Vadique Myself

PHYSICS *of* ELASTIC CONTINUA

FOREWORD

I presented the following models of an elastic continuum in this book: the nonlinear and the linear ones, the micropolar and the classical momentless; the three-dimensional, the two-dimensional (shells and plates) and the one-dimensional (rods, including thin-walled ones). I also explained the fundamentals of dynamics — oscillations, waves and stability. For the thermoelasticity and the magnetoelasticity, I gave the summary of the classical theories of the thermodynamics and the electrodynamics. The dynamics of destruction is described via the theories of defects and fractures. The approaches to modeling of human-made inhomogeneous materials, "composites", are also shown.

The word "continua" in the title says that a body (a medium) is modeled here not as a discrete collection of particles, but as a continuous field of location vectors, a continuous matter. It gives a large convenience, because the apparatus of calculus of infinitesimals can be used for such models.

When I just began writing this book, I thought of a reader who is pretty acquainted with "higher" mathematics. But later I decided to conduct such an acquaintance by myself, and yet, as a side effect, every reader with any knowledge of math can comprehend the content of the book.

The book is written using the compact and elegant direct indexless tensor notation. The mathematical apparatus for interpreting the direct tensor relations is located in the first chapter.

I am writing this book simultaneously in the two languages, English and Russian. The reader is free to pick any language of the two.

github.com/VadiqueMe/PhysicsOfElasticContinua

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chapter 1
MATHEMATICAL APPARATUS

Mathematics, or math for short, is abstract. Abstract is the adjective of math, math is the noun of abstract. “Abstract”, “theoretical” and “mathematical” are synonyms. When someone is doing math, he’s playing a game in the far-far-away magical world of imagination.

For example, numbers are not real entities at all. They are purely imaginary concepts, nonsense. We cannot experience, sensate numbers, can’t touch or smell them. Yep, one can compose stories about them, such as $1 + 1 = 2$. But no one can ever feel, perceive such an operation, since there are no such things as *one* and *two**.

And “synthesized by imagination”— it’s not only about numbers. Geometric objects, be it a point, a line, a triangle or a plane, and all kinds of adventures with them are mind derived as well.

§1. The ancient but intuitive geometry

Εὐκλείδης Eὐkleídēs Euclid, ευκλείδειος euclidean

the plane geometry, or the two-dimensional euclidean geometry

Στοιχεῖα Stoikheîa Elements, Principles

(1.1) Points

* I’m not about *two apples* or *two similar bananas* for *a couple of days*, but about the number “two” itself.

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Oρος α′ (1) Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.	Term α′ (1) A point is that which has no part.

This description shows that Euclid imagines a point as an indivisible location, without width, length or breadth.

(1.2) Lines, curved and straight

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Oρος β′ (2) Γραμμὴ δὲ μῆκος ἀπλατές.	Term β′ (2) A line is breadthless length.

"Line" is the second primitive term in the Elements. "Breadthless length" says that a line will have one dimension, length, but it won't have breadth. The terms "length" and "breadth" are not defined in the Elements.

Linear lines

(1.3) A relation between lines and points

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Oρος γ′ (3) Γραμμῆς δὲ πέρατα σημεῖα.	Term γ′ (3) The ends of a line are points.

This statement doesn't mention how many ends a line can have.

(1.4) Do straight lines exists?

The hypothesis of the existence of straight lines.

The existence of Euclidean straight lines in space.

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Ορος δʹ (4) Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται.	Term δʹ (4) A straight line is a line which lies evenly with the points on itself.

To draw a straight line by hand is absolutely impossible.

(1.5) Vectors. Lines and vectors

(1.6) The existence of vectors. Do vectors exist?

(1.7) Continuity of line

(1.8) A point of reference

(1.9) Translation as the easiest kind of motion. Translations and vectors

(1.10) Straight line and vector

A (geometric) vector may be like a straight line with an arrow at one of its ends. Then it is fully described (characterized) by the magnitude and the direction.

Within the abstract algebra, the word *vector* is about any object which can be summed with similar objects and scaled (multiplied) by scalars, and vector space is a synonym of linear space. Therefore I clarify that in this book *vector* is nothing else than three-dimensional geometric (Ευκλείδειος, Euclidean) vector.

Why are vectors always straight (linear)?

(a) Vectors are linear (straight), they cannot be curved.

(b) Vectors are neither straight nor curved. A vector has the magnitude and the direction. A vector is not a line or a curve, albeit it can be represented by a straight line.

Vector can't be thought of as a line.

(1.11) The line which figures real numbers

often just "number line"

(1.12) What is a distance?

(1.13) Plane and more dimensional space

(1.14) Distance on plane or more dimensional space

(1.15) What is an angle?

angle $\equiv$ inclination /slope, slant/ of two lines

two lines sharing a common point are usually called intersecting lines

angle $\equiv$ the amount of rotation of line or plane within space

angle $\equiv$ the result of the dot product of two unit vectors gives angle's cosine

(1.16) Differentiation of continuous into small differential chunks

small differential chunks

infinitesimal (infinitely small)

A mention of tensors may scare away the reader, commonly avoiding needless complications. Don't be afraid: tensors are used just due to their wonderful property of the invariance — the independence from a coordinate system.

§ 2. Vector

I propose to begin familiarizing with tensors via memoirs about such a phenomenon as a vector.

- ✓ A *point* has position in space. The only characteristic that distinguishes one point from another is its position.
- ✓ A *vector* has both magnitude and direction, but no specific position in space.

(2.1) What is a vector?

What is "linear"?
(1) straight
(2) relating to, resembling, or having a graph that is a straight line

All vectors are linear objects.

Examples of vectors:

✓ A force acts on an object.

✓ The velocity of an object describes what's happening with this object at an instant.

Multiplication of a vector by a scalar

Multiplication by the minus one

The Newton's action–reaction principle "действие равно противодействию по магнитуде и обратно ему по направлению".

Each mechanical interaction of two objects is characterized by two forces that act on both interacting objects. These forces can be represented as two vectors that are equal in magnitude and reverse in direction.

Multiplying a vector by the negative one -1 reverses the vector's direction but doesn't change its magnitude.

(2.2) The addition and subtraction

The sum (combination) of two or more vectors is the new "resultant" vector. There are two similar methods to calculate the resultant vector geometrically.

The "head to tail method" involves lining up the head of one vector with the tail of another. Here the resultant goes from the initial point (the "tail") of the first addend to the end point (the "head") of the second addend when the tail (the initial point) of the second one coincides with the head (the end point) of the first one.

[.... figure here ....]

The "parallelogram method" ...

[.... figure here ....]

The vector addition is commutative

$$\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}.$$

....

$$\boldsymbol{p} + \boldsymbol{q},$$
$$\boldsymbol{p} - \boldsymbol{q} = \boldsymbol{p} + \left(-\boldsymbol{q}\right) = \boldsymbol{p} + \left(-1\right)\boldsymbol{q}.$$

For every action, there's an equal (in magnitude) and opposite (in direction) reaction force.

A vector may be also represented as the sum (combination) of some trio of other vectors, called "basis", when the each of the three is scaled by a number (coefficient). Such a representation is called a "linear combination" of basis vectors. A list (array, tuple) of coefficients alone, without basis vectors, is not enough and can't represent a vector.

....

To get the numerical relations from the vector ones, a coordinate system is introduced, and on its axes the vector relations are projected.

....

У сами́х по себе векторов как элементов векторного пространства компонент нет. Vector components appear only when a certain basis is chosen, then any vector can be decomposed into components. В разных базисах компоненты одного и того же вектора отличаются друг от друга.

Here it is — a vector, $\boldsymbol{v}$ looks like a suitable name for it.

Like all geometric vectors, $\boldsymbol{v}$ is pretty well characterized by the two mutually independent properties: its length (magnitude, norm, modulus) and its direction in space. This characterization is complete, so some two vectors with the same magnitude and the same direction are considered equal.

Every vector exists objectively by itself, independently of methods and units of measurement of both lengths and directions, including any abstractions of such units and methods.

(2.3) The method of coordinates

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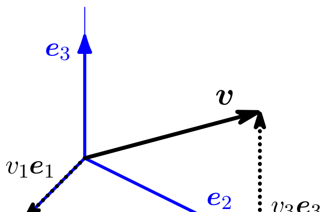

By choosing some mutually perpendicular unit vectors $\boldsymbol{e}_i$ as the basis for measurements, I introduce the rectangular ("cartesian") coordinates.

Three $(i = 1, 2, 3)$ basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are needed for a three-dimensional — 3D — space.

Within such a system, "$\bullet$"-products of the basis vectors are equal to the Kronecker delta

$$\boldsymbol{e}_i \bullet \boldsymbol{e}_j = \delta_{ij} = \begin{cases} 1, \; i = j \\ 0, \; i \neq j \end{cases}$$

for any orthonormal basis.

Decomposing vector $\boldsymbol{v}$ in some orthonormal basis $\boldsymbol{e}_i$ $(i = 1, 2, 3)$, we get coefficients v_i — the components of vector $\boldsymbol{v}$ in that basis (fig. 1)

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 \equiv \sum_{i=1}^{3} v_i\boldsymbol{e}_i \equiv v_i\boldsymbol{e}_i\,, \;\; v_i = \boldsymbol{v} \bullet \boldsymbol{e}_i\,. \tag{2.1}$$

Here and hereinafter, the Einstein's summation convention is accepted: an index repeated twice (and no more than twice) in a single term implies a summation over this index. And a non-repeating index is called "free", and it is identical in the both parts of the equation. These are examples:

$$\sigma = \tau_{ii}\,, \;\; p_j = n_i\tau_{ij}\,, \;\; m_i = e_{ijk}x_jf_k\,, \;\; a_i = \lambda b_i + \mu c_i\,.$$

(But equations $a = b_{kkk}$, $c = f_i + g_k$, $a_{ij} = k_i\gamma_{ij}$ are incorrect.)

Having components of a vector in an orthonormal basis, the length of this vector is retrieved by the "Πυθαγόρας' equation"

$$\boldsymbol{v} \bullet \boldsymbol{v} = v_i\boldsymbol{e}_i \bullet v_j\boldsymbol{e}_j = v_i\delta_{ij}v_j = v_iv_i\,, \;\; \|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \bullet \boldsymbol{v}} = \sqrt{v_iv_i}\,. \tag{2.2}$$

The magnitude represents the length independent of direction. The direction of a vector in space is measured by the three angles (cosines of angles) between this vector and each of the basis ones:

$$\cos \measuredangle\big(\boldsymbol{v}\,\widehat{\;}\,\boldsymbol{e}_i\big) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \bullet \boldsymbol{e}_i = \frac{v_i}{\sqrt{v_jv_j}} \;\; \Leftrightarrow \;\; \underbrace{v_i}_{\boldsymbol{v} \bullet \boldsymbol{e}_i} = \|\boldsymbol{v}\| \cos \measuredangle(\boldsymbol{v}, \boldsymbol{e}_i)\,. \tag{2.3}$$

Measurement of angles. The cosine of an angle between two vectors is the same as the dot product of these vectors if their magnitudes are equal to the one unit of length

When the magnitudes of two vectors are equal to the one unit of length, then the cosine of the least angle between them is the same as the dot product of these vectors. Any vector with the non-unit magnitude (but the null vector) can be "normalized" via dividing a vector by its magnitude.

$$\cos \measuredangle(\boldsymbol{v}, \boldsymbol{w}) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \bullet \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}\,.$$

To accompany the magnitude, which represents the length independent of direction, there's a way to represent the direction of a vector independent of its length. For this purpose, the unit vectors (the vectors with the magnitude of 1) are used.

A rotation matrix is just a transform that expresses the basis vectors of the input space in a different orientation. The length of the basis vectors will be the same, and the origin will not change. Also, the angle between the basis vectors will not change. All that changes is the relative direction of all of the basis vectors.

Therefore, a rotation matrix is not really just a "rotation" matrix; it is an orientation matrix.

There are also pseudovectors, waiting for the reader below in § 7.

The angle between two random vectors. According to (2.3)

$$\cos \measuredangle(\boldsymbol{v}, \boldsymbol{e}_m) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \bullet \boldsymbol{e}_m = \frac{v_m}{\sqrt{v_j v_j}}\,,$$
$$\cos \measuredangle(\boldsymbol{w}, \boldsymbol{e}_n) = \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} \bullet \boldsymbol{e}_n = \frac{w_n}{\sqrt{w_k w_k}}\,.$$

The length (2.2) and the direction in space (2.3), that can be measured by the means of the trio of basic vectors, describe a vector. And every vector possesses these properties*. However, this is not enough ("not sufficient" in jargon of the math books).

* And what is the direction of the null vector ("(vanishing) vector") $\mathbf{0}$ with the zero length $\|\mathbf{0}\| = 0$? (The zero vector without a magnitude ends exactly where it begins and thus it is not directed anywhere, its direction is *undefined*.)

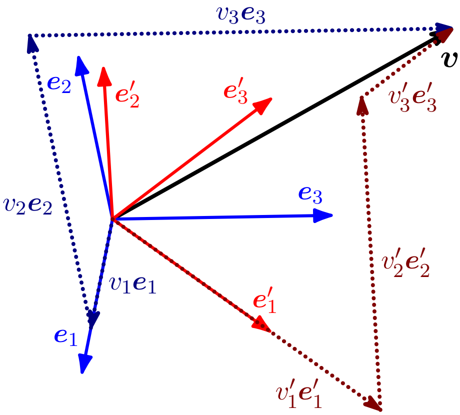

figure 2

A vector is not just a collection of components in some basis.

A triple of pairwise perpendicular unit vectors can only rotate and thereby it can characterize the angular orientation of other vectors.

The decomposition of the same vector $\boldsymbol{v}$ in the two cartesian systems with basis unit vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}'_i$ (fig. 2) gives

$$\boldsymbol{v} = v_i \boldsymbol{e}_i = v'_i \boldsymbol{e}'_i ,$$

where

$$v_i = \boldsymbol{v} \cdot \boldsymbol{e}_i = v'_k \boldsymbol{e}'_k \cdot \boldsymbol{e}_i ,$$
$$v'_i = \boldsymbol{v} \cdot \boldsymbol{e}'_i = v_k \boldsymbol{e}_k \cdot \boldsymbol{e}'_i .$$

Appeared here two-index objects (the two-dimensional arrays) $o_{k'i} \equiv \boldsymbol{e}'_k \cdot \boldsymbol{e}_i$ and $o_{ki'} \equiv \boldsymbol{e}_k \cdot \boldsymbol{e}'_i$ are used to shorten the formulas.

Написáть о пассивном повороте, описанном ниже, и об активном повороте из § 12 ....

The "•"-product (dot product) of two vectors is commutative — that is, the swapping of multipliers doesn't change the result. Thus

$$o_{k'i} = \boldsymbol{e}'_k \cdot \boldsymbol{e}_i = \cos \measuredangle(\boldsymbol{e}'_k , \boldsymbol{e}_i) = \cos \measuredangle(\boldsymbol{e}_i , \boldsymbol{e}'_k) = \boldsymbol{e}_i \cdot \boldsymbol{e}'_k = o_{ik'} , \tag{2.3a}$$
$$o_{ki'} = \boldsymbol{e}_k \cdot \boldsymbol{e}'_i = \cos \measuredangle(\boldsymbol{e}_k , \boldsymbol{e}'_i) = \cos \measuredangle(\boldsymbol{e}'_i , \boldsymbol{e}_k) = \boldsymbol{e}'_i \cdot \boldsymbol{e}_k = o_{i'k} . \tag{2.3b}$$

The lines of equalities (2.3a) and (2.3b) are mutually reciprocal by multiplication

$$o_{k'i} o_{ki'} = o_{ki'} o_{k'i} = 1 , \;\; o_{k'i} o_{i'k} = o_{i'k} o_{k'i} = 1 .$$

Multiplying of an orthogonal matrix by the components of any vector retains the length of this vector:

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = v'_i v'_i = o_{i'k} v_k \, o_{i'n} v_n = v_n v_n$$

— this conclusion leans on (??).

Orthogonal transformation of the vector components

$$\boldsymbol{v} \bullet \boldsymbol{e}'_i = v_k \boldsymbol{e}_k \bullet \boldsymbol{e}'_i = \boldsymbol{e}'_i \bullet \boldsymbol{e}_k v_k = o_{i'k} v_k = v'_i \tag{2.4}$$

is sometimes used for defining a vector itself. If in each orthonormal basis $\boldsymbol{e}_i$ a triplet of numbers v_i is known, and with a rotation of the basis as a whole it is transformed according to (2.4). then this triplet of components represents an invariant object — vector $\boldsymbol{v}$.

§ 3. Tensor and its components

When in each orthonormal basis $\boldsymbol{e}_i$ we have a set of nine ($3^2 = 9$) numbers B_{ij} $(i, j = 1, 2, 3)$, and this set is transformed during a transition to a new (rotated) orthonormal basis $\boldsymbol{e}'_i$ as

$$B'_{ij} = \boldsymbol{e}'_i \bullet \boldsymbol{e}_m B_{mn} \boldsymbol{e}_n \bullet \boldsymbol{e}'_j = \boldsymbol{e}'_i \bullet \boldsymbol{e}_m \boldsymbol{e}'_j \bullet \boldsymbol{e}_n B_{mn} = o_{i'm} o_{j'n} B_{mn}, \tag{3.1}$$

then this set of components presents an invariant object — a tensor of a second complexity (of a second valence, bivalent) ${}^2\boldsymbol{B}$.

In other words, tensor ${}^2\boldsymbol{B}$ reveals in every basis as a collection of its components B_{ij}, changing along with a basis according to (3.1).

The key example of a second complexity tensor is a dyad. Having two vectors $\boldsymbol{a} = a_i \boldsymbol{e}_i$ and $\boldsymbol{b} = b_i \boldsymbol{e}_i$, in each basis $\boldsymbol{e}_i$ assume $d_{ij} \equiv a_i b_j$. It's easy to see how components d_{ij} transform according to (3.1):

$$a'_i = o_{i'm} a_m, \; b'_j = o_{j'n} b_n \;\Rightarrow\; d'_{ij} = a'_i b'_j = o_{i'm} a_m o_{j'n} b_n = o_{i'm} o_{j'n} d_{mn}.$$

Resulting tensor ${}^2\boldsymbol{d}$ is called a dyadic product or just dyad and is written as $\boldsymbol{a} \otimes \boldsymbol{b}$ or $\boldsymbol{ab}$. I choose the notation "${}^2\boldsymbol{d} = \boldsymbol{ab}$", without the $\otimes$ symbol.

Essential exemplar of a bivalent tensor is the unit tensor (other names are unit dyad, identity tensor and metric tensor). Let for any orthonormal (cartesian) basis $E_{ij} \equiv \boldsymbol{e}_i \bullet \boldsymbol{e}_j = \delta_{ij}$. These are really components of tensor, (3.1) is actual: $E'_{mn} = o_{m'i} o_{n'j} E_{ij} = o_{m'i} o_{n'i} = \delta_{mn}$. I write this tensor as $\boldsymbol{E}$ (other popular choices are $\boldsymbol{I}$ and ${}^2\mathbf{1}$).

Invariableness of components upon any rotation makes the tensor $\boldsymbol{E}$ isotropic. There are no non-null (nonvanishing) isotropic vectors (all components of the null vector ("(vanishing) vector") $\mathbf{0}$ are equal to zero in any basis).

The next example is related to a linear transformation (a linear mapping) of vectors.

If $\boldsymbol{b} = b_i \boldsymbol{e}_i$ is linear (preserving addition and multiplication by number) function of $\boldsymbol{a} = a_j \boldsymbol{e}_j$, then $b_i = c_{ij} a_j$ in every basis. Transformation coefficients c_{ij} alter when a basis rotates:

$$b'_i = c'_{ij} a'_j = o_{i'k} b_k = o_{i'k} c_{kn} a_n, \; a_n = o_{j'n} a'_j \;\Rightarrow\; c'_{ij} = o_{i'k} o_{j'n} c_{kn}.$$

It turns out that a set of two-index objects c_{ij}, c'_{ij}, ..., describing the same linear mapping $\boldsymbol{a} \mapsto \boldsymbol{b}$, but in various bases, represents a single invariant object — a tensor of second complexity ${}^2\boldsymbol{c}$. And many book authors introduce tensors in that way, by means of linear mappings (linear transformations).

And the last example is a bilinear form $\digamma(\boldsymbol{a}, \boldsymbol{b}) = f_{ij} a_i b_j$, where f_{ij} are coefficients, a_i and b_j are components of vector arguments $\boldsymbol{a} = a_i \boldsymbol{e}_i$ and $\boldsymbol{b} = b_j \boldsymbol{e}_j$. The result $\digamma$ is invariant (independent of basis) with the transformation (3.1) for coefficients f_{ij}:

$$\digamma' = f'_{ij} a'_i b'_j = f_{mn} \underbrace{a_m}_{o_{i'm} a'_i} \underbrace{b_n}_{o_{j'n} b'_j} = \digamma \;\Leftrightarrow\; f'_{ij} = o_{i'm} o_{j'n} f_{mn}.$$

If $f_{ij} = \delta_{ij}$, then $\digamma = \delta_{ij} a_i b_j = a_i b_i$ — the "$\bullet$"-product (dot product, scalar product) of two vectors. When both arguments are the same, such a homogeneous polynomial of second degree (quadratic) of one vector's components $\digamma(\boldsymbol{a}, \boldsymbol{a}) = f_{ij} a_i a_j$ is called a quadratic form.

Now about more complex tensors (of valence larger than two). Tensor of third complexity ${}^3\boldsymbol{C}$ is represented by a collection of $3^3 = 27$ numbers C_{ijk}, changing with a rotation of basis as

$$C'_{ijk} = \boldsymbol{e}'_i \bullet \boldsymbol{e}_p \boldsymbol{e}'_j \bullet \boldsymbol{e}_q \boldsymbol{e}'_k \bullet \boldsymbol{e}_r C_{pqr} = o_{i'p} o_{j'q} o_{k'r} C_{pqr}. \tag{3.2}$$

The primary example is a triad of three vectors $\boldsymbol{a} = a_i \boldsymbol{e}_i$, $\boldsymbol{b} = b_j \boldsymbol{e}_j$ and $\boldsymbol{c} = c_k \boldsymbol{e}_k$

$$t_{ijk} \equiv a_i b_j c_k \;\Leftrightarrow\; {}^3\boldsymbol{t} = \boldsymbol{abc}.$$

It is seen that orthogonal transformations (3.2) and (3.1) are results of "repeating" vector's (2.4). The reader will easily compose a transformation of components for tensor of any complexity and will write a corresponding polyad as an example.

Vectors with transformation (2.4) are tensors of the first complexity (monovalent tensors).

The least complex objects are scalars or tensors of the zeroth complexity. A scalar is a single ($3^0 = 1$) number, which doesn't depend on a basis: the energy, the mass, the temperature et al. But what are components, for example, of vector $\boldsymbol{v} = v_i \boldsymbol{e}_i$, $v_i = \boldsymbol{v} \cdot \boldsymbol{e}_i$? If not scalars, then what? Here could be no simple answer. In each particular basis, $\boldsymbol{e}_i$ are vectors and v_i are scalars.

§ 4. Tensor algebra, or operations with tensors

The whole tensor algebra can be built on the only five* operations (or actions). This section is just about them.

Equality

The first (or the zeroth) is **the equality** "=". This operation shows whether one tensor "on the left" is equal to another tensor "on the right". Tensors can be equal only when their complexities (valencies) are the same. Tensors of different valencies cannot be equal or not equal.

$$..... \tag{4.1}$$

....

Linear combination

The next operation is **the linear combination**. It aggregates the addition and the multiplication by a number (by a scalar, or, in another

* The four without the equality.

word, scaling). The arguments of this operation and the result are of the same complexity. For a pair of tensors

$$\lambda a_{ij\ldots} + \mu b_{ij\ldots} = c_{ij\ldots} \;\Leftrightarrow\; \lambda \boldsymbol{a} + \mu \boldsymbol{b} = \boldsymbol{c}. \tag{4.2}$$

Here λ and μ are scalar coefficients; $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are tensors of the same complexity. It's easy to show that the components of the result $\boldsymbol{c}$ satisfy an orthogonal transformation like (3.1).

The decomposition of a vector by some basis, that is the representation of a vector as the sum $\boldsymbol{v} = v_i \boldsymbol{e}_i$, is nothing else but the linear combination of the basis vectors $\boldsymbol{e}_i$ with the coefficients v_i.

This operation is *linear* because the only two kinds of motion are possible on the line: the translation (the movement along a straight line) and the reflexion (the backward movement).

Multiplication of tensors

One more operation — **the multiplication (the tensor product, the direct product)**. It takes arguments of any complexities, returning the result of the cumulative complexity. Examples:

$$\begin{aligned} v_i a_{jk} = C_{ijk} \;&\Leftrightarrow\; \boldsymbol{v}\, {}^2\boldsymbol{a} = {}^3\boldsymbol{C}, \\ a_{ij} B_{abc} = D_{ijabc} \;&\Leftrightarrow\; {}^2\boldsymbol{a}\, {}^3\boldsymbol{B} = {}^5\boldsymbol{D}. \end{aligned} \tag{4.3}$$

Transformation of a collection of result's components, such as $C_{ijk} = v_i a_{jk}$, during a rotation of basis is orthogonal, similar to (3.2), thus here's no doubt that such a collection is a set of tensor components.

Primary and already known (from § 3) subtype of multiplication is the dyadic product of two vectors ${}^2\boldsymbol{A} = \boldsymbol{b}\boldsymbol{c}$.

Contraction

The fourth (or the third) operation is called **the contraction**. It applies to bivalent and more complex tensors. This operation acts upon a single tensor, without other "participants". Roughly speaking, contracting a tensor is summing of its components over some pair of indices. As a result, tensor's complexity decreases by two.

For a trivalent tensor ${}^3\boldsymbol{D}$ there are the three possible contractions. They give vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ with components

$$a_i = D_{kki}\,, \quad b_i = D_{kik}\,, \quad c_i = D_{ikk}\,. \tag{4.4}$$

A rotation of basis

$$a'_i = D'_{kki} = \underbrace{o_{k'p}\, o_{k'q}}_{\delta_{pq}}\, o_{i'r}\, D_{pqr} = o_{i'r}\, D_{ppr} = o_{i'r}\, a_r$$

shows "the tensorial nature" as the result of contraction.

For a tensor of second complexity, the only one variant of contraction is possible. It gives a scalar, known as the trace

$$\boldsymbol{B}_{\bullet} \equiv \operatorname{trace} \boldsymbol{B} \equiv \mathrm{I}(\boldsymbol{B}) = B_{kk}\,.$$

The trace of the unit tensor ("contraction of the Kronecker delta") is equal to the dimension of space

$$\operatorname{trace} \boldsymbol{E} = \boldsymbol{E}_{\bullet} = \delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

Index juggling, transposing

The last operation is applicable to a single tensor of the second and bigger complexities. It is named as **the index swap, index juggling, transposing**. From components of a tensor, the new collection is emerged with another sequence of indices, the result's complexity stays the same. For example, trivalent tensor ${}^3\boldsymbol{D}$ can give tensors ${}^3\boldsymbol{A}$, ${}^3\boldsymbol{B}$, ${}^3\boldsymbol{C}$ with components

$$\begin{aligned} {}^3\boldsymbol{A} &= {}^3\boldsymbol{D}_{1\rightleftarrows 2} \;\Leftrightarrow\; A_{ijk} = D_{jik}\,, \\ {}^3\boldsymbol{B} &= {}^3\boldsymbol{D}_{1\rightleftarrows 3} \;\Leftrightarrow\; B_{ijk} = D_{kji}\,, \\ {}^3\boldsymbol{C} &= {}^3\boldsymbol{D}_{2\rightleftarrows 3} \;\Leftrightarrow\; C_{ijk} = D_{ikj}\,. \end{aligned} \tag{4.5}$$

For a bivalent tensor, the only one transposition is possible: $\boldsymbol{A}^{\mathsf{T}} \equiv \boldsymbol{A}_{1\rightleftarrows 2} = \boldsymbol{B} \;\Leftrightarrow\; B_{ij} = A_{ji}$. Obviously, $\big(\boldsymbol{A}^{\mathsf{T}}\big)^{\mathsf{T}} = \boldsymbol{A}$.

For the dyadic multiplication of two vectors, $\boldsymbol{ab} = \boldsymbol{ba}^{\mathsf{T}}$.

The four presented operations (actions) can be combined in various sequences.

The combination of multiplication (4.3) and contraction (4.4) — the "•"-product (dot product) — is the most frequently used. In the direct indexless notation this is denoted by large dot "•", which shows the contraction by adjacent indices:

$$\boldsymbol{a} = \boldsymbol{B} \bullet \boldsymbol{c} \Leftrightarrow a_i = B_{ij} c_j , \quad \boldsymbol{A} = \boldsymbol{B} \bullet \boldsymbol{C} \Leftrightarrow A_{ij} = B_{ik} C_{kj} . \tag{4.6}$$

The defining property of the unit tensor — it is the neutral element for the dyadic product with the subsequent contraction by adjacent indices ("•"-product)

$$^{\mathrm{n}}\boldsymbol{a} \bullet \boldsymbol{E} = \boldsymbol{E} \bullet {}^{\mathrm{n}}\boldsymbol{a} = {}^{\mathrm{n}}\boldsymbol{a} \quad \forall\, {}^{\mathrm{n}}\boldsymbol{a} \;\; \forall \mathrm{n} > 0. \tag{4.7}$$

In the commutative scalar product of two vectors, the dot represents the same: the dyadic product and the subsequent contraction

$$\boldsymbol{a} \bullet \boldsymbol{b} = (\boldsymbol{ab})_{\bullet} = a_i b_i = b_i a_i = (\boldsymbol{ba})_{\bullet} = \boldsymbol{b} \bullet \boldsymbol{a}. \tag{4.8}$$

The following identity describes how to swap multipliers for the "•"-product (dot product) of two second complexity tensors

$$\begin{aligned} \boldsymbol{B} \bullet \boldsymbol{Q} &= \left(\boldsymbol{Q}^{\mathsf{T}} \bullet \boldsymbol{B}^{\mathsf{T}}\right)^{\mathsf{T}} \\ \left(\boldsymbol{B} \bullet \boldsymbol{Q}\right)^{\mathsf{T}} &= \boldsymbol{Q}^{\mathsf{T}} \bullet \boldsymbol{B}^{\mathsf{T}}. \end{aligned} \tag{4.9}$$

For two dyads $\boldsymbol{B} = \boldsymbol{bd}$ and $\boldsymbol{Q} = \boldsymbol{pq}$

$$\begin{aligned} \left(\boldsymbol{bd} \bullet \boldsymbol{pq}\right)^{\mathsf{T}} &= \boldsymbol{pq}^{\mathsf{T}} \bullet \boldsymbol{bd}^{\mathsf{T}} \\ d_i p_i \, \boldsymbol{bq}^{\mathsf{T}} &= \boldsymbol{qp} \bullet \boldsymbol{db} \\ d_i p_i \, \boldsymbol{qb} &= p_i d_i \, \boldsymbol{qb}. \end{aligned}$$

For a vector and a bivalent tensor

$$\boldsymbol{c} \bullet \boldsymbol{B} = \boldsymbol{B}^{\mathsf{T}} \bullet \boldsymbol{c}, \quad \boldsymbol{B} \bullet \boldsymbol{c} = \boldsymbol{c} \bullet \boldsymbol{B}^{\mathsf{T}}. \tag{4.10}$$

Contraction can be repeated two times or more: $\big(\boldsymbol{A} \cdot \boldsymbol{B}\big)_{\bullet} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{B} = A_{ij} B_{ji}$, and here are useful equations for second complexity tensors

$$
\begin{gathered}
\boldsymbol{A} \cdot\cdot\, \boldsymbol{B} = \boldsymbol{B} \cdot\cdot\, \boldsymbol{A}, \quad \boldsymbol{d} \cdot \boldsymbol{A} \cdot \boldsymbol{b} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{b}\boldsymbol{d} = \boldsymbol{b}\boldsymbol{d} \cdot\cdot\, \boldsymbol{A} = b_j d_i A_{ij}\,, \\
\boldsymbol{A} \cdot\cdot\, \boldsymbol{B} = \boldsymbol{A}^{\mathsf{T}} \cdot\cdot\, \boldsymbol{B}^{\mathsf{T}} = A_{ij} B_{ji}\,, \quad \boldsymbol{A} \cdot\cdot\, \boldsymbol{B}^{\mathsf{T}} = \boldsymbol{A}^{\mathsf{T}} \cdot\cdot\, \boldsymbol{B} = A_{ij} B_{ij}\,, \\
\boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = \boldsymbol{E} \cdot\cdot\, \boldsymbol{A} = \boldsymbol{A}_{\bullet} = A_{jj}\,, \\
\boldsymbol{A} \cdot \boldsymbol{B} \cdot\cdot\, \boldsymbol{E} = A_{ij} B_{jk} \delta_{ki} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{B}, \quad \boldsymbol{A} \cdot \boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{A}, \\
\boldsymbol{A} \cdot\cdot\, \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} \cdot\cdot\, \boldsymbol{C} = \boldsymbol{C} \cdot\cdot\, \boldsymbol{A} \cdot \boldsymbol{B} = A_{ij} B_{jk} C_{ki}\,, \\
\boldsymbol{A} \cdot\cdot\, \boldsymbol{B} \cdot \boldsymbol{C} \cdot \boldsymbol{D} = \boldsymbol{A} \cdot \boldsymbol{B} \cdot\cdot\, \boldsymbol{C} \cdot \boldsymbol{D} = \boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{C} \cdot\cdot\, \boldsymbol{D} = \\
= \boldsymbol{D} \cdot\cdot\, \boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{C} = A_{ij} B_{jk} C_{kh} D_{hi}\,.
\end{gathered}
\tag{4.11}
$$

§ 5. Polyadic representation (decomposition)

Before in § 3, a tensor was presented as some invariant object, showing itself in every basis as a collection of numbers (components). Such a presentation is typical for majority of books about tensors. Index notation can be convenient, especially when only rectangular coordinates are used, but very often it is not. And the relevant case is physics of elastic continua: it needs more elegant, more powerful and perfect apparatus of the direct tensor calculus, operating with indexless invariant objects.

Linear combination $\boldsymbol{v} = v_i \boldsymbol{e}_i$ from decomposition (2.1) connects vector $\boldsymbol{v}$ with basis $\boldsymbol{e}_i$ and vector's components v_i in that basis. Soon we will get a similar relation for a tensor of any complexity.

Any bivalent tensor ${}^2\boldsymbol{B}$ has nine components B_{ij} in each basis. The number of various dyads $\boldsymbol{e}_i \boldsymbol{e}_j$ for the same basis is nine (3^2) too. Linear combining these dyads with coefficients B_{ij} gives the sum $B_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$. This is tensor, but what are its components, and how this representation changes or doesn't change with a rotation of basis?

Components of the constructed sum

$$
\big(B_{ij} \boldsymbol{e}_i \boldsymbol{e}_j\big)_{pq} = B_{ij} \delta_{ip} \delta_{jq} = B_{pq}
$$

are components of tensor ${}^2\boldsymbol{B}$. And with a rotation of basis

$$B'_{ij}\boldsymbol{e}'_i\boldsymbol{e}'_j = o_{i'p}o_{j'q}B_{pq}o_{i'n}\boldsymbol{e}_n o_{j'm}\boldsymbol{e}_m = \delta_{pn}\delta_{qm}B_{pq}\boldsymbol{e}_n\boldsymbol{e}_m = B_{pq}\boldsymbol{e}_p\boldsymbol{e}_q\,.$$

Doubts are dropped: a tensor of second complexity can be (re)presented as the linear combination

$${}^2\boldsymbol{B} = B_{ij}\,\boldsymbol{e}_i\boldsymbol{e}_j \tag{5.1}$$

— the dyadic decomposition of a bivalent tensor.

For the unit tensor

$$\boldsymbol{E} = E_{ij}\,\boldsymbol{e}_i\boldsymbol{e}_j = \delta_{ij}\,\boldsymbol{e}_i\boldsymbol{e}_j = \boldsymbol{e}_i\boldsymbol{e}_i = \boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3\,,$$

that's why $\boldsymbol{E}$ is called the unit dyad.

Polyadic representations like (5.1) help to operate with the tensors much easier:

$$\boldsymbol{v} \boldsymbol{\cdot} {}^2\boldsymbol{B} = v_i\boldsymbol{e}_i \boldsymbol{\cdot} \boldsymbol{e}_j B_{jk}\boldsymbol{e}_k = v_i\delta_{ij}B_{jk}\boldsymbol{e}_k = v_iB_{ik}\boldsymbol{e}_k\,,$$

$$\boldsymbol{e}_i \boldsymbol{\cdot} {}^2\boldsymbol{B} \boldsymbol{\cdot} \boldsymbol{e}_j = \boldsymbol{e}_i \boldsymbol{\cdot} B_{pq}\boldsymbol{e}_p\boldsymbol{e}_q \boldsymbol{\cdot} \boldsymbol{e}_j = B_{pq}\delta_{ip}\delta_{qj} = B_{ij} = {}^2\boldsymbol{B} \boldsymbol{\cdot\cdot}\, \boldsymbol{e}_j\boldsymbol{e}_i\,. \tag{5.2}$$

§ 6. Matrices, permutations and determinants

The matrices are the convenient tool for solving the systems of linear equations and for arranging of elements. Any matrix has the same number of elements in each row and the same number of elements in each column.

Do you need the two-dimensional arrays? The matrices can be presented as tables full of rows and columns.

Do you want a rectangular arrangement of your elements? Matrices are full of numbers and expressions in the rows and columns.

Do you know that matrices are sometimes called arrays?

Matrix dimensions

Matrices come in all sizes that are dimensions .

The dimension of a matrix consists of the number of rows, then a multiplication sign ("×" is used the most often), and then the number of columns.

Examples.

$$\underset{3\times 3}{[\mathcal{A}]} = \ldots\ldots$$

Matrix [A] is a 3×3 matrix, because it has 3 rows and 3 columns. Matrix [B] has 2 rows and 4 columns, so its dimension is 2×4. Matrix [C] is a column matrix (that is a matrix with just one column), and its dimension is 3×1. And [D] is a row matrix with dimension 1×6.

The matrix algebra

The matrix algebra includes the linear operations — the addition of matrices and the multiplication by scalar.

The dimension of a matrix is essential for the binary operations, that is for operations involving the two matrices.

An addition or subtraction of the two matrices is possible only if they have the same sizes.

(6.1) The multiplication of matrices

....

$$\underset{m\times n}{[\mathcal{A}]} = \ldots$$

The matrix of the result, known as "the matrix product", has the number of rows of the first multiplier matrix and the number of columns of the second matrix.

....

Square matrices

....

The two indices of a table is more than the single index of a one-dimensional array. Due to this, an one-dimensional array could be presented as a table of rows or as a table of columns.

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix},$$

or the vertical tables

$$\begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}.$$

$\det\limits_{i,j} \delta_{ij} = 1$

...

the permutation parity symbols via the determinant

$$e_{pqr} = e_{ijk}\delta_{pi}\delta_{qj}\delta_{rk} = e_{ijk}\delta_{ip}\delta_{jq}\delta_{kr}\,,$$

$$e_{pqr} = \det \begin{bmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{bmatrix} = \det \begin{bmatrix} \delta_{p1} & \delta_{p2} & \delta_{p3} \\ \delta_{q1} & \delta_{q2} & \delta_{q3} \\ \delta_{r1} & \delta_{r2} & \delta_{r3} \end{bmatrix}. \tag{6.1}$$

...

A determinant is not sensitive to transposing:

$$\det_{i,j} A_{ij} = \det_{i,j} A_{ji} = \det_{j,i} A_{ij}\,.$$

...

“The determinant of the matrix product of the two matrices is equal to the product of determinants of these matrices”

$$\det_{i,k} B_{ik} \det_{k,j} C_{kj} = \det_{i,j} B_{ik}C_{kj} \tag{6.2}$$

$$e_{fgh} \det_{m,n} B_{ms}C_{sn} = e_{pqr} B_{fi}C_{ip}B_{gj}C_{jq}B_{hk}C_{kr}$$

$$e_{fgh} \det_{m,s} B_{ms} = e_{ijk} B_{fi}B_{gj}B_{hk}$$

$$e_{ijk} \det_{s,n} C_{sn} = e_{pqr} C_{ip} C_{jq} C_{kr}$$

$$e_{fgh}\, e_{ijk} \det_{m,s} B_{ms} \det_{s,n} C_{sn} = e_{ijk}\, e_{pqr} B_{fi} B_{gj} B_{hk} C_{ip} C_{jq} C_{kr}$$

...

Определитель компонент of a bivalent tensor is invariant, он не меняется с поворотом базиса

$$A'_{ij} = o_{i'm} o_{j'n} A_{mn}$$

§ 7. The cross product

By common notions, the "×"-product (the "cross product", the "vector product", sometimes the "oriented area product") of the two vectors is the vector, heading perpendicular to the plane of multipliers, whose length is equal to the area of the parallelogram, spanned by the multipliers

$$\| \boldsymbol{a} \times \boldsymbol{b} \| = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \sin \measuredangle(\boldsymbol{a}, \boldsymbol{b}).$$

However, a "×"-product isn't quite a vector, since it is not completely invariant.

The multipliers of the "×"-product $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ determine the result's direction in space, with an accuracy up to the sign fig. 3.

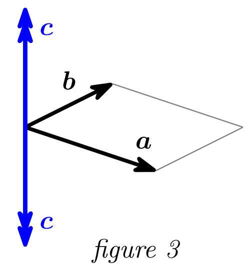

figure 3

Once you pick as the positive the " right-chiral " ("right-handed") or the " left-chiral " ("left-handed") orientation of space, the one direction from the possible two, then the results of the "× "-products become completely determined.

"The chiral" means asymmetric in such a way that. the thing and its mirror image are not superimposable, a picture cannot be superposed on its mirror image by any combination of rotations and translations.

An object is chiral if it is distinguishable from its mirror image.

Vectors are usually measured via some a basis $\boldsymbol{e}_i$. They are decomposed into linear combinations like $\boldsymbol{a} = a_i \boldsymbol{e}_i$. So the orientation of space is equivalent to the orientation of the sequential triple of basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$. It means that the sequence of basis vectors becomes significant (for linear combinations, the sequence of addends doesn't affect anything).

If two bases consist of different sequences of the same vectors, then orientations of these bases differ by some permutation.

The orientation of the space is a (kind of) asymmetry. This asymmetry makes it impossible to replicate a reflection by the means of any rotations*

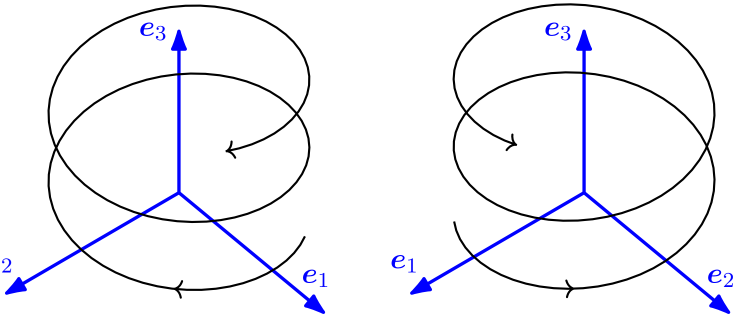

figure 4

A pseudovector is a vector-like object, invariant under any rotation. **.

... put the figure here ...

Except in the rare cases, the direction of a fully invariant (polar) vector will change with a reflection.

A pseudovector (an axial vector), unlike a polar vector, doesn't change the component orthogonal to the plane of reflection, and turns

* Applying only rotations, it's impossible to replace the left hand of a human figure into the right hand. But it is possible by reflection of a figure in a mirror.

** Rotations cannot change the orientation of a triple of basis vectors, only a reflection can.

out to be flipped relatively to the polar vectors and the geometry of the entire space. This happens because the sign (and, accordingly, the direction) of each axial vector changes along with changing the sign of the "$\times$"-product — which corresponds to reflection.

The otherness of pseudovectors narrows the variety of formulas: a pseudovector is not additive with a vector. Formula $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}$ from an absolutely rigid undeformable body's kinematics is correct, because $\boldsymbol{\omega}$ is pseudovector there, and with the cross product the two "pseudo" give $(-1)^2 = 1$, mutually compensating each other.

The parity of permutations tensor is the volumetric tensor of third complexity

$$ {}^3\boldsymbol{\epsilon} = \in_{[ijk]} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k \, , \;\; \in_{[ijk]} \equiv \boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k \tag{7.1} $$

with the components $\in_{[ijk]}$ equal to the "triple" (the "mixed", the "cross-dot") products of the basis vectors.

The absolute value (the modulus) of each nonzero component of ${}^3\boldsymbol{\epsilon}$ is equal to the volume $\sqrt{g}$ of a parallelopiped drew upon a basis. For a basis $\boldsymbol{e}_i$ of pairwise perpendicular one unit long vectors $\sqrt{g} = 1$.

The tensor ${}^3\boldsymbol{\epsilon}$ is isotropic, its components are constant and independent of any rotations of a basis. But a reflection — a change in the orientation of a triple of basis vectors (a change in "the direction of screw") — changes the sign of ${}^3\boldsymbol{\epsilon}$, so this is a pseudotensor (an axial tensor).

If $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$ without the "minus" sign, then the basis triple $\boldsymbol{e}_i$ is oriented positively. The positive orientation (or "the positive direction") is chosen for different reasons from the two possible ones (fig. 3). For a positively oriented basis triplet, the components of ${}^3\boldsymbol{\epsilon}$ are equal to the parity of permutations symbol $\in_{[ijk]} = e_{ijk}$. And when $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = -\boldsymbol{e}_3$, then the basis triple $\boldsymbol{e}_i$ is oriented negatively, or "mirrored". For mirrored triples $\in_{[ijk]} = -e_{ijk}$ (and $e_{ijk} = -\boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k$).

With the Levi-Civita tensor ${}^3\boldsymbol{\epsilon}$ it is possible to take a fresh look at the cross product:

$$ \in_{[ijk]} = \boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k \;\Leftrightarrow\; \boldsymbol{e}_i \times \boldsymbol{e}_j = \in_{[ijk]} \boldsymbol{e}_k \, , $$

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} = a_i \boldsymbol{e}_i \times b_j \boldsymbol{e}_j = a_i b_j \boldsymbol{e}_i \times \boldsymbol{e}_j = a_i b_j \in_{[ijk]} \boldsymbol{e}_k = \\ = b_j a_i \boldsymbol{e}_j \boldsymbol{e}_i \cdot\cdot \in_{[mnk]} \boldsymbol{e}_m \boldsymbol{e}_n \boldsymbol{e}_k = \boldsymbol{b}\boldsymbol{a} \cdot\cdot {}^3\!\boldsymbol{\epsilon}, \\ = a_i \in_{[ijk]} \boldsymbol{e}_k b_j = -a_i \in_{[ikj]} \boldsymbol{e}_k b_j = -\boldsymbol{a} \cdot {}^3\!\boldsymbol{\epsilon} \cdot \boldsymbol{b}.\end{aligned} \tag{7.2}$$

So that, the cross product is not another new, entirely distinct operation. With the Levi-Civita tensor it reduces to the four already described (§ 4) and is applicable to tensors of any complexity.

"The cross product" is just the dot product — the combination of multiplication and contraction (§ 4) — involving tensor ${}^3\!\boldsymbol{\epsilon}$. Such combinations are possible with any tensors:

$$\boldsymbol{a} \times {}^2\boldsymbol{B} = a_i \boldsymbol{e}_i \times B_{jk} \boldsymbol{e}_j \boldsymbol{e}_k = \underbrace{a_i B_{jk} \in_{[ijn]}}_{-a_i \in_{[inj]} B_{jk}} \boldsymbol{e}_n \boldsymbol{e}_k = -\boldsymbol{a} \cdot {}^3\!\boldsymbol{\epsilon} \cdot {}^2\boldsymbol{B},$$

$$\begin{aligned}{}^2\boldsymbol{C} \times \boldsymbol{d}\boldsymbol{b} = C_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \times d_p b_q \boldsymbol{e}_p \boldsymbol{e}_q = \boldsymbol{e}_i C_{ij} d_p \underbrace{\in_{[jpk]}}_{-\in_{[pjk]} = -\in_{[jkp]}} \boldsymbol{e}_k b_q \boldsymbol{e}_q = \\ = -{}^2\boldsymbol{C}\boldsymbol{d} \cdot\cdot {}^3\!\boldsymbol{\epsilon}\boldsymbol{b} = -{}^2\boldsymbol{C} \cdot {}^3\!\boldsymbol{\epsilon} \cdot \boldsymbol{d}\boldsymbol{b},\end{aligned}$$

$$\boldsymbol{E} \times \boldsymbol{E} = \boldsymbol{e}_i \boldsymbol{e}_i \times \boldsymbol{e}_j \boldsymbol{e}_j = \underbrace{- \in_{[ijk]} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k}_{+\in_{[ijk]} \boldsymbol{e}_i \boldsymbol{e}_k \boldsymbol{e}_j} = -{}^3\!\boldsymbol{\epsilon}. \tag{7.3}$$

The last equation connects the isotropic tensors of second and third complexities.

Generalizing to all tensors of nonzero complexity

$${}^{\mathrm{n}}\boldsymbol{\xi} \times {}^{\mathrm{m}}\boldsymbol{\zeta} = -{}^{\mathrm{n}}\boldsymbol{\xi} \cdot {}^3\!\boldsymbol{\epsilon} \cdot {}^{\mathrm{m}}\boldsymbol{\zeta} \quad \forall\, {}^{\mathrm{n}}\boldsymbol{\xi}, {}^{\mathrm{m}}\boldsymbol{\zeta} \;\; \forall n > 0,\, m > 0. \tag{7.4}$$

When one of the operands is the unit (metric) tensor, from (7.4) and (4.7) $\forall\, {}^{\mathrm{n}}\boldsymbol{\Upsilon}\; \forall\, \mathrm{n} > 0$

$$\begin{aligned}\boldsymbol{E} \times {}^{\mathrm{n}}\boldsymbol{\Upsilon} = -\boldsymbol{E} \cdot {}^3\!\boldsymbol{\epsilon} \cdot {}^{\mathrm{n}}\boldsymbol{\Upsilon} = -{}^3\!\boldsymbol{\epsilon} \cdot {}^{\mathrm{n}}\boldsymbol{\Upsilon}, \\ {}^{\mathrm{n}}\boldsymbol{\Upsilon} \times \boldsymbol{E} = -{}^{\mathrm{n}}\boldsymbol{\Upsilon} \cdot {}^3\!\boldsymbol{\epsilon} \cdot \boldsymbol{E} = -{}^{\mathrm{n}}\boldsymbol{\Upsilon} \cdot {}^3\!\boldsymbol{\epsilon}.\end{aligned}$$

The cross product of the two vectors is not commutative, but is anticommutative:

$$\begin{gathered}
\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{E}) = (\boldsymbol{a} \times \boldsymbol{E}) \cdot \boldsymbol{b} = -\boldsymbol{ab} \cdot\cdot\, {}^3\!\boldsymbol{\epsilon} = -{}^3\!\boldsymbol{\epsilon} \cdot\cdot\, \boldsymbol{ab}, \\
\boldsymbol{b} \times \boldsymbol{a} = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{E}) = (\boldsymbol{b} \times \boldsymbol{E}) \cdot \boldsymbol{a} = -\boldsymbol{ba} \cdot\cdot\, {}^3\!\boldsymbol{\epsilon} = -{}^3\!\boldsymbol{\epsilon} \cdot\cdot\, \boldsymbol{ba}, \\
\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{ab} \cdot\cdot\, {}^3\!\boldsymbol{\epsilon} = \boldsymbol{ba} \cdot\cdot\, {}^3\!\boldsymbol{\epsilon} \;\Rightarrow\; \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}.
\end{gathered} \tag{7.5}$$

For any bivalent tensor ${}^2\boldsymbol{B}$ and a tensor of first complexity (vector) $\boldsymbol{a}$

$$ {}^2\boldsymbol{B} \times \boldsymbol{a} = \boldsymbol{e}_i B_{ij} \boldsymbol{e}_j \times a_k \boldsymbol{e}_k = \left(-a_k \boldsymbol{e}_k \times \boldsymbol{e}_j B_{ij} \boldsymbol{e}_i\right)^{\mathsf{T}} = -\left(\boldsymbol{a} \times {}^2\boldsymbol{B}^{\mathsf{T}}\right)^{\mathsf{T}}. $$

However, in the particular case of the unit tensor $\boldsymbol{E}$ and a vector

$$\begin{gathered}
\boldsymbol{E} \times \boldsymbol{a} = -\left(\boldsymbol{a} \times \boldsymbol{E}^{\mathsf{T}}\right)^{\mathsf{T}} = -\left(\boldsymbol{a} \times \boldsymbol{E}\right)^{\mathsf{T}} = \boldsymbol{a} \times \boldsymbol{E}, \\
\boldsymbol{E} \times \boldsymbol{a} = \boldsymbol{a} \times \boldsymbol{E} = -\boldsymbol{a} \cdot {}^3\!\boldsymbol{\epsilon} = -{}^3\!\boldsymbol{\epsilon} \cdot \boldsymbol{a}.
\end{gathered} \tag{7.6}$$

Справедливо такое соотношение

$$ e_{ijk} e_{pqr} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix} \tag{7.7}$$

○ Доказательство начнём с представления символов чётности перестановки как определителей (6.1). $e_{ijk} = \pm\, \boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k$ по строкам, $e_{pqr} = \pm\, \boldsymbol{e}_p \times \boldsymbol{e}_q \cdot \boldsymbol{e}_r$ по столбцам, с “−” для “левой” тройки

$$ e_{ijk} = \det \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}, \quad e_{pqr} = \det \begin{bmatrix} \delta_{p1} & \delta_{q1} & \delta_{r1} \\ \delta_{p2} & \delta_{q2} & \delta_{r2} \\ \delta_{p3} & \delta_{q3} & \delta_{r3} \end{bmatrix}. $$

Левая часть (7.7) есть произведение $e_{ijk} e_{pqr}$ этих определителей. Но $\det(AB) = (\det A)(\det B)$ — определитель произведения матриц равен произведению определителей (6.2). В матрице-произведении элемент $[\cdots]_{11}$ равен $\delta_{is}\delta_{ps} = \delta_{ip}$, как и в (7.7); легко проверить и другие фрагменты. ●

The contraction of (7.7) приводит к полезным формулам

$$e_{ijk}e_{pqk} = \det\begin{bmatrix}\delta_{ip} & \delta_{iq} & \delta_{ik}\\ \delta_{jp} & \delta_{jq} & \delta_{jk}\\ \delta_{kp} & \delta_{kq} & \delta_{kk}\end{bmatrix} = \det\begin{bmatrix}\delta_{ip} & \delta_{iq} & \delta_{ik}\\ \delta_{jp} & \delta_{jq} & \delta_{jk}\\ \delta_{kp} & \delta_{kq} & 3\end{bmatrix} =$$
$$= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jk}\delta_{kp} + \delta_{ik}\delta_{jp}\delta_{kq} - \delta_{ik}\delta_{jq}\delta_{kp} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jk}\delta_{kq} =$$
$$= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} + \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} =$$
$$= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\,,$$

$$e_{ijk}e_{pjk} = \delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip}\,,$$

$$e_{ijk}e_{ijk} = 2\delta_{ii} = 6\,.$$

Or in short

$$e_{ijk}e_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\,, \quad e_{ijk}e_{pjk} = 2\delta_{ip}\,, \quad e_{ijk}e_{ijk} = 6\,. \tag{7.8}$$

The first of these formulas даёт представление двойного векторного произведения

$$\begin{aligned}\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= a_i\boldsymbol{e}_i \times \in_{[pqj]} b_p c_q \boldsymbol{e}_j = \in_{[kij]}\in_{[pqj]} a_i b_p c_q \boldsymbol{e}_k = \\ &= (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip})\, a_i b_p c_q \boldsymbol{e}_k = a_i b_k c_i \boldsymbol{e}_k - a_i b_i c_k \boldsymbol{e}_k = \\ &= \boldsymbol{a}\cdot\boldsymbol{cb} - \boldsymbol{a}\cdot\boldsymbol{bc} = \boldsymbol{a}\cdot\big(\boldsymbol{cb}-\boldsymbol{bc}\big) = \boldsymbol{a}\cdot\boldsymbol{cb} - \boldsymbol{cb}\cdot\boldsymbol{a}.\end{aligned} \tag{7.9}$$

By another interpretation, the dot product of a dyad and a vector is not commutative: $\boldsymbol{bd}\cdot\boldsymbol{c} \neq \boldsymbol{c}\cdot\boldsymbol{bd}$, and this difference can be expressed as

$$\boldsymbol{bd}\cdot\boldsymbol{c} - \boldsymbol{c}\cdot\boldsymbol{bd} = \boldsymbol{c}\times\big(\boldsymbol{b}\times\boldsymbol{d}\big). \tag{7.10}$$

$\boldsymbol{a}\cdot\boldsymbol{bc} = \boldsymbol{cb}\cdot\boldsymbol{a} = \boldsymbol{ca}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{ac}$

$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = -\,\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{c}\times(\boldsymbol{b}\times\boldsymbol{a})$

The same way it may be derived that

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \big(\boldsymbol{ba}-\boldsymbol{ab}\big)\cdot\boldsymbol{c} = \boldsymbol{ba}\cdot\boldsymbol{c} - \boldsymbol{ab}\cdot\boldsymbol{c}. \tag{7.11}$$

And following identities for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{E} = \in_{[ijk]} a_i b_j \boldsymbol{e}_k \times \boldsymbol{e}_n \boldsymbol{e}_n = a_i b_j \in_{[ijk]} \in_{[knq]} \boldsymbol{e}_q \boldsymbol{e}_n = \\ = a_i b_j \big(\delta_{in}\delta_{jq} - \delta_{iq}\delta_{jn}\big) \boldsymbol{e}_q \boldsymbol{e}_n = a_i b_j \boldsymbol{e}_j \boldsymbol{e}_i - a_i b_j \boldsymbol{e}_i \boldsymbol{e}_j = \\ = \boldsymbol{ba} - \boldsymbol{ab},\end{aligned} \quad (7.12)$$

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{E})\cdot(\boldsymbol{b}\times\boldsymbol{E}) = (\boldsymbol{a}\cdot{}^3\!\boldsymbol{\epsilon})\cdot(\boldsymbol{b}\cdot{}^3\!\boldsymbol{\epsilon}) = \\ = a_i \in_{[ipn]} \boldsymbol{e}_p \boldsymbol{e}_n \cdot b_j \in_{[jsk]} \boldsymbol{e}_s \boldsymbol{e}_k = a_i b_j \in_{[ipn]} \in_{[nkj]} \boldsymbol{e}_p \boldsymbol{e}_k = \\ = a_i b_j \big(\delta_{ik}\delta_{pj} - \delta_{ij}\delta_{pk}\big) \boldsymbol{e}_p \boldsymbol{e}_k = a_i b_j \boldsymbol{e}_j \boldsymbol{e}_i - a_i b_i \boldsymbol{e}_k \boldsymbol{e}_k = \\ = \boldsymbol{ba} - \boldsymbol{a}\cdot\boldsymbol{b}\boldsymbol{E}.\end{aligned} \quad (7.13)$$

Finally, the one more correlation between the isotropic tensors of the second and third complexities:

$${}^3\!\boldsymbol{\epsilon} \cdot\cdot\, {}^3\!\boldsymbol{\epsilon} = \in_{[ijk]} \boldsymbol{e}_i \in_{[kjn]} \boldsymbol{e}_n = -2\,\delta_{in} \boldsymbol{e}_i \boldsymbol{e}_n = -2\boldsymbol{E}. \quad (7.14)$$

§ 8. Symmetric and antisymmetric tensors

A tensor that does not change upon a permutation of some pair of its indices is called symmetric for that pair of indices. And if a tensor alternates the sign $(+/-)$*upon a permutation of some pair of indices, then it is called antisymmetric or skew-symmetric for that pair of indices.

The tensor of the parity of permutations ${}^3\!\boldsymbol{\epsilon}$ is antisymmetric by any pair of indices, it is completely (absolutely) antisymmetric (skew-symmetric).

Tensor of the second complexity $\boldsymbol{B}$ is symmetric if $\boldsymbol{B} = \boldsymbol{B}^{\mathsf{T}}$. When the transposing changes the sign of a tensor $\boldsymbol{A}^{\mathsf{T}} = -\boldsymbol{A}$, then it is antisymmetric (skew-symmetric).

$$\begin{gathered}\boldsymbol{C} = \boldsymbol{C}^{\mathsf{S}} + \boldsymbol{C}^{\mathsf{A}},\ \boldsymbol{C}^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{S}} - \boldsymbol{C}^{\mathsf{A}}; \\ \boldsymbol{C}^{\mathsf{S}} \equiv \frac{1}{2}\big(\boldsymbol{C} + \boldsymbol{C}^{\mathsf{T}}\big),\ \boldsymbol{C}^{\mathsf{A}} \equiv \frac{1}{2}\big(\boldsymbol{C} - \boldsymbol{C}^{\mathsf{T}}\big).\end{gathered} \quad (8.1)$$

* $\cdot(-1)$

For a dyad $\boldsymbol{cd} = \boldsymbol{cd}^{\mathsf{S}} + \boldsymbol{cd}^{\mathsf{A}} = \frac{1}{2}\,(\boldsymbol{cd} + \boldsymbol{dc}) + \frac{1}{2}\,(\boldsymbol{cd} - \boldsymbol{dc})$.

Произведение двух симметричных тензоров $\boldsymbol{C}^{\mathsf{S}} \cdot \boldsymbol{D}^{\mathsf{S}}$ симметрично далеко не всегда, а лишь когда $\boldsymbol{D}^{\mathsf{S}} \cdot \boldsymbol{C}^{\mathsf{S}} = \boldsymbol{C}^{\mathsf{S}} \cdot \boldsymbol{D}^{\mathsf{S}}$, ведь по (**??**) $\left(\boldsymbol{C}^{\mathsf{S}} \cdot \boldsymbol{D}^{\mathsf{S}}\right)^{\mathsf{T}} = \boldsymbol{D}^{\mathsf{S}} \cdot \boldsymbol{C}^{\mathsf{S}}$.

В нечётномерных пространствах любой антисимметричный тензор второй сложности необрати́м, определитель матрицы компонент для него — нулевой.

Существует взаимно-однозначное соответствие между антисимметричными тензорами второй сложности и (псевдо)векторами. Компоненты кососимметричного тензора полностью определяются тройкой чисел (диагональные элементы матрицы компонент — нули, недиагональные — попарно противоположны). Dot product кососимметричного $\boldsymbol{A}$ и какого-нибудь тензора ${}^{\mathrm{n}}\boldsymbol{\xi}$ однозначно соответствует cross product'у псевдовектора $\boldsymbol{a}$ и того же тензора ${}^{\mathrm{n}}\boldsymbol{\xi}$

$$\begin{aligned} \boldsymbol{b} &= \boldsymbol{A} \cdot {}^{\mathrm{n}}\boldsymbol{\xi} \;\Leftrightarrow\; \boldsymbol{a} \times {}^{\mathrm{n}}\boldsymbol{\xi} = \boldsymbol{b} \;\; \forall \boldsymbol{A} = \boldsymbol{A}^{\mathsf{A}} \;\; \forall\, {}^{\mathrm{n}}\boldsymbol{\xi} \;\; \forall\, \mathrm{n} > 0, \\ \boldsymbol{d} &= {}^{\mathrm{n}}\boldsymbol{\xi} \cdot \boldsymbol{A} \;\Leftrightarrow\; {}^{\mathrm{n}}\boldsymbol{\xi} \times \boldsymbol{a} = \boldsymbol{d} \;\; \forall \boldsymbol{A} = \boldsymbol{A}^{\mathsf{A}} \;\; \forall\, {}^{\mathrm{n}}\boldsymbol{\xi} \;\; \forall\, \mathrm{n} > 0. \end{aligned} \tag{8.2}$$

Раскроем это соответствие $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{a})$:

$$\begin{aligned} \boldsymbol{A} \cdot {}^{\mathrm{n}}\boldsymbol{\xi} &= \boldsymbol{a} \times {}^{\mathrm{n}}\boldsymbol{\xi} \\ A_{hi} \boldsymbol{e}_h \boldsymbol{e}_i \cdot \xi_{jk\ldots q}\, \boldsymbol{e}_j \boldsymbol{e}_k \ldots \boldsymbol{e}_q &= a_i \boldsymbol{e}_i \times \xi_{jk\ldots q}\, \boldsymbol{e}_j \boldsymbol{e}_k \ldots \boldsymbol{e}_q \\ A_{hj}\, \xi_{jk\ldots q}\, \boldsymbol{e}_h \boldsymbol{e}_k \ldots \boldsymbol{e}_q &= a_i \in_{[ijh]} \xi_{jk\ldots q}\, \boldsymbol{e}_h \boldsymbol{e}_k \ldots \boldsymbol{e}_q \\ A_{hj} &= a_i \in_{[ijh]} \\ A_{hj} &= -\, a_i \in_{[ihj]} \\ \boldsymbol{A} &= -\, \boldsymbol{a} \cdot {}^{3}\boldsymbol{\epsilon} \end{aligned}$$

Так же из ${}^{\mathrm{n}}\boldsymbol{\xi} \cdot \boldsymbol{A} = {}^{\mathrm{n}}\boldsymbol{\xi} \times \boldsymbol{a}$ получается $\boldsymbol{A} = -\,{}^{3}\boldsymbol{\epsilon} \cdot \boldsymbol{a}$.

Или проще, согласно (7.4)

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{A} \cdot \boldsymbol{E} = \boldsymbol{a} \times \boldsymbol{E} = -\, \boldsymbol{a} \cdot {}^{3}\boldsymbol{\epsilon}, \\ \boldsymbol{A} &= \boldsymbol{E} \cdot \boldsymbol{A} = \boldsymbol{E} \times \boldsymbol{a} = -\,{}^{3}\boldsymbol{\epsilon} \cdot \boldsymbol{a}. \end{aligned}$$

(Псевдо)вектор $\boldsymbol{a}$ называется сопутствующим для тензора $\boldsymbol{A}$.

В общем, для взаимно-однозначного соответствия между $\boldsymbol{A}$ and $\boldsymbol{a}$ имеем

$$\begin{aligned} \boldsymbol{A} &= -\boldsymbol{a} \cdot {}^3\boldsymbol{\epsilon} = \boldsymbol{a} \times \boldsymbol{E} = -{}^3\boldsymbol{\epsilon} \cdot \boldsymbol{a} = \boldsymbol{E} \times \boldsymbol{a}, \\ \boldsymbol{a} &= \boldsymbol{a} \cdot \boldsymbol{E} = \boldsymbol{a} \cdot \left(-\tfrac{1}{2}\, {}^3\boldsymbol{\epsilon} \cdot\cdot\, {}^3\boldsymbol{\epsilon}\right) = \tfrac{1}{2}\, \boldsymbol{A} \cdot\cdot\, {}^3\boldsymbol{\epsilon}. \end{aligned} \tag{8.3}$$

The components of a skew-symmetric tensor $\boldsymbol{A}$ thru the components of the accompanying pseudovector $\boldsymbol{a}$

$$\boldsymbol{A} = -\, {}^3\boldsymbol{\epsilon} \cdot \boldsymbol{a} = -\in_{[ijk]} \boldsymbol{e}_i \boldsymbol{e}_j a_k,$$

$$A_{ij} = -\in_{[ijk]} a_k = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

and vice versa

$$\boldsymbol{a} = \tfrac{1}{2}\, \boldsymbol{A} \cdot\cdot\, {}^3\boldsymbol{\epsilon} = \tfrac{1}{2}\, A_{jk} \in_{[kji]} \boldsymbol{e}_i,$$

$$a_i = \tfrac{1}{2} \in_{[ikj]} A_{jk} = \frac{1}{2} \begin{bmatrix} \in_{[123]} A_{32} + \in_{[132]} A_{23} \\ \in_{[213]} A_{31} + \in_{[231]} A_{13} \\ \in_{[312]} A_{21} + \in_{[321]} A_{12} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}.$$

The easy to memorize "pseudovector invariant" $\boldsymbol{A}_\times$ comes from the original tensor $\boldsymbol{A}$ via replacing the dyadic product by the cross product

$$\begin{aligned} \boldsymbol{A}_\times &\equiv A_{ij}\, \boldsymbol{e}_i \times \boldsymbol{e}_j = -\boldsymbol{A} \cdot\cdot\, {}^3\boldsymbol{\epsilon}, \\ \boldsymbol{A}_\times = \left(\boldsymbol{a} \times \boldsymbol{E}\right)_\times &= -2\boldsymbol{a}, \;\; \boldsymbol{a} = -\tfrac{1}{2}\, \boldsymbol{A}_\times = -\tfrac{1}{2} \left(\boldsymbol{a} \times \boldsymbol{E}\right)_\times. \end{aligned} \tag{8.4}$$

Explaination:

$$\boldsymbol{a} \times \boldsymbol{E} = -\tfrac{1}{2}\, \boldsymbol{A}_\times \times \boldsymbol{E} = -\tfrac{1}{2}\, A_{ij} \underbrace{\left(\boldsymbol{e}_i \times \boldsymbol{e}_j\right)}_{\in_{[ijn]} \boldsymbol{e}_n} \times \boldsymbol{e}_k \boldsymbol{e}_k$$

$$= -\tfrac{1}{2}\, A_{ij} \underbrace{\in_{[nij]} \in_{[nkp]}}_{\delta_{jp}\delta_{ik} - \delta_{ip}\delta_{jk}} \boldsymbol{e}_p \boldsymbol{e}_k = -\tfrac{1}{2}\, A_{ij} \left(\boldsymbol{e}_j \boldsymbol{e}_i - \boldsymbol{e}_i \boldsymbol{e}_j\right)$$

$$= -\tfrac{1}{2} \left(\boldsymbol{A}^{\mathsf{T}} - \boldsymbol{A}\right) = \boldsymbol{A}^{\mathsf{A}} = \boldsymbol{A}.$$

The accompanying vector can be introduced for any bivalent tensor. But only the asymmetric part contributes here: $\boldsymbol{C}^{\mathsf{A}} = -\frac{1}{2}\boldsymbol{C}_{\times} \times \boldsymbol{E}$.

For a symmetric tensor, the accompanying vector is zero:

$$\boldsymbol{B}_{\times} = \boldsymbol{0} \Leftrightarrow \boldsymbol{B} = \boldsymbol{B}^{\mathsf{T}} = \boldsymbol{B}.$$

With (8.4) the decomposition of some tensor $\boldsymbol{C}$ on the symmetric and the antisymmetric parts looks like

$$\boldsymbol{C} = \boldsymbol{C}^{\mathsf{S}} - \tfrac{1}{2}\boldsymbol{C}_{\times} \times \boldsymbol{E}. \tag{8.5}$$

For a dyad

$$(7.12) \Rightarrow (\boldsymbol{c} \times \boldsymbol{d}) \times \boldsymbol{E} = \boldsymbol{dc} - \boldsymbol{cd} = -2\boldsymbol{cd}^{\mathsf{A}}, \;\; (\boldsymbol{cd})_{\times} = \boldsymbol{c} \times \boldsymbol{d},$$

and its decomposition

$$\boldsymbol{cd} = \frac{1}{2}(\boldsymbol{cd} + \boldsymbol{dc}) - \frac{1}{2}(\boldsymbol{c} \times \boldsymbol{d}) \times \boldsymbol{E}. \tag{8.6}$$

§ 9. Polar decomposition

Any tensor of the second complexity $\boldsymbol{F}$ with $\det F_{ij} \neq 0$ (not singular) can be decomposed as

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Example. Polar decompose tensor $\boldsymbol{C} = C_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$, where $\boldsymbol{e}_k$ are pairwise perpendicular unit vectors and C_{ij} are the tensor's components.

$$C_{ij} = \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix}$$

$$\boldsymbol{O} = O_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = \boldsymbol{O}_1 \cdot \boldsymbol{O}_2$$

$$O_{ij} = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$\boldsymbol{C} = \boldsymbol{O} \cdot \boldsymbol{S}_{\mathsf{R}}, \;\; \boldsymbol{O}^{\mathsf{T}} \cdot \boldsymbol{C} = \boldsymbol{S}_{\mathsf{R}}$$

$$\boldsymbol{C} = \boldsymbol{S}_{\mathsf{L}} \cdot \boldsymbol{O}, \;\; \boldsymbol{C} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{S}_{\mathsf{L}}$$

$$S_{\mathsf{R}ij} = \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix}$$

$$S_{\mathsf{L}ij} = \begin{bmatrix} 104/5 & 47/5 & 5 \\ 47/5 & -129/5 & -10 \\ 5 & -10 & 3 \end{bmatrix}$$

...

§ 10. Eigenvectors and eigenvalues

If for some tensor ${}^2\boldsymbol{B}$ and the nonzero vector $\boldsymbol{a}$

$$ {}^2\boldsymbol{B} \cdot \boldsymbol{a} = \eta \boldsymbol{a}, \;\; \boldsymbol{a} \neq \boldsymbol{0} \tag{10.1}$$

$$ {}^2\boldsymbol{B} \cdot \boldsymbol{a} = \eta \boldsymbol{E} \cdot \boldsymbol{a}, \;\; \big({}^2\boldsymbol{B} - \eta \boldsymbol{E}\big) \cdot \boldsymbol{a} = \boldsymbol{0},$$

then η is called the eigenvalue (or the characteristic value) of tensor ${}^2\boldsymbol{B}$, and the axis (direction) of eigenvector $\boldsymbol{a}$ is called its characteristic axis (direction).

In components, this is an eigenvalue problem for a matrix. A homogeneous system of linear equations $(B_{ij} - \eta\delta_{ij})\, a_j = 0$ has a non-zero solution if the determinant of a matrix of components

$$\det_{\mathrm{i,\,j}} (B_{ij} - \eta\delta_{ij})$$

is equal to zero:

$$\det\begin{bmatrix} B_{11}-\eta & B_{12} & B_{13} \\ B_{21} & B_{22}-\eta & B_{23} \\ B_{31} & B_{32} & B_{33}-\eta \end{bmatrix} = -\eta^3 + \mathrm{chaI}\,\eta^2 - \mathrm{chaII}\,\eta + \mathrm{chaIII} = 0; \tag{10.2}$$

$$\begin{aligned} \mathrm{chaI} &= \operatorname{trace} {}^2\boldsymbol{B} = B_{kk} = B_{11} + B_{22} + B_{33}, \\ \mathrm{chaII} &= B_{11}B_{22} - B_{12}B_{21} + B_{11}B_{33} - B_{13}B_{31} + B_{22}B_{33} - B_{23}B_{32}, \\ \mathrm{chaIII} &= \det {}^2\boldsymbol{B} = \det_{i,j} B_{ij} = e_{ijk}B_{1i}B_{2j}B_{3k} = e_{ijk}B_{i1}B_{j2}B_{k3}. \end{aligned} \tag{10.3}$$

The roots of the characteristic equation (10.2) — the eigenvalues — don't depend on the basis and therefore are invariants.

The coefficients of (10.3) also don't depend on the basis; they are called the first, the second and the third characteristic invariants of a tensor. The first invariant chaI is the trace. It was described earlier in § 4. The second invariant chaII is the trace of the adjugate matrix — the transpose of the cofactor matrix (of the matrix of algebraic complements)

$$\mathrm{chaII}({}^2\boldsymbol{B}) \equiv \operatorname{trace}(\operatorname{adj} B_{ij})$$

(it's hard, yeah). Or

$$\mathrm{chaII}({}^2\boldsymbol{B}) \equiv \tfrac{1}{2}\Big[({}^2\boldsymbol{B}_{\bullet})^2 - {}^2\boldsymbol{B} \cdot\cdot {}^2\boldsymbol{B}\Big] = \tfrac{1}{2}\Big[(B_{kk})^2 - B_{ij}B_{ji}\Big].$$

And the third invariant chaIII is the determinant of a matrix of tensor components: $\mathrm{chaIII}({}^2\boldsymbol{B}) \equiv \det {}^2\boldsymbol{B}$.

This applies to all second complexity tensors. Besides that, in case of a symmetric tensor, the following is true:

1° The eigenvalues of a symmetric bivalent tensor are real numbers.

2° The characteristic axes (directions) for different eigenvalues are orthogonal to each other.

○ The first statement is proved by contradiction. If η is a complex root of (10.2) corresponding to eigenvector $\boldsymbol{a}$, then conjugate number $\overline{\eta}$

will also be the root of (10.2). Eigenvector $\overline{\boldsymbol{a}}$ with the conjugate components corresponds to it. And then

$$(10.1) \;\Rightarrow\; (\overline{\boldsymbol{a}}\,\boldsymbol{\cdot})\; {}^2\!\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a} = \eta\boldsymbol{a},\;\; (\boldsymbol{a}\,\boldsymbol{\cdot})\; {}^2\!\boldsymbol{B}\boldsymbol{\cdot}\overline{\boldsymbol{a}} = \overline{\eta}\,\overline{\boldsymbol{a}} \;\Rightarrow$$
$$\Rightarrow\; \overline{\boldsymbol{a}}\boldsymbol{\cdot}{}^2\!\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{\cdot}{}^2\!\boldsymbol{B}\boldsymbol{\cdot}\overline{\boldsymbol{a}} = (\eta - \overline{\eta}\,)\,\boldsymbol{a}\boldsymbol{\cdot}\overline{\boldsymbol{a}}.$$

Here on the left is zero, because $\boldsymbol{a}\boldsymbol{\cdot}{}^2\!\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{c} = \boldsymbol{c}\boldsymbol{\cdot}{}^2\!\boldsymbol{B}^{\mathsf{T}}\boldsymbol{\cdot}\boldsymbol{a}$ and ${}^2\!\boldsymbol{B} = {}^2\!\boldsymbol{B}^{\mathsf{T}}$. Thence $\eta = \overline{\eta}$, that is a real number.

Just as simple looks the proof of 2°:

$$\underbrace{\boldsymbol{a}_2\boldsymbol{\cdot}{}^2\!\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a}_1 - \boldsymbol{a}_1\boldsymbol{\cdot}{}^2\!\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a}_2}_{=\,0} = (\eta_1 - \eta_2)\,\boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{a}_2,\; \eta_1 \neq \eta_2 \;\Rightarrow$$
$$\Rightarrow\; \boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{a}_2 = 0. \qquad \bullet$$

If the roots of the characteristic equation (the eigenvalues) are different, then one unit long eigenvectors $\boldsymbol{\ae}_i$ compose an orthonormal basis. What are tensor components in such a basis?

$${}^2\!\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\ae}_k = \cancel{\sum_k}\,\eta_k\,\boldsymbol{\ae}_k,\quad k = 1, 2, 3$$
$${}^2\!\boldsymbol{B}\boldsymbol{\cdot}\underbrace{\boldsymbol{\ae}_k\boldsymbol{\ae}_k}_{\boldsymbol{E}} = \sum_k \eta_k\,\boldsymbol{\ae}_k\boldsymbol{\ae}_k$$

In a common case $B_{ij} = \boldsymbol{e}_i\boldsymbol{\cdot}{}^2\!\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{e}_j$. In the basis $\boldsymbol{\ae}_1$, $\boldsymbol{\ae}_2$, $\boldsymbol{\ae}_3$ of mutually perpendicular one unit long $\boldsymbol{\ae}_i\boldsymbol{\cdot}\boldsymbol{\ae}_j = \delta_{ij}$ eigenvectors of a symmetric tensor

$$B_{11} = \boldsymbol{\ae}_1\boldsymbol{\cdot}(\eta_1\boldsymbol{\ae}_1\boldsymbol{\ae}_1 + \eta_2\boldsymbol{\ae}_2\boldsymbol{\ae}_2 + \eta_3\boldsymbol{\ae}_3\boldsymbol{\ae}_3)\boldsymbol{\cdot}\boldsymbol{\ae}_1 = \eta_1,$$
$$B_{12} = \boldsymbol{\ae}_1\boldsymbol{\cdot}(\eta_1\boldsymbol{\ae}_1\boldsymbol{\ae}_1 + \eta_2\boldsymbol{\ae}_2\boldsymbol{\ae}_2 + \eta_3\boldsymbol{\ae}_3\boldsymbol{\ae}_3)\boldsymbol{\cdot}\boldsymbol{\ae}_2 = 0,$$
$$\ldots$$

The matrix of components is diagonal and ${}^2\!\boldsymbol{B} = \sum \eta_i\,\boldsymbol{\ae}_i\boldsymbol{\ae}_i$.

Here goes a summation over the three repeating indices, because the special basis is used.

The case of multiplicity of the eigenvalues is considered in the limit.

If simpler $\eta_2 \to \eta_1$, then any linear combination of vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in the limit satisfies the equation (10.1).

Then any axis in the plane $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ becomes characteristic.

When the three eigenvalues coincide, then any axis in the space is characteristic.

Then ${}^{2}\boldsymbol{B} = \eta \boldsymbol{E}$, such tensors are called isotropic or "spherical".

§ 11. Collections of invariants of a symmetric bivalent tensor

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"The algebraic" invariants

...

"The characteristic" invariants

These are coefficients of a characteristic equation (10.1).

...

"The research" invariants

...

"The harmonic" invariants

...

§ 12. Rotations in 3-dimensional space: rotation tensors

The relation between two "right" (or two "left" orthonormal bases $\boldsymbol{e}_i$ and $\mathring{\boldsymbol{e}}_i$ can be described by a two-index array represented as a matrix (§ 2, § 6)

$$\boldsymbol{e}_i = \boldsymbol{e}_i \cdot \underbrace{\mathring{\boldsymbol{e}}_j \mathring{\boldsymbol{e}}_j}_{\boldsymbol{E}} = o_{i\mathring{j}}\, \mathring{\boldsymbol{e}}_j, \;\; o_{i\mathring{j}} \equiv \boldsymbol{e}_i \cdot \mathring{\boldsymbol{e}}_j$$

("a matrix of cosines").

Also, a rotation of tensor can be described by another tensor, called a rotation tensor $\boldsymbol{O}$

$$\boldsymbol{e}_i = \boldsymbol{e}_j \underbrace{\mathring{\boldsymbol{e}}_j \cdot \mathring{\boldsymbol{e}}_i}_{\delta_{ji}} = \boldsymbol{O} \cdot \mathring{\boldsymbol{e}}_i, \;\; \boldsymbol{O} \equiv \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j = \boldsymbol{e}_1 \mathring{\boldsymbol{e}}_1 + \boldsymbol{e}_2 \mathring{\boldsymbol{e}}_2 + \boldsymbol{e}_3 \mathring{\boldsymbol{e}}_3. \tag{12.1}$$

Components of $\boldsymbol{O}$ both in an initial $\mathring{\boldsymbol{e}}_i$ and in a rotated $\boldsymbol{e}_i$ bases are the same

$$\begin{aligned} \boldsymbol{e}_i \cdot \boldsymbol{O} \cdot \boldsymbol{e}_j &= \underbrace{\boldsymbol{e}_i \cdot \boldsymbol{e}_k}_{\delta_{ik}} \mathring{\boldsymbol{e}}_k \cdot \boldsymbol{e}_j = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j, \\ \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{O} \cdot \mathring{\boldsymbol{e}}_j &= \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_k \underbrace{\mathring{\boldsymbol{e}}_k \cdot \mathring{\boldsymbol{e}}_j}_{\delta_{kj}} = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j. \end{aligned} \tag{12.2}$$

In the matrix notation, these components present the matrix of cosines $o_{\underset{ji}{\circ}} = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j$:

$$\boldsymbol{O} = o_{\underset{ji}{\circ}} \boldsymbol{e}_i \boldsymbol{e}_j = o_{\underset{ji}{\circ}} \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j .$$

Spatial transformations in the 3-dimensional Euclidean space $\mathbb{R}^3$ are distinguished into active or alibi transformations, and passive or alias transformations. An active transformation is a transformation which actually changes the physical position (alibi, elsewhere) of objects, which can be defined in the absence of a coordinate system; whereas a passive transformation is merely a change in the coordinate system in which the object is described (alias, other name) (change of coordinates, or change of basis). By transformation, math texts usually refer to active transformations.

Tensor $\boldsymbol{O}$ relates the two vectors — "before rotation" $\mathring{\boldsymbol{r}} = \rho_i \mathring{\boldsymbol{e}}_i$ and "after rotation" $\boldsymbol{r} = \rho_i \boldsymbol{e}_i$. Components $\rho_i = \mathsf{constant}$ of $\boldsymbol{r}$ in rotated basis $\boldsymbol{e}_i$ are the same as of $\mathring{\boldsymbol{r}}$ in immobile basis $\mathring{\boldsymbol{e}}_i$. So that the rotation tensor describes the rotation of the vector together with the basis. And since $\boldsymbol{e}_i = \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j \cdot \mathring{\boldsymbol{e}}_i \;\Leftrightarrow\; \rho_i \boldsymbol{e}_i = \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j \cdot \rho_i \mathring{\boldsymbol{e}}_i$, then

$$\boldsymbol{r} = \boldsymbol{O} \cdot \mathring{\boldsymbol{r}} \tag{12.3}$$

(this is the Rodrigues rotation formula).

Olinde Rodrigues. Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendants des causes qui peuvent les produire. *Journal de mathématiques pures et appliquées*, tome 5 (1840), pages 380–440.

For a second complexity tensor $\mathring{\boldsymbol{C}} = C_{ij}\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j$, a rotation into the current position $\boldsymbol{C} = C_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$ looks like

$$\boldsymbol{e}_i C_{ij} \boldsymbol{e}_j = \boldsymbol{e}_i \mathring{\boldsymbol{e}}_i \cdot \mathring{\boldsymbol{e}}_p C_{pq} \mathring{\boldsymbol{e}}_q \cdot \mathring{\boldsymbol{e}}_j \boldsymbol{e}_j \;\Leftrightarrow\; \boldsymbol{C} = \boldsymbol{O} \cdot \mathring{\boldsymbol{C}} \cdot \boldsymbol{O}^{\mathsf{T}}. \tag{12.4}$$

The essential property of a rotation tensor — the orthogonality — is expressed as

$$\underbrace{\boldsymbol{O}}_{\boldsymbol{e}_i\mathring{\boldsymbol{e}}_i} \cdot \underbrace{\boldsymbol{O}^{\mathsf{T}}}_{\mathring{\boldsymbol{e}}_j\boldsymbol{e}_j} = \underbrace{\boldsymbol{O}^{\mathsf{T}}}_{\mathring{\boldsymbol{e}}_i\boldsymbol{e}_i} \cdot \underbrace{\boldsymbol{O}}_{\boldsymbol{e}_j\mathring{\boldsymbol{e}}_j} = \underbrace{\boldsymbol{E}}_{\substack{\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_i \\ \boldsymbol{e}_i\boldsymbol{e}_i}}, \tag{12.5}$$

that is the transposed tensor coincides with the reciprocal tensor: $\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{O}^{-1} \Leftrightarrow \boldsymbol{O} = \boldsymbol{O}^{-\mathsf{T}}$.

An orthogonal tensor retains lengths and angles (the metric) because it does not change the "$\cdot$"-product of vectors

$$(\boldsymbol{O} \cdot \boldsymbol{a}) \cdot (\boldsymbol{O} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{O}^{\mathsf{T}} \cdot \boldsymbol{O} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{E} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}. \tag{12.6}$$

For all orthogonal tensors $(\det \boldsymbol{Q})^2 = 1$:

$$1 = \det \boldsymbol{E} = \det\left(\boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathsf{T}}\right) = (\det \boldsymbol{Q})\left(\det \boldsymbol{Q}^{\mathsf{T}}\right) = (\det \boldsymbol{Q})^2 .$$

A rotation tensor is an orthogonal tensor with $\det \boldsymbol{O} = 1$.

But not only rotation tensors possess the property of orthogonality. When in (12.1) the first basis is "left", and the second one is "right", then there's a combination of a rotation and a reflection ("rotoreflection") $\boldsymbol{O} = -\boldsymbol{E} \cdot \boldsymbol{O}$ with $\det(-\boldsymbol{E} \cdot \boldsymbol{O}) = -1$.

У любого бивалентного тензора в трёхмерном пространстве как минимум одно собственное число — the root of (10.2) is non-complex (real). For a rotation tensor, it is equal to one

$$\boldsymbol{O} \cdot \boldsymbol{a} = \eta \boldsymbol{a} \;\Rightarrow\; \boldsymbol{a} \cdot \boldsymbol{O}^{\mathsf{T}} \cdot \boldsymbol{O} \cdot \boldsymbol{a} = \eta \boldsymbol{a} \cdot \eta \boldsymbol{a} \;\Rightarrow\; \eta^2 = 1.$$

Соответствующая собственная ось называется осью поворота. Теорема Euler'а о конечном повороте в том и состоит, что такая ось существует [ПУБИКАЦИЯ ЭЙЛЕРА ПРО ЭТО]. Если $\boldsymbol{k}$ — орт этой оси, а ϑ — конечная величина угла поворота, то тензор поворота представи́м как

$$\boldsymbol{O}(\boldsymbol{k}, \vartheta) = \boldsymbol{E} \cos\vartheta + \boldsymbol{k} \times \boldsymbol{E} \sin\vartheta + \boldsymbol{k}\boldsymbol{k}\,(1 - \cos\vartheta). \tag{12.7}$$

Доказывается эта формула так. Направление $\boldsymbol{k}$ при повороте не меняется ($\boldsymbol{O} \boldsymbol{\cdot} \boldsymbol{k} = \boldsymbol{k}$), поэтому на оси поворота $\mathring{\boldsymbol{e}}_3 = \boldsymbol{e}_3 = \boldsymbol{k}$. В перпендикулярной плоскости (рис. 5) $\mathring{\boldsymbol{e}}_1 = \boldsymbol{e}_1 \cos\vartheta - \boldsymbol{e}_2 \sin\vartheta$, $\mathring{\boldsymbol{e}}_2 = \boldsymbol{e}_1 \sin\vartheta + \boldsymbol{e}_2 \cos\vartheta$, $\boldsymbol{O} = \boldsymbol{e}_i \mathring{\boldsymbol{e}}_i \;\Rightarrow\; (12.7)$.

Из (12.7) и (12.3) получаем формулу поворота Родрига в параметрах $\boldsymbol{k}$ и ϑ:

$$\boldsymbol{r} \;=\; \mathring{\boldsymbol{r}} \cos\vartheta \;+\; \boldsymbol{k} \times \mathring{\boldsymbol{r}} \sin\vartheta \;+\; \boldsymbol{k}\boldsymbol{k} \boldsymbol{\cdot} \mathring{\boldsymbol{r}}\,(1 - \cos\vartheta).$$

В параметрах конечного поворота транспонирование, оно же обращение, тензора $\boldsymbol{O}$ эквивалентно перемене направления поворота — знака угла ϑ

$$\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{O}\big|_{\vartheta = -\vartheta} = \boldsymbol{E} \cos\vartheta - \boldsymbol{k} \times \boldsymbol{E} \sin\vartheta + \boldsymbol{k}\boldsymbol{k}\,(1 - \cos\vartheta).$$

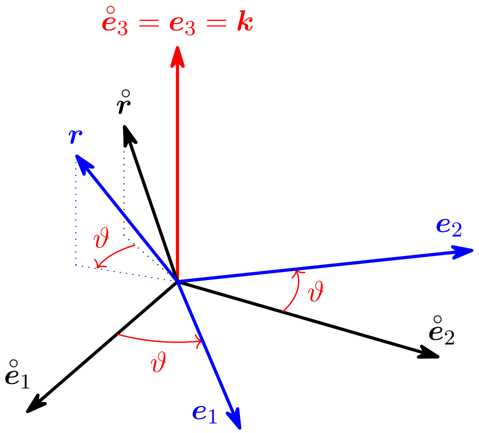
Пусть теперь тензор поворота меняется со временем: $\boldsymbol{O} = \boldsymbol{O}(t)$. Псевдовектор угловой скорости $\boldsymbol{\omega}$ вводится через тензор поворота $\boldsymbol{O}$ таким путём. Дифференцируем тождество ортогональности (12.5) по времени*

$$\dot{\boldsymbol{O}} \boldsymbol{\cdot} \boldsymbol{O}^{\mathsf{T}} + \boldsymbol{O} \boldsymbol{\cdot} \dot{\boldsymbol{O}}^{\mathsf{T}} = {}^2\mathbf{0}.$$

* Various notations are used to designate the time derivative. In addition to the Leibniz's notation ${}^{dx}/_{dt}$, the very popular one is the "dot above" Newton's notation $\dot{x}$.

$$\mathring{\boldsymbol{e}}_i = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j \boldsymbol{e}_j$$

$$\begin{bmatrix} \mathring{\boldsymbol{e}}_1 \\ \mathring{\boldsymbol{e}}_2 \\ \mathring{\boldsymbol{e}}_3 \end{bmatrix} = \begin{bmatrix} \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{bmatrix}$$



$$\begin{bmatrix} \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & \cos(90^\circ + \vartheta) & \cos 90^\circ \\ \cos(90^\circ - \vartheta) & \cos\vartheta & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 0^\circ \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{gathered} \mathring{\boldsymbol{e}}_1 = \boldsymbol{e}_1 \cos\vartheta - \boldsymbol{e}_2 \sin\vartheta \\ \mathring{\boldsymbol{e}}_2 = \boldsymbol{e}_1 \sin\vartheta + \boldsymbol{e}_2 \cos\vartheta \\ \mathring{\boldsymbol{e}}_3 = \boldsymbol{e}_3 = \boldsymbol{k} \end{gathered}$$

$$\boldsymbol{O} = \boldsymbol{e}_1 \mathring{\boldsymbol{e}}_1 + \boldsymbol{e}_2 \mathring{\boldsymbol{e}}_2 + \boldsymbol{e}_3 \mathring{\boldsymbol{e}}_3 =$$

$$= \overbrace{\boldsymbol{e}_1 \boldsymbol{e}_1 \cos\vartheta - \boldsymbol{e}_1 \boldsymbol{e}_2 \sin\vartheta}^{\boldsymbol{e}_1 \mathring{\boldsymbol{e}}_1} + \overbrace{\boldsymbol{e}_2 \boldsymbol{e}_1 \sin\vartheta + \boldsymbol{e}_2 \boldsymbol{e}_2 \cos\vartheta}^{\boldsymbol{e}_2 \mathring{\boldsymbol{e}}_2} + \overbrace{\boldsymbol{k}\boldsymbol{k}}^{\boldsymbol{e}_3 \mathring{\boldsymbol{e}}_3} =$$

$$= \boldsymbol{E} \cos\vartheta - \underbrace{\boldsymbol{e}_3 \boldsymbol{e}_3}_{\boldsymbol{k}\boldsymbol{k}} \cos\vartheta + \underbrace{(\boldsymbol{e}_2 \boldsymbol{e}_1 - \boldsymbol{e}_1 \boldsymbol{e}_2)}_{\boldsymbol{e}_3 \times \boldsymbol{e}_i \boldsymbol{e}_i = {}^3\!\boldsymbol{\epsilon}_{3ij} \boldsymbol{e}_j \boldsymbol{e}_i} \sin\vartheta + \boldsymbol{k}\boldsymbol{k} =$$

$$= \boldsymbol{E} \cos\vartheta + \boldsymbol{k} \times \boldsymbol{E} \sin\vartheta + \boldsymbol{k}\boldsymbol{k}\,(1 - \cos\vartheta)$$

рисунок 5
“Finite rotation”

Тензор $\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}}$ (по (??) $\left(\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}}\right)^{\mathsf{T}} = \boldsymbol{O} \cdot \dot{\boldsymbol{O}}^{\mathsf{T}}$) оказался антисимметричным. Поэтому согласно (8.3) он представи́м сопутствующим вектором как $\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\omega} \times \boldsymbol{E} = \boldsymbol{\omega} \times \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}}$. То есть

$$\dot{\boldsymbol{O}} = \boldsymbol{\omega} \times \boldsymbol{O}, \quad \boldsymbol{\omega} \equiv -\frac{1}{2}\left(\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}}\right)_{\times} \tag{12.8}$$

Помимо этого общего представления вектора $\boldsymbol{\omega}$, для него есть и другие. Например, через параметры конечного поворота.

Производная $\dot{\boldsymbol{O}}$ в параметрах конечного поворота в общем случае (оба параметра — и единичный вектор $\boldsymbol{k}$, и угол ϑ — переменны во времени):

$$\dot{\boldsymbol{O}} = \left(\boldsymbol{O}^{\mathsf{S}} + \boldsymbol{O}^{\mathsf{A}}\right)^{\bullet} = \Big(\overbrace{\boldsymbol{E}\cos\vartheta + \boldsymbol{kk}\,(1-\cos\vartheta)}^{\boldsymbol{O}^{\mathsf{S}}} + \overbrace{\boldsymbol{k} \times \boldsymbol{E}\sin\vartheta}^{\boldsymbol{O}^{\mathsf{A}}}\Big)^{\bullet} =$$
$$= \underbrace{(\boldsymbol{kk} - \boldsymbol{E})\,\dot{\vartheta}\sin\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})\,(1-\cos\vartheta)}_{\dot{\boldsymbol{O}}^{\mathsf{S}}} +$$
$$+ \underbrace{\boldsymbol{k} \times \boldsymbol{E}\,\dot{\vartheta}\cos\vartheta + \dot{\boldsymbol{k}} \times \boldsymbol{E}\sin\vartheta}_{\dot{\boldsymbol{O}}^{\mathsf{A}}}.$$

Находим

$$\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}} = (\dot{\boldsymbol{O}}^{\mathsf{S}} + \dot{\boldsymbol{O}}^{\mathsf{A}}) \cdot (\boldsymbol{O}^{\mathsf{S}} - \boldsymbol{O}^{\mathsf{A}}) =$$
$$= \dot{\boldsymbol{O}}^{\mathsf{S}} \cdot \boldsymbol{O}^{\mathsf{S}} + \dot{\boldsymbol{O}}^{\mathsf{A}} \cdot \boldsymbol{O}^{\mathsf{S}} - \dot{\boldsymbol{O}}^{\mathsf{S}} \cdot \boldsymbol{O}^{\mathsf{A}} - \dot{\boldsymbol{O}}^{\mathsf{A}} \cdot \boldsymbol{O}^{\mathsf{A}},$$

using

$$\boldsymbol{k} \cdot \boldsymbol{k} = 1 = \mathsf{constant} \;\Rightarrow\; \boldsymbol{k} \cdot \dot{\boldsymbol{k}} + \dot{\boldsymbol{k}} \cdot \boldsymbol{k} = 0 \;\Leftrightarrow\; \dot{\boldsymbol{k}} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \dot{\boldsymbol{k}} = 0,$$
$$\boldsymbol{kk} \cdot \boldsymbol{kk} = \boldsymbol{kk}, \;\; \dot{\boldsymbol{k}}\boldsymbol{k} \cdot \boldsymbol{kk} = \dot{\boldsymbol{k}}\boldsymbol{k}, \;\; \boldsymbol{k}\dot{\boldsymbol{k}} \cdot \boldsymbol{kk} = {}^2\mathbf{0},$$
$$(\boldsymbol{kk} - \boldsymbol{E}) \cdot \boldsymbol{k} = \boldsymbol{k} - \boldsymbol{k} = \mathbf{0}, \;\; (\boldsymbol{kk} - \boldsymbol{E}) \cdot \boldsymbol{kk} = \boldsymbol{kk} - \boldsymbol{kk} = {}^2\mathbf{0},$$
$$\boldsymbol{k} \cdot (\boldsymbol{k} \times \boldsymbol{E}) = (\boldsymbol{k} \times \boldsymbol{E}) \cdot \boldsymbol{k} = \boldsymbol{k} \times \boldsymbol{k} = \mathbf{0}, \;\; \boldsymbol{kk} \cdot (\boldsymbol{k} \times \boldsymbol{E}) = (\boldsymbol{k} \times \boldsymbol{E}) \cdot \boldsymbol{kk} = {}^2\mathbf{0},$$
$$(\boldsymbol{kk} - \boldsymbol{E}) \cdot (\boldsymbol{k} \times \boldsymbol{E}) = -\boldsymbol{k} \times \boldsymbol{E},$$
$$(\boldsymbol{a} \times \boldsymbol{E}) \cdot \boldsymbol{b} = \boldsymbol{a} \times (\boldsymbol{E} \cdot \boldsymbol{b}) = \boldsymbol{a} \times \boldsymbol{b} \;\Rightarrow\; (\dot{\boldsymbol{k}} \times \boldsymbol{E}) \cdot \boldsymbol{kk} = \dot{\boldsymbol{k}} \times \boldsymbol{kk},$$

$$(7.13) \;\Rightarrow\; (\boldsymbol{k} \times \boldsymbol{E}) \cdot (\boldsymbol{k} \times \boldsymbol{E}) = \boldsymbol{kk} - \boldsymbol{E}, \;\; (\dot{\boldsymbol{k}} \times \boldsymbol{E}) \cdot (\boldsymbol{k} \times \boldsymbol{E}) = \boldsymbol{k}\dot{\boldsymbol{k}} - \cancel{\dot{\boldsymbol{k}} \cdot \boldsymbol{k}\boldsymbol{E}},$$
$$(7.12) \;\Rightarrow\; \dot{\boldsymbol{k}}\boldsymbol{k} - \boldsymbol{k}\dot{\boldsymbol{k}} = (\boldsymbol{k} \times \dot{\boldsymbol{k}}) \times \boldsymbol{E}, \;\; (\dot{\boldsymbol{k}} \times \boldsymbol{k})\boldsymbol{k} - \boldsymbol{k}(\dot{\boldsymbol{k}} \times \boldsymbol{k}) = \boldsymbol{k} \times (\dot{\boldsymbol{k}} \times \boldsymbol{k}) \times \boldsymbol{E}$$

$$\dot{\boldsymbol{P}}^{\mathsf{S}} \cdot \boldsymbol{P}^{\mathsf{S}} =$$

$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\,\dot\vartheta \sin\vartheta \cdot \boldsymbol{E}\cos\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta) \cdot \boldsymbol{E}\cos\vartheta \,+$$

$$+ \cancel{(\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\,\dot\vartheta \sin\vartheta \cdot \boldsymbol{k}\boldsymbol{k}\,(1-\cos\vartheta)} + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta) \cdot \boldsymbol{k}\boldsymbol{k}\,(1-\cos\vartheta) =$$

$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\,\dot\vartheta \sin\vartheta\cos\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})\cos\vartheta\,(1-\cos\vartheta) + (\cancel{\boldsymbol{k}\dot{\boldsymbol{k}} \cdot \boldsymbol{k}\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k} \cdot \boldsymbol{k}\boldsymbol{k})\,(1-\cos\vartheta)^2 =$$

$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\,\dot\vartheta \sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta\,(1-\cos\vartheta) \,+$$

$$+ \dot{\boldsymbol{k}}\boldsymbol{k}\cos\vartheta - \dot{\boldsymbol{k}}\boldsymbol{k}\cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k} - 2\,\dot{\boldsymbol{k}}\boldsymbol{k}\cos\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\cos^2\vartheta =$$

$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\,\dot\vartheta \sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}}\cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\,(1-\cos\vartheta)\,,$$

$$\dot{\boldsymbol{P}}^{\mathsf{A}} \cdot \boldsymbol{P}^{\mathsf{S}} =$$

$$= (\boldsymbol{k}\times\boldsymbol{E}) \cdot \boldsymbol{E}\,\dot\vartheta\cos^2\vartheta + (\dot{\boldsymbol{k}}\times\boldsymbol{E}) \cdot \boldsymbol{E}\sin\vartheta\cos\vartheta \,+$$

$$+ \cancel{(\boldsymbol{k}\times\boldsymbol{E}) \cdot \boldsymbol{k}\boldsymbol{k}\,\dot\vartheta\cos\vartheta\,(1-\cos\vartheta)} + (\dot{\boldsymbol{k}}\times\boldsymbol{E}) \cdot \boldsymbol{k}\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta) =$$

$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot\vartheta\cos^2\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{k}\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta)\,,$$

$$\dot{\boldsymbol{P}}^{\mathsf{S}} \cdot \boldsymbol{P}^{\mathsf{A}} =$$

$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\,\dot\vartheta\sin\vartheta \cdot (\boldsymbol{k}\times\boldsymbol{E})\sin\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta) \cdot (\boldsymbol{k}\times\boldsymbol{E})\sin\vartheta =$$

$$= \cancel{\boldsymbol{k}\boldsymbol{k} \cdot (\boldsymbol{k}\times\boldsymbol{E})\,\dot\vartheta\sin^2\vartheta} - \boldsymbol{E} \cdot (\boldsymbol{k}\times\boldsymbol{E})\,\dot\vartheta\sin^2\vartheta + \Big(\boldsymbol{k}\dot{\boldsymbol{k}} \cdot (\boldsymbol{k}\times\boldsymbol{E}) + \cancel{\dot{\boldsymbol{k}}\boldsymbol{k} \cdot (\boldsymbol{k}\times\boldsymbol{E})}\Big)\sin\vartheta\,(1-\cos\vartheta) =$$

$$= -\boldsymbol{k}\times\boldsymbol{E}\,\dot\vartheta\sin^2\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\times\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta)\,,$$

$$\dot{\boldsymbol{P}}^{\mathsf{A}} \cdot \boldsymbol{P}^{\mathsf{A}} = (\boldsymbol{k}\times\boldsymbol{E})\,\dot\vartheta\cos\vartheta \cdot (\boldsymbol{k}\times\boldsymbol{E})\sin\vartheta + (\dot{\boldsymbol{k}}\times\boldsymbol{E}) \cdot (\boldsymbol{k}\times\boldsymbol{E})\sin^2\vartheta =$$

$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\,\dot\vartheta\sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\sin^2\vartheta\,;$$

$$\dot{\boldsymbol{P}} \cdot \boldsymbol{P}^{\mathsf{T}} = \dot{\boldsymbol{P}}^{\mathsf{S}} \cdot \boldsymbol{P}^{\mathsf{S}} + \dot{\boldsymbol{P}}^{\mathsf{A}} \cdot \boldsymbol{P}^{\mathsf{S}} - \dot{\boldsymbol{P}}^{\mathsf{S}} \cdot \boldsymbol{P}^{\mathsf{A}} - \dot{\boldsymbol{P}}^{\mathsf{A}} \cdot \boldsymbol{P}^{\mathsf{A}} =$$

$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\,\dot\vartheta\sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}}\cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\,(1-\cos\vartheta) \,+$$

$$+ \boldsymbol{k}\times\boldsymbol{E}\,\dot\vartheta\cos^2\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{k}\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta) \,+$$

$$+ \boldsymbol{k}\times\boldsymbol{E}\,\dot\vartheta\sin^2\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}}\times\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta) - (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\,\dot\vartheta\sin\vartheta\cos\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}}\sin^2\vartheta =$$

$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot\vartheta + (\dot{\boldsymbol{k}}\boldsymbol{k} - \boldsymbol{k}\dot{\boldsymbol{k}})(1-\cos\vartheta) + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + (\dot{\boldsymbol{k}}\times\boldsymbol{k}\boldsymbol{k} - \boldsymbol{k}\dot{\boldsymbol{k}}\times\boldsymbol{k})\sin\vartheta\,(1-\cos\vartheta) =$$

$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot\vartheta + \boldsymbol{k}\times\dot{\boldsymbol{k}}\times\boldsymbol{E}\,(1-\cos\vartheta) + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + \boldsymbol{k}\times(\dot{\boldsymbol{k}}\times\boldsymbol{k})\times\boldsymbol{E}\sin\vartheta\,(1-\cos\vartheta) =$$

$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + (\dot{\boldsymbol{k}}\boldsymbol{k} \cdot \boldsymbol{k} - \cancel{\boldsymbol{k}\dot{\boldsymbol{k}} \cdot \boldsymbol{k}})\times\boldsymbol{E}\sin\vartheta\,(1-\cos\vartheta) + \boldsymbol{k}\times\dot{\boldsymbol{k}}\times\boldsymbol{E}\,(1-\cos\vartheta) =$$

$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta + \boldsymbol{k}\times\dot{\boldsymbol{k}}\times\boldsymbol{E}\,(1-\cos\vartheta)\,.$$

Этот результат, подставленный в определение (12.8) псевдовектора $\boldsymbol{\omega}$, даёт

$$\boldsymbol{\omega} = \boldsymbol{k}\dot{\vartheta} + \dot{\boldsymbol{k}}\sin\vartheta + \boldsymbol{k}\times\dot{\boldsymbol{k}}\,(1-\cos\vartheta). \tag{12.9}$$

Вектор $\boldsymbol{\omega}$ получился разложенным по трём взаимно ортогональным направлениям — $\boldsymbol{k}$, $\dot{\boldsymbol{k}}$ и $\boldsymbol{k}\times\dot{\boldsymbol{k}}$. При неподвижной оси поворота $\dot{\boldsymbol{k}} = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{\omega} = \boldsymbol{k}\dot{\vartheta}$.

Ещё одно представление $\boldsymbol{\omega}$ связано с компонентами тензора поворота (12.2). Поскольку $\boldsymbol{P} = o_{j\mathring{i}}\,\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j$, $\boldsymbol{P}^{\mathsf{T}} = o_{i\mathring{j}}\,\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j$, а векторы начального базиса $\mathring{\boldsymbol{e}}_i$ неподвижны (со временем не меняются), то

$$\dot{\boldsymbol{P}} = \dot{o}_{j\mathring{i}}\,\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j,\;\; \dot{\boldsymbol{P}}\cdot\boldsymbol{P}^{\mathsf{T}} = \dot{o}_{n\mathring{i}}\,o_{n\mathring{j}}\,\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j,$$

$$\boldsymbol{\omega} = -\tfrac{1}{2}\,\dot{o}_{n\mathring{i}}\,o_{n\mathring{j}}\,\mathring{\boldsymbol{e}}_i\times\mathring{\boldsymbol{e}}_j = \tfrac{1}{2}\in_{[jik]} o_{n\mathring{j}}\,\dot{o}_{n\mathring{i}}\,\mathring{\boldsymbol{e}}_k\,. \tag{12.10}$$

Отметим и формулы

$$\begin{gathered}(12.8) \;\Rightarrow\; \dot{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_i = \boldsymbol{\omega}\times\boldsymbol{e}_i\mathring{\boldsymbol{e}}_i \;\Rightarrow\; \dot{\boldsymbol{e}}_i = \boldsymbol{\omega}\times\boldsymbol{e}_i\,,\\ (12.8) \;\Rightarrow\; \boldsymbol{\omega} = -\tfrac{1}{2}\left(\dot{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_i\cdot\mathring{\boldsymbol{e}}_j\boldsymbol{e}_j\right)_{\times} = -\tfrac{1}{2}\left(\dot{\boldsymbol{e}}_i\boldsymbol{e}_i\right)_{\times} = \tfrac{1}{2}\,\boldsymbol{e}_i\times\dot{\boldsymbol{e}}_i\,.\end{gathered} \tag{12.11}$$

Не всё то вектор, что имеет величину и направление. Поворот тела вокруг оси представляет, казалось бы, вектор: его численное значение равно углу поворота, а направление совпадает с направлением оси вращения. Однако, повороты не складываются как векторы*.

На са́мом же деле последовательные повороты не складываются, а умножаются.

Можно ли складывать угловые скорости? — Да, ведь угол поворота в $\dot{\vartheta}$ бесконечномалый. — Но только при вращении вокруг неподвижной оси?

...

Варьируя тождество (12.5), получим $\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O}\cdot\delta\boldsymbol{O}^{\mathsf{T}}$. Этот тензор антисимметричен, и потому выражается через свой сопут-

* Когда углы поворота не бесконечно-ма́лые.

ствующий вектор $\boldsymbol{\delta o}$ как $\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\delta o} \times \boldsymbol{E}$. Приходим к соотношениям

$$\delta \boldsymbol{O} = \boldsymbol{\delta o} \times \boldsymbol{O}, \;\; \boldsymbol{\delta o} = -\frac{1}{2}\Big(\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}}\Big)_{\times}, \tag{12.12}$$

аналогичным (12.8). Вектор бесконечно малого поворота $\boldsymbol{\delta o}$ это не "вариация **o**", но единый символ (в отличие от $\delta \boldsymbol{O}$).

Малый поворот определяется вектором $\boldsymbol{\delta o}$, но конечный поворот тоже возможно представить как вектор.

...

§ 13. Rotations in 3-dimensional space: quaternions

The other way to describe a rotation (or orientation) in space is via quaternions. It is very popular for computer graphics.

Lorem ipsum ....

...

§ 14. Variations

Further we will pretty often use the operation of varying. It is similar to the differentiation.

The variations are seen as the infinitesimal displacements, compatible with the constraints. If there are no restrictions for the variable x, then the variations δx are completely random. But when

$$x = x(y)$$

is the function of the independent argument y, then

$$\delta x = x'(y)\,\delta y.$$

Variations are similar to differentials. As example, if δx and δy are variations of x and y, u and v are the finite values, then we write $u\delta x + v\delta y = \delta w$ even if δw is not a variation of w.

In this case δw is a single symbol. Surely if $u = u(x, y)$, $v = v(x, y)$ and $\partial_x v = \partial_y u$ ($\frac{\partial}{\partial x} v = \frac{\partial}{\partial y} u$), then the sum $\delta w = u \delta x + v \delta y$ will be a variation of some w.

Varying the identity (12.5), we get

$$\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O} \cdot \delta \boldsymbol{O}^{\mathsf{T}}.$$

This tensor is antisymmetric, and thus is representable via its companion pseudovector $\boldsymbol{\delta o}$ as

$$\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\delta o} \times \boldsymbol{E}.$$

We have the following relations

$$\delta \boldsymbol{O} = \boldsymbol{\delta o} \times \boldsymbol{O}, \quad \boldsymbol{\delta o} = -\frac{1}{2} \Big(\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} \Big)_{\times}, \tag{14.1}$$

similar to (12.8). Vector $\boldsymbol{\delta o}$ of an infinitesimal rotation is not “a variation of $\mathbf{o}$”, but a single symbol.

An infinitesimal rotation is defined by vector $\boldsymbol{\delta o}$, but a finite rotation is also possible to represent as a vector

...

§ 15. Polar decomposition

Any tensor of the second complexity $\boldsymbol{F}$ with $\det F_{ij} \neq 0$, that is a not singular tensor, can be decomposed as

...

Example. Polar decompose tensor $\boldsymbol{C} = C_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$, where $\boldsymbol{e}_k$ are mutually perpendicular unit vectors of basis, and C_{ij} are tensor's components

$$C_{ij} = \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix}$$

$$\boldsymbol{O} = O_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = \boldsymbol{O}_1 \cdot \boldsymbol{O}_2$$

$$O_{ij} = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$\boldsymbol{C} = \boldsymbol{O} \cdot \boldsymbol{S}_{\mathsf{R}}, \;\; \boldsymbol{O}^{\mathsf{T}} \cdot \boldsymbol{C} = \boldsymbol{S}_{\mathsf{R}}$$

$$\boldsymbol{C} = \boldsymbol{S}_{\mathsf{L}} \cdot \boldsymbol{O}, \;\; \boldsymbol{C} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{S}_{\mathsf{L}}$$

$$S_{\mathsf{R}ij} = \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix}$$

$$S_{\mathsf{L}ij} = \begin{bmatrix} 104/5 & 47/5 & 5 \\ 47/5 & -129/5 & -10 \\ 5 & -10 & 3 \end{bmatrix}$$

...

§ 16. In the oblique basis

Until now a basis of the three mutually perpendicular unit vectors $\boldsymbol{e}_i$ was used. Now we will take a basis of any three linearly independent (non-coplanar) vectors $\boldsymbol{a}_i$.

The decomposition of vector $\boldsymbol{v}$ in the basis $\boldsymbol{a}_i$ (fig. 6) is the linear combination

$$\boldsymbol{v} = v^i \boldsymbol{a}_i. \qquad (16.1)$$

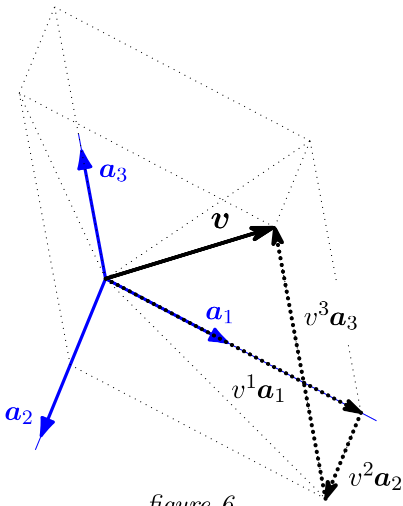


figure 6

The summation convention gains the new conditions: a summation index is repeated at different levels of the same monomial, and a free index stays at the equal height in every part of the expression ($a_i = b_{ij} c^j$ is correct, $a_i = b^i_{kk}$ is wrong twice).

В таком базисе ужé $\boldsymbol{v} \bullet \boldsymbol{a}_i = v^k \boldsymbol{a}_k \bullet \boldsymbol{a}_i \neq v^i$, ведь тут $\boldsymbol{a}_i \bullet \boldsymbol{a}_k \neq \delta_{ik}$.

Дополним же базис $\boldsymbol{a}_i$ ещё другой тройкой векторов $\boldsymbol{a}^i$, называемых кобазисом или взаимным базисом, чтобы

$$\boldsymbol{a}_i \bullet \boldsymbol{a}^j = \delta_i^j, \;\; \boldsymbol{a}^i \bullet \boldsymbol{a}_j = \delta_j^i, \quad \boldsymbol{E} = \boldsymbol{a}^i \boldsymbol{a}_i = \boldsymbol{a}_i \boldsymbol{a}^i. \qquad (16.2)$$

Это — основное свойство кобазиса. Ортонормированный (ортонормальный) базис может быть определён как совпада́ющий со своим кобазисом: $\boldsymbol{e}^i = \boldsymbol{e}_i$.

Для, к примеру, первого вектора кобазиса $\boldsymbol{a}^1$

$$\begin{cases} \boldsymbol{a}^1 \bullet \boldsymbol{a}_1 = 1 \\ \boldsymbol{a}^1 \bullet \boldsymbol{a}_2 = 0 \\ \boldsymbol{a}^1 \bullet \boldsymbol{a}_3 = 0 \end{cases} \Rightarrow \begin{cases} \boldsymbol{a}^1 \bullet \boldsymbol{a}_1 = 1 \\ \gamma \boldsymbol{a}^1 = \boldsymbol{a}_2 \times \boldsymbol{a}_3 \end{cases} \Rightarrow \begin{cases} \boldsymbol{a}^1 = {}^1\!/\gamma \, \boldsymbol{a}_2 \times \boldsymbol{a}_3 \\ \gamma = \boldsymbol{a}_2 \times \boldsymbol{a}_3 \bullet \boldsymbol{a}_1 \end{cases}$$

Коэффициент γ получился равным (с точностью до знака для “левой” тройки $\boldsymbol{a}_i$) объёму параллелепипеда, построенного на векторах $\boldsymbol{a}_i$. In §7 the same volume was presented as $\sqrt{g}$, and this is not without reason, because it coincides with the square root of gramian $g \equiv \det g_{ij}$ — determinant of the symmetric J. P. Gram matrix $g_{ij} \equiv \boldsymbol{a}_i \bullet \boldsymbol{a}_j$.

○ The proof resembles the derivation of (7.7). The "triple product" $\boldsymbol{a}_i \times \boldsymbol{a}_j \bullet \boldsymbol{a}_k$ in some orthonormal basis $\boldsymbol{e}_i$ вычисли́мо как детерминант (с "−" для "левой" тройки $\boldsymbol{a}_i$) по строкам

$$\in_{[ijk]} \equiv \boldsymbol{a}_i \times \boldsymbol{a}_j \bullet \boldsymbol{a}_k = \pm \det \begin{bmatrix} \boldsymbol{a}_i \bullet \boldsymbol{e}_1 & \boldsymbol{a}_i \bullet \boldsymbol{e}_2 & \boldsymbol{a}_i \bullet \boldsymbol{e}_3 \\ \boldsymbol{a}_j \bullet \boldsymbol{e}_1 & \boldsymbol{a}_j \bullet \boldsymbol{e}_2 & \boldsymbol{a}_j \bullet \boldsymbol{e}_3 \\ \boldsymbol{a}_k \bullet \boldsymbol{e}_1 & \boldsymbol{a}_k \bullet \boldsymbol{e}_2 & \boldsymbol{a}_k \bullet \boldsymbol{e}_3 \end{bmatrix}$$

или по столбцам

$$\in_{[pqr]} \equiv \boldsymbol{a}_p \times \boldsymbol{a}_q \bullet \boldsymbol{a}_r = \pm \det \begin{bmatrix} \boldsymbol{a}_p \bullet \boldsymbol{e}_1 & \boldsymbol{a}_q \bullet \boldsymbol{e}_1 & \boldsymbol{a}_r \bullet \boldsymbol{e}_1 \\ \boldsymbol{a}_p \bullet \boldsymbol{e}_2 & \boldsymbol{a}_q \bullet \boldsymbol{e}_2 & \boldsymbol{a}_r \bullet \boldsymbol{e}_2 \\ \boldsymbol{a}_p \bullet \boldsymbol{e}_3 & \boldsymbol{a}_q \bullet \boldsymbol{e}_3 & \boldsymbol{a}_r \bullet \boldsymbol{e}_3 \end{bmatrix}.$$

Произведение определителей $\in_{[ijk]}\in_{[pqr]}$ равно определителю произведения матриц, and elements of the latter are sums like $\boldsymbol{a}_i \bullet \boldsymbol{e}_s \boldsymbol{a}_p \bullet \boldsymbol{e}_s = \boldsymbol{a}_i \bullet \boldsymbol{e}_s \boldsymbol{e}_s \bullet \boldsymbol{a}_p = \boldsymbol{a}_i \bullet \boldsymbol{E} \bullet \boldsymbol{a}_p = \boldsymbol{a}_i \bullet \boldsymbol{a}_p$, в результате

$$\in_{[ijk]}\in_{[pqr]} = \det \begin{bmatrix} \boldsymbol{a}_i \bullet \boldsymbol{a}_p & \boldsymbol{a}_i \bullet \boldsymbol{a}_q & \boldsymbol{a}_i \bullet \boldsymbol{a}_r \\ \boldsymbol{a}_j \bullet \boldsymbol{a}_p & \boldsymbol{a}_j \bullet \boldsymbol{a}_q & \boldsymbol{a}_j \bullet \boldsymbol{a}_r \\ \boldsymbol{a}_k \bullet \boldsymbol{a}_p & \boldsymbol{a}_k \bullet \boldsymbol{a}_q & \boldsymbol{a}_k \bullet \boldsymbol{a}_r \end{bmatrix};$$

$i=p=1,\ j=q=2,\ k=r=3 \ \Rightarrow \in_{[123]}\in_{[123]} = \det\limits_{i,j} (\boldsymbol{a}_i \bullet \boldsymbol{a}_j) = \det\limits_{i,j} g_{ij}.$ ●

Representing $\boldsymbol{a}^1$ and other cobasis vectors as the sum

$$\pm 2\sqrt{g}\,\boldsymbol{a}^1 = \boldsymbol{a}_2 \times \boldsymbol{a}_3 \overbrace{- \boldsymbol{a}_3 \times \boldsymbol{a}_2}^{+\,\boldsymbol{a}_2 \times \boldsymbol{a}_3},$$

приходим к общей формуле (с "−" для "левой" тройки $\boldsymbol{a}_i$)

$$\boldsymbol{a}^i = \pm \frac{1}{2\sqrt{g}}\, e^{ijk} \boldsymbol{a}_j \times \boldsymbol{a}_k, \quad \sqrt{g} \equiv \pm\, \boldsymbol{a}_1 \times \boldsymbol{a}_2 \bullet \boldsymbol{a}_3 > 0. \tag{16.3}$$

Здесь e^{ijk} по-прежнему символ перестановки Veblen'a (± 1 или 0): $e^{ijk} \equiv e_{ijk}$. Произведение $\boldsymbol{a}_j \times \boldsymbol{a}_k = \in_{[jkn]} \boldsymbol{a}^n$, компоненты тензора Ле́ви-Чиви́ты $\in_{[jkn]} = \pm\, e_{jkn} \sqrt{g}$, and by (7.8) $e^{ijk} e_{jkn} = 2\delta^i_n$. Thus

$$\boldsymbol{a}^1 = \pm {}^1\!/\!\sqrt{g}\, (\boldsymbol{a}_2 \times \boldsymbol{a}_3), \;\; \boldsymbol{a}^2 = \pm {}^1\!/\!\sqrt{g}\, (\boldsymbol{a}_3 \times \boldsymbol{a}_1), \;\; \boldsymbol{a}^3 = \pm {}^1\!/\!\sqrt{g}\, (\boldsymbol{a}_1 \times \boldsymbol{a}_2).$$

Example. Get cobasis for basis $\boldsymbol{a}_i$ when

$$\begin{aligned} \boldsymbol{a}_1 &= \boldsymbol{e}_1 + \boldsymbol{e}_2, \\ \boldsymbol{a}_2 &= \boldsymbol{e}_1 + \boldsymbol{e}_3, \\ \boldsymbol{a}_3 &= \boldsymbol{e}_2 + \boldsymbol{e}_3. \end{aligned}$$

$$\sqrt{g} = -\, \boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 = -\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 2;$$

$$-\, \boldsymbol{a}_2 \times \boldsymbol{a}_3 = \det \begin{bmatrix} 1 & \boldsymbol{e}_1 & 0 \\ 0 & \boldsymbol{e}_2 & 1 \\ 1 & \boldsymbol{e}_3 & 1 \end{bmatrix} = \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3,$$

$$-\, \boldsymbol{a}_3 \times \boldsymbol{a}_1 = \det \begin{bmatrix} 0 & \boldsymbol{e}_1 & 1 \\ 1 & \boldsymbol{e}_2 & 1 \\ 1 & \boldsymbol{e}_3 & 0 \end{bmatrix} = \boldsymbol{e}_1 + \boldsymbol{e}_3 - \boldsymbol{e}_2,$$

$$-\, \boldsymbol{a}_1 \times \boldsymbol{a}_2 = \det \begin{bmatrix} 1 & \boldsymbol{e}_1 & 1 \\ 1 & \boldsymbol{e}_2 & 0 \\ 0 & \boldsymbol{e}_3 & 1 \end{bmatrix} = \boldsymbol{e}_2 + \boldsymbol{e}_3 - \boldsymbol{e}_1$$

and finally

$$\begin{aligned} \boldsymbol{a}^1 &= \tfrac{1}{2} \big(\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\big), \\ \boldsymbol{a}^2 &= \tfrac{1}{2} \big(\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3\big), \\ \boldsymbol{a}^3 &= \tfrac{1}{2} \big(-\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3\big). \end{aligned}$$

Имея кобазис, возможно не только разложить по нему любой вектор (рис. 7), но и найти коэффициенты разложения (16.1):

$$\begin{gathered} \boldsymbol{v} = v^i \boldsymbol{a}_i = v_i \boldsymbol{a}^i, \\ \boldsymbol{v} \cdot \boldsymbol{a}^i = v^k \boldsymbol{a}_k \cdot \boldsymbol{a}^i = v^i, \;\; v_i = \boldsymbol{v} \cdot \boldsymbol{a}_i. \end{gathered} \tag{16.4}$$

Коэффициенты v_i называются ковариантными компонентами вектора $\boldsymbol{v}$, а v^i — его контравариантными* компонентами.

Есть литература о тензорах, где introducing existence and различают ковариантные и контравариантные... векторы (and "covectors", "dual vectors"). Не сто́ит вводить читателя в заблуждение: вектор-то один и тот же, просто разложение по двум разным базисам даёт два набора компонент.

От векторов перейдём к тензорам второй сложности. Имеем четыре комплекта диад: $\boldsymbol{a}_i \boldsymbol{a}_j$, $\boldsymbol{a}^i \boldsymbol{a}^j$, $\boldsymbol{a}_i \boldsymbol{a}^j$, $\boldsymbol{a}^i \boldsymbol{a}_j$. Согласующиеся коэффициенты в декомпозиции тензора называются его контравариантными, ковариантными и смешанными компонентами:

$$
\begin{gathered}
{}^2\boldsymbol{B} = B^{ij} \boldsymbol{a}_i \boldsymbol{a}_j = B_{ij} \boldsymbol{a}^i \boldsymbol{a}^j = B^{i}_{\cdot j} \boldsymbol{a}_i \boldsymbol{a}^j = B_{i}^{\cdot j} \boldsymbol{a}^i \boldsymbol{a}_j \,, \\
B^{ij} = \boldsymbol{a}^i \cdot {}^2\boldsymbol{B} \cdot \boldsymbol{a}^j , \;\; B_{ij} = \boldsymbol{a}_i \cdot {}^2\boldsymbol{B} \cdot \boldsymbol{a}_j \,, \\
B^{i}_{\cdot j} = \boldsymbol{a}^i \cdot {}^2\boldsymbol{B} \cdot \boldsymbol{a}_j \,, \;\; B_{i}^{\cdot j} = \boldsymbol{a}_i \cdot {}^2\boldsymbol{B} \cdot \boldsymbol{a}^j .
\end{gathered}
\tag{16.5}
$$

Для двух видов смешанных компонент точка в индексе это просто свободное место: у $B^{i}_{\cdot j}$ верхний индекс "i" — первый, а нижний "j" — второй.

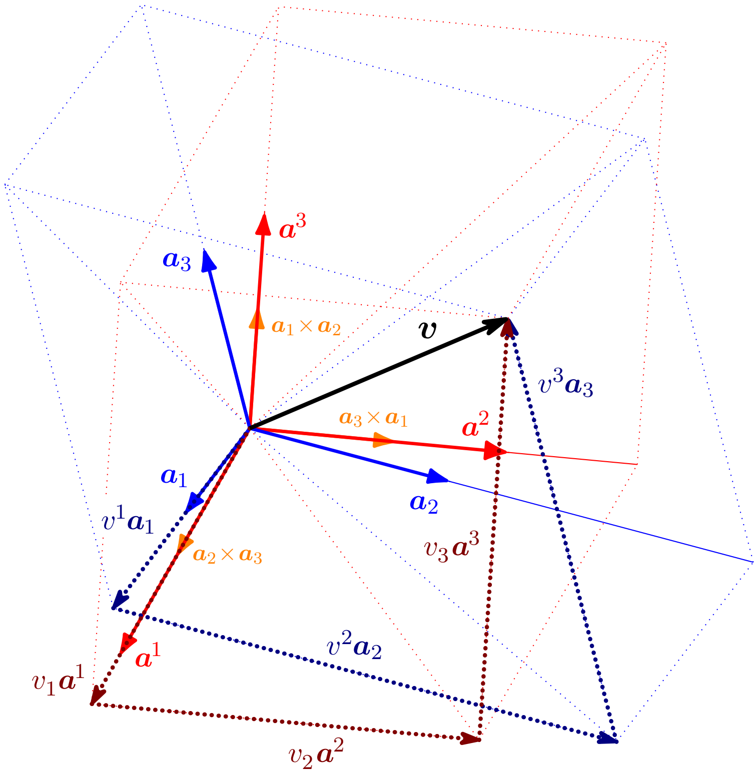
Компоненты единичного ("метрического") тензора $\boldsymbol{E}$

$$
\begin{gathered}
\boldsymbol{E} = \boldsymbol{a}^k \boldsymbol{a}_k = \boldsymbol{a}_k \boldsymbol{a}^k = g_{jk} \boldsymbol{a}^j \boldsymbol{a}^k = g^{jk} \boldsymbol{a}_j \boldsymbol{a}_k : \\
\boldsymbol{a}_i \cdot \boldsymbol{E} \cdot \boldsymbol{a}^j = \boldsymbol{a}_i \cdot \boldsymbol{a}^j = \delta_i^j , \;\; \boldsymbol{a}^i \cdot \boldsymbol{E} \cdot \boldsymbol{a}_j = \boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta_j^i , \\
\boldsymbol{a}_i \cdot \boldsymbol{E} \cdot \boldsymbol{a}_j = \boldsymbol{a}_i \cdot \boldsymbol{a}_j \equiv g_{ij} , \;\; \boldsymbol{a}^i \cdot \boldsymbol{E} \cdot \boldsymbol{a}^j = \boldsymbol{a}^i \cdot \boldsymbol{a}^j \equiv g^{ij} ; \\
\boldsymbol{E} \cdot \boldsymbol{E} = g_{ij} \boldsymbol{a}^i \boldsymbol{a}^j \cdot g^{nk} \boldsymbol{a}_n \boldsymbol{a}_k = g_{ij} g^{jk} \boldsymbol{a}^i \boldsymbol{a}_k = \boldsymbol{E} \;\Rightarrow\; g_{ij} g^{jk} = \delta_i^k .
\end{gathered}
\tag{16.6}
$$

Вдобавок к (16.2) и (16.3) открылся ещё один способ найти векторы кобазиса — через матрицу g^{ij}, обратную матрице Грама g_{ij}. И наоборот:

$$
\begin{aligned}
\boldsymbol{a}^i &= \boldsymbol{E} \cdot \boldsymbol{a}^i = g^{jk} \boldsymbol{a}_j \boldsymbol{a}_k \cdot \boldsymbol{a}^i = g^{jk} \boldsymbol{a}_j \delta_k^i = g^{ji} \boldsymbol{a}_j \,, \\
\boldsymbol{a}_i &= \boldsymbol{E} \cdot \boldsymbol{a}_i = g_{jk} \boldsymbol{a}^j \boldsymbol{a}^k \cdot \boldsymbol{a}_i = g_{jk} \boldsymbol{a}^j \delta_i^k = g_{ji} \boldsymbol{a}^j .
\end{aligned}
\tag{16.7}
$$

* Потому что они меняются обратно (contra) изменению длин базисных векторов $\boldsymbol{a}_i$.



$$\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 = \sqrt{g} = 0.56274$$
$$1/\sqrt{g} = 1.77703$$

$$\boldsymbol{a}_i \cdot \boldsymbol{a}^j = \begin{bmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{a}^1 & \boldsymbol{a}_1 \cdot \boldsymbol{a}^2 & \boldsymbol{a}_1 \cdot \boldsymbol{a}^3 \\ \boldsymbol{a}_2 \cdot \boldsymbol{a}^1 & \boldsymbol{a}_2 \cdot \boldsymbol{a}^2 & \boldsymbol{a}_2 \cdot \boldsymbol{a}^3 \\ \boldsymbol{a}_3 \cdot \boldsymbol{a}^1 & \boldsymbol{a}_3 \cdot \boldsymbol{a}^2 & \boldsymbol{a}_3 \cdot \boldsymbol{a}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_i^j$$

рисунок 7
"Decomposition of vector in oblique basis"

Example. Using reversed Gram matrix, get cobasis for basis $\boldsymbol{a}_i$ when

$$\boldsymbol{a}_1 = \boldsymbol{e}_1 + \boldsymbol{e}_2,$$
$$\boldsymbol{a}_2 = \boldsymbol{e}_1 + \boldsymbol{e}_3,$$
$$\boldsymbol{a}_3 = \boldsymbol{e}_2 + \boldsymbol{e}_3.$$

$$g_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \;\; \det g_{ij} = 4,$$

$$\operatorname{adj} g_{ij} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}^{\mathsf{T}},$$

$$g^{ij} = g_{ij}^{-1} = \frac{\operatorname{adj} g_{ij}}{\det g_{ij}} = \frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

Using $\boldsymbol{a}^i = g^{ij} \boldsymbol{a}_j$

$$\boldsymbol{a}^1 = g^{11}\boldsymbol{a}_1 + g^{12}\boldsymbol{a}_2 + g^{13}\boldsymbol{a}_3 = \tfrac{1}{2}\boldsymbol{e}_1 + \tfrac{1}{2}\boldsymbol{e}_2 - \tfrac{1}{2}\boldsymbol{e}_3,$$
$$\boldsymbol{a}^2 = g^{21}\boldsymbol{a}_1 + g^{22}\boldsymbol{a}_2 + g^{23}\boldsymbol{a}_3 = \tfrac{1}{2}\boldsymbol{e}_1 - \tfrac{1}{2}\boldsymbol{e}_2 + \tfrac{1}{2}\boldsymbol{e}_3,$$
$$\boldsymbol{a}^3 = g^{31}\boldsymbol{a}_1 + g^{32}\boldsymbol{a}_2 + g^{33}\boldsymbol{a}_3 = -\tfrac{1}{2}\boldsymbol{e}_1 + \tfrac{1}{2}\boldsymbol{e}_2 + \tfrac{1}{2}\boldsymbol{e}_3.$$

...

Единичный тензор (unit tensor, identity tensor, metric tensor)

$\boldsymbol{E} \cdot \boldsymbol{\xi} = \boldsymbol{\xi} \cdot \boldsymbol{E} = \boldsymbol{\xi} \quad \forall \boldsymbol{\xi}$

$\boldsymbol{E} \cdot\cdot\, \boldsymbol{ab} = \boldsymbol{ab} \cdot\cdot\, \boldsymbol{E} = \boldsymbol{a} \cdot \boldsymbol{E} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}$

$\boldsymbol{E} \cdot\cdot\, \boldsymbol{A} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = \operatorname{trace} \boldsymbol{A}$

$\boldsymbol{E} \cdot\cdot\, \boldsymbol{A} = \boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = \operatorname{trace} \boldsymbol{A} \neq \text{notanymore}\, A_{jj}$

Thus for, say, trace of some tensor $\boldsymbol{A} = A_{ij} \boldsymbol{r}^i \boldsymbol{r}^j$: $\boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = \operatorname{trace} \boldsymbol{A}$, you have

$\boldsymbol{A} \cdot\cdot\, \boldsymbol{E} = A_{ij} \boldsymbol{r}^i \boldsymbol{r}^j \cdot\cdot\, \boldsymbol{r}_{\partial k} \boldsymbol{r}^k = A_{ij} \boldsymbol{r}^i \cdot \boldsymbol{r}^j = A_{ij} g^{ij}$

...

Тензор поворота (the rotation tensor)

$\boldsymbol{P} = \boldsymbol{a}_i \mathring{\boldsymbol{a}}^i = \boldsymbol{a}^i \mathring{\boldsymbol{a}}_i = \boldsymbol{P}^{-\mathsf{T}}$

$\boldsymbol{P}^{-1} = \mathring{\boldsymbol{a}}_i \boldsymbol{a}^i = \mathring{\boldsymbol{a}}^i \boldsymbol{a}_i = \boldsymbol{P}^{\mathsf{T}}$

$\boldsymbol{P}^{\mathsf{T}} = \mathring{\boldsymbol{a}}^i \boldsymbol{a}_i = \mathring{\boldsymbol{a}}_i \boldsymbol{a}^i = \boldsymbol{P}^{-1}$

...

... Характеристическое уравнение (10.2) быстро приводит к тождеству Кэ́ли–Га́мильтона (Cayley–Hamilton)

$$\begin{gathered} -\boldsymbol{B} \cdot \boldsymbol{B} \cdot \boldsymbol{B} + \mathrm{I}\,\boldsymbol{B} \cdot \boldsymbol{B} - \mathrm{II}\,\boldsymbol{B} + \mathrm{III}\,\boldsymbol{E} = {}^2\mathbf{0}, \\ -\boldsymbol{B}^3 + \mathrm{I}\,\boldsymbol{B}^2 - \mathrm{II}\,\boldsymbol{B} + \mathrm{III}\,\boldsymbol{E} = {}^2\mathbf{0}. \end{gathered} \tag{16.8}$$

§ 17. Tensor functions

In the concept of function $y = f(x)$ as of mapping (morphism) $f\colon x \mapsto y$, an input (argument) x and an output (result) y may be tensors of any complexities.

Consider at least a scalar function of a bivalent tensor $\varphi = \varphi(\boldsymbol{B})$. Examples are $\boldsymbol{B} \cdot\cdot\, \boldsymbol{\Phi}$ (or $\boldsymbol{p} \cdot \boldsymbol{B} \cdot \boldsymbol{q}$) and $\boldsymbol{B} \cdot\cdot\, \boldsymbol{B}$. Then in each basis $\boldsymbol{a}_i$ paired with cobasis $\boldsymbol{a}^i$ we have function $\varphi(B_{ij})$ of nine numeric arguments — components B_{ij} of tensor $\boldsymbol{B}$. For example

$$\varphi(\boldsymbol{B}) = \boldsymbol{B} \cdot\cdot\, \boldsymbol{\Phi} = B_{ij}\, \boldsymbol{a}^i \boldsymbol{a}^j \cdot\cdot\, \boldsymbol{a}_m \boldsymbol{a}_n \Phi^{mn} = B_{ij} \Phi^{ji} = \varphi(B_{ij}).$$

With any transition to a new basis, the result doesn't change: $\varphi(B_{ij}) = \varphi(B'_{ij}) = \varphi(\boldsymbol{B})$.

Differentiation of $\varphi(\boldsymbol{B})$ looks like

$$d\varphi = \frac{\partial \varphi}{\partial B_{ij}}\, dB_{ij} = \frac{\partial \varphi}{\partial \boldsymbol{B}} \cdot\cdot\, d\boldsymbol{B}^{\mathsf{T}}. \tag{17.1}$$

Tensor ${}^{\partial\varphi}/_{\partial \boldsymbol{B}}$ is called the derivative of function φ by argument $\boldsymbol{B}$; $d\boldsymbol{B}$ is the differential of tensor $\boldsymbol{B}$, $d\boldsymbol{B} = dB_{ij}\, \boldsymbol{a}^i \boldsymbol{a}^j$; ${}^{\partial\varphi}/_{\partial B_{ij}}$ are components (contravariant ones) of ${}^{\partial\varphi}/_{\partial \boldsymbol{B}}$

$$\boldsymbol{a}^i \cdot \frac{\partial \varphi}{\partial \boldsymbol{B}} \cdot \boldsymbol{a}^j = \frac{\partial \varphi}{\partial \boldsymbol{B}} \cdot\cdot\, \boldsymbol{a}^j \boldsymbol{a}^i = \frac{\partial \varphi}{\partial B_{ij}} \;\Leftrightarrow\; \frac{\partial \varphi}{\partial \boldsymbol{B}} = \frac{\partial \varphi}{\partial B_{ij}}\, \boldsymbol{a}_i \boldsymbol{a}_j .$$

...

$$\begin{gathered} \varphi(\boldsymbol{B}) = \boldsymbol{B} \cdot\cdot\, \boldsymbol{\Phi} \\ d\varphi = d(\boldsymbol{B} \cdot\cdot\, \boldsymbol{\Phi}) = d\boldsymbol{B} \cdot\cdot\, \boldsymbol{\Phi} = \boldsymbol{\Phi} \cdot\cdot\, d\boldsymbol{B} = \boldsymbol{\Phi}^{\mathsf{T}} \cdot\cdot\, d\boldsymbol{B}^{\mathsf{T}} \\ d\varphi = \frac{\partial \varphi}{\partial \boldsymbol{B}} \cdot\cdot\, d\boldsymbol{B}^{\mathsf{T}}, \;\; \frac{\partial\, (\boldsymbol{B} \cdot\cdot\, \boldsymbol{\Phi})}{\partial \boldsymbol{B}} = \boldsymbol{\Phi}^{\mathsf{T}} \end{gathered}$$

$\boldsymbol{p} \cdot \boldsymbol{B} \cdot \boldsymbol{q} = \boldsymbol{B} \cdot\cdot\, \boldsymbol{q}\boldsymbol{p}$

$$\frac{\partial \, (\boldsymbol{p} \cdot \boldsymbol{B} \cdot \boldsymbol{q})}{\partial \boldsymbol{B}} = \boldsymbol{p}\boldsymbol{q}$$

...

$$\varphi(\boldsymbol{B}) = \boldsymbol{B} \cdot\cdot\, \boldsymbol{B}$$
$$d\varphi = d\,(\boldsymbol{B} \cdot\cdot\, \boldsymbol{B}) = d...$$

...

Но согласно опять-таки (16.8) $-\boldsymbol{B}^2 + \mathrm{I}\boldsymbol{B} - \mathrm{II}\boldsymbol{E} + \mathrm{III}\boldsymbol{B}^{-1} = {}^2\boldsymbol{0}$, поэтому

....

Скалярная функция $\varphi(\boldsymbol{B})$ называется изотропной, если она не чувствительна к повороту аргумента:

$$\varphi(\boldsymbol{B}) = \varphi(\boldsymbol{O} \cdot \mathring{\boldsymbol{B}} \cdot \boldsymbol{O}^{\mathsf{T}}) = \varphi(\mathring{\boldsymbol{B}}) \;\; \forall \boldsymbol{O} = \boldsymbol{a}_i \mathring{\boldsymbol{a}}^i = \boldsymbol{a}^i \mathring{\boldsymbol{a}}_i = \boldsymbol{O}^{-\mathsf{T}}$$

для любого ортогонального тензора $\boldsymbol{O}$ (тензора поворота, § 12).

Симметричный тензор $\boldsymbol{B}^{\mathsf{S}}$ полностью определяется тройкой инвариантов и угловой ориентацией собственных осей (они же взаимно ортогональны, § 10). Ясно, что изотропная функция $\varphi(\boldsymbol{B}^{\mathsf{S}})$ симметричного аргумента является функцией, входы-аргументы которой — только инварианты $\mathrm{I}(\boldsymbol{B}^{\mathsf{S}})$, $\mathrm{II}(\boldsymbol{B}^{\mathsf{S}})$, $\mathrm{III}(\boldsymbol{B}^{\mathsf{S}})$. Дифференцируется такая функция согласно (**??**), где транспонирование излишне.

§ 18. Spatial differentiation

««« rename: remove fields

Tensor field is a tensor varying from point to point (variable in space, coordinate dependent).

Пусть at each point of some region of a three-dimensional space определена величина ς. Тогда говорят, что есть тензорное поле $\varsigma = \varsigma(\boldsymbol{r})$, where $\boldsymbol{r}$ is location vector (radius vector) of a point in space.

Величина ς может быть тензором любой сложности. Пример скалярного поля — поле температуры в среде, векторного поля — скорости частиц жидкости.

Концепт тензорного поля никак не связан с концептом поля с операциями $+$ и $*$ с 11 свойствами этих операций.

Не только для решения прикладных задач, но нередко и в “чистой теории” вместо аргумента $\boldsymbol{r}$ используется набор (какая-либо тройка) криволинейных координат q^i. Если непрерывно менять лишь одну координату из трёх, получается координатная линия. Каждая точка трёхмерного пространства лежит на пересечении трёх координатных линий (рис. 8). Вектор положения точки выражается через набор координат as relation $\boldsymbol{r} = \boldsymbol{r}(q^i)$.

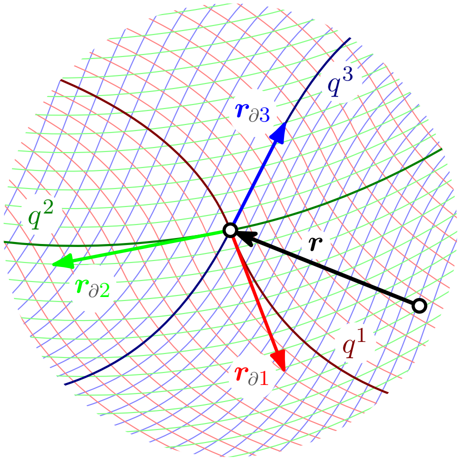


figure 8

Commonly used sets of coordinates«« « Rectangular (“cartesian”), spherical and cylindrical coordinates are

Curvilinear coordinates may be derived from a set of rectangular (“cartesian”) coordinates by using a transformation that is locally invertible (a one-to-one map) at each point. Therefore rectangular coordinates of any point of space can be converted to some curvilinear coordinates and vice versa.

...

The differential of a function presents a change in the linearization of this function.

...

partial derivative

$$\partial_i \equiv \frac{\partial}{\partial q^i}$$

...

differential of $\varsigma(q^i)$

$$d\varsigma = \frac{\partial \varsigma}{\partial q^i} dq^i = \partial_i \varsigma \, dq^i \tag{18.1}$$

...

Linearity

$$\partial_i \big(\lambda p + \mu q\big) = \lambda \big(\partial_i p\big) + \mu \big(\partial_i q\big) \tag{18.2}$$

"Product rule"

$$\partial_i \big(p \circ q\big) = \big(\partial_i p\big) \circ q + p \circ \big(\partial_i q\big) \tag{18.3}$$

...

Local basis $\boldsymbol{r}_{\partial i}$

The differential of location vector $\boldsymbol{r}(q^i)$ is

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial q^i} dq^i = dq^i \boldsymbol{r}_{\partial i}, \;\; \boldsymbol{r}_{\partial i} \equiv \frac{\partial \boldsymbol{r}}{\partial q^i} \equiv \partial_i \boldsymbol{r} \tag{18.4}$$

...

Local cobasis $\boldsymbol{r}^i$, $\boldsymbol{r}^i \bullet \boldsymbol{r}_{\partial j} = \delta^i_j$

...

$$\frac{\partial \varsigma}{\partial \boldsymbol{r}} = \frac{\partial \varsigma}{\partial q^i} \boldsymbol{r}^i = \partial_i \varsigma \, \boldsymbol{r}^i$$

$$d\varsigma = \frac{\partial \varsigma}{\partial \boldsymbol{r}} \bullet d\boldsymbol{r} = \partial_i \varsigma \, \boldsymbol{r}^i \bullet dq^j \boldsymbol{r}_{\partial j} = \partial_i \varsigma \, dq^i \tag{18.5}$$

...

The bivalent unit tensor (metric tensor) $\boldsymbol{E}$, which is neutral (4.7) to the "$\bullet$"-product (dot product), can be represented as

$$\boldsymbol{E} = \boldsymbol{r}^i \boldsymbol{r}_{\partial i} = \boldsymbol{r}^i \partial_i \boldsymbol{r} = \boldsymbol{\nabla} \boldsymbol{r}, \tag{18.6}$$

where appears the differential "nabla" operator

$$\boldsymbol{\nabla} \equiv \boldsymbol{r}^i \partial_i \,. \tag{18.7}$$

...

$$d\varsigma = \frac{\partial \varsigma}{\partial \boldsymbol{r}} \cdot d\boldsymbol{r} = d\boldsymbol{r} \cdot \boldsymbol{\nabla} \varsigma = \partial_i \varsigma \, dq^i \tag{18.8}$$

$d\boldsymbol{r} = d\boldsymbol{r} \cdot \overbrace{\boldsymbol{\nabla} \boldsymbol{r}}^{\boldsymbol{E}}$

...

Divergence of the dyadic product of two vectors

$$\begin{aligned}\boldsymbol{\nabla} \cdot \big(\boldsymbol{a}\boldsymbol{b}\big) = \boldsymbol{r}^i \partial_i \cdot \big(\boldsymbol{a}\boldsymbol{b}\big) = \boldsymbol{r}^i \cdot \partial_i \big(\boldsymbol{a}\boldsymbol{b}\big) = \boldsymbol{r}^i \cdot \big(\partial_i \boldsymbol{a}\big) \boldsymbol{b} + \boldsymbol{r}^i \cdot \boldsymbol{a} \big(\partial_i \boldsymbol{b}\big) = \\ = \big(\boldsymbol{r}^i \cdot \partial_i \boldsymbol{a}\big) \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{r}^i \big(\partial_i \boldsymbol{b}\big) = \big(\boldsymbol{r}^i \partial_i \cdot \boldsymbol{a}\big) \boldsymbol{b} + \boldsymbol{a} \cdot \big(\boldsymbol{r}^i \partial_i \boldsymbol{b}\big) = \\ = \big(\boldsymbol{\nabla} \cdot \boldsymbol{a}\big) \boldsymbol{b} + \boldsymbol{a} \cdot \big(\boldsymbol{\nabla} \boldsymbol{b}\big)\end{aligned} \tag{18.9}$$

— here's no need to expand vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, expanding just differential operator $\boldsymbol{\nabla}$.

...

Gradient of cross product of two vectors, applying "product rule" (18.3) and relation (7.5) for any two vectors (partial derivative ∂_i of some vector by scalar coordinate q^i is a vector too)

$$\begin{aligned}\boldsymbol{\nabla} (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{r}^i \partial_i (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{r}^i (\partial_i \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \partial_i \boldsymbol{b}) = \\ = \boldsymbol{r}^i (\partial_i \boldsymbol{a} \times \boldsymbol{b} - \partial_i \boldsymbol{b} \times \boldsymbol{a}) = \boldsymbol{r}^i \partial_i \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{r}^i \partial_i \boldsymbol{b} \times \boldsymbol{a} = \\ = \boldsymbol{\nabla} \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{\nabla} \boldsymbol{b} \times \boldsymbol{a} \,.\end{aligned} \tag{18.10}$$

...

Gradient of dot product of two vectors

$$\begin{aligned}\boldsymbol{\nabla} \big(\boldsymbol{a} \cdot \boldsymbol{b}\big) = \boldsymbol{r}^i \partial_i \big(\boldsymbol{a} \cdot \boldsymbol{b}\big) = \boldsymbol{r}^i \big(\partial_i \boldsymbol{a}\big) \cdot \boldsymbol{b} + \boldsymbol{r}^i \boldsymbol{a} \cdot \big(\partial_i \boldsymbol{b}\big) = \\ = \big(\boldsymbol{r}^i \partial_i \boldsymbol{a}\big) \cdot \boldsymbol{b} + \boldsymbol{r}^i \big(\partial_i \boldsymbol{b}\big) \cdot \boldsymbol{a} = \big(\boldsymbol{\nabla} \boldsymbol{a}\big) \cdot \boldsymbol{b} + \big(\boldsymbol{\nabla} \boldsymbol{b}\big) \cdot \boldsymbol{a} \,.\end{aligned} \tag{18.11}$$

$\underbrace{\quad}_{\boldsymbol{\nabla}}$

§ 19. The integral theorems

Для векторных полей известны интегральные теоремы Gauss'а и Stokes'а.

Gauss' theorem (divergence theorem) enables an integral taken over a volume to be replaced by one taken over the closed surface bounding that volume, and vice versa.

Stokes' theorem enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve. Stokes' theorem relates a line integral around a closed path to a surface integral over what is called a *capping surface* of the path.

Теорема Gauss'а о дивергенции

This theorem is about how to replace a volume integral with a surface one (and vice versa). В этой теореме рассматривается поток (ef)flux вектора через ограничивающую объём V за́мкнутую поверхность $\mathcal{O}(\partial V)$. Единичный вектор внешней нормали $\boldsymbol{n}$ к поверхности $\mathcal{O}(\partial V)$

$$\oint\limits_{\mathcal{O}(\partial V)} \boldsymbol{n} \cdot \boldsymbol{a} \, d\mathcal{O} = \int\limits_{V} \boldsymbol{\nabla} \cdot \boldsymbol{a} \, dV. \tag{19.1}$$

Объём V нарезается тремя семействами координатных поверхностей на множество бесконечно малых элементов. Поток через поверхность $\mathcal{O}(\partial V)$ равен сумме потоков через края получившихся элементов. В бесконечной малости каждый такой элемент — маленький локальный дифференциальный кубик (параллелепипед). ... Поток вектора $\boldsymbol{a}$ через грани малого кубика с объёмом dV равен $\sum_{i=1}^{6} \boldsymbol{n}_i \cdot \boldsymbol{a} \mathcal{O}_i$, а поток через сам этот объём равен $\boldsymbol{\nabla} \cdot \boldsymbol{a} \, dV$.

Похожая трактовка этой теоремы есть, для примера, в курсе Richard'а Feynman'а [86].

(рисунок с кубиками)

to dice — нареза́ть кубиками

small cube, little cube

локально ортонормальные координаты $\boldsymbol{\xi} = \xi_i \boldsymbol{n}_i$, $d\boldsymbol{\xi} = d\xi_i \boldsymbol{n}_i$, $\boldsymbol{\nabla} = \boldsymbol{n}_i \partial_i$

разложение вектора $\boldsymbol{a} = a_i \boldsymbol{n}_i$

Теорема Stokes'а о циркуляции

Эта теорема выражается равенством

...

§ 20. Curvature tensors

The *Riemann curvature tensor* or *Riemann–Christoffel tensor* (after **Bernhard Riemann** and **Elwin Bruno Christoffel**) is the most common method used to express the curvature of Riemannian manifolds. It's a tensor field, it assigns a tensor to each point of a Riemannian manifold, that measures the extent to which the metric tensor is not locally isometric to that of "flat" space. The curvature tensor measures noncommutativity of the covariant derivative, and as such is the integrability obstruction for the existence of an isometry with "flat" space.

Рассматривая тензорные поля в криволинейных координатах (§ 18), мы исходили из представления вектора-радиуса (вектора положения) точки функцией этих координат: $\boldsymbol{r} = \boldsymbol{r}(q^i)$. Этим отношением порождаются выражения

- ✓ векторов локального касательного базиса $\boldsymbol{r}_{\partial i} \equiv {}^{\partial \boldsymbol{r}}\!/_{\partial q^i} \equiv \partial_i \boldsymbol{r}$,
- ✓ компонент $g_{ij} \equiv \boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j}$ и $g^{ij} \equiv \boldsymbol{r}^i \cdot \boldsymbol{r}^j = g_{ij}^{-1}$ единичного "метрического" тензора $\boldsymbol{E} = \boldsymbol{r}_{\partial i} \boldsymbol{r}^i = \boldsymbol{r}^i \boldsymbol{r}_{\partial i} = g_{jk} \boldsymbol{r}^j \boldsymbol{r}^k = g^{jk} \boldsymbol{r}_{\partial j} \boldsymbol{r}_{\partial k}$,
- ✓ векторов локального взаимного кокасательного базиса $\boldsymbol{r}^i \cdot \boldsymbol{r}_{\partial j} = \delta_j^i$, $\boldsymbol{r}^i = g^{ij} \boldsymbol{r}_{\partial j}$,
- ✓ дифференциального набла-оператора Hamilton'а $\boldsymbol{\nabla} \equiv \boldsymbol{r}^i \partial_i$, $\boldsymbol{E} = \boldsymbol{\nabla} \boldsymbol{r}$,
- ✓ полного дифференциала $d\boldsymbol{\xi} = d\boldsymbol{r} \cdot \boldsymbol{\nabla} \boldsymbol{\xi}$,
- ✓ частных производных касательного базиса (вторых частных производных $\boldsymbol{r}$) $\boldsymbol{r}_{\partial i \partial j} \equiv \partial_i \partial_j \boldsymbol{r} = \partial_i \boldsymbol{r}_{\partial j}$,
- ✓ символов "связности" Христоффеля (Christoffel symbols) $\Gamma_{ij}^k \equiv \boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}^k$ и $\Gamma_{ij\underset{\bullet}{k}} \equiv \boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}_{\partial k}$.

Представим теперь, что функция $\boldsymbol{r}(q^k)$ не известна, но зато́ в каждой точке пространства известны шесть независимых компонент положительно определённой (all Gram matrices are non-negative definite) симметричной метрической матрицы Gram $g_{ij}(q^k)$.

the Gram matrix (or Gramian)

Билинейная форма ...

...

Поскольку шесть функций $g_{ij}(q^k)$ происходят от векторной функции $\boldsymbol{r}(q^k)$, то между элементами g_{ij} существуют некие соотношения.

Differential $d\boldsymbol{r}$ (18.4) is exact. This is true if and only if second partial derivatives commute:

$$d\boldsymbol{r} = \boldsymbol{r}_{\partial k} dq^k \;\Leftrightarrow\; \partial_i \boldsymbol{r}_{\partial j} = \partial_j \boldsymbol{r}_{\partial i} \text{ or } \boldsymbol{r}_{\partial i \partial j} = \boldsymbol{r}_{\partial j \partial i}.$$

Но это условие ужé обеспечено симметрией g_{ij}

...

metric (“affine”) connection ∇_i, её же называют “covariant derivative”

$$\boldsymbol{r}_{\partial i \partial j} = \underbrace{\boldsymbol{r}_{\partial i \partial j} \cdot \overbrace{\boldsymbol{r}^k}^{}}_{\Gamma_{ij}^{k}} \!\!\overset{\boldsymbol{E}}{\overbrace{}}\!\! \boldsymbol{r}_{\partial k} = \underbrace{\boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}_{\partial k}}_{\Gamma_{ijk\bullet}} \boldsymbol{r}^k$$

$\Gamma_{ij}^{k} \boldsymbol{r}_{\partial k} = \boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}^k \boldsymbol{r}_{\partial k} = \boldsymbol{r}_{\partial i \partial j}$

covariant derivative (affine connection) is only defined for vector fields

$\boldsymbol{\nabla v} = \boldsymbol{r}^i \partial_i \big(v^j \boldsymbol{r}_{\partial j} \big) = \boldsymbol{r}^i \big(\partial_i v^j \boldsymbol{r}_{\partial j} + v^j \boldsymbol{r}_{\partial i \partial j} \big)$

$\boldsymbol{\nabla v} = \boldsymbol{r}^i \boldsymbol{r}_{\partial j} \nabla_i v^j, \;\; \nabla_i v^j \equiv \partial_i v^j + \Gamma_{in}^{j} v^n$

$\boldsymbol{\nabla r}_{\partial i} = \boldsymbol{r}^k \partial_k \boldsymbol{r}_{\partial i} = \boldsymbol{r}^k \boldsymbol{r}_{\partial k \partial i} = \boldsymbol{r}^k \boldsymbol{r}_{\partial n} \Gamma_{ki}^{n}, \;\; \nabla_i \boldsymbol{r}_{\partial n} = \Gamma_{in}^{k} \boldsymbol{r}_{\partial k}$

Christoffel symbols describe a metric (“affine”) connection, that is how the basis changes from point to point.

символы Christoffel’я это “components of connection” in local coordinates

...

torsion tensor ${}^3\boldsymbol{\mathfrak{T}}$ with components

$$\mathfrak{T}_{ij}^{k} = \Gamma_{ij}^{k} - \Gamma_{ji}^{k}$$

determines the antisymmetric part of a connection

...

симметрия $\Gamma_{ij\underset{\bullet}{k}} = \Gamma_{ji\underset{\bullet}{k}}$, поэтому $3^3 - 3 \cdot 3 = 18$ разных (независимых) $\Gamma_{ij\underset{\bullet}{k}}$

$$\begin{gathered}\Gamma_{ij}^{n} g_{nk} = \Gamma_{ij\underset{\bullet}{k}} = \boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}_{\partial k} = \\ = \tfrac{1}{2}\big(\boldsymbol{r}_{\partial i \partial j} + \boldsymbol{r}_{\partial j \partial i}\big) \cdot \boldsymbol{r}_{\partial k} + \tfrac{1}{2}\big(\boldsymbol{r}_{\partial j \partial k} - \boldsymbol{r}_{\partial k \partial j}\big) \cdot \boldsymbol{r}_{\partial i} + \tfrac{1}{2}\big(\boldsymbol{r}_{\partial i \partial k} - \boldsymbol{r}_{\partial k \partial i}\big) \cdot \boldsymbol{r}_{\partial j} = \\ = \tfrac{1}{2}\big(\boldsymbol{r}_{\partial i \partial j} \cdot \boldsymbol{r}_{\partial k} + \boldsymbol{r}_{\partial i \partial k} \cdot \boldsymbol{r}_{\partial j}\big) + \tfrac{1}{2}\big(\boldsymbol{r}_{\partial j \partial i} \cdot \boldsymbol{r}_{\partial k} + \boldsymbol{r}_{\partial j \partial k} \cdot \boldsymbol{r}_{\partial i}\big) - \tfrac{1}{2}\big(\boldsymbol{r}_{\partial k \partial i} \cdot \boldsymbol{r}_{\partial j} + \boldsymbol{r}_{\partial k \partial j} \cdot \boldsymbol{r}_{\partial i}\big) = \\ = \frac{1}{2}\Big(\partial_i(\boldsymbol{r}_{\partial j} \cdot \boldsymbol{r}_{\partial k}) + \partial_j(\boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial k}) - \partial_k(\boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j})\Big) = \\ = \frac{1}{2}\big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\big). \qquad (20.1)\end{gathered}$$

Все символы Christoffel'я тождественно равны нулю лишь в ортонормальной (декартовой) системе. (А какие они для косоугольной?)

Дальше: $d\boldsymbol{r}_{\partial i} = d\boldsymbol{r} \cdot \boldsymbol{\nabla} \boldsymbol{r}_{\partial i} = dq^k \partial_k \boldsymbol{r}_{\partial i} = \boldsymbol{r}_{\partial k \partial i}\, dq^k$ — тоже полные дифференциалы.

$$d\boldsymbol{r}_{\partial k} = \partial_i \boldsymbol{r}_{\partial k}\, dq^i = \frac{\partial \boldsymbol{r}_{\partial k}}{\partial q^1} dq^1 + \frac{\partial \boldsymbol{r}_{\partial k}}{\partial q^2} dq^2 + \frac{\partial \boldsymbol{r}_{\partial k}}{\partial q^3} dq^3$$

Поэтому $\partial_i \partial_j \boldsymbol{r}_{\partial k} = \partial_j \partial_i \boldsymbol{r}_{\partial k}$, $\partial_i \boldsymbol{r}_{\partial j \partial k} = \partial_j \boldsymbol{r}_{\partial i \partial k}$, и трёхиндексный объект из векторов третьих частных производных

$$\boldsymbol{r}_{\partial i \partial j \partial k} \equiv \partial_i \partial_j \partial_k \boldsymbol{r} = \partial_i \boldsymbol{r}_{\partial j \partial k} \qquad (20.2)$$

симметричен по первому и второму индексам (а не только по второму и третьему). И тогда равен нулю ${}^4\mathbf{0}$ следующий тензор четвёртой сложности — *Riemann curvature tensor* (or *Riemann–Christoffel tensor*)

$${}^4\boldsymbol{\mathfrak{R}} = \mathfrak{R}_{hijk} \boldsymbol{r}^h \boldsymbol{r}^i \boldsymbol{r}^j \boldsymbol{r}^k, \;\; \mathfrak{R}_{hijk} \equiv \boldsymbol{r}_{\partial h} \cdot (\boldsymbol{r}_{\partial j \partial i \partial k} - \boldsymbol{r}_{\partial i \partial j \partial k}). \qquad (20.3)$$

Выразим компоненты $\mathfrak{R}_{ijkn}$ через метрическую матрицу g_{ij}. Начнём с дифференцирования локального кобазиса:

$$\boldsymbol{r}^i \bullet \boldsymbol{r}_{\partial k} = \delta^i_k \;\Rightarrow\; \partial_j \boldsymbol{r}^i \bullet \boldsymbol{r}_{\partial k} + \boldsymbol{r}^i \bullet \boldsymbol{r}_{\partial j \partial k} = 0 \;\Rightarrow\; \partial_j \boldsymbol{r}^i = -\Gamma^i_{jk} \boldsymbol{r}^k.$$

...

Шесть независимых компонент: $\mathfrak{R}_{1212}$, $\mathfrak{R}_{1213}$, $\mathfrak{R}_{1223}$, $\mathfrak{R}_{1313}$, $\mathfrak{R}_{1323}$, $\mathfrak{R}_{2323}$.

...

Symmetric bivalent *Ricci curvature tensor*

$$\boldsymbol{\mathscr{R}} \equiv \frac{1}{4} \mathfrak{R}_{abij} \boldsymbol{r}^a \times \boldsymbol{r}^b \boldsymbol{r}^i \times \boldsymbol{r}^j = \frac{1}{4} \in^{[abp]} \in^{[ijq]} \mathfrak{R}_{abij} \boldsymbol{r}_{\partial p} \boldsymbol{r}_{\partial q} = \mathscr{R}^{pq} \boldsymbol{r}_{\partial p} \boldsymbol{r}_{\partial q}$$

(coefficient $\frac{1}{4}$ is used here for convenience) with components

$$\begin{aligned}
&\mathscr{R}^{11} = \frac{1}{g} \mathfrak{R}_{2323}\,, \\
&\mathscr{R}^{21} = \frac{1}{g} \mathfrak{R}_{1323}\,,\;\; \mathscr{R}^{22} = \frac{1}{g} \mathfrak{R}_{1313}\,, \\
&\mathscr{R}^{31} = \frac{1}{g} \mathfrak{R}_{1223}\,,\;\; \mathscr{R}^{32} = \frac{1}{g} \mathfrak{R}_{1213}\,,\;\; \mathscr{R}^{33} = \frac{1}{g} \mathfrak{R}_{1212}\,.
\end{aligned}$$

Равенство тензора Риччи нулю $\boldsymbol{\mathscr{R}} = {}^2\boldsymbol{0}$ (в компонентах это шесть уравнений $\mathscr{R}^{ij} = \mathscr{R}^{ji} = 0$) is the necessary condition of integrability ("compatibility") для нахождения вектора-радиуса $\boldsymbol{r}(q^k)$ по полю $g_{ij}(q^k)$.

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There are many books which describe only the apparatus of tensor calculus [98, 99, 100, 16, **?**, 101].

However, the index notation (it's when the tensors are presented as the sets of components) is still more popular than the direct indexless notation.

The direct notation is widely used, for example, in the appendices to the books by Anatoliy I. Lurie (Анатолий И. Лурье) [29, 30].

In "Теории упругости" ("The theory of elasticity") by Вениамин Блох (Veniamin Blokh) [7] the direct indexless notation is used too.

The R. Feynman's lectures [86] contain the bright description of the vector fields theory.
Also, information about the tensor calculus is the part of the unusual and interesting book by C. Truesdell [61].

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