Vadique Myself

PHYSICS *of* ELASTIC CONTINUA

FOREWORD

In this book I am trying to guide the reader* through the ways of constructing the models of an elastic continuum. These models are : the nonlinear and the linear ones, the micropolar and the classical momentless; the three-dimensional, the two-dimensional (shells and plates) and the one-dimensional (rods, including thin-walled ones). I also explained the fundamentals of dynamics — oscillations, waves and stability. For the thermoelasticity and the magnetoelasticity, I gave the summary of the classical theories of the thermodynamics and the electrodynamics. The dynamics of destruction is described via the theories of defects and fractures. The approaches to modeling of human-made inhomogeneous materials, "composites", are also shown.

The word "continua" in the title says that an object (a body, a medium) is modeled here not as a discrete collection of particles, but as a continuous space of location vectors, a continuous matter. It gives a large convenience, because the apparatus of calculus of infinitesimals can be used for such models.

When I just began writing this book, I thought of a reader who is pretty acquainted with "higher" mathematics. But later I decided to conduct such an acquaintance by myself, and yet, as a side effect, every reader with any knowledge of math can comprehend the content of the book.

* and myself as well

The book is written using the compact and elegant direct indexless tensor notation. The mathematical apparatus for interpreting the direct tensor relations is located in the first chapter.

I am writing this book simultaneously in the two languages, English and Russian. The reader is free to pick any language of the two.

Vadique

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chapter 1
MATHEMATICAL APPARATUS

Mathematics, or math for short, is abstract. Abstract is the adjective of math, math is the noun of abstract. “Abstract”, “theoretical” and “mathematical” are synonyms. When someone is doing math, he’s playing a game in the far-far-away magical world of imagination.

For example, numbers are not real entities at all. They are purely imaginary concepts. We cannot experience, sensate numbers, can’t see, touch or smell them. Yep, one can compose stories about them, such as $1+1=2$ — mathematical relations between imaginary entities. Nevertheless, no one can ever feel, perceive it, since there are no such things as *one* and *two*[*].

And “synthesized by imagination” — it’s not only about numbers. Geometric objects, be it a point, a line, a triangle or a plane, and all kinds of adventures with them are mind derived as well.

§1. The ancient but intuitive geometry

For nearly two thousand years, the freedom of human’s thought was limited by the fairy tale about imaginary perfectly straight one-dimensional lines between some two absolutely dimensionless points and beyond, on and on to the very ἄπειρο (infinity) down and up the both sides, about imaginary and completely flat planes-trigons (“triangles”, sometimes “tripoints”), with shortest distances between points in straight lines, with always equal to each other “straight” or “~~st~~r~~a~~ight” angles, as well as about many other pretty funny mythical characters and piquant relations between them. Over two thousand years people were in captivity, in slavery to the idea about the existænce of the only one ευκλείδειος γεωμετρίας (the εὐκλείδ’ean geometry), and

[*] I’m not about *two apples* or *two similar bananas* for *a couple of days*, but about the number “two” itself.

the magic world, described by it, was equated in the past with the real space around themselves.

Εὐκλείδης Eὐkleídēs Euclid, ευκλείδειος euclidean

ευκλείδεια γεωμετρία

the plane geometry, or the two-dimensional euclidean geometry

Στοιχεῖα Stoikheîa Elements, Principles

(1.1) Points

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Ορος α′ (1) Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.	Term α′ (1) A point is that which has no part.

This description shows that Euclid imagines a point as an indivisible location, without width, length or breadth.

(1.2) Lines, curved and straight

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Ορος β′ (2) Γραμμὴ δὲ μῆκος ἀπλατές.	Term β′ (2) A line is breadthless length.

"Line" is the second primitive term in the Elements. "Breadthless length" says that a line will have one dimension, length, but it won't have breadth. The terms "length" and "breadth" are not defined in the Elements.

Linear lines

(1.3) A relation between lines and points

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Ορος γ′ (3) Γραμμῆς δὲ πέρατα σημεῖα.	Term γ′ (3) The ends of a line are points.

This statement doesn't mention how many ends a line can have.

(1.4) Do straight lines exists?

The hypothesis on the existænce of straight lines.

The existænce of Euclidean straight lines in space.

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Ορος δ′ (4) Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ′ ἑαυτῆς σημείοις κεῖται.	Term δ′ (4) A straight line is a line which lies evenly with the points on itself.

To draw a straight line by hand is absolutely impossible.

(1.5) Vectors. Lines and vectors

(1.6) The existænce of vectors. Do vectors exist?

(1.7) Continuity of line

(1.8) A point of reference

(1.9) Translation as the easiest kind of motion. Translations and vectors

(1.10) Straight line and vector

A (geometric) vector may be like a straight line with an arrow at one of its ends. Then it is fully described (characterized) by the magnitude and the direction.

Within the abstract algebra, the word *vector* is about any object which can be summed with similar objects and scaled (multiplied) by scalars, and vector space is a synonym of linear space. Therefore I clarify that in this book *vector* is nothing else than three-dimensional geometric (Ευκλείδειος, Euclidean) vector.

Why are vectors always straight (linear)?

(a) Vectors are linear (straight), they cannot be curved.

(b) Vectors are neither straight nor curved. A vector has the magnitude and the direction. A vector is not a line or a curve, albeit it can be represented by a straight line.

Vector can't be thought of as a line.

(1.11) The line which figures real numbers

often just "number line"

(1.12) What is a distance?

(1.13) Plane and more dimensional space

(1.14) Distance on plane or more dimensional space

(1.15) What is an angle?

angle ≡ inclination /slope, slant/ of two lines

two lines sharing a common point are usually called intersecting lines

angle ≡ the amount of rotation of line or plane within space

angle ≡ the result of the dot product of two unit vectors gives angle's cosine

(1.16) Differentiation of continuous into small differential chunks

small differential chunks

infinitesimal (infinitely small)

A mention of tensors may scare away the reader, commonly avoiding needless complications. Don't be afraid: tensors are used just due to their wonderful property of the invariance — the independence from a coordinate system.

§ 2. Vector

I propose to begin familiarizing with tensors via memoirs about such a phenomenon as a vector.

- ✓ A *point* has position in space. The only characteristic that distinguishes one point from another is its position.
- ✓ A *vector* has both magnitude and direction, but no specific position in space.

(2.1) What is a vector?

What is “linear”?
(1) straight
(2) relating to, resembling, or having a graph that is a straight line

All vectors are linear objects.

Examples of vectors:

- ✓ A force acts on an object.
- ✓ The velocity of an object describes what’s happening with this object at an instant.

Multiplication of a vector by a scalar

Multiplication by the minus one

The Newton’s action–reaction principle “действие равно противодействию по магнитуде и обратно ему по направлению”.

Each mechanical interaction of two objects is characterized by two forces that act on both interacting objects. These forces can be represented as two vectors that are equal in magnitude and reverse in direction.

Multiplying a vector by the negative one -1 reverses the vector’s direction but doesn’t change its magnitude.

(2.2) The addition and subtraction

The sum (combination) of two or more vectors is the new “resultant” vector. There are two similar methods to calculate the resultant vector geometrically.

The “head to tail method” involves lining up the head of one vector with the tail of another. Here the resultant goes from the initial point (the “tail”) of the first addend to the end point (the “head”) of the second addend when the tail (the initial point) of the second one coincides with the head (the end point) of the first one.

[.... figure here ....]

The “parallelogram method” ...

[.... figure here ....]

The vector addition is commutative

$$\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}.$$

....

$$\boldsymbol{p} + \boldsymbol{q},$$
$$\boldsymbol{p} - \boldsymbol{q} = \boldsymbol{p} + (-\boldsymbol{q}) = \boldsymbol{p} + (-1)\boldsymbol{q}.$$

For every action, there’s an equal (in magnitude) and opposite (in direction) reaction force.

A vector may be also represented as the sum (combination) of some trio of other vectors, called “basis”, when the each of the three is scaled by a number (coefficient). Such a representation is called a “linear combination” of basis vectors. A list (array, tuple) of coefficients alone, without basis vectors, is not enough and can’t represent a vector.

....

To get the numerical relations from the vector ones, a coordinate system is introduced, and on its axes the vector relations are projected.

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Vectors themselves (as elements of a vector space) do not have components. Vector components appear only when a certain basis is chosen, then any vector can be “decomposed” — represented as as the sum of basis vectors, premultiplied by coefficients (“components)” is just another name for coefficients of a linear combination). The same vector in different bases has different components.

Here it is — a vector, $\boldsymbol{v}$ looks like a suitable name for it.

Like all geometric vectors, $\boldsymbol{v}$ is pretty well characterized by the two mutually independent properties: its length (magnitude, norm, modulus) and its direction in space. This characterization is complete, so some two vectors with the same magnitude and the same direction are considered equal.

Every vector exists objectively by itself, independently of methods and units of measurement of both lengths and directions, including any abstractions of such units and methods.

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Not everything is a vector that has a magnitude and direction. Поворот тела вокруг оси, казалось бы, обладает всеми аттрибутами вектора : у него есть численное значение, равное углу поворота, и направление оси вращения. However, when the rotation angles are not infinitely small, the rotations don't sum like vectors*.

Складываются ли угловые скорости? — Да, ведь угол поворота в $\dot{\vartheta}$ бесконечно-малый. — Но только при вращении вокруг неподвижной оси?

(2.3) The method of coordinates

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By choosing some mutually perpendicular unit vectors $\boldsymbol{e}_i$ as the basis for measurements, I introduce the rectangular ("cartesian") coordinates.

Three $(i = 1, 2, 3)$ basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are needed for a three-dimensional — 3D — space.

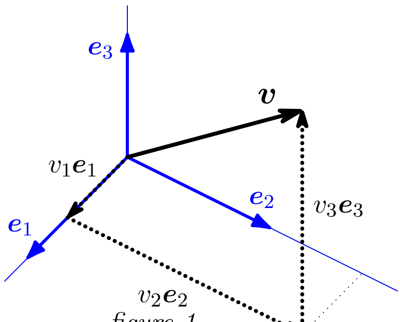


figure 1

Within such a system, "•"-products of the basis vectors are equal to the Kronecker delta

$$\boldsymbol{e}_i \bullet \boldsymbol{e}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for any orthonormal basis.

* Actually, the sequential rotations are not added, but multiplied.

Decomposing vector $\boldsymbol{v}$ in some orthonormal basis $\boldsymbol{e}_i$ $(i = 1, 2, 3)$, we get coefficients v_i — the components of vector $\boldsymbol{v}$ in that basis (figure 1)

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 \equiv \sum_{i=1}^{3} v_i\boldsymbol{e}_i \equiv v_i\boldsymbol{e}_i\,, \;\; v_i = \boldsymbol{v} \cdot \boldsymbol{e}_i\,. \tag{2.1}$$

Here and hereinafter, the Einstein's summation convention is accepted : an index repeated twice (and no more than twice) in a single term implies a summation over this index. And a non-repeating index is called "free", and it is identical in the both parts of equality. These are examples

$$a_i = \lambda b_i + \mu c_i\,, \;\; \sigma = \tau_{ii} = \sum_i \tau_{ii}\,,$$
$$p_j = n_i \tau_{ij} = \sum_i n_i \tau_{ij}\,, \;\; m_i = e_{ijk}x_j f_k = \sum_{j,k} e_{ijk} x_j f_k\,.$$

(But equalities $a = b_{kkk}\,,\; c = f_i + g_k\,,\; d_{ij} = k_i q_{ij}$ are incorrect.)

Having components of a vector in an orthonormal basis, the length of this vector is retrieved by the "Πυθαγόρας' equation"

$$\boldsymbol{v} \cdot \boldsymbol{v} = v_i\boldsymbol{e}_i \cdot v_j\boldsymbol{e}_j = v_i\delta_{ij}v_j = v_i v_i\,, \;\; \|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \sqrt{v_i v_i}\,. \tag{2.2}$$

The magnitude represents the length independent of direction. The direction of a vector in space is measured by the three angles (cosines of angles) between this vector and each of the basis ones:

$$\cos\measuredangle\big(\boldsymbol{v}^\frown\boldsymbol{e}_i\big) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\cdot\boldsymbol{e}_i = \frac{v_i}{\sqrt{v_j v_j}} \;\;\Leftrightarrow\;\; \underbrace{v_i}_{\boldsymbol{v}\cdot\boldsymbol{e}_i} = \|\boldsymbol{v}\|\cos\measuredangle(\boldsymbol{v}, \boldsymbol{e}_i)\,. \tag{2.3}$$

Measurement of angles. The cosine of an angle between two vectors is the same as the dot product of these vectors if their magnitudes are equal to the one unit of length

When the magnitudes of two vectors are equal to the one unit of length, then the cosine of the least angle between them is the same as the dot product of these vectors. Any vector with the non-unit magnitude (but the null vector) can be "normalized" via dividing a vector by its magnitude.

$$\cos\measuredangle(\boldsymbol{v}, \boldsymbol{w}) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\cdot\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}\,.$$

To accompany the magnitude, which represents the length independent of direction, there's a way to represent the direction of a vector independent of its length. For this purpose, the unit vectors (the vectors with the magnitude of 1) are used.

A rotation matrix is just a transform that expresses the basis vectors of the input space in a different orientation. The length of the basis vectors will be the same, and the origin will not change. Also, the angle between the basis vectors will not change. All that changes is the relative direction of all of the basis vectors.

Therefore, a rotation matrix is not really just a "rotation" matrix; it is an orientation matrix.

There are also pseudovectors, waiting for the reader below in § 7.

The angle between two random vectors. According to (2.3)

$$\cos\measuredangle(\boldsymbol{v}, \boldsymbol{e}_m) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \bullet \boldsymbol{e}_m = \frac{v_m}{\sqrt{v_j v_j}},$$
$$\cos\measuredangle(\boldsymbol{w}, \boldsymbol{e}_n) = \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} \bullet \boldsymbol{e}_n = \frac{w_n}{\sqrt{w_k w_k}}.$$

The length (2.2) and the direction in space (2.3), that can be measured by the means of the trio of basic vectors, describe a vector. And every vector possesses these properties*. However, this is not enough ("not sufficient" in jargon of the math books).

* And what is the direction of the null vector ("(vanishing) vector") $\mathbf{0}$ with the zero length $\|\mathbf{0}\| = 0$? (The zero vector without a magnitude ends exactly where it begins and thus it is not directed anywhere, its direction is *undefined*.)

A vector is not just a collection of components in some basis.

A triple of pairwise perpendicular unit vectors can only rotate and thereby it can characterize the angular orientation of other vectors.

The decomposition of the same vector $\boldsymbol{v}$ in the two cartesian systems with basis unit vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}'_i$ (figure 2) gives

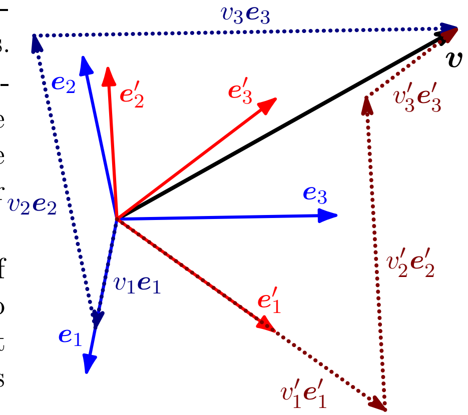


figure 2

$$\boldsymbol{v} = v_i \boldsymbol{e}_i = v'_i \boldsymbol{e}'_i,$$

where

$$v_i = \boldsymbol{v} \bullet \boldsymbol{e}_i = v'_k \boldsymbol{e}'_k \bullet \boldsymbol{e}_i,$$
$$v'_i = \boldsymbol{v} \bullet \boldsymbol{e}'_i = v_k \boldsymbol{e}_k \bullet \boldsymbol{e}'_i.$$

Appeared here two-index objects (the two-dimensional arrays) $o_{k'i} \equiv \boldsymbol{e}'_k \bullet \boldsymbol{e}_i$ and $o_{ki'} \equiv \boldsymbol{e}_k \bullet \boldsymbol{e}'_i$ are used to shorten the formulas.

Написа́ть о пассивном повороте, описанном ниже, и об активном повороте из § 11 ....

The "•"-product (dot product) of two vectors is commutative — that is, the swapping of multipliers doesn't change the result. Thus

$$o_{k'i} = \boldsymbol{e}'_k \bullet \boldsymbol{e}_i = \cos\measuredangle(\boldsymbol{e}'_k, \boldsymbol{e}_i) = \cos\measuredangle(\boldsymbol{e}_i, \boldsymbol{e}'_k) = \boldsymbol{e}_i \bullet \boldsymbol{e}'_k = o_{ik'}, \quad (2.3a)$$
$$o_{ki'} = \boldsymbol{e}_k \bullet \boldsymbol{e}'_i = \cos\measuredangle(\boldsymbol{e}_k, \boldsymbol{e}'_i) = \cos\measuredangle(\boldsymbol{e}'_i, \boldsymbol{e}_k) = \boldsymbol{e}'_i \bullet \boldsymbol{e}_k = o_{i'k}. \quad (2.3b)$$

Lines (2.3a) and (2.3b) are mutually reciprocal by multiplication

$$o_{k'i}\, o_{ki'} = o_{ki'}\, o_{k'i} = 1, \;\; o_{k'i}\, o_{i'k} = o_{i'k}\, o_{k'i} = 1.$$

Multiplying of an orthogonal matrix by the components of any vector retains the length of this vector:

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \bullet \boldsymbol{v} = v'_i v'_i = o_{i'k} v_k\, o_{i'n} v_n = v_n v_n$$

— this conclusion leans on (**??**).

Orthogonal transformation of the vector components

$$\boldsymbol{v} \cdot \boldsymbol{e}'_i = v_k \boldsymbol{e}_k \cdot \boldsymbol{e}'_i = \boldsymbol{e}'_i \cdot \boldsymbol{e}_k v_k = o_{i'k} v_k = v'_i \tag{2.4}$$

is sometimes used for defining a vector itself. If in each orthonormal basis $\boldsymbol{e}_i$ a triplet of numbers v_i is known, and with a rotation of the basis as a whole it is transformed according to (2.4). then this triplet of components represents an invariant object — vector $\boldsymbol{v}$.

§ 3. Tensor and its components

When in each orthonormal basis $\boldsymbol{e}_i$ we have a set of nine ($3^2 = 9$) numbers B_{ij} $(i, j = 1, 2, 3)$, and this set is transformed during a transition to a new (rotated) orthonormal basis $\boldsymbol{e}'_i$ as

$$B'_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_m B_{mn} \boldsymbol{e}_n \cdot \boldsymbol{e}'_j = \boldsymbol{e}'_i \cdot \boldsymbol{e}_m \boldsymbol{e}'_j \cdot \boldsymbol{e}_n B_{mn} = o_{i'm} o_{j'n} B_{mn}, \tag{3.1}$$

then this set of components presents an invariant object — a tensor ${}^2\boldsymbol{B}$ of the second complexity (of the second valence, bivalent).

In other words, tensor ${}^2\boldsymbol{B}$ reveals in every basis as a collection of its components B_{ij}, changing along with a basis according to (3.1).

The key example of a second complexity tensor is a dyad. Having two vectors $\boldsymbol{a} = a_i \boldsymbol{e}_i$ and $\boldsymbol{b} = b_i \boldsymbol{e}_i$, in each basis $\boldsymbol{e}_i$ assume $d_{ij} \equiv a_i b_j$. It's easy to see how components d_{ij} transform according to (3.1):

$$a'_i = o_{i'm} a_m, \; b'_j = o_{j'n} b_n \Rightarrow d'_{ij} = a'_i b'_j = o_{i'm} a_m o_{j'n} b_n = o_{i'm} o_{j'n} d_{mn}.$$

A resulting tensor ${}^2\boldsymbol{d}$ is called a dyadic product or just dyad and is written as $\boldsymbol{a} \otimes \boldsymbol{b}$ or $\boldsymbol{ab}$. I choose the notation "${}^2\boldsymbol{d} = \boldsymbol{ab}$", without the $\otimes$ symbol.

When some bivalent tensor ${}^2\boldsymbol{B}$ is a dyad $\boldsymbol{\beta b}$, its components $B_{ij} = \beta_i b_j$ satisfy the equality $B_{pq} B_{mn} = B_{mq} B_{pn}$ to get commutativity of multiplication $\beta_p b_q \beta_m b_n = \beta_m b_q \beta_p b_n$. Here $p \neq m$, or else the equality becomes the identity.

The essential bivalent tensor is the unit tensor (other names are unit dyad, identity tensor and metric tensor). Let for any orthonormal (cartesian) basis $E_{ij} \equiv \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$. These are really components of tensor, (3.1) is actual: $E'_{mn} = o_{m'i} o_{n'j} E_{ij} = o_{m'i} o_{n'i} = \delta_{mn}$. I write this tensor as $\boldsymbol{E}$ (other popular choices are $\boldsymbol{I}$ and ${}^2\mathbf{1}$).

Invariableness of components upon any rotation makes the tensor $\boldsymbol{E}$ isotropic. There are no non-null (nonvanishing) isotropic vectors (all components of the null, or "vanishing", vector $\mathbf{0}$ are equal to zero within any basis).

The next example is related to a linear transformation (a linear mapping) of vectors.

If $\boldsymbol{b} = b_i \boldsymbol{e}_i$ is linear (preserving addition and multiplication by number) function of $\boldsymbol{a} = a_j \boldsymbol{e}_j$, then $b_i = c_{ij} a_j$ in every basis. Transformation coefficients c_{ij} alter when a basis rotates :

$$b'_i = c'_{ij} a'_j = o_{i'k} b_k = o_{i'k} c_{kn} a_n,\; a_n = o_{j'n} a'_j \;\Rightarrow\; c'_{ij} = o_{i'k} o_{j'n} c_{kn}.$$

It turns out that a set of two-index objects c_{ij}, c'_{ij}, ..., describing the same linear mapping $\boldsymbol{a} \mapsto \boldsymbol{b}$, but in various bases, represents a single invariant object — a tensor of second complexity ${}^2\boldsymbol{c}$. And many book authors introduce tensors in that way, by means of linear mappings (linear transformations).

And the last example is a bilinear form $\digamma(\boldsymbol{a}, \boldsymbol{b}) = f_{ij}\, a_i b_j$, where f_{ij} are coefficients, a_i and b_j are components of vector arguments $\boldsymbol{a} = a_i \boldsymbol{e}_i$ and $\boldsymbol{b} = b_j \boldsymbol{e}_j$. The result $\digamma$ is invariant (independent of basis) with the transformation (3.1) for coefficients f_{ij}:

$$\digamma' = f'_{ij}\, a'_i b'_j = f_{mn}\, \underbrace{a_m}_{o_{i'm} a'_i} \underbrace{b_n}_{o_{j'n} b'_j} = \digamma \;\;\Leftrightarrow\;\; f'_{ij} = o_{i'm} o_{j'n} f_{mn}.$$

If $f_{ij} = \delta_{ij}$, then $\digamma = \delta_{ij}\, a_i b_j = a_i b_i$ — the "$\bullet$"-product (dot product, scalar product) of two vectors. When both arguments are the same, such a homogeneous polynomial of second degree (quadratic) of one vector's components $\digamma(\boldsymbol{a}, \boldsymbol{a}) = f_{ij}\, a_i a_j$ is called a quadratic form.

Now about more complex tensors (of valence larger than two). Tensor of third complexity ${}^3\boldsymbol{C}$ is represented by a collection of $3^3 = 27$ numbers C_{ijk}, changing with a rotation of basis as

$$C'_{ijk} = \boldsymbol{e}'_i \bullet \boldsymbol{e}_p \boldsymbol{e}'_j \bullet \boldsymbol{e}_q \boldsymbol{e}'_k \bullet \boldsymbol{e}_r C_{pqr} = o_{i'p} o_{j'q} o_{k'r} C_{pqr}. \tag{3.2}$$

The primary example is a triad of three vectors $\boldsymbol{a} = a_i \boldsymbol{e}_i$, $\boldsymbol{b} = b_j \boldsymbol{e}_j$ and $\boldsymbol{c} = c_k \boldsymbol{e}_k$

$$t_{ijk} \equiv a_i b_j c_k \;\Leftrightarrow\; {}^3\boldsymbol{t} = \boldsymbol{abc}.$$

It is seen that orthogonal transformations (3.2) and (3.1) are results of "repeating" vector's (2.4). The reader will easily compose a transformation of components for tensor of any complexity and will write a corresponding polyad as an example.

Vectors with transformation (2.4) are tensors of the first complexity (monovalent tensors).

The least complex objects are scalars or tensors of the zeroth complexity. A scalar is a single ($3^0 = 1$) number, which doesn't depend on a basis: the energy, the mass, the temperature et al. But what are components, for example, of vector $\boldsymbol{v} = v_i \boldsymbol{e}_i$, $v_i = \boldsymbol{v} \bullet \boldsymbol{e}_i$? If not scalars, then what? Here could be no simple answer. In each particular basis, $\boldsymbol{e}_i$ are vectors and v_i are scalars.

§ 4. Tensor algebra, or operations with tensors

The whole tensor algebra can be built on the only five* operations (or actions). This section is just about them.

Equality

The first (or the zeroth) is **the equality** "=". This operation shows whether one tensor "on the left" is equal to another tensor "on the right". Tensors can be equal only when their complexities (valencies) are the same. Tensors of different valencies cannot be equal or not equal.

$$\ldots\ldots \tag{4.1}$$

....

* The four without the equality.

Linear combination

The next operation is **the linear combination**. It aggregates the addition and the multiplication by a number (by a scalar, or, in another word, scaling). The arguments of this operation and the result are of the same complexity. For a pair of tensors

$$\lambda a_{ij\ldots} + \mu b_{ij\ldots} = c_{ij\ldots} \;\Leftrightarrow\; \lambda \boldsymbol{a} + \mu \boldsymbol{b} = \boldsymbol{c}. \tag{4.2}$$

Here λ and μ are scalar coefficients; $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are tensors of the same complexity. It's easy to show that the components of the result $\boldsymbol{c}$ satisfy an orthogonal transformation like (3.1).

The decomposition of a vector by some basis, that is the representation of a vector as the sum $\boldsymbol{v} = v_i \boldsymbol{e}_i$, is nothing else but the linear combination of the basis vectors $\boldsymbol{e}_i$ with the coefficients v_i.

This operation is *linear* because the only two atomary kinds of motion are possible on a line: the translation (the movement along a straight line) and the reflexion (mirroring) (the backward movement).

Multiplication of tensors

One more operation — **the multiplication (the tensor product, the direct product)**. It takes arguments of any complexities, returning the result of the cumulative complexity. Examples:

$$\begin{aligned} v_i a_{jk} = C_{ijk} \;&\Leftrightarrow\; \boldsymbol{v}\,{}^2\boldsymbol{a} = {}^3\boldsymbol{C}, \\ a_{ij} B_{abc} = D_{ijabc} \;&\Leftrightarrow\; {}^2\boldsymbol{a}\,{}^3\boldsymbol{B} = {}^5\boldsymbol{D}. \end{aligned} \tag{4.3}$$

Transformation of a collection of result's components, such as $C_{ijk} = v_i a_{jk}$, during a rotation of basis is orthogonal, similar to (3.2), thus here's no doubt that such a collection is a set of tensor components.

The primary and already known (from § 3) subtype of multiplication is the dyadic product of two vectors ${}^2\boldsymbol{A} = \boldsymbol{bc}$.

Contraction

The fourth (or the third) operation is called **the contraction**. It applies to bivalent and more complex tensors. This operation acts upon a single tensor, without other “participants”. Roughly speaking, contracting a tensor is summing of its components over some pair of indices. As a result, the tensor’s complexity decreases by two.

For a trivalent tensor ${}^3\boldsymbol{D}$ there are the three possible contractions. They give vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ with components

$$a_i = D_{kki}\,, \quad b_i = D_{kik}\,, \quad c_i = D_{ikk}\,. \tag{4.4}$$

A rotation of basis

$$a'_i = D'_{kki} = \underbrace{o_{k'p}\,o_{k'q}}_{\delta_{pq}}\,o_{i'r}\,D_{pqr} = o_{i'r}\,D_{ppr} = o_{i'r}\,a_r$$

shows “the tensorial nature” of the result of contraction.

For a tensor of second complexity, the only one kind of contraction is possible. It gives a scalar, known as *“trace”*

$$\boldsymbol{B}_{\bullet} \equiv \operatorname{trace}\boldsymbol{B} \equiv \mathrm{I}(\boldsymbol{B}) = B_{kk}\,.$$

The trace of the unit tensor (“contraction of the Kronecker delta”) is equal to the dimension of space

$$\operatorname{trace}\boldsymbol{E} = \boldsymbol{E}_{\bullet} = \delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

Index juggling, transposing

The last operation is applicable to a single tensor of the second* and bigger complexities. It is named as **the index swap, index juggling, transposing**. From components of a tensor, the new collection emerges with another sequence of indices, and the result’s complexity stays the same. For example, a trivalent tensor ${}^3\boldsymbol{D}$ can give tensors ${}^3\boldsymbol{A}$, ${}^3\boldsymbol{B}$, ${}^3\boldsymbol{C}$ with components

$$\begin{aligned} {}^3\boldsymbol{A} &= {}^3\boldsymbol{D}_{1\rightleftarrows 2} \Leftrightarrow A_{ijk} = D_{jik}\,, \\ {}^3\boldsymbol{B} &= {}^3\boldsymbol{D}_{1\rightleftarrows 3} \Leftrightarrow B_{ijk} = D_{kji}\,, \\ {}^3\boldsymbol{C} &= {}^3\boldsymbol{D}_{2\rightleftarrows 3} \Leftrightarrow C_{ijk} = D_{ikj}\,. \end{aligned} \tag{4.5}$$

* Transposing a vector makes no sense.

For a bivalent tensor, the only one transposition is possible: $\boldsymbol{A}^{\mathsf{T}} \equiv \boldsymbol{A}_{1\rightleftarrows 2} = \boldsymbol{B} \Leftrightarrow B_{ij} = A_{ji}$. Obviously, $\left(\boldsymbol{A}^{\mathsf{T}}\right)^{\mathsf{T}} = \boldsymbol{A}$.

For the dyadic product of two vectors, $\boldsymbol{ab} = \boldsymbol{ba}^{\mathsf{T}}$.

Combining operations

The four presented algebraic operations (actions) can be combined in various sequences.

The combination of multiplication (4.3) and contraction (4.4) — the "•"-product (dot product) — is the most frequently used. In the direct indexless notation this is denoted by the large dot "•", which shows the contraction by adjacent indices:

$$\boldsymbol{a} = \boldsymbol{B} \cdot \boldsymbol{c} \Leftrightarrow a_i = B_{ij} c_j, \;\; \boldsymbol{A} = \boldsymbol{B} \cdot \boldsymbol{C} \Leftrightarrow A_{ij} = B_{ik} C_{kj}. \tag{4.6}$$

The defining property of the unit tensor — the neutrality (it is the "identity element") for the "•"-product (the tensor product with the subsequent contraction by adjacent indices)

$$^{\mathrm{n}}\boldsymbol{a} \cdot \boldsymbol{E} = \boldsymbol{E} \cdot {}^{\mathrm{n}}\boldsymbol{a} = {}^{\mathrm{n}}\boldsymbol{a} \;\; \forall\, {}^{\mathrm{n}}\boldsymbol{a} \;\; \forall \mathrm{n} > 0. \tag{4.7}$$

In the (commutative) scalar product of two vectors, the dot represents the same: the dyadic product and the subsequent contraction

$$\boldsymbol{a} \cdot \boldsymbol{b} = (\boldsymbol{ab})_{\bullet} = a_i b_i = b_i a_i = (\boldsymbol{ba})_{\bullet} = \boldsymbol{b} \cdot \boldsymbol{a}. \tag{4.8}$$

And here's how the multipliers of the "•"-product (dot product) of two second complexity tensors are swapped

$$\begin{aligned} \boldsymbol{B} \cdot \boldsymbol{Q} &= \left(\boldsymbol{Q}^{\mathsf{T}} \cdot \boldsymbol{B}^{\mathsf{T}}\right)^{\mathsf{T}} \\ \left(\boldsymbol{B} \cdot \boldsymbol{Q}\right)^{\mathsf{T}} &= \boldsymbol{Q}^{\mathsf{T}} \cdot \boldsymbol{B}^{\mathsf{T}}. \end{aligned} \tag{4.9}$$

For two dyads $\boldsymbol{B} = \boldsymbol{bd}$ and $\boldsymbol{Q} = \boldsymbol{pq}$

$$\begin{aligned} \left(\boldsymbol{bd} \cdot \boldsymbol{pq}\right)^{\mathsf{T}} &= \boldsymbol{pq}^{\mathsf{T}} \cdot \boldsymbol{bd}^{\mathsf{T}} \\ d_i p_i \, \boldsymbol{bq}^{\mathsf{T}} &= \boldsymbol{qp} \cdot \boldsymbol{db} \\ d_i p_i \, \boldsymbol{qb} &= p_i d_i \, \boldsymbol{qb}. \end{aligned}$$

For a vector and a bivalent tensor

$$\boldsymbol{c} \cdot \boldsymbol{B} = \boldsymbol{B}^{\mathsf{T}} \cdot \boldsymbol{c}, \quad \boldsymbol{B} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{B}^{\mathsf{T}}. \tag{4.10}$$

Contraction can be repeated for two or more adjacent indices,

$$\big(\boldsymbol{A} \cdot \boldsymbol{B}\big)_{\bullet} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{B} = A_{ij} B_{ji} \,. \tag{4.11}$$

The double contraction of a bivalent tensor with the unit dyad gives the trace

$$\boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = \boldsymbol{E} \cdot\cdot\, \boldsymbol{A} = \boldsymbol{A}_{\bullet} = \operatorname{trace} \boldsymbol{A} = A_{jj} \,. \tag{4.12}$$

The commutativity is guaranteed

$$\boldsymbol{A} \cdot\cdot\, \boldsymbol{B} = A_{ij} B_{ji} = B_{ji} A_{ij} = \boldsymbol{B} \cdot\cdot\, \boldsymbol{A} \tag{4.13}$$

for any two bivalent tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, contracted twice.

In other texts, a double contraction may be written "vertically" as $\boldsymbol{A} \boldsymbol{:} \boldsymbol{B}$. $\boldsymbol{A} \boldsymbol{:} \boldsymbol{1} = \boldsymbol{1} \boldsymbol{:} \boldsymbol{A}$ is nothing but trace $\boldsymbol{A}$ or A_{ii} (4.12) in "$\boldsymbol{:}$"-notation and $\boldsymbol{1}$ for the unit tensor. To further confuse the reader, such a colon can denote either $\boldsymbol{A} \boldsymbol{:} \boldsymbol{B} \overset{(1)}{=} A_{ij} B_{ji}$ or $\boldsymbol{A} \boldsymbol{:} \boldsymbol{B} \overset{(2)}{=} A_{ij} B_{ij}$ with the additional transposition of one of the tensors. But don't worry, in this book you can meet the "$\boldsymbol{:}$"-product only in this paragraph, and when $\boldsymbol{B}$ is transposed, then it is $\boldsymbol{A} \cdot\cdot\, \boldsymbol{B}^{\mathsf{T}}$. Or $\boldsymbol{A}^{\mathsf{T}} \cdot\cdot\, \boldsymbol{B}$, because these are equal

$$\begin{aligned} \boldsymbol{A} \cdot\cdot\, \boldsymbol{B}^{\mathsf{T}} &= \boldsymbol{A}^{\mathsf{T}} \cdot\cdot\, \boldsymbol{B} = A_{ij} B_{ij} \,, \\ \boldsymbol{A} \cdot\cdot\, \boldsymbol{B} &= \boldsymbol{A}^{\mathsf{T}} \cdot\cdot\, \boldsymbol{B}^{\mathsf{T}} = A_{ij} B_{ji} \,. \end{aligned} \tag{4.14}$$

And as a bonus, here are more useful equalities for bivalent tensors

$$\begin{gathered} \boldsymbol{d} \cdot \boldsymbol{A} \cdot \boldsymbol{b} = d_i A_{ij} b_j = \boldsymbol{A} \cdot\cdot\, \boldsymbol{b}\boldsymbol{d} = \boldsymbol{b}\boldsymbol{d} \cdot\cdot\, \boldsymbol{A} = b_j d_i A_{ij} \,, \\ \boldsymbol{A} \cdot \boldsymbol{B} \cdot\cdot\, \boldsymbol{E} = A_{ij} B_{jk} \delta_{ki} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{B}, \;\; \boldsymbol{A} \cdot \boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{A}, \\ \boldsymbol{A} \cdot\cdot\, \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} \cdot\cdot\, \boldsymbol{C} = \boldsymbol{C} \cdot\cdot\, \boldsymbol{A} \cdot \boldsymbol{B} = A_{ij} B_{jk} C_{ki} \,, \\ \boldsymbol{A} \cdot\cdot\, \boldsymbol{B} \cdot \boldsymbol{C} \cdot \boldsymbol{D} = \boldsymbol{A} \cdot \boldsymbol{B} \cdot\cdot\, \boldsymbol{C} \cdot \boldsymbol{D} = \boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{C} \cdot\cdot\, \boldsymbol{D} \\ = \boldsymbol{D} \cdot\cdot\, \boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{C} = A_{ij} B_{jk} C_{kh} D_{hi} \,. \end{gathered} \tag{4.15}$$

§ 5. Polyadic representation (decomposition)

Before in § 3, a tensor was presented as some invariant object, revealing itself in every basis as a collection of numbers (components). Such a presentation is typical for the majority of books about tensors. The index notation can be convenient, especially when only rectangular coordinates are used, but quite often it is not. And the relevant case is *physics of continua, elastic and not very elastic* : it needs more elegant, more powerful and perfect apparatus of the direct tensor calculus, operating with indexless invariant objects.

The linear combination like $\boldsymbol{v} = v_i \boldsymbol{e}_i$ from (2.1) connects the vector $\boldsymbol{v}$ with the basis $\boldsymbol{e}_i$ and the vector's components v_i in that basis. Is there a similar relation for tensor of any complexity?

Any bivalent tensor ${}^2\boldsymbol{B}$ has nine (3^2) components B_{ij} in each basis. The number of different dyads $\boldsymbol{e}_i \boldsymbol{e}_j$ for the same basis is nine too. Linear combining these dyads with coefficients B_{ij} gives $B_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$. Yes, this is a tensor, like any linear combination of tensors. Yet what are its components, and how such a representation changes or doesn't change with a rotation of basis?

The components of combination

$$\big(B_{ij} \boldsymbol{e}_i \boldsymbol{e}_j\big)_{pq} \equiv B_{ij} \big(\boldsymbol{e}_i \cdot \boldsymbol{e}_p\big) \big(\boldsymbol{e}_j \cdot \boldsymbol{e}_q\big) = B_{ij} \delta_{ip} \delta_{jq} = B_{pq}$$

are the components of tensor ${}^2\boldsymbol{B}$. And with a rotation of basis

$$B'_{ij} \boldsymbol{e}'_i \boldsymbol{e}'_j = o_{i'p} o_{j'q} B_{pq} o_{i'n} \boldsymbol{e}_n o_{j'm} \boldsymbol{e}_m = \delta_{pn} \delta_{qm} B_{pq} \boldsymbol{e}_n \boldsymbol{e}_m = B_{pq} \boldsymbol{e}_p \boldsymbol{e}_q .$$

Doubts are dropped : a tensor of second complexity can be (re)presented as a linear combination

$${}^2\boldsymbol{B} = B_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \tag{5.1}$$

— the dyadic decomposition of a bivalent tensor.

For the unit tensor

$$\boldsymbol{E} = E_{ij} \, \boldsymbol{e}_i \boldsymbol{e}_j = \delta_{ij} \, \boldsymbol{e}_i \boldsymbol{e}_j = \boldsymbol{e}_i \boldsymbol{e}_i = \boldsymbol{e}_1 \boldsymbol{e}_1 + \boldsymbol{e}_2 \boldsymbol{e}_2 + \boldsymbol{e}_3 \boldsymbol{e}_3 ,$$

that's why $\boldsymbol{E}$ is called the unit dyad.

Using polyadic representations like (5.1), tensors are much easier to handle :

$$\boldsymbol{v} \cdot {}^2\boldsymbol{B} = v_i \boldsymbol{e}_i \cdot \boldsymbol{e}_j B_{jk} \boldsymbol{e}_k = v_i \delta_{ij} B_{jk} \boldsymbol{e}_k = v_i B_{ik} \boldsymbol{e}_k \,,$$

$$\boldsymbol{e}_i \cdot {}^2\boldsymbol{B} \cdot \boldsymbol{e}_j = \boldsymbol{e}_i \cdot B_{pq} \boldsymbol{e}_p \boldsymbol{e}_q \cdot \boldsymbol{e}_j = B_{pq} \delta_{ip} \delta_{qj} = B_{ij} = {}^2\boldsymbol{B} \cdot\cdot\, \boldsymbol{e}_j \boldsymbol{e}_i \,. \qquad (5.2)$$

The last line here is quite interesting : the tensor components are presented through the tensor itself. An orthogonal transformation of components with a rotation of basis (3.1) turns out to be just a version of (5.2).

And any tensor, of any complexity above zero, may be decomposed into the basis polyads. For a trivalent tensor

$$\begin{gathered} {}^3\boldsymbol{C} = C_{ijk} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k \,, \\ C_{ijk} = {}^3\boldsymbol{C} \cdot\cdot\cdot\, \boldsymbol{e}_k \boldsymbol{e}_j \boldsymbol{e}_i = \boldsymbol{e}_i \cdot {}^3\boldsymbol{C} \cdot\cdot\, \boldsymbol{e}_k \boldsymbol{e}_j = \boldsymbol{e}_j \boldsymbol{e}_i \cdot\cdot\, {}^3\boldsymbol{C} \cdot \boldsymbol{e}_k \,. \end{gathered} \qquad (5.3)$$

Now it's pretty easy to see the actuality of property (4.7) — the "unitness" of tensor $\boldsymbol{E}$:

$$\begin{gathered} {}^{\mathrm{n}}\boldsymbol{a} = a_{ij\ldots q}\, \boldsymbol{e}_i \boldsymbol{e}_j \ldots \boldsymbol{e}_q, \;\; \boldsymbol{E} = \boldsymbol{e}_e \boldsymbol{e}_e \\ {}^{\mathrm{n}}\boldsymbol{a} \cdot \boldsymbol{E} = a_{ij\ldots q}\, \boldsymbol{e}_i \boldsymbol{e}_j \ldots \underbrace{\boldsymbol{e}_q \cdot \boldsymbol{e}_e}_{\delta_{eq}} \boldsymbol{e}_e = a_{ij\ldots q}\, \boldsymbol{e}_i \boldsymbol{e}_j \ldots \boldsymbol{e}_q = {}^{\mathrm{n}}\boldsymbol{a}, \\ \boldsymbol{E} \cdot {}^{\mathrm{n}}\boldsymbol{a} = \boldsymbol{e}_e \boldsymbol{e}_e \cdot a_{ij\ldots q}\, \boldsymbol{e}_i \boldsymbol{e}_j \ldots \boldsymbol{e}_q = a_{ij\ldots q} \delta_{ei} \boldsymbol{e}_e \boldsymbol{e}_j \ldots \boldsymbol{e}_q = {}^{\mathrm{n}}\boldsymbol{a}. \end{gathered}$$

The polyadic representation links the direct and index notations together. It's not worth contraposing one another. The direct notation is compact, elegant, it much more than others suits for final relations. But, sometimes, the index notation is very convenient too, as it is for cumbersome manipulations with tensors.

§ 6. Matrices, permutations and determinants

Matrices are the convenient tool for arranging of elements and for solving systems of linear equations.

Does the reader know that matrices are sometimes called "arrays"? Does someone need two-dimensional arrays? Matrices can be presented as tables full of rows and columns. Any matrix has the same number of elements in each row and the same number of elements in each column. Rectangular arrangement of items, anyone? Matrices are full of numbers and expressions in rows and columns. A column arranges elements vertically from top to bottom, while a row arranges horizontally from left to right.

Matrix dimensions

Matrices come in all sizes, or "dimensions".

By convention, to avoid confusion, the rows are listed first, and the columns second. The dimension of a matrix is written as the number of rows, then a multiplication sign ("$\times$" is used the most often), and then the number of columns.

Here are examples

$$\underset{3\times 3}{\left[\mathcal{A}\right]} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

3 rows and 3 columns
a 3×3 matrix
it's a *"square"* matrix

the number of rows
is the same
as of columns

$$\underset{2\times 4}{\left[\mathcal{B}\right]} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \end{bmatrix}$$

2 rows and 4 columns
dimension 2×4

$$\underset{3\times 1}{\left[\mathcal{C}\right]} = \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix}$$

3 rows, 1 column
3×1
a *"column matrix"*

a matrix
with just one
column

$$\underset{1\times 6}{\left[\mathcal{D}\right]} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \end{bmatrix}$$

1 row, 6 columns
1×6
a *"row matrix"*

a matrix
with just one
row

The matrix algebra

The matrix algebra includes the linear operations — the addition of matrices and the multiplication by scalar.

The dimension of a matrix is essential for binary operations, that is for operations involving two matrices.

An addition or subtraction of the two matrices is possible only if they have the same sizes.

The multiplication of matrices

....

$$\underset{m\times n}{[\mathcal{A}]} = \ldots$$

The matrix of the result, known as “the matrix product”, has the number of rows of the first multiplier matrix and the number of columns of the second matrix.

....

Square matrices

....

Matrices and one-dimensional arrays

Two indices of a table are more than the single index of a one-dimensional array. Due to this, a one-dimensional array could be presented “vertically” or “horizontally”, either as a table of rows in one column

$$\begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix},$$

or as a table of columns in one row

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix}.$$

....

Permutation parity symbols

To write permutations, the “parity symbols” e_{ijk} are introduced. They are often associated with the names of Oswald Veblen and Tullio Levi-Civita.

...

the permutation parity symbols via the determinant

$$e_{pqr} = e_{ijk}\delta_{pi}\delta_{qj}\delta_{rk} = e_{ijk}\delta_{ip}\delta_{jq}\delta_{kr}\,,$$

$$e_{pqr} = \det \begin{bmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{bmatrix} = \det \begin{bmatrix} \delta_{p1} & \delta_{p2} & \delta_{p3} \\ \delta_{q1} & \delta_{q2} & \delta_{q3} \\ \delta_{r1} & \delta_{r2} & \delta_{r3} \end{bmatrix}. \tag{6.1}$$

....

A determinant is not sensitive to transposing,

$$\det_{i,j} A_{ij} = \det_{i,j} A_{ji} = \det_{j,i} A_{ij} .$$

....

The determinant of the matrix product of two matrices is equal to the product of the determinants of each of these matrices

$$\det_{i,k} B_{ik} \det_{k,j} C_{kj} = \det_{i,j} B_{ik} C_{kj} \tag{6.2}$$

$$e_{fgh} \det_{m,n} B_{ms} C_{sn} = e_{pqr} B_{fi} C_{ip} B_{gj} C_{jq} B_{hk} C_{kr}$$

$$e_{fgh} \det_{m,s} B_{ms} = e_{ijk} B_{fi} B_{gj} B_{hk}$$

$$e_{ijk} \det_{s,n} C_{sn} = e_{pqr} C_{ip} C_{jq} C_{kr}$$

$$e_{fgh} e_{ijk} \det_{m,s} B_{ms} \det_{s,n} C_{sn} = e_{ijk} e_{pqr} B_{fi} B_{gj} B_{hk} C_{ip} C_{jq} C_{kr}$$

...

There's the following equality

$$e_{ijk} e_{pqr} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix} \tag{6.3}$$

○ Representing the permutation parity symbols via determinants (6.1), e_{ijk} by rows and e_{pqr} by columns

$$e_{ijk} = \det \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}, \quad e_{pqr} = \det \begin{bmatrix} \delta_{p1} & \delta_{q1} & \delta_{r1} \\ \delta_{p2} & \delta_{q2} & \delta_{r2} \\ \delta_{p3} & \delta_{q3} & \delta_{r3} \end{bmatrix},$$

$e_{ijk}e_{pqr}$ on the left side of (6.3) appears as the product of these determinants. And then $\det(AB) = (\det A)(\det B)$ — the determinant of matrix product is equal to the product of determinants (6.2). In the product matrix, the item $[\cdots]_{11}$ equals $\delta_{is}\delta_{ps} = \delta_{ip}$, just like on the right side of (6.3) ; same for all the other items. ●

The contraction of (6.3) leads to the useful formulas

$$e_{ijk}e_{pqk} = \det\begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & \delta_{kk} \end{bmatrix} = \det\begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & 3 \end{bmatrix}$$

$$= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jk}\delta_{kp} + \delta_{ik}\delta_{jp}\delta_{kq} - \delta_{ik}\delta_{jq}\delta_{kp} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jk}\delta_{kq}$$
$$= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} + \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq}$$
$$= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\,,$$

$$e_{ijk}e_{pjk} = \delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip}\,,$$

$$e_{ijk}e_{ijk} = 2\delta_{ii} = 6$$

or in short

$$e_{ijk}e_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\,,\quad e_{ijk}e_{pjk} = 2\delta_{ip}\,,\quad e_{ijk}e_{ijk} = 6\,. \tag{6.4}$$

....

$\det\limits_{i,j} \delta_{ij} = 3$

.....

The determinant of components of a bivalent tensor is invariant, it doesn't change with a rotation of the basis

$$A'_{ij} = o_{i'm}o_{j'n}A_{mn}$$

....

§ 7. The cross product

By common notions, the "×"-product (the "cross product", the "vector product", sometimes the "oriented area product") of two vectors is the vector, heading perpendicular to the plane of multipliers, whose length is equal to the area of the parallelogram, spanned by the multipliers

$$\| \boldsymbol{a} \times \boldsymbol{b} \| = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \sin \measuredangle(\boldsymbol{a}, \boldsymbol{b}).$$

However, a "×"-product isn't quite a vector, since it is not completely invariant.

The multipliers of the "×"-product $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ determine the result's direction in space, with an accuracy up to the sign figure 3.

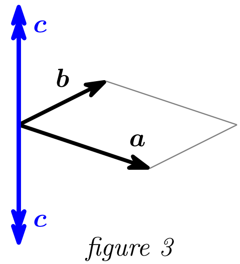


figure 3

Once you pick as the positive the " right-chiral " ("right-handed") or the " left-chiral " ("left-handed") orientation of space, the one direction from the possible two, then the results of the "× "-products become completely determined.

"The chiral" means asymmetric in such a way that the thing and its mirror image are not superimposable, a picture cannot be superposed on its mirror image by any combination of rotations and translations.

An object is chiral if it is distinguishable from its mirror image.

Vectors are usually measured via some basis $\boldsymbol{e}_i$. They are decomposed into linear combinations like $\boldsymbol{a} = a_i \boldsymbol{e}_i$. So the orientation of space is equivalent to the orientation of the sequential triple of basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$. It means that the sequence of basis vectors becomes significant (for linear combinations, the sequence of addends doesn't affect anything).

If two bases consist of different sequences of the same vectors, then orientations of these bases differ by some permutation.

The orientation of the space is a (kind of) asymmetry. This asymmetry makes it impossible to replicate mirroring by the means of any rotations*

* Applying only rotations, it's impossible to replace the left hand with the right hand. But it is possible by mirroring.

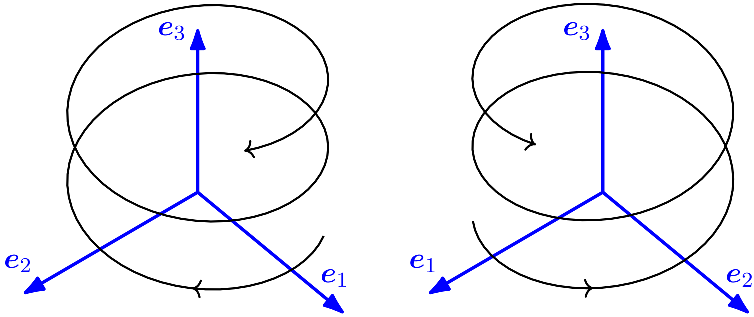


figure 4

A pseudovector is a vector-like object, invariant under any rotation. *.

... put the figure here ...

Except on rare cases, mirroring changes the direction of a fully invariant (polar) vector.

A pseudovector (an axial vector), unlike a polar vector, doesn't change the component that is perpendicular to the mirroring plane, and turns out to be flipped relatively to the polar vectors and the geometry of the entire space. This happens because the sign (and, accordingly, the direction) of each axial vector changes along with changing the sign of the "×"-product — which corresponds to mirroring.

The otherness of pseudovectors narrows the variety of formulas: a pseudovector is not additive with a vector. The formula $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}$ from an absolutely rigid undeformable body's kinematics is correct, because $\boldsymbol{\omega}$ is pseudovector there, and with the cross product the two "pseudo" give $(-1)^2 = 1$, mutually compensating each other.

....

$e_{ijk} = \pm\, \boldsymbol{e}_i \times \boldsymbol{e}_j \bullet \boldsymbol{e}_k$

$e_{pqr} = \pm\, \boldsymbol{e}_p \times \boldsymbol{e}_q \bullet \boldsymbol{e}_r$

* Rotations cannot change the orientation of a triple of basis vectors, it is possible only via mirroring.

with "−" for "left" triple

...

The permutations parity tensor is the volumetric tensor of third complexity

$$ {}^3\boldsymbol{\epsilon} = \in_{ijk} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k, \;\; \in_{ijk} \equiv \boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k \tag{7.1} $$

with the components $\in_{ijk}$ equal to the "triple" (the "mixed", the "cross-dot") products of the basis vectors.

The absolute value (the modulus) of each nonzero component of ${}^3\boldsymbol{\epsilon}$ is equal to the volume $\sqrt{g}$ of a parallelopiped drew upon a basis. For a basis $\boldsymbol{e}_i$ of pairwise perpendicular one unit long vectors $\sqrt{g} = 1$.

The tensor ${}^3\boldsymbol{\epsilon}$ is isotropic, its components are constant and independent of any rotations of a basis. But mirroring — a change in the orientation of a triple of basis vectors (a change in "the direction of screw") — changes the sign of ${}^3\boldsymbol{\epsilon}$, so this is a pseudotensor (an axial tensor).

If $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$ without the "minus" sign, then the basis triple $\boldsymbol{e}_i$ is oriented positively. The positive orientation (or "the positive direction") is chosen for different reasons from the two possible ones (figure 3). For a positively oriented basis triplet, the components of ${}^3\boldsymbol{\epsilon}$ are equal to the permutation parity symbols $\in_{ijk} = e_{ijk}$. And when $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = -\boldsymbol{e}_3$, then the basis triple $\boldsymbol{e}_i$ is oriented negatively, or "mirrored". For mirrored triples $\in_{ijk} = -e_{ijk}$ (and $e_{ijk} = -\boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k$).

With the permutations parity tensor ${}^3\boldsymbol{\epsilon}$ it's possible to take the fresh look at the cross "×"-product :

$$ \in_{ijk} = \boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k \;\Leftrightarrow\; \boldsymbol{e}_i \times \boldsymbol{e}_j = \in_{ijk} \boldsymbol{e}_k, $$

$$ \begin{aligned} \boldsymbol{a} \times \boldsymbol{b} = a_i \boldsymbol{e}_i \times b_j \boldsymbol{e}_j = a_i b_j \boldsymbol{e}_i \times \boldsymbol{e}_j = a_i b_j \in_{ijk} \boldsymbol{e}_k = \\ = b_j a_i \boldsymbol{e}_j \boldsymbol{e}_i \cdot\cdot \in_{mnk} \boldsymbol{e}_m \boldsymbol{e}_n \boldsymbol{e}_k = \boldsymbol{b}\boldsymbol{a} \cdot\cdot {}^3\boldsymbol{\epsilon}, \\ = a_i \in_{ijk} \boldsymbol{e}_k b_j = -a_i \in_{ikj} \boldsymbol{e}_k b_j = -\boldsymbol{a} \cdot {}^3\boldsymbol{\epsilon} \cdot \boldsymbol{b}. \end{aligned} \tag{7.2} $$

So that, the cross product is not another new, entirely distinct operation. With the permutations parity tensor it reduces to the four already described (§ 4) and is applicable to tensors of any complexity.

"The cross product" is just the dot product — the combination of multiplication and contraction (§ 4) — involving tensor ${}^3\boldsymbol{\epsilon}$. Such combinations are possible with any tensors :

$$\boldsymbol{a} \times {}^2\boldsymbol{B} = a_i \boldsymbol{e}_i \times B_{jk} \boldsymbol{e}_j \boldsymbol{e}_k = \underbrace{a_i B_{jk} \in_{ijn}}_{-a_i \in_{inj} B_{jk}} \boldsymbol{e}_n \boldsymbol{e}_k = -\boldsymbol{a} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^2\boldsymbol{B},$$

$$\begin{aligned} {}^2\boldsymbol{C} \times \boldsymbol{d}\boldsymbol{b} = C_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \times d_p b_q \boldsymbol{e}_p \boldsymbol{e}_q = \boldsymbol{e}_i C_{ij} d_p \underbrace{\in_{jpk}}_{-\in_{pjk} = -\in_{jkp}} \boldsymbol{e}_k b_q \boldsymbol{e}_q = \\ = -{}^2\boldsymbol{C}\boldsymbol{d} \cdot\cdot {}^3\boldsymbol{\epsilon}\boldsymbol{b} = -{}^2\boldsymbol{C} \cdot {}^3\boldsymbol{\epsilon} \cdot \boldsymbol{d}\boldsymbol{b}, \end{aligned}$$

$$\boldsymbol{E} \times \boldsymbol{E} = \boldsymbol{e}_i \boldsymbol{e}_i \times \boldsymbol{e}_j \boldsymbol{e}_j = \underbrace{- \in_{ijk} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k}_{+\in_{ijk} \boldsymbol{e}_i \boldsymbol{e}_k \boldsymbol{e}_j} = -{}^3\boldsymbol{\epsilon}. \tag{7.3}$$

The last equation links the isotropic tensor of the second complexity and the isotropic tensor of the third complexity.

Generalizing to all tensors of nonzero complexity

$${}^{\mathrm{n}}\boldsymbol{\xi} \times {}^{\mathrm{m}}\boldsymbol{\zeta} = -{}^{\mathrm{n}}\boldsymbol{\xi} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^{\mathrm{m}}\boldsymbol{\zeta} \quad \forall\, {}^{\mathrm{n}}\boldsymbol{\xi}, {}^{\mathrm{m}}\boldsymbol{\zeta} \;\; \forall n > 0,\, m > 0. \tag{7.4}$$

When one of the operands is the unit (metric) tensor, from (7.4) and (4.7) for $\forall\, {}^{\mathrm{n}}\boldsymbol{\Upsilon}\ \forall \mathrm{n} > 0$

$$\begin{aligned} \boldsymbol{E} \times {}^{\mathrm{n}}\boldsymbol{\Upsilon} = -\boldsymbol{E} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^{\mathrm{n}}\boldsymbol{\Upsilon} = -{}^3\boldsymbol{\epsilon} \cdot {}^{\mathrm{n}}\boldsymbol{\Upsilon}, \\ {}^{\mathrm{n}}\boldsymbol{\Upsilon} \times \boldsymbol{E} = -{}^{\mathrm{n}}\boldsymbol{\Upsilon} \cdot {}^3\boldsymbol{\epsilon} \cdot \boldsymbol{E} = -{}^{\mathrm{n}}\boldsymbol{\Upsilon} \cdot {}^3\boldsymbol{\epsilon}. \end{aligned} \tag{7.5}$$

The cross product of two vectors is not commutative, but is anti-commutative :

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{E}) = (\boldsymbol{a} \times \boldsymbol{E}) \cdot \boldsymbol{b} = \overbrace{-\ \boldsymbol{a}\boldsymbol{b} \cdot\cdot {}^3\boldsymbol{\epsilon} = -\ {}^3\boldsymbol{\epsilon} \cdot\cdot \boldsymbol{a}\boldsymbol{b}}^{\in_{jik} = \in_{kji}\, \Rightarrow\, a_i b_j \in_{jik} \boldsymbol{e}_k = \in_{kji} a_i b_j \boldsymbol{e}_k}, \\ \boldsymbol{b} \times \boldsymbol{a} = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{E}) = (\boldsymbol{b} \times \boldsymbol{E}) \cdot \boldsymbol{a} = -\boldsymbol{b}\boldsymbol{a} \cdot\cdot {}^3\boldsymbol{\epsilon} = -{}^3\boldsymbol{\epsilon} \cdot\cdot \boldsymbol{b}\boldsymbol{a}, \\ \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{a}\boldsymbol{b} \cdot\cdot {}^3\boldsymbol{\epsilon} = \boldsymbol{b}\boldsymbol{a} \cdot\cdot {}^3\boldsymbol{\epsilon} \;\; \Rightarrow \;\; \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}. \end{aligned} \tag{7.6}$$

For any bivalent tensor ${}^2\boldsymbol{B}$ and any vector $\boldsymbol{a}$

$$\begin{aligned} {}^2\boldsymbol{B} \times \boldsymbol{a} = \boldsymbol{e}_i B_{ij} \boldsymbol{e}_j \times a_k \boldsymbol{e}_k = -\boldsymbol{e}_i B_{ij}\, a_k \boldsymbol{e}_k \times \boldsymbol{e}_j \\ = \big(-a_k \boldsymbol{e}_k \times \boldsymbol{e}_j B_{ij} \boldsymbol{e}_i\big)^{\mathsf{T}} = -\big(\boldsymbol{a} \times {}^2\boldsymbol{B}^{\mathsf{T}}\big)^{\mathsf{T}}, \end{aligned} \tag{7.7}$$

and only for the unit dyad and a vector, the "$\times$"-product is commutative

$$-\boldsymbol{E}\times\boldsymbol{a}=\overbrace{{}^{3}\boldsymbol{\epsilon}\cdot\boldsymbol{a}=\boldsymbol{a}\cdot{}^{3}\boldsymbol{\epsilon}}^{\in_{ijk}=\in_{kij}\;\Rightarrow\;\in_{ijk}a_k\boldsymbol{e}_i\boldsymbol{e}_j=a_k\in_{kij}\boldsymbol{e}_i\boldsymbol{e}_j}=-\boldsymbol{a}\times\boldsymbol{E}, \tag{7.8}$$

plus as the particular case of (7.7)

$$\boldsymbol{E}\times\boldsymbol{a}=-\big(\boldsymbol{a}\times\boldsymbol{E}^{\mathsf{T}}\big)^{\mathsf{T}}=-\big(\boldsymbol{a}\times\boldsymbol{E}\big)^{\mathsf{T}}. \tag{7.9}$$

The first of (6.4) formulas gives the following representation for the double "$\times$"-product

$$\begin{aligned}\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})&=a_i\boldsymbol{e}_i\times\in_{pqj}b_pc_q\boldsymbol{e}_j=\in_{kij}\in_{pqj}a_ib_pc_q\boldsymbol{e}_k=\\&=(\delta_{kp}\delta_{iq}-\delta_{kq}\delta_{ip})\,a_ib_pc_q\boldsymbol{e}_k=a_ib_kc_i\boldsymbol{e}_k-a_ib_ic_k\boldsymbol{e}_k=\\&=\boldsymbol{a}\cdot\boldsymbol{c}\boldsymbol{b}-\boldsymbol{a}\cdot\boldsymbol{b}\boldsymbol{c}=\boldsymbol{a}\cdot\big(\boldsymbol{c}\boldsymbol{b}-\boldsymbol{b}\boldsymbol{c}\big)=\boldsymbol{a}\cdot\boldsymbol{c}\boldsymbol{b}-\boldsymbol{c}\boldsymbol{b}\cdot\boldsymbol{a}.\end{aligned} \tag{7.10}$$

By another interpretation, the dot product of a dyad and a vector is not commutative : $\boldsymbol{bd}\cdot\boldsymbol{c}\neq\boldsymbol{c}\cdot\boldsymbol{bd}$, and the difference can be rendered as

$$\boldsymbol{bd}\cdot\boldsymbol{c}-\boldsymbol{c}\cdot\boldsymbol{bd}=\boldsymbol{c}\times\big(\boldsymbol{b}\times\boldsymbol{d}\big). \tag{7.11}$$

$\boldsymbol{a}\cdot\boldsymbol{bc}=\boldsymbol{cb}\cdot\boldsymbol{a}=\boldsymbol{ca}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{ac}$
$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=-\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{c}\times(\boldsymbol{b}\times\boldsymbol{a})$
The same way it may be derived that

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\big(\boldsymbol{ba}-\boldsymbol{ab}\big)\cdot\boldsymbol{c}=\boldsymbol{ba}\cdot\boldsymbol{c}-\boldsymbol{ab}\cdot\boldsymbol{c}. \tag{7.12}$$

And the following identities for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\begin{aligned}\big(\boldsymbol{a}\times\boldsymbol{b}\big)\times\boldsymbol{E}&=\in_{ijk}a_ib_j\boldsymbol{e}_k\times\boldsymbol{e}_n\boldsymbol{e}_n=a_ib_j\in_{ijk}\in_{knq}\boldsymbol{e}_q\boldsymbol{e}_n=\\&=a_ib_j\big(\delta_{in}\delta_{jq}-\delta_{iq}\delta_{jn}\big)\boldsymbol{e}_q\boldsymbol{e}_n=a_ib_j\boldsymbol{e}_j\boldsymbol{e}_i-a_ib_j\boldsymbol{e}_i\boldsymbol{e}_j=\\&=\boldsymbol{ba}-\boldsymbol{ab},\end{aligned} \tag{7.13}$$

$$\begin{aligned}\big(\boldsymbol{a}\times\boldsymbol{E}\big)\cdot\big(\boldsymbol{b}\times\boldsymbol{E}\big)&=\big(\boldsymbol{a}\cdot{}^{3}\boldsymbol{\epsilon}\big)\cdot\big(\boldsymbol{b}\cdot{}^{3}\boldsymbol{\epsilon}\big)=\\&=a_i\in_{ipn}\boldsymbol{e}_p\boldsymbol{e}_n\cdot b_j\in_{jsk}\boldsymbol{e}_s\boldsymbol{e}_k=a_ib_j\in_{ipn}\in_{nkj}\boldsymbol{e}_p\boldsymbol{e}_k=\\&=a_ib_j\big(\delta_{ik}\delta_{pj}-\delta_{ij}\delta_{pk}\big)\boldsymbol{e}_p\boldsymbol{e}_k=a_ib_j\boldsymbol{e}_j\boldsymbol{e}_i-a_ib_i\boldsymbol{e}_k\boldsymbol{e}_k=\\&=\boldsymbol{ba}-\boldsymbol{a}\cdot\boldsymbol{b}\boldsymbol{E}.\end{aligned} \tag{7.14}$$

Finally, the direct relation between the isotropic tensors of the second and third complexities

$$ {}^3\!\boldsymbol{\epsilon} \cdot\cdot\, {}^3\!\boldsymbol{\epsilon} = \boldsymbol{e}_i \in_{ijk} \in_{kjn} \boldsymbol{e}_n = -2\,\delta_{in}\, \boldsymbol{e}_i \boldsymbol{e}_n = -2\boldsymbol{E}. \tag{7.15}$$

§ 8. Symmetric and skewsymmetric tensors

A tensor that does not change upon a permutation of some pair of its indices is called symmetric for that pair of indices. And when a permutation of some pair of indices alternates the sign "+/−" of a tensor, then this tensor is called anti-symmetric or skew-symmetric for that pair of indices.

As example, the tensor of parity of permutations ${}^3\!\boldsymbol{\epsilon}$ (7.1) is antisymmetric for any & every pair of indices, it is completely (absolutely) skewsymmetric.

Tensor of the second complexity $\boldsymbol{B}$ is symmetric when $\boldsymbol{B} = \boldsymbol{B}^{\mathsf{T}}$. If transposing changes the tensor's sign, that is $\boldsymbol{A}^{\mathsf{T}} = -\boldsymbol{A}$, then tensor $\boldsymbol{A}$ is skewsymmetric (antisymmetric).

The sum of a bivalent tensor $\boldsymbol{C}$ with the transpose $\boldsymbol{C}^{\mathsf{T}}$ is always symmetric : $\bigl(\boldsymbol{C} + \boldsymbol{C}^{\mathsf{T}}\bigr)^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{T}} + \boldsymbol{C} = \boldsymbol{C} + \boldsymbol{C}^{\mathsf{T}}\ \forall \boldsymbol{C}$, while the difference $\bigl(\boldsymbol{C} - \boldsymbol{C}^{\mathsf{T}}\bigr)^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{T}} - \boldsymbol{C} = -\bigl(\boldsymbol{C} - \boldsymbol{C}^{\mathsf{T}}\bigr)$ is always $\forall \boldsymbol{C}$ antisymmetric.

Denoting

$$\boldsymbol{C}^{\mathsf{S}} \equiv \tfrac{1}{2}\bigl(\boldsymbol{C} + \boldsymbol{C}^{\mathsf{T}}\bigr),\ \ \boldsymbol{C}^{\mathsf{A}} \equiv \tfrac{1}{2}\bigl(\boldsymbol{C} - \boldsymbol{C}^{\mathsf{T}}\bigr) \tag{8.1}$$

— the symmetric $\boldsymbol{C}^{\mathsf{S}}$ and the antisymmetric $\boldsymbol{C}^{\mathsf{A}}$ parts of some bivalent tensor $\boldsymbol{C}$, any bivalent tensor can be presented as the sum of these parts

$$\boldsymbol{C} = \boldsymbol{C}^{\mathsf{S}} + \boldsymbol{C}^{\mathsf{A}},\ \ \boldsymbol{C}^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{S}} - \boldsymbol{C}^{\mathsf{A}}. \tag{8.2}$$

For a dyad

$$\boldsymbol{cd} = \overbrace{\frac{1}{2}\left(\boldsymbol{cd} + \boldsymbol{dc}\right)}^{\boldsymbol{cd}^{\mathsf{S}}} + \overbrace{\frac{1}{2}\left(\boldsymbol{cd} - \boldsymbol{dc}\right)}^{\boldsymbol{cd}^{\mathsf{A}}}.$$

> The product $\boldsymbol{C}^{\mathsf{S}} \cdot \boldsymbol{D}^{\mathsf{S}}$ of two symmetric tensors $\boldsymbol{C}^{\mathsf{S}}$ and $\boldsymbol{D}^{\mathsf{S}}$ is symmetric not always, but only when $\boldsymbol{D}^{\mathsf{S}} \cdot \boldsymbol{C}^{\mathsf{S}} = \boldsymbol{C}^{\mathsf{S}} \cdot \boldsymbol{D}^{\mathsf{S}}$, because by (4.9) $\bigl(\boldsymbol{C}^{\mathsf{S}} \cdot \boldsymbol{D}^{\mathsf{S}}\bigr)^{\mathsf{T}} = \boldsymbol{D}^{\mathsf{S}} \cdot \boldsymbol{C}^{\mathsf{S}}$.

With (7.8) and (7.9), the skew symmetry of the "×"-product for the unit dyad and a vector is obvious

$$
\begin{aligned}
\big(\boldsymbol{E} \times \boldsymbol{a}\big)^{\mathsf{T}} = \big(\boldsymbol{e}_j \boldsymbol{e}_j \times a_i \boldsymbol{e}_i\big)^{\mathsf{T}} = \big(-\boldsymbol{e}_j a_i \boldsymbol{e}_i \times \boldsymbol{e}_j\big)^{\mathsf{T}} = -a_i \boldsymbol{e}_i \times \boldsymbol{e}_j \boldsymbol{e}_j \\
= -\boldsymbol{a} \times \boldsymbol{E} = -\boldsymbol{E} \times \boldsymbol{a} = \big(\boldsymbol{a} \times \boldsymbol{E}\big)^{\mathsf{T}}.
\end{aligned}
\tag{8.3}
$$

In search for a case when a bivalent tensor $\boldsymbol{A}$ can be represented by just a single vector $\boldsymbol{a}$, in such a way that an action of vector $\boldsymbol{a}$ on other objects is exactly like an action of bivalent $\boldsymbol{A}$ on the same objects, perhaps there's a chance to find such $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{a})$ that for $\forall^{\mathrm{n}}\boldsymbol{\xi}\ \forall \mathrm{n} > 0$

$$
\begin{aligned}
\boldsymbol{b} = \boldsymbol{A} \cdot {}^{\mathrm{n}}\boldsymbol{\xi} \ &\Leftrightarrow\ \boldsymbol{a} \times {}^{\mathrm{n}}\boldsymbol{\xi} = \boldsymbol{b} \quad \forall \boldsymbol{b}, \\
\boldsymbol{d} = {}^{\mathrm{n}}\boldsymbol{\xi} \cdot \boldsymbol{A} \ &\Leftrightarrow\ {}^{\mathrm{n}}\boldsymbol{\xi} \times \boldsymbol{a} = \boldsymbol{d} \quad \forall \boldsymbol{d}
\end{aligned}
$$

or, in words, the "•"-product of bivalent $\boldsymbol{A}$ and some other tensor ${}^{\mathrm{n}}\boldsymbol{\xi}$ is equal to the "×"-product of pseudovector $\boldsymbol{a}$ and the same tensor ${}^{\mathrm{n}}\boldsymbol{\xi}$.

The relation $\boldsymbol{a} \mapsto \boldsymbol{A}$ can be derived from (4.7) and (7.5)

$$
\begin{aligned}
\boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{E} = \boldsymbol{a} \times \boldsymbol{E} = -\boldsymbol{a} \cdot {}^{3}\boldsymbol{\epsilon}, \\
\boldsymbol{A} = \boldsymbol{E} \cdot \boldsymbol{A} = \boldsymbol{E} \times \boldsymbol{a} = -{}^{3}\boldsymbol{\epsilon} \cdot \boldsymbol{a}.
\end{aligned}
\tag{8.4}
$$

Or, delving into components,

$$
\begin{aligned}
\boldsymbol{A} \cdot {}^{\mathrm{n}}\boldsymbol{\xi} &= \boldsymbol{a} \times {}^{\mathrm{n}}\boldsymbol{\xi} \\
A_{hi} \boldsymbol{e}_h \boldsymbol{e}_i \cdot \xi_{jk\ldots q} \boldsymbol{e}_j \boldsymbol{e}_k \ldots \boldsymbol{e}_q &= a_i \boldsymbol{e}_i \times \xi_{jk\ldots q} \boldsymbol{e}_j \boldsymbol{e}_k \ldots \boldsymbol{e}_q \\
A_{hj} \xi_{jk\ldots q} \boldsymbol{e}_h \boldsymbol{e}_k \ldots \boldsymbol{e}_q &= a_i \in_{ijh} \xi_{jk\ldots q} \boldsymbol{e}_h \boldsymbol{e}_k \ldots \boldsymbol{e}_q \\
A_{hj} &= a_i \in_{ijh} \\
A_{hj} &= -a_i \in_{ihj} \\
\boldsymbol{A} &= -\boldsymbol{a} \cdot {}^{3}\boldsymbol{\epsilon}
\end{aligned}
$$

and by similar way from ${}^{\mathrm{n}}\boldsymbol{\xi} \cdot \boldsymbol{A} = {}^{\mathrm{n}}\boldsymbol{\xi} \times \boldsymbol{a}$ follows $\boldsymbol{A} = -{}^{3}\boldsymbol{\epsilon} \cdot \boldsymbol{a}$.

(Pseudo)vector $\boldsymbol{a}$ is sometimes named as "accompanying" or "companion" for tensor $\boldsymbol{A}$.

To components

$$
\begin{aligned}
\boldsymbol{A} &= -{}^{3}\boldsymbol{\epsilon} \cdot \boldsymbol{a} \\
A_{ij} \boldsymbol{e}_i \boldsymbol{e}_j &= -\in_{ijk} \boldsymbol{e}_i \boldsymbol{e}_j a_k \\
A_{ij}(a_k):\ \ A_{ij} &= -\in_{ijk} a_k
\end{aligned}
$$

or, written as a matrix,

$$\underset{3\times3}{\left[A_{ij}\right]} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & -\in_{123} a_3 & -\in_{132} a_2 \\ -\in_{213} a_3 & 0 & -\in_{231} a_1 \\ -\in_{312} a_2 & -\in_{321} a_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

That the bivalent $\boldsymbol{A} = \boldsymbol{a} \times \boldsymbol{E} = \boldsymbol{E} \times \boldsymbol{a}$ is skewsymmetric was clear since (8.3). In the three-dimensional space, any antisymmetric tensor of the second complexity has only three independent components out of 9 : $A_{ij} = -A_{ji}$ and $A_{jj} = 0$.

The uniqueness of $\boldsymbol{a}$ for the unique $\boldsymbol{A}$, that is if $\boldsymbol{a}' \times \boldsymbol{E} = \boldsymbol{A}$ and $\boldsymbol{a}'' \times \boldsymbol{E} = \boldsymbol{A}$ (or $\boldsymbol{a}' \times \boldsymbol{E} - \boldsymbol{a}'' \times \boldsymbol{E} = \boldsymbol{A} - \boldsymbol{A}$) then $\boldsymbol{a}' = \boldsymbol{a}''$ or $\boldsymbol{a}' - \boldsymbol{a}'' = \boldsymbol{0}$

$$\left(\boldsymbol{a}' - \boldsymbol{a}''\right) \times \boldsymbol{E} = {}^2\boldsymbol{0}$$
$$\left(\boldsymbol{a}' - \boldsymbol{a}''\right) \boldsymbol{\cdot} {}^3\boldsymbol{\epsilon} = \boldsymbol{0} \boldsymbol{\cdot} {}^3\boldsymbol{\epsilon}$$

follows from the equal zeros $\boldsymbol{0} \boldsymbol{\cdot} {}^3\boldsymbol{\epsilon} = {}^2\boldsymbol{0}$ and the uniqueness of the "•"-product's result ($\boldsymbol{b} \boldsymbol{\cdot} \boldsymbol{c} = \boldsymbol{d} \boldsymbol{\cdot} \boldsymbol{c}$, $\boldsymbol{c} \neq \boldsymbol{0} \Leftrightarrow \boldsymbol{b} = \boldsymbol{d}$, including $\boldsymbol{b} \boldsymbol{\cdot} \boldsymbol{c} = \boldsymbol{0} \boldsymbol{\cdot} \boldsymbol{c}$, $\boldsymbol{c} \neq \boldsymbol{0} \Leftrightarrow \boldsymbol{b} = \boldsymbol{0}$). For $\boldsymbol{a} = \boldsymbol{0}$, $\boldsymbol{A}(\boldsymbol{0}) = \boldsymbol{0} \times \boldsymbol{E} = {}^2\boldsymbol{0}$.

✓ $\boldsymbol{a}$ is unique for $\boldsymbol{A}$

And yet about the reciprocal relation $\boldsymbol{A} \mapsto \boldsymbol{a} \,:\, \boldsymbol{a} = \boldsymbol{a}(\boldsymbol{A})$. By (7.15), the unit dyad $\boldsymbol{E}$ via ${}^3\boldsymbol{\epsilon}$

$$\boldsymbol{E} = -\tfrac{1}{2}\, {}^3\boldsymbol{\epsilon} \boldsymbol{\cdot\cdot} {}^3\boldsymbol{\epsilon},$$

and it is neutral (4.7) for the "•"-product

$$\boldsymbol{a} = \underbrace{\boldsymbol{a} \boldsymbol{\cdot} \left(-\tfrac{1}{2}\, {}^3\boldsymbol{\epsilon} \boldsymbol{\cdot\cdot} {}^3\boldsymbol{\epsilon}\right)}_{-1/2\, a_a \in_{abc} \in_{cbm} \boldsymbol{e}_m} = \underbrace{\left(-\tfrac{1}{2}\, {}^3\boldsymbol{\epsilon} \boldsymbol{\cdot\cdot} {}^3\boldsymbol{\epsilon}\right) \boldsymbol{\cdot} \boldsymbol{a}}_{-1/2\, \boldsymbol{e}_h \in_{hij} \in_{jik} a_k}$$

or without brackets

$$\boldsymbol{a} = -\tfrac{1}{2}\, {}^3\boldsymbol{\epsilon} \boldsymbol{\cdot\cdot} {}^3\boldsymbol{\epsilon} \boldsymbol{\cdot} \boldsymbol{a} = -\tfrac{1}{2}\, \boldsymbol{a} \boldsymbol{\cdot} {}^3\boldsymbol{\epsilon} \boldsymbol{\cdot\cdot} {}^3\boldsymbol{\epsilon}.$$

Bivalent $\boldsymbol{A}$ can be introduced here as in (8.4), $-\boldsymbol{A} = \boldsymbol{a} \cdot {}^3\!\boldsymbol{\epsilon} = {}^3\!\boldsymbol{\epsilon} \cdot \boldsymbol{a}$, and then

$$\boldsymbol{a}(\boldsymbol{A}) = \tfrac{1}{2}\, \boldsymbol{A} \cdot\cdot\, {}^3\!\boldsymbol{\epsilon} = \tfrac{1}{2}\, {}^3\!\boldsymbol{\epsilon} \cdot\cdot\, \boldsymbol{A}. \tag{8.5}$$

....

$$a_i \boldsymbol{e}_i = \tfrac{1}{2}\, A_{jk} \in_{kji} \boldsymbol{e}_i = \tfrac{1}{2} \in_{ikj} A_{jk}\, \boldsymbol{e}_i\,,$$

$$a_i = \frac{1}{2} \in_{ikj} A_{jk} = \frac{1}{2} \begin{bmatrix} \in_{123} A_{32} + \in_{132} A_{23} \\ \in_{213} A_{31} + \in_{231} A_{13} \\ \in_{312} A_{21} + \in_{321} A_{12} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}.$$

....

$$-2\,a_1 = \overbrace{\in_{123}}^{(+1)} \underbrace{\overbrace{\in_{321}}^{(-1)} a_1}_{-A_{32}} + \overbrace{\in_{132}}^{(-1)} \underbrace{\overbrace{\in_{231}}^{(+1)} a_1}_{-A_{23}}$$

$$-2\,a_2 = \overbrace{\in_{213}}^{-1} \underbrace{\overbrace{\in_{312}}^{+1} a_2}_{-A_{31}} + \overbrace{\in_{231}}^{+1} \underbrace{\overbrace{\in_{132}}^{-1} a_2}_{-A_{13}}$$

$$-2\,a_3 = \overbrace{\in_{312}}^{(+1)} \underbrace{\overbrace{\in_{213}}^{(-1)} a_3}_{-A_{21}} + \overbrace{\in_{321}}^{(-1)} \underbrace{\overbrace{\in_{123}}^{(+1)} a_3}_{-A_{12}}$$

....

$\boldsymbol{a}' \times \boldsymbol{E} = \boldsymbol{A}'$ and $\boldsymbol{a}'' \times \boldsymbol{E} = \boldsymbol{A}''$

PROVE IT $\boldsymbol{A}$ is unique for $\boldsymbol{a}$ **PROVE IT**

....

There is **PROVE THAT ONLY FOR SKEWSYMMETRIC** $\forall \boldsymbol{A} = \boldsymbol{A}^{\mathsf{A}}$ **IT IS BIJECTION** a bijection* between antisymmetric bivalent tensors and (pseudo)vectors. The components of a skewsymmetric tensor are fully described by the three numbers xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

....

* "a bijective relation", "a reciprocally reversible mapping", "a one-to-one correspondence"

All in all, here is the bijection $\boldsymbol{A} \leftrightarrow \boldsymbol{a}$

$$\boldsymbol{A}(\boldsymbol{a}) = -\,\boldsymbol{a} \cdot {}^3\!\boldsymbol{\epsilon} = \boldsymbol{a} \times \boldsymbol{E} = -\,{}^3\!\boldsymbol{\epsilon} \cdot \boldsymbol{a} = \boldsymbol{E} \times \boldsymbol{a}, \tag{8.6$^{A(a)}$}$$

$$\boldsymbol{a}(\boldsymbol{A}) = \tfrac{1}{2}\,\boldsymbol{A} \cdot\cdot\, {}^3\!\boldsymbol{\epsilon} = \tfrac{1}{2}\,{}^3\!\boldsymbol{\epsilon} \cdot\cdot\, \boldsymbol{A}. \tag{8.6$^{a(A)}$}$$

Easy to memorize, the "pseudovector invariant" $\boldsymbol{A}_\times$ comes from the original tensor $\boldsymbol{A}$ by replacing a dyadic product with a cross product

$$\begin{gathered} \boldsymbol{A}_\times \equiv A_{ij}\,\boldsymbol{e}_i \times \boldsymbol{e}_j = -\,\boldsymbol{A} \cdot\cdot\, {}^3\!\boldsymbol{\epsilon}, \\ \boldsymbol{A}_\times = \big(\boldsymbol{a} \times \boldsymbol{E}\big)_\times = -2\boldsymbol{a}, \;\; \boldsymbol{a} = -\tfrac{1}{2}\,\boldsymbol{A}_\times = -\tfrac{1}{2}\,\big(\boldsymbol{a} \times \boldsymbol{E}\big)_\times. \end{gathered} \tag{8.7}$$

Explaination:

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{E} = -\tfrac{1}{2}\,\boldsymbol{A}_\times \times \boldsymbol{E} &= -\tfrac{1}{2}\,A_{ij}\,(\underbrace{\boldsymbol{e}_i \times \boldsymbol{e}_j}_{\in_{ijn}\boldsymbol{e}_n}) \times \boldsymbol{e}_k\boldsymbol{e}_k \\ = -\tfrac{1}{2}\,A_{ij}\underbrace{\in_{nij}\in_{nkp}}_{\delta_{jp}\delta_{ik}\,-\,\delta_{ip}\delta_{jk}}\boldsymbol{e}_p\boldsymbol{e}_k &= -\tfrac{1}{2}\,A_{ij}\,(\boldsymbol{e}_j\boldsymbol{e}_i - \boldsymbol{e}_i\boldsymbol{e}_j) \\ &= -\tfrac{1}{2}\,\big(\boldsymbol{A}^{\mathsf{T}} - \boldsymbol{A}\big) = \boldsymbol{A}^{\mathsf{A}} = \boldsymbol{A}. \end{aligned}$$

The companion vector can be introduced for any bivalent tensor. But only the asymmetric part contributes here: $\boldsymbol{C}^{\mathsf{A}} = -\frac{1}{2}\,\boldsymbol{C}_\times \times \boldsymbol{E}$.

For a symmetric tensor, the companion vector is zero :

$$\boldsymbol{B}_\times = \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{B} = \boldsymbol{B}^{\mathsf{T}} = \boldsymbol{B}^{\mathsf{S}}.$$

With (8.7) the decomposition of some tensor $\boldsymbol{C}$ into the symmetric and antisymmetric parts looks like

$$\boldsymbol{C} = \boldsymbol{C}^{\mathsf{S}} - \tfrac{1}{2}\,\boldsymbol{C}_\times \times \boldsymbol{E}. \tag{8.8}$$

For a dyad

$$(7.13) \;\Rightarrow\; (\boldsymbol{c} \times \boldsymbol{d}) \times \boldsymbol{E} = \boldsymbol{d}\boldsymbol{c} - \boldsymbol{c}\boldsymbol{d} = -2\boldsymbol{c}\boldsymbol{d}^{\mathsf{A}}, \;\; (\boldsymbol{c}\boldsymbol{d})_\times = \boldsymbol{c} \times \boldsymbol{d},$$

and its decomposition

$$\boldsymbol{c}\boldsymbol{d} = \tfrac{1}{2}\,(\boldsymbol{c}\boldsymbol{d} + \boldsymbol{d}\boldsymbol{c}) - \tfrac{1}{2}\,(\boldsymbol{c} \times \boldsymbol{d}) \times \boldsymbol{E}. \tag{8.9}$$

§ 9. Polar decomposition

Any tensor of the second complexity $\boldsymbol{F}$ with $\det F_{ij} \neq 0$ (not singular) can be decomposed as

...

Example. Polar decompose tensor $\boldsymbol{C} = C_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$, where $\boldsymbol{e}_k$ are pairwise perpendicular unit vectors and C_{ij} are the tensor's components.

$$C_{ij} = \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix}$$

$$\boldsymbol{O} = O_{ij} \boldsymbol{e}_i \boldsymbol{e}_j = \boldsymbol{O}_1 \cdot \boldsymbol{O}_2$$

$$O_{ij} = \begin{bmatrix} 0 & {}^{3}\!/_{5} & {}^{4}\!/_{5} \\ 0 & {}^{4}\!/_{5} & -{}^{3}\!/_{5} \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & {}^{4}\!/_{5} & -{}^{3}\!/_{5} \\ 0 & {}^{3}\!/_{5} & {}^{4}\!/_{5} \end{bmatrix}$$

$$\boldsymbol{C} = \boldsymbol{O} \cdot \boldsymbol{S}_{\mathsf{R}}, \;\; \boldsymbol{O}^{\mathsf{T}} \cdot \boldsymbol{C} = \boldsymbol{S}_{\mathsf{R}}$$

$$\boldsymbol{C} = \boldsymbol{S}_{\mathsf{L}} \cdot \boldsymbol{O}, \;\; \boldsymbol{C} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{S}_{\mathsf{L}}$$

$$S_{\mathsf{R}ij} = \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix}$$

$$S_{\mathsf{L}ij} = \begin{bmatrix} {}^{104}\!/_{5} & {}^{47}\!/_{5} & 5 \\ {}^{47}\!/_{5} & -{}^{129}\!/_{5} & -10 \\ 5 & -10 & 3 \end{bmatrix}$$

...

§ 10. Eigenvectors and eigenvalues

If for some tensor ${}^2\boldsymbol{B}$ and the nonzero vector $\boldsymbol{a}$

$$^2\boldsymbol{B} \cdot \boldsymbol{a} = \eta \boldsymbol{a}, \;\; \boldsymbol{a} \neq \boldsymbol{0} \tag{10.1}$$

$$^2\boldsymbol{B} \cdot \boldsymbol{a} = \eta \boldsymbol{E} \cdot \boldsymbol{a}, \;\; \big({}^2\boldsymbol{B} - \eta \boldsymbol{E}\big) \cdot \boldsymbol{a} = \boldsymbol{0},$$

then η is called the eigenvalue (or the characteristic value) of tensor ${}^2\boldsymbol{B}$, and the axis (direction) of eigenvector $\boldsymbol{a}$ is called its characteristic axis (or direction).

In components, this is the eigenvalue problem for a matrix. A homogeneous system of linear equations $(B_{ij} - \eta \delta_{ij}) a_j = 0$ has a non-zero solution if the determinant of a matrix of components

$$\det_{i,j}\,(B_{ij} - \eta\delta_{ij})$$

is equal to zero:

$$\det \begin{bmatrix} B_{11}-\eta & B_{12} & B_{13} \\ B_{21} & B_{22}-\eta & B_{23} \\ B_{31} & B_{32} & B_{33}-\eta \end{bmatrix} = -\eta^3 + {}_{\text{cha}}\mathrm{I}\,\eta^2 - {}_{\text{cha}}\mathrm{II}\,\eta + {}_{\text{cha}}\mathrm{III} = 0; \tag{10.2}$$

$$\begin{aligned} {}_{\text{cha}}\mathrm{I} &= \operatorname{trace} {}^2\boldsymbol{B} = B_{kk} = B_{11} + B_{22} + B_{33}, \\ {}_{\text{cha}}\mathrm{II} &= B_{11}B_{22} - B_{12}B_{21} + B_{11}B_{33} - B_{13}B_{31} + B_{22}B_{33} - B_{23}B_{32}, \\ {}_{\text{cha}}\mathrm{III} &= \det {}^2\boldsymbol{B} = \det_{i,j} B_{ij} = e_{ijk}B_{1i}B_{2j}B_{3k} = e_{ijk}B_{i1}B_{j2}B_{k3}. \end{aligned} \tag{10.3}$$

The roots of the characteristic equation (10.2) — the eigenvalues — don't depend on basis and therefore are invariant.

The coefficients of (10.3) also don't depend on the basis; they are called the first, the second and the third characteristic invariants of a tensor. The first invariant ${}_{\text{cha}}\mathrm{I}$ is the trace. It was described earlier in § 4. The second characteristic invariant ${}_{\text{cha}}\mathrm{II}$ is the trace of the adjugate matrix — the transpose of the cofactor matrix (of the matrix of algebraic complements)

$${}_{\text{cha}}\mathrm{II}({}^2\boldsymbol{B}) \equiv \operatorname{trace}(\operatorname{adj} B_{ij})$$

(it's hard, yeah). Or

$${}_{\text{cha}}\mathrm{II}({}^2\boldsymbol{B}) \equiv \tfrac{1}{2}\Big[({}^2\boldsymbol{B}_{\bullet})^2 - {}^2\boldsymbol{B} \cdot\cdot\, {}^2\boldsymbol{B}\Big] = \tfrac{1}{2}\Big[(B_{kk})^2 - B_{ij}B_{ji}\Big].$$

And the third invariant ${}_{\text{cha}}\mathrm{III}$ is the determinant of a matrix of tensor components: ${}_{\text{cha}}\mathrm{III}({}^2\boldsymbol{B}) \equiv \det {}^2\boldsymbol{B}$.

This applies to all second complexity tensors. Besides that, in case of a symmetric tensor, the following is true:

1° The eigenvalues of a symmetric bivalent tensor are real numbers.

2° The characteristic axes (directions) for different eigenvalues are orthogonal to each other.

○ The first statement is proved by contradiction. If η is a complex root of (10.2) corresponding to eigenvector $\boldsymbol{a}$, then conjugate number $\overline{\eta}$ will also be the root of (10.2). Eigenvector $\overline{\boldsymbol{a}}$ with the conjugate components corresponds to it. And then

$$(10.1) \;\Rightarrow\; (\overline{\boldsymbol{a}}\,\boldsymbol{\cdot})\;{}^2\boldsymbol{B} \boldsymbol{\cdot} \boldsymbol{a} = \eta \boldsymbol{a}, \;\; (\boldsymbol{a}\,\boldsymbol{\cdot})\;{}^2\boldsymbol{B} \boldsymbol{\cdot} \overline{\boldsymbol{a}} = \overline{\eta}\,\overline{\boldsymbol{a}} \;\Rightarrow$$
$$\Rightarrow\; \overline{\boldsymbol{a}} \boldsymbol{\cdot} {}^2\boldsymbol{B} \boldsymbol{\cdot} \boldsymbol{a} - \boldsymbol{a} \boldsymbol{\cdot} {}^2\boldsymbol{B} \boldsymbol{\cdot} \overline{\boldsymbol{a}} = (\eta - \overline{\eta})\, \boldsymbol{a} \boldsymbol{\cdot} \overline{\boldsymbol{a}}.$$

Here on the left is zero, because $\boldsymbol{a} \boldsymbol{\cdot} {}^2\boldsymbol{B} \boldsymbol{\cdot} \boldsymbol{c} = \boldsymbol{c} \boldsymbol{\cdot} {}^2\boldsymbol{B}^{\mathsf{T}} \boldsymbol{\cdot} \boldsymbol{a}$ and ${}^2\boldsymbol{B} = {}^2\boldsymbol{B}^{\mathsf{T}}$. Thence $\eta = \overline{\eta}$, that is a real number.

Just as simple looks the proof of 2°:

$$\underbrace{\boldsymbol{a}_2 \boldsymbol{\cdot} {}^2\boldsymbol{B} \boldsymbol{\cdot} \boldsymbol{a}_1 - \boldsymbol{a}_1 \boldsymbol{\cdot} {}^2\boldsymbol{B} \boldsymbol{\cdot} \boldsymbol{a}_2}_{=0} = (\eta_1 - \eta_2)\, \boldsymbol{a}_1 \boldsymbol{\cdot} \boldsymbol{a}_2, \; \eta_1 \neq \eta_2 \;\Rightarrow$$
$$\Rightarrow\; \boldsymbol{a}_1 \boldsymbol{\cdot} \boldsymbol{a}_2 = 0. \qquad \bullet$$

If the roots of the characteristic equation (the eigenvalues) are different, then the one unit long eigenvectors $\boldsymbol{æ}_i$ compose an orthonormal basis. What are the tensor components in such a basis?

$${}^2\boldsymbol{B} \boldsymbol{\cdot} \boldsymbol{æ}_k = \cancel{\sum_k} \eta_k \boldsymbol{æ}_k, \quad k = 1, 2, 3$$
$${}^2\boldsymbol{B} \boldsymbol{\cdot} \underbrace{\boldsymbol{æ}_k \boldsymbol{æ}_k}_{\boldsymbol{E}} = \sum_k \eta_k \boldsymbol{æ}_k \boldsymbol{æ}_k$$

For a common case $B_{ij} = \boldsymbol{e}_i \boldsymbol{\cdot} {}^2\boldsymbol{B} \boldsymbol{\cdot} \boldsymbol{e}_j$. In the basis $\boldsymbol{æ}_1$, $\boldsymbol{æ}_2$, $\boldsymbol{æ}_3$ of mutually perpendicular one unit long $\boldsymbol{æ}_i \boldsymbol{\cdot} \boldsymbol{æ}_j = \delta_{ij}$ eigenvectors of a symmetric tensor

$$B_{11} = \boldsymbol{æ}_1 \boldsymbol{\cdot} (\eta_1 \boldsymbol{æ}_1 \boldsymbol{æ}_1 + \eta_2 \boldsymbol{æ}_2 \boldsymbol{æ}_2 + \eta_3 \boldsymbol{æ}_3 \boldsymbol{æ}_3) \boldsymbol{\cdot} \boldsymbol{æ}_1 = \eta_1,$$
$$B_{12} = \boldsymbol{æ}_1 \boldsymbol{\cdot} (\eta_1 \boldsymbol{æ}_1 \boldsymbol{æ}_1 + \eta_2 \boldsymbol{æ}_2 \boldsymbol{æ}_2 + \eta_3 \boldsymbol{æ}_3 \boldsymbol{æ}_3) \boldsymbol{\cdot} \boldsymbol{æ}_2 = 0,$$
$$\dots$$

The matrix of components is diagonal and ${}^2\boldsymbol{B} = \sum_i \eta_i \boldsymbol{æ}_i \boldsymbol{æ}_i$. Yes, here is the summation over the three repeating indices, because the special basis is used.

The transition to a coincidence case of eigenvalues can be acquired via limit calculation. If $\eta_2 \to \eta_1$, then any linear combination of vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in the limit satisfies the equation (10.1). And then any axis in the plane $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ becomes characteristic.

When all the three eigenvalues coincide, then any axis in space is characteristic. Then ${}^2\boldsymbol{B} = \eta \boldsymbol{E}$, such tensors are called isotropic or "spherical".

Collections of invariants of a symmetric bivalent tensor

....

The "algebraic" invariants

...

The "characteristic" invariants. These are coefficients of the characteristic equation (10.2) of the eigenvalues problem (10.1).

.....

The "research" invariants

...

The "harmonic" invariants

...

§ 11. Rotations via rotation tensors

The relation between two "right" (or two "left" orthonormal bases $\boldsymbol{e}_i$ and $\mathring{\boldsymbol{e}}_i$ can be described by a two-index array represented as a matrix (§ 2, § 6)

$$\boldsymbol{e}_i = \boldsymbol{e}_i \cdot \underbrace{\mathring{\boldsymbol{e}}_j \mathring{\boldsymbol{e}}_j}_{\boldsymbol{E}} = o_{i\mathring{j}} \mathring{\boldsymbol{e}}_j, \;\; o_{i\mathring{j}} \equiv \boldsymbol{e}_i \cdot \mathring{\boldsymbol{e}}_j$$

("a matrix of cosines").

Also, a rotation of tensor can be described by another tensor, called a rotation tensor $\boldsymbol{O}$

$$\boldsymbol{e}_i = \boldsymbol{e}_j \underbrace{\mathring{\boldsymbol{e}}_j \cdot \mathring{\boldsymbol{e}}_i}_{\delta_{ji}} = \boldsymbol{O} \cdot \mathring{\boldsymbol{e}}_i, \;\; \boldsymbol{O} \equiv \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j = \boldsymbol{e}_1 \mathring{\boldsymbol{e}}_1 + \boldsymbol{e}_2 \mathring{\boldsymbol{e}}_2 + \boldsymbol{e}_3 \mathring{\boldsymbol{e}}_3. \tag{11.1}$$

Components of $\boldsymbol{O}$ both in the initial $\mathring{\boldsymbol{e}}_i$ and in the rotated $\boldsymbol{e}_i$ bases are the same

$$\begin{aligned} \boldsymbol{e}_i \cdot \boldsymbol{O} \cdot \boldsymbol{e}_j &= \underbrace{\boldsymbol{e}_i \cdot \boldsymbol{e}_k}_{\delta_{ik}} \mathring{\boldsymbol{e}}_k \cdot \boldsymbol{e}_j = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j , \\ \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{O} \cdot \mathring{\boldsymbol{e}}_j &= \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_k \underbrace{\mathring{\boldsymbol{e}}_k \cdot \mathring{\boldsymbol{e}}_j}_{\delta_{kj}} = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j . \end{aligned} \tag{11.2}$$

These components present the matrix of cosines $o_{\underset{ji}{\circ}} = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j$

$$\boldsymbol{O} = o_{\underset{ji}{\circ}} \boldsymbol{e}_i \boldsymbol{e}_j = o_{\underset{ji}{\circ}} \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j .$$

Spatial transformations are distinguished into active or alibi transformations, and passive or alias transformations. An active transformation is a transformation which actually changes the physical position (alibi, elsewhere) of objects, which can be defined in the absence of a coordinate system; whereas a passive transformation (alias, other name) is merely a change in the coordinate system in which the object is described (change of coordinates, or change of basis).

Tensor $\boldsymbol{O}$ relates the two vectors, "before rotation" $\mathring{\boldsymbol{r}} = \rho_i \mathring{\boldsymbol{e}}_i$ and "after rotation" $\boldsymbol{r} = \rho_i \boldsymbol{e}_i$. Components $\rho_i = \mathsf{constant}$ are the same for both $\boldsymbol{r}$ in the rotated basis $\boldsymbol{e}_i$ and $\mathring{\boldsymbol{r}}$ in the immobile basis $\mathring{\boldsymbol{e}}_i$. So that the rotation tensor describes the rotation of the vector together with the basis. And since $\boldsymbol{e}_i = \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j \cdot \mathring{\boldsymbol{e}}_i \;\Leftrightarrow\; \rho_i \boldsymbol{e}_i = \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j \cdot \rho_i \mathring{\boldsymbol{e}}_i$, then

$$\boldsymbol{r} = \boldsymbol{O} \cdot \mathring{\boldsymbol{r}} \tag{11.3}$$

(this is the Rodrigues rotation formula*).

For a second complexity tensor $\mathring{\boldsymbol{C}} = C_{ij} \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j$, a rotation into the current position $\boldsymbol{C} = C_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$ looks like

$$\boldsymbol{e}_i C_{ij} \boldsymbol{e}_j = \boldsymbol{e}_i \mathring{\boldsymbol{e}}_i \cdot \mathring{\boldsymbol{e}}_p C_{pq} \mathring{\boldsymbol{e}}_q \cdot \mathring{\boldsymbol{e}}_j \boldsymbol{e}_j \;\Leftrightarrow\; \boldsymbol{C} = \boldsymbol{O} \cdot \mathring{\boldsymbol{C}} \cdot \boldsymbol{O}^{\mathsf{T}} . \tag{11.4}$$

* **Olinde Rodrigues**. Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendants des causes qui peuvent les produire. *Journal de mathématiques pures et appliquées*, tome 5 (1840), pages 380–440.

The essential property of a rotation tensor — the orthogonality — is expressed as

$$\underbrace{\boldsymbol{O}}_{\boldsymbol{e}_i\mathring{\boldsymbol{e}}_i} \cdot \underbrace{\boldsymbol{O}^{\mathsf{T}}}_{\mathring{\boldsymbol{e}}_j\boldsymbol{e}_j} = \underbrace{\boldsymbol{O}^{\mathsf{T}}}_{\mathring{\boldsymbol{e}}_i\boldsymbol{e}_i} \cdot \underbrace{\boldsymbol{O}}_{\boldsymbol{e}_j\mathring{\boldsymbol{e}}_j} = \underbrace{\boldsymbol{E}}_{\substack{\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_i \\ \boldsymbol{e}_i\boldsymbol{e}_i}}\,, \tag{11.5}$$

that is the transposed tensor coincides with the reciprocal tensor: $\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{O}^{-1} \Leftrightarrow \boldsymbol{O} = \boldsymbol{O}^{-\mathsf{T}}$.

An orthogonal tensor retains lengths and angles ("the metric") because it does not change the "$\boldsymbol{\cdot}$"-product of vectors

$$(\boldsymbol{O} \cdot \boldsymbol{a}) \cdot (\boldsymbol{O} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{O}^{\mathsf{T}} \cdot \boldsymbol{O} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{E} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}. \tag{11.6}$$

For all orthogonal tensors $(\det \boldsymbol{Q})^2 = 1$:

$$1 = \det \boldsymbol{E} = \det\left(\boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathsf{T}}\right) = (\det \boldsymbol{Q})\left(\det \boldsymbol{Q}^{\mathsf{T}}\right) = (\det \boldsymbol{Q})^2.$$

A rotation tensor is an orthogonal tensor with $\det \boldsymbol{O} = +1$.

But not only rotation tensors possess the property of orthogonality. When in (11.1) the first basis is "left", and the second one is "right", then there's a combination of rotating and mirroring (a "rotoreflexion") $\boldsymbol{O} = -\boldsymbol{E} \cdot \boldsymbol{O}$ with $\det(-\boldsymbol{E} \cdot \boldsymbol{O}) = -1$.

Any bivalent tensor in the three-dimensional (3D) space has at least one non-complex (real) eigenvalue — the root of (10.2). For a rotation tensor, it is equal to one

$$\boldsymbol{O} \cdot \boldsymbol{a} = \eta \boldsymbol{a} \;\Rightarrow\; \eta \boldsymbol{a} \cdot \eta \boldsymbol{a} = \overbrace{\boldsymbol{a} \cdot \boldsymbol{O}^{\mathsf{T}}}^{\boldsymbol{O} \cdot \boldsymbol{a}} \!\!\!\!\!\underbrace{\phantom{\boldsymbol{O}^{\mathsf{T}}} \cdot \boldsymbol{O}}_{\boldsymbol{E}} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{a} \;\Rightarrow\; \eta^2 = 1.$$

The corresponding eigenvector is called the axis of rotation. The Euler's theorem on finite rotation (Euler's rotation theorem) is just about that such an axis exists. If $\boldsymbol{k}$ is the unit vector of that axis, and ϑ is the finite angle of rotation, then the rotation tensor is representable as

$$\boldsymbol{O}(\boldsymbol{k}, \vartheta) = \boldsymbol{E} \cos\vartheta + \boldsymbol{k} \times \boldsymbol{E} \sin\vartheta + \boldsymbol{k}\boldsymbol{k}\,(1 - \cos\vartheta)\,. \tag{11.7}$$

This formula is proved like this. During rotation $\boldsymbol{k}$ doesn't change ($\boldsymbol{O} \cdot \boldsymbol{k} = \boldsymbol{k}$), therefore on the axis of rotation $\mathring{\boldsymbol{e}}_3 = \boldsymbol{e}_3 = \boldsymbol{k}$.

In the perpendicular plane (figure 5) $\mathring{\boldsymbol{e}}_1 = \boldsymbol{e}_1 \cos\vartheta - \boldsymbol{e}_2 \sin\vartheta$, $\mathring{\boldsymbol{e}}_2 = \boldsymbol{e}_1 \sin\vartheta + \boldsymbol{e}_2 \cos\vartheta$. Composing the tensor $\boldsymbol{O} = \boldsymbol{e}_i \mathring{\boldsymbol{e}}_i \Rightarrow$ (11.7).

From (11.7) and (11.3) follows the Rodrigues rotation formula in parameters $\boldsymbol{k}$ and ϑ :

$$\boldsymbol{r} = \mathring{\boldsymbol{r}} \cos\vartheta + \boldsymbol{k} \times \mathring{\boldsymbol{r}} \sin\vartheta + \boldsymbol{k}\boldsymbol{k} \cdot \mathring{\boldsymbol{r}} \, (1 - \cos\vartheta).$$

In the parameters of finite rotation, transposing (which is here also inversing) the tensor $\boldsymbol{O}$ is equivalent to changing the direction of rotation — the sign of angle ϑ

$$\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{O}\big|_{\vartheta = -\vartheta} = \boldsymbol{E} \cos\vartheta - \boldsymbol{k} \times \boldsymbol{E} \sin\vartheta + \boldsymbol{k}\boldsymbol{k} \, (1 - \cos\vartheta).$$

Now let the rotation tensor change over time, $\boldsymbol{O} = \boldsymbol{O}(t)$. The time derivative $\dot{\boldsymbol{O}} = \left(\boldsymbol{e}_i \mathring{\boldsymbol{e}}_i\right)^{\bullet} = \dot{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_i$ (since $\mathring{\boldsymbol{e}}_i = \text{constant}$), $\dot{\boldsymbol{e}}_i = \dot{\boldsymbol{O}} \cdot \mathring{\boldsymbol{e}}_i$.

For a unit vector $\boldsymbol{e}$ that doesn't change length $\boldsymbol{e} \cdot \boldsymbol{e} = 1^2 = \text{constant}$, and the derivative $\dot{\boldsymbol{e}}$ is orthogonal (perpendicular) to it

$$\left(\boldsymbol{e} \cdot \boldsymbol{e}\right)^{\bullet} = \dot{\boldsymbol{e}} \cdot \boldsymbol{e} + \boldsymbol{e} \cdot \dot{\boldsymbol{e}} = 0 \;\Rightarrow\; \dot{\boldsymbol{e}} \cdot \boldsymbol{e} = 0. \qquad (11.8)$$

....

The pseudovector of angular velocity $\boldsymbol{\omega}$ comes out from the rotation tensor $\boldsymbol{O}$ in this way. Differentiating the orthogonality identity (11.5) by time* gives

$$\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}} + \boldsymbol{O} \cdot \dot{\boldsymbol{O}}^{\mathsf{T}} = {}^2\boldsymbol{0}.$$

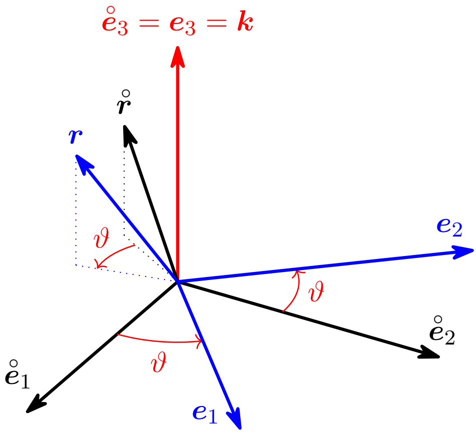
By (4.9) $\left(\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}}\right)^{\mathsf{T}} = \boldsymbol{O} \cdot \dot{\boldsymbol{O}}^{\mathsf{T}}$, therefore tensor $\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}}$ happens to be antisymmetric. Then, according to ($8.6^{A(a)}$), it can be represented via the companion vector as $\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\omega} \times \boldsymbol{E} = \boldsymbol{\omega} \times \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}}$. That is

$$\dot{\boldsymbol{O}} = \boldsymbol{\omega} \times \boldsymbol{O}, \quad \boldsymbol{\omega} \equiv -\frac{1}{2} \left(\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}}\right)_{\times} \qquad (11.9)$$

* Various notations are used to designate the time derivative. In addition to the Leibniz's notation $^{dx}/_{dt}$, the very popular one is the "dot above" Newton's notation $\dot{x}$.

$$\mathring{\boldsymbol{e}}_i = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j \boldsymbol{e}_j$$

$$\begin{bmatrix} \mathring{\boldsymbol{e}}_1 \\ \mathring{\boldsymbol{e}}_2 \\ \mathring{\boldsymbol{e}}_3 \end{bmatrix} = \begin{bmatrix} \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{bmatrix}$$



$$\begin{bmatrix} \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & \cos\left({}^{\pi}\!/_{2} + \vartheta\right) & \cos {}^{\pi}\!/_{2} \\ \cos\left({}^{\pi}\!/_{2} - \vartheta\right) & \cos\vartheta & \cos {}^{\pi}\!/_{2} \\ \cos {}^{\pi}\!/_{2} & \cos {}^{\pi}\!/_{2} & \cos 0 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{gathered} \mathring{\boldsymbol{e}}_1 = \boldsymbol{e}_1 \cos\vartheta - \boldsymbol{e}_2 \sin\vartheta \\ \mathring{\boldsymbol{e}}_2 = \boldsymbol{e}_1 \sin\vartheta + \boldsymbol{e}_2 \cos\vartheta \\ \mathring{\boldsymbol{e}}_3 = \boldsymbol{e}_3 = \boldsymbol{k} \end{gathered}$$

$$\boldsymbol{O} = \boldsymbol{e}_1 \mathring{\boldsymbol{e}}_1 + \boldsymbol{e}_2 \mathring{\boldsymbol{e}}_2 + \boldsymbol{e}_3 \mathring{\boldsymbol{e}}_3 =$$

$$= \overbrace{\boldsymbol{e}_1 \boldsymbol{e}_1 \cos\vartheta - \boldsymbol{e}_1 \boldsymbol{e}_2 \sin\vartheta}^{\boldsymbol{e}_1 \mathring{\boldsymbol{e}}_1} + \overbrace{\boldsymbol{e}_2 \boldsymbol{e}_1 \sin\vartheta + \boldsymbol{e}_2 \boldsymbol{e}_2 \cos\vartheta}^{\boldsymbol{e}_2 \mathring{\boldsymbol{e}}_2} + \overbrace{\boldsymbol{k}\boldsymbol{k}}^{\boldsymbol{e}_3 \mathring{\boldsymbol{e}}_3} =$$

$$= \left(\boldsymbol{E} \cos\vartheta - \underbrace{\boldsymbol{e}_3 \boldsymbol{e}_3}_{\boldsymbol{k}\boldsymbol{k}} \cos\vartheta\right) + \underbrace{\left(\boldsymbol{e}_2 \boldsymbol{e}_1 - \boldsymbol{e}_1 \boldsymbol{e}_2\right)}_{\boldsymbol{e}_3 \times \boldsymbol{e}_i \boldsymbol{e}_i = \in_{3ij} \boldsymbol{e}_j \boldsymbol{e}_i} \sin\vartheta + \boldsymbol{k}\boldsymbol{k} =$$

$$= \boldsymbol{E} \cos\vartheta + \boldsymbol{k} \times \boldsymbol{E} \sin\vartheta + \boldsymbol{k}\boldsymbol{k}\left(1 - \cos\vartheta\right)$$

figure 5
"Finite rotation"

Besides this generic representation for the (pseudo)vector $\boldsymbol{\omega}$, there are others too. As, for example, via the two finite rotation parameters.

The derivative $\dot{\boldsymbol{O}}$ in the finite rotation parameters in the broadest case, when both parameters — and the unit vector $\boldsymbol{k}$, and the angle ϑ — are variable over time, is

$$\begin{aligned}\dot{\boldsymbol{O}} &= \left(\boldsymbol{O}^{\mathsf{S}} + \boldsymbol{O}^{\mathsf{A}}\right)^{\bullet} = \Big(\overbrace{\boldsymbol{E}\cos\vartheta + \boldsymbol{k}\boldsymbol{k}\,(1-\cos\vartheta)}^{\boldsymbol{O}^{\mathsf{S}}} + \overbrace{\boldsymbol{k}\times\boldsymbol{E}\sin\vartheta}^{\boldsymbol{O}^{\mathsf{A}}}\Big)^{\bullet} \\ &= \underbrace{(\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})\,(1-\cos\vartheta)}_{\dot{\boldsymbol{O}}^{\mathsf{S}}} \\ &\qquad + \underbrace{\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\cos\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta}_{\dot{\boldsymbol{O}}^{\mathsf{A}}}.\end{aligned}$$

Находим

$$\begin{aligned}\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}} &= \left(\dot{\boldsymbol{O}}^{\mathsf{S}} + \dot{\boldsymbol{O}}^{\mathsf{A}}\right)\cdot\left(\boldsymbol{O}^{\mathsf{S}} - \boldsymbol{O}^{\mathsf{A}}\right) \\ &= \dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{S}} + \dot{\boldsymbol{O}}^{\mathsf{A}}\cdot\boldsymbol{O}^{\mathsf{S}} - \dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{A}} - \dot{\boldsymbol{O}}^{\mathsf{A}}\cdot\boldsymbol{O}^{\mathsf{A}},\end{aligned}$$

using

$$\begin{gathered}\boldsymbol{k}\cdot\boldsymbol{k} = 1 = \mathsf{constant} \;\Rightarrow\; \boldsymbol{k}\cdot\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\cdot\boldsymbol{k} = 0 \;\Leftrightarrow\; \dot{\boldsymbol{k}}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\dot{\boldsymbol{k}} = 0, \\ \boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{k} = \boldsymbol{k}\boldsymbol{k},\;\; \dot{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{k} = \dot{\boldsymbol{k}}\boldsymbol{k},\;\; \boldsymbol{k}\dot{\boldsymbol{k}}\cdot\boldsymbol{k}\boldsymbol{k} = {}^2\boldsymbol{0}, \\ (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\cdot\boldsymbol{k} = \boldsymbol{k}-\boldsymbol{k} = \boldsymbol{0},\;\; (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k} = \boldsymbol{k}\boldsymbol{k}-\boldsymbol{k}\boldsymbol{k} = {}^2\boldsymbol{0}, \\ \boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{E}) = (\boldsymbol{k}\times\boldsymbol{E})\cdot\boldsymbol{k} = \boldsymbol{k}\times\boldsymbol{k} = \boldsymbol{0},\;\; \boldsymbol{k}\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{E}) = (\boldsymbol{k}\times\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k} = {}^2\boldsymbol{0}, \\ (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E}) = -\boldsymbol{k}\times\boldsymbol{E}, \\ (\boldsymbol{a}\times\boldsymbol{E})\cdot\boldsymbol{b} = \boldsymbol{a}\times(\boldsymbol{E}\cdot\boldsymbol{b}) = \boldsymbol{a}\times\boldsymbol{b} \;\;\Rightarrow\;\; (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k} = \dot{\boldsymbol{k}}\times\boldsymbol{k}\boldsymbol{k},\end{gathered}$$

$$\begin{aligned}&(7.14) \;\Rightarrow\; (\boldsymbol{k}\times\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E}) = \boldsymbol{k}\boldsymbol{k}-\boldsymbol{E},\;\; (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E}) = \boldsymbol{k}\dot{\boldsymbol{k}} - \cancel{\dot{\boldsymbol{k}}\cdot\boldsymbol{k}\boldsymbol{E}}, \\ &(7.13) \;\Rightarrow\; \dot{\boldsymbol{k}}\boldsymbol{k} - \boldsymbol{k}\dot{\boldsymbol{k}} = (\boldsymbol{k}\times\dot{\boldsymbol{k}})\times\boldsymbol{E},\;\; (\dot{\boldsymbol{k}}\times\boldsymbol{k})\boldsymbol{k} - \boldsymbol{k}(\dot{\boldsymbol{k}}\times\boldsymbol{k}) = \boldsymbol{k}\times(\dot{\boldsymbol{k}}\times\boldsymbol{k})\times\boldsymbol{E}\end{aligned}$$

$$\begin{aligned}&\dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{S}} = \\ &= (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cdot\boldsymbol{E}\cos\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}}+\dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta)\cdot\boldsymbol{E}\cos\vartheta + \\ &\qquad + \cancel{(\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cdot\boldsymbol{k}\boldsymbol{k}\,(1-\cos\vartheta)} + (\boldsymbol{k}\dot{\boldsymbol{k}}+\dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta)\cdot\boldsymbol{k}\boldsymbol{k}\,(1-\cos\vartheta) = \\ &= (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cos\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}}+\dot{\boldsymbol{k}}\boldsymbol{k})\cos\vartheta\,(1-\cos\vartheta) + (\cancel{\boldsymbol{k}\dot{\boldsymbol{k}}\cdot\boldsymbol{k}\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{k})\,(1-\cos\vartheta)^2 = \\ &= (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta\,(1-\cos\vartheta) + \\ &\qquad\qquad + \dot{\boldsymbol{k}}\boldsymbol{k}\cos\vartheta - \dot{\boldsymbol{k}}\boldsymbol{k}\cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k} - 2\,\dot{\boldsymbol{k}}\boldsymbol{k}\cos\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\cos^2\vartheta = \\ &\qquad = (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}}\cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\,(1-\cos\vartheta),\end{aligned}$$

$$
\begin{aligned}
&\dot{\boldsymbol{O}}^{\mathsf{A}} \cdot \boldsymbol{O}^{\mathsf{S}} = \\
&= (\boldsymbol{k} \times \boldsymbol{E}) \cdot \boldsymbol{E}\, \dot{\vartheta} \cos^2\vartheta + (\dot{\boldsymbol{k}} \times \boldsymbol{E}) \cdot \boldsymbol{E} \sin\vartheta \cos\vartheta + \\
&\qquad + \cancel{(\boldsymbol{k} \times \boldsymbol{E}) \cdot \boldsymbol{k}\boldsymbol{k}\, \dot{\vartheta} \cos\vartheta\, (1 - \cos\vartheta)} + (\dot{\boldsymbol{k}} \times \boldsymbol{E}) \cdot \boldsymbol{k}\boldsymbol{k} \sin\vartheta\, (1 - \cos\vartheta) = \\
&\qquad = \boldsymbol{k} \times \boldsymbol{E}\, \dot{\vartheta} \cos^2\vartheta + \dot{\boldsymbol{k}} \times \boldsymbol{E} \sin\vartheta \cos\vartheta + \dot{\boldsymbol{k}} \times \boldsymbol{k}\boldsymbol{k} \sin\vartheta\, (1 - \cos\vartheta)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\dot{\boldsymbol{O}}^{\mathsf{S}} \cdot \boldsymbol{O}^{\mathsf{A}} = \\
&= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\, \dot{\vartheta} \sin\vartheta \cdot (\boldsymbol{k} \times \boldsymbol{E}) \sin\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})(1 - \cos\vartheta) \cdot (\boldsymbol{k} \times \boldsymbol{E}) \sin\vartheta = \\
&= \cancel{\boldsymbol{k}\boldsymbol{k} \cdot (\boldsymbol{k} \times \boldsymbol{E})\, \dot{\vartheta} \sin^2\vartheta} - \boldsymbol{E} \cdot (\boldsymbol{k} \times \boldsymbol{E})\, \dot{\vartheta} \sin^2\vartheta + \Big(\boldsymbol{k}\dot{\boldsymbol{k}} \cdot (\boldsymbol{k} \times \boldsymbol{E}) + \cancel{\dot{\boldsymbol{k}}\boldsymbol{k} \cdot (\boldsymbol{k} \times \boldsymbol{E})}\Big) \sin\vartheta\, (1 - \cos\vartheta) = \\
&\qquad = -\boldsymbol{k} \times \boldsymbol{E}\, \dot{\vartheta} \sin^2\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}} \times \boldsymbol{k} \sin\vartheta\, (1 - \cos\vartheta)\,,
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{O}}^{\mathsf{A}} \cdot \boldsymbol{O}^{\mathsf{A}} &= (\boldsymbol{k} \times \boldsymbol{E})\, \dot{\vartheta} \cos\vartheta \cdot (\boldsymbol{k} \times \boldsymbol{E}) \sin\vartheta + (\dot{\boldsymbol{k}} \times \boldsymbol{E}) \cdot (\boldsymbol{k} \times \boldsymbol{E}) \sin^2\vartheta = \\
&= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\, \dot{\vartheta} \sin\vartheta \cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}} \sin^2\vartheta\,;
\end{aligned}
$$

$$
\begin{aligned}
&\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}} = \dot{\boldsymbol{O}}^{\mathsf{S}} \cdot \boldsymbol{O}^{\mathsf{S}} + \dot{\boldsymbol{O}}^{\mathsf{A}} \cdot \boldsymbol{O}^{\mathsf{S}} - \dot{\boldsymbol{O}}^{\mathsf{S}} \cdot \boldsymbol{O}^{\mathsf{A}} - \dot{\boldsymbol{O}}^{\mathsf{A}} \cdot \boldsymbol{O}^{\mathsf{A}} = \\
&\qquad = (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\, \dot{\vartheta} \sin\vartheta \cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}} \cos\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}} \cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\, (1 - \cos\vartheta) + \\
&\qquad + \boldsymbol{k} \times \boldsymbol{E}\, \dot{\vartheta} \cos^2\vartheta + \dot{\boldsymbol{k}} \times \boldsymbol{E} \sin\vartheta \cos\vartheta + \dot{\boldsymbol{k}} \times \boldsymbol{k}\boldsymbol{k} \sin\vartheta\, (1 - \cos\vartheta) + \\
&\quad + \boldsymbol{k} \times \boldsymbol{E}\, \dot{\vartheta} \sin^2\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}} \times \boldsymbol{k} \sin\vartheta\, (1 - \cos\vartheta) - (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\, \dot{\vartheta} \sin\vartheta \cos\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}} \sin^2\vartheta = \\
&= \boldsymbol{k} \times \boldsymbol{E}\, \dot{\vartheta} + (\dot{\boldsymbol{k}}\boldsymbol{k} - \boldsymbol{k}\dot{\boldsymbol{k}})(1 - \cos\vartheta) + \dot{\boldsymbol{k}} \times \boldsymbol{E} \sin\vartheta \cos\vartheta + (\dot{\boldsymbol{k}} \times \boldsymbol{k}\boldsymbol{k} - \boldsymbol{k}\dot{\boldsymbol{k}} \times \boldsymbol{k}) \sin\vartheta\, (1 - \cos\vartheta) = \\
&= \boldsymbol{k} \times \boldsymbol{E}\, \dot{\vartheta} + \boldsymbol{k} \times \dot{\boldsymbol{k}} \times \boldsymbol{E}\, (1 - \cos\vartheta) + \dot{\boldsymbol{k}} \times \boldsymbol{E} \sin\vartheta \cos\vartheta + \boldsymbol{k} \times (\dot{\boldsymbol{k}} \times \boldsymbol{k}) \times \boldsymbol{E} \sin\vartheta\, (1 - \cos\vartheta) = \\
&= \boldsymbol{k} \times \boldsymbol{E}\, \dot{\vartheta} + \dot{\boldsymbol{k}} \times \boldsymbol{E} \sin\vartheta \cos\vartheta + (\dot{\boldsymbol{k}}\boldsymbol{k} \cdot \boldsymbol{k} - \cancel{\boldsymbol{k}\dot{\boldsymbol{k}} \cdot \boldsymbol{k}}) \times \boldsymbol{E} \sin\vartheta\, (1 - \cos\vartheta) + \boldsymbol{k} \times \dot{\boldsymbol{k}} \times \boldsymbol{E}\, (1 - \cos\vartheta) = \\
&\qquad = \boldsymbol{k} \times \boldsymbol{E}\, \dot{\vartheta} + \dot{\boldsymbol{k}} \times \boldsymbol{E} \sin\vartheta + \boldsymbol{k} \times \dot{\boldsymbol{k}} \times \boldsymbol{E}\, (1 - \cos\vartheta).
\end{aligned}
$$

Этот результат, подставленный в определение (11.9) псевдовектора $\boldsymbol{\omega}$, даёт

$$
\boldsymbol{\omega} = \boldsymbol{k}\dot{\vartheta} + \dot{\boldsymbol{k}} \sin\vartheta + \boldsymbol{k} \times \dot{\boldsymbol{k}}\, (1 - \cos\vartheta). \tag{11.10}
$$

Вектор $\boldsymbol{\omega}$ получился разложенным по трём взаимно ортогональным направлениям — $\boldsymbol{k}$, $\dot{\boldsymbol{k}}$ и $\boldsymbol{k} \times \dot{\boldsymbol{k}}$. При неподвижной оси поворота $\dot{\boldsymbol{k}} = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{\omega} = \boldsymbol{k}\dot{\vartheta}$.

Ещё одно представление $\boldsymbol{\omega}$ связано с компонентами тензора поворота (11.2). Поскольку $\boldsymbol{O} = o_{\overset{\circ}{ji}}\, \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j$, $\boldsymbol{O}^{\mathsf{T}} = o_{\overset{\circ}{ij}}\, \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j$, а векторы начального базиса $\mathring{\boldsymbol{e}}_i$ неподвижны (со временем не меняются), то

$$\dot{\boldsymbol{O}} = \dot{o}_{\underset{ji}{\circ\circ}} \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j\,, \;\; \dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}} = \dot{o}_{\underset{ni}{\circ\circ}} o_{\underset{nj}{\circ\circ}} \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j\,,$$

$$\boldsymbol{\omega} = -\tfrac{1}{2}\, \dot{o}_{\underset{ni}{\circ\circ}} o_{\underset{nj}{\circ\circ}} \mathring{\boldsymbol{e}}_i \times \mathring{\boldsymbol{e}}_j = \tfrac{1}{2} \in_{jik} o_{\underset{nj}{\circ\circ}} \dot{o}_{\underset{ni}{\circ\circ}} \mathring{\boldsymbol{e}}_k\,. \tag{11.11}$$

Отметим и формулы

$$\begin{gathered} (11.9) \;\Rightarrow\; \dot{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \mathring{\boldsymbol{e}}_i \;\Rightarrow\; \dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i\,, \\ (11.9) \;\Rightarrow\; \boldsymbol{\omega} = -\tfrac{1}{2} \left(\dot{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_i \cdot \mathring{\boldsymbol{e}}_j \boldsymbol{e}_j \right)_{\times} = -\tfrac{1}{2} \left(\dot{\boldsymbol{e}}_i \boldsymbol{e}_i \right)_{\times} = \tfrac{1}{2}\, \boldsymbol{e}_i \times \dot{\boldsymbol{e}}_i\,. \end{gathered} \tag{11.12}$$

...

Варьируя тождество (11.5), получим $\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O} \cdot \delta \boldsymbol{O}^{\mathsf{T}}$. Этот тензор антисимметричен, и потому выражается через свой сопутствующий вектор $\boldsymbol{\delta\mathrm{o}}$ как $\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\delta\mathrm{o}} \times \boldsymbol{E}$. Приходим к соотношениям

$$\delta \boldsymbol{O} = \boldsymbol{\delta\mathrm{o}} \times \boldsymbol{O}, \;\; \boldsymbol{\delta\mathrm{o}} = -\frac{1}{2} \Big(\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} \Big)_{\times}, \tag{11.13}$$

аналогичным (11.9). Вектор бесконечно малого поворота $\boldsymbol{\delta\mathrm{o}}$ это не “вариация **o**”, но единый символ (в отличие от $\delta \boldsymbol{O}$).

Малый поворот определяется вектором $\boldsymbol{\delta\mathrm{o}}$, но и конечный поворот тоже возможно представить как вектор.

...

§ 12. Rotations via quaternions

The other way to describe a rotation in 3-dimensional space is using quaternions. Meanwhile, they are very popular for computer graphics.

Quaternions were invented by William Rowan Hamilton in 1843*.

....

* *On Quaternions; or on a new System of Imaginaries in Algebra* by **William Rowan Hamilton** appeared in 18 publications in ‘The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science’, volumes xxv–xxxvi, 3rd series, 1844–1850.

$$p, q \in \mathbb{C}$$
$$p = e^{i\varphi} = \cos\varphi + i\sin\varphi$$
$$q = e^{i\psi} = \cos\psi + i\sin\psi$$

the composition of two rotations

$$pq = e^{i\varphi}e^{i\psi} = e^{i(\varphi+\psi)}$$

$$\begin{aligned} e^{i(\varphi+\psi)} &= \cos(\varphi+\psi) + i\sin(\varphi+\psi) \\ e^{i\varphi}e^{i\psi} &= (\cos\varphi + i\sin\varphi)(\cos\psi + i\sin\psi) \\ &= (\cos\varphi\cos\psi - \sin\varphi\sin\psi) \\ &\qquad + i\,(\sin\varphi\cos\psi + \cos\varphi\sin\psi) \end{aligned}$$

from there also follows

$$\cos(\varphi+\psi) = \cos\varphi\cos\psi - \sin\varphi\sin\psi$$
$$\sin(\varphi+\psi) = \sin\varphi\cos\psi + \cos\varphi\sin\psi$$

If the reader doubts that the multiplication of two exponentiations $e^{i\varphi}e^{i\psi}$ and the exponentiation of the sum of arguments $e^{i(\varphi+\psi)}$ are equal

$$e^{i\varphi}e^{i\psi} = e^{i(\varphi+\psi)},$$

then here's quite convincing proof of that using the linearity of differentiation.

○ At first, I'll define what an exponentiation e^w, sometimes written as $\exp(w)$, is (w for "whatever it is", something of absolutely any kind).

> e^w is such and only such function whose differential $d(e^w)$ is $e^w dw$. When for some function $\substack{\text{omnino}\\\text{aliquid}} \mapsto f$ the differential df equals to $fd\left(\substack{\text{omnino}\\\text{aliquid}}\right)$, then such a function is an exponentiation — $\exp\left(\substack{\text{omnino}\\\text{aliquid}}\right)$.

The differential of a product of functions can be found as

$$d(f_1 f_2) = (df_1)f_2 + f_1(df_2)$$

— the (Leibniz') "product rule". For two exponentiations (the imaginary unit i is constant, thus $di = 0$ and $\forall\xi\ d(i\xi) = (di)\xi + i(d\xi) = id\xi$)

$$d\big(e^{i\varphi}e^{i\psi}\big) = \big(de^{i\varphi}\big)e^{i\psi} + e^{i\varphi}\big(de^{i\psi}\big)$$
$$= e^{i\varphi}d(i\varphi)e^{i\psi} + e^{i\varphi}e^{i\psi}d(i\psi) = e^{i\varphi}e^{i\psi}\,i\big(d\varphi + d\psi\big).$$

For an exponentiation of the sum of arguments, the differential is

$$d\big(e^{i(\varphi+\psi)}\big) = e^{i(\varphi+\psi)}\,i\,d\big(\varphi+\psi\big).$$

.....

....

All rotations are a "group" in the sense of the group theory in abstract algebra.

Évariste Galois

permutations are group too

A *group* $\{\mathcal{G},\ \text{“}\circ\text{”}\}$ is a set $\mathcal{G}$ with a binary operation "$\circ$", where *(i)* "$\circ$" is associative, $(h \circ g) \circ f = h \circ (g \circ f)$ for any $h, g, f \in \mathcal{G}$, *(ii)* $\mathcal{G}$ has a two-sided identity element, $\exists\, e \in \mathcal{G} \colon g \circ e = e \circ g = g \;\; \forall g \in \mathcal{G}$, and *(iii)* every element of $\mathcal{G}$ has an inverse element, $\forall g \in \mathcal{G}\;\exists g^{-1} \colon g \circ g^{-1} = g^{-1} \circ g = e$.

Group operation "$\circ$" can be commutative or non-commutative. A group with a commutative operation is called commutative group, or abelian group. If there exists at least one pair of elements $a, b \in \mathcal{G}$ for which $a \circ b \neq b \circ a$, then this group is non-abelian (non-commutative).

A *field* $\{\mathcal{F},\ \text{“}+\text{”},\ \text{“}*\text{”}\}$ is a set $\mathcal{F}$ with two binary operations "$+$" and "$*$", when *(i)* $\{\mathcal{F}, \text{“}+\text{”}\}$ is commutative group, $a + b = b + a$ $\forall a, b \in \mathcal{F}$, with "additive" identity $e_0 \in \mathcal{F}$, *(ii)* $\{\mathcal{F} \backslash e_0, \text{“}*\text{”}\}$ is commutative group, $a * b = b * a \;\forall a, b \in \mathcal{F} \backslash e_0$, with "multiplicative" identity $e_1 \in \mathcal{F} \backslash e_0$ $(e_1 \neq e_0)$, $a, b \in \mathcal{F} \backslash e_0 \Rightarrow a * b \neq e_0$ for $\forall a \neq e_0$ and $\forall b \neq e_0$ *(iii)* ......

....

Two imaginary units $j \neq i$

$$(a + bi)(c + dj) = ac + cbi + adj + bdij$$

By introducing third imaginary unit $k:\ ij = k$

$$i^2 = j^2 = k^2 = ijk = -1. \tag{12.1}$$

and making multiplication not commutative but anticommutative

$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik, \tag{12.2}$$

meet quaternions.

In fact there's only two different imaginary units, and a third one is there just for short notation ($k = ij$), say they're only i and j, $i^2 = j^2 = -1$, $i \neq j$, and a quaternion is $q = a + bi + cj + dij$, with the really-easy-to-memorize equalities for multiplication $jij = i$, $iji = j$ and obvious $ijj = -i$, $iij = -j$, from there follows the anticommutativity itself, $ji = -ij$.

The following are absolutely the same, just a result of denoting $k = ij$, $ijij = ijk = k^2$ and the "really-easy-to-memorize" are derivable as well $ijij = (iji)j = j^2 = i(jij) = i^2$.

...

§ 13. Variations

Further in this book pretty often will be used the operation of varying. It is similar to the differentiation.

Variations are seen as infinitesimal displacements, compatible with the constraints. If there are no constraints for variable x, then variations δx are completely random. But when

$$y \mapsto x : \; x = x(y)$$

is a function of some independent argument y, then

$$\delta x = x'(y)\,\delta y.$$

Variations are similar to differentials. As example, for δx and δy as variations of x and y, with finite u and v, $u\delta x + v\delta y = \delta w$ is written even when δw is not a variation of w. In such a case, δw is a single symbol.

Surely if $u = u(x, y)$, $v = v(x, y)$ and $\partial_x v = \partial_y u$ ($\frac{\partial}{\partial x} v = \frac{\partial}{\partial y} u$), then the sum $\delta w = u\delta x + v\delta y$ will be a variation of some w.

Varying the identity (11.5), we get

$$\delta\boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O} \cdot \delta\boldsymbol{O}^{\mathsf{T}}.$$

This tensor is antisymmetric, and thus is representable via its companion pseudovector $\boldsymbol{\delta o}$ as

$$\delta\boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\delta o} \times \boldsymbol{E}.$$

We have the following relations

$$\delta\boldsymbol{O} = \boldsymbol{\delta o} \times \boldsymbol{O}, \;\; \boldsymbol{\delta o} = -\frac{1}{2}\Big(\delta\boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}}\Big)_{\times}, \tag{13.1}$$

similar to (11.9). Vector $\boldsymbol{\delta o}$ of an infinitesimal rotation is not "a variation of $\mathbf{o}$", but a single symbol.

An infinitesimal rotation is defined by vector $\boldsymbol{\delta o}$, but a finite rotation is also possible to represent as a vector

...

§ 14. Polar decomposition

Any tensor of the second complexity $\boldsymbol{F}$ with $\det F_{ij} \neq 0$, that is a not singular tensor, can be decomposed as

...

Example. Polar decompose tensor $\boldsymbol{C} = C_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$, where $\boldsymbol{e}_k$ are mutually perpendicular unit vectors of basis, and C_{ij} are tensor's components

$$C_{ij} = \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix}$$

$$\boldsymbol{O} = O_{ij} \boldsymbol{e}_i \boldsymbol{e}_j = \boldsymbol{O}_1 \cdot \boldsymbol{O}_2$$

$$O_{ij} = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$\boldsymbol{C} = \boldsymbol{O} \cdot \boldsymbol{S}_{\mathsf{R}}, \;\; \boldsymbol{O}^{\mathsf{T}} \cdot \boldsymbol{C} = \boldsymbol{S}_{\mathsf{R}}$$

$$\boldsymbol{C} = \boldsymbol{S}_{\mathsf{L}} \cdot \boldsymbol{O}, \;\; \boldsymbol{C} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{S}_{\mathsf{L}}$$

$$S_{\mathsf{R}ij} = \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix}$$

$$S_{\mathsf{L}ij} = \begin{bmatrix} 104/5 & 47/5 & 5 \\ 47/5 & -129/5 & -10 \\ 5 & -10 & 3 \end{bmatrix}$$

...

§ 15. In the oblique basis

Until now, a basis of the three mutually perpendicular unit vectors $\boldsymbol{e}_i$ was used. However, such a basis is not the only possible one. Any linearly independent (non-coplanar) vectors $\boldsymbol{a}_i$ can be chosen as the basis ones.

Then, some vector $\boldsymbol{v}$ can be represented as a linear combination of basis vectors $\boldsymbol{a}_i$ with scalar multipliers v^i

$$\boldsymbol{v} = v^i \boldsymbol{a}_i. \qquad (15.1)$$

The summation convention gains the new conditions : a summation index is repeated at the different levels of the same monomial, and a free index stays at the same height in any part of the expression ($a_i = b_{ij} c^j$ is correct, $a_i = b^i_{kk}$ is wrong twice).

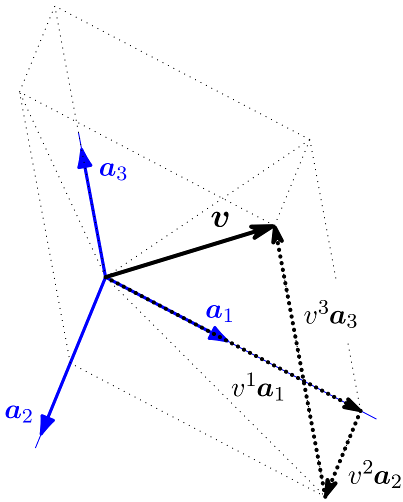


figure 6

Here the "•"-product $\boldsymbol{v} \cdot \boldsymbol{a}_i$ is no longer equal to the component v^i : $\boldsymbol{v} \cdot \boldsymbol{a}_i = v^k \boldsymbol{a}_k \cdot \boldsymbol{a}_i \neq v^i$, since $\boldsymbol{a}_i \cdot \boldsymbol{a}_k \neq \delta_{ik}$.

Fortunately, for any $\boldsymbol{a}_i$ there's another — the "superscript" — triple of vectors $\boldsymbol{a}^i$, such that

$$\begin{gathered} \boldsymbol{a}_i \cdot \boldsymbol{a}^j = \delta_i^j, \;\; \boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j, \\ \boldsymbol{E} = \boldsymbol{a}^i \boldsymbol{a}_i = \boldsymbol{a}_i \boldsymbol{a}^i. \end{gathered} \qquad (15.2)$$

The triple $\boldsymbol{a}^i$ is called cobasis or reciprocal (dual) basis. (15.2) is the defining property of the cobasis.

The case when a basis coincides with its cobasis $\boldsymbol{e}^i = \boldsymbol{e}_i$ is possible only when the basis vectors are mutually perpendicular to one another and all are one unit long. Such a basis is called "cartesian".

To get, for example, the first cobasis vector $\boldsymbol{a}^1$

$$\begin{cases} \boldsymbol{a}^1 \cdot \boldsymbol{a}_1 = 1 \\ \boldsymbol{a}^1 \cdot \boldsymbol{a}_2 = 0 \\ \boldsymbol{a}^1 \cdot \boldsymbol{a}_3 = 0 \end{cases} \Rightarrow \begin{cases} \boldsymbol{a}^1 \cdot \boldsymbol{a}_1 = 1 \\ \gamma \boldsymbol{a}^1 = \boldsymbol{a}_2 \times \boldsymbol{a}_3 \end{cases} \Rightarrow \begin{cases} \boldsymbol{a}^1 = {}^1\!/\gamma \, \boldsymbol{a}_2 \times \boldsymbol{a}_3 \\ \gamma = \boldsymbol{a}_2 \times \boldsymbol{a}_3 \cdot \boldsymbol{a}_1 \end{cases}$$

Here, the coefficient γ turned out to be equal (up to a sign) to the volume of parallelepiped built on vectors $\boldsymbol{a}_i$. The coincidence that in § 7 the same volume was presented as $\sqrt{g}$ is not accidental. Because γ equals to the square root of the gramian $g \equiv \det g_{ij}$ — the determinant of the symmetric J. P. Gram matrix $g_{ij} \equiv \boldsymbol{a}_i \cdot \boldsymbol{a}_j$.

○ The proof resembles the derivation of (6.3). The "triple product" $\boldsymbol{a}_i \times \boldsymbol{a}_j \cdot \boldsymbol{a}_k$ in some orthonormal basis $\boldsymbol{e}_i$ can be calculated as the determinant (with "−" for a "left" triplet $\boldsymbol{a}_i$) by the rows

$$\in_{ijk} \equiv \boldsymbol{a}_i \times \boldsymbol{a}_j \cdot \boldsymbol{a}_k = \pm \det \begin{bmatrix} \boldsymbol{a}_i \cdot \boldsymbol{e}_1 & \boldsymbol{a}_i \cdot \boldsymbol{e}_2 & \boldsymbol{a}_i \cdot \boldsymbol{e}_3 \\ \boldsymbol{a}_j \cdot \boldsymbol{e}_1 & \boldsymbol{a}_j \cdot \boldsymbol{e}_2 & \boldsymbol{a}_j \cdot \boldsymbol{e}_3 \\ \boldsymbol{a}_k \cdot \boldsymbol{e}_1 & \boldsymbol{a}_k \cdot \boldsymbol{e}_2 & \boldsymbol{a}_k \cdot \boldsymbol{e}_3 \end{bmatrix}$$

or by the columns

$$\in_{pqr} \equiv \boldsymbol{a}_p \times \boldsymbol{a}_q \cdot \boldsymbol{a}_r = \pm \det \begin{bmatrix} \boldsymbol{a}_p \cdot \boldsymbol{e}_1 & \boldsymbol{a}_q \cdot \boldsymbol{e}_1 & \boldsymbol{a}_r \cdot \boldsymbol{e}_1 \\ \boldsymbol{a}_p \cdot \boldsymbol{e}_2 & \boldsymbol{a}_q \cdot \boldsymbol{e}_2 & \boldsymbol{a}_r \cdot \boldsymbol{e}_2 \\ \boldsymbol{a}_p \cdot \boldsymbol{e}_3 & \boldsymbol{a}_q \cdot \boldsymbol{e}_3 & \boldsymbol{a}_r \cdot \boldsymbol{e}_3 \end{bmatrix}.$$

As proven in (6.2), the determinant of the matrix product of matrices is equal to the multiplication of the determinants of each of these matrices, here $\in_{ijk} \in_{pqr}$. The elements of the matrix product are the sums like $\boldsymbol{a}_i \cdot \boldsymbol{e}_s \boldsymbol{a}_p \cdot \boldsymbol{e}_s = \boldsymbol{a}_i \cdot \boldsymbol{e}_s \boldsymbol{e}_s \cdot \boldsymbol{a}_p = \boldsymbol{a}_i \cdot \boldsymbol{E} \cdot \boldsymbol{a}_p = \boldsymbol{a}_i \cdot \boldsymbol{a}_p$, therefore

$$\in_{ijk} \in_{pqr} = \det \begin{bmatrix} \boldsymbol{a}_i \cdot \boldsymbol{a}_p & \boldsymbol{a}_i \cdot \boldsymbol{a}_q & \boldsymbol{a}_i \cdot \boldsymbol{a}_r \\ \boldsymbol{a}_j \cdot \boldsymbol{a}_p & \boldsymbol{a}_j \cdot \boldsymbol{a}_q & \boldsymbol{a}_j \cdot \boldsymbol{a}_r \\ \boldsymbol{a}_k \cdot \boldsymbol{a}_p & \boldsymbol{a}_k \cdot \boldsymbol{a}_q & \boldsymbol{a}_k \cdot \boldsymbol{a}_r \end{bmatrix};$$

$i{=}p{=}1,\ j{=}q{=}2,\ k{=}r{=}3 \ \Rightarrow \in_{123} \in_{123} = \det\limits_{i,j} (\boldsymbol{a}_i \cdot \boldsymbol{a}_j) = \det\limits_{i,j} g_{ij}.$ ●

Representing $\boldsymbol{a}^1$ and the other cobasis vectors as the sum

$$\pm 2 \sqrt{g} \, \boldsymbol{a}^1 = \boldsymbol{a}_2 \times \boldsymbol{a}_3 - \boldsymbol{a}_3 \times \boldsymbol{a}_2 \,,$$

приходим к общей формуле (with "$-$" for "левой" тройки $\boldsymbol{a}_i$)

$$\boldsymbol{a}^i = \pm \frac{1}{2\sqrt{g}}\, e^{ijk} \boldsymbol{a}_j \times \boldsymbol{a}_k, \quad \sqrt{g} \equiv \pm\, \boldsymbol{a}_1 \times \boldsymbol{a}_2 \bullet \boldsymbol{a}_3 > 0\,. \tag{15.3}$$

Here e^{ijk} по-прежнему символы чётности перестановки (± 1 or 0): $e^{ijk} \equiv e_{ijk}$. Произведение $\boldsymbol{a}_j \times \boldsymbol{a}_k = \in_{jkn} \boldsymbol{a}^n$, компоненты тензора Ле́ви-Чиви́ты $\in_{jkn} = \pm\, e_{jkn}\sqrt{g}$, and by (6.4) $e^{ijk} e_{jkn} = 2\delta^i_n$. Thus

$$\boldsymbol{a}^1 = \pm {}^1\!/\!\sqrt{g}\,(\boldsymbol{a}_2 \times \boldsymbol{a}_3), \;\; \boldsymbol{a}^2 = \pm {}^1\!/\!\sqrt{g}\,(\boldsymbol{a}_3 \times \boldsymbol{a}_1), \;\; \boldsymbol{a}^3 = \pm {}^1\!/\!\sqrt{g}\,(\boldsymbol{a}_1 \times \boldsymbol{a}_2).$$

Example. Get cobasis for basis $\boldsymbol{a}_i$ when

$$\begin{aligned} \boldsymbol{a}_1 &= \boldsymbol{e}_1 + \boldsymbol{e}_2, \\ \boldsymbol{a}_2 &= \boldsymbol{e}_1 + \boldsymbol{e}_3, \\ \boldsymbol{a}_3 &= \boldsymbol{e}_2 + \boldsymbol{e}_3. \end{aligned}$$

$$\sqrt{g} = -\,\boldsymbol{a}_1 \times \boldsymbol{a}_2 \bullet \boldsymbol{a}_3 = -\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 2;$$

$$-\,\boldsymbol{a}_2 \times \boldsymbol{a}_3 = \det \begin{bmatrix} 1 & \boldsymbol{e}_1 & 0 \\ 0 & \boldsymbol{e}_2 & 1 \\ 1 & \boldsymbol{e}_3 & 1 \end{bmatrix} = \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3,$$

$$-\,\boldsymbol{a}_3 \times \boldsymbol{a}_1 = \det \begin{bmatrix} 0 & \boldsymbol{e}_1 & 1 \\ 1 & \boldsymbol{e}_2 & 1 \\ 1 & \boldsymbol{e}_3 & 0 \end{bmatrix} = \boldsymbol{e}_1 + \boldsymbol{e}_3 - \boldsymbol{e}_2,$$

$$-\,\boldsymbol{a}_1 \times \boldsymbol{a}_2 = \det \begin{bmatrix} 1 & \boldsymbol{e}_1 & 1 \\ 1 & \boldsymbol{e}_2 & 0 \\ 0 & \boldsymbol{e}_3 & 1 \end{bmatrix} = \boldsymbol{e}_2 + \boldsymbol{e}_3 - \boldsymbol{e}_1$$

and finally

$$\begin{aligned} \boldsymbol{a}^1 &= \tfrac{1}{2}\big(\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\big), \\ \boldsymbol{a}^2 &= \tfrac{1}{2}\big(\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3\big), \\ \boldsymbol{a}^3 &= \tfrac{1}{2}\big(-\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3\big). \end{aligned}$$

Имея кобазис, возможно не только разложить по нему любой вектор (рисунок 7), но и найти коэффициенты разложения (15.1):

$$\begin{aligned} \boldsymbol{v} &= v^i \boldsymbol{a}_i = v_i \boldsymbol{a}^i, \\ \boldsymbol{v} \bullet \boldsymbol{a}^i &= v^k \boldsymbol{a}_k \bullet \boldsymbol{a}^i = v^i, \quad v_i = \boldsymbol{v} \bullet \boldsymbol{a}_i. \end{aligned} \tag{15.4}$$

Коэффициенты v_i называются ковариантными компонентами вектора $\boldsymbol{v}$, а v^i — его контравариантными* компонентами.

Есть литература о тензорах, где introducing existænce and различают ковариантные и контравариантные... векторы (and "covectors", "dual vectors"). Не сто́ит вводить читателя в заблуждение : вектор-то один и тот же, просто разложение по двум разным базисам даёт два набора компонент.

От векторов перейдём к тензорам второй сложности. Имеем четыре комплекта диад: $\boldsymbol{a}_i \boldsymbol{a}_j$, $\boldsymbol{a}^i \boldsymbol{a}^j$, $\boldsymbol{a}_i \boldsymbol{a}^j$, $\boldsymbol{a}^i \boldsymbol{a}_j$. Согласующиеся коэффициенты в декомпозиции тензора называются его контравариантными, ковариантными и смешанными компонентами:

$$
\begin{gathered}
{}^2\boldsymbol{B} = B^{ij} \boldsymbol{a}_i \boldsymbol{a}_j = B_{ij} \boldsymbol{a}^i \boldsymbol{a}^j = B^{i}_{\cdot j} \boldsymbol{a}_i \boldsymbol{a}^j = B_{i}^{\cdot j} \boldsymbol{a}^i \boldsymbol{a}_j \,, \\
B^{ij} = \boldsymbol{a}^i \cdot {}^2\boldsymbol{B} \cdot \boldsymbol{a}^j, \;\; B_{ij} = \boldsymbol{a}_i \cdot {}^2\boldsymbol{B} \cdot \boldsymbol{a}_j \,, \\
B^{i}_{\cdot j} = \boldsymbol{a}^i \cdot {}^2\boldsymbol{B} \cdot \boldsymbol{a}_j \,, \;\; B_{i}^{\cdot j} = \boldsymbol{a}_i \cdot {}^2\boldsymbol{B} \cdot \boldsymbol{a}^j.
\end{gathered}
\tag{15.5}
$$

For двух видов смешанных компонент точка в индексе это просто свободное место: у $B^{i}_{\cdot j}$ верхний индекс "i" — первый, а нижний "j" — второй.

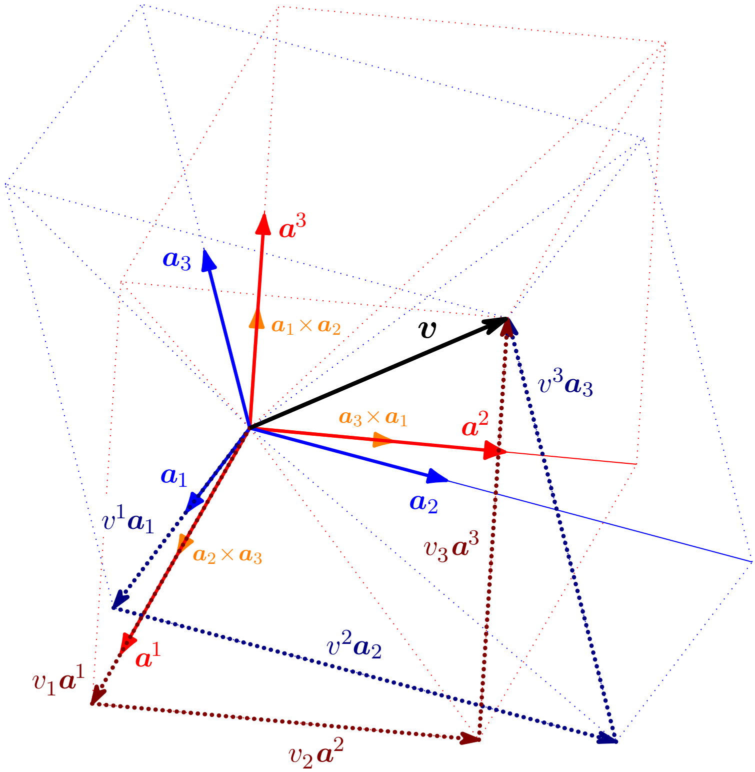
The components of the unit ("metric") tensor $\boldsymbol{E}$

$$
\begin{gathered}
\boldsymbol{E} = \boldsymbol{a}^k \boldsymbol{a}_k = \boldsymbol{a}_k \boldsymbol{a}^k = g_{jk} \boldsymbol{a}^j \boldsymbol{a}^k = g^{jk} \boldsymbol{a}_j \boldsymbol{a}_k : \\
\boldsymbol{a}_i \cdot \boldsymbol{E} \cdot \boldsymbol{a}^j = \boldsymbol{a}_i \cdot \boldsymbol{a}^j = \delta_i^j, \;\; \boldsymbol{a}^i \cdot \boldsymbol{E} \cdot \boldsymbol{a}_j = \boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j \,, \\
\boldsymbol{a}_i \cdot \boldsymbol{E} \cdot \boldsymbol{a}_j = \boldsymbol{a}_i \cdot \boldsymbol{a}_j \equiv g_{ij}, \;\; \boldsymbol{a}^i \cdot \boldsymbol{E} \cdot \boldsymbol{a}^j = \boldsymbol{a}^i \cdot \boldsymbol{a}^j \equiv g^{ij} ; \\
\boldsymbol{E} \cdot \boldsymbol{E} = g_{ij} \boldsymbol{a}^i \boldsymbol{a}^j \cdot g^{nk} \boldsymbol{a}_n \boldsymbol{a}_k = g_{ij} g^{jk} \boldsymbol{a}^i \boldsymbol{a}_k = \boldsymbol{E} \;\Rightarrow\; g_{ij} g^{jk} = \delta_i^k.
\end{gathered}
\tag{15.6}
$$

Besides (15.2) and (15.3), there's one more way to find the cobasis vectors — via matrix g^{ij}, which is the inverse of the Gram matrix g_{ij}. And vice versa :

$$
\begin{gathered}
\boldsymbol{a}^i = \boldsymbol{E} \cdot \boldsymbol{a}^i = g^{jk} \boldsymbol{a}_j \boldsymbol{a}_k \cdot \boldsymbol{a}^i = g^{jk} \boldsymbol{a}_j \, \delta_k^i = g^{ji} \boldsymbol{a}_j \,, \\
\boldsymbol{a}_i = \boldsymbol{E} \cdot \boldsymbol{a}_i = g_{jk} \boldsymbol{a}^j \boldsymbol{a}^k \cdot \boldsymbol{a}_i = g_{jk} \boldsymbol{a}^j \, \delta_i^k = g_{ji} \boldsymbol{a}^j.
\end{gathered}
\tag{15.7}
$$

* Потому что они меняются обратно (contra) изменению длин базисных векторов $\boldsymbol{a}_i$.



$$\boldsymbol{a}_1 \times \boldsymbol{a}_2 \bullet \boldsymbol{a}_3 = \sqrt{g} = 0.56274$$
$$1/\sqrt{g} = 1.77703$$

$$\boldsymbol{a}_i \bullet \boldsymbol{a}^j = \begin{bmatrix} \boldsymbol{a}_1 \bullet \boldsymbol{a}^1 & \boldsymbol{a}_1 \bullet \boldsymbol{a}^2 & \boldsymbol{a}_1 \bullet \boldsymbol{a}^3 \\ \boldsymbol{a}_2 \bullet \boldsymbol{a}^1 & \boldsymbol{a}_2 \bullet \boldsymbol{a}^2 & \boldsymbol{a}_2 \bullet \boldsymbol{a}^3 \\ \boldsymbol{a}_3 \bullet \boldsymbol{a}^1 & \boldsymbol{a}_3 \bullet \boldsymbol{a}^2 & \boldsymbol{a}_3 \bullet \boldsymbol{a}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_i^j$$

figure 7
"The decomposition of a vector in an oblique basis"

Example. Using the inverse of the Gram matrix, get the cobasis for basis $\boldsymbol{a}_i$ if

$$\boldsymbol{a}_1 = \boldsymbol{e}_1 + \boldsymbol{e}_2,$$
$$\boldsymbol{a}_2 = \boldsymbol{e}_1 + \boldsymbol{e}_3,$$
$$\boldsymbol{a}_3 = \boldsymbol{e}_2 + \boldsymbol{e}_3.$$

$$g_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \det g_{ij} = 4,$$

$$\operatorname{adj} g_{ij} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}^{\mathsf{T}},$$

$$g^{ij} = g_{ij}^{-1} = \frac{\operatorname{adj} g_{ij}}{\det g_{ij}} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

Using $\boldsymbol{a}^i = g^{ij} \boldsymbol{a}_j$

$$\boldsymbol{a}^1 = g^{11}\boldsymbol{a}_1 + g^{12}\boldsymbol{a}_2 + g^{13}\boldsymbol{a}_3 = \tfrac{1}{2}\boldsymbol{e}_1 + \tfrac{1}{2}\boldsymbol{e}_2 - \tfrac{1}{2}\boldsymbol{e}_3,$$
$$\boldsymbol{a}^2 = g^{21}\boldsymbol{a}_1 + g^{22}\boldsymbol{a}_2 + g^{23}\boldsymbol{a}_3 = \tfrac{1}{2}\boldsymbol{e}_1 - \tfrac{1}{2}\boldsymbol{e}_2 + \tfrac{1}{2}\boldsymbol{e}_3,$$
$$\boldsymbol{a}^3 = g^{31}\boldsymbol{a}_1 + g^{32}\boldsymbol{a}_2 + g^{33}\boldsymbol{a}_3 = -\tfrac{1}{2}\boldsymbol{e}_1 + \tfrac{1}{2}\boldsymbol{e}_2 + \tfrac{1}{2}\boldsymbol{e}_3.$$

....

Единичный тензор (unit tensor, identity tensor, metric tensor)

$\boldsymbol{E} \cdot \boldsymbol{\xi} = \boldsymbol{\xi} \cdot \boldsymbol{E} = \boldsymbol{\xi} \quad \forall \boldsymbol{\xi}$

$\boldsymbol{E} \cdot\cdot\, \boldsymbol{ab} = \boldsymbol{ab} \cdot\cdot\, \boldsymbol{E} = \boldsymbol{a} \cdot \boldsymbol{E} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}$

$\boldsymbol{E} \cdot\cdot\, \boldsymbol{A} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = \operatorname{trace} \boldsymbol{A}$

$\boldsymbol{E} \cdot\cdot\, \boldsymbol{A} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = \operatorname{trace} \boldsymbol{A} \neq \text{notanymore}\, A_{jj}$

Thus for, say, trace of some tensor $\boldsymbol{A} = A_{ij} \boldsymbol{r}^i \boldsymbol{r}^j$: $\boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = \operatorname{trace} \boldsymbol{A}$, you have

$\boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = A_{ij} \boldsymbol{r}^i \boldsymbol{r}^j \cdot\cdot\, \boldsymbol{r}_{\partial k} \boldsymbol{r}^k = A_{ij} \boldsymbol{r}^i \cdot \boldsymbol{r}^j = A_{ij} g^{ij}$

...

Тензор поворота (the rotation tensor)

$\boldsymbol{P} = \boldsymbol{a}_i \mathring{\boldsymbol{a}}^i = \boldsymbol{a}^i \mathring{\boldsymbol{a}}_i = \boldsymbol{P}^{-\mathsf{T}}$

$\boldsymbol{P}^{-1} = \mathring{\boldsymbol{a}}_i \boldsymbol{a}^i = \mathring{\boldsymbol{a}}^i \boldsymbol{a}_i = \boldsymbol{P}^{\mathsf{T}}$

$\boldsymbol{P}^{\mathsf{T}} = \mathring{\boldsymbol{a}}^i \boldsymbol{a}_i = \mathring{\boldsymbol{a}}_i \boldsymbol{a}^i = \boldsymbol{P}^{-1}$

...

... Характеристическое уравнение (10.2) быстро приводит к тождеству Кэ́ли–Га́мильтона (Cayley–Hamilton)

$$\begin{gathered} -\boldsymbol{B}\cdot\boldsymbol{B}\cdot\boldsymbol{B}+\mathrm{I}\,\boldsymbol{B}\cdot\boldsymbol{B}-\mathrm{II}\,\boldsymbol{B}+\mathrm{III}\,\boldsymbol{E}={}^2\mathbf{0}, \\ -\boldsymbol{B}^3+\mathrm{I}\,\boldsymbol{B}^2-\mathrm{II}\,\boldsymbol{B}+\mathrm{III}\,\boldsymbol{E}={}^2\mathbf{0}. \end{gathered} \tag{15.8}$$

§ 16. Tensor functions

In the concept of function $y = f(x)$ as of mapping (morphism) $f\colon x \mapsto y$, an input (argument) x and an output (result) y may be tensors of any complexities.

Consider at least a scalar function of a bivalent tensor $\varphi = \varphi(\boldsymbol{B})$. Examples are $\boldsymbol{B} \cdot\cdot\, \boldsymbol{\Phi}$ (or $\boldsymbol{p}\cdot\boldsymbol{B}\cdot\boldsymbol{q}$) and $\boldsymbol{B}\cdot\cdot\,\boldsymbol{B}$. Then in each basis $\boldsymbol{a}_i$ paired with cobasis $\boldsymbol{a}^i$ we have function $\varphi(B_{ij})$ of nine numeric arguments — components B_{ij} of tensor $\boldsymbol{B}$. For example

$$\varphi(\boldsymbol{B}) = \boldsymbol{B}\cdot\cdot\,\boldsymbol{\Phi} = B_{ij}\,\boldsymbol{a}^i\boldsymbol{a}^j \cdot\cdot\, \boldsymbol{a}_m\boldsymbol{a}_n\Phi^{mn} = B_{ij}\Phi^{ji} = \varphi(B_{ij}).$$

With any transition to a new basis, the result doesn't change: $\varphi(B_{ij}) = \varphi(B'_{ij}) = \varphi(\boldsymbol{B})$.

Differentiation of $\varphi(\boldsymbol{B})$ looks like

$$d\varphi = \frac{\partial\varphi}{\partial B_{ij}}\,dB_{ij} = \frac{\partial\varphi}{\partial\boldsymbol{B}}\cdot\cdot\,d\boldsymbol{B}^{\mathsf{T}}. \tag{16.1}$$

Tensor ${}^{\partial\varphi}\!/\!_{\partial\boldsymbol{B}}$ is called the derivative of function φ by argument $\boldsymbol{B}$; $d\boldsymbol{B}$ is the differential of tensor $\boldsymbol{B}$, $d\boldsymbol{B} = dB_{ij}\,\boldsymbol{a}^i\boldsymbol{a}^j$; ${}^{\partial\varphi}\!/\!_{\partial B_{ij}}$ are components (contravariant ones) of ${}^{\partial\varphi}\!/\!_{\partial\boldsymbol{B}}$

$$\boldsymbol{a}^i\cdot\frac{\partial\varphi}{\partial\boldsymbol{B}}\cdot\boldsymbol{a}^j = \frac{\partial\varphi}{\partial\boldsymbol{B}}\cdot\cdot\,\boldsymbol{a}^j\boldsymbol{a}^i = \frac{\partial\varphi}{\partial B_{ij}} \;\Leftrightarrow\; \frac{\partial\varphi}{\partial\boldsymbol{B}} = \frac{\partial\varphi}{\partial B_{ij}}\,\boldsymbol{a}_i\boldsymbol{a}_j\,.$$

...

$$\begin{gathered} \varphi(\boldsymbol{B}) = \boldsymbol{B}\cdot\cdot\,\boldsymbol{\Phi} \\ d\varphi = d(\boldsymbol{B}\cdot\cdot\,\boldsymbol{\Phi}) = d\boldsymbol{B}\cdot\cdot\,\boldsymbol{\Phi} = \boldsymbol{\Phi}\cdot\cdot\,d\boldsymbol{B} = \boldsymbol{\Phi}^{\mathsf{T}}\cdot\cdot\,d\boldsymbol{B}^{\mathsf{T}} \\ d\varphi = \frac{\partial\varphi}{\partial\boldsymbol{B}}\cdot\cdot\,d\boldsymbol{B}^{\mathsf{T}},\;\; \frac{\partial\,(\boldsymbol{B}\cdot\cdot\,\boldsymbol{\Phi})}{\partial\boldsymbol{B}} = \boldsymbol{\Phi}^{\mathsf{T}} \end{gathered}$$

$\boldsymbol{p} \cdot \boldsymbol{B} \cdot \boldsymbol{q} = \boldsymbol{B} \cdot\cdot\, \boldsymbol{q}\boldsymbol{p}$

$$\frac{\partial\, (\boldsymbol{p} \cdot \boldsymbol{B} \cdot \boldsymbol{q})}{\partial \boldsymbol{B}} = \boldsymbol{p}\boldsymbol{q}$$

...

$$\varphi(\boldsymbol{B}) = \boldsymbol{B} \cdot\cdot\, \boldsymbol{B}$$
$$d\varphi = d\,(\boldsymbol{B} \cdot\cdot\, \boldsymbol{B}) = d...$$

...

Но согласно опять-таки (15.8) $-\boldsymbol{B}^2 + \mathrm{I}\boldsymbol{B} - \mathrm{II}\boldsymbol{E} + \mathrm{III}\boldsymbol{B}^{-1} = {}^2\mathbf{0}$, поэтому

....

Скалярная функция $\varphi(\boldsymbol{B})$ называется изотропной, если она не чувствительна к повороту аргумента:

$$\varphi(\boldsymbol{B}) = \varphi(\boldsymbol{O} \cdot \mathring{\boldsymbol{B}} \cdot \boldsymbol{O}^{\mathsf{T}}) = \varphi(\mathring{\boldsymbol{B}}) \quad \forall \boldsymbol{O} = \boldsymbol{a}_i \mathring{\boldsymbol{a}}^i = \boldsymbol{a}^i \mathring{\boldsymbol{a}}_i = \boldsymbol{O}^{-\mathsf{T}}$$

для любого ортогонального тензора $\boldsymbol{O}$ (тензора поворота, § 11).

Симметричный тензор $\boldsymbol{B}^{\mathsf{S}}$ полностью определяется тройкой инвариантов и угловой ориентацией собственных осей (они же взаимно ортогональны, § 10). Ясно, что изотропная функция $\varphi(\boldsymbol{B}^{\mathsf{S}})$ симметричного аргумента является функцией, входы-аргументы которой — только инварианты $\mathrm{I}(\boldsymbol{B}^{\mathsf{S}})$, $\mathrm{II}(\boldsymbol{B}^{\mathsf{S}})$, $\mathrm{III}(\boldsymbol{B}^{\mathsf{S}})$. Дифференцируется такая функция согласно (**??**), где транспонирование излишне.

§ 17. Spatial differentiation

Let at each point of some region of a three-dimensional space the variable ς be defined. Then it's said that there is a *tensor field* $\varsigma = \varsigma(\boldsymbol{r})$, where $\boldsymbol{r}$ is the location vector ("radius" vector) of a point in space. The variable ς can be a tensor of any complexity. An example of scalar field — the temperature field in a medium, of vector field — the velocities of liquid particles.

So, a *tensor field* is a tensor varying from point to point (variable in space, coordinate dependent). The concept of a tensor field is not about a field in algebra that considers operations "+" and "$*$" with the specific properties.

Not only for solving applied problems, but often in "pure theory", instead of the argument $\boldsymbol{r}$, the set of curvilinear coordinates q^i is used. If only one coordinate from the set continuously changes, this gives a coordinate line. In three-dimensional space, a point is an intersection of three coordinate lines (figure 8). The location vector of a point is a function from the set of coordinates

$$\boldsymbol{r} = \boldsymbol{r}(q^i). \tag{17.1}$$

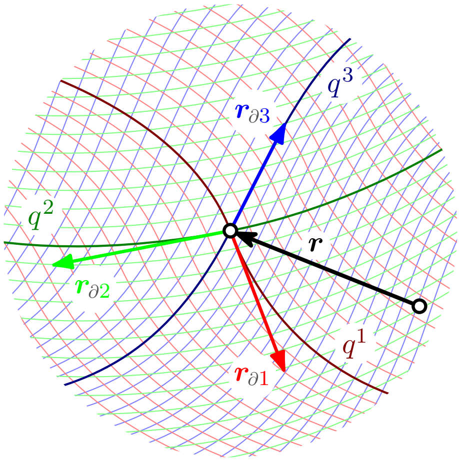


figure 8

The most often used sets of coordinates are rectangular ("cartesian") coordinates, spherical coordinates and cylindrical coordinates. Any curvilinear coordinates can be converted to the rectangular coordinates and vice versa by means of a locally invertible ("one-to-one") mapping at each point.

...

The differential of a function presents the change in linearization of that function.

....

partial derivatives

$$\partial_i \equiv \frac{\partial}{\partial q^i}$$

...

the differential of $\varsigma(q^i)$

$$d\varsigma = \frac{\partial \varsigma}{\partial q^i} dq^i = \partial_i \varsigma \, dq^i \tag{17.2}$$

...

Linearity

$$\partial_i \bigl(\lambda p + \mu q\bigr) = \lambda \bigl(\partial_i p\bigr) + \mu \bigl(\partial_i q\bigr) \tag{17.3}$$

The “product rule” is about the differentiation of $p \circ q$, the product of p and q :

$$d\bigl(p \circ q\bigr) = \bigl(p + dp\bigr) \circ \bigl(q + dq\bigr) - p \circ q = dp \circ q + p \circ dq + dp \circ dq\,,$$

where $dp \circ dq = \infty^{-1}\bigl(dp, dq\bigr)$ — the product $dp \circ dq$ of two infinitesimals is considered to be infinitesimally smaller than either infinitesimal (dp or dq) alone. Thus, the term $dp \circ dq$ can be dropped as negligible compared to terms with one infinitesimal. And then

$$d\bigl(p \circ q\bigr) = \bigl(dp\bigr) \circ q + p \circ \bigl(dq\bigr). \tag{17.4}$$

For a partial derivative $\partial_i \equiv \dfrac{\partial}{\partial q^i}$

$$\partial_i \bigl(p \circ q\bigr) = \bigl(\partial_i p\bigr) \circ q + p \circ \bigl(\partial_i q\bigr). \tag{17.5}$$

...

Local basis $\boldsymbol{r}_{\partial i}$

The differential of the location vector $\boldsymbol{r}(q^i)$ is

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial q^i}\, dq^i = dq^i \boldsymbol{r}_{\partial i}\,, \quad \boldsymbol{r}_{\partial i} \equiv \frac{\partial \boldsymbol{r}}{\partial q^i} \equiv \partial_i \boldsymbol{r} \tag{17.6}$$

...

Local cobasis $\boldsymbol{r}^i$

$\boldsymbol{r}^i \bullet \boldsymbol{r}_{\partial j} = \delta^i_j$

The (spatial) differential of something

$$\begin{gathered} \frac{\partial \varsigma}{\partial \boldsymbol{r}} = \frac{\partial \varsigma}{\partial q^i} \boldsymbol{r}^i = \partial_i \varsigma\, \boldsymbol{r}^i \\ d\varsigma = \frac{\partial \varsigma}{\partial \boldsymbol{r}} \bullet d\boldsymbol{r} = \partial_i \varsigma\, \boldsymbol{r}^i \bullet \boldsymbol{r}_{\partial j}\, dq^j = \partial_i \varsigma\, dq^i \end{gathered} \tag{17.7}$$

...

The unit dyad (metric tensor) $\boldsymbol{E}$, neutral (4.7) for the "$\cdot$"-product, can be represented as

$$\boldsymbol{E} = \boldsymbol{r}^i \boldsymbol{r}_{\partial i} = \underbrace{\boldsymbol{r}^i \partial_i}_{\boldsymbol{\nabla}} \boldsymbol{r} = \boldsymbol{\nabla} \boldsymbol{r}, \tag{17.8}$$

where appears the differential "nabla" operator

$$\boldsymbol{\nabla} \equiv \boldsymbol{r}^i \partial_i . \tag{17.9}$$

...

With $\boldsymbol{\nabla}$, the spatial differential of something is

$$\begin{gathered} \frac{\partial \zeta}{\partial \boldsymbol{r}} = \partial_i \boldsymbol{\zeta} \boldsymbol{r}^i \\ d\varsigma = \frac{\partial \varsigma}{\partial \boldsymbol{r}} \cdot d\boldsymbol{r} = d\boldsymbol{r} \cdot \boldsymbol{\nabla} \varsigma = \partial_i \varsigma \, dq^i \end{gathered} \tag{17.10}$$

$d\boldsymbol{r} = d\boldsymbol{r} \cdot \overbrace{\boldsymbol{\nabla} \boldsymbol{r}}^{\boldsymbol{E}}$

...

Divergence of the dyadic product of two vectors

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\boldsymbol{a}\boldsymbol{b}) = \boldsymbol{r}^i \partial_i \cdot (\boldsymbol{a}\boldsymbol{b}) = \boldsymbol{r}^i \cdot \partial_i (\boldsymbol{a}\boldsymbol{b}) = \boldsymbol{r}^i \cdot (\partial_i \boldsymbol{a}) \boldsymbol{b} + \boldsymbol{r}^i \cdot \boldsymbol{a} (\partial_i \boldsymbol{b}) = \\ = (\boldsymbol{r}^i \cdot \partial_i \boldsymbol{a}) \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{r}^i (\partial_i \boldsymbol{b}) = (\boldsymbol{r}^i \partial_i \cdot \boldsymbol{a}) \boldsymbol{b} + \boldsymbol{a} \cdot (\boldsymbol{r}^i \partial_i \boldsymbol{b}) = \\ = (\boldsymbol{\nabla} \cdot \boldsymbol{a}) \boldsymbol{b} + \boldsymbol{a} \cdot (\boldsymbol{\nabla} \boldsymbol{b}) \end{aligned} \tag{17.11}$$

— here's no need to expand vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, expanding just the differential operator $\boldsymbol{\nabla}$.

....

The gradient of the "$\times$"-product of two vectors, applying "the product rule" (17.5) and relation (7.6) for any two vectors (a partial derivative ∂_i of some vector by scalar coordinate q^i is a vector too)

$$\begin{aligned} \boldsymbol{\nabla} (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{r}^i \partial_i (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{r}^i (\partial_i \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \partial_i \boldsymbol{b}) = \\ = \boldsymbol{r}^i (\partial_i \boldsymbol{a} \times \boldsymbol{b} - \partial_i \boldsymbol{b} \times \boldsymbol{a}) = \boldsymbol{r}^i \partial_i \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{r}^i \partial_i \boldsymbol{b} \times \boldsymbol{a} = \\ = \boldsymbol{\nabla} \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{\nabla} \boldsymbol{b} \times \boldsymbol{a} . \end{aligned} \tag{17.12}$$

...

The gradient of the "$\boldsymbol{\cdot}$"-product of two vectors

$$\boldsymbol{\nabla}\big(\boldsymbol{a}\cdot\boldsymbol{b}\big) = \boldsymbol{r}^i\partial_i\big(\boldsymbol{a}\cdot\boldsymbol{b}\big) = \boldsymbol{r}^i\big(\partial_i\boldsymbol{a}\big)\cdot\boldsymbol{b} + \boldsymbol{r}^i\boldsymbol{a}\cdot\big(\partial_i\boldsymbol{b}\big) = \\ = \big(\boldsymbol{r}^i\partial_i\boldsymbol{a}\big)\cdot\boldsymbol{b} + \boldsymbol{r}^i\big(\partial_i\boldsymbol{b}\big)\cdot\boldsymbol{a} = \big(\boldsymbol{\nabla}\boldsymbol{a}\big)\cdot\boldsymbol{b} + \big(\boldsymbol{\nabla}\boldsymbol{b}\big)\cdot\boldsymbol{a}. \quad (17.13)$$

§ 18. The integral theorems

For vector fields, the integral theorems are known — the Gauss' or Ostrogradsky's (Остроградского) divergence theorem and Stokes' circulation theorem.

Gauss' theorem (divergence theorem) enables an integral taken over a volume to be replaced by one taken over the closed surface bounding that volume, and vice versa.

Stokes' theorem enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve. Stokes' theorem relates a line integral around a closed path to a surface integral over what is called a *capping surface* of the path.

The Gauss's divergence theorem

(wikipedia) Divergence theorem

This theorem is about how to replace a volume integral with a surface one (and vice versa).

В этой теореме рассматривается поток (ef)flux вектора через ограничивающую объём $\mathcal{V}$ за́мкнутую поверхность $\mathcal{O}(\partial\mathcal{V})$...

$$\oint\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{n}\cdot\boldsymbol{a}\, d\mathcal{O} = \int\limits_{\mathcal{V}} \boldsymbol{\nabla}\cdot\boldsymbol{a}\, d\mathcal{V}, \quad (18.1)$$

the "$\boldsymbol{\cdot}$"-product always commutes

$$\oint\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{a}\cdot\boldsymbol{n}\, d\mathcal{O} = \int\limits_{\mathcal{V}} \boldsymbol{a}\cdot\boldsymbol{\nabla}\, d\mathcal{V},$$

$\boldsymbol{n}\,d\mathcal{O} = d\boldsymbol{\mathcal{O}}$

$$\oint\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{a}\cdot d\boldsymbol{\mathcal{O}} = \int\limits_{\mathcal{V}} \boldsymbol{\nabla}\cdot\boldsymbol{a}\, d\mathcal{V},$$

$\boldsymbol{n}$ is the unit vector of outward normal to surface $\mathcal{O}(\partial\mathcal{V})$.

Volume $\mathcal{V}$ нарезается тремя семействами координатных поверхностей на множество бесконечно малых элементов. Поток через поверхность $\mathcal{O}(\partial\mathcal{V})$ равен сумме потоков через края получившихся элементов. В бесконечной малости каждый такой элемент — малюсенький локальный дифференциальный кубик (параллелепипед). .... The flux of vector $\boldsymbol{a}$ through the faces of a small cube is equal to $\sum_{i=1}^{6} \boldsymbol{n}_i \cdot \boldsymbol{a} \mathcal{O}_i$, а поток через объём $d\mathcal{V}$ этого малого кубика равен $\boldsymbol{\nabla} \cdot \boldsymbol{a}\, d\mathcal{V}$.

A similar interpretation of this theorem is given, for example, in Richard Feynman's lectures [86].

(рисунок с кубиками)

to dice — нареза́ть кубиками

small cube, little cube

локально ортонормальные координаты $\boldsymbol{\xi} = \xi_i \boldsymbol{n}_i$, $d\boldsymbol{\xi} = d\xi_i \boldsymbol{n}_i$, $\boldsymbol{\nabla} = \boldsymbol{n}_i \partial_i$

разложение вектора $\boldsymbol{a} = a_i \boldsymbol{n}_i$

The Stokes' circulation theorem

(wikipedia) Stokes' theorem

This theorem is formulated as the equality

$$\oint\limits_{\mathcal{C}(\partial\mathcal{O})} \boldsymbol{a} \cdot d\boldsymbol{\mathcal{C}} = \int\limits_{\mathcal{O}} \boldsymbol{n} \cdot (\boldsymbol{\nabla} \times \boldsymbol{a})\, d\mathcal{O}. \tag{18.2}$$

$$\oint\limits_{\mathcal{C}(\partial\mathcal{O})} \boldsymbol{a} \cdot d\boldsymbol{\mathcal{C}} = \int\limits_{\mathcal{O}} (\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot \boldsymbol{n}\, d\mathcal{O}.$$

$$\oint\limits_{\mathcal{C}(\partial\mathcal{O})} \boldsymbol{a} \cdot d\boldsymbol{\mathcal{C}} = \int\limits_{\mathcal{O}} (\boldsymbol{\nabla} \times \boldsymbol{a}) \cdot d\boldsymbol{\mathcal{O}}.$$

...

§ 19. Curvature tensors

The *Riemann curvature tensor* or *Riemann–Christoffel tensor* (after **Bernhard Riemann** and **Elwin Bruno Christoffel**) is the most common method used to express the curvature of Riemannian manifolds. It's a tensor field, it assigns a tensor to each point of a Riemannian manifold, that measures the extent to which the metric tensor is not locally isometric to that of "flat" space. The curvature tensor measures noncommutativity of the covariant derivative, and as such is the integrability obstruction for the existænce of an isometry with "flat" space.

Dealing with tensor fields in curvilinear coordinates (§ 17), the location vector (radius vector) of a point was introduced as a function (17.1) of these coordinates, $\boldsymbol{r} = \boldsymbol{r}(q^i)$. From this relation originate many others, such as

- ✓ the vectors of local tangent basis $\boldsymbol{r}_{\partial i} \equiv {}^{\partial \boldsymbol{r}}\!/\!_{\partial q^i} \equiv \partial_i \boldsymbol{r}$,
- ✓ the components $g_{ij} \equiv \boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j}$ and $g^{ij} \equiv \boldsymbol{r}^i \cdot \boldsymbol{r}^j = g_{ij}^{-1}$ of the unit ("metric") tensor $\boldsymbol{E} = \boldsymbol{r}_{\partial i}\boldsymbol{r}^i = \boldsymbol{r}^i\boldsymbol{r}_{\partial i} = g_{jk}\boldsymbol{r}^j\boldsymbol{r}^k = g^{jk}\boldsymbol{r}_{\partial j}\boldsymbol{r}_{\partial k}$,
- ✓ the vectors of local dual (reciprocal) cotangent basis $\boldsymbol{r}^i \cdot \boldsymbol{r}_{\partial j} = \delta^i_j$, $\boldsymbol{r}^i = g^{ij}\boldsymbol{r}_{\partial j}$,
- ✓ the differential nabla-operator (Hamilton's operator, Hamiltonian) $\boldsymbol{\nabla} \equiv \boldsymbol{r}^i \partial_i$, $\boldsymbol{E} = \boldsymbol{\nabla}\boldsymbol{r}$,
- ✓ the full differential $d\boldsymbol{\xi} = d\boldsymbol{r} \cdot \boldsymbol{\nabla}\boldsymbol{\xi}$,
- ✓ the partial derivatives of tangent basis vectors (the second partial derivatives of $\boldsymbol{r}$) $\boldsymbol{r}_{\partial i \partial j} \equiv \partial_i \partial_j \boldsymbol{r} = \partial_i \boldsymbol{r}_{\partial j}$,
- ✓ the Christoffel symbols of metric connection $\Gamma^k_{ij} \equiv \boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}^k$ and $\Gamma_{ij\dot{k}} \equiv \boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}_{\partial k}$.

Представим теперь, что функция $\boldsymbol{r}(q^k)$ не известна, но затó в каждой точке пространства известны шесть независимых компонент положительно определённой (all Gram matrices are non-negative definite) симметричной метрической матрицы Gram $g_{ij}(q^k)$.

the Gram matrix (or Gramian)

Билинейная форма ...

...

Поскольку шесть функций $g_{ij}(q^k)$ происходят от векторной функции $\boldsymbol{r}(q^k)$, то между элементами g_{ij} существуют некие соотношения.

Differential $d\boldsymbol{r}$ (17.6) is exact. This is true if and only if second partial derivatives commute:

$$d\boldsymbol{r} = \boldsymbol{r}_{\partial k}\, dq^k \;\;\Leftrightarrow\;\; \partial_i \boldsymbol{r}_{\partial j} = \partial_j \boldsymbol{r}_{\partial i} \;\text{ or }\; \boldsymbol{r}_{\partial i \partial j} = \boldsymbol{r}_{\partial j \partial i}.$$

Но это условие ужé обеспечено симметрией g_{ij}

...

metric (“affine”) connection ∇_i, её же называют “covariant derivative”

$$\boldsymbol{r}_{\partial i \partial j} = \underbrace{\boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}^k}_{\Gamma_{ij}^k} \overbrace{\phantom{\boldsymbol{r}^k}\!\!\!\!\boldsymbol{r}_{\partial k}}^{\boldsymbol{E}} = \underbrace{\boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}_{\partial k}}_{\Gamma_{ijk}} \boldsymbol{r}^k$$

$\Gamma_{ij}^{k} \boldsymbol{r}_{\partial k} = \boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}^k \boldsymbol{r}_{\partial k} = \boldsymbol{r}_{\partial i \partial j}$

covariant derivative (affine connection) is only defined for vector fields

$\boldsymbol{\nabla v} = \boldsymbol{r}^i \partial_i \big(v^j \boldsymbol{r}_{\partial j} \big) = \boldsymbol{r}^i \big(\partial_i v^j \boldsymbol{r}_{\partial j} + v^j \boldsymbol{r}_{\partial i \partial j} \big)$

$\boldsymbol{\nabla v} = \boldsymbol{r}^i \boldsymbol{r}_{\partial j} \nabla_i v^j, \;\; \nabla_i v^j \equiv \partial_i v^j + \Gamma_{in}^{j} v^n$

$\boldsymbol{\nabla} \boldsymbol{r}_{\partial i} = \boldsymbol{r}^k \partial_k \boldsymbol{r}_{\partial i} = \boldsymbol{r}^k \boldsymbol{r}_{\partial k \partial i} = \boldsymbol{r}^k \boldsymbol{r}_{\partial n} \Gamma_{ki}^{n}, \;\; \nabla_i \boldsymbol{r}_{\partial n} = \Gamma_{in}^{k} \boldsymbol{r}_{\partial k}$

Christoffel symbols describe a metric (“affine”) connection, that is how the basis changes from point to point.

символы Christoffel’я это “components of connection” in local coordinates

...

torsion tensor ${}^3\mathfrak{T}$ with components

$$\mathfrak{T}_{ij}^{k} = \Gamma_{ij}^{k} - \Gamma_{ji}^{k}$$

determines the antisymmetric part of a connection

...

симметрия $\Gamma_{ij\bullet k} = \Gamma_{ji\bullet k}$, поэтому $3^3 - 3 \cdot 3 = 18$ разных (независимых) $\Gamma_{ij\bullet k}$

$$\begin{aligned}
&\Gamma_{ij}^{n} g_{nk} = \Gamma_{ij\bullet k} = \boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}_{\partial k} = \\
&= \tfrac{1}{2}\big(\boldsymbol{r}_{\partial i \partial j} + \boldsymbol{r}_{\partial j \partial i}\big) \cdot \boldsymbol{r}_{\partial k} + \tfrac{1}{2}\big(\boldsymbol{r}_{\partial j \partial k} - \boldsymbol{r}_{\partial k \partial j}\big) \cdot \boldsymbol{r}_{\partial i} + \tfrac{1}{2}\big(\boldsymbol{r}_{\partial i \partial k} - \boldsymbol{r}_{\partial k \partial i}\big) \cdot \boldsymbol{r}_{\partial j} = \\
&= \tfrac{1}{2}\big(\boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}_{\partial k} + \boldsymbol{r}_{\partial i \partial k} \cdot \boldsymbol{r}_{\partial j}\big) + \tfrac{1}{2}\big(\boldsymbol{r}_{\partial j \partial i} \cdot \boldsymbol{r}_{\partial k} + \boldsymbol{r}_{\partial j \partial k} \cdot \boldsymbol{r}_{\partial i}\big) - \tfrac{1}{2}\big(\boldsymbol{r}_{\partial k \partial i} \cdot \boldsymbol{r}_{\partial j} + \boldsymbol{r}_{\partial k \partial j} \cdot \boldsymbol{r}_{\partial i}\big) = \\
&\qquad = \frac{1}{2}\Big(\partial_i(\boldsymbol{r}_{\partial j} \cdot \boldsymbol{r}_{\partial k}) + \partial_j(\boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial k}) - \partial_k(\boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j})\Big) = \\
&\qquad\qquad = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right).
\end{aligned} \tag{19.1}$$

Все символы Christoffel'я тождественно равны нулю лишь в ортонормальной (декартовой) системе. (А какие они для косоугольной?)

Дальше: $d\boldsymbol{r}_{\partial i} = d\boldsymbol{r} \cdot \boldsymbol{\nabla} \boldsymbol{r}_{\partial i} = dq^k \partial_k \boldsymbol{r}_{\partial i} = \boldsymbol{r}_{\partial k \partial i} dq^k$ — тоже полные дифференциалы.

$$d\boldsymbol{r}_{\partial k} = \partial_i \boldsymbol{r}_{\partial k} dq^i = \frac{\partial \boldsymbol{r}_{\partial k}}{\partial q^1} dq^1 + \frac{\partial \boldsymbol{r}_{\partial k}}{\partial q^2} dq^2 + \frac{\partial \boldsymbol{r}_{\partial k}}{\partial q^3} dq^3$$

Поэтому $\partial_i \partial_j \boldsymbol{r}_{\partial k} = \partial_j \partial_i \boldsymbol{r}_{\partial k}$, $\partial_i \boldsymbol{r}_{\partial j \partial k} = \partial_j \boldsymbol{r}_{\partial i \partial k}$, и трёхиндексный объект из векторов третьих частных производных

$$\boldsymbol{r}_{\partial i \partial j \partial k} \equiv \partial_i \partial_j \partial_k \boldsymbol{r} = \partial_i \boldsymbol{r}_{\partial j \partial k} \tag{19.2}$$

симметричен по первому и второму индексам (а не только по второму и третьему). И тогда равен нулю ${}^4\mathbf{0}$ следующий тензор четвёртой сложности — *Riemann curvature tensor* (or *Riemann–Christoffel tensor*)

$$^4\boldsymbol{\mathfrak{R}} = \mathfrak{R}_{hijk} \boldsymbol{r}^h \boldsymbol{r}^i \boldsymbol{r}^j \boldsymbol{r}^k, \;\; \mathfrak{R}_{hijk} \equiv \boldsymbol{r}_{\partial h} \cdot \left(\boldsymbol{r}_{\partial j \partial i \partial k} - \boldsymbol{r}_{\partial i \partial j \partial k}\right). \tag{19.3}$$

Выразим компоненты $\mathfrak{R}_{ijkn}$ через метрическую матрицу g_{ij}. Начнём с дифференцирования локального кобазиса:

$$\boldsymbol{r}^i \cdot \boldsymbol{r}_{\partial k} = \delta_k^i \;\Rightarrow\; \partial_j \boldsymbol{r}^i \cdot \boldsymbol{r}_{\partial k} + \boldsymbol{r}^i \cdot \boldsymbol{r}_{\partial j \partial k} = 0 \;\Rightarrow\; \partial_j \boldsymbol{r}^i = -\Gamma_{jk}^i \boldsymbol{r}^k.$$

...

The six independent components $\mathfrak{R}_{1212}$, $\mathfrak{R}_{1213}$, $\mathfrak{R}_{1223}$, $\mathfrak{R}_{1313}$, $\mathfrak{R}_{1323}$, $\mathfrak{R}_{2323}$.

...

The symmetric bivalent *Ricci curvature tensor*

$$\boldsymbol{\mathscr{R}} \equiv \frac{1}{4}\,\mathfrak{R}_{abij}\,\boldsymbol{r}^a \times \boldsymbol{r}^b \boldsymbol{r}^i \times \boldsymbol{r}^j = \frac{1}{4}\in^{abp}\in^{ijq}\mathfrak{R}_{abij}\,\boldsymbol{r}_{\partial p}\boldsymbol{r}_{\partial q} = \mathscr{R}^{pq}\boldsymbol{r}_{\partial p}\boldsymbol{r}_{\partial q}$$

(coefficient $\frac{1}{4}$ is used here for convenience) with components

$$\begin{aligned}
&\mathscr{R}^{11} = \frac{1}{g}\,\mathfrak{R}_{2323}\,,\\
&\mathscr{R}^{21} = \frac{1}{g}\,\mathfrak{R}_{1323}\,,\;\; \mathscr{R}^{22} = \frac{1}{g}\,\mathfrak{R}_{1313}\,,\\
&\mathscr{R}^{31} = \frac{1}{g}\,\mathfrak{R}_{1223}\,,\;\; \mathscr{R}^{32} = \frac{1}{g}\,\mathfrak{R}_{1213}\,,\;\; \mathscr{R}^{33} = \frac{1}{g}\,\mathfrak{R}_{1212}\,.
\end{aligned}$$

The equality of the Ricci tensor to zero $\boldsymbol{\mathscr{R}} = {}^2\mathbf{0}$ (in components these are six equations $\mathscr{R}^{ij} = \mathscr{R}^{ji} = 0$) is the necessary condition of integrability ("compatibility") for determining the location ("radius") vector $\boldsymbol{r}(q^k)$ by the known field $g_{ij}(q^k)$.

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There are many books worth mentioning that cover only the apparatus of tensor calculus [98, 99, 100, 101, 102]. However, the index notation (it's when tensors are seen as the sets of components) is still more popular than the direct indexless look. The direct notation is widely used, for example, in the appendices to the books by Anatoliy I. Lurie (Анатолий И. Лурье) [27, 28]. "Теория упругости" ("The theory of elasticity") by Вениамин Блох (Veniamin Blokh) [7] is as well written in the direct indexless notation. The vivid description of the vector fields theory is presented in R. Feynman's lectures [86]. Also, information about tensor calculus is the part of the unusual and interesting book by C. Truesdell [55].

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