - \cos \vartheta) + \mathbf{k} \times \dot{\mathbf{k}} \times \mathbf{E} (1 - \cos \vartheta) = \\
&= \mathbf{k} \times \mathbf{E} \dot{\vartheta} + \dot{\mathbf{k}} \times \mathbf{E} \sin \vartheta + \mathbf{k} \times \dot{\mathbf{k}} \times \mathbf{E} (1 - \cos \vartheta).
\end{aligned}$$

Этот результат, подставленный в определение (12.8) псевдовектора $\boldsymbol{\omega}$, даёт

$$\boldsymbol{\omega} = \mathbf{k}\dot{\vartheta} + \dot{\mathbf{k}} \sin \vartheta + \mathbf{k} \times \dot{\mathbf{k}} (1 - \cos \vartheta). \quad (12.9)$$

Вектор $\boldsymbol{\omega}$ получился разложенным по трём взаимно ортогональным направлениям — \mathbf{k} , $\dot{\mathbf{k}}$ и $\mathbf{k} \times \dot{\mathbf{k}}$. При неподвижной оси поворота $\dot{\mathbf{k}} = \mathbf{0} \Rightarrow \boldsymbol{\omega} = \mathbf{k}\dot{\vartheta}$.

Ещё одно представление $\boldsymbol{\omega}$ связано с компонентами тензора поворота (12.2). Поскольку $\mathbf{O} = o_{ji} \overset{\circ}{e}_i \overset{\circ}{e}_j$, $\mathbf{O}^\top = o_{ij} \overset{\circ}{e}_i \overset{\circ}{e}_j$, а векторы начального базиса $\overset{\circ}{e}_i$ неподвижны (со временем не меняются), то

$$\begin{aligned} \dot{\mathbf{O}} &= \dot{o}_{ji} \overset{\circ}{e}_i \overset{\circ}{e}_j, \quad \dot{\mathbf{O}} \cdot \mathbf{O}^\top = \dot{o}_{ni} o_{nj} \overset{\circ}{e}_i \overset{\circ}{e}_j, \\ \boldsymbol{\omega} &= -\frac{1}{2} \dot{o}_{ni} o_{nj} \overset{\circ}{e}_i \times \overset{\circ}{e}_j = \frac{1}{2} \epsilon_{jik} o_{nj} \dot{o}_{ni} \overset{\circ}{e}_k. \end{aligned} \quad (12.10)$$

Отметим и формулы

$$\begin{aligned} (12.8) &\Rightarrow \dot{\mathbf{e}}_i \overset{\circ}{e}_i = \boldsymbol{\omega} \times \mathbf{e}_i \overset{\circ}{e}_i \Rightarrow \dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i, \\ (12.8) &\Rightarrow \boldsymbol{\omega} = -\frac{1}{2} (\dot{\mathbf{e}}_i \overset{\circ}{e}_i \cdot \overset{\circ}{e}_j \mathbf{e}_j)_{\times} = -\frac{1}{2} (\dot{\mathbf{e}}_i \mathbf{e}_i)_{\times} = \frac{1}{2} \mathbf{e}_i \times \dot{\mathbf{e}}_i. \end{aligned} \quad (12.11)$$

Не всё то вектор, что имеет величину и направление. Поворот тела вокруг оси представляет, казалось бы, вектор: его численное значение равно углу поворота, а направление совпадает с направлением оси вращения. Однако, повороты не складываются как векторы*.

На самом же деле последовательные повороты не складываются, а умножаются.

Можно ли складывать угловые скорости? — Да, ведь угол поворота в $\dot{\vartheta}$ бесконечномалый. — Но только при вращении вокруг неподвижной оси?

...

Варьируя тождество (12.5), получим $\delta \mathbf{O} \cdot \mathbf{O}^\top = -\mathbf{O} \cdot \delta \mathbf{O}^\top$. Этот тензор антисимметричен, и потому выражается через свой сопут-

* Когда углы поворота не бесконечно-малые.

ствующий вектор $\delta \mathbf{o}$ как $\delta \mathbf{O} \cdot \mathbf{O}^\top = \delta \mathbf{o} \times \mathbf{E}$. Приходим к соотношениям

$$\delta \mathbf{O} = \delta \mathbf{o} \times \mathbf{O}, \quad \delta \mathbf{o} = -\frac{1}{2} \left(\delta \mathbf{O} \cdot \mathbf{O}^\top \right)_{\times}, \quad (12.12)$$

аналогичным (12.8). Вектор бесконечно малого поворота $\delta \mathbf{o}$ это не “вариация \mathbf{o} ”, но единый символ (в отличие от $\delta \mathbf{O}$).

Малый поворот определяется вектором $\delta \mathbf{o}$, но конечный поворот тоже возможно представить как вектор.

...

§ 13. Rotations via quaternions

The other way to describe a rotation in 3-dimensional space is using quaternions. They are very popular for computer graphics.

Quaternions were invented by William Rowan Hamilton in 1843*.

Lorem ipsum

$$i^2 = j^2 = k^2 = ijk = -1. \quad (13.1)$$

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik, \quad (13.2)$$

...

§ 14. Variations

Further in this book pretty often will be used the operation of varying. It is similar to the differentiation.

Variations are seen as infinitesimal displacements, compatible with the constraints. If there are no constraints for variable x , then

* *On Quaternions; or on a new System of Imaginaries in Algebra* by William Rowan Hamilton appeared in 18 publications in ‘The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science’, volumes xxv–xxxvi, 3rd Series, 1844–1850.

variations δx are completely random. But when

$$y \mapsto x : x = x(y)$$

is a function of some independent argument y , then

$$\delta x = x'(y)\delta y.$$

Variations are similar to differentials. As example, for δx and δy as variations of x and y , with finite u and v , $u\delta x + v\delta y = \delta w$ is written even when δw is not a variation of w . In such a case, δw is a single symbol.

Surely if $u = u(x, y)$, $v = v(x, y)$ and $\partial_x v = \partial_y u$ ($\frac{\partial}{\partial x} v = \frac{\partial}{\partial y} u$), then the sum $\delta w = u\delta x + v\delta y$ will be a variation of some w .

Varying the identity (12.5), we get

$$\delta \mathbf{O} \cdot \mathbf{O}^\top = -\mathbf{O} \cdot \delta \mathbf{O}^\top.$$

This tensor is antisymmetric, and thus is representable via its companion pseudovector $\delta \mathbf{o}$ as

$$\delta \mathbf{O} \cdot \mathbf{O}^\top = \delta \mathbf{o} \times \mathbf{E}.$$

We have the following relations

$$\delta \mathbf{O} = \delta \mathbf{o} \times \mathbf{O}, \quad \delta \mathbf{o} = -\frac{1}{2} \left(\delta \mathbf{O} \cdot \mathbf{O}^\top \right)_\times, \quad (14.1)$$

similar to (12.8). Vector $\delta \mathbf{o}$ of an infinitesimal rotation is not “a variation of \mathbf{o} ”, but a single symbol.

An infinitesimal rotation is defined by vector $\delta \mathbf{o}$, but a finite rotation is also possible to represent as a vector

...

§ 15. Polar decomposition

Any tensor of the second complexity \mathbf{F} with $\det F_{ij} \neq 0$, that is a not singular tensor, can be decomposed as

...

Example. Polar decompose tensor $\mathbf{C} = C_{ij}\mathbf{e}_i\mathbf{e}_j$, where \mathbf{e}_k are mutually perpendicular unit vectors of basis, and C_{ij} are tensor's components

$$C_{ij} = \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix}$$

$$\mathbf{O} = O_{ij}\mathbf{e}_i\mathbf{e}_j = \mathbf{O}_1 \cdot \mathbf{O}_2$$

$$O_{ij} = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$\mathbf{C} = \mathbf{O} \cdot \mathbf{S}_R, \quad \mathbf{O}^\top \cdot \mathbf{C} = \mathbf{S}_R$$

$$\mathbf{C} = \mathbf{S}_L \cdot \mathbf{O}, \quad \mathbf{C} \cdot \mathbf{O}^\top = \mathbf{S}_L$$

$$S_{Rij} = \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix}$$

$$S_{Lij} = \begin{bmatrix} 104/5 & 47/5 & 5 \\ 47/5 & -129/5 & -10 \\ 5 & -10 & 3 \end{bmatrix}$$

...

§16. In the oblique basis

Until now a basis of the three mutually perpendicular unit vectors \mathbf{e}_i was used. Now we will take a basis of any three linearly independent (non-coplanar) vectors \mathbf{a}_i .

The decomposition of vector \mathbf{v} in the basis \mathbf{a}_i (figure 6) is the linear combination

$$\mathbf{v} = v^i \mathbf{a}_i. \quad (16.1)$$

The summation convention gains the new conditions: a summation index is repeated at different levels of the same monomial, and a free index stays at the same height in any part of the expression ($a_i = b_{ij}c^j$ is correct, $a_i = b_{kk}^i$ is wrong twice).

В таком базисе уже $\mathbf{v} \cdot \mathbf{a}_i = v^k \mathbf{a}_k \cdot \mathbf{a}_i \neq v^i$, ведь тут $\mathbf{a}_i \cdot \mathbf{a}_k \neq \delta_{ik}$.

Дополним же базис \mathbf{a}_i ещё другой тройкой векторов \mathbf{a}^i , называемых кобазисом или взаимным базисом, чтобы

$$\begin{aligned} \mathbf{a}_i \cdot \mathbf{a}^j &= \delta_i^j, \quad \mathbf{a}^i \cdot \mathbf{a}_j = \delta_j^i, \\ \mathbf{E} &= \mathbf{a}^i \mathbf{a}_i = \mathbf{a}_i \mathbf{a}^i. \end{aligned} \quad (16.2)$$

Это — основное свойство кобазиса. Ортонормированный (ортонормальный) базис может быть определён как совпадающий со своим кобазисом: $\mathbf{e}^i = \mathbf{e}_i$.

For, as example, the first cobasis vector \mathbf{a}^1

$$\begin{cases} \mathbf{a}^1 \cdot \mathbf{a}_1 = 1 \\ \mathbf{a}^1 \cdot \mathbf{a}_2 = 0 \\ \mathbf{a}^1 \cdot \mathbf{a}_3 = 0 \end{cases} \Rightarrow \begin{cases} \mathbf{a}^1 \cdot \mathbf{a}_1 = 1 \\ \gamma \mathbf{a}^1 = \mathbf{a}_2 \times \mathbf{a}_3 \end{cases} \Rightarrow \begin{cases} \mathbf{a}^1 = 1/\gamma \mathbf{a}_2 \times \mathbf{a}_3 \\ \gamma = \mathbf{a}_2 \times \mathbf{a}_3 \cdot \mathbf{a}_1 \end{cases}$$

Коэффициент γ получился равным (с точностью до знака для “левой” тройки \mathbf{a}_i) объёму параллелепипеда, построенного на век-

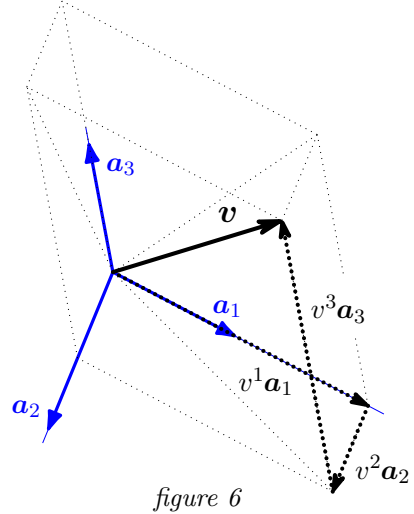


figure 6

topax \mathbf{a}_i . In §7 the same volume was presented as \sqrt{g} , and this is not without reason, because it coincides with the square root of gramian $g \equiv \det g_{ij}$ — determinant of the symmetric J. P. Gram matrix $g_{ij} \equiv \mathbf{a}_i \cdot \mathbf{a}_j$.

○ The proof resembles the derivation of (7.10). The “triple product” $\mathbf{a}_i \times \mathbf{a}_j \cdot \mathbf{a}_k$ in some orthonormal basis \mathbf{e}_i вычисли́мо как детерминант (с “—” для “левой” тройки \mathbf{a}_i) по строкам

$$\epsilon_{ijk} \equiv \mathbf{a}_i \times \mathbf{a}_j \cdot \mathbf{a}_k = \pm \det \begin{bmatrix} \mathbf{a}_i \cdot \mathbf{e}_1 & \mathbf{a}_i \cdot \mathbf{e}_2 & \mathbf{a}_i \cdot \mathbf{e}_3 \\ \mathbf{a}_j \cdot \mathbf{e}_1 & \mathbf{a}_j \cdot \mathbf{e}_2 & \mathbf{a}_j \cdot \mathbf{e}_3 \\ \mathbf{a}_k \cdot \mathbf{e}_1 & \mathbf{a}_k \cdot \mathbf{e}_2 & \mathbf{a}_k \cdot \mathbf{e}_3 \end{bmatrix}$$

или по столбцам

$$\epsilon_{pqr} \equiv \mathbf{a}_p \times \mathbf{a}_q \cdot \mathbf{a}_r = \pm \det \begin{bmatrix} \mathbf{a}_p \cdot \mathbf{e}_1 & \mathbf{a}_q \cdot \mathbf{e}_1 & \mathbf{a}_r \cdot \mathbf{e}_1 \\ \mathbf{a}_p \cdot \mathbf{e}_2 & \mathbf{a}_q \cdot \mathbf{e}_2 & \mathbf{a}_r \cdot \mathbf{e}_2 \\ \mathbf{a}_p \cdot \mathbf{e}_3 & \mathbf{a}_q \cdot \mathbf{e}_3 & \mathbf{a}_r \cdot \mathbf{e}_3 \end{bmatrix}.$$

Произведение определителей $\epsilon_{ijk}\epsilon_{pqr}$ равно определителю произведения матриц, and elements of the latter are sums like $\mathbf{a}_i \cdot \mathbf{e}_s \mathbf{a}_p \cdot \mathbf{e}_s = \mathbf{a}_i \cdot \mathbf{e}_s \mathbf{e}_s \cdot \mathbf{a}_p = \mathbf{a}_i \cdot \mathbf{E} \cdot \mathbf{a}_p = \mathbf{a}_i \cdot \mathbf{a}_p$, в результате

$$\epsilon_{ijk}\epsilon_{pqr} = \det \begin{bmatrix} \mathbf{a}_i \cdot \mathbf{a}_p & \mathbf{a}_i \cdot \mathbf{a}_q & \mathbf{a}_i \cdot \mathbf{a}_r \\ \mathbf{a}_j \cdot \mathbf{a}_p & \mathbf{a}_j \cdot \mathbf{a}_q & \mathbf{a}_j \cdot \mathbf{a}_r \\ \mathbf{a}_k \cdot \mathbf{a}_p & \mathbf{a}_k \cdot \mathbf{a}_q & \mathbf{a}_k \cdot \mathbf{a}_r \end{bmatrix};$$

$$i=p=1, j=q=2, k=r=3 \Rightarrow \epsilon_{123}\epsilon_{123} = \det_{i,j}(\mathbf{a}_i \cdot \mathbf{a}_j) = \det_{i,j} g_{ij}. \quad \bullet$$

Representing \mathbf{a}^1 and other cobasis vectors as the sum

$$\pm 2\sqrt{g}\mathbf{a}^1 = \mathbf{a}_2 \times \mathbf{a}_3 \overbrace{- \mathbf{a}_3 \times \mathbf{a}_2}^{+ \mathbf{a}_2 \times \mathbf{a}_3},$$

приходим к общей формуле (с “—” для “левой” тройки \mathbf{a}_i)

$$\mathbf{a}^i = \pm \frac{1}{2\sqrt{g}} e^{ijk} \mathbf{a}_j \times \mathbf{a}_k, \quad \sqrt{g} \equiv \pm \mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 > 0. \quad (16.3)$$

Здесь e^{ijk} по-прежнему символ перестановки Veblen'a (± 1 или 0): $e^{ijk} \equiv e_{ijk}$. Произведение $\mathbf{a}_j \times \mathbf{a}_k = \epsilon_{jkn} \mathbf{a}^n$, компоненты тензора Лёви-Чивиты $\epsilon_{jkn} = \pm e_{jkn} \sqrt{g}$, and by (7.11) $e^{ijk} e_{jkn} = 2\delta_n^i$. Thus

$$\mathbf{a}^1 = \pm 1/\sqrt{g} (\mathbf{a}_2 \times \mathbf{a}_3), \quad \mathbf{a}^2 = \pm 1/\sqrt{g} (\mathbf{a}_3 \times \mathbf{a}_1), \quad \mathbf{a}^3 = \pm 1/\sqrt{g} (\mathbf{a}_1 \times \mathbf{a}_2).$$

Example. Get cobasis for basis \mathbf{a}_i when

$$\mathbf{a}_1 = \mathbf{e}_1 + \mathbf{e}_2,$$

$$\mathbf{a}_2 = \mathbf{e}_1 + \mathbf{e}_3,$$

$$\mathbf{a}_3 = \mathbf{e}_2 + \mathbf{e}_3.$$

$$\sqrt{g} = -\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 = -\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 2;$$

$$-\mathbf{a}_2 \times \mathbf{a}_3 = \det \begin{bmatrix} 1 & \mathbf{e}_1 & 0 \\ 0 & \mathbf{e}_2 & 1 \\ 1 & \mathbf{e}_3 & 1 \end{bmatrix} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3,$$

$$-\mathbf{a}_3 \times \mathbf{a}_1 = \det \begin{bmatrix} 0 & \mathbf{e}_1 & 1 \\ 1 & \mathbf{e}_2 & 1 \\ 1 & \mathbf{e}_3 & 0 \end{bmatrix} = \mathbf{e}_1 + \mathbf{e}_3 - \mathbf{e}_2,$$

$$-\mathbf{a}_1 \times \mathbf{a}_2 = \det \begin{bmatrix} 1 & \mathbf{e}_1 & 1 \\ 1 & \mathbf{e}_2 & 0 \\ 0 & \mathbf{e}_3 & 1 \end{bmatrix} = \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_1$$

and finally

$$\mathbf{a}^1 = \frac{1}{2} (\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3),$$

$$\mathbf{a}^2 = \frac{1}{2} (\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3),$$

$$\mathbf{a}^3 = \frac{1}{2} (-\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3).$$

Имея кобазис, возможно не только разложить по нему любой вектор (рисунок 7), но и найти коэффициенты разложения (16.1):

$$\begin{aligned} \mathbf{v} &= v^i \mathbf{a}_i = v_i \mathbf{a}^i, \\ \mathbf{v} \cdot \mathbf{a}^i &= v^k \mathbf{a}_k \cdot \mathbf{a}^i = v^i, \quad v_i = \mathbf{v} \cdot \mathbf{a}_i. \end{aligned} \tag{16.4}$$

Коэффициенты v_i называются ковариантными компонентами вектора \mathbf{v} , а v^i — его контравариантными* компонентами.

Есть литература о тензорах, где introducing existænce and разли-чают ковариантные и контравариантные... векторы (and “covectors”, “dual vectors”). Не сто́ит вводить читателя в заблуждение: вектор-то один и тот же, просто разложение по двум разным базисам даёт два набора компонент.

От векторов перейдём к тензорам второй сложности. Имеем четыре комплекта диад: $\mathbf{a}_i \mathbf{a}_j$, $\mathbf{a}^i \mathbf{a}^j$, $\mathbf{a}_i \mathbf{a}^j$, $\mathbf{a}^i \mathbf{a}_j$. Сопасаующиеся коэффициенты в декомпозиции тензора называются его контравариантными, ковариантными и смешанными компонентами:

$$\begin{aligned} {}^2\mathbf{B} &= B^{ij} \mathbf{a}_i \mathbf{a}_j = B_{ij} \mathbf{a}^i \mathbf{a}^j = B_{\dot{j}}^i \mathbf{a}_i \mathbf{a}^j = B_i^{\dot{j}} \mathbf{a}^i \mathbf{a}_j, \\ B^{ij} &= \mathbf{a}^i \cdot {}^2\mathbf{B} \cdot \mathbf{a}^j, \quad B_{ij} = \mathbf{a}_i \cdot {}^2\mathbf{B} \cdot \mathbf{a}_j, \\ B_{\dot{j}}^i &= \mathbf{a}^i \cdot {}^2\mathbf{B} \cdot \mathbf{a}_j, \quad B_i^{\dot{j}} = \mathbf{a}_i \cdot {}^2\mathbf{B} \cdot \mathbf{a}^j. \end{aligned} \quad (16.5)$$

Для двух видов смешанных компонент точка в индексе это просто свободное место: у $B_{\dot{j}}^i$ верхний индекс “ i ” — первый, а нижний “ j ” — второй.

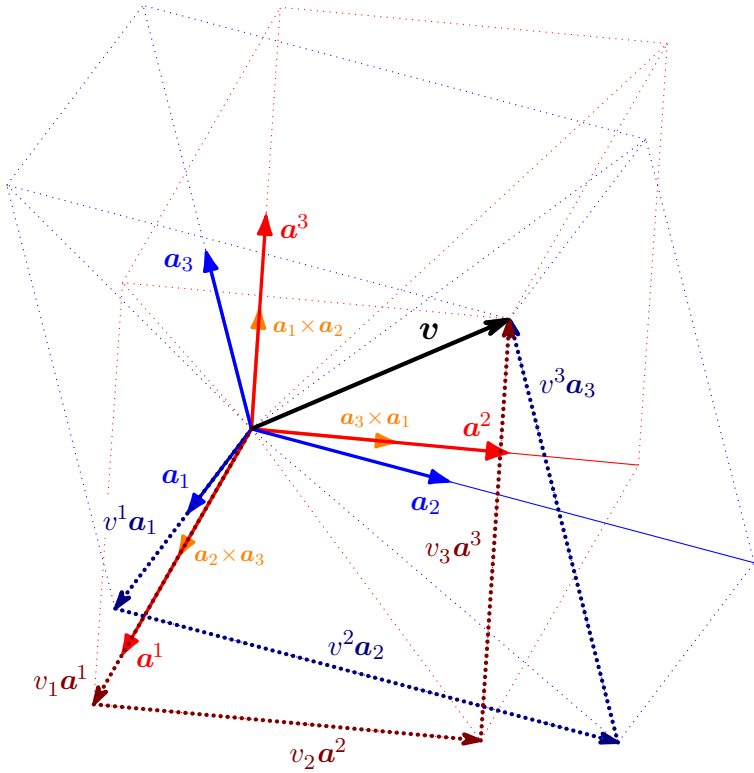
Компоненты единичного (“метрического”) тензора \mathbf{E}

$$\begin{aligned} \mathbf{E} &= \mathbf{a}^k \mathbf{a}_k = \mathbf{a}_k \mathbf{a}^k = g_{jk} \mathbf{a}^j \mathbf{a}^k = g^{jk} \mathbf{a}_j \mathbf{a}_k: \\ \mathbf{a}_i \cdot \mathbf{E} \cdot \mathbf{a}^j &= \mathbf{a}_i \cdot \mathbf{a}^j = \delta_i^j, \quad \mathbf{a}^i \cdot \mathbf{E} \cdot \mathbf{a}_j = \mathbf{a}^i \cdot \mathbf{a}_j = \delta_j^i, \\ \mathbf{a}_i \cdot \mathbf{E} \cdot \mathbf{a}_j &= \mathbf{a}_i \cdot \mathbf{a}_j \equiv g_{ij}, \quad \mathbf{a}^i \cdot \mathbf{E} \cdot \mathbf{a}^j = \mathbf{a}^i \cdot \mathbf{a}^j \equiv g^{ij}; \\ \mathbf{E} \cdot \mathbf{E} &= g_{ij} \mathbf{a}^i \mathbf{a}^j \cdot g^{nk} \mathbf{a}_n \mathbf{a}_k = g_{ij} g^{jk} \mathbf{a}^i \mathbf{a}_k = \mathbf{E} \Rightarrow g_{ij} g^{jk} = \delta_i^k. \end{aligned} \quad (16.6)$$

Вдобавок к (16.2) и (16.3) открылся ещё один способ найти векторы кобазиса — через матрицу g^{ij} , обратную матрице Грама g_{ij} . И наоборот:

$$\begin{aligned} \mathbf{a}^i &= \mathbf{E} \cdot \mathbf{a}^i = g^{jk} \mathbf{a}_j \mathbf{a}_k \cdot \mathbf{a}^i = g^{jk} \mathbf{a}_j \delta_k^i = g^{ji} \mathbf{a}_j, \\ \mathbf{a}_i &= \mathbf{E} \cdot \mathbf{a}_i = g_{jk} \mathbf{a}^j \mathbf{a}^k \cdot \mathbf{a}_i = g_{jk} \mathbf{a}^j \delta_i^k = g_{ji} \mathbf{a}^j. \end{aligned} \quad (16.7)$$

* Потому что они меняются обратно (contra) изменению длин базисных векторов \mathbf{a}_i .



$$\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 = \sqrt{g} = 0.56274$$

$$1/\sqrt{g} = 1.77703$$

$$\mathbf{a}_i \cdot \mathbf{a}^j = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}^1 & \mathbf{a}_1 \cdot \mathbf{a}^2 & \mathbf{a}_1 \cdot \mathbf{a}^3 \\ \mathbf{a}_2 \cdot \mathbf{a}^1 & \mathbf{a}_2 \cdot \mathbf{a}^2 & \mathbf{a}_2 \cdot \mathbf{a}^3 \\ \mathbf{a}_3 \cdot \mathbf{a}^1 & \mathbf{a}_3 \cdot \mathbf{a}^2 & \mathbf{a}_3 \cdot \mathbf{a}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_i^j$$

рисунок 7
 “Decomposition of vector in oblique basis”

Example. Using reversed Gram matrix, get cobasis for basis \mathbf{a}_i when

$$\mathbf{a}_1 = \mathbf{e}_1 + \mathbf{e}_2,$$

$$\mathbf{a}_2 = \mathbf{e}_1 + \mathbf{e}_3,$$

$$\mathbf{a}_3 = \mathbf{e}_2 + \mathbf{e}_3.$$

$$g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \det g_{ij} = 4,$$

$$\text{adj } g_{ij} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}^T,$$

$$g^{ij} = g_{ij}^{-1} = \frac{\text{adj } g_{ij}}{\det g_{ij}} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

Using $\mathbf{a}^i = g^{ij} \mathbf{a}_j$

$$\mathbf{a}^1 = g^{11} \mathbf{a}_1 + g^{12} \mathbf{a}_2 + g^{13} \mathbf{a}_3 = \frac{1}{2} \mathbf{e}_1 + \frac{1}{2} \mathbf{e}_2 - \frac{1}{2} \mathbf{e}_3,$$

$$\mathbf{a}^2 = g^{21} \mathbf{a}_1 + g^{22} \mathbf{a}_2 + g^{23} \mathbf{a}_3 = \frac{1}{2} \mathbf{e}_1 - \frac{1}{2} \mathbf{e}_2 + \frac{1}{2} \mathbf{e}_3,$$

$$\mathbf{a}^3 = g^{31} \mathbf{a}_1 + g^{32} \mathbf{a}_2 + g^{33} \mathbf{a}_3 = -\frac{1}{2} \mathbf{e}_1 + \frac{1}{2} \mathbf{e}_2 + \frac{1}{2} \mathbf{e}_3.$$

...

Едини́чный тензор (unit tensor, identity tensor, metric tensor)

$$\mathbf{E} \cdot \boldsymbol{\xi} = \boldsymbol{\xi} \cdot \mathbf{E} = \boldsymbol{\xi} \quad \forall \boldsymbol{\xi}$$

$$\mathbf{E} \cdot \mathbf{a} \mathbf{b} = \mathbf{a} \mathbf{b} \cdot \mathbf{E} = \mathbf{a} \cdot \mathbf{E} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$$

$$\mathbf{E} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{E} = \text{trace } \mathbf{A}$$

$$\mathbf{E} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{E} = \text{trace } \mathbf{A} \neq \text{not anymore } A_{jj}$$

Thus for, say, trace of some tensor $\mathbf{A} = A_{ij} \mathbf{r}^i \mathbf{r}^j$: $\mathbf{A} \cdot \mathbf{E} = \text{trace } \mathbf{A}$, you have

$$\mathbf{A} \cdot \mathbf{E} = A_{ij} \mathbf{r}^i \mathbf{r}^j \cdot \mathbf{r}_{\partial k} \mathbf{r}^k = A_{ij} \mathbf{r}^i \cdot \mathbf{r}^j = A_{ij} g^{ij}$$

...

Тензор поворо́та (the rotation tensor)

$$\mathbf{P} = \mathbf{a}_i \hat{\mathbf{a}}^i = \hat{\mathbf{a}}^i \mathbf{a}_i = \mathbf{P}^{-\top}$$

$$\mathbf{P}^{-1} = \hat{\mathbf{a}}_i \mathbf{a}^i = \hat{\mathbf{a}}^i \mathbf{a}_i = \mathbf{P}^{\top}$$

$$\mathbf{P}^{\top} = \hat{\mathbf{a}}^i \mathbf{a}_i = \hat{\mathbf{a}}_i \mathbf{a}^i = \mathbf{P}^{-1}$$

...

... Характеристическое уравнение (10.2) быстро приводит к тождеству Кэли–Гамильтона (Cayley–Hamilton)

$$\begin{aligned} -\mathbf{B} \cdot \mathbf{B} \cdot \mathbf{B} + \mathbf{I} \mathbf{B} \cdot \mathbf{B} - \mathbf{II} \mathbf{B} + \mathbf{III} \mathbf{E} &= {}^2\mathbf{0}, \\ -\mathbf{B}^3 + \mathbf{I} \mathbf{B}^2 - \mathbf{II} \mathbf{B} + \mathbf{III} \mathbf{E} &= {}^2\mathbf{0}. \end{aligned} \quad (16.8)$$

§ 17. Tensor functions

In the concept of function $y=f(x)$ as of mapping (morphism) $f: x \mapsto y$, an input (argument) x and an output (result) y may be tensors of any complexities.

Consider at least a scalar function of a bivalent tensor $\varphi=\varphi(\mathbf{B})$. Examples are $\mathbf{B} \cdot \cdot \Phi$ (or $\mathbf{p} \cdot \mathbf{B} \cdot \mathbf{q}$) and $\mathbf{B} \cdot \cdot \mathbf{B}$. Then in each basis \mathbf{a}_i paired with cobasis \mathbf{a}^i we have function $\varphi(B_{ij})$ of nine numeric arguments — components B_{ij} of tensor \mathbf{B} . For example

$$\varphi(\mathbf{B}) = \mathbf{B} \cdot \cdot \Phi = B_{ij} \mathbf{a}^i \mathbf{a}^j \cdot \cdot \mathbf{a}_m \mathbf{a}_n \Phi^{mn} = B_{ij} \Phi^{ji} = \varphi(B_{ij}).$$

With any transition to a new basis, the result doesn't change: $\varphi(B_{ij}) = \varphi(B'_{ij}) = \varphi(\mathbf{B})$.

Differentiation of $\varphi(\mathbf{B})$ looks like

$$d\varphi = \frac{\partial \varphi}{\partial B_{ij}} dB_{ij} = \frac{\partial \varphi}{\partial \mathbf{B}} \cdot \cdot d\mathbf{B}^\top. \quad (17.1)$$

Tensor $\partial\varphi/\partial\mathbf{B}$ is called the derivative of function φ by argument \mathbf{B} ; $d\mathbf{B}$ is the differential of tensor \mathbf{B} , $d\mathbf{B} = dB_{ij} \mathbf{a}^i \mathbf{a}^j$; $\partial\varphi/\partial B_{ij}$ are components (contravariant ones) of $\partial\varphi/\partial\mathbf{B}$

$$\mathbf{a}^i \cdot \frac{\partial \varphi}{\partial \mathbf{B}} \cdot \mathbf{a}^j = \frac{\partial \varphi}{\partial \mathbf{B}} \cdot \cdot \mathbf{a}^j \mathbf{a}^i = \frac{\partial \varphi}{\partial B_{ij}} \Leftrightarrow \frac{\partial \varphi}{\partial \mathbf{B}} = \frac{\partial \varphi}{\partial B_{ij}} \mathbf{a}_i \mathbf{a}_j.$$

...

$$\begin{aligned} \varphi(\mathbf{B}) &= \mathbf{B} \cdot \cdot \Phi \\ d\varphi &= d(\mathbf{B} \cdot \cdot \Phi) = d\mathbf{B} \cdot \cdot \Phi = \Phi \cdot \cdot d\mathbf{B} = \Phi^\top \cdot \cdot d\mathbf{B}^\top \\ d\varphi &= \frac{\partial \varphi}{\partial \mathbf{B}} \cdot \cdot d\mathbf{B}^\top, \quad \frac{\partial(\mathbf{B} \cdot \cdot \Phi)}{\partial \mathbf{B}} = \Phi^\top \end{aligned}$$

$$p \bullet B \bullet q = B \bullet \bullet qp$$

$$\frac{\partial (p \bullet B \bullet q)}{\partial B} = pq$$

...

$$\begin{aligned}\varphi(B) &= B \bullet \bullet B \\ d\varphi &= d(B \bullet \bullet B) = d\ldots\end{aligned}$$

...

Но согласно опять-таки (16.8) $-B^2 + \text{I}B - \text{II}E + \text{III}B^{-1} = {}^2\mathbf{0}$, поэтому

....

Скалярная функция $\varphi(B)$ называется изотропной, если она не чувствительна к повороту аргумента:

$$\varphi(B) = \varphi(O \bullet \overset{\circ}{B} \bullet O^{\top}) = \varphi(\overset{\circ}{B}) \quad \forall O = a_i \overset{\circ}{a}^i = a^i \overset{\circ}{a}_i = O^{-\top}$$

для любого ортогонального тензора O (тензора поворота, § 12).

Симметричный тензор B^S полностью определяется тройкой инвариантов и угловой ориентацией собственных осей (они же взаимно ортогональны, § 10). Ясно, что изотропная функция $\varphi(B^S)$ симметричного аргумента является функцией, входные-аргументы которой — только инварианты $\text{I}(B^S)$, $\text{II}(B^S)$, $\text{III}(B^S)$. Дифференцируется такая функция согласно (??), где транспонирование излишне.

§ 18. Spatial differentiation

««« rename: remove fields

Tensor field is a tensor varying from point to point (variable in space, coordinate dependent).

Пусть at each point of some region of a three-dimensional space определена величина ς . Тогда говорят, что есть тензорное поле $\varsigma = \varsigma(\mathbf{r})$, where \mathbf{r} is location vector (radius vector) of a point in space.

Величина ς может быть тензором любой сложности. Пример скалярного поля — поле температуры в среде, векторного поля — скорости частиц жидкости.

Концепт тензорного поля никак не связан с концептом поля с операциями $+$ и $*$ с 11 свойствами этих операций.

Не только для решения прикладных задач, но нередко и в “чистой теории” вместо аргумента \mathbf{r} используется набор (какая-либо тройка) криволинейных координат q^i . Если непрерывно менять лишь одну координату из трёх, получается координатная линия. Каждая точка трёхмерного пространства лежит на пересечении трёх координатных линий (рисунок 8). Вектор положения точки выражается через набор координат as relation $\mathbf{r} = \mathbf{r}(q^i)$.

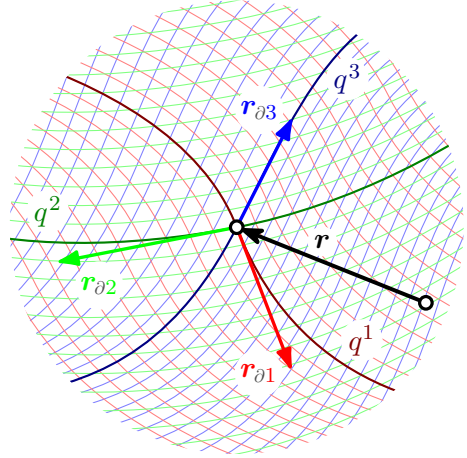


figure 8

Commonly used sets of coordinates «Rectangular (“cartesian”), spherical and cylindrical coordinates are

Curvilinear coordinates may be derived from a set of rectangular (“cartesian”) coordinates by using a transformation that is locally invertible (a one-to-one map) at each point. Therefore rectangular coordinates of any point of space can be converted to some curvilinear coordinates and vice versa.

...

The differential of a function presents a change in the linearization of this function.

...

partial derivative

$$\partial_i \equiv \frac{\partial}{\partial q^i}$$

...

differential of $\varsigma(q^i)$

$$d\varsigma = \frac{\partial \varsigma}{\partial q^i} dq^i = \partial_i \varsigma dq^i \quad (18.1)$$

...

Linearity

$$\partial_i(\lambda p + \mu q) = \lambda(\partial_i p) + \mu(\partial_i q) \quad (18.2)$$

“Product rule”

$$\partial_i(p \circ q) = (\partial_i p) \circ q + p \circ (\partial_i q) \quad (18.3)$$

...

Local basis $\mathbf{r}_{\partial i}$

The differential of location vector $\mathbf{r}(q^i)$ is

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial q^i} dq^i = dq^i \mathbf{r}_{\partial i}, \quad \mathbf{r}_{\partial i} \equiv \frac{\partial \mathbf{r}}{\partial q^i} \equiv \partial_i \mathbf{r} \quad (18.4)$$

...

Local cobasis \mathbf{r}^i , $\mathbf{r}^i \bullet \mathbf{r}_{\partial j} = \delta_j^i$

...

$$\frac{\partial \varsigma}{\partial \mathbf{r}} = \frac{\partial \varsigma}{\partial q^i} \mathbf{r}^i = \partial_i \varsigma \mathbf{r}^i$$

$$d\varsigma = \frac{\partial \varsigma}{\partial \mathbf{r}} \bullet d\mathbf{r} = \partial_i \varsigma \mathbf{r}^i \bullet dq^j \mathbf{r}_{\partial j} = \partial_i \varsigma dq^i \quad (18.5)$$

...

The unit dyad (metric tensor) \mathbf{E} , neutral (4.7) for the “ \bullet ”-product, can be represented as

$$\mathbf{E} = \mathbf{r}^i \mathbf{r}_{\partial i} = \mathbf{r}^i \partial_i \mathbf{r} = \nabla \mathbf{r}, \quad (18.6)$$

where appears the differential “nabla” operator

$$\nabla \equiv \mathbf{r}^i \partial_i. \quad (18.7)$$

...

$$d\zeta = \frac{\partial \zeta}{\partial \mathbf{r}} \cdot d\mathbf{r} = d\mathbf{r} \cdot \nabla \zeta = \partial_i \zeta dq^i \quad (18.8)$$

$$d\mathbf{r} = d\mathbf{r} \cdot \overbrace{\nabla \mathbf{r}}^{\mathbf{E}}$$

...

Divergence of the dyadic product of two vectors

$$\begin{aligned} \nabla \cdot (\mathbf{a}\mathbf{b}) &= \mathbf{r}^i \partial_i \cdot (\mathbf{a}\mathbf{b}) = \mathbf{r}^i \cdot \partial_i (\mathbf{a}\mathbf{b}) = \mathbf{r}^i \cdot (\partial_i \mathbf{a})\mathbf{b} + \mathbf{r}^i \cdot \mathbf{a}(\partial_i \mathbf{b}) = \\ &= (\mathbf{r}^i \cdot \partial_i \mathbf{a})\mathbf{b} + \mathbf{a} \cdot \mathbf{r}^i (\partial_i \mathbf{b}) = (\mathbf{r}^i \partial_i \cdot \mathbf{a})\mathbf{b} + \mathbf{a} \cdot (\mathbf{r}^i \partial_i \mathbf{b}) = \\ &= (\nabla \cdot \mathbf{a})\mathbf{b} + \mathbf{a} \cdot (\nabla \mathbf{b}) \end{aligned} \quad (18.9)$$

— here’s no need to expand vectors \mathbf{a} and \mathbf{b} , expanding just differential operator ∇ .

...

Gradient of cross product of two vectors, applying “product rule” (18.3) and relation (7.6) for any two vectors (partial derivative ∂_i of some vector by scalar coordinate q^i is a vector too)

$$\begin{aligned} \nabla (\mathbf{a} \times \mathbf{b}) &= \mathbf{r}^i \partial_i (\mathbf{a} \times \mathbf{b}) = \mathbf{r}^i (\partial_i \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \partial_i \mathbf{b}) = \\ &= \mathbf{r}^i (\partial_i \mathbf{a} \times \mathbf{b} - \partial_i \mathbf{b} \times \mathbf{a}) = \mathbf{r}^i \partial_i \mathbf{a} \times \mathbf{b} - \mathbf{r}^i \partial_i \mathbf{b} \times \mathbf{a} = \\ &= \nabla \mathbf{a} \times \mathbf{b} - \nabla \mathbf{b} \times \mathbf{a}. \end{aligned} \quad (18.10)$$

...

Gradient of dot product of two vectors

$$\begin{aligned} \nabla (\mathbf{a} \cdot \mathbf{b}) &= \mathbf{r}^i \partial_i (\mathbf{a} \cdot \mathbf{b}) = \mathbf{r}^i (\partial_i \mathbf{a}) \cdot \mathbf{b} + \mathbf{r}^i \mathbf{a} \cdot (\partial_i \mathbf{b}) = \\ &= (\mathbf{r}^i \partial_i \mathbf{a}) \cdot \mathbf{b} + \mathbf{r}^i (\partial_i \mathbf{b}) \cdot \mathbf{a} = (\nabla \mathbf{a}) \cdot \mathbf{b} + (\nabla \mathbf{b}) \cdot \mathbf{a}. \end{aligned} \quad (18.11)$$

⏟
∇

§ 19. The integral theorems

Для векторных полей известны интегральные теоремы Gauss'a и Stokes'a.

Gauss' theorem (divergence theorem) enables an integral taken over a volume to be replaced by one taken over the closed surface bounding that volume, and vice versa.

Stokes' theorem enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve. Stokes' theorem relates a line integral around a closed path to a surface integral over what is called a *capping surface* of the path.

Теорема Gauss'a о дивергенции

This theorem is about how to replace a volume integral with a surface one (and vice versa). В этой теореме рассматривается поток (ef)flux вектора через ограничивающую объём V замкнутую поверхность $\mathcal{O}(\partial V)$. Единичный вектор внешней нормали \mathbf{n} к поверхности $\mathcal{O}(\partial V)$

$$\oint_{\mathcal{O}(\partial V)} \mathbf{n} \cdot \mathbf{a} d\mathcal{O} = \int_V \nabla \cdot \mathbf{a} dV. \quad (19.1)$$

Объём V нарезается тремя семействами координатных поверхностей на множество бесконечно малых элементов. Поток через поверхность $\mathcal{O}(\partial V)$ равен сумме потоков через края получившихся элементов. В бесконечной малости каждый такой элемент — маленький локальный дифференциальный кубик (параллелепипед). ... Поток вектора \mathbf{a} через грани малого кубика с объёмом dV равен $\sum_{i=1}^6 \mathbf{n}_i \cdot \mathbf{a} \mathcal{O}_i$, а поток через сам этот объём равен $\nabla \cdot \mathbf{a} dV$.

Похожая трактовка этой теоремы есть, для примера, в курсе Richard'a Feynman'a [86].

(рисунок с кубиками)

to dice — нарезать кубиками

small cube, little cube

локально ортонормальные координаты $\boldsymbol{\xi} = \xi_i \mathbf{n}_i$, $d\boldsymbol{\xi} = d\xi_i \mathbf{n}_i$,
 $\nabla = \mathbf{n}_i \partial_i$

разложение вектора $\mathbf{a} = a_i \mathbf{n}_i$

Теорема Stokes'a о циркуляции

Эта теорема выражается равенством

...

§ 20. Curvature tensors

The *Riemann curvature tensor* or *Riemann-Christoffel tensor* (after **Bernhard Riemann** and **Elwin Bruno Christoffel**) is the most common method used to express the curvature of Riemannian manifolds. It's a tensor field, it assigns a tensor to each point of a Riemannian manifold, that measures the extent to which the metric tensor is not locally isometric to that of “flat” space. The curvature tensor measures noncommutativity of the covariant derivative, and as such is the integrability obstruction for the existence of an isometry with “flat” space.

Рассматривая тензорные поля в криволинейных координатах (§ 18), мы исходили из представления вектора-радиуса (вектора положения) точки функцией этих координат: $\mathbf{r} = \mathbf{r}(q^i)$. Этим отношением порождаются выражения

- ✓ векторов локального касательного базиса $\mathbf{r}_{\partial i} \equiv \partial \mathbf{r} / \partial q^i \equiv \partial_i \mathbf{r}$,
- ✓ компонент $g_{ij} \equiv \mathbf{r}_{\partial i} \cdot \mathbf{r}_{\partial j}$ и $g^{ij} \equiv \mathbf{r}^i \cdot \mathbf{r}^j = g_{ij}^{-1}$ единичного “метрического” тензора $\mathbf{E} = \mathbf{r}_{\partial i} \mathbf{r}^i = \mathbf{r}^i \mathbf{r}_{\partial i} = g_{jk} \mathbf{r}^j \mathbf{r}^k = g^{jk} \mathbf{r}_{\partial j} \mathbf{r}_{\partial k}$,
- ✓ векторов локального взаимного кокасательного базиса $\mathbf{r}^i \cdot \mathbf{r}_{\partial j} = \delta_j^i$, $\mathbf{r}^i = g^{ij} \mathbf{r}_{\partial j}$,
- ✓ дифференциального набла-оператора Hamilton'a $\nabla \equiv \mathbf{r}^i \partial_i$, $\mathbf{E} = \nabla \mathbf{r}$,
- ✓ полного дифференциала $d\xi = d\mathbf{r} \cdot \nabla \xi$,
- ✓ частных производных касательного базиса (вторых частных производных \mathbf{r}) $\mathbf{r}_{\partial i \partial j} \equiv \partial_i \partial_j \mathbf{r} = \partial_i \mathbf{r}_{\partial j}$,
- ✓ символов “связности” Христоффеля (Christoffel symbols) $\Gamma_{ij}^k \equiv \mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}^k$ и $\Gamma_{ijk} \equiv \mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}_{\partial k}$.

Представим теперь, что функция $\mathbf{r}(q^k)$ не известна, но зато в каждой точке пространства известны шесть независимых

компонент положительно определённой (all Gram matrices are non-negative definite) симметричной метрической матрицы Gram $g_{ij}(q^k)$.

the Gram matrix (or Gramian)

Билинейная форма ...

...

Поскольку шесть функций $g_{ij}(q^k)$ происходят от векторной функции $\mathbf{r}(q^k)$, то между элементами g_{ij} существуют некие соотношения.

Differential $d\mathbf{r}$ (18.4) is exact. This is true if and only if second partial derivatives commute:

$$d\mathbf{r} = \mathbf{r}_{\partial k} dq^k \Leftrightarrow \partial_i \mathbf{r}_{\partial j} = \partial_j \mathbf{r}_{\partial i} \text{ or } \mathbf{r}_{\partial i \partial j} = \mathbf{r}_{\partial j \partial i}.$$

Но это условие уже обеспечено симметрией g_{ij}

...

metric (“affine”) connection ∇_i , её же называют “covariant derivative”

$$\mathbf{r}_{\partial i \partial j} = \underbrace{\mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}^k}_{\Gamma_{ij}^k} \mathbf{r}_{\partial k} = \underbrace{\mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}_{\partial k}}_{\Gamma_{ij\bullet}^k} \mathbf{r}^k$$

$$\Gamma_{ij}^k \mathbf{r}_{\partial k} = \mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}^k \mathbf{r}_{\partial k} = \mathbf{r}_{\partial i \partial j}$$

covariant derivative (affine connection) is only defined for vector fields

$$\nabla \mathbf{v} = \mathbf{r}^i \partial_i (v^j \mathbf{r}_{\partial j}) = \mathbf{r}^i (\partial_i v^j \mathbf{r}_{\partial j} + v^j \mathbf{r}_{\partial i \partial j})$$

$$\nabla \mathbf{v} = \mathbf{r}^i \mathbf{r}_{\partial j} \nabla_i v^j, \quad \nabla_i v^j \equiv \partial_i v^j + \Gamma_{in}^j v^n$$

$$\nabla \mathbf{r}_{\partial i} = \mathbf{r}^k \partial_k \mathbf{r}_{\partial i} = \mathbf{r}^k \mathbf{r}_{\partial k \partial i} = \mathbf{r}^k \mathbf{r}_{\partial n} \Gamma_{ki}^n, \quad \nabla_i \mathbf{r}_{\partial n} = \Gamma_{in}^k \mathbf{r}_{\partial k}$$

Christoffel symbols describe a metric (“affine”) connection, that is how the basis changes from point to point.

символы Christoffel’я это “components of connection” in local coordinates

...

torsion tensor ${}^3\mathfrak{T}$ with components

$$\mathfrak{T}_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$$

determines the antisymmetric part of a connection

...

симметрия $\Gamma_{ij\dot{k}} = \Gamma_{ji\dot{k}}$, поэтому $3^3 - 3 \cdot 3 = 18$ разных (независимых) $\Gamma_{ij\dot{k}}$

$$\begin{aligned} \Gamma_{ij}^n g_{nk} &= \Gamma_{ij\dot{k}} = \mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}_{\partial k} = \\ &= \frac{1}{2}(\mathbf{r}_{\partial i \partial j} + \mathbf{r}_{\partial j \partial i}) \cdot \mathbf{r}_{\partial k} + \frac{1}{2}(\mathbf{r}_{\partial j \partial k} - \mathbf{r}_{\partial k \partial j}) \cdot \mathbf{r}_{\partial i} + \frac{1}{2}(\mathbf{r}_{\partial i \partial k} - \mathbf{r}_{\partial k \partial i}) \cdot \mathbf{r}_{\partial j} = \\ &= \frac{1}{2}(\mathbf{r}_{\partial i \partial j} \cdot \mathbf{r}_{\partial k} + \mathbf{r}_{\partial i \partial k} \cdot \mathbf{r}_{\partial j}) + \frac{1}{2}(\mathbf{r}_{\partial j \partial i} \cdot \mathbf{r}_{\partial k} + \mathbf{r}_{\partial j \partial k} \cdot \mathbf{r}_{\partial i}) - \frac{1}{2}(\mathbf{r}_{\partial k \partial i} \cdot \mathbf{r}_{\partial j} + \mathbf{r}_{\partial k \partial j} \cdot \mathbf{r}_{\partial i}) = \\ &= \frac{1}{2}(\partial_i(\mathbf{r}_{\partial j} \cdot \mathbf{r}_{\partial k}) + \partial_j(\mathbf{r}_{\partial i} \cdot \mathbf{r}_{\partial k}) - \partial_k(\mathbf{r}_{\partial i} \cdot \mathbf{r}_{\partial j})) = \\ &= \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}). \quad (20.1) \end{aligned}$$

Все символы Christoffel'я тождественно равны нулю лишь в ортонормальной (декартовой) системе. (А какие они для ко-соугольной?)

Дальше: $d\mathbf{r}_{\partial i} = d\mathbf{r} \cdot \nabla \mathbf{r}_{\partial i} = dq^k \partial_k \mathbf{r}_{\partial i} = \mathbf{r}_{\partial k \partial i} dq^k$ — тоже полные дифференциалы.

$$d\mathbf{r}_{\partial k} = \partial_i \mathbf{r}_{\partial k} dq^i = \frac{\partial \mathbf{r}_{\partial k}}{\partial q^1} dq^1 + \frac{\partial \mathbf{r}_{\partial k}}{\partial q^2} dq^2 + \frac{\partial \mathbf{r}_{\partial k}}{\partial q^3} dq^3$$

Поэтому $\partial_i \partial_j \mathbf{r}_{\partial k} = \partial_j \partial_i \mathbf{r}_{\partial k}$, $\partial_i \mathbf{r}_{\partial j \partial k} = \partial_j \mathbf{r}_{\partial i \partial k}$, и трёхиндексный объект из векторов третьих частных производных

$$\mathbf{r}_{\partial i \partial j \partial k} \equiv \partial_i \partial_j \partial_k \mathbf{r} = \partial_i \mathbf{r}_{\partial j \partial k} \quad (20.2)$$

симметричен по первому и второму индексам (а не только по второму и третьему). И тогда равен нулю⁴ следующий тензор четвёртой сложности — *Riemann curvature tensor* (or *Riemann-Christoffel tensor*)

$${}^4\mathfrak{R} = \mathfrak{R}_{hijk} \mathbf{r}^h \mathbf{r}^i \mathbf{r}^j \mathbf{r}^k, \quad \mathfrak{R}_{hijk} \equiv \mathbf{r}_{\partial h} \cdot (\mathbf{r}_{\partial j \partial i \partial k} - \mathbf{r}_{\partial i \partial j \partial k}). \quad (20.3)$$

Выразим компоненты \mathfrak{R}_{ijkn} через метрическую матрицу g_{ij} . Начнём с дифференцирования локального кобазиса:

$$\mathbf{r}^i \cdot \mathbf{r}_{\partial k} = \delta_k^i \Rightarrow \partial_j \mathbf{r}^i \cdot \mathbf{r}_{\partial k} + \mathbf{r}^i \cdot \mathbf{r}_{\partial j \partial k} = 0 \Rightarrow \partial_j \mathbf{r}^i = -\Gamma_{jk}^i \mathbf{r}^k.$$

...

Шесть независимых компонент: \mathfrak{R}_{1212} , \mathfrak{R}_{1213} , \mathfrak{R}_{1223} , \mathfrak{R}_{1313} , \mathfrak{R}_{1323} , \mathfrak{R}_{2323} .

...

Symmetric bivalent *Ricci curvature tensor*

$$\mathcal{R} \equiv \frac{1}{4} \mathfrak{R}_{abij} \mathbf{r}^a \times \mathbf{r}^b \mathbf{r}^i \times \mathbf{r}^j = \frac{1}{4} \epsilon^{abp} \epsilon^{ijq} \mathfrak{R}_{abij} \mathbf{r}_{\partial p} \mathbf{r}_{\partial q} = \mathcal{R}^{pq} \mathbf{r}_{\partial p} \mathbf{r}_{\partial q}$$

(coefficient $\frac{1}{4}$ is used here for convenience) with components

$$\begin{aligned} \mathcal{R}^{11} &= \frac{1}{g} \mathfrak{R}_{2323}, \\ \mathcal{R}^{21} &= \frac{1}{g} \mathfrak{R}_{1323}, \quad \mathcal{R}^{22} = \frac{1}{g} \mathfrak{R}_{1313}, \\ \mathcal{R}^{31} &= \frac{1}{g} \mathfrak{R}_{1223}, \quad \mathcal{R}^{32} = \frac{1}{g} \mathfrak{R}_{1213}, \quad \mathcal{R}^{33} = \frac{1}{g} \mathfrak{R}_{1212}. \end{aligned}$$

Равенство тензора Риччи нулю $\mathcal{R} = {}^2\mathbf{0}$ (в компонентах это шесть уравнений $\mathcal{R}^{ij} = \mathcal{R}^{ji} = 0$) is the **necessary** condition of integrability (“compatibility”) для нахождения вектора радиуса $\mathbf{r}(q^k)$ по полю $g_{ij}(q^k)$.

Bibliography

There are many books which describe only the apparatus of tensor calculus [98, 99, 100, 16, ?, 101].

However, the index notation (it’s when the tensors are presented as the sets of components) is still more popular than the direct indexless notation.

The direct notation is widely used, for example, in the appendices to the books by Anatoliy I. Lurie (Анатолий И. Лурье) [29, 30].

In “Теории упругости” (“The theory of elasticity”) by Вениамин Блох (Veniamin Blokh) [7] the direct indexless notation is used too.

The R. Feynman's lectures [86] contain the bright description of the vector fields theory.

Also, information about the tensor calculus is the part of the unusual and interesting book by C. Truesdell [61].

LIST OF PUBLICATIONS

1. **Antman, Stuart S.** The theory of rods. In: Truesdell C. (editor) Mechanics of solids. Volume II. Linear theories of elasticity and thermoelasticity. Linear and nonlinear theories of rods, plates, and shells. Springer-Verlag, 1973. Pages 641–703.
2. **Алфутов Н. А.** Основы расчета на устойчивость упругих систем. Издание 2-е. М.: Машиностроение, 1991. 336 с.
3. **Артоболевский И. И., Бобровницкий Ю. И., Генкин М. Д.** Введение в акустическую динамику машин. «Наука», 1979. 296 с.
4. **Ахтырец Г. П., Короткин В. И.** Использование МКЭ при решении контактной задачи теории упругости с переменной зоной контакта // Известия северо-кавказского научного центра высшей школы (СКНЦ ВШ). Серия естественные науки. Ростов-на-Дону: Издательство РГУ, 1984. № 1. С. 38–42.
5. **Ахтырец Г. П., Короткин В. И.** К решению контактной задачи с помощью метода конечных элементов // Механика сплошной среды. Ростов-на-Дону: Издательство РГУ, 1988. С. 43–48.
6. **Бидерман В. Л.** Механика тонкостенных конструкций. М.: Машиностроение, 1977. 488 с.
7. **Вениамин И. Блох.** Теория упругости. Харьков: Издательство Харьковского Государственного Университета, 1964. 484 с.
8. **Власов В. З.** Тонкостенные упругие стержни. М.: Физматгиз, 1959. 568 с.
9. **Гольденвейзер А. Л.** Теория упругих тонких оболочек. «Наука», 1976. 512 с.
10. **Гольденвейзер А. Л., Лидский В. Б., Товстик П. Е.** Свободные колебания тонких упругих оболочек. «Наука», 1979. 383 с.
11. **Gordon, James E.** Structures, or Why things don't fall down. Penguin Books, 1978. 395 pages. *Перевод: Гордон Дж.* Конструкции, или почему не ломаются вещи. «Мир», 1980. 390 с.

12. **Gordon, James E.** The new science of strong materials, or Why you don't fall through the floor. Penguin Books, 1968. 269 pages. *Перевод: Гордон Дж.* Почему мы не проваливаемся сквозь пол. «Мир», 1971. 272 с.
13. **Александр Н. Гузь.** Устойчивость упругих тел при конечных деформациях. Киев: “Наукова думка”, 1973. 271 с.
14. *Перевод: Де Вит Р.* Континуальная теория дисклинаций. «Мир», 1977. 208 с.
15. **Джанелидзе Г. Ю., Пановко Я. Г.** Статика упругих тонкостенных стержней. Л., М.: Гостехиздат, 1948. 208 с.
16. **Димитриенко Ю. И.** Тензорное исчисление: Учебное пособие для вузов. М.: “Высшая школа”, 2001. 575 с.
17. **Dorin Ieşan.** Classical and generalized models of elastic rods. 2nd edition. CRC Press, Taylor & Francis Group, 2009. 369 pages
18. **Владимир В. Елисеев.** Одномерные и трёхмерные модели в механике упругих стержней. Диссертация на соискание учёной степени доктора физико-математических наук. ЛГТУ, 1991. 300 с.
19. **Eshelby, John D.** The continuum theory of lattice defects // Solid State Physics, Academic Press, vol. 3, 1956, pp. 79–144. *Перевод: Эшелби Дж.* Континуальная теория дислокаций. М.: ИИЛ, 1963. 247 с.
20. **Журавлёв В. Ф.** Основы теоретической механики. 3-е издание, переработанное. М.: ФИЗМАТЛИТ, 2008. 304 с.
21. **Зубов Л. М.** Методы нелинейной теории упругости в теории оболочек. Изд-во Ростовского ун-та, 1982. 144 с.
22. **Кац, Арнольд М.** Теория упругости. 2-е издание, стереотипное. Санкт-Петербург: Издательство «Лань», 2002. 208 с.
23. **Качанов Л. М.** Основы механики разрушения. «Наука», 1974. 312 с.
24. **Керштейн И. М., Ключников В. Д., Ломакин Е. В., Шестериков С. А.** Основы экспериментальной механики разрушения. Изд-во МГУ, 1989. 140 с.
25. **Cosserat E. et Cosserat F.** Théorie des corps déformables. Paris: A. Hermann et Fils, 1909. 226 p.
26. **Cottrell, Alan.** Theory of crystal dislocations. Gordon and Breach (Documents on Modern Physics), 1964. 94 p. *Перевод: Коттрел А.* Теория дислокаций. «Мир», 1969. 96 с.

27. **Kröner, Ekkehart** (i) *Kontinuumstheorie der Versetzungen und Eigenspannungen*. Springer-Verlag, 1958. 180 pages. (ii) *Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen* // *Archive for Rational Mechanics and Analysis*. Volume 4, Issue 1 (January 1959), pp. 273–334. *Перевод: Крёнер Э.* Общая континуальная теория дислокаций и собственных напряжений. «Мир», 1965. 104 с.
28. **Augustus Edward Hough Love**. A treatise on the mathematical theory of elasticity. Volume I. Cambridge, 1892. 354 p. Volume II. Cambridge, 1893. 327 p. 4th edition. Cambridge, 1927. Dover, 1944. 643 p. *Перевод: Аугустус Ляв* Математическая теория упругости. М.: ОНТИ, 1935. 674 с.
29. **Лурье А. И.** Нелинейная теория упругости. «Наука», 1980. 512 с. *Translation: Lurie, A. I.* Nonlinear Theory of Elasticity: translated from the Russian by K. A. Lurie. Elsevier Science Publishers B.V., 1990. 617 p.
30. **Лурье А. И.** Теория упругости. «Наука», 1970. 940 с. *Translation: Lurie, A. I.* Theory of Elasticity (translated by A. Belyaev). Springer-Verlag, 2005. 1050 p.
31. **Лурье А. И.** Пространственные задачи теории упругости. М.: Гостехиздат, 1955. 492 с.
32. **Лурье А. И.** Статика тонкостенных упругих оболочек. М., Л.: Гостехиздат, 1947. 252 с.
33. **George E. Mase**. *Schaum's outline of theory and problems of continuum mechanics* (Schaum's outline series). McGraw-Hill, 1970. 221 p. *Перевод: Джордж Мейз.* Теория и задачи механики сплошных сред. Издание 3-е. URSS, 2010. 320 с.
34. **Ernst Melan, Heinz Parkus**. *Wärmespannungen infolge stationärer Temperaturfelder*. Wein, Springer-Verlag, 1953. 114 Seiten. *Перевод: Мелан Э., Паркус Г.* Термоупругие напряжения, вызываемые стационарными температурными полями. М.: Физматгиз, 1958. 167 с.
35. **Меркин Д. Р.** Введение в механику гибкой нити. «Наука», 1980. 240 с.
36. **Меркин Д. Р.** Введение в теорию устойчивости движения. 3-е издание. «Наука», 1987. 304 с.

37. **Mindlin, Raymond David and Tiersten, Harry F.** Effects of couple-stresses in linear elasticity // Archive for Rational Mechanics and Analysis. Volume 11, Issue 1 (January 1962), pp. 415–448. *Перевод: Миндлин Р. Д., Тирстен Г. Ф.* Эффекты моментных напряжений в линейной теории упругости // Механика: Сборник переводов и обзоров иностранной периодической литературы. «Мир», 1964. № 4 (86). С. 80–114.
38. **Морозов Н. Ф.** Математические вопросы теории трещин. «Наука», 1984. 256 с.
39. **Naghdi P. M.** The theory of shells and plates. In: Truesdell C. (editor) Mechanics of solids. Volume II. Linear theories of elasticity and thermoelasticity. Linear and nonlinear theories of rods, plates, and shells. Springer-Verlag, 1973. Pages 425–640.
40. **Witold Nowacki.** Dynamiczne zagadnienia termosprężystości. Warszawa: Państwowe wydawnictwo naukowe, 1966. 366 stron. *Translation: Nowacki, Witold.* Dynamic problems of thermoelasticity. Leyden: Noordhoff international publishing, 1975. 436 pages. *Перевод: Витольд Новацкий.* Динамические задачи термоупругости. «Мир», 1970. 256 с.
41. **Witold Nowacki.** Teoria sprężystości. Warszawa: Państwowe wydawnictwo naukowe, 1970. 769 stron. *Перевод: Новацкий Витольд.* Теория упругости. «Мир», 1975. 872 с.
42. **Witold Nowacki.** Efekty elektromagnetyczne w stałych ciałach odkształcalnych. Państwowe wydawnictwo naukowe, 1983. 147 stron. *Перевод: Новацкий В.* Электромагнитные эффекты в твёрдых телах. «Мир», 1986. 160 с.
43. **Новожилов В. В.** Теория тонких оболочек. 2-е издание. Л.: Судпромгиз, 1962. 431 с.
44. **Пановко Я. Г., Бейлин Е. А.** Тонкостенные стержни и системы, составленные из тонкостенных стержней. В сборнике: Рабинович И. М. (редактор) Строительная механика в СССР 1917–1967. М.: Стройиздат, 1969. С. 75–98.
45. **Пановко Я. Г., Губанова И. И.** Устойчивость и колебания упругих систем. Современные концепции, парадоксы и ошибки. 4-е издание. «Наука», 1987. 352 с.
46. **Heinz Parkus.** Instationäre Wärmespannungen. Springer-Verlag, 1959. 176 Seiten. *Перевод: Паркус Г.* Неустановившиеся температурные напряжения. М.: Физматгиз, 1963. 252 с.

47. **Партон В. З.** Механика разрушения: от теории к практике. «Наука», 1990. 240 с.
48. **Партон В. З., Кудрявцев Б. А.** Электромагнитоупругость пьезоэлектрических и электропроводных тел. «Наука», 1988. 472 с.
49. **Партон В. З., Морозов Е. М.** Механика упругопластического разрушения. 2-е издание. «Наука», 1985. 504 с.
50. **Подстригач Я. С., Бурак Я. И., Кондрат В. Ф.** Магнитотермоупругость электропроводных тел. Киев: Наукова думка, 1982. 296 с.
51. **Поручиков В. Б.** Методы динамической теории упругости. «Наука», 1986. 328 с.
52. **Southwell, Richard V.** An introduction to the theory of elasticity for engineers and physicists. Dover Publications, 1970. 509 pages. *Перевод: Саусвелл Р. В.* Введение в теорию упругости для инженеров и физиков. М.: ИИЛ, 1948. 675 с.
53. **Седов Л. И.** Механика сплошной среды. Том 2. 6-е издание. «Лань», 2004. 560 с.
54. **Ciarlet, Philippe G.** Mathematical elasticity. Volume 1: Three-dimensional elasticity. Elsevier Science Publishers B. V., 1988. xlii + 452 pp. *Перевод: Филипп Сьярле* Математическая теория упругости. «Мир», 1992. 472 с.
55. **Adhémar-Jean-Claude Barré de Saint-Venant.** Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exerçant simultanément. Memoires presentes par divers savants a l'Academie des sciences, t. 14, année 1856. 327 pages. *Перевод на русский язык: Сен-Венан Б.* Мемуар о кручении призм. Мемуар об изгибе призм. М.: Физматгиз, 1961. 518 страниц.
56. **Adhémar-Jean-Claude Barré de Saint-Venant.** Mémoire sur la flexion des prismes Journal de mathematiques pures et appliquees, publie par J. Liouville. 2me serie, t. 1, année 1856. *Перевод на русский язык: Сен-Венан Б.* Мемуар о кручении призм. Мемуар об изгибе призм. М.: Физматгиз, 1961. 518 страниц.
57. **Cristian Teodosiu.** Elastic models of crystal defects. Springer-Verlag, 1982. 336 pages. *Перевод: Теодосиу К.* Упругие модели дефектов в кристаллах. «Мир», 1985. 352 с.
58. **Тимошенко Степан П.** Устойчивость стержней, пластин и оболочек. «Наука», 1971. 808 с.

59. **Тимошенко Степан П., Войновский-Кригер С.** Пластинки и оболочки. «Наука», 1966. 635 с.
60. **Stephen P. Timoshenko and James N. Goodier.** Theory of Elasticity. 2nd edition. McGraw–Hill, 1951. 506 pages. 3rd edition. McGraw–Hill, 1970. 567 pages. *Перевод: Тимошенко Степан П., Джеймс Гудьер.* Теория упругости. 2-е издание. «Наука», 1979. 560 с.
61. **Truesdell, Clifford A.** A first course in rational continuum mechanics. Volume 1: General concepts. 2nd edition. Academic Press, 1991. 391 pages. *Перевод: Трусделл К.* Первоначальный курс рациональной механики сплошных сред. «Мир», 1975. 592 с.
62. **Феодосьев В. И.** Десять лекций-бесед по сопротивлению материалов. 2-е издание. «Наука», 1975. 173 с.
63. *Перевод: Хеллан К.* Введение в механику разрушения. «Мир», 1988. 364 с.
64. *Перевод: Циглер Г.* Основы теории устойчивости конструкций. «Мир», 1971. 192 с.
65. **Черепанов Г. П.** Механика хрупкого разрушения. «Наука», 1974. 640 с.
66. **Черных К. Ф.** Введение в анизотропную упругость. «Наука», 1988. 192 с.
67. **Шермергор Т. Д.** Теория упругости микронеоднородных сред. «Наука», 1977. 400 с.

Oscillations and waves

68. **Timoshenko, Stephen P.; Young, Donovan H.; William Weaver, jr.** Vibration problems in engineering. 5th edition. John Wiley & Sons, 1990. 624 pages. *Перевод: Тимошенко Степан П., Янг Донован Х., Уильям Уивер.* Колебания в инженерном деле. М.: Машиностроение, 1985. 472 с.
69. **Бабаков И. М.** Теория колебаний. 4-е издание. «Дрофа», 2004. 592 с.
70. **Бидерман В. Л.** Теория механических колебаний. М.: Высшая школа, 1980. 408 с.
71. **Болотин В. В.** Случайные колебания упругих систем. «Наука», 1979. 336 с.
72. **Гринченко В. Т., Мелешко В. В.** Гармонические колебания и волны в упругих телах. Киев: Наукова думка, 1981. 284 с.

73. **Whitham, Gerald B.** Linear and nonlinear waves. John Wiley & Sons, 1974. 636 pages. *Перевод: Узием Дж.* Линейные и нелинейные волны. «Мир», 1977. 624 с.
74. **Kolsky, Herbert.** Stress waves in solids. Oxford, Clarendon Press, 1953. 211 p. 2nd edition. Dover Publications, 2012. 224 p. *Перевод: Кольский Г.* Волны напряжения в твёрдых телах. М.: ИИЛ, 1955. 192 с.
75. **Энгельбрехт Ю. К., Нигул У. К.** Нелинейные волны деформации. «Наука», 1981. 256 с.
76. **Слепян Л. И.** Нестационарные упругие волны. Л.: Судостроение, 1972. 376 с.
77. **Григолюк Э. И., Селезов И. Т.** Неклассические теории колебаний стержней, пластин и оболочек. (Итоги науки и техники. Механика твёрдых деформируемых тел. Том 5.) М.: ВИНТИ, 1973. 272 с.

Composites

78. **Christensen, Richard M.** Mechanics of composite materials. New York: Wiley, 1979. 348 p. *Перевод: Кристенсен Р.* Введение в механику композитов. «Мир», 1982. 336 с.
79. **Кравчук А. С., Майборода В. П., Уржумцев Ю. С.** Механика полимерных и композиционных материалов. Экспериментальные и численные методы. «Наука», 1985. 304 с.
80. **Победря Б. Е.** Механика композиционных материалов. Изд-во Моск. ун-та, 1984. 336 с.
81. **Черепанов Г. П.** Механика разрушения композиционных материалов. «Наука», 1983. 296 с.
82. **Бахвалов Н. С., Панасенко Г. П.** Осреднение процессов в периодических средах. Математические задачи механики композиционных материалов. «Наука», 1984. 352 с.
83. **Bensoussan A., Lions J.-L., Papanicolaou G.** Asymptotic analysis for periodic structures. Amsterdam: North-Holland, 1978. 700 p.

The finite element method

84. **Зенкевич О., Морган К.** Конечные элементы и аппроксимация. «Мир», 1986. 318 с.
85. **Шабров Н. Н.** Метод конечных элементов в расчётах деталей тепловых двигателей. Л.: Машиностроение, 1983. 212 с.

Mechanics, thermodynamics, electromagnetism

86. **Feynman, Richard Ph. • Leighton, Robert B. • Sands, Matthew.** The Feynman Lectures on Physics. New millennium edition. Volume II: Mainly electromagnetism and matter. Basic Books, 2011. 566 pages. *Online: The Feynman Lectures on Physics. Online edition.*
87. **Goldstein, Herbert; Poole, Charles P.; Saffo, John L.** Classical Mechanics. 3rd edition. Addison–Wesley, 2001. 638 pages. *Перевод: Голдстейн Г., Пул Ч., Сафко Дж.* Классическая механика. URSS, 2012. 828 с.
88. **Pars, Leopold A.** A treatise on analytical dynamics. London: Heinemann, 1965. 641 pages. *Перевод: Парс Л. А.* Аналитическая динамика. «Наука», 1971. 636 с.
89. **Ter Haar, Dirk.** Elements of hamiltonian mechanics. 2nd edition. Pergamon Press, 1971. 201 pages. *Перевод: Тер Хаар Д.* Основы гамильтоновой механики. «Наука», 1974. 223 с.
90. **Беляев Н. М., Рядно А. А.** Методы теории теплопроводности. М.: Высшая школа, 1982. В 2-х томах. Том 1, 328 с. Том 2, 304 с.
91. **Бредов М. М., Румянцев В. В., Топтыгин И. Н.** Классическая электродинамика. «Наука», 1985. 400 с.
92. **Феликс Р. Гантмахер** Лекции по аналитической механике. Издание 2-е. «Наука», 1966. 300 с.
93. **Ландау Л. Д., Лифшиц Е. М.** Краткий курс теоретической физики. Книга 1. Механика. Электродинамика. «Наука», 1969. 271 с.
94. **Лойцянский Л. Г., Лурье А. И.** Курс теоретической механики: В 2-х томах. «Дрофа», 2006. Том 1: Статика и кинематика. 9-е издание. 447 с. Том 2: Динамика. 7-е издание. 719 с.
95. **Лурье А. И.** Аналитическая механика. М.: Физматгиз, 1961. 824 с.
96. **Ольховский И. И.** Курс теоретической механики для физиков. 3-е издание. Изд-во МГУ, 1978. 575 с.
97. **Тамм И. Е.** Основы теории электричества. 11-е издание. М.: Физматлит, 2003. 616 с.

Tensors and tensor calculus

98. **McConnell, Albert Joseph.** Applications of tensor analysis. New York: Dover Publications, 1957. 318 pages. *Перевод: Мак-Коннел А. Дж.* Введение в тензорный анализ с приложениями к геометрии, механике и физике. М.: Физматгиз, 1963. 412 с.

99. **Schouten, Jan A.** Tensor analysis for physicists. 2nd edition. Dover Publications, 2011. 320 pages. *Перевод: Схоутен Я. А.* Тензорный анализ для физиков. «Наука», 1965. 456 с.
100. **Sokolnikoff, I. S.** Tensor analysis: Theory and applications to geometry and mechanics of continua. 2nd edition. John Wiley & Sons, 1965. 361 pages. *Перевод: Сокольников И. С.* Тензорный анализ (с приложениями к геометрии и механике сплошных сред). «Наука», 1971. 376 с.
101. **Рашевский П. К.** Риманова геометрия и тензорный анализ. Издание 3-е. «Наука», 1967. 664 с.

Variational methods

102. **Karel Rektorys.** Variační metody v inženýrských problémech a v problémech matematické fyziky. SNTL (Státní nakladatelství technické literatury), 1974. 593 s. *Translation: Rektorys, Karel.* Variational Methods in Mathematics, Science and Engineering. Second edition. D. Reidel Publishing Company, 1980. 571 p. *Перевод: Ректорис К.* Вариационные методы в математической физике. «Мир», 1985. 590 с.
103. **Washizu, Kyuichiro.** Variational methods in elasticity and plasticity. 3rd edition. Pergamon Press, Oxford, 1982. 630 pages. *Перевод: Васидзу К.* Вариационные методы в теории упругости и пластичности. «Мир», 1987. 542 с.
104. **Бердичевский В. Л.** Вариационные принципы механики сплошной среды. «Наука», 1983. 448 с.
105. **Михлин С. Г.** Вариационные методы в математической физике. Издание 2-е. «Наука», 1970. 512 с.

Perturbation methods (asymptotic methods)

106. **Cole, Julian D.** Perturbation methods in applied mathematics. Blaisdell Publishing Co., 1968. 260 pages. *Перевод: Коул Дж.* Методы возмущений в прикладной математике. «Мир», 1972. 274 с.
107. **Nayfeh, Ali H.** Introduction to perturbation techniques. Wiley, 1981. 536 pages. *Перевод: Найфэ Али Х.* Введение в методы возмущений. «Мир», 1984. 535 с.
108. **Nayfeh, Ali H.** Perturbation methods. Wiley-VCH, 2004. 425 pages.
109. **Боголюбов Н. Н., Митропольский Ю. А.** Асимптотические методы в теории нелинейных колебаний. «Наука», 1974. 504 с.

110. **Васильева А. Б., Бутузов В. Ф.** Асимптотические методы в теории сингулярных возмущений. М.: Высшая школа, 1990. 208 с.
111. **Зино И. Е., Тропш Э. А.** Асимптотические методы в задачах теории теплопроводности и термоупругости. Изд-во ЛГУ, 1978. 224 с.
112. **Моисеев Н. Н.** Асимптотические методы нелинейной механики. 2-е издание. «Наука», 1981. 400 с.
113. **Товстик П. Е.** Устойчивость тонких оболочек: асимптотические методы. «Наука», 1995. 319 с.

Other topics of mathematics

114. **Collatz, Lothar.** Eigenwertaufgaben mit technischen Anwendungen. 2. Auflage. Akademische Verlagsgesellschaft Geest & Portig, Leipzig, 1963. 500 Seiten. *Перевод: Коллатц Л.* Задачи на собственные значения (с техническими приложениями). «Наука», 1968. 504 с.
115. **Dwight, Herbert Bristol.** Tables of integrals and other mathematical data. 4th edition. The Macmillan Co., 1961. 336 pages. *Перевод: Двайт Г. Б.* Таблицы интегралов и другие математические формулы. Издание 4-е. «Наука», 1973. 228 с.
116. **Kamke, Erich.** Differentialgleichungen, Lösungsmethoden und Lösungen. Bd. I. Gewöhnliche Differentialgleichungen. 10. Auflage. Teubner Verlag, 1977. 670 Seiten. *Перевод: Камке Э.* Справочник по обыкновенным дифференциальным уравнениям. 6-е издание. «Лань», 2003. 576 с.
117. **Korn, Granino A. and Korn, Theresa M.** Mathematical handbook for scientists and engineers: definitions, theorems, and formulas for reference and review. Revised edition. Dover Publications, 2013. 1152 pages. *Перевод: Корн Г., Корн Т.* Справочник по математике для научных работников и инженеров. «Наука», 1974. 832 с.
118. **Лаврентьев М. А., Шабат Б. В.** Методы теории функций комплексного переменного. 4-е издание. «Наука», 1973. 736 с.
119. **Погорелов А. В.** Дифференциальная геометрия. Издание 6-е. «Наука», 1974. 176 с.