Vadique Myself

PHYSICS *of* ELASTIC CONTINUA

FOREWORD

In this book I am trying to guide the reader* through the ways of constructing the models of an elastic continuum. These models are : the nonlinear and the linear ones, the micropolar and the classical momentless; the three-dimensional, the two-dimensional (shells and plates) and the one-dimensional (rods, including thin-walled ones). I also explained the fundamentals of dynamics — oscillations, waves and stability. For the thermoelasticity and the magnetoelasticity, I gave the summary of the classical theories of the thermodynamics and the electrodynamics. The dynamics of destruction is described via the theories of defects and fractures. The approaches to modeling of human-made inhomogeneous materials, "composites", are also shown.

The word "continua" in the title says that an object (a body, a medium) is modeled here not as a discrete collection of particles, but as a continuous space of location vectors, a continuous matter. It gives a large convenience, because the apparatus of calculus of infinitesimals can be used for such models.

When I just began writing this book, I thought of a reader who is pretty acquainted with "higher" mathematics. But later I decided to conduct such an acquaintance by myself, and yet, as a side effect, every reader with any knowledge of math can comprehend the content of the book.

* and myself as well

The book is written using the compact and elegant direct indexless tensor notation. The mathematical apparatus for interpreting the direct tensor relations is located in the first chapter.

I am writing this book simultaneously in the two languages, English and Russian. The reader is free to pick any language of the two.

Vadique

CONTENTS

chapter 1

MATHEMATICAL APPARATUS

Mathematics, or math for short, is abstract. Abstract is the adjective of math, math is the noun of abstract. "Abstract", "theoretical" and "mathematical" are synonyms. When someone is doing math, he's playing a game in the far-far-away magical world of imagination.

For example, numbers are not real entities at all. They are purely imaginary concepts. We cannot experience, sensate numbers, can't see, touch or smell them. Yep, one can compose stories about them, such as $1+1=2$ — mathematical relations between imaginary entities. Nevertheless, no one can ever feel, perceive it, since there are no such things as *one* and *two*[*].

And "synthesized by imagination" — it's not only about numbers. Geometric objects, be it a point, a line, a triangle or a plane, and all kinds of adventures with them are mind derived as well.

§1. The ancient but intuitive geometry

For nearly two thousand years, the freedom of human's thought was limited by the fairy tale about imaginary perfectly straight one-dimensional lines between some two absolutely dimensionless points and beyond, on and on to the very άπειρο (infinity) down and up the both sides, about imaginary and completely flat planes-trigons ("triangles", sometimes "tripoints"), with shortest distances between points in straight lines, with always equal to each other "straight" or "~~st~~r~~a~~ight" angles, as well as about many other pretty funny mythical characters and piquant relations between them. Over two thousand years people were in captivity, in slavery to the idea about the existænce of the only one ευκλείδειος γεωμετρίας (the εὐκλείδ'ean geometry), and

[*] I'm not about *two apples* or *two similar bananas* for *a couple of days*, but about the number "two" itself.

the magic world, described by it, was equated in the past with the real space around themselves.

Εὐκλείδης Eúkleídēs Euclid, ευκλείδειος euclidean

ευκλείδεια γεωμετρία

the plane geometry, or the two-dimensional euclidean geometry

Στοιχεῖα Stoikheîa Elements, Principles

(1.1) Points

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Ορος α΄ (1)	Term α΄ (1)
Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.	A point is that which has no part.

This description shows that Euclid imagines a point as an indivisible location, without width, length or breadth.

(1.2) Lines, curved and straight

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Ορος β΄ (2)	Term β΄ (2)
Γραμμὴ δὲ μῆκος ἀπλατές.	A line is breadthless length.

"Line" is the second primitive term in the Elements. "Breadthless length" says that a line will have one dimension, length, but it won't have breadth. The terms "length" and "breadth" are not defined in the Elements.

Linear lines

(1.3) A relation between lines and points

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Ορος γ′ (3)	Term γ′ (3)
Γραμμῆς δὲ πέρατα σημεῖα.	The ends of a line are points.

This statement doesn't mention how many ends a line can have.

(1.4) Do straight lines exists?

The hypothesis on the existænce of straight lines.

The existænce of Euclidean straight lines in space.

Στοιχεῖα Εὐκλείδου Βιβλίον I	Euclid's Elements Book I
Ορος δ′ (4)	Term δ′ (4)
Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ′ ἑαυτῆς σημείοις κεῖται.	A straight line is a line which lies evenly with the points on itself.

To draw a straight line by hand is absolutely impossible.

(1.5) Vectors. Lines and vectors

(1.6) The existænce of vectors. Do vectors exist?

(1.7) Continuity of line

(1.8) A point of reference

(1.9) Translation as the easiest kind of motion. Translations and vectors

(1.10) Straight line and vector

A (geometric) vector may be like a straight line with an arrow at one of its ends. Then it is fully described (characterized) by the magnitude and the direction.

Within the abstract algebra, the word *vector* is about any object which can be summed with similar objects and scaled (multiplied) by scalars, and vector space is a synonym of linear space. Therefore I clarify that in this book *vector* is nothing else than three-dimensional geometric (Ευκλείδειος, Euclidean) vector.

Why are vectors always straight (linear)?

(a) Vectors are linear (straight), they cannot be curved.

(b) Vectors are neither straight nor curved. A vector has the magnitude and the direction. A vector is not a line or a curve, albeit it can be represented by a straight line.

Vector can't be thought of as a line.

(1.11) The line which figures real numbers

often just "number line"

(1.12) What is a distance?

(1.13) Plane and more dimensional space

(1.14) Distance on plane or more dimensional space

(1.15) What is an angle?

angle ≡ inclination /slope, slant/ of two lines

two lines sharing a common point are usually called intersecting lines

angle ≡ the amount of rotation of line or plane within space

angle ≡ the result of the dot product of two unit vectors gives angle's cosine

(1.16) Differentiation of continuous into small differential chunks

small differential chunks

infinitesimal (infinitely small)

A mention of tensors may scare away the reader, commonly avoiding needless complications. Don't be afraid: tensors are used just due to their wonderful property of the invariance — the independence from a coordinate system.

§ 2. Vector

I propose to begin familiarizing with tensors via memoirs about such a phenomenon as a vector.

- ✓ A *point* has position in space. The only characteristic that distinguishes one point from another is its position.
- ✓ A *vector* has both magnitude and direction, but no specific position in space.

(2.1) What is a vector?

What is "linear"?
(1) straight
(2) relating to, resembling, or having a graph that is a straight line

All vectors are linear objects.

Examples of vectors:

- ✓ A force acts on an object.
- ✓ The velocity of an object describes what's happening with this object at an instant.

Multiplication of a vector by a scalar

Multiplication by the minus one

The Newton's action–reaction principle "действие равно противодействию по магнитуде и обратно ему по направлению".

Each mechanical interaction of two objects is characterized by two forces that act on both interacting objects. These forces can be represented as two vectors that are equal in magnitude and reverse in direction.

Multiplying a vector by the negative one -1 reverses the vector's direction but doesn't change its magnitude.

(2.2) The addition and subtraction

The sum (combination) of two or more vectors is the new "resultant" vector. There are two similar methods to calculate the resultant vector geometrically.

The "head to tail method" involves lining up the head of one vector with the tail of another. Here the resultant goes from the initial point

(the "tail") of the first addend to the end point (the "head") of the second addend when the tail (the initial point) of the second one coincides with the head (the end point) of the first one.

[.... figure here ....]

The "parallelogram method" ...

[.... figure here ....]

The vector addition is commutative

$$\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}.$$

....

$$\boldsymbol{p} + \boldsymbol{q},$$
$$\boldsymbol{p} - \boldsymbol{q} = \boldsymbol{p} + \big(-\boldsymbol{q}\big) = \boldsymbol{p} + \big(-1\big)\boldsymbol{q}.$$

For every action, there's an equal (in magnitude) and opposite (in direction) reaction force.

A vector may be also represented as the sum (combination) of some trio of other vectors, called "basis", when the each of the three is scaled by a number (coefficient). Such a representation is called a "linear combination" of basis vectors. A list (array, tuple) of coefficients alone, without basis vectors, is not enough and can't represent a vector.

....

To get the numerical relations from the vector ones, a coordinate system is introduced, and on its axes the vector relations are projected.

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Vectors themselves (as elements of a vector space) do not have components. Vector components appear only when a certain basis is chosen, then any vector can be "decomposed"— represented as as the sum of basis vectors, premultiplied by coefficients ("components)" is just another name for coefficients of a linear combination). The same vector in different bases has different components.

Here it is — a vector, $\boldsymbol{v}$ looks like a suitable name for it.

Like all geometric vectors, $\boldsymbol{v}$ is pretty well characterized by the two mutually independent properties: its length (magnitude, norm,

modulus) and its direction in space. This characterization is complete, so some two vectors with the same magnitude and the same direction are considered equal.

Every vector exists objectively by itself, independently of methods and units of measurement of both lengths and directions, including any abstractions of such units and methods.

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Not everything is a vector that has a magnitude and direction. Поворот тела вокруг оси, казалось бы, обладает всеми аттрибутами вектора : у него есть численное значение, равное углу поворота, и направление оси вращения. However, when the rotation angles are not infinitely small, the rotations don't sum like vectors*.

Складываются ли угловые скорости? — Да, ведь угол поворота в $\dot{\vartheta}$ бесконечно-малый. — Но только при вращении вокруг неподвижной оси?

(2.3) The method of coordinates

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By choosing some mutually perpendicular unit vectors $\boldsymbol{e}_i$ as the basis for measurements, I introduce the rectangular ("cartesian") coordinates.

Three $(i = 1, 2, 3)$ basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are needed for a three-dimensional — 3D — space.

Within such a system, "•"-products of the basis vectors are equal to the Kronecker delta

$$\boldsymbol{e}_i \bullet \boldsymbol{e}_j = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for any orthonormal basis.

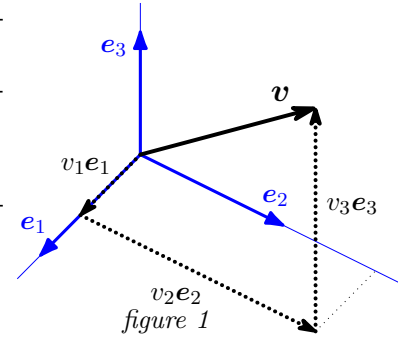

figure 1

* Actually, the sequential rotations are not added, but multiplied.

Decomposing vector $\boldsymbol{v}$ in some orthonormal basis $\boldsymbol{e}_i$ $(i = 1, 2, 3)$, we get coefficients v_i — the components of vector $\boldsymbol{v}$ in that basis (figure 1)

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 \equiv \sum_{i=1}^{3} v_i\boldsymbol{e}_i \equiv v_i\boldsymbol{e}_i, \;\; v_i = \boldsymbol{v}\bullet\boldsymbol{e}_i. \qquad (2.1)$$

Here and hereinafter, the Einstein's summation convention is accepted : an index repeated twice (and no more than twice) in a single term implies a summation over this index. And a non-repeating index is called "free", and it is identical in the both parts of equality. These are examples

$$a_i = \lambda b_i + \mu c_i, \;\; \sigma = \tau_{ii} = \sum_i \tau_{ii},$$
$$p_j = n_i\tau_{ij} = \sum_i n_i\tau_{ij}, \;\; m_i = e_{ijk}x_j f_k = \sum_{j,k} e_{ijk}x_j f_k.$$

(But equalities $a = b_{kkk}$, $c = f_i + g_k$, $d_{ij} = k_i\, q_{ij}$ are incorrect.)

Having components of a vector in an orthonormal basis, the length of this vector is retrieved by the "Πυθαγόρας' equation"

$$\boldsymbol{v}\bullet\boldsymbol{v} = v_i\boldsymbol{e}_i\bullet v_j\boldsymbol{e}_j = v_i\delta_{ij}v_j = v_iv_i, \;\; \|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}\bullet\boldsymbol{v}} = \sqrt{v_iv_i}. \qquad (2.2)$$

The magnitude represents the length independent of direction. The direction of a vector in space is measured by the three angles (cosines of angles) between this vector and each of the basis ones:

$$\cos\measuredangle\big(\boldsymbol{v}\,\widehat{}\,\boldsymbol{e}_i\big) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\bullet\boldsymbol{e}_i = \frac{v_i}{\sqrt{v_jv_j}} \;\Leftrightarrow\; \underbrace{v_i}_{\boldsymbol{v}\bullet\boldsymbol{e}_i} = \|\boldsymbol{v}\|\cos\measuredangle(\boldsymbol{v},\boldsymbol{e}_i). \qquad (2.3)$$

Measurement of angles. The cosine of an angle between two vectors is the same as the dot product of these vectors if their magnitudes are equal to the one unit of length

When the magnitudes of two vectors are equal to the one unit of length, then the cosine of the least angle between them is the same as the dot product of these vectors. Any vector with the non-unit magnitude (but the null vector) can be "normalized" via dividing a vector by its magnitude.

$$\cos\measuredangle(\boldsymbol{v},\boldsymbol{w}) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\bullet\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}.$$

To accompany the magnitude, which represents the length independent of direction, there's a way to represent the direction of a vector independent of

its length. For this purpose, the unit vectors (the vectors with the magnitude of 1) are used.

A rotation matrix is just a transform that expresses the basis vectors of the input space in a different orientation. The length of the basis vectors will be the same, and the origin will not change. Also, the angle between the basis vectors will not change. All that changes is the relative direction of all of the basis vectors.

Therefore, a rotation matrix is not really just a "rotation" matrix; it is an orientation matrix.

There are also pseudovectors, waiting for the reader below in § 7.

The angle between two random vectors. According to (2.3)

$$\cos \measuredangle(\boldsymbol{v}, \boldsymbol{e}_m) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \boldsymbol{e}_m = \frac{v_m}{\sqrt{v_j v_j}},$$
$$\cos \measuredangle(\boldsymbol{w}, \boldsymbol{e}_n) = \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} \cdot \boldsymbol{e}_n = \frac{w_n}{\sqrt{w_k w_k}}.$$

The length (2.2) and the direction in space (2.3), that can be measured by the means of the trio of basic vectors, describe a vector. And every vector possesses these properties*. However, this is not enough ("not sufficient" in jargon of the math books).

* And what is the direction of the null vector ("(vanishing) vector") $\mathbf{0}$ with the zero length $\|\mathbf{0}\| = 0$? (The zero vector without a magnitude ends exactly where it begins and thus it is not directed anywhere, its direction is *undefined*.)

A vector is not just a collection of components in some basis.

A triple of pairwise perpendicular unit vectors can only rotate and thereby it can characterize the angular orientation of other vectors.

The decomposition of the same vector $\boldsymbol{v}$ in the two cartesian systems with basis unit vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}'_i$ (figure 2) gives

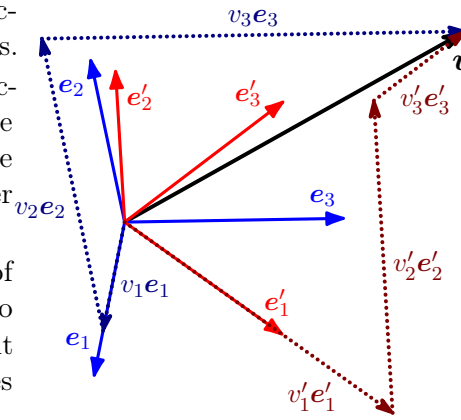

figure 2

$$\boldsymbol{v} = v_i\boldsymbol{e}_i = v'_i\boldsymbol{e}'_i,$$

where

$$v_i = \boldsymbol{v}\cdot\boldsymbol{e}_i = v'_k\boldsymbol{e}'_k\cdot\boldsymbol{e}_i,$$
$$v'_i = \boldsymbol{v}\cdot\boldsymbol{e}'_i = v_k\boldsymbol{e}_k\cdot\boldsymbol{e}'_i.$$

Appeared here two-index objects (the two-dimensional arrays) $o_{k'i} \equiv \boldsymbol{e}'_k\cdot\boldsymbol{e}_i$ and $o_{ki'} \equiv \boldsymbol{e}_k\cdot\boldsymbol{e}'_i$ are used to shorten the formulas.

Написáть о пассивном повороте, описанном ниже, и об активном повороте из § 11 .....

The "$\cdot$"-product (dot product) of two vectors is commutative — that is, the swapping of multipliers doesn't change the result. Thus

$$o_{k'i} = \boldsymbol{e}'_k\cdot\boldsymbol{e}_i = \cos\measuredangle(\boldsymbol{e}'_k,\boldsymbol{e}_i) = \cos\measuredangle(\boldsymbol{e}_i,\boldsymbol{e}'_k) = \boldsymbol{e}_i\cdot\boldsymbol{e}'_k = o_{ik'}, \quad (2.3a)$$
$$o_{ki'} = \boldsymbol{e}_k\cdot\boldsymbol{e}'_i = \cos\measuredangle(\boldsymbol{e}_k,\boldsymbol{e}'_i) = \cos\measuredangle(\boldsymbol{e}'_i,\boldsymbol{e}_k) = \boldsymbol{e}'_i\cdot\boldsymbol{e}_k = o_{i'k}. \quad (2.3b)$$

Lines (2.3a) and (2.3b) are mutually reciprocal by multiplication

$$o_{k'i}o_{ki'} = o_{ki'}o_{k'i} = 1, \;\; o_{k'i}o_{i'k} = o_{i'k}o_{k'i} = 1.$$

Multiplying of an orthogonal matrix by the components of any vector retains the length of this vector:

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v} = v'_iv'_i = o_{i'k}v_ko_{i'n}v_n = v_nv_n$$

— this conclusion leans on (**??**).

Orthogonal transformation of the vector components

$$\boldsymbol{v}\cdot\boldsymbol{e}'_i = v_k\boldsymbol{e}_k\cdot\boldsymbol{e}'_i = \boldsymbol{e}'_i\cdot\boldsymbol{e}_k v_k = o_{i'k}v_k = v'_i \tag{2.4}$$

is sometimes used for defining a vector itself. If in each orthonormal basis $\boldsymbol{e}_i$ a triplet of numbers v_i is known, and with a rotation of the basis as a whole it is transformed according to (2.4). then this triplet of components represents an invariant object — vector $\boldsymbol{v}$.

§ 3. Tensor and its components

When in each orthonormal basis $\boldsymbol{e}_i$ we have a set of nine ($3^2=9$) numbers B_{ij} ($i,j=1,2,3$), and this set is transformed during a transition to a new (rotated) orthonormal basis $\boldsymbol{e}'_i$ as

$$B'_{ij} = \boldsymbol{e}'_i\cdot\boldsymbol{e}_m B_{mn}\boldsymbol{e}_n\cdot\boldsymbol{e}'_j = \boldsymbol{e}'_i\cdot\boldsymbol{e}_m\boldsymbol{e}'_j\cdot\boldsymbol{e}_n B_{mn} = o_{i'm}o_{j'n}B_{mn}, \tag{3.1}$$

then this set of components presents an invariant object — a tensor ${}^2\boldsymbol{B}$ of the second complexity (of the second valence, bivalent).

In other words, tensor ${}^2\boldsymbol{B}$ reveals in every basis as a collection of its components B_{ij}, changing along with a basis according to (3.1).

The key example of a second complexity tensor is a dyad. Having two vectors $\boldsymbol{a}=a_i\boldsymbol{e}_i$ and $\boldsymbol{b}=b_i\boldsymbol{e}_i$, in each basis $\boldsymbol{e}_i$ assume $d_{ij}\equiv a_ib_j$. It's easy to see how components d_{ij} transform according to (3.1):

$$a'_i = o_{i'm}a_m,\ b'_j = o_{j'n}b_n \ \Rightarrow\ d'_{ij} = a'_ib'_j = o_{i'm}a_mo_{j'n}b_n = o_{i'm}o_{j'n}d_{mn}.$$

A resulting tensor ${}^2\boldsymbol{d}$ is called a dyadic product or just dyad and is written as $\boldsymbol{a}\otimes\boldsymbol{b}$ or $\boldsymbol{ab}$. I choose the notation "${}^2\boldsymbol{d}=\boldsymbol{ab}$", without the $\otimes$ symbol.

When some bivalent tensor ${}^2\boldsymbol{B}$ is a dyad $\boldsymbol{\beta b}$, its components $B_{ij}=\beta_ib_j$ satisfy the equality $B_{pq}B_{mn}=B_{mq}B_{pn}$ to get commutativity of multiplication $\beta_pb_q\beta_mb_n=\beta_mb_q\beta_pb_n$. Here $p\neq m$, or else the equality becomes the identity.

The essential bivalent tensor is the unit tensor (other names are unit dyad, identity tensor and metric tensor). Let for any orthonormal (cartesian) basis $E_{ij}\equiv\boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij}$. These are really components

of tensor, (3.1) is actual: $E'_{mn} = o_{m'i}o_{n'j}E_{ij} = o_{m'i}o_{n'i} = \delta_{mn}$. I write this tensor as $\boldsymbol{E}$ (other popular choices are $\boldsymbol{I}$ and ${}^2\mathbf{1}$).

Invariableness of components upon any rotation makes the tensor $\boldsymbol{E}$ isotropic. There are no non-null (nonvanishing) isotropic vectors (all components of the null, or "vanishing", vector $\mathbf{0}$ are equal to zero within any basis).

The next example is related to a linear transformation (a linear mapping) of vectors.

If $\boldsymbol{b} = b_i\boldsymbol{e}_i$ is linear (preserving addition and multiplication by number) function of $\boldsymbol{a} = a_j\boldsymbol{e}_j$, then $b_i = c_{ij}a_j$ in every basis. Transformation coefficients c_{ij} alter when a basis rotates :

$$b'_i = c'_{ij}a'_j = o_{i'k}b_k = o_{i'k}c_{kn}a_n,\ a_n = o_{j'n}a'_j \ \Rightarrow\ c'_{ij} = o_{i'k}o_{j'n}c_{kn}.$$

It turns out that a set of two-index objects c_{ij}, c'_{ij}, ..., describing the same linear mapping $\boldsymbol{a} \mapsto \boldsymbol{b}$, but in various bases, represents a single invariant object — a tensor of second complexity ${}^2\boldsymbol{c}$. And many book authors introduce tensors in that way, by means of linear mappings (linear transformations).

And the last example is a bilinear form $F(\boldsymbol{a},\boldsymbol{b}) = f_{ij}\,a_ib_j$, where f_{ij} are coefficients, a_i and b_j are components of vector arguments $\boldsymbol{a} = a_i\boldsymbol{e}_i$ and $\boldsymbol{b} = b_j\boldsymbol{e}_j$. The result F is invariant (independent of basis) with the transformation (3.1) for coefficients f_{ij}:

$$F' = f'_{ij}\,a'_ib'_j = f_{mn}\underbrace{a_m}_{o_{i'm}a'_i}\underbrace{b_n}_{o_{j'n}b'_j} = F \ \Leftrightarrow\ f'_{ij} = o_{i'm}o_{j'n}f_{mn}.$$

If $f_{ij} = \delta_{ij}$, then $F = \delta_{ij}a_ib_j = a_ib_i$ — the "$\bullet$"-product (dot product, scalar product) of two vectors. When both arguments are the same, such a homogeneous polynomial of second degree (quadratic) of one vector's components $F(\boldsymbol{a},\boldsymbol{a}) = f_{ij}\,a_ia_j$ is called a quadratic form.

Now about more complex tensors (of valence larger than two). Tensor of third complexity ${}^3\boldsymbol{C}$ is represented by a collection of $3^3 = 27$ numbers C_{ijk}, changing with a rotation of basis as

$$C'_{ijk} = \boldsymbol{e}'_i \bullet \boldsymbol{e}_p\boldsymbol{e}'_j \bullet \boldsymbol{e}_q\boldsymbol{e}'_k \bullet \boldsymbol{e}_r C_{pqr} = o_{i'p}o_{j'q}o_{k'r}C_{pqr}. \qquad (3.2)$$

The primary example is a triad of three vectors $\boldsymbol{a} = a_i \boldsymbol{e}_i$, $\boldsymbol{b} = b_j \boldsymbol{e}_j$ and $\boldsymbol{c} = c_k \boldsymbol{e}_k$

$$t_{ijk} \equiv a_i b_j c_k \;\Leftrightarrow\; {}^3\boldsymbol{t} = \boldsymbol{abc}.$$

It is seen that orthogonal transformations (3.2) and (3.1) are results of "repeating" vector's (2.4). The reader will easily compose a transformation of components for tensor of any complexity and will write a corresponding polyad as an example.

Vectors with transformation (2.4) are tensors of the first complexity (monovalent tensors).

The least complex objects are scalars or tensors of the zeroth complexity. A scalar is a single ($3^0 = 1$) number, which doesn't depend on a basis: the energy, the mass, the temperature et al. But what are components, for example, of vector $\boldsymbol{v} = v_i \boldsymbol{e}_i$, $v_i = \boldsymbol{v} \bullet \boldsymbol{e}_i$? If not scalars, then what? Here could be no simple answer. In each particular basis, $\boldsymbol{e}_i$ are vectors and v_i are scalars.

§ 4. Tensor algebra, or operations with tensors

The whole tensor algebra can be built on the only five* operations (or actions). This section is just about them.

Equality

The first (or the zeroth) is **the equality** "=". This operation shows whether one tensor "on the left" is equal to another tensor "on the right". Tensors can be equal only when their complexities (valencies) are the same. Tensors of different valencies cannot be equal or not equal.

$$\ldots\ldots \tag{4.1}$$

....

* The four without the equality.

Linear combination

The next operation is **the linear combination**. It aggregates the addition and the multiplication by a number (by a scalar, or, in another word, scaling). The arguments of this operation and the result are of the same complexity. For a pair of tensors

$$\lambda a_{ij\dots} + \mu b_{ij\dots} = c_{ij\dots} \;\Leftrightarrow\; \lambda \boldsymbol{a} + \mu \boldsymbol{b} = \boldsymbol{c}. \tag{4.2}$$

Here λ and μ are scalar coefficients; $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are tensors of the same complexity. It's easy to show that the components of the result $\boldsymbol{c}$ satisfy an orthogonal transformation like (3.1).

The decomposition of a vector by some basis, that is the representation of a vector as the sum $\boldsymbol{v} = v_i \boldsymbol{e}_i$, is nothing else but the linear combination of the basis vectors $\boldsymbol{e}_i$ with the coefficients v_i.

This operation is *linear* because the only two atomary kinds of motion are possible on a line: the translation (the movement along a straight line) and the reflexion (mirroring) (the backward movement).

Multiplication of tensors

One more operation — **the multiplication (the tensor product, the direct product)**. It takes arguments of any complexities, returning the result of the cumulative complexity. Examples:

$$\begin{aligned} v_i a_{jk} = C_{ijk} \;&\Leftrightarrow\; \boldsymbol{v}\,{}^2\boldsymbol{a} = {}^3\boldsymbol{C}, \\ a_{ij} B_{abc} = D_{ijabc} \;&\Leftrightarrow\; {}^2\boldsymbol{a}\,{}^3\boldsymbol{B} = {}^5\boldsymbol{D}. \end{aligned} \tag{4.3}$$

Transformation of a collection of result's components, such as $C_{ijk} = v_i a_{jk}$, during a rotation of basis is orthogonal, similar to (3.2), thus here's no doubt that such a collection is a set of tensor components.

The primary and already known (from § 3) subtype of multiplication is the dyadic product of two vectors ${}^2\boldsymbol{A} = \boldsymbol{bc}$.

Contraction

The fourth (or the third) operation is called **the contraction**. It applies to bivalent and more complex tensors. This operation acts upon

a single tensor, without other "participants". Roughly speaking, contracting a tensor is summing of its components over some pair of indices. As a result, the tensor's complexity decreases by two.

For a trivalent tensor ${}^3\boldsymbol{D}$ there are the three possible contractions. They give vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ with components

$$a_i = D_{kki}, \quad b_i = D_{kik}, \quad c_i = D_{ikk}. \tag{4.4}$$

A rotation of basis

$$a'_i = D'_{kki} = \underbrace{o_{k'p} o_{k'q}}_{\delta_{pq}} o_{i'r} D_{pqr} = o_{i'r} D_{ppr} = o_{i'r} a_r$$

shows "the tensorial nature" of the result of contraction.

For a tensor of second complexity, the only one kind of contraction is possible. It gives a scalar, known as *"trace"*

$$\boldsymbol{B}_\bullet \equiv \operatorname{trace} \boldsymbol{B} \equiv \mathrm{I}(\boldsymbol{B}) = B_{kk}.$$

The trace of the unit tensor ("contraction of the Kronecker delta") is equal to the dimension of space

$$\operatorname{trace} \boldsymbol{E} = \boldsymbol{E}_\bullet = \delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

Index juggling, transposing

The last operation is applicable to a single tensor of the second* and bigger complexities. It is named as **the index swap, index juggling, transposing**. From components of a tensor, the new collection emerges with another sequence of indices, and the result's complexity stays the same. For example, a trivalent tensor ${}^3\boldsymbol{D}$ can give tensors ${}^3\boldsymbol{A}$, ${}^3\boldsymbol{B}$, ${}^3\boldsymbol{C}$ with components

$$\begin{aligned} {}^3\boldsymbol{A} &= {}^3\boldsymbol{D}_{1\rightleftarrows 2} \Leftrightarrow A_{ijk} = D_{jik}, \\ {}^3\boldsymbol{B} &= {}^3\boldsymbol{D}_{1\rightleftarrows 3} \Leftrightarrow B_{ijk} = D_{kji}, \\ {}^3\boldsymbol{C} &= {}^3\boldsymbol{D}_{2\rightleftarrows 3} \Leftrightarrow C_{ijk} = D_{ikj}. \end{aligned} \tag{4.5}$$

* Transposing a vector makes no sense.

For a bivalent tensor, the only one transposition is possible: $\boldsymbol{A}^{\intercal} \equiv \boldsymbol{A}_{1\rightleftarrows 2} = \boldsymbol{B} \Leftrightarrow B_{ij} = A_{ji}$. Obviously, $\left(\boldsymbol{A}^{\intercal}\right)^{\intercal} = \boldsymbol{A}$.

For the dyadic product of two vectors, $\boldsymbol{ab} = \boldsymbol{ba}^{\intercal}$.

Combining operations

The four presented algebraic operations (actions) can be combined in various sequences.

The combination of multiplication (4.3) and contraction (4.4) — the "•"-product (dot product) — is the most frequently used. In the direct indexless notation this is denoted by the large dot "•", which shows the contraction by adjacent indices:

$$\boldsymbol{a} = \boldsymbol{B} \cdot \boldsymbol{c} \Leftrightarrow a_i = B_{ij} c_j, \quad \boldsymbol{A} = \boldsymbol{B} \cdot \boldsymbol{C} \Leftrightarrow A_{ij} = B_{ik} C_{kj}. \tag{4.6}$$

The defining property of the unit tensor — the neutrality (it is the "identity element") for the "•"-product (the tensor product with the subsequent contraction by adjacent indices)

$$^{\mathrm{n}}\boldsymbol{a} \cdot \boldsymbol{E} = \boldsymbol{E} \cdot {}^{\mathrm{n}}\boldsymbol{a} = {}^{\mathrm{n}}\boldsymbol{a} \quad \forall\, {}^{\mathrm{n}}\boldsymbol{a} \quad \forall \mathrm{n} > 0. \tag{4.7}$$

In the (commutative) scalar product of two vectors, the dot represents the same: the dyadic product and the subsequent contraction

$$\boldsymbol{a} \cdot \boldsymbol{b} = (\boldsymbol{ab})_{\bullet} = a_i b_i = b_i a_i = (\boldsymbol{ba})_{\bullet} = \boldsymbol{b} \cdot \boldsymbol{a}. \tag{4.8}$$

And here's how the multipliers of the "•"-product (dot product) of two second complexity tensors are swapped

$$\begin{aligned} \boldsymbol{B} \cdot \boldsymbol{Q} &= \left(\boldsymbol{Q}^{\intercal} \cdot \boldsymbol{B}^{\intercal}\right)^{\intercal} \\ \left(\boldsymbol{B} \cdot \boldsymbol{Q}\right)^{\intercal} &= \boldsymbol{Q}^{\intercal} \cdot \boldsymbol{B}^{\intercal}. \end{aligned} \tag{4.9}$$

For two dyads $\boldsymbol{B} = \boldsymbol{bd}$ and $\boldsymbol{Q} = \boldsymbol{pq}$

$$\begin{aligned} \left(\boldsymbol{bd} \cdot \boldsymbol{pq}\right)^{\intercal} &= \boldsymbol{pq}^{\intercal} \cdot \boldsymbol{bd}^{\intercal} \\ d_i p_i \boldsymbol{bq}^{\intercal} &= \boldsymbol{qp} \cdot \boldsymbol{db} \\ d_i p_i \boldsymbol{qb} &= p_i d_i \boldsymbol{qb}. \end{aligned}$$

For a vector and a bivalent tensor

$$\boldsymbol{c}\cdot\boldsymbol{B}=\boldsymbol{B}^{\mathsf{T}}\cdot\boldsymbol{c},\quad \boldsymbol{B}\cdot\boldsymbol{c}=\boldsymbol{c}\cdot\boldsymbol{B}^{\mathsf{T}}. \tag{4.10}$$

Contraction can be repeated for two or more adjacent indices,

$$\big(\boldsymbol{A}\cdot\boldsymbol{B}\big)_{\bullet}=\boldsymbol{A}\cdot\!\cdot\,\boldsymbol{B}=A_{ij}B_{ji}. \tag{4.11}$$

The double contraction of a bivalent tensor with the unit dyad gives the trace

$$\boldsymbol{A}\cdot\!\cdot\,\boldsymbol{E}=\boldsymbol{E}\cdot\!\cdot\,\boldsymbol{A}=\boldsymbol{A}_{\bullet}=\operatorname{trace}\boldsymbol{A}=A_{jj}. \tag{4.12}$$

The commutativity is guaranteed

$$\boldsymbol{A}\cdot\!\cdot\,\boldsymbol{B}=A_{ij}B_{ji}=B_{ji}A_{ij}=\boldsymbol{B}\cdot\!\cdot\,\boldsymbol{A} \tag{4.13}$$

for any two bivalent tensors $\boldsymbol{A}$ and $\boldsymbol{B}$, contracted twice.

In other texts, a double contraction may be written "vertically" as $\boldsymbol{A}\,\boldsymbol{:}\,\boldsymbol{B}$. $\boldsymbol{A}\,\boldsymbol{:}\,\boldsymbol{1}=\boldsymbol{1}\,\boldsymbol{:}\,\boldsymbol{A}$ is nothing but trace $\boldsymbol{A}$ or A_{ii} (4.12) in "$\boldsymbol{:}$"-notation and $\boldsymbol{1}$ for the unit tensor. To further confuse the reader, such a colon can denote either $\boldsymbol{A}\,\boldsymbol{:}\,\boldsymbol{B}\overset{(1)}{=}A_{ij}B_{ji}$ or $\boldsymbol{A}\,\boldsymbol{:}\,\boldsymbol{B}\overset{(2)}{=}A_{ij}B_{ij}$ with the additional transposition of one of the tensors. But don't worry, in this book you can meet the "$\boldsymbol{:}$"-product only in this paragraph, and when $\boldsymbol{B}$ is transposed, then it is $\boldsymbol{A}\cdot\!\cdot\,\boldsymbol{B}^{\mathsf{T}}$. Or $\boldsymbol{A}^{\mathsf{T}}\cdot\!\cdot\,\boldsymbol{B}$, because these are equal

$$\begin{aligned}\boldsymbol{A}\cdot\!\cdot\,\boldsymbol{B}^{\mathsf{T}}&=\boldsymbol{A}^{\mathsf{T}}\cdot\!\cdot\,\boldsymbol{B}=A_{ij}B_{ij},\\ \boldsymbol{A}\cdot\!\cdot\,\boldsymbol{B}&=\boldsymbol{A}^{\mathsf{T}}\cdot\!\cdot\,\boldsymbol{B}^{\mathsf{T}}=A_{ij}B_{ji}.\end{aligned} \tag{4.14}$$

And as a bonus, here are more useful equalities for bivalent tensors

$$\begin{aligned}\boldsymbol{d}\cdot\boldsymbol{A}\cdot\boldsymbol{b}&=d_iA_{ij}b_j=\boldsymbol{A}\cdot\!\cdot\,\boldsymbol{b}\boldsymbol{d}=\boldsymbol{b}\boldsymbol{d}\cdot\!\cdot\,\boldsymbol{A}=b_jd_iA_{ij},\\ \boldsymbol{A}\cdot\boldsymbol{B}\cdot\!\cdot\,\boldsymbol{E}&=A_{ij}B_{jk}\delta_{ki}=\boldsymbol{A}\cdot\!\cdot\,\boldsymbol{B},\quad \boldsymbol{A}\cdot\boldsymbol{A}\cdot\!\cdot\,\boldsymbol{E}=\boldsymbol{A}\cdot\!\cdot\,\boldsymbol{A},\\ \boldsymbol{A}\cdot\!\cdot\,\boldsymbol{B}\cdot\boldsymbol{C}&=\boldsymbol{A}\cdot\boldsymbol{B}\cdot\!\cdot\,\boldsymbol{C}=\boldsymbol{C}\cdot\!\cdot\,\boldsymbol{A}\cdot\boldsymbol{B}=A_{ij}B_{jk}C_{ki},\\ \boldsymbol{A}\cdot\!\cdot\,\boldsymbol{B}\cdot\boldsymbol{C}\cdot\boldsymbol{D}&=\boldsymbol{A}\cdot\boldsymbol{B}\cdot\!\cdot\,\boldsymbol{C}\cdot\boldsymbol{D}=\boldsymbol{A}\cdot\boldsymbol{B}\cdot\boldsymbol{C}\cdot\!\cdot\,\boldsymbol{D}\\ &=\boldsymbol{D}\cdot\!\cdot\,\boldsymbol{A}\cdot\boldsymbol{B}\cdot\boldsymbol{C}=A_{ij}B_{jk}C_{kh}D_{hi}.\end{aligned} \tag{4.15}$$

§ 5. Polyadic representation (decomposition)

Before in § 3, a tensor was presented as some invariant object, revealing itself in every basis as a collection of numbers (components). Such a presentation is typical for the majority of books about tensors. The index notation can be convenient, especially when only rectangular coordinates are used, but quite often it is not. And the relevant case is *physics of continua, elastic and not very elastic* : it needs more elegant, more powerful and perfect apparatus of the direct tensor calculus, operating with indexless invariant objects.

The linear combination like $\boldsymbol{v} = v_i \boldsymbol{e}_i$ from (2.1) connects the vector $\boldsymbol{v}$ with the basis $\boldsymbol{e}_i$ and the vector's components v_i in that basis. Is there a similar relation for tensor of any complexity?

Any bivalent tensor ${}^2\boldsymbol{B}$ has nine (3^2) components B_{ij} in each basis. The number of different dyads $\boldsymbol{e}_i\boldsymbol{e}_j$ for the same basis is nine too. Linear combining these dyads with coefficients B_{ij} gives $B_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$. Yes, this is a tensor, like any linear combination of tensors. Yet what are its components, and how such a representation changes or doesn't change with a rotation of basis?

The components of combination

$$\big(B_{ij}\,\boldsymbol{e}_i\boldsymbol{e}_j\big)_{pq} \equiv B_{ij}\big(\boldsymbol{e}_i \cdot \boldsymbol{e}_p\big)\big(\boldsymbol{e}_j \cdot \boldsymbol{e}_q\big) = B_{ij}\,\delta_{ip}\delta_{jq} = B_{pq}$$

are the components of tensor ${}^2\boldsymbol{B}$. And with a rotation of basis

$$B'_{ij}\boldsymbol{e}'_i\boldsymbol{e}'_j = o_{i'p}o_{j'q}B_{pq}o_{i'n}\boldsymbol{e}_n o_{j'm}\boldsymbol{e}_m = \delta_{pn}\delta_{qm}B_{pq}\boldsymbol{e}_n\boldsymbol{e}_m = B_{pq}\boldsymbol{e}_p\boldsymbol{e}_q\,.$$

Doubts are dropped : a tensor of second complexity can be (re)presented as a linear combination

$${}^2\boldsymbol{B} = B_{ij}\,\boldsymbol{e}_i\boldsymbol{e}_j \tag{5.1}$$

— the dyadic decomposition of a bivalent tensor.

For the unit tensor

$$\boldsymbol{E} = E_{ij}\,\boldsymbol{e}_i\boldsymbol{e}_j = \delta_{ij}\,\boldsymbol{e}_i\boldsymbol{e}_j = \boldsymbol{e}_i\boldsymbol{e}_i = \boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3\,,$$

that's why $\boldsymbol{E}$ is called the unit dyad.

Using polyadic representations like (5.1), tensors are much easier to handle :

$$\boldsymbol{v}\bullet{}^2\boldsymbol{B} = v_i\boldsymbol{e}_i\bullet\boldsymbol{e}_jB_{jk}\boldsymbol{e}_k = v_i\delta_{ij}B_{jk}\boldsymbol{e}_k = v_iB_{ik}\boldsymbol{e}_k,$$

$$\boldsymbol{e}_i\bullet{}^2\boldsymbol{B}\bullet\boldsymbol{e}_j = \boldsymbol{e}_i\bullet B_{pq}\boldsymbol{e}_p\boldsymbol{e}_q\bullet\boldsymbol{e}_j = B_{pq}\delta_{ip}\delta_{qj} = B_{ij} = {}^2\boldsymbol{B}\bullet\bullet\,\boldsymbol{e}_j\boldsymbol{e}_i. \qquad (5.2)$$

The last line here is quite interesting : the tensor components are presented through the tensor itself. An orthogonal transformation of components with a rotation of basis (3.1) turns out to be just a version of (5.2).

And any tensor, of any complexity above zero, may be decomposed into the basis polyads. For a trivalent tensor

$$\begin{gathered}{}^3\boldsymbol{C} = C_{ijk}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k,\\ C_{ijk} = {}^3\boldsymbol{C}\bullet\bullet\bullet\,\boldsymbol{e}_k\boldsymbol{e}_j\boldsymbol{e}_i = \boldsymbol{e}_i\bullet{}^3\boldsymbol{C}\bullet\bullet\,\boldsymbol{e}_k\boldsymbol{e}_j = \boldsymbol{e}_j\boldsymbol{e}_i\bullet\bullet\,{}^3\boldsymbol{C}\bullet\boldsymbol{e}_k.\end{gathered} \qquad (5.3)$$

Now it's pretty easy to see the actuality of property (4.7) — the "unitness" of tensor $\boldsymbol{E}$:

$${}^n\boldsymbol{a} = a_{ij\ldots q}\,\boldsymbol{e}_i\boldsymbol{e}_j\ldots\boldsymbol{e}_q,\ \ \boldsymbol{E} = \boldsymbol{e}_e\boldsymbol{e}_e$$

$${}^n\boldsymbol{a}\bullet\boldsymbol{E} = a_{ij\ldots q}\,\boldsymbol{e}_i\boldsymbol{e}_j\ldots\underbrace{\boldsymbol{e}_q\bullet\boldsymbol{e}_e}_{\delta_{eq}}\boldsymbol{e}_e = a_{ij\ldots q}\,\boldsymbol{e}_i\boldsymbol{e}_j\ldots\boldsymbol{e}_q = {}^n\boldsymbol{a},$$

$$\boldsymbol{E}\bullet{}^n\boldsymbol{a} = \boldsymbol{e}_e\boldsymbol{e}_e\bullet a_{ij\ldots q}\,\boldsymbol{e}_i\boldsymbol{e}_j\ldots\boldsymbol{e}_q = a_{ij\ldots q}\delta_{ei}\boldsymbol{e}_e\boldsymbol{e}_j\ldots\boldsymbol{e}_q = {}^n\boldsymbol{a}.$$

The polyadic representation links the direct and index notations together. It's not worth contraposing one another. The direct notation is compact, elegant, it much more than others suits for final relations. But, sometimes, the index notation is very convenient too, as it is for cumbersome manipulations with tensors.

§ 6. Matrices, permutations and determinants

Matrices are the convenient tool for arranging of elements and for solving systems of linear equations.

Does the reader know that matrices are sometimes called "arrays"? Does someone need two-dimensional arrays? Matrices can be presented

as tables full of rows and columns. Any matrix has the same number of elements in each row and the same number of elements in each column. Rectangular arrangement of items, anyone? Matrices are full of numbers and expressions in rows and columns. A column arranges elements vertically from top to bottom, while a row arranges horizontally from left to right.

Matrix dimensions

Matrices come in all sizes, or "dimensions".

By convention, to avoid confusion, the rows are listed first, and the columns second. The dimension of a matrix is written as the number of rows, then a multiplication sign ("×" is used the most often), and then the number of columns.

Here are examples

$$\underset{3\times3}{\left[\mathcal{A}\right]} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

3 rows and 3 columns
a 3×3 matrix
it's a *"square"* matrix

the number of rows is the same as of columns

$$\underset{2\times4}{\left[\mathcal{B}\right]} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \end{bmatrix}$$

2 rows and 4 columns
dimension 2×4

$$\underset{3\times1}{\left[\mathcal{C}\right]} = \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix}$$

3 rows, 1 column
3×1
a *"column matrix"*

a matrix with just one column

$$\underset{1\times6}{\left[\mathcal{D}\right]} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \end{bmatrix}$$

1 row, 6 columns
1×6
a *"row matrix"*

a matrix with just one row

The matrix algebra

The matrix algebra includes the linear operations — the addition of matrices and the multiplication by scalar.

The dimension of a matrix is essential for binary operations, that is for operations involving two matrices.

An addition or subtraction of the two matrices is possible only if they have the same sizes.

.....

$$\underset{m\times n}{\big[\mathcal{A}\big]} = \dots$$

The matrix of the result, known as "the matrix product", has the number of rows of the first multiplier matrix and the number of columns of the second matrix.

.....

Square matrices

....

Matrices and one-dimensional arrays

Two indices of a table are more than the single index of a one-dimensional array. Due to this, a one-dimensional array could be presented "vertically" or "horizontally", either as a table of rows in one column

$$\begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix},$$

or as a table of columns in one row

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix}.$$

....

Permutation parity symbols

To write permutations, the "parity symbols" e_{ijk} are introduced. They are often associated with the names of Oswald Veblen and Tullio Levi-Civita.

...

the permutation parity symbols via the determinant

$$e_{pqr} = e_{ijk}\delta_{pi}\delta_{qj}\delta_{rk} = e_{ijk}\delta_{ip}\delta_{jq}\delta_{kr},$$

$$e_{pqr} = \det \begin{bmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{bmatrix} = \det \begin{bmatrix} \delta_{p1} & \delta_{p2} & \delta_{p3} \\ \delta_{q1} & \delta_{q2} & \delta_{q3} \\ \delta_{r1} & \delta_{r2} & \delta_{r3} \end{bmatrix}. \tag{6.1}$$

....

A determinant is not sensitive to transposing,

$$\det_{i,j} A_{ij} = \det_{i,j} A_{ji} = \det_{j,i} A_{ij}.$$

....

The determinant of the matrix product of two matrices is equal to the product of the determinants of each of these matrices

$$\det_{i,k} B_{ik} \det_{k,j} C_{kj} = \det_{i,j} B_{ik} C_{kj} \tag{6.2}$$

$$e_{fgh} \det_{m,n} B_{ms} C_{sn} = e_{pqr} B_{fi} C_{ip} B_{gj} C_{jq} B_{hk} C_{kr}$$

$$e_{fgh} \det_{m,s} B_{ms} = e_{ijk} B_{fi} B_{gj} B_{hk}$$

$$e_{ijk} \det_{s,n} C_{sn} = e_{pqr} C_{ip} C_{jq} C_{kr}$$

$$e_{fgh}\, e_{ijk} \det_{m,s} B_{ms} \det_{s,n} C_{sn} = e_{ijk}\, e_{pqr} B_{fi} B_{gj} B_{hk} C_{ip} C_{jq} C_{kr}$$

...

There's the following equality

$$e_{ijk} e_{pqr} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix} \tag{6.3}$$

○ Representing the permutation parity symbols via determinants (6.1), e_{ijk} by rows and e_{pqr} by columns

$$e_{ijk} = \det \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}, \quad e_{pqr} = \det \begin{bmatrix} \delta_{p1} & \delta_{q1} & \delta_{r1} \\ \delta_{p2} & \delta_{q2} & \delta_{r2} \\ \delta_{p3} & \delta_{q3} & \delta_{r3} \end{bmatrix},$$

$e_{ijk}e_{pqr}$ on the left side of (6.3) appears as the product of these determinants. And then $\det(AB) = (\det A)(\det B)$ — the determinant of matrix product is equal to the product of determinants (6.2). In the product matrix, the item $[\cdots]_{11}$ equals $\delta_{is}\delta_{ps} = \delta_{ip}$, just like on the right side of (6.3); same for all the other items. ●

The contraction of (6.3) leads to the useful formulas

$$e_{ijk}e_{pqk} = \det\begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & \delta_{kk} \end{bmatrix} = \det\begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & 3 \end{bmatrix}$$

$$= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jk}\delta_{kp} + \delta_{ik}\delta_{jp}\delta_{kq} - \delta_{ik}\delta_{jq}\delta_{kp} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jk}\delta_{kq}$$

$$= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} + \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq}$$

$$= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp},$$

$$e_{ijk}e_{pjk} = \delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip},$$

$$e_{ijk}e_{ijk} = 2\delta_{ii} = 6$$

or in short

$$e_{ijk}e_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \quad e_{ijk}e_{pjk} = 2\delta_{ip}, \quad e_{ijk}e_{ijk} = 6. \tag{6.4}$$

....

$$\det_{i,j} \delta_{ij} = 3$$

.....

The determinant of components of a bivalent tensor is invariant, it doesn't change with a rotation of the basis

$$A'_{ij} = o_{i'm}o_{j'n}A_{mn}$$

....

§ 7. The cross product

By common notions, the "×"-product (the "cross product", the "vector product", sometimes the "oriented area product") of two vectors

is the vector, heading perpendicular to the plane of multipliers, whose length is equal to the area of the parallelogram, spanned by the multipliers

$$\| \boldsymbol{a} \times \boldsymbol{b} \| = \|\boldsymbol{a}\| \, \|\boldsymbol{b}\| \sin \measuredangle(\boldsymbol{a}, \boldsymbol{b}).$$

However, a "×"-product isn't quite a vector, since it is not completely invariant.

The multipliers of the "×"-product $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ determine the result's direction in space, with an accuracy up to the sign figure 3.

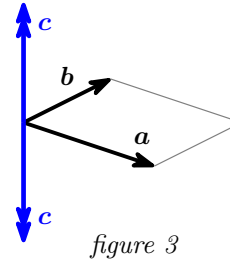

figure 3

Once you pick as the positive the " right-chiral " ("right-handed") or the " left-chiral " ("left-handed") orientation of space, the one direction from the possible two, then the results of the "× "-products become completely determined.

"The chiral" means asymmetric in such a way that the thing and its mirror image are not superimposable, a picture cannot be superposed on its mirror image by any combination of rotations and translations.

An object is chiral if it is distinguishable from its mirror image.

Vectors are usually measured via some basis $\boldsymbol{e}_i$. They are decomposed into linear combinations like $\boldsymbol{a} = a_i \boldsymbol{e}_i$. So the orientation of space is equivalent to the orientation of the sequential triple of basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$. It means that the sequence of basis vectors becomes significant (for linear combinations, the sequence of addends doesn't affect anything).

If two bases consist of different sequences of the same vectors, then orientations of these bases differ by some permutation.

The orientation of the space is a (kind of) asymmetry. This asymmetry makes it impossible to replicate mirroring by the means of any rotations*

* Applying only rotations, it's impossible to replace the left hand with the right hand. But it is possible by mirroring.

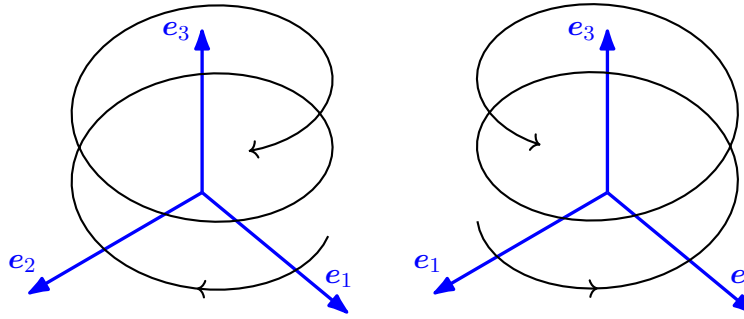

figure 4

A pseudovector is a vector-like object, invariant under any rotation.
*.

... put the figure here ...

Except on rare cases, mirroring changes the direction of a fully invariant (polar) vector.

A pseudovector (an axial vector), unlike a polar vector, doesn't change the component that is perpendicular to the mirroring plane, and turns out to be flipped relatively to the polar vectors and the geometry of the entire space. This happens because the sign (and, accordingly, the direction) of each axial vector changes along with changing the sign of the "×"-product — which corresponds to mirroring.

The otherness of pseudovectors narrows the variety of formulas: a pseudovector is not additive with a vector. The formula $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{\omega} \times \boldsymbol{r}$ from an absolutely rigid undeformable body's kinematics is correct, because $\boldsymbol{\omega}$ is pseudovector there, and with the cross product the two "pseudo" give $(-1)^2 = 1$, mutually compensating each other.

....

$e_{ijk} = \pm\, \boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k$

$e_{pqr} = \pm\, \boldsymbol{e}_p \times \boldsymbol{e}_q \cdot \boldsymbol{e}_r$

* Rotations cannot change the orientation of a triple of basis vectors, it is possible only via mirroring.

with "−" for "left" triple

...

The permutations parity tensor is the volumetric tensor of third complexity

$$ {}^3\boldsymbol{\epsilon} = \in_{ijk} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k, \;\; \in_{ijk} \equiv \boldsymbol{e}_i \times \boldsymbol{e}_j \bullet \boldsymbol{e}_k \tag{7.1} $$

with the components $\in_{ijk}$ equal to the "triple" (the "mixed", the "cross-dot") products of the basis vectors.

The absolute value (the modulus) of each nonzero component of ${}^3\boldsymbol{\epsilon}$ is equal to the volume $\sqrt{g}$ of a parallelopiped drew upon a basis. For a basis $\boldsymbol{e}_i$ of pairwise perpendicular one unit long vectors $\sqrt{g} = 1$.

The tensor ${}^3\boldsymbol{\epsilon}$ is isotropic, its components are constant and independent of any rotations of a basis. But mirroring — a change in the orientation of a triple of basis vectors (a change in "the direction of screw") — changes the sign of ${}^3\boldsymbol{\epsilon}$, so this is a pseudotensor (an axial tensor).

If $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$ without the "minus" sign, then the basis triple $\boldsymbol{e}_i$ is oriented positively. The positive orientation (or "the positive direction") is chosen for different reasons from the two possible ones (figure 3). For a positively oriented basis triplet, the components of ${}^3\boldsymbol{\epsilon}$ are equal to the permutation parity symbols $\in_{ijk} = e_{ijk}$. And when $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = -\boldsymbol{e}_3$, then the basis triple $\boldsymbol{e}_i$ is oriented negatively, or "mirrored". For mirrored triples $\in_{ijk} = -e_{ijk}$ (and $e_{ijk} = -\boldsymbol{e}_i \times \boldsymbol{e}_j \bullet \boldsymbol{e}_k$).

With the permutations parity tensor ${}^3\boldsymbol{\epsilon}$ it's possible to take the fresh look at the cross "×"-product :

$$ \in_{ijk} = \boldsymbol{e}_i \times \boldsymbol{e}_j \bullet \boldsymbol{e}_k \;\Leftrightarrow\; \boldsymbol{e}_i \times \boldsymbol{e}_j = \in_{ijk} \boldsymbol{e}_k, $$

$$ \begin{aligned} \boldsymbol{a} \times \boldsymbol{b} = a_i \boldsymbol{e}_i \times b_j \boldsymbol{e}_j = a_i b_j \boldsymbol{e}_i \times \boldsymbol{e}_j = a_i b_j \in_{ijk} \boldsymbol{e}_k = \\ = b_j a_i \boldsymbol{e}_j \boldsymbol{e}_i \bullet\bullet \in_{mnk} \boldsymbol{e}_m \boldsymbol{e}_n \boldsymbol{e}_k = \boldsymbol{b}\boldsymbol{a} \bullet\bullet {}^3\boldsymbol{\epsilon}, \\ = a_i \in_{ijk} \boldsymbol{e}_k b_j = -a_i \in_{ikj} \boldsymbol{e}_k b_j = -\boldsymbol{a} \bullet {}^3\boldsymbol{\epsilon} \bullet \boldsymbol{b}. \end{aligned} \tag{7.2} $$

So that, the cross product is not another new, entirely distinct operation. With the permutations parity tensor it reduces to the four already described (§ 4) and is applicable to tensors of any complexity.

"The cross product" is just the dot product — the combination of multiplication and contraction (§ 4) — involving tensor ${}^3\boldsymbol{\epsilon}$. Such combinations are possible with any tensors :

$$\begin{aligned}
\boldsymbol{a} \times {}^2\boldsymbol{B} &= a_i \boldsymbol{e}_i \times B_{jk} \boldsymbol{e}_j \boldsymbol{e}_k = \underbrace{a_i B_{jk} \in_{ijn}}_{-a_i \in_{inj} B_{jk}} \boldsymbol{e}_n \boldsymbol{e}_k = -\boldsymbol{a} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^2\boldsymbol{B}, \\
{}^2\boldsymbol{C} \times \boldsymbol{d}\boldsymbol{b} &= C_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \times d_p b_q \boldsymbol{e}_p \boldsymbol{e}_q = \boldsymbol{e}_i C_{ij} d_p \underbrace{\in_{jpk}}_{-\in_{pjk} = -\in_{jkp}} \boldsymbol{e}_k b_q \boldsymbol{e}_q = \\
&= -{}^2\boldsymbol{C}\boldsymbol{d} \bullet\bullet {}^3\boldsymbol{\epsilon}\boldsymbol{b} = -{}^2\boldsymbol{C} \cdot {}^3\boldsymbol{\epsilon} \cdot \boldsymbol{d}\boldsymbol{b},
\end{aligned}$$

$$\boldsymbol{E} \times \boldsymbol{E} = \boldsymbol{e}_i \boldsymbol{e}_i \times \boldsymbol{e}_j \boldsymbol{e}_j = \underbrace{-\in_{ijk} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k}_{+\in_{ijk} \boldsymbol{e}_i \boldsymbol{e}_k \boldsymbol{e}_j} = -{}^3\boldsymbol{\epsilon}. \tag{7.3}$$

The last equation links the isotropic tensor of the second complexity and the isotropic tensor of the third complexity.

Generalizing to all tensors of nonzero complexity

$${}^{\mathrm{n}}\boldsymbol{\xi} \times {}^{\mathrm{m}}\boldsymbol{\zeta} = -{}^{\mathrm{n}}\boldsymbol{\xi} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^{\mathrm{m}}\boldsymbol{\zeta} \quad \forall\, {}^{\mathrm{n}}\boldsymbol{\xi}, {}^{\mathrm{m}}\boldsymbol{\zeta} \quad \forall n > 0,\, m > 0. \tag{7.4}$$

When one of the operands is the unit (metric) tensor, from (7.4) and (4.7) for $\forall\, {}^{\mathrm{n}}\boldsymbol{\Upsilon}\ \forall\, \mathrm{n} > 0$

$$\begin{aligned}
\boldsymbol{E} \times {}^{\mathrm{n}}\boldsymbol{\Upsilon} &= -\boldsymbol{E} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^{\mathrm{n}}\boldsymbol{\Upsilon} = -{}^3\boldsymbol{\epsilon} \cdot {}^{\mathrm{n}}\boldsymbol{\Upsilon}, \\
{}^{\mathrm{n}}\boldsymbol{\Upsilon} \times \boldsymbol{E} &= -{}^{\mathrm{n}}\boldsymbol{\Upsilon} \cdot {}^3\boldsymbol{\epsilon} \cdot \boldsymbol{E} = -{}^{\mathrm{n}}\boldsymbol{\Upsilon} \cdot {}^3\boldsymbol{\epsilon}.
\end{aligned} \tag{7.5}$$

The cross product of two vectors is not commutative, but is anti-commutative :

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{E}) = (\boldsymbol{a} \times \boldsymbol{E}) \cdot \boldsymbol{b} &= \overbrace{-\boldsymbol{a}\boldsymbol{b} \bullet\bullet {}^3\boldsymbol{\epsilon}}^{\in_{jik} = \in_{kji} \Rightarrow a_i b_j \in_{jik} \boldsymbol{e}_k = \in_{kji} a_i b_j \boldsymbol{e}_k} = -{}^3\boldsymbol{\epsilon} \bullet\bullet \boldsymbol{a}\boldsymbol{b}, \\
\boldsymbol{b} \times \boldsymbol{a} = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{E}) = (\boldsymbol{b} \times \boldsymbol{E}) \cdot \boldsymbol{a} &= -\boldsymbol{b}\boldsymbol{a} \bullet\bullet {}^3\boldsymbol{\epsilon} = -{}^3\boldsymbol{\epsilon} \bullet\bullet \boldsymbol{b}\boldsymbol{a}, \\
\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{a}\boldsymbol{b} \bullet\bullet {}^3\boldsymbol{\epsilon} = \boldsymbol{b}\boldsymbol{a} \bullet\bullet {}^3\boldsymbol{\epsilon} &\ \Rightarrow\ \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}.
\end{aligned} \tag{7.6}$$

For any bivalent tensor ${}^2\boldsymbol{B}$ and any vector $\boldsymbol{a}$

$$\begin{aligned}
{}^2\boldsymbol{B} \times \boldsymbol{a} &= \boldsymbol{e}_i B_{ij} \boldsymbol{e}_j \times a_k \boldsymbol{e}_k = -\boldsymbol{e}_i B_{ij} a_k \boldsymbol{e}_k \times \boldsymbol{e}_j \\
&= \left(-a_k \boldsymbol{e}_k \times \boldsymbol{e}_j B_{ij} \boldsymbol{e}_i\right)^{\mathsf{T}} = -\left(\boldsymbol{a} \times {}^2\boldsymbol{B}^{\mathsf{T}}\right)^{\mathsf{T}},
\end{aligned} \tag{7.7}$$

and only for the unit dyad and a vector, the "×"-product is commutative

$$\in_{ijk}=\in_{kij}\Rightarrow \in_{ijk}a_k\boldsymbol{e}_i\boldsymbol{e}_j=a_k\in_{kij}\boldsymbol{e}_i\boldsymbol{e}_j$$
$$-\boldsymbol{E}\times\boldsymbol{a}=\overbrace{{}^3\boldsymbol{\epsilon}\cdot\boldsymbol{a}=\boldsymbol{a}\cdot{}^3\boldsymbol{\epsilon}}=-\boldsymbol{a}\times\boldsymbol{E}, \tag{7.8}$$

plus as the particular case of (7.7)

$$\boldsymbol{E}\times\boldsymbol{a}=-\big(\boldsymbol{a}\times\boldsymbol{E}^{\mathsf{T}}\big)^{\mathsf{T}}=-\big(\boldsymbol{a}\times\boldsymbol{E}\big)^{\mathsf{T}}. \tag{7.9}$$

The first of (6.4) formulas gives the following representation for the double "×"-product

$$\begin{aligned}\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})&=a_i\boldsymbol{e}_i\times\in_{pqj}b_pc_q\boldsymbol{e}_j=\in_{kij}\in_{pqj}a_ib_pc_q\boldsymbol{e}_k=\\&=(\delta_{kp}\delta_{iq}-\delta_{kq}\delta_{ip})\,a_ib_pc_q\boldsymbol{e}_k=a_ib_kc_i\boldsymbol{e}_k-a_ib_ic_k\boldsymbol{e}_k=\\&=\boldsymbol{a}\cdot\boldsymbol{cb}-\boldsymbol{a}\cdot\boldsymbol{bc}=\boldsymbol{a}\cdot\big(\boldsymbol{cb}-\boldsymbol{bc}\big)=\boldsymbol{a}\cdot\boldsymbol{cb}-\boldsymbol{cb}\cdot\boldsymbol{a}.\end{aligned} \tag{7.10}$$

By another interpretation, the dot product of a dyad and a vector is not commutative: $\boldsymbol{bd}\cdot\boldsymbol{c}\neq\boldsymbol{c}\cdot\boldsymbol{bd}$, and the difference can be rendered as

$$\boldsymbol{bd}\cdot\boldsymbol{c}-\boldsymbol{c}\cdot\boldsymbol{bd}=\boldsymbol{c}\times\big(\boldsymbol{b}\times\boldsymbol{d}\big). \tag{7.11}$$

$\boldsymbol{a}\cdot\boldsymbol{bc}=\boldsymbol{cb}\cdot\boldsymbol{a}=\boldsymbol{ca}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{ac}$
$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=-\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{c}\times(\boldsymbol{b}\times\boldsymbol{a})$
The same way it may be derived that

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\big(\boldsymbol{ba}-\boldsymbol{ab}\big)\cdot\boldsymbol{c}=\boldsymbol{ba}\cdot\boldsymbol{c}-\boldsymbol{ab}\cdot\boldsymbol{c}. \tag{7.12}$$

And the following identities for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\begin{aligned}\big(\boldsymbol{a}\times\boldsymbol{b}\big)\times\boldsymbol{E}&=\in_{ijk}a_ib_j\boldsymbol{e}_k\times\boldsymbol{e}_n\boldsymbol{e}_n=a_ib_j\in_{ijk}\in_{knq}\boldsymbol{e}_q\boldsymbol{e}_n=\\&=a_ib_j\big(\delta_{in}\delta_{jq}-\delta_{iq}\delta_{jn}\big)\boldsymbol{e}_q\boldsymbol{e}_n=a_ib_j\boldsymbol{e}_j\boldsymbol{e}_i-a_ib_j\boldsymbol{e}_i\boldsymbol{e}_j=\\&=\boldsymbol{ba}-\boldsymbol{ab},\end{aligned} \tag{7.13}$$

$$\begin{aligned}\big(\boldsymbol{a}\times\boldsymbol{E}\big)\cdot\big(\boldsymbol{b}\times\boldsymbol{E}\big)&=\big(\boldsymbol{a}\cdot{}^3\boldsymbol{\epsilon}\big)\cdot\big(\boldsymbol{b}\cdot{}^3\boldsymbol{\epsilon}\big)=\\&=a_i\in_{ipn}\boldsymbol{e}_p\boldsymbol{e}_n\cdot b_j\in_{jsk}\boldsymbol{e}_s\boldsymbol{e}_k=a_ib_j\in_{ipn}\in_{nkj}\boldsymbol{e}_p\boldsymbol{e}_k=\\&=a_ib_j\big(\delta_{ik}\delta_{pj}-\delta_{ij}\delta_{pk}\big)\boldsymbol{e}_p\boldsymbol{e}_k=a_ib_j\boldsymbol{e}_j\boldsymbol{e}_i-a_ib_i\boldsymbol{e}_k\boldsymbol{e}_k=\\&=\boldsymbol{ba}-\boldsymbol{a}\cdot\boldsymbol{b}\boldsymbol{E}.\end{aligned} \tag{7.14}$$

Finally, the direct relation between the isotropic tensors of the second and third complexities

$$ {}^3\boldsymbol{\epsilon} \bullet\bullet {}^3\boldsymbol{\epsilon} = \boldsymbol{e}_i \in_{ijk} \in_{kjn} \boldsymbol{e}_n = -2\,\delta_{in}\,\boldsymbol{e}_i\boldsymbol{e}_n = -2\boldsymbol{E}. \tag{7.15} $$

§ 8. Symmetric and skewsymmetric tensors

A tensor that does not change upon a permutation of some pair of its indices is called symmetric for that pair of indices. And when a permutation of some pair of indices alternates the sign "+/−" of a tensor, then this tensor is called anti-symmetric or skew-symmetric for that pair of indices.

As example, the tensor of parity of permutations ${}^3\boldsymbol{\epsilon}$ (7.1) is antisymmetric for any & every pair of indices, it is completely (absolutely) skewsymmetric.

Tensor of the second complexity $\boldsymbol{B}$ is symmetric when $\boldsymbol{B} = \boldsymbol{B}^{\mathsf{T}}$. If transposing changes the tensor's sign, that is $\boldsymbol{A}^{\mathsf{T}} = -\boldsymbol{A}$, then tensor $\boldsymbol{A}$ is skewsymmetric (antisymmetric).

The sum of a bivalent tensor $\boldsymbol{C}$ with the transpose $\boldsymbol{C}^{\mathsf{T}}$ is always symmetric : $\big(\boldsymbol{C}+\boldsymbol{C}^{\mathsf{T}}\big)^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{T}}+\boldsymbol{C} = \boldsymbol{C}+\boldsymbol{C}^{\mathsf{T}}\ \forall \boldsymbol{C}$, while the difference $\big(\boldsymbol{C}-\boldsymbol{C}^{\mathsf{T}}\big)^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{T}}-\boldsymbol{C} = -\big(\boldsymbol{C}-\boldsymbol{C}^{\mathsf{T}}\big)$ is always $\forall \boldsymbol{C}$ antisymmetric.

Denoting

$$ \boldsymbol{C}^{\mathsf{S}} \equiv \tfrac{1}{2}\big(\boldsymbol{C}+\boldsymbol{C}^{\mathsf{T}}\big),\ \ \boldsymbol{C}^{\mathsf{A}} \equiv \tfrac{1}{2}\big(\boldsymbol{C}-\boldsymbol{C}^{\mathsf{T}}\big) \tag{8.1} $$

— the symmetric $\boldsymbol{C}^{\mathsf{S}}$ and the antisymmetric $\boldsymbol{C}^{\mathsf{A}}$ parts of some bivalent tensor $\boldsymbol{C}$, any bivalent tensor can be presented as the sum of these parts

$$ \boldsymbol{C} = \boldsymbol{C}^{\mathsf{S}} + \boldsymbol{C}^{\mathsf{A}},\ \ \boldsymbol{C}^{\mathsf{T}} = \boldsymbol{C}^{\mathsf{S}} - \boldsymbol{C}^{\mathsf{A}}. \tag{8.2} $$

For a dyad

$$ \boldsymbol{cd} = \overbrace{\tfrac{1}{2}\,(\boldsymbol{cd}+\boldsymbol{dc})}^{\boldsymbol{cd}^{\mathsf{S}}} + \overbrace{\tfrac{1}{2}\,(\boldsymbol{cd}-\boldsymbol{dc})}^{\boldsymbol{cd}^{\mathsf{A}}}. $$

The product $\boldsymbol{C}^{\mathsf{S}}\cdot\boldsymbol{D}^{\mathsf{S}}$ of two symmetric tensors $\boldsymbol{C}^{\mathsf{S}}$ and $\boldsymbol{D}^{\mathsf{S}}$ is symmetric not always, but only when $\boldsymbol{D}^{\mathsf{S}}\cdot\boldsymbol{C}^{\mathsf{S}} = \boldsymbol{C}^{\mathsf{S}}\cdot\boldsymbol{D}^{\mathsf{S}}$, because by (4.9) $\big(\boldsymbol{C}^{\mathsf{S}}\cdot\boldsymbol{D}^{\mathsf{S}}\big)^{\mathsf{T}} = \boldsymbol{D}^{\mathsf{S}}\cdot\boldsymbol{C}^{\mathsf{S}}$.

With (7.8) and (7.9), the skew symmetry of the "×"-product for the unit dyad and a vector is obvious

$$\begin{aligned}\big(\boldsymbol{E}\times\boldsymbol{a}\big)^{\mathsf{T}}=\big(\boldsymbol{e}_j\boldsymbol{e}_j\times a_i\boldsymbol{e}_i\big)^{\mathsf{T}}=\big(-\boldsymbol{e}_j a_i\boldsymbol{e}_i\times\boldsymbol{e}_j\big)^{\mathsf{T}}=-a_i\boldsymbol{e}_i\times\boldsymbol{e}_j\boldsymbol{e}_j\\ =-\boldsymbol{a}\times\boldsymbol{E}=-\boldsymbol{E}\times\boldsymbol{a}=\big(\boldsymbol{a}\times\boldsymbol{E}\big)^{\mathsf{T}}.\end{aligned}\tag{8.3}$$

In search for a case when a bivalent tensor $\boldsymbol{A}$ can be represented by just a single vector $\boldsymbol{a}$, in such a way that an action of vector $\boldsymbol{a}$ on other objects is exactly like an action of bivalent $\boldsymbol{A}$ on the same objects, perhaps there's a chance to find such $\boldsymbol{A}=\boldsymbol{A}(\boldsymbol{a})$ that for $\forall^{\mathrm{n}}\boldsymbol{\xi}\ \forall\mathrm{n}>0$

$$\begin{aligned}\boldsymbol{b}=\boldsymbol{A}\bullet{}^{\mathrm{n}}\boldsymbol{\xi}\ \Leftrightarrow\ \boldsymbol{a}\times{}^{\mathrm{n}}\boldsymbol{\xi}=\boldsymbol{b}\ \ \forall\boldsymbol{b},\\ \boldsymbol{d}={}^{\mathrm{n}}\boldsymbol{\xi}\bullet\boldsymbol{A}\ \Leftrightarrow\ {}^{\mathrm{n}}\boldsymbol{\xi}\times\boldsymbol{a}=\boldsymbol{d}\ \ \forall\boldsymbol{d}\end{aligned}$$

or, in words, the "•"-product of bivalent $\boldsymbol{A}$ and some other tensor ${}^{\mathrm{n}}\boldsymbol{\xi}$ is equal to the "×"-product of pseudovector $\boldsymbol{a}$ and the same tensor ${}^{\mathrm{n}}\boldsymbol{\xi}$.

The relation $\boldsymbol{a}\mapsto\boldsymbol{A}$ can be derived from (4.7) and (7.5)

$$\begin{aligned}\boldsymbol{A}=\boldsymbol{A}\bullet\boldsymbol{E}=\boldsymbol{a}\times\boldsymbol{E}=-\boldsymbol{a}\bullet{}^{3}\boldsymbol{\epsilon},\\ \boldsymbol{A}=\boldsymbol{E}\bullet\boldsymbol{A}=\boldsymbol{E}\times\boldsymbol{a}=-{}^{3}\boldsymbol{\epsilon}\bullet\boldsymbol{a}.\end{aligned}\tag{8.4}$$

Or, delving into components,

$$\begin{aligned}\boldsymbol{A}\bullet{}^{\mathrm{n}}\boldsymbol{\xi}&=\boldsymbol{a}\times{}^{\mathrm{n}}\boldsymbol{\xi}\\ A_{hi}\boldsymbol{e}_h\boldsymbol{e}_i\bullet\xi_{jk\ldots q}\boldsymbol{e}_j\boldsymbol{e}_k\ldots\boldsymbol{e}_q&=a_i\boldsymbol{e}_i\times\xi_{jk\ldots q}\boldsymbol{e}_j\boldsymbol{e}_k\ldots\boldsymbol{e}_q\\ A_{hj}\xi_{jk\ldots q}\boldsymbol{e}_h\boldsymbol{e}_k\ldots\boldsymbol{e}_q&=a_i\in_{ijh}\xi_{jk\ldots q}\boldsymbol{e}_h\boldsymbol{e}_k\ldots\boldsymbol{e}_q\\ A_{hj}&=a_i\in_{ijh}\\ A_{hj}&=-a_i\in_{ihj}\\ \boldsymbol{A}&=-\boldsymbol{a}\bullet{}^{3}\boldsymbol{\epsilon}\end{aligned}$$

and by similar way from ${}^{\mathrm{n}}\boldsymbol{\xi}\bullet\boldsymbol{A}={}^{\mathrm{n}}\boldsymbol{\xi}\times\boldsymbol{a}$ follows $\boldsymbol{A}=-{}^{3}\boldsymbol{\epsilon}\bullet\boldsymbol{a}$.

(Pseudo)vector $\boldsymbol{a}$ is sometimes named as "accompanying" or "companion" for tensor $\boldsymbol{A}$.

To components

$$\begin{aligned}\boldsymbol{A}&=-{}^{3}\boldsymbol{\epsilon}\bullet\boldsymbol{a}\\ A_{ij}\boldsymbol{e}_i\boldsymbol{e}_j&=-\in_{ijk}\boldsymbol{e}_i\boldsymbol{e}_j a_k\\ A_{ij}(a_k):\ \ A_{ij}&=-\in_{ijk}a_k\end{aligned}$$

or, written as a matrix,

$$\underset{3\times3}{\big[A_{ij}\big]} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & -\in_{123}a_3 & -\in_{132}a_2 \\ -\in_{213}a_3 & 0 & -\in_{231}a_1 \\ -\in_{312}a_2 & -\in_{321}a_1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

That the bivalent $\boldsymbol{A} = \boldsymbol{a}\times\boldsymbol{E} = \boldsymbol{E}\times\boldsymbol{a}$ is skewsymmetric was clear since (8.3). In the three-dimensional space, any antisymmetric tensor of the second complexity has only three independent components out of 9 : $A_{ij} = -A_{ji}$ and $A_{jj} = 0$.

The uniqueness of $\boldsymbol{a}$ for the unique $\boldsymbol{A}$, that is if $\boldsymbol{a}'\times\boldsymbol{E} = \boldsymbol{A}$ and $\boldsymbol{a}''\times\boldsymbol{E} = \boldsymbol{A}$ (or $\boldsymbol{a}'\times\boldsymbol{E} - \boldsymbol{a}''\times\boldsymbol{E} = \boldsymbol{A} - \boldsymbol{A}$) then $\boldsymbol{a}' = \boldsymbol{a}''$ or $\boldsymbol{a}' - \boldsymbol{a}'' = \boldsymbol{0}$

$$\big(\boldsymbol{a}' - \boldsymbol{a}''\big)\times\boldsymbol{E} = {}^2\boldsymbol{0}$$
$$\big(\boldsymbol{a}' - \boldsymbol{a}''\big)\cdot{}^3\boldsymbol{\epsilon} = \boldsymbol{0}\cdot{}^3\boldsymbol{\epsilon}$$

follows from the equal zeros $\boldsymbol{0}\cdot{}^3\boldsymbol{\epsilon} = {}^2\boldsymbol{0}$ and the uniqueness of the "$\cdot$"-product's result ($\boldsymbol{b}\cdot\boldsymbol{c} = \boldsymbol{d}\cdot\boldsymbol{c}$, $\boldsymbol{c}\neq\boldsymbol{0} \Leftrightarrow \boldsymbol{b} = \boldsymbol{d}$, including $\boldsymbol{b}\cdot\boldsymbol{c} = \boldsymbol{0}\cdot\boldsymbol{c}$, $\boldsymbol{c}\neq\boldsymbol{0} \Leftrightarrow \boldsymbol{b} = \boldsymbol{0}$). For $\boldsymbol{a} = \boldsymbol{0}$, $\boldsymbol{A}(\boldsymbol{0}) = \boldsymbol{0}\times\boldsymbol{E} = {}^2\boldsymbol{0}$.

✓ $\boldsymbol{a}$ is unique for $\boldsymbol{A}$

And yet about the reciprocal relation $\boldsymbol{A}\mapsto\boldsymbol{a}$: $\boldsymbol{a}{=}\boldsymbol{a}(\boldsymbol{A})$. By (7.15), the unit dyad $\boldsymbol{E}$ via ${}^3\boldsymbol{\epsilon}$

$$\boldsymbol{E} = -\tfrac{1}{2}\,{}^3\boldsymbol{\epsilon}\bullet\bullet{}^3\boldsymbol{\epsilon},$$

and it is neutral (4.7) for the "$\cdot$"-product

$$\boldsymbol{a} = \underbrace{\boldsymbol{a}\cdot\Big(-\tfrac{1}{2}\,{}^3\boldsymbol{\epsilon}\bullet\bullet{}^3\boldsymbol{\epsilon}\Big)}_{-1/2\,a_a\in_{abc}\in_{cbm}\boldsymbol{e}_m} = \underbrace{\Big(-\tfrac{1}{2}\,{}^3\boldsymbol{\epsilon}\bullet\bullet{}^3\boldsymbol{\epsilon}\Big)\cdot\boldsymbol{a}}_{-1/2\,\boldsymbol{e}_h\in_{hij}\in_{jik}a_k}$$

or without brackets

$$\boldsymbol{a} = -\tfrac{1}{2}\,{}^3\boldsymbol{\epsilon}\bullet\bullet{}^3\boldsymbol{\epsilon}\cdot\boldsymbol{a} = -\tfrac{1}{2}\,\boldsymbol{a}\cdot{}^3\boldsymbol{\epsilon}\bullet\bullet{}^3\boldsymbol{\epsilon}.$$

Bivalent $\boldsymbol{A}$ can be introduced here as in (8.4), $-\boldsymbol{A} = \boldsymbol{a} \bullet {}^3\boldsymbol{\epsilon} = {}^3\boldsymbol{\epsilon} \bullet \boldsymbol{a}$, and then

$$\boldsymbol{a}(\boldsymbol{A}) = \tfrac{1}{2}\, \boldsymbol{A} \bullet\bullet {}^3\boldsymbol{\epsilon} = \tfrac{1}{2}\, {}^3\boldsymbol{\epsilon} \bullet\bullet \boldsymbol{A}. \tag{8.5}$$

.....

$$a_i \boldsymbol{e}_i = \tfrac{1}{2}\, A_{jk} \in_{kji} \boldsymbol{e}_i = \tfrac{1}{2}\, \in_{ikj} A_{jk} \boldsymbol{e}_i\,,$$

$$a_i = \tfrac{1}{2}\, \in_{ikj} A_{jk} = \frac{1}{2} \begin{bmatrix} \in_{123} A_{32} + \in_{132} A_{23} \\ \in_{213} A_{31} + \in_{231} A_{13} \\ \in_{312} A_{21} + \in_{321} A_{12} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}.$$

.....

$$-2a_1 = \overbrace{\in_{123}}^{(+1)} \underbrace{\overbrace{\in_{321}}^{(-1)} a_1}_{-A_{32}} + \overbrace{\in_{132}}^{(-1)} \underbrace{\overbrace{\in_{231}}^{(+1)} a_1}_{-A_{23}}$$

$$-2a_2 = \overbrace{\in_{213}}^{-1} \underbrace{\overbrace{\in_{312}}^{+1} a_2}_{-A_{31}} + \overbrace{\in_{231}}^{+1} \underbrace{\overbrace{\in_{132}}^{-1} a_2}_{-A_{13}}$$

$$-2a_3 = \overbrace{\in_{312}}^{(+1)} \underbrace{\overbrace{\in_{213}}^{(-1)} a_3}_{-A_{21}} + \overbrace{\in_{321}}^{(-1)} \underbrace{\overbrace{\in_{123}}^{(+1)} a_3}_{-A_{12}}$$

.....

$\boldsymbol{a}' \times \boldsymbol{E} = \boldsymbol{A}'$ and $\boldsymbol{a}'' \times \boldsymbol{E} = \boldsymbol{A}''$

PROVE IT $\boldsymbol{A}$ is unique for $\boldsymbol{a}$ **PROVE IT**

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There is **PROVE THAT ONLY FOR SKEWSYMMETRIC** $\forall \boldsymbol{A} = \boldsymbol{A}^{\mathsf{A}}$ **IT IS BIJECTION** a bijection* between antisymmetric bivalent tensors and (pseudo)vectors. The components of a skewsymmetric tensor are fully described by the three numbers xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

.....

*"a bijective relation", "a reciprocally reversible mapping", "a one-to-one correspondence"

All in all, here is the bijection $\boldsymbol{A} \leftrightarrow \boldsymbol{a}$

$$\boldsymbol{A}(\boldsymbol{a}) = -\boldsymbol{a} \centerdot {}^3\boldsymbol{\epsilon} = \boldsymbol{a} \times \boldsymbol{E} = -{}^3\boldsymbol{\epsilon} \centerdot \boldsymbol{a} = \boldsymbol{E} \times \boldsymbol{a}, \qquad (8.6^{A(a)})$$

$$\boldsymbol{a}(\boldsymbol{A}) = \tfrac{1}{2}\, \boldsymbol{A} \centerdot\centerdot {}^3\boldsymbol{\epsilon} = \tfrac{1}{2}\, {}^3\boldsymbol{\epsilon} \centerdot\centerdot \boldsymbol{A}. \qquad (8.6^{a(A)})$$

Easy to memorize, the "pseudovector invariant" $\boldsymbol{A}_\times$ comes from the original tensor $\boldsymbol{A}$ by replacing a dyadic product with a cross product

$$\begin{gathered} \boldsymbol{A}_\times \equiv A_{ij}\, \boldsymbol{e}_i \times \boldsymbol{e}_j = -\boldsymbol{A} \centerdot\centerdot {}^3\boldsymbol{\epsilon}, \\ \boldsymbol{A}_\times = \big(\boldsymbol{a} \times \boldsymbol{E}\big)_\times = -2\boldsymbol{a}, \;\; \boldsymbol{a} = -\tfrac{1}{2}\, \boldsymbol{A}_\times = -\tfrac{1}{2}\, \big(\boldsymbol{a} \times \boldsymbol{E}\big)_\times. \end{gathered} \qquad (8.7)$$

Explaination:

$$\boldsymbol{a} \times \boldsymbol{E} = -\tfrac{1}{2}\, \boldsymbol{A}_\times \times \boldsymbol{E} = -\tfrac{1}{2}\, A_{ij} \underbrace{(\boldsymbol{e}_i \times \boldsymbol{e}_j)}_{\in_{ijn} \boldsymbol{e}_n} \times \boldsymbol{e}_k \boldsymbol{e}_k$$

$$= -\tfrac{1}{2}\, A_{ij} \underbrace{\in_{nij} \in_{nkp}}_{\delta_{jp}\delta_{ik} - \delta_{ip}\delta_{jk}} \boldsymbol{e}_p \boldsymbol{e}_k = -\tfrac{1}{2}\, A_{ij}\, (\boldsymbol{e}_j \boldsymbol{e}_i - \boldsymbol{e}_i \boldsymbol{e}_j)$$

$$= -\tfrac{1}{2}\, \big(\boldsymbol{A}^{\mathsf{T}} - \boldsymbol{A}\big) = \boldsymbol{A}^{\mathsf{A}} = \boldsymbol{A}.$$

The companion vector can be introduced for any bivalent tensor. But only the asymmetric part contributes here: $\boldsymbol{C}^{\mathsf{A}} = -\frac{1}{2}\, \boldsymbol{C}_\times \times \boldsymbol{E}$.

For a symmetric tensor, the companion vector is zero :

$$\boldsymbol{B}_\times = \boldsymbol{0} \;\; \Leftrightarrow \;\; \boldsymbol{B} = \boldsymbol{B}^{\mathsf{T}} = \boldsymbol{B}^{\mathsf{S}}.$$

With (8.7) the decomposition of some tensor $\boldsymbol{C}$ into the symmetric and antisymmetric parts looks like

$$\boldsymbol{C} = \boldsymbol{C}^{\mathsf{S}} - \tfrac{1}{2}\, \boldsymbol{C}_\times \times \boldsymbol{E}. \qquad (8.8)$$

For a dyad

$$(7.13) \;\Rightarrow\; (\boldsymbol{c} \times \boldsymbol{d}) \times \boldsymbol{E} = \boldsymbol{d}\boldsymbol{c} - \boldsymbol{c}\boldsymbol{d} = -2\boldsymbol{c}\boldsymbol{d}^{\mathsf{A}}, \;\; (\boldsymbol{c}\boldsymbol{d})_\times = \boldsymbol{c} \times \boldsymbol{d},$$

and its decomposition

$$\boldsymbol{c}\boldsymbol{d} = \tfrac{1}{2}\, (\boldsymbol{c}\boldsymbol{d} + \boldsymbol{d}\boldsymbol{c}) - \tfrac{1}{2}\, (\boldsymbol{c} \times \boldsymbol{d}) \times \boldsymbol{E}. \qquad (8.9)$$

§ 9. Polar decomposition

Any tensor of the second complexity $\boldsymbol{F}$ with $\det F_{ij} \neq 0$ (not singular) can be decomposed as

...

Example. Polar decompose tensor $\boldsymbol{C} = C_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$, where $\boldsymbol{e}_k$ are pairwise perpendicular unit vectors and C_{ij} are the tensor's components.

$$C_{ij} = \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix}$$

$$\boldsymbol{O} = O_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = \boldsymbol{O}_1 \cdot \boldsymbol{O}_2$$

$$O_{ij} = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$\boldsymbol{C} = \boldsymbol{O} \cdot \boldsymbol{S}_{\mathsf{R}}, \;\; \boldsymbol{O}^{\mathsf{T}} \cdot \boldsymbol{C} = \boldsymbol{S}_{\mathsf{R}}$$

$$\boldsymbol{C} = \boldsymbol{S}_{\mathsf{L}} \cdot \boldsymbol{O}, \;\; \boldsymbol{C} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{S}_{\mathsf{L}}$$

$$S_{\mathsf{R}ij} = \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix}$$

$$S_{\mathsf{L}ij} = \begin{bmatrix} 104/5 & 47/5 & 5 \\ 47/5 & -129/5 & -10 \\ 5 & -10 & 3 \end{bmatrix}$$

...

§ 10. Eigenvectors and eigenvalues

If for some tensor ${}^2\boldsymbol{B}$ and the nonzero vector $\boldsymbol{a}$

$$ {}^2\boldsymbol{B} \cdot \boldsymbol{a} = \eta \boldsymbol{a}, \;\; \boldsymbol{a} \neq \boldsymbol{0} \tag{10.1}$$

$$ {}^2\boldsymbol{B} \cdot \boldsymbol{a} = \eta \boldsymbol{E} \cdot \boldsymbol{a}, \;\; \left({}^2\boldsymbol{B} - \eta \boldsymbol{E}\right) \cdot \boldsymbol{a} = \boldsymbol{0},$$

then η is called the eigenvalue (or the characteristic value) of tensor ${}^2\boldsymbol{B}$, and the axis (direction) of eigenvector $\boldsymbol{a}$ is called its characteristic axis (or direction).

In components, this is the eigenvalue problem for a matrix. A homogeneous system of linear equations $(B_{ij} - \eta\delta_{ij})\, a_j = 0$ has

a non-zero solution if the determinant of a matrix of components

$$\det_{i,j}(B_{ij}-\eta\delta_{ij})$$

is equal to zero:

$$\det\begin{bmatrix} B_{11}-\eta & B_{12} & B_{13} \\ B_{21} & B_{22}-\eta & B_{23} \\ B_{31} & B_{32} & B_{33}-\eta \end{bmatrix} = -\eta^3 + {}_{\text{cha}}\text{I}\,\eta^2 - {}_{\text{cha}}\text{II}\,\eta + {}_{\text{cha}}\text{III} = 0; \tag{10.2}$$

$$\begin{aligned} {}_{\text{cha}}\text{I} &= \operatorname{trace}{}^2\boldsymbol{B} = B_{kk} = B_{11}+B_{22}+B_{33}, \\ {}_{\text{cha}}\text{II} &= B_{11}B_{22}-B_{12}B_{21}+B_{11}B_{33}-B_{13}B_{31}+B_{22}B_{33}-B_{23}B_{32}, \\ {}_{\text{cha}}\text{III} &= \det{}^2\boldsymbol{B} = \det_{i,j} B_{ij} = e_{ijk}B_{1i}B_{2j}B_{3k} = e_{ijk}B_{i1}B_{j2}B_{k3}. \end{aligned} \tag{10.3}$$

The roots of the characteristic equation (10.2) — the eigenvalues — don't depend on basis and therefore are invariant.

The coefficients of (10.3) also don't depend on the basis; they are called the first, the second and the third characteristic invariants of a tensor. The first invariant ${}_{\text{cha}}\text{I}$ is the trace. It was described earlier in § 4. The second characteristic invariant ${}_{\text{cha}}\text{II}$ is the trace of the adjugate matrix — the transpose of the cofactor matrix (of the matrix of algebraic complements)

$${}_{\text{cha}}\text{II}({}^2\boldsymbol{B}) \equiv \operatorname{trace}(\operatorname{adj} B_{ij})$$

(it's hard, yeah). Or

$${}_{\text{cha}}\text{II}({}^2\boldsymbol{B}) \equiv \frac{1}{2}\Big[({}^2\boldsymbol{B}_\bullet)^2 - {}^2\boldsymbol{B}\bullet\bullet{}^2\boldsymbol{B}\Big] = \frac{1}{2}\Big[(B_{kk})^2 - B_{ij}B_{ji}\Big].$$

And the third invariant ${}_{\text{cha}}\text{III}$ is the determinant of a matrix of tensor components: ${}_{\text{cha}}\text{III}({}^2\boldsymbol{B}) \equiv \det{}^2\boldsymbol{B}$.

This applies to all second complexity tensors. Besides that, in case of a symmetric tensor, the following is true:

1° The eigenvalues of a symmetric bivalent tensor are real numbers.

2° The characteristic axes (directions) for different eigenvalues are orthogonal to each other.

○ The first statement is proved by contradiction. If η is a complex root of (10.2) corresponding to eigenvector $\boldsymbol{a}$, then conjugate number $\overline{\eta}$ will also be the root of (10.2). Eigenvector $\overline{\boldsymbol{a}}$ with the conjugate components corresponds to it. And then

$$(10.1) \;\Rightarrow\; (\overline{\boldsymbol{a}}\,\boldsymbol{\cdot})\,{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a}=\eta\boldsymbol{a},\;\; (\boldsymbol{a}\,\boldsymbol{\cdot})\,{}^2\boldsymbol{B}\boldsymbol{\cdot}\overline{\boldsymbol{a}}=\overline{\eta}\,\overline{\boldsymbol{a}} \;\Rightarrow$$
$$\Rightarrow\;\; \overline{\boldsymbol{a}}\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a}-\boldsymbol{a}\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\overline{\boldsymbol{a}}=(\eta-\overline{\eta}\,)\boldsymbol{a}\boldsymbol{\cdot}\overline{\boldsymbol{a}}.$$

Here on the left is zero, because $\boldsymbol{a}\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{c}=\boldsymbol{c}\boldsymbol{\cdot}{}^2\boldsymbol{B}^{\mathsf{T}}\boldsymbol{\cdot}\boldsymbol{a}$ and ${}^2\boldsymbol{B}={}^2\boldsymbol{B}^{\mathsf{T}}$. Thence $\eta=\overline{\eta}$, that is a real number.

Just as simple looks the proof of 2°:

$$\underbrace{\boldsymbol{a}_2\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a}_1-\boldsymbol{a}_1\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a}_2}_{=0}=(\eta_1-\eta_2)\,\boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{a}_2,\;\eta_1\neq\eta_2\;\Rightarrow$$
$$\Rightarrow\;\boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{a}_2=0. \qquad \bullet$$

If the roots of the characteristic equation (the eigenvalues) are different, then the one unit long eigenvectors $\boldsymbol{æ}_i$ compose an orthonormal basis. What are the tensor components in such a basis?

$${}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{æ}_k=\cancel{\sum_k}\,\eta_k\boldsymbol{æ}_k,\;\; k=1,2,3$$
$${}^2\boldsymbol{B}\boldsymbol{\cdot}\underbrace{\boldsymbol{æ}_k\boldsymbol{æ}_k}_{\boldsymbol{E}}=\sum_k\eta_k\boldsymbol{æ}_k\boldsymbol{æ}_k$$

For a common case $B_{ij}=\boldsymbol{e}_i\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{e}_j$. In the basis $\boldsymbol{æ}_1$, $\boldsymbol{æ}_2$, $\boldsymbol{æ}_3$ of mutually perpendicular one unit long $\boldsymbol{æ}_i\boldsymbol{\cdot}\boldsymbol{æ}_j=\delta_{ij}$ eigenvectors of a symmetric tensor

$$B_{11}=\boldsymbol{æ}_1\boldsymbol{\cdot}(\eta_1\boldsymbol{æ}_1\boldsymbol{æ}_1+\eta_2\boldsymbol{æ}_2\boldsymbol{æ}_2+\eta_3\boldsymbol{æ}_3\boldsymbol{æ}_3)\boldsymbol{\cdot}\boldsymbol{æ}_1=\eta_1,$$
$$B_{12}=\boldsymbol{æ}_1\boldsymbol{\cdot}(\eta_1\boldsymbol{æ}_1\boldsymbol{æ}_1+\eta_2\boldsymbol{æ}_2\boldsymbol{æ}_2+\eta_3\boldsymbol{æ}_3\boldsymbol{æ}_3)\boldsymbol{\cdot}\boldsymbol{æ}_2=0,$$
$$\ldots$$

The matrix of components is diagonal and ${}^2\boldsymbol{B}=\sum_i\eta_i\boldsymbol{æ}_i\boldsymbol{æ}_i$. Yes, here is the summation over the three repeating indices, because the special basis is used.

The transition to a coincidence case of eigenvalues can be acquired via limit calculation. If $\eta_2\to\eta_1$, then any linear combination of

vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in the limit satisfies the equation (10.1). And then any axis in the plane $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ becomes characteristic.

When all the three eigenvalues coincide, then any axis in space is characteristic.

Then ${}^2\boldsymbol{B} = \eta \boldsymbol{E}$, such tensors are called isotropic or "spherical".

Collections of invariants of a symmetric bivalent tensor

....

The "algebraic" invariants

...

The "characteristic" invariants. These are coefficients of the characteristic equation (10.2) of the eigenvalues problem (10.1).

.....

The "research" invariants

...

The "harmonic" invariants

...

§ 11. Rotations via rotation tensors

The relation between two "right" (or two "left" orthonormal bases $\boldsymbol{e}_i$ and $\mathring{\boldsymbol{e}}_i$ can be described by a two-index array represented as a matrix (§ 2, § 6)

$$\boldsymbol{e}_i = \boldsymbol{e}_i \cdot \underbrace{\mathring{\boldsymbol{e}}_j \mathring{\boldsymbol{e}}_j}_{\boldsymbol{E}} = o_{ij} \mathring{\boldsymbol{e}}_j, \;\; o_{ij} \equiv \boldsymbol{e}_i \cdot \mathring{\boldsymbol{e}}_j$$

("a matrix of cosines").

Also, a rotation of tensor can be described by another tensor, called a rotation tensor $\boldsymbol{O}$

$$\boldsymbol{e}_i = \boldsymbol{e}_j \underbrace{\mathring{\boldsymbol{e}}_j \cdot \mathring{\boldsymbol{e}}_i}_{\delta_{ji}} = \boldsymbol{O} \cdot \mathring{\boldsymbol{e}}_i, \;\; \boldsymbol{O} \equiv \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j = \boldsymbol{e}_1 \mathring{\boldsymbol{e}}_1 + \boldsymbol{e}_2 \mathring{\boldsymbol{e}}_2 + \boldsymbol{e}_3 \mathring{\boldsymbol{e}}_3. \qquad (11.1)$$

Components of $\boldsymbol{O}$ both in an initial $\mathring{\boldsymbol{e}}_i$ and in a rotated $\boldsymbol{e}_i$ bases are the same

$$\begin{aligned} \boldsymbol{e}_i \cdot \boldsymbol{O} \cdot \boldsymbol{e}_j &= \underbrace{\boldsymbol{e}_i \cdot \boldsymbol{e}_k}_{\delta_{ik}} \mathring{\boldsymbol{e}}_k \cdot \boldsymbol{e}_j = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j, \\ \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{O} \cdot \mathring{\boldsymbol{e}}_j &= \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_k \underbrace{\mathring{\boldsymbol{e}}_k \cdot \mathring{\boldsymbol{e}}_j}_{\delta_{kj}} = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j. \end{aligned} \tag{11.2}$$

In the matrix notation, these components present the matrix of cosines $o_{ji}^{\circ} = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j$:

$$\boldsymbol{O} = o_{ji}^{\circ} \boldsymbol{e}_i \boldsymbol{e}_j = o_{ji}^{\circ} \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j.$$

Spatial transformations in the 3-dimensional Euclidean space $\mathbb{R}^3$ are distinguished into active or alibi transformations, and passive or alias transformations. An active transformation is a transformation which actually changes the physical position (alibi, elsewhere) of objects, which can be defined in the absence of a coordinate system; whereas a passive transformation is merely a change in the coordinate system in which the object is described (alias, other name) (change of coordinates, or change of basis). By transformation, math texts usually refer to active transformations.

Tensor $\boldsymbol{O}$ relates the two vectors — "before rotation" $\mathring{\boldsymbol{r}} = \rho_i \mathring{\boldsymbol{e}}_i$ and "after rotation" $\boldsymbol{r} = \rho_i \boldsymbol{e}_i$. Components $\rho_i = \mathsf{constant}$ of $\boldsymbol{r}$ in rotated basis $\boldsymbol{e}_i$ are the same as of $\mathring{\boldsymbol{r}}$ in immobile basis $\mathring{\boldsymbol{e}}_i$. So that the rotation tensor describes the rotation of the vector together with the basis. And since $\boldsymbol{e}_i = \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j \cdot \mathring{\boldsymbol{e}}_i \;\Leftrightarrow\; \rho_i \boldsymbol{e}_i = \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j \cdot \rho_i \mathring{\boldsymbol{e}}_i$, then

$$\boldsymbol{r} = \boldsymbol{O} \cdot \mathring{\boldsymbol{r}} \tag{11.3}$$

(this is the Rodrigues rotation formula).

Olinde Rodrigues. Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendants des causes qui peuvent les produire. *Journal de mathématiques pures et appliquées*, tome 5 (1840), pages 380–440.

For a second complexity tensor $\mathring{\boldsymbol{C}} = C_{ij} \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j$, a rotation into the current position $\boldsymbol{C} = C_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$ looks like

$$\boldsymbol{e}_i C_{ij} \boldsymbol{e}_j = \boldsymbol{e}_i \mathring{\boldsymbol{e}}_i \cdot \mathring{\boldsymbol{e}}_p C_{pq} \mathring{\boldsymbol{e}}_q \cdot \mathring{\boldsymbol{e}}_j \boldsymbol{e}_j \;\Leftrightarrow\; \boldsymbol{C} = \boldsymbol{O} \cdot \mathring{\boldsymbol{C}} \cdot \boldsymbol{O}^{\mathsf{T}}. \tag{11.4}$$

The essential property of a rotation tensor — the orthogonality — is expressed as

$$\underbrace{\boldsymbol{O}}_{\boldsymbol{e}_i\mathring{\boldsymbol{e}}_i}\cdot\underbrace{\boldsymbol{O}^{\mathsf{T}}}_{\mathring{\boldsymbol{e}}_j\boldsymbol{e}_j}=\underbrace{\boldsymbol{O}^{\mathsf{T}}}_{\mathring{\boldsymbol{e}}_i\boldsymbol{e}_i}\cdot\underbrace{\boldsymbol{O}}_{\boldsymbol{e}_j\mathring{\boldsymbol{e}}_j}=\underbrace{\boldsymbol{E}}_{\substack{\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_i\\ \boldsymbol{e}_i\boldsymbol{e}_i}}, \tag{11.5}$$

that is the transposed tensor coincides with the reciprocal tensor: $\boldsymbol{O}^{\mathsf{T}}=\boldsymbol{O}^{-1}\Leftrightarrow\boldsymbol{O}=\boldsymbol{O}^{-\mathsf{T}}$.

An orthogonal tensor retains lengths and angles (the metric) because it does not change the "$\cdot$"-product of vectors

$$(\boldsymbol{O}\cdot\boldsymbol{a})\cdot(\boldsymbol{O}\cdot\boldsymbol{b})=\boldsymbol{a}\cdot\boldsymbol{O}^{\mathsf{T}}\cdot\boldsymbol{O}\cdot\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{E}\cdot\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{b}. \tag{11.6}$$

For all orthogonal tensors $(\det\boldsymbol{Q})^2=1$:

$$1=\det\boldsymbol{E}=\det\left(\boldsymbol{Q}\cdot\boldsymbol{Q}^{\mathsf{T}}\right)=(\det\boldsymbol{Q})\left(\det\boldsymbol{Q}^{\mathsf{T}}\right)=(\det\boldsymbol{Q})^2.$$

A rotation tensor is an orthogonal tensor with $\det\boldsymbol{O}=+1$.

But not only rotation tensors possess the property of orthogonality. When in (11.1) the first basis is "left", and the second one is "right", then there's a combination of rotating and mirroring (a "rotoreflexion") $\boldsymbol{O}=-\boldsymbol{E}\cdot\boldsymbol{O}$ with $\det(-\boldsymbol{E}\cdot\boldsymbol{O})=-1$.

Any bivalent tensor in the three-dimensional (3D) space has at least one eigenvalue — the root of (10.2) — is non-complex, or real. For a rotation tensor, it is equal to one

$$\boldsymbol{O}\cdot\boldsymbol{a}=\eta\boldsymbol{a}\;\Rightarrow\;\overbrace{\boldsymbol{a}\cdot\underbrace{\boldsymbol{O}^{\mathsf{T}}\cdot\boldsymbol{O}}_{\boldsymbol{E}}}^{\boldsymbol{O}\cdot\boldsymbol{a}}\cdot\boldsymbol{a}=\eta\boldsymbol{a}\cdot\eta\boldsymbol{a}\;\Rightarrow\;\eta^2=1.$$

The corresponding eigenvector is called the axis of rotation. The Euler's theorem on finite rotation is just about that such an axis exists [http://eulerarchive.maa.org/docs/originals/E478.pdf]. If $\boldsymbol{k}$ is the unit vector of that axis, and ϑ is the finite angle of rotation, then the rotation tensor is representable as

$$\boldsymbol{O}(\boldsymbol{k},\vartheta)=\boldsymbol{E}\cos\vartheta+\boldsymbol{k}\times\boldsymbol{E}\sin\vartheta+\boldsymbol{k}\boldsymbol{k}(1-\cos\vartheta). \tag{11.7}$$

Доказывается эта формула так. Направление $\boldsymbol{k}$ во вре́мя поворота не меняется ($\boldsymbol{O}\cdot\boldsymbol{k}=\boldsymbol{k}$), поэтому на оси поворота $\mathring{\boldsymbol{e}}_3=\boldsymbol{e}_3=\boldsymbol{k}$. В перпендикулярной плоскости (рисунок 5) $\mathring{\boldsymbol{e}}_1=\boldsymbol{e}_1\cos\vartheta-\boldsymbol{e}_2\sin\vartheta$, $\mathring{\boldsymbol{e}}_2=\boldsymbol{e}_1\sin\vartheta+\boldsymbol{e}_2\cos\vartheta$, $\boldsymbol{O}=\boldsymbol{e}_i\mathring{\boldsymbol{e}}_i\ \Rightarrow$ (11.7).

Из (11.7) и (11.3) получаем формулу поворота Родрига в параметрах $\boldsymbol{k}$ и ϑ:

$$\boldsymbol{r}\ =\ \mathring{\boldsymbol{r}}\cos\vartheta\ +\ \boldsymbol{k}\times\mathring{\boldsymbol{r}}\sin\vartheta\ +\ \boldsymbol{k}\boldsymbol{k}\cdot\mathring{\boldsymbol{r}}\,(1-\cos\vartheta).$$

В параметрах конечного поворота транспонирование, оно же обращение, тензора $\boldsymbol{O}$ эквивалентно перемене направления поворота — знака угла ϑ

$$\boldsymbol{O}^{\mathsf{T}}=\boldsymbol{O}\big|_{\vartheta=-\vartheta}=\boldsymbol{E}\cos\vartheta-\boldsymbol{k}\times\boldsymbol{E}\sin\vartheta+\boldsymbol{k}\boldsymbol{k}\,(1-\cos\vartheta).$$

Пусть теперь тензор поворота меняется со временем: $\boldsymbol{O}=\boldsymbol{O}(t)$. Псевдовектор угловой скорости $\boldsymbol{\omega}$ вводится через тензор поворота $\boldsymbol{O}$ таким путём. Дифференцируем тождество ортогональности (11.5) по времени*

$$\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}}+\boldsymbol{O}\cdot\dot{\boldsymbol{O}}^{\mathsf{T}}={}^2\mathbf{0}.$$

По (4.9) $\left(\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}}\right)^{\mathsf{T}}=\boldsymbol{O}\cdot\dot{\boldsymbol{O}}^{\mathsf{T}}$, поэтому тензор $\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}}$ оказывается антисимметричным. Тогда согласно ($8.6^{A(a)}$) он представи́м через сопутствующий вектор как $\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}}=\boldsymbol{\omega}\times\boldsymbol{E}=\boldsymbol{\omega}\times\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}}$. То есть

$$\dot{\boldsymbol{O}}=\boldsymbol{\omega}\times\boldsymbol{O},\quad \boldsymbol{\omega}\equiv-\frac{1}{2}\left(\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}}\right)_{\times}\tag{11.8}$$

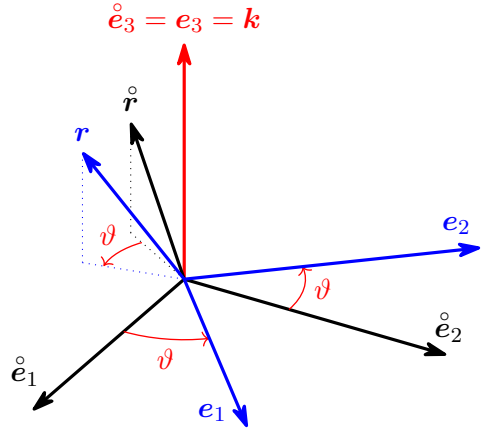
Besides this generic representation for the (pseudo)vector $\boldsymbol{\omega}$, there are others too. For example, via two parameters of the finite rotation.

Производная $\dot{\boldsymbol{O}}$ в параметрах конечного поворота в общем случае, когда оба параметра — и единичный вектор $\boldsymbol{k}$, и угол ϑ — переменны во времени :

* Various notations are used to designate the time derivative. In addition to the Leibniz's notation $^{dx}/_{dt}$, the very popular one is the "dot above" Newton's notation $\dot{x}$.

$$\mathring{\boldsymbol{e}}_i = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j \boldsymbol{e}_j$$

$$\begin{bmatrix} \mathring{\boldsymbol{e}}_1 \\ \mathring{\boldsymbol{e}}_2 \\ \mathring{\boldsymbol{e}}_3 \end{bmatrix} = \begin{bmatrix} \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{bmatrix}$$



$$\begin{bmatrix} \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & \cos(\pi/2 + \vartheta) & \cos\pi/2 \\ \cos(\pi/2 - \vartheta) & \cos\vartheta & \cos\pi/2 \\ \cos\pi/2 & \cos\pi/2 & \cos 0 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathring{\boldsymbol{e}}_1 = \boldsymbol{e}_1 \cos\vartheta - \boldsymbol{e}_2 \sin\vartheta$$
$$\mathring{\boldsymbol{e}}_2 = \boldsymbol{e}_1 \sin\vartheta + \boldsymbol{e}_2 \cos\vartheta$$
$$\mathring{\boldsymbol{e}}_3 = \boldsymbol{e}_3 = \boldsymbol{k}$$

$$\boldsymbol{O} = \boldsymbol{e}_1\mathring{\boldsymbol{e}}_1 + \boldsymbol{e}_2\mathring{\boldsymbol{e}}_2 + \boldsymbol{e}_3\mathring{\boldsymbol{e}}_3 =$$

$$= \overbrace{\boldsymbol{e}_1\boldsymbol{e}_1 \cos\vartheta - \boldsymbol{e}_1\boldsymbol{e}_2 \sin\vartheta}^{\boldsymbol{e}_1\mathring{\boldsymbol{e}}_1} + \overbrace{\boldsymbol{e}_2\boldsymbol{e}_1 \sin\vartheta + \boldsymbol{e}_2\boldsymbol{e}_2 \cos\vartheta}^{\boldsymbol{e}_2\mathring{\boldsymbol{e}}_2} + \overbrace{\boldsymbol{k}\boldsymbol{k}}^{\boldsymbol{e}_3\mathring{\boldsymbol{e}}_3} =$$

$$= \big(\boldsymbol{E}\cos\vartheta - \underbrace{\boldsymbol{e}_3\boldsymbol{e}_3}_{\boldsymbol{k}\boldsymbol{k}}\cos\vartheta\big) + \underbrace{\big(\boldsymbol{e}_2\boldsymbol{e}_1 - \boldsymbol{e}_1\boldsymbol{e}_2\big)}_{\boldsymbol{e}_3 \times \boldsymbol{e}_i\boldsymbol{e}_i = \in_{3ij}\boldsymbol{e}_j\boldsymbol{e}_i} \sin\vartheta + \boldsymbol{k}\boldsymbol{k} =$$

$$= \boldsymbol{E}\cos\vartheta + \boldsymbol{k} \times \boldsymbol{E}\sin\vartheta + \boldsymbol{k}\boldsymbol{k}\big(1 - \cos\vartheta\big)$$

рисунок 5
“Finite rotation”

$$\dot{\boldsymbol{O}} = \big(\boldsymbol{O}^{\mathsf{S}} + \boldsymbol{O}^{\mathsf{A}}\big)^{\bullet} = \big(\overbrace{\boldsymbol{E}\cos\vartheta + \boldsymbol{k}\boldsymbol{k}\,(1-\cos\vartheta)}^{\boldsymbol{O}^{\mathsf{S}}} + \overbrace{\boldsymbol{k}\times\boldsymbol{E}\sin\vartheta}^{\boldsymbol{O}^{\mathsf{A}}}\big)^{\bullet} =$$
$$= \underbrace{(\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}}+\dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta)}_{\dot{\boldsymbol{O}}^{\mathsf{S}}} +$$
$$+ \underbrace{\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\cos\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta}_{\dot{\boldsymbol{O}}^{\mathsf{A}}}.$$

Находим

$$\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}} = (\dot{\boldsymbol{O}}^{\mathsf{S}} + \dot{\boldsymbol{O}}^{\mathsf{A}})\cdot(\boldsymbol{O}^{\mathsf{S}} - \boldsymbol{O}^{\mathsf{A}}) =$$
$$= \dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{S}} + \dot{\boldsymbol{O}}^{\mathsf{A}}\cdot\boldsymbol{O}^{\mathsf{S}} - \dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{A}} - \dot{\boldsymbol{O}}^{\mathsf{A}}\cdot\boldsymbol{O}^{\mathsf{A}},$$

using

$$\boldsymbol{k}\cdot\boldsymbol{k} = 1 = \text{constant} \;\Rightarrow\; \boldsymbol{k}\cdot\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\cdot\boldsymbol{k} = 0 \;\Leftrightarrow\; \dot{\boldsymbol{k}}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\dot{\boldsymbol{k}} = 0,$$
$$\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{k} = \boldsymbol{k}\boldsymbol{k},\;\; \dot{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{k} = \dot{\boldsymbol{k}}\boldsymbol{k},\;\; \boldsymbol{k}\dot{\boldsymbol{k}}\cdot\boldsymbol{k}\boldsymbol{k} = {}^{2}\boldsymbol{0},$$
$$(\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\cdot\boldsymbol{k} = \boldsymbol{k}-\boldsymbol{k} = \boldsymbol{0},\;\; (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k} = \boldsymbol{k}\boldsymbol{k}-\boldsymbol{k}\boldsymbol{k} = {}^{2}\boldsymbol{0},$$
$$\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{E}) = (\boldsymbol{k}\times\boldsymbol{E})\cdot\boldsymbol{k} = \boldsymbol{k}\times\boldsymbol{k} = \boldsymbol{0},\;\; \boldsymbol{k}\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{E}) = (\boldsymbol{k}\times\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k} = {}^{2}\boldsymbol{0},$$
$$(\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E}) = -\boldsymbol{k}\times\boldsymbol{E},$$
$$(\boldsymbol{a}\times\boldsymbol{E})\cdot\boldsymbol{b} = \boldsymbol{a}\times(\boldsymbol{E}\cdot\boldsymbol{b}) = \boldsymbol{a}\times\boldsymbol{b} \;\Rightarrow\; (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k} = \dot{\boldsymbol{k}}\times\boldsymbol{k}\boldsymbol{k},$$

$$(7.14) \;\Rightarrow\; (\boldsymbol{k}\times\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E}) = \boldsymbol{k}\boldsymbol{k}-\boldsymbol{E},\;\; (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E}) = \boldsymbol{k}\dot{\boldsymbol{k}} - \cancel{\dot{\boldsymbol{k}}\cdot\boldsymbol{k}\,\boldsymbol{E}},$$
$$(7.13) \;\Rightarrow\; \dot{\boldsymbol{k}}\boldsymbol{k} - \boldsymbol{k}\dot{\boldsymbol{k}} = (\boldsymbol{k}\times\dot{\boldsymbol{k}})\times\boldsymbol{E},\;\; (\dot{\boldsymbol{k}}\times\boldsymbol{k})\boldsymbol{k} - \boldsymbol{k}(\dot{\boldsymbol{k}}\times\boldsymbol{k}) = \boldsymbol{k}\times(\dot{\boldsymbol{k}}\times\boldsymbol{k})\times\boldsymbol{E}$$

$$\dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{S}} =$$
$$= (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cdot\boldsymbol{E}\cos\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}}+\dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta)\cdot\boldsymbol{E}\cos\vartheta +$$
$$+ \cancel{(\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cdot\boldsymbol{k}\boldsymbol{k}\,(1-\cos\vartheta)} + (\boldsymbol{k}\dot{\boldsymbol{k}}+\dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta)\cdot\boldsymbol{k}\boldsymbol{k}\,(1-\cos\vartheta) =$$
$$= (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cos\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}}+\dot{\boldsymbol{k}}\boldsymbol{k})\cos\vartheta\,(1-\cos\vartheta) + \cancel{(\boldsymbol{k}\dot{\boldsymbol{k}}\cdot\boldsymbol{k}\boldsymbol{k} + \dot{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{k})\,(1-\cos\vartheta)^2} =$$
$$= (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta\,(1-\cos\vartheta) +$$
$$+ \dot{\boldsymbol{k}}\boldsymbol{k}\cos\vartheta - \dot{\boldsymbol{k}}\boldsymbol{k}\cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k} - 2\,\dot{\boldsymbol{k}}\boldsymbol{k}\cos\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\cos^2\vartheta =$$
$$= (\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}}\cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\,(1-\cos\vartheta),$$

$$\dot{\boldsymbol{O}}^{\mathsf{A}}\cdot\boldsymbol{O}^{\mathsf{S}} =$$
$$= (\boldsymbol{k}\times\boldsymbol{E})\cdot\boldsymbol{E}\,\dot{\vartheta}\cos^2\vartheta + (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot\boldsymbol{E}\sin\vartheta\cos\vartheta +$$
$$+ \cancel{(\boldsymbol{k}\times\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k}\,\dot{\vartheta}\cos\vartheta\,(1-\cos\vartheta)} + (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta) =$$
$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\cos^2\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{k}\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta),$$

$$\dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{A}}=$$
$$=(\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\dot{\vartheta}\sin\vartheta\cdot(\boldsymbol{k}\times\boldsymbol{E})\sin\vartheta+(\boldsymbol{k}\dot{\boldsymbol{k}}+\dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta)\cdot(\boldsymbol{k}\times\boldsymbol{E})\sin\vartheta=$$
$$=\cancel{\boldsymbol{k}\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{E})\dot{\vartheta}\sin^2\vartheta}-\boldsymbol{E}\cdot(\boldsymbol{k}\times\boldsymbol{E})\dot{\vartheta}\sin^2\vartheta+\Big(\boldsymbol{k}\dot{\boldsymbol{k}}\cdot(\boldsymbol{k}\times\boldsymbol{E})+\cancel{\dot{\boldsymbol{k}}\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{E})}\Big)\sin\vartheta(1-\cos\vartheta)=$$
$$=-\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\sin^2\vartheta+\boldsymbol{k}\dot{\boldsymbol{k}}\times\boldsymbol{k}\sin\vartheta(1-\cos\vartheta),$$

$$\dot{\boldsymbol{O}}^{\mathsf{A}}\cdot\boldsymbol{O}^{\mathsf{A}}=(\boldsymbol{k}\times\boldsymbol{E})\dot{\vartheta}\cos\vartheta\cdot(\boldsymbol{k}\times\boldsymbol{E})\sin\vartheta+(\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E})\sin^2\vartheta=$$
$$=(\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\dot{\vartheta}\sin\vartheta\cos\vartheta+\boldsymbol{k}\dot{\boldsymbol{k}}\sin^2\vartheta;$$

$$\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}}=\dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{S}}+\dot{\boldsymbol{O}}^{\mathsf{A}}\cdot\boldsymbol{O}^{\mathsf{S}}-\dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{A}}-\dot{\boldsymbol{O}}^{\mathsf{A}}\cdot\boldsymbol{O}^{\mathsf{A}}=$$
$$=(\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\dot{\vartheta}\sin\vartheta\cos\vartheta+\boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta-\boldsymbol{k}\dot{\boldsymbol{k}}\cos^2\vartheta+\dot{\boldsymbol{k}}\boldsymbol{k}(1-\cos\vartheta)+$$
$$+\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\cos^2\vartheta+\dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta+\dot{\boldsymbol{k}}\times\boldsymbol{k}\boldsymbol{k}\sin\vartheta(1-\cos\vartheta)+$$
$$+\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\sin^2\vartheta-\boldsymbol{k}\dot{\boldsymbol{k}}\times\boldsymbol{k}\sin\vartheta(1-\cos\vartheta)-(\boldsymbol{k}\boldsymbol{k}-\boldsymbol{E})\dot{\vartheta}\sin\vartheta\cos\vartheta-\boldsymbol{k}\dot{\boldsymbol{k}}\sin^2\vartheta=$$
$$=\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}+(\dot{\boldsymbol{k}}\boldsymbol{k}-\boldsymbol{k}\dot{\boldsymbol{k}})(1-\cos\vartheta)+\dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta+(\dot{\boldsymbol{k}}\times\boldsymbol{k}\boldsymbol{k}-\boldsymbol{k}\dot{\boldsymbol{k}}\times\boldsymbol{k})\sin\vartheta(1-\cos\vartheta)=$$
$$=\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}+\boldsymbol{k}\times\dot{\boldsymbol{k}}\times\boldsymbol{E}(1-\cos\vartheta)+\dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta+\boldsymbol{k}\times(\dot{\boldsymbol{k}}\times\boldsymbol{k})\times\boldsymbol{E}\sin\vartheta(1-\cos\vartheta)=$$
$$=\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}+\dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta+(\dot{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{k}-\dot{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{k})\times\boldsymbol{E}\sin\vartheta(1-\cos\vartheta)+\boldsymbol{k}\times\dot{\boldsymbol{k}}\times\boldsymbol{E}(1-\cos\vartheta)=$$
$$=\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}+\dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta+\boldsymbol{k}\times\dot{\boldsymbol{k}}\times\boldsymbol{E}(1-\cos\vartheta).$$

Этот результат, подставленный в определение (11.8) псевдовектора $\boldsymbol{\omega}$, даёт

$$\boldsymbol{\omega}=\boldsymbol{k}\dot{\vartheta}+\dot{\boldsymbol{k}}\sin\vartheta+\boldsymbol{k}\times\dot{\boldsymbol{k}}(1-\cos\vartheta).\qquad(11.9)$$

Вектор $\boldsymbol{\omega}$ получился разложенным по трём взаимно ортогональным направлениям — $\boldsymbol{k}$, $\dot{\boldsymbol{k}}$ и $\boldsymbol{k}\times\dot{\boldsymbol{k}}$. При неподвижной оси поворота $\dot{\boldsymbol{k}}=\boldsymbol{0}\;\Rightarrow\;\boldsymbol{\omega}=\boldsymbol{k}\dot{\vartheta}$.

Ещё одно представление $\boldsymbol{\omega}$ связано с компонентами тензора поворота (11.2). Поскольку $\boldsymbol{O}=o_{ji}\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j$, $\boldsymbol{O}^{\mathsf{T}}=o_{ij}\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j$, а векторы начального базиса $\mathring{\boldsymbol{e}}_i$ неподвижны (со временем не меняются), то

$$\dot{\boldsymbol{O}}=\dot{o}_{ji}\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j,\;\;\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}}=\dot{o}_{ni}o_{nj}\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j,$$

$$\boldsymbol{\omega}=-\frac{1}{2}\dot{o}_{ni}o_{nj}\mathring{\boldsymbol{e}}_i\times\mathring{\boldsymbol{e}}_j=\frac{1}{2}\in_{jik}o_{nj}\dot{o}_{ni}\mathring{\boldsymbol{e}}_k.\qquad(11.10)$$

Отметим и формулы

$$\begin{aligned} (11.8) \;&\Rightarrow\; \dot{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \mathring{\boldsymbol{e}}_i \;\Rightarrow\; \dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i, \\ (11.8) \;&\Rightarrow\; \boldsymbol{\omega} = -\tfrac{1}{2}\left(\dot{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_i \cdot \mathring{\boldsymbol{e}}_j \boldsymbol{e}_j\right)_\times = -\tfrac{1}{2}\left(\dot{\boldsymbol{e}}_i \boldsymbol{e}_i\right)_\times = \tfrac{1}{2}\, \boldsymbol{e}_i \times \dot{\boldsymbol{e}}_i. \end{aligned} \tag{11.11}$$

...

Варьируя тождество (11.5), получим $\delta\boldsymbol{O} \cdot \boldsymbol{O}^\mathsf{T} = -\boldsymbol{O} \cdot \delta\boldsymbol{O}^\mathsf{T}$. Этот тензор антисимметричен, и потому выражается через свой сопутствующий вектор $\boldsymbol{\delta o}$ как $\delta\boldsymbol{O} \cdot \boldsymbol{O}^\mathsf{T} = \boldsymbol{\delta o} \times \boldsymbol{E}$. Приходим к соотношениям

$$\delta\boldsymbol{O} = \boldsymbol{\delta o} \times \boldsymbol{O}, \quad \boldsymbol{\delta o} = -\frac{1}{2}\Big(\delta\boldsymbol{O} \cdot \boldsymbol{O}^\mathsf{T}\Big)_\times, \tag{11.12}$$

аналогичным (11.8). Вектор бесконечно малого поворота $\boldsymbol{\delta o}$ это не "вариация $\mathbf{o}$", но единый символ (в отличие от $\delta\boldsymbol{O}$).

Малый поворот определяется вектором $\boldsymbol{\delta o}$, но и конечный поворот тоже возможно представить как вектор.

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§ 12. Rotations via quaternions

The other way to describe a rotation in 3-dimensional space is using quaternions. Meanwhile, they are very popular for computer graphics.

Quaternions were invented by William Rowan Hamilton in 1843*.

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Complex numbers

$$p, q \in \mathbb{C}$$
$$p = e^{i\varphi} = \cos\varphi + i\sin\varphi$$
$$q = e^{i\psi} = \cos\psi + i\sin\psi$$

* *On Quaternions; or on a new System of Imaginaries in Algebra* by **William Rowan Hamilton** appeared in 18 publications in 'The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science', volumes xxv–xxxvi, 3rd series, 1844–1850.

the composition of two rotations

$$pq = e^{i\varphi}e^{i\psi} = e^{i(\varphi+\psi)}$$

$$\begin{aligned} e^{i(\varphi+\psi)} &= \cos(\varphi+\psi) + i\sin(\varphi+\psi) \\ e^{i\varphi}e^{i\psi} &= (\cos\varphi + i\sin\varphi)(\cos\psi + i\sin\psi) \\ &= (\cos\varphi\cos\psi - \sin\varphi\sin\psi) \\ &\qquad + i(\sin\varphi\cos\psi + \cos\varphi\sin\psi) \end{aligned}$$

from there also follows

$$\begin{aligned} \cos(\varphi+\psi) &= \cos\varphi\cos\psi - \sin\varphi\sin\psi \\ \sin(\varphi+\psi) &= \sin\varphi\cos\psi + \cos\varphi\sin\psi \end{aligned}$$

If the reader doubts that the multiplication of two exponentiations $e^{i\varphi}e^{i\psi}$ and the exponentiation of the sum of arguments $e^{i(\varphi+\psi)}$ are equal

$$e^{i\varphi}e^{i\psi} = e^{i(\varphi+\psi)},$$

then here's quite convincing proof of that using the linearity of differentiation.

○ At first, I'll define what an exponentiation e^w, sometimes written as $\exp(w)$, is (w for "whatever it is", something of absolutely any kind).

e^w is such and only such function whose differential $d(e^w)$ is $e^w dw$. When for some function $\overset{\text{omnino}}{\text{aliquid}} \mapsto f$ the differential df equals to $fd\left(\overset{\text{omnino}}{\text{aliquid}}\right)$, then such a function is an exponentiation — $\exp\left(\overset{\text{omnino}}{\text{aliquid}}\right)$.

The differential of a product of functions can be found as

$$d(f_1 f_2) = (df_1)f_2 + f_1(df_2)$$

— the (Leibniz') "product rule". For two exponentiations (the imaginary unit i is constant, thus $di = 0$ and $\forall\xi\ d(i\xi) = (di)\xi + i(d\xi) = id\xi$)

$$\begin{aligned} d\left(e^{i\varphi}e^{i\psi}\right) &= \left(de^{i\varphi}\right)e^{i\psi} + e^{i\varphi}\left(de^{i\psi}\right) \\ &= e^{i\varphi}d(i\varphi)e^{i\psi} + e^{i\varphi}e^{i\psi}d(i\psi) = e^{i\varphi}e^{i\psi}i\left(d\varphi + d\psi\right). \end{aligned}$$

For an exponentiation of the sum of arguments, the differential is

$$d\left(e^{i(\varphi+\psi)}\right) = e^{i(\varphi+\psi)}\,i\,d\left(\varphi+\psi\right).$$

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All rotations are a “group” in the sense of the group theory in abstract algebra.

Évariste Galois

permutations are group too

A *group* $\{\mathcal{G}, \text{“}\circ\text{”}\}$ is a set $\mathcal{G}$ with a binary operation “$\circ$”, where *(i)* “$\circ$” is associative, $(h\circ g)\circ f = h\circ(g\circ f)$ for any $h, g, f \in \mathcal{G}$, *(ii)* $\mathcal{G}$ has a two-sided identity element, $\exists\, e\in\mathcal{G}\colon g\circ e = e\circ g = g\ \forall g\in\mathcal{G}$, and *(iii)* every element of $\mathcal{G}$ has an inverse element, $\forall g\in\mathcal{G}\ \exists g^{-1}\colon g\circ g^{-1} = g^{-1}\circ g = e$.

Group operation “$\circ$” can be commutative or non-commutative. A group with a commutative operation is called commutative group, or abelian group. If there exists at least one pair of elements $a, b \in \mathcal{G}$ for which $a\circ b \neq b\circ a$, then this group is non-abelian (non-commutative).

A *field* $\{\mathcal{F}, \text{“}+\text{”}, \text{“}*\text{”}\}$ is a set $\mathcal{F}$ with two binary operations “$+$” and “$*$”, when *(i)* $\{\mathcal{F}, \text{“}+\text{”}\}$ is commutative group, $a+b=b+a$ $\forall a,b\in\mathcal{F}$, with “additive” identity $e_0\in\mathcal{F}$, *(ii)* $\{\mathcal{F}\backslash e_0, \text{“}*\text{”}\}$ is commutative group, $a*b=b*a\ \forall a,b\in\mathcal{F}\backslash e_0$, with “multiplicative” identity $e_1\in\mathcal{F}\backslash e_0$ $(e_1\neq e_0)$, $a,b\in\mathcal{F}\backslash e_0 \Rightarrow a*b\neq e_0$ for $\forall a\neq e_0$ and $\forall b\neq e_0$ *(iii)* .....

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Two imaginary units $j \neq i$

$$(a+bi)(c+dj) = ac + cbi + adj + bdij$$

By introducing third imaginary unit $k\colon\ ij = k$

$$i^2 = j^2 = k^2 = ijk = -1\,. \tag{12.1}$$

and making multiplication not commutative but anticommutative

$$ij = k = -ji,\quad jk = i = -kj,\quad ki = j = -ik, \tag{12.2}$$

meet quaternions.

In fact there's only two different imaginary units, and a third one is there just for short notation ($k = ij$), say they're only i and j, $i^2 = j^2 = -1$, $i \neq j$, and a quaternion is $q = a + bi + cj + dij$, with the really-easy-to-memorize equalities for multiplication $jij = i$, $iji = j$ and obvious $ijj = -i$, $iij = -j$, from there follows the anticommutativity itself, $ji = -ij$.

The following are absolutely the same, just a result of denoting $k = ij$, $ijij = ijk = k^2$ and the "really-easy-to-memorize" are derivable as well $ijij = (iji)j = j^2 = i(jij) = i^2$.

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§ 13. Variations

Further in this book pretty often will be used the operation of varying. It is similar to the differentiation.

Variations are seen as infinitesimal displacements, compatible with the constraints. If there are no constraints for variable x, then variations δx are completely random. But when

$$y \mapsto x : \ x = x(y)$$

is a function of some independent argument y, then

$$\delta x = x'(y)\delta y.$$

Variations are similar to differentials. As example, for δx and δy as variations of x and y, with finite u and v, $u\delta x + v\delta y = \delta w$ is written even when δw is not a variation of w. In such a case, δw is a single symbol.

Surely if $u = u(x, y)$, $v = v(x, y)$ and $\partial_x v = \partial_y u$ ($\frac{\partial}{\partial x} v = \frac{\partial}{\partial y} u$), then the sum $\delta w = u\delta x + v\delta y$ will be a variation of some w.

Varying the identity (11.5), we get

$$\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O} \cdot \delta \boldsymbol{O}^{\mathsf{T}}.$$

This tensor is antisymmetric, and thus is representable via its companion pseudovector $\boldsymbol{\delta o}$ as

$$\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\delta o} \times \boldsymbol{E}.$$

We have the following relations

$$\delta \boldsymbol{O} = \boldsymbol{\delta o} \times \boldsymbol{O}, \quad \boldsymbol{\delta o} = -\frac{1}{2}\Big(\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}}\Big)_{\times}, \tag{13.1}$$

similar to (11.8). Vector $\boldsymbol{\delta o}$ of an infinitesimal rotation is not "a variation of $\mathbf{o}$", but a single symbol.

An infinitesimal rotation is defined by vector $\boldsymbol{\delta o}$, but a finite rotation is also possible to represent as a vector

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§ 14. Polar decomposition

Any tensor of the second complexity $\boldsymbol{F}$ with $\det F_{ij} \neq 0$, that is a not singular tensor, can be decomposed as

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Example. Polar decompose tensor $\boldsymbol{C} = C_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$, where $\boldsymbol{e}_k$ are mutually perpendicular unit vectors of basis, and C_{ij} are tensor's components

$$C_{ij} = \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix}$$

$$\boldsymbol{O} = O_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = \boldsymbol{O}_1 \cdot \boldsymbol{O}_2$$

$$O_{ij} = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$\boldsymbol{C} = \boldsymbol{O} \cdot \boldsymbol{S}_{\mathsf{R}}, \;\; \boldsymbol{O}^{\mathsf{T}} \cdot \boldsymbol{C} = \boldsymbol{S}_{\mathsf{R}}$$

$$\boldsymbol{C} = \boldsymbol{S}_{\mathsf{L}} \cdot \boldsymbol{O}, \;\; \boldsymbol{C} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{S}_{\mathsf{L}}$$

$$S_{\mathsf{R}ij} = \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix}$$

$$S_{\mathsf{L}ij} = \begin{bmatrix} 104/5 & 47/5 & 5 \\ 47/5 & -129/5 & -10 \\ 5 & -10 & 3 \end{bmatrix}$$

...

§ 15. In the oblique basis

Until now, a basis of the three mutually perpendicular unit vectors $\boldsymbol{e}_i$ was used. However, such a basis is not the only possible one. Any linearly independent (non-coplanar) vectors $\boldsymbol{a}_i$ can be chosen as the basis ones.

Then, some vector $\boldsymbol{v}$ can be represented as a linear combination of basis vectors $\boldsymbol{a}_i$ with scalar multipliers v^i

$$\boldsymbol{v} = v^i \boldsymbol{a}_i. \qquad (15.1)$$

The summation convention gains the new conditions: a summation index is repeated at the different levels of the same monomial, and a free index stays at the same height in any part of the expression ($a_i = b_{ij}c^j$ is correct, $a_i = b^i_{kk}$ is wrong twice).

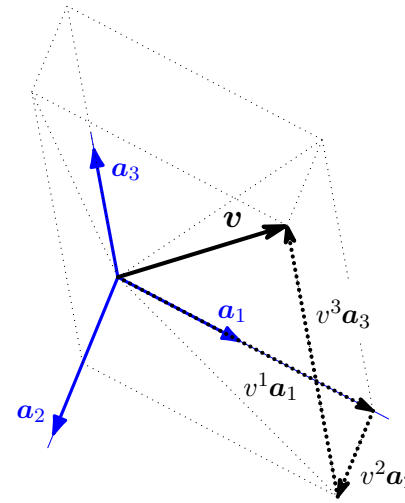


figure 6

Here the "•"-product $\boldsymbol{v} \bullet \boldsymbol{a}_i$ is no longer equal to the component v^i: $\boldsymbol{v} \bullet \boldsymbol{a}_i = v^k \boldsymbol{a}_k \bullet \boldsymbol{a}_i \neq v^i$, since $\boldsymbol{a}_i \bullet \boldsymbol{a}_k \neq \delta_{ik}$.

Fortunately, for any $\boldsymbol{a}_i$ there's another — the "superscript" — triple of vectors $\boldsymbol{a}^i$, such that

$$\begin{gathered} \boldsymbol{a}_i \bullet \boldsymbol{a}^j = \delta_i^j, \;\; \boldsymbol{a}^i \bullet \boldsymbol{a}_j = \delta_j^i, \\ \boldsymbol{E} = \boldsymbol{a}^i \boldsymbol{a}_i = \boldsymbol{a}_i \boldsymbol{a}^i. \end{gathered} \qquad (15.2)$$

The triple $\boldsymbol{a}^i$ is called cobasis or reciprocal (dual) basis. (15.2) is the defining property of the cobasis.

The case when a basis coincides with its cobasis $\boldsymbol{e}^i = \boldsymbol{e}_i$ is possible only when the basis vectors are mutually perpendicular to one another and all are one unit long. Such a basis is called "cartesian".

To get, for example, the first cobasis vector $\boldsymbol{a}^1$

$$\begin{cases} \boldsymbol{a}^1 \cdot \boldsymbol{a}_1 = 1 \\ \boldsymbol{a}^1 \cdot \boldsymbol{a}_2 = 0 \\ \boldsymbol{a}^1 \cdot \boldsymbol{a}_3 = 0 \end{cases} \Rightarrow \begin{cases} \boldsymbol{a}^1 \cdot \boldsymbol{a}_1 = 1 \\ \gamma \boldsymbol{a}^1 = \boldsymbol{a}_2 \times \boldsymbol{a}_3 \end{cases} \Rightarrow \begin{cases} \boldsymbol{a}^1 = {}^1\!/\gamma\, \boldsymbol{a}_2 \times \boldsymbol{a}_3 \\ \gamma = \boldsymbol{a}_2 \times \boldsymbol{a}_3 \cdot \boldsymbol{a}_1 \end{cases}$$

Here, the coefficient γ turned out to be equal (up to a sign) to the volume of parallelepiped built on vectors $\boldsymbol{a}_i$. The coincidence that in § 7 the same volume was presented as $\sqrt{g}$ is not accidental. Because γ equals to the square root of the gramian $g \equiv \det g_{ij}$ — the determinant of the symmetric J. P. Gram matrix $g_{ij} \equiv \boldsymbol{a}_i \cdot \boldsymbol{a}_j$.

○ The proof resembles the derivation of (6.3). The "triple product" $\boldsymbol{a}_i \times \boldsymbol{a}_j \cdot \boldsymbol{a}_k$ in some orthonormal basis $\boldsymbol{e}_i$ can be calculated as the determinant (with "−" for a "left" triplet $\boldsymbol{a}_i$) by the rows

$$\in_{ijk} \equiv \boldsymbol{a}_i \times \boldsymbol{a}_j \cdot \boldsymbol{a}_k = \pm \det \begin{bmatrix} \boldsymbol{a}_i \cdot \boldsymbol{e}_1 & \boldsymbol{a}_i \cdot \boldsymbol{e}_2 & \boldsymbol{a}_i \cdot \boldsymbol{e}_3 \\ \boldsymbol{a}_j \cdot \boldsymbol{e}_1 & \boldsymbol{a}_j \cdot \boldsymbol{e}_2 & \boldsymbol{a}_j \cdot \boldsymbol{e}_3 \\ \boldsymbol{a}_k \cdot \boldsymbol{e}_1 & \boldsymbol{a}_k \cdot \boldsymbol{e}_2 & \boldsymbol{a}_k \cdot \boldsymbol{e}_3 \end{bmatrix}$$

or by the columns

$$\in_{pqr} \equiv \boldsymbol{a}_p \times \boldsymbol{a}_q \cdot \boldsymbol{a}_r = \pm \det \begin{bmatrix} \boldsymbol{a}_p \cdot \boldsymbol{e}_1 & \boldsymbol{a}_q \cdot \boldsymbol{e}_1 & \boldsymbol{a}_r \cdot \boldsymbol{e}_1 \\ \boldsymbol{a}_p \cdot \boldsymbol{e}_2 & \boldsymbol{a}_q \cdot \boldsymbol{e}_2 & \boldsymbol{a}_r \cdot \boldsymbol{e}_2 \\ \boldsymbol{a}_p \cdot \boldsymbol{e}_3 & \boldsymbol{a}_q \cdot \boldsymbol{e}_3 & \boldsymbol{a}_r \cdot \boldsymbol{e}_3 \end{bmatrix}.$$

As proven in (6.2), the determinant of the matrix product of matrices is equal to the multiplication of the determinants of each of these matrices, here $\in_{ijk} \in_{pqr}$. The elements of the matrix product are the sums like $\boldsymbol{a}_i \cdot \boldsymbol{e}_s \boldsymbol{a}_p \cdot \boldsymbol{e}_s = \boldsymbol{a}_i \cdot \boldsymbol{e}_s \boldsymbol{e}_s \cdot \boldsymbol{a}_p = \boldsymbol{a}_i \cdot \boldsymbol{E} \cdot \boldsymbol{a}_p = \boldsymbol{a}_i \cdot \boldsymbol{a}_p$, therefore

$$\in_{ijk} \in_{pqr} = \det \begin{bmatrix} \boldsymbol{a}_i \cdot \boldsymbol{a}_p & \boldsymbol{a}_i \cdot \boldsymbol{a}_q & \boldsymbol{a}_i \cdot \boldsymbol{a}_r \\ \boldsymbol{a}_j \cdot \boldsymbol{a}_p & \boldsymbol{a}_j \cdot \boldsymbol{a}_q & \boldsymbol{a}_j \cdot \boldsymbol{a}_r \\ \boldsymbol{a}_k \cdot \boldsymbol{a}_p & \boldsymbol{a}_k \cdot \boldsymbol{a}_q & \boldsymbol{a}_k \cdot \boldsymbol{a}_r \end{bmatrix};$$

$i{=}p{=}1,\ j{=}q{=}2,\ k{=}r{=}3 \ \Rightarrow \in_{123} \in_{123} = \det\limits_{i,j} (\boldsymbol{a}_i \cdot \boldsymbol{a}_j) = \det\limits_{i,j} g_{ij}$. ●

Representing $\boldsymbol{a}^1$ and the other cobasis vectors as the sum

$$\pm 2\sqrt{g}\, \boldsymbol{a}^1 = \boldsymbol{a}_2 \times \boldsymbol{a}_3 - \boldsymbol{a}_3 \times \boldsymbol{a}_2,$$

приходим к общей формуле (with "–" for "левой" тройки $\boldsymbol{a}_i$)

$$\boldsymbol{a}^i = \pm\frac{1}{2\sqrt{g}}\, e^{ijk}\boldsymbol{a}_j \times \boldsymbol{a}_k,\ \ \sqrt{g} \equiv \pm\,\boldsymbol{a}_1 \times \boldsymbol{a}_2 \bullet \boldsymbol{a}_3 > 0. \tag{15.3}$$

Here e^{ijk} по-прежнему символы чётности перестановки (±1 or 0): $e^{ijk} \equiv e_{ijk}$. Произведение $\boldsymbol{a}_j \times \boldsymbol{a}_k = \in_{jkn} \boldsymbol{a}^n$, компоненты тензора Ле́ви-Чиви́ты $\in_{jkn} = \pm\, e_{jkn}\sqrt{g}$, and by (6.4) $e^{ijk}e_{jkn} = 2\delta^i_n$. Thus

$$\boldsymbol{a}^1 = \pm 1/\sqrt{g}\,(\boldsymbol{a}_2 \times \boldsymbol{a}_3),\ \ \boldsymbol{a}^2 = \pm 1/\sqrt{g}\,(\boldsymbol{a}_3 \times \boldsymbol{a}_1),\ \ \boldsymbol{a}^3 = \pm 1/\sqrt{g}\,(\boldsymbol{a}_1 \times \boldsymbol{a}_2).$$

Example. Get cobasis for basis $\boldsymbol{a}_i$ when

$$\begin{aligned}\boldsymbol{a}_1 &= \boldsymbol{e}_1 + \boldsymbol{e}_2,\\ \boldsymbol{a}_2 &= \boldsymbol{e}_1 + \boldsymbol{e}_3,\\ \boldsymbol{a}_3 &= \boldsymbol{e}_2 + \boldsymbol{e}_3.\end{aligned}$$

$$\sqrt{g} = -\,\boldsymbol{a}_1 \times \boldsymbol{a}_2 \bullet \boldsymbol{a}_3 = -\det\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix} = 2;$$

$$-\,\boldsymbol{a}_2 \times \boldsymbol{a}_3 = \det\begin{bmatrix}1 & \boldsymbol{e}_1 & 0\\ 0 & \boldsymbol{e}_2 & 1\\ 1 & \boldsymbol{e}_3 & 1\end{bmatrix} = \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3,$$

$$-\,\boldsymbol{a}_3 \times \boldsymbol{a}_1 = \det\begin{bmatrix}0 & \boldsymbol{e}_1 & 1\\ 1 & \boldsymbol{e}_2 & 1\\ 1 & \boldsymbol{e}_3 & 0\end{bmatrix} = \boldsymbol{e}_1 + \boldsymbol{e}_3 - \boldsymbol{e}_2,$$

$$-\,\boldsymbol{a}_1 \times \boldsymbol{a}_2 = \det\begin{bmatrix}1 & \boldsymbol{e}_1 & 1\\ 1 & \boldsymbol{e}_2 & 0\\ 0 & \boldsymbol{e}_3 & 1\end{bmatrix} = \boldsymbol{e}_2 + \boldsymbol{e}_3 - \boldsymbol{e}_1$$

and finally

$$\begin{aligned}\boldsymbol{a}^1 &= \tfrac{1}{2}\big(\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\big),\\ \boldsymbol{a}^2 &= \tfrac{1}{2}\big(\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3\big),\\ \boldsymbol{a}^3 &= \tfrac{1}{2}\big(-\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3\big).\end{aligned}$$

Имея кобазис, возможно не только разложить по нему любой вектор (рисунок 7), но и найти коэффициенты разложения (15.1):

$$\begin{aligned}\boldsymbol{v} &= v^i\boldsymbol{a}_i = v_i\boldsymbol{a}^i,\ \boldsymbol{a}_2\times\boldsymbol{a}_3\\ \boldsymbol{v}\bullet\boldsymbol{a}^i &= v^k\boldsymbol{a}_k\bullet\boldsymbol{a}^i = v^i,\ \overbrace{\ v_i = \boldsymbol{v}\bullet\boldsymbol{a}_i}.\end{aligned} \tag{15.4}$$

Коэффициенты v_i называются ковариантными компонентами вектора $\boldsymbol{v}$, а v^i — его контравариантными* компонентами.

Есть литература о тензорах, где introducing existænce and различают ковариантные и контравариантные... векторы (and "covectors", "dual vectors"). Не сто́ит вводить читателя в заблуждение: вектор-то один и тот же, просто разложение по двум разным базисам даёт два набора компонент.

От векторов перейдём к тензорам второй сложности. Имеем четыре комплекта диад: $\boldsymbol{a}_i\boldsymbol{a}_j$, $\boldsymbol{a}^i\boldsymbol{a}^j$, $\boldsymbol{a}_i\boldsymbol{a}^j$, $\boldsymbol{a}^i\boldsymbol{a}_j$. Согласующиеся коэффициенты в декомпозиции тензора называются его контравариантными, ковариантными и смешанными компонентами:

$$\begin{gathered}{}^2\boldsymbol{B} = B^{ij}\boldsymbol{a}_i\boldsymbol{a}_j = B_{ij}\boldsymbol{a}^i\boldsymbol{a}^j = B^i_{\cdot j}\boldsymbol{a}_i\boldsymbol{a}^j = B^{\cdot j}_i\boldsymbol{a}^i\boldsymbol{a}_j,\\ B^{ij} = \boldsymbol{a}^i\cdot{}^2\boldsymbol{B}\cdot\boldsymbol{a}^j,\ \ B_{ij} = \boldsymbol{a}_i\cdot{}^2\boldsymbol{B}\cdot\boldsymbol{a}_j,\\ B^i_{\cdot j} = \boldsymbol{a}^i\cdot{}^2\boldsymbol{B}\cdot\boldsymbol{a}_j,\ \ B^{\cdot j}_i = \boldsymbol{a}_i\cdot{}^2\boldsymbol{B}\cdot\boldsymbol{a}^j.\end{gathered} \tag{15.5}$$

For двух видов смешанных компонент точка в индексе это просто свободное место: у $B^i_{\cdot j}$ верхний индекс "i" — первый, а нижний "j" — второй.

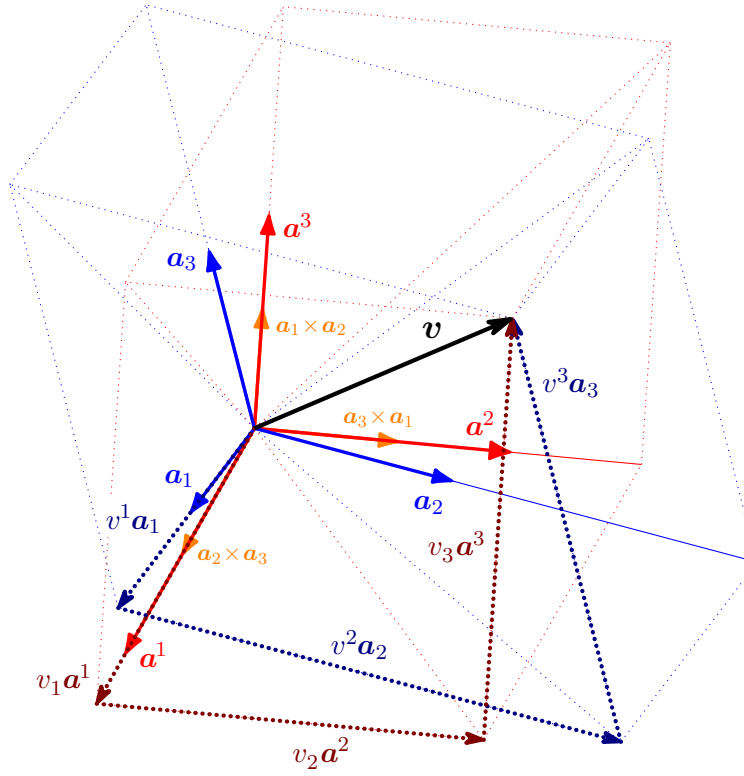
The components of the unit ("metric") tensor $\boldsymbol{E}$

$$\begin{gathered}\boldsymbol{E} = \boldsymbol{a}^k\boldsymbol{a}_k = \boldsymbol{a}_k\boldsymbol{a}^k = g_{jk}\boldsymbol{a}^j\boldsymbol{a}^k = g^{jk}\boldsymbol{a}_j\boldsymbol{a}_k:\\ \boldsymbol{a}_i\cdot\boldsymbol{E}\cdot\boldsymbol{a}^j = \boldsymbol{a}_i\cdot\boldsymbol{a}^j = \delta_i^j,\ \ \boldsymbol{a}^i\cdot\boldsymbol{E}\cdot\boldsymbol{a}_j = \boldsymbol{a}^i\cdot\boldsymbol{a}_j = \delta^i_j,\\ \boldsymbol{a}_i\cdot\boldsymbol{E}\cdot\boldsymbol{a}_j = \boldsymbol{a}_i\cdot\boldsymbol{a}_j \equiv g_{ij},\ \ \boldsymbol{a}^i\cdot\boldsymbol{E}\cdot\boldsymbol{a}^j = \boldsymbol{a}^i\cdot\boldsymbol{a}^j \equiv g^{ij};\\ \boldsymbol{E}\cdot\boldsymbol{E} = g_{ij}\boldsymbol{a}^i\boldsymbol{a}^j\cdot g^{nk}\boldsymbol{a}_n\boldsymbol{a}_k = g_{ij}g^{jk}\boldsymbol{a}^i\boldsymbol{a}_k = \boldsymbol{E}\ \Rightarrow\ g_{ij}g^{jk} = \delta_i^k.\end{gathered} \tag{15.6}$$

Besides (15.2) and (15.3), there's one more way to find the cobasis vectors — via matrix g^{ij}, which is the inverse of the Gram matrix g_{ij}. And vice versa:

$$\begin{aligned}\boldsymbol{a}^i &= \boldsymbol{E}\cdot\boldsymbol{a}^i = g^{jk}\boldsymbol{a}_j\boldsymbol{a}_k\cdot\boldsymbol{a}^i = g^{ji}\boldsymbol{a}_j,\\ \boldsymbol{a}_i &= \boldsymbol{E}\cdot\boldsymbol{a}_i = g_{jk}\boldsymbol{a}^j\boldsymbol{a}^k\cdot\boldsymbol{a}_i = g_{ji}\boldsymbol{a}^j.\end{aligned} \tag{15.7}$$

* Потому что они меняются обратно (contra) изменению длин базисных векторов $\boldsymbol{a}_i$.



$$\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 = \sqrt{g} = 0.56274$$
$$1/\sqrt{g} = 1.77703$$

$$\boldsymbol{a}_i \cdot \boldsymbol{a}^j = \begin{bmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{a}^1 & \boldsymbol{a}_1 \cdot \boldsymbol{a}^2 & \boldsymbol{a}_1 \cdot \boldsymbol{a}^3 \\ \boldsymbol{a}_2 \cdot \boldsymbol{a}^1 & \boldsymbol{a}_2 \cdot \boldsymbol{a}^2 & \boldsymbol{a}_2 \cdot \boldsymbol{a}^3 \\ \boldsymbol{a}_3 \cdot \boldsymbol{a}^1 & \boldsymbol{a}_3 \cdot \boldsymbol{a}^2 & \boldsymbol{a}_3 \cdot \boldsymbol{a}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_i^j$$

figure 7
"The decomposition of a vector in an oblique basis"

Example. Using the inverse of the Gram matrix, get the cobasis for basis $\boldsymbol{a}_i$ if

$$\begin{aligned}\boldsymbol{a}_1 &= \boldsymbol{e}_1 + \boldsymbol{e}_2,\\ \boldsymbol{a}_2 &= \boldsymbol{e}_1 + \boldsymbol{e}_3,\\ \boldsymbol{a}_3 &= \boldsymbol{e}_2 + \boldsymbol{e}_3.\end{aligned}$$

$$g_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j = \begin{bmatrix} 2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix},\ \det g_{ij} = 4,$$

$$\operatorname{adj} g_{ij} = \begin{bmatrix} 3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}^{\mathsf{T}},$$

$$g^{ij} = g_{ij}^{-1} = \frac{\operatorname{adj} g_{ij}}{\det g_{ij}} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}.$$

Using $\boldsymbol{a}^i = g^{ij}\boldsymbol{a}_j$

$$\begin{aligned}\boldsymbol{a}^1 &= g^{11}\boldsymbol{a}_1 + g^{12}\boldsymbol{a}_2 + g^{13}\boldsymbol{a}_3 = \tfrac{1}{2}\boldsymbol{e}_1 + \tfrac{1}{2}\boldsymbol{e}_2 - \tfrac{1}{2}\boldsymbol{e}_3,\\ \boldsymbol{a}^2 &= g^{21}\boldsymbol{a}_1 + g^{22}\boldsymbol{a}_2 + g^{23}\boldsymbol{a}_3 = \tfrac{1}{2}\boldsymbol{e}_1 - \tfrac{1}{2}\boldsymbol{e}_2 + \tfrac{1}{2}\boldsymbol{e}_3,\\ \boldsymbol{a}^3 &= g^{31}\boldsymbol{a}_1 + g^{32}\boldsymbol{a}_2 + g^{33}\boldsymbol{a}_3 = -\tfrac{1}{2}\boldsymbol{e}_1 + \tfrac{1}{2}\boldsymbol{e}_2 + \tfrac{1}{2}\boldsymbol{e}_3.\end{aligned}$$

....

Единичный тензор (unit tensor, identity tensor, metric tensor)

$\boldsymbol{E}\cdot\boldsymbol{\xi} = \boldsymbol{\xi}\cdot\boldsymbol{E} = \boldsymbol{\xi} \quad \forall\boldsymbol{\xi}$

$\boldsymbol{E}\cdot\cdot\,\boldsymbol{ab} = \boldsymbol{ab}\cdot\cdot\,\boldsymbol{E} = \boldsymbol{a}\cdot\boldsymbol{E}\cdot\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{b}$

$\boldsymbol{E}\cdot\cdot\,\boldsymbol{A} = \boldsymbol{A}\cdot\cdot\,\boldsymbol{E} = \operatorname{trace}\boldsymbol{A}$

$\boldsymbol{E}\cdot\cdot\,\boldsymbol{A} = \boldsymbol{A}\cdot\cdot\,\boldsymbol{E} = \operatorname{trace}\boldsymbol{A} \neq \text{notanymore } A_{jj}$

Thus for, say, trace of some tensor $\boldsymbol{A} = A_{ij}\boldsymbol{r}^i\boldsymbol{r}^j$: $\boldsymbol{A}\cdot\cdot\,\boldsymbol{E} = \operatorname{trace}\boldsymbol{A}$, you have

$\boldsymbol{A}\cdot\cdot\,\boldsymbol{E} = A_{ij}\boldsymbol{r}^i\boldsymbol{r}^j\cdot\cdot\,\boldsymbol{r}_{\partial k}\boldsymbol{r}^k = A_{ij}\boldsymbol{r}^i\cdot\boldsymbol{r}^j = A_{ij}g^{ij}$

...

Тензор поворота (the rotation tensor)

$\boldsymbol{P} = \boldsymbol{a}_i\mathring{\boldsymbol{a}}^i = \boldsymbol{a}^i\mathring{\boldsymbol{a}}_i = \boldsymbol{P}^{-\mathsf{T}}$

$\boldsymbol{P}^{-1} = \mathring{\boldsymbol{a}}_i\boldsymbol{a}^i = \mathring{\boldsymbol{a}}^i\boldsymbol{a}_i = \boldsymbol{P}^{\mathsf{T}}$

$\boldsymbol{P}^{\mathsf{T}} = \mathring{\boldsymbol{a}}^i\boldsymbol{a}_i = \mathring{\boldsymbol{a}}_i\boldsymbol{a}^i = \boldsymbol{P}^{-1}$

...

... Характеристическое уравнение (10.2) быстро приводит к тождеству Кэ́ли–Га́мильтона (Cayley–Hamilton)

$$\begin{gathered} -\boldsymbol{B}\cdot\boldsymbol{B}\cdot\boldsymbol{B} + \mathrm{I}\boldsymbol{B}\cdot\boldsymbol{B} - \mathrm{II}\boldsymbol{B} + \mathrm{III}\boldsymbol{E} = {}^2\boldsymbol{0}, \\ -\boldsymbol{B}^3 + \mathrm{I}\boldsymbol{B}^2 - \mathrm{II}\boldsymbol{B} + \mathrm{III}\boldsymbol{E} = {}^2\boldsymbol{0}. \end{gathered} \tag{15.8}$$

§ 16. Tensor functions

In the concept of function $y{=}f(x)$ as of mapping (morphism) $f\colon x \mapsto y$, an input (argument) x and an output (result) y may be tensors of any complexities.

Consider at least a scalar function of a bivalent tensor $\varphi{=}\varphi(\boldsymbol{B})$. Examples are $\boldsymbol{B}\cdot\cdot\boldsymbol{\Phi}$ (or $\boldsymbol{p}\cdot\boldsymbol{B}\cdot\boldsymbol{q}$) and $\boldsymbol{B}\cdot\cdot\boldsymbol{B}$. Then in each basis $\boldsymbol{a}_i$ paired with cobasis $\boldsymbol{a}^i$ we have function $\varphi(B_{ij})$ of nine numeric arguments — components B_{ij} of tensor $\boldsymbol{B}$. For example

$$\varphi(\boldsymbol{B}) = \boldsymbol{B}\cdot\cdot\boldsymbol{\Phi} = B_{ij}\,\boldsymbol{a}^i\boldsymbol{a}^j\cdot\cdot\,\boldsymbol{a}_m\boldsymbol{a}_n\boldsymbol{\Phi}^{mn} = B_{ij}\,\boldsymbol{\Phi}^{ji} = \varphi(B_{ij}).$$

With any transition to a new basis, the result doesn't change: $\varphi(B_{ij}) = \varphi(B'_{ij}) = \varphi(\boldsymbol{B})$.

Differentiation of $\varphi(\boldsymbol{B})$ looks like

$$d\varphi = \frac{\partial\varphi}{\partial B_{ij}}\,dB_{ij} = \frac{\partial\varphi}{\partial\boldsymbol{B}}\cdot\cdot\,d\boldsymbol{B}^{\mathsf{T}}. \tag{16.1}$$

Tensor $\partial\varphi/\partial\boldsymbol{B}$ is called the derivative of function φ by argument $\boldsymbol{B}$; $d\boldsymbol{B}$ is the differential of tensor $\boldsymbol{B}$, $d\boldsymbol{B} = dB_{ij}\,\boldsymbol{a}^i\boldsymbol{a}^j$; $\partial\varphi/\partial B_{ij}$ are components (contravariant ones) of $\partial\varphi/\partial\boldsymbol{B}$

$$\boldsymbol{a}^i\cdot\frac{\partial\varphi}{\partial\boldsymbol{B}}\cdot\boldsymbol{a}^j = \frac{\partial\varphi}{\partial\boldsymbol{B}}\cdot\cdot\,\boldsymbol{a}^j\boldsymbol{a}^i = \frac{\partial\varphi}{\partial B_{ij}} \;\Leftrightarrow\; \frac{\partial\varphi}{\partial\boldsymbol{B}} = \frac{\partial\varphi}{\partial B_{ij}}\,\boldsymbol{a}_i\boldsymbol{a}_j.$$

...

$$\begin{gathered} \varphi(\boldsymbol{B}) = \boldsymbol{B}\cdot\cdot\boldsymbol{\Phi} \\ d\varphi = d(\boldsymbol{B}\cdot\cdot\boldsymbol{\Phi}) = d\boldsymbol{B}\cdot\cdot\boldsymbol{\Phi} = \boldsymbol{\Phi}\cdot\cdot\,d\boldsymbol{B} = \boldsymbol{\Phi}^{\mathsf{T}}\cdot\cdot\,d\boldsymbol{B}^{\mathsf{T}} \\ d\varphi = \frac{\partial\varphi}{\partial\boldsymbol{B}}\cdot\cdot\,d\boldsymbol{B}^{\mathsf{T}},\;\; \frac{\partial(\boldsymbol{B}\cdot\cdot\boldsymbol{\Phi})}{\partial\boldsymbol{B}} = \boldsymbol{\Phi}^{\mathsf{T}} \end{gathered}$$

$\boldsymbol{p}\cdot\boldsymbol{B}\cdot\boldsymbol{q}=\boldsymbol{B}\cdot\cdot\boldsymbol{qp}$

$$\frac{\partial(\boldsymbol{p}\cdot\boldsymbol{B}\cdot\boldsymbol{q})}{\partial\boldsymbol{B}}=\boldsymbol{pq}$$

...

$$\varphi(\boldsymbol{B})=\boldsymbol{B}\cdot\cdot\boldsymbol{B}$$
$$d\varphi=d(\boldsymbol{B}\cdot\cdot\boldsymbol{B})=d...$$

...

Но согласно опять-таки (15.8) $-\boldsymbol{B}^2+\mathrm{I}\boldsymbol{B}-\mathrm{II}\boldsymbol{E}+\mathrm{III}\boldsymbol{B}^{-1}={}^2\boldsymbol{0}$, поэтому

....

Скалярная функция $\varphi(\boldsymbol{B})$ называется изотропной, если она не чувствительна к повороту аргумента:

$$\varphi(\boldsymbol{B})=\varphi(\boldsymbol{O}\cdot\mathring{\boldsymbol{B}}\cdot\boldsymbol{O}^{\mathsf{T}})=\varphi(\mathring{\boldsymbol{B}})\quad\forall\boldsymbol{O}=\boldsymbol{a}_i\mathring{\boldsymbol{a}}^i=\boldsymbol{a}^i\mathring{\boldsymbol{a}}_i=\boldsymbol{O}^{-\mathsf{T}}$$

для любого ортогонального тензора $\boldsymbol{O}$ (тензора поворота, § 11).

Симметричный тензор $\boldsymbol{B}^{\mathsf{S}}$ полностью определяется тройкой инвариантов и угловой ориентацией собственных осей (они же взаимно ортогональны, § 10). Ясно, что изотропная функция $\varphi(\boldsymbol{B}^{\mathsf{S}})$ симметричного аргумента является функцией, входы-аргументы которой — только инварианты $\mathrm{I}(\boldsymbol{B}^{\mathsf{S}})$, $\mathrm{II}(\boldsymbol{B}^{\mathsf{S}})$, $\mathrm{III}(\boldsymbol{B}^{\mathsf{S}})$. Дифференцируется такая функция согласно (**??**), где транспонирование излишне.

§ 17. Spatial differentiation

Let at each point of some region of a three-dimensional space the variable ς be defined. Then it's said that there is a *tensor field* $\varsigma=\varsigma(\boldsymbol{r})$, where $\boldsymbol{r}$ is the location vector ("radius" vector) of a point in space. The variable ς can be a tensor of any complexity. An example of scalar field — the temperature field in a medium, of vector field — the velocities of liquid particles.

So, a *tensor field* is a tensor varying from point to point (variable in space, coordinate dependent). The concept of a tensor field is not

about a field in algebra that considers operations "+" and "∗" with the specific properties.

Not only for solving applied problems, but often in "pure theory", instead of the argument $\boldsymbol{r}$, the set of curvilinear coordinates q^i is used. If only one coordinate from the set continuously changes, this gives a coordinate line. In three-dimensional space, a point is an intersection of three coordinate lines (figure 8). The location vector of a point is a function from the set of coordinates

$$\boldsymbol{r}=\boldsymbol{r}(q^i). \qquad (17.1)$$

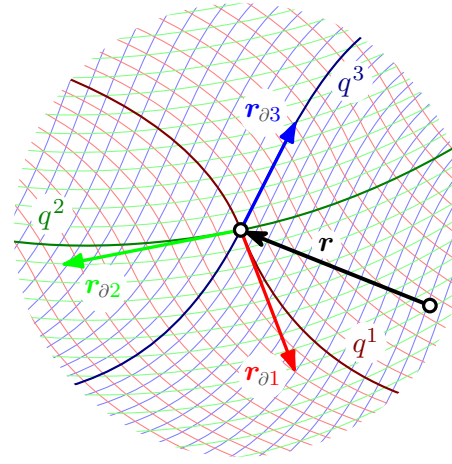


figure 8

The most often used sets of coordinates are rectangular ("cartesian") coordinates, spherical coordinates and cylindrical coordinates. Any curvilinear coordinates can be converted to the rectangular coordinates and vice versa by means of a locally invertible ("one-to-one") mapping at each point.

...

The differential of a function presents the change in linearization of that function.

....

partial derivatives

$$\partial_i \equiv \frac{\partial}{\partial q^i}$$

...

the differential of $\varsigma(q^i)$

$$d\varsigma = \frac{\partial \varsigma}{\partial q^i} dq^i = \partial_i \varsigma \, dq^i \qquad (17.2)$$

...

Linearity

$$\partial_i\big(\lambda p+\mu q\big)=\lambda\big(\partial_i p\big)+\mu\big(\partial_i q\big) \tag{17.3}$$

The "product rule" is about the differentiation of $p\circ q$, the product of p and q:

$$d\big(p\circ q\big)=\big(p+dp\big)\circ\big(q+dq\big)-p\circ q=dp\circ q+p\circ dq+dp\circ dq,$$

where $dp\circ dq=\infty^{-1}\big(dp,dq\big)$ — the product $dp\circ dq$ of two infinitesimals is considered to be infinitesimally smaller than either infinitesimal (dp or dq) alone. Thus, the term $dp\circ dq$ can be dropped as negligible compared to terms with one infinitesimal. And then

$$d\big(p\circ q\big)=\big(dp\big)\circ q+p\circ\big(dq\big). \tag{17.4}$$

For a partial derivative $\partial_i\equiv\dfrac{\partial}{\partial q^i}$

$$\partial_i\big(p\circ q\big)=\big(\partial_i p\big)\circ q+p\circ\big(\partial_i q\big). \tag{17.5}$$

...

Local basis $\boldsymbol{r}_{\partial i}$

The differential of the location vector $\boldsymbol{r}(q^i)$ is

$$d\boldsymbol{r}=\frac{\partial\boldsymbol{r}}{\partial q^i}\,dq^i=dq^i\boldsymbol{r}_{\partial i},\ \ \boldsymbol{r}_{\partial i}\equiv\frac{\partial\boldsymbol{r}}{\partial q^i}\equiv\partial_i\boldsymbol{r} \tag{17.6}$$

...

Local cobasis $\boldsymbol{r}^i$

$\boldsymbol{r}^i\bullet\boldsymbol{r}_{\partial j}=\delta^i_j$

The (spatial) differential of something

$$\begin{aligned}\frac{\partial\varsigma}{\partial\boldsymbol{r}}&=\frac{\partial\varsigma}{\partial q^i}\boldsymbol{r}^i=\partial_i\varsigma\,\boldsymbol{r}^i\\ d\varsigma=\frac{\partial\varsigma}{\partial\boldsymbol{r}}\bullet d\boldsymbol{r}&=\partial_i\varsigma\,\boldsymbol{r}^i\bullet\boldsymbol{r}_{\partial j}\,dq^j=\partial_i\varsigma\,dq^i\end{aligned} \tag{17.7}$$

...

The unit dyad (metric tensor) $\boldsymbol{E}$, neutral (4.7) for the "$\bullet$"-product, can be represented as

$$\boldsymbol{E} = \boldsymbol{r}^i \boldsymbol{r}_{\partial i} = \underbrace{\boldsymbol{r}^i \partial_i}_{\boldsymbol{\nabla}} \boldsymbol{r} = \boldsymbol{\nabla} \boldsymbol{r}, \tag{17.8}$$

where appears the differential "nabla" operator

$$\boldsymbol{\nabla} \equiv \boldsymbol{r}^i \partial_i . \tag{17.9}$$

...

With $\boldsymbol{\nabla}$, the spatial differential of something is

$$\begin{gathered} \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{r}} = \partial_i \boldsymbol{\zeta} \boldsymbol{r}^i \\ d\varsigma = \frac{\partial \varsigma}{\partial \boldsymbol{r}} \bullet d\boldsymbol{r} = d\boldsymbol{r} \bullet \boldsymbol{\nabla} \varsigma = \partial_i \varsigma \, dq^i \end{gathered} \tag{17.10}$$

$$d\boldsymbol{r} = d\boldsymbol{r} \bullet \overbrace{\boldsymbol{\nabla} \boldsymbol{r}}^{\boldsymbol{E}}$$

...

Divergence of the dyadic product of two vectors

$$\begin{aligned} \boldsymbol{\nabla} \bullet (\boldsymbol{a}\boldsymbol{b}) = \boldsymbol{r}^i \partial_i \bullet (\boldsymbol{a}\boldsymbol{b}) = \boldsymbol{r}^i \bullet \partial_i (\boldsymbol{a}\boldsymbol{b}) = \boldsymbol{r}^i \bullet (\partial_i \boldsymbol{a}) \boldsymbol{b} + \boldsymbol{r}^i \bullet \boldsymbol{a} (\partial_i \boldsymbol{b}) = \\ = (\boldsymbol{r}^i \bullet \partial_i \boldsymbol{a}) \boldsymbol{b} + \boldsymbol{a} \bullet \boldsymbol{r}^i (\partial_i \boldsymbol{b}) = (\boldsymbol{r}^i \partial_i \bullet \boldsymbol{a}) \boldsymbol{b} + \boldsymbol{a} \bullet (\boldsymbol{r}^i \partial_i \boldsymbol{b}) = \\ = (\boldsymbol{\nabla} \bullet \boldsymbol{a}) \boldsymbol{b} + \boldsymbol{a} \bullet (\boldsymbol{\nabla} \boldsymbol{b}) \end{aligned} \tag{17.11}$$

— here's no need to expand vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, expanding just the differential operator $\boldsymbol{\nabla}$.

....

The gradient of the "$\times$"-product of two vectors, applying "the product rule" (17.5) and relation (7.6) for any two vectors (a partial derivative ∂_i of some vector by scalar coordinate q^i is a vector too)

$$\begin{aligned} \boldsymbol{\nabla} (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{r}^i \partial_i (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{r}^i (\partial_i \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \partial_i \boldsymbol{b}) = \\ = \boldsymbol{r}^i (\partial_i \boldsymbol{a} \times \boldsymbol{b} - \partial_i \boldsymbol{b} \times \boldsymbol{a}) = \boldsymbol{r}^i \partial_i \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{r}^i \partial_i \boldsymbol{b} \times \boldsymbol{a} = \\ = \boldsymbol{\nabla} \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{\nabla} \boldsymbol{b} \times \boldsymbol{a} . \end{aligned} \tag{17.12}$$

...

The gradient of the "•"-product of two vectors

$$\boldsymbol{\nabla}\big(\boldsymbol{a}\cdot\boldsymbol{b}\big)=\boldsymbol{r}^i\partial_i\big(\boldsymbol{a}\cdot\boldsymbol{b}\big)=\boldsymbol{r}^i\big(\partial_i\boldsymbol{a}\big)\cdot\boldsymbol{b}+\boldsymbol{r}^i\boldsymbol{a}\cdot\big(\partial_i\boldsymbol{b}\big)= \\ =\big(\boldsymbol{r}^i\partial_i\boldsymbol{a}\big)\cdot\boldsymbol{b}+\boldsymbol{r}^i\big(\partial_i\boldsymbol{b}\big)\cdot\boldsymbol{a}=\big(\boldsymbol{\nabla}\boldsymbol{a}\big)\cdot\boldsymbol{b}+\big(\boldsymbol{\nabla}\boldsymbol{b}\big)\cdot\boldsymbol{a}. \quad (17.13)$$

§ 18. The integral theorems

For vector fields, the integral theorems are known — the Gauss' or Ostrogradsky's (Остроградского) divergence theorem and Stokes' circulation theorem.

Gauss' theorem (divergence theorem) enables an integral taken over a volume to be replaced by one taken over the closed surface bounding that volume, and vice versa.

Stokes' theorem enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve. Stokes' theorem relates a line integral around a closed path to a surface integral over what is called a *capping surface* of the path.

The Gauss's divergence theorem

(wikipedia) Divergence theorem

This theorem is about how to replace a volume integral with a surface one (and vice versa).

В этой теореме рассматривается поток (ef)flux вектора через ограничивающую объём $\mathcal{V}$ за́мкнутую поверхность $\mathcal{O}(\partial\mathcal{V})$...

$$\oint\limits_{\mathcal{O}(\partial\mathcal{V})}\boldsymbol{n}\cdot\boldsymbol{a}\,d\mathcal{O}=\int\limits_{\mathcal{V}}\boldsymbol{\nabla}\cdot\boldsymbol{a}\,d\mathcal{V}, \quad (18.1)$$

the "•"-product always commutes

$$\oint\limits_{\mathcal{O}(\partial\mathcal{V})}\boldsymbol{a}\cdot\boldsymbol{n}\,d\mathcal{O}=\int\limits_{\mathcal{V}}\boldsymbol{a}\cdot\boldsymbol{\nabla}d\mathcal{V},$$

$\boldsymbol{n}\,d\mathcal{O}=d\boldsymbol{\mathcal{O}}$

$$\oint\limits_{\mathcal{O}(\partial\mathcal{V})}\boldsymbol{a}\cdot d\boldsymbol{\mathcal{O}}=\int\limits_{\mathcal{V}}\boldsymbol{\nabla}\cdot\boldsymbol{a}\,d\mathcal{V},$$

$\boldsymbol{n}$ is the unit vector of outward normal to surface $\mathcal{O}(\partial\mathcal{V})$.

Volume $\mathcal{V}$ нарезается тремя семействами координатных поверхностей на множество бесконечно малых элементов. Поток через поверхность $\mathcal{O}(\partial\mathcal{V})$ равен сумме потоков через края получившихся элементов. В бесконечной малости каждый такой элемент — малюсенький локальный дифференциальный кубик (параллелепипед). .... The flux of vector $\boldsymbol{a}$ through the faces of a small cube is equal to $\sum_{i=1}^{6} \boldsymbol{n}_i \bullet \boldsymbol{a}\mathcal{O}_i$, а поток через объём $d\mathcal{V}$ этого малого кубика равен $\boldsymbol{\nabla} \bullet \boldsymbol{a}\, d\mathcal{V}$.

A similar interpretation of this theorem is given, for example, in Richard Feynman's lectures [86].

(рисунок с кубиками)

to dice — нарезáть кубиками

small cube, little cube

локально ортонормальные координаты $\boldsymbol{\xi} = \xi_i \boldsymbol{n}_i$, $d\boldsymbol{\xi} = d\xi_i \boldsymbol{n}_i$, $\boldsymbol{\nabla} = \boldsymbol{n}_i \partial_i$

разложение вектора $\boldsymbol{a} = a_i \boldsymbol{n}_i$

The Stokes' circulation theorem

(wikipedia) Stokes' theorem

This theorem is formulated as the equality

$$\oint\limits_{\mathcal{C}(\partial\mathcal{O})} \boldsymbol{a} \bullet d\boldsymbol{\mathcal{C}} = \int\limits_{\mathcal{O}} \boldsymbol{n} \bullet \big(\boldsymbol{\nabla} \times \boldsymbol{a}\big)\, d\mathcal{O}. \tag{18.2}$$

$$\oint\limits_{\mathcal{C}(\partial\mathcal{O})} \boldsymbol{a} \bullet d\boldsymbol{\mathcal{C}} = \int\limits_{\mathcal{O}} \big(\boldsymbol{\nabla} \times \boldsymbol{a}\big) \bullet \boldsymbol{n}\, d\mathcal{O}.$$

$$\oint\limits_{\mathcal{C}(\partial\mathcal{O})} \boldsymbol{a} \bullet d\boldsymbol{\mathcal{C}} = \int\limits_{\mathcal{O}} \big(\boldsymbol{\nabla} \times \boldsymbol{a}\big) \bullet d\boldsymbol{\mathcal{O}}.$$

...

§ 19. Curvature tensors

The *Riemann curvature tensor* or *Riemann–Christoffel tensor* (after **Bernhard Riemann** and **Elwin Bruno Christoffel**) is the most common method used to express the curvature of Riemannian manifolds. It's a tensor field, it assigns a tensor to each point of a Riemannian manifold, that measures the extent to which the metric tensor is not locally isometric to that of "flat" space. The curvature tensor measures noncommutativity of the covariant derivative, and as such is the integrability obstruction for the existænce of an isometry with "flat" space.

Dealing with tensor fields in curvilinear coordinates (§ 17), the location vector (radius vector) of a point was introduced as a function (17.1) of these coordinates, $\boldsymbol{r}=\boldsymbol{r}(q^i)$. From this relation originate many others, such as

- ✓ the vectors of local tangent basis $\boldsymbol{r}_{\partial i} \equiv {}^{\partial \boldsymbol{r}}\!/_{\partial q^i} \equiv \partial_i \boldsymbol{r}$,
- ✓ the components $g_{ij} \equiv \boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j}$ and $g^{ij} \equiv \boldsymbol{r}^i \cdot \boldsymbol{r}^j = g_{ij}^{-1}$ of the unit ("metric") tensor $\boldsymbol{E} = \boldsymbol{r}_{\partial i}\boldsymbol{r}^i = \boldsymbol{r}^i\boldsymbol{r}_{\partial i} = g_{jk}\boldsymbol{r}^j\boldsymbol{r}^k = g^{jk}\boldsymbol{r}_{\partial j}\boldsymbol{r}_{\partial k}$,
- ✓ the vectors of local dual (reciprocal) cotangent basis $\boldsymbol{r}^i \cdot \boldsymbol{r}_{\partial j} = \delta^i_j$, $\boldsymbol{r}^i = g^{ij}\boldsymbol{r}_{\partial j}$,
- ✓ the differential nabla-operator (Hamilton's operator, Hamiltonian) $\boldsymbol{\nabla} \equiv \boldsymbol{r}^i\partial_i$, $\boldsymbol{E} = \boldsymbol{\nabla}\boldsymbol{r}$,
- ✓ the full differential $d\boldsymbol{\xi} = d\boldsymbol{r} \cdot \boldsymbol{\nabla}\boldsymbol{\xi}$,
- ✓ the partial derivatives of tangent basis vectors (the second partial derivatives of $\boldsymbol{r}$) $\boldsymbol{r}_{\partial i\partial j} \equiv \partial_i\partial_j\boldsymbol{r} = \partial_i\boldsymbol{r}_{\partial j}$,
- ✓ the Christoffel symbols of metric connection $\Gamma^k_{ij} \equiv \boldsymbol{r}_{\partial i\partial j} \cdot \boldsymbol{r}^k$ and $\Gamma_{ijk} \equiv \boldsymbol{r}_{\partial i\partial j} \cdot \boldsymbol{r}_{\partial k}$.

Представим теперь, что функция $\boldsymbol{r}(q^k)$ не известна, но зато́ в каждой точке пространства известны шесть независимых компонент положительно определённой (all Gram matrices are non-negative definite) симметричной метрической матрицы Gram $g_{ij}(q^k)$.

the Gram matrix (or Gramian)

Билинейная форма ...

...

Поскольку шесть функций $g_{ij}(q^k)$ происходят от векторной функции $\boldsymbol{r}(q^k)$, то между элементами g_{ij} существуют некие соотношения.

Differential $d\boldsymbol{r}$ (17.6) is exact. This is true if and only if second partial derivatives commute:

$$d\boldsymbol{r} = \boldsymbol{r}_{\partial k}\, dq^k \;\;\Leftrightarrow\;\; \partial_i \boldsymbol{r}_{\partial j} = \partial_j \boldsymbol{r}_{\partial i} \;\text{ or }\; \boldsymbol{r}_{\partial i \partial j} = \boldsymbol{r}_{\partial j \partial i}.$$

Но это условие уже́ обеспечено симметрией g_{ij}

...

metric (“affine”) connection ∇_i, её же называют “covariant derivative”

$$\boldsymbol{r}_{\partial i \partial j} = \underbrace{\boldsymbol{r}_{\partial i \partial j} \bullet \overbrace{\boldsymbol{r}^k}^{\boldsymbol{E}}}_{\Gamma^k_{ij}} \boldsymbol{r}_{\partial k} = \underbrace{\boldsymbol{r}_{\partial i \partial j} \bullet \overbrace{\boldsymbol{r}_{\partial k}}^{\boldsymbol{E}}}_{\Gamma_{ij}{}^{\bullet}_{k}} \boldsymbol{r}^k$$

$\Gamma^k_{ij}\, \boldsymbol{r}_{\partial k} = \boldsymbol{r}_{\partial i \partial j} \bullet \boldsymbol{r}^k \boldsymbol{r}_{\partial k} = \boldsymbol{r}_{\partial i \partial j}$

covariant derivative (affine connection) is only defined for vector fields

$\boldsymbol{\nabla v} = \boldsymbol{r}^i \partial_i \big(v^j \boldsymbol{r}_{\partial j}\big) = \boldsymbol{r}^i \big(\partial_i v^j \boldsymbol{r}_{\partial j} + v^j \boldsymbol{r}_{\partial i \partial j}\big)$

$\boldsymbol{\nabla v} = \boldsymbol{r}^i \boldsymbol{r}_{\partial j} \nabla_i v^j, \;\; \nabla_i v^j \equiv \partial_i v^j + \Gamma^j_{in} v^n$

$\boldsymbol{\nabla r}_{\partial i} = \boldsymbol{r}^k \partial_k \boldsymbol{r}_{\partial i} = \boldsymbol{r}^k \boldsymbol{r}_{\partial k \partial i} = \boldsymbol{r}^k \boldsymbol{r}_{\partial n} \Gamma^n_{ki}, \;\; \nabla_i \boldsymbol{r}_{\partial n} = \Gamma^k_{in} \boldsymbol{r}_{\partial k}$

Christoffel symbols describe a metric (“affine”) connection, that is how the basis changes from point to point.

символы Christoffel’я это “components of connection” in local coordinates

...

torsion tensor ${}^3\boldsymbol{\mathfrak{T}}$ with components

$$\mathfrak{T}^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$$

determines the antisymmetric part of a connection

...

симметрия $\Gamma_{ij\dot{k}} = \Gamma_{ji\dot{k}}$, поэтому $3^3 - 3\cdot 3 = 18$ разных (независимых) $\Gamma_{ij\dot{k}}$

$$\Gamma^n_{ij}\, g_{nk} = \Gamma_{ij\dot{k}} = \boldsymbol{r}_{\partial i\partial j}\bullet \boldsymbol{r}_{\partial k} =$$
$$= \tfrac{1}{2}\big(\boldsymbol{r}_{\partial i\partial j} + \boldsymbol{r}_{\partial j\partial i}\big)\bullet \boldsymbol{r}_{\partial k} + \tfrac{1}{2}\big(\boldsymbol{r}_{\partial j\partial k} - \boldsymbol{r}_{\partial k\partial j}\big)\bullet \boldsymbol{r}_{\partial i} + \tfrac{1}{2}\big(\boldsymbol{r}_{\partial i\partial k} - \boldsymbol{r}_{\partial k\partial i}\big)\bullet \boldsymbol{r}_{\partial j} =$$
$$= \tfrac{1}{2}\big(\boldsymbol{r}_{\partial i\partial j}\bullet \boldsymbol{r}_{\partial k} + \boldsymbol{r}_{\partial i\partial k}\bullet \boldsymbol{r}_{\partial j}\big) + \tfrac{1}{2}\big(\boldsymbol{r}_{\partial j\partial i}\bullet \boldsymbol{r}_{\partial k} + \boldsymbol{r}_{\partial j\partial k}\bullet \boldsymbol{r}_{\partial i}\big) - \tfrac{1}{2}\big(\boldsymbol{r}_{\partial k\partial i}\bullet \boldsymbol{r}_{\partial j} + \boldsymbol{r}_{\partial k\partial j}\bullet \boldsymbol{r}_{\partial i}\big) =$$
$$= \tfrac{1}{2}\Big(\partial_i(\boldsymbol{r}_{\partial j}\bullet \boldsymbol{r}_{\partial k}) + \partial_j(\boldsymbol{r}_{\partial i}\bullet \boldsymbol{r}_{\partial k}) - \partial_k(\boldsymbol{r}_{\partial i}\bullet \boldsymbol{r}_{\partial j})\Big) =$$
$$= \tfrac{1}{2}\big(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\big). \quad (19.1)$$

Все символы Christoffel'я тождественно равны нулю лишь в ортонормальной (декартовой) системе. (А какие они для косоугольной?)

Дальше: $d\boldsymbol{r}_{\partial i} = d\boldsymbol{r}\bullet\boldsymbol{\nabla}\boldsymbol{r}_{\partial i} = dq^k\partial_k\boldsymbol{r}_{\partial i} = \boldsymbol{r}_{\partial k\partial i}\,dq^k$ — тоже полные дифференциалы.

$$d\boldsymbol{r}_{\partial k} = \partial_i\boldsymbol{r}_{\partial k}\,dq^i = \frac{\partial\boldsymbol{r}_{\partial k}}{\partial q^1}\,dq^1 + \frac{\partial\boldsymbol{r}_{\partial k}}{\partial q^2}\,dq^2 + \frac{\partial\boldsymbol{r}_{\partial k}}{\partial q^3}\,dq^3$$

Поэтому $\partial_i\partial_j\boldsymbol{r}_{\partial k} = \partial_j\partial_i\boldsymbol{r}_{\partial k}$, $\partial_i\boldsymbol{r}_{\partial j\partial k} = \partial_j\boldsymbol{r}_{\partial i\partial k}$, и трёхиндексный объект из векторов третьих частных производных

$$\boldsymbol{r}_{\partial i\partial j\partial k} \equiv \partial_i\partial_j\partial_k\boldsymbol{r} = \partial_i\boldsymbol{r}_{\partial j\partial k} \quad (19.2)$$

симметричен по первому и второму индексам (а не только по второму и третьему). И тогда равен нулю ${}^4\mathbf{0}$ следующий тензор четвёртой сложности — *Riemann curvature tensor* (or *Riemann–Christoffel tensor*)

$${}^4\boldsymbol{\mathfrak{R}} = \mathfrak{R}_{hijk}\boldsymbol{r}^h\boldsymbol{r}^i\boldsymbol{r}^j\boldsymbol{r}^k, \;\; \mathfrak{R}_{hijk} \equiv \boldsymbol{r}_{\partial h}\bullet(\boldsymbol{r}_{\partial j\partial i\partial k} - \boldsymbol{r}_{\partial i\partial j\partial k}). \quad (19.3)$$

Выразим компоненты $\mathfrak{R}_{ijkn}$ через метрическую матрицу g_{ij}. Начнём с дифференцирования локального кобазиса:

$$\boldsymbol{r}^i\bullet\boldsymbol{r}_{\partial k} = \delta^i_k \;\Rightarrow\; \partial_j\boldsymbol{r}^i\bullet\boldsymbol{r}_{\partial k} + \boldsymbol{r}^i\bullet\boldsymbol{r}_{\partial j\partial k} = 0 \;\Rightarrow\; \partial_j\boldsymbol{r}^i = -\Gamma^i_{jk}\boldsymbol{r}^k.$$

...

The six independent components $\mathfrak{R}_{1212}$, $\mathfrak{R}_{1213}$, $\mathfrak{R}_{1223}$, $\mathfrak{R}_{1313}$, $\mathfrak{R}_{1323}$, $\mathfrak{R}_{2323}$.

...

The symmetric bivalent *Ricci curvature tensor*

$$\boldsymbol{\mathscr{R}} \equiv \frac{1}{4}\,\mathfrak{R}_{abij}\,\boldsymbol{r}^a \times \boldsymbol{r}^b \boldsymbol{r}^i \times \boldsymbol{r}^j = \frac{1}{4} \in^{abp} \in^{ijq} \mathfrak{R}_{abij}\,\boldsymbol{r}_{\partial p} \boldsymbol{r}_{\partial q} = \mathscr{R}^{pq}\,\boldsymbol{r}_{\partial p} \boldsymbol{r}_{\partial q}$$

(coefficient $\frac{1}{4}$ is used here for convenience) with components

$$\begin{aligned}
&\mathscr{R}^{11} = \frac{1}{g}\,\mathfrak{R}_{2323}\,,\\
&\mathscr{R}^{21} = \frac{1}{g}\,\mathfrak{R}_{1323}\,,\quad \mathscr{R}^{22} = \frac{1}{g}\,\mathfrak{R}_{1313}\,,\\
&\mathscr{R}^{31} = \frac{1}{g}\,\mathfrak{R}_{1223}\,,\quad \mathscr{R}^{32} = \frac{1}{g}\,\mathfrak{R}_{1213}\,,\quad \mathscr{R}^{33} = \frac{1}{g}\,\mathfrak{R}_{1212}\,.
\end{aligned}$$

The equality of the Ricci tensor to zero $\boldsymbol{\mathscr{R}} = {}^2\mathbf{0}$ (in components these are six equations $\mathscr{R}^{ij} = \mathscr{R}^{ji} = 0$) is the necessary condition of integrability ("compatibility") for determining the location ("radius") vector $\boldsymbol{r}(q^k)$ by the known field $g_{ij}(q^k)$.

Bibliography

There are many books worth mentioning that cover only the apparatus of tensor calculus [98, 99, 100, 101, 102]. However, the index notation (it's when tensors are seen as the sets of components) is still more popular than the direct indexless look. The direct notation is widely used, for example, in the appendices to the books by Anatoliy I. Lurie (Анатолий И. Лурье) [27, 28]. "Теория упругости" ("The theory of elasticity") by Вениамин Блох (Veniamin Blokh) [7] is as well written in the direct indexless notation. The vivid description of the vector fields theory is presented in R. Feynman's lectures [86]. Also, information about tensor calculus is the part of the unusual and interesting book by C. Truesdell [55].

chapter 2

CLASSICAL MECHANICS

When relativistic mechanics (for the very fast) and quantum mechanics (for the very small) emerged at the beginning of the XX$^{\text{th}}$ century, the equations of mechanics existed prior to that, still perfectly suitable for describing objects of everyday sizes and speeds, needed a new name. So the "classical" in mechanics doesn't refer to antiquity. This was just chosen as the name for description of reality without any quantum and relativistic effects influencing it.

§1. Discrete collection of particles

Classical mechanics models physical objects by discretizing them into a collection of particles ("pointlike masses", "material points"*).

In a collection of N particles, each k-th particle has its nonzero mass $m_k = \mathsf{constant} > 0$ and the motion function $\boldsymbol{r}_k(t)$. The function $\boldsymbol{r}_k(t)$ is measured relative to the chosen reference system.

The "reference system" (or "reference frame") consists of (figure 9)

- ✓ some "null" reference point o,
- ✓ a set of coordinates, which give the units of spatial measurements,
- ✓ a clock.

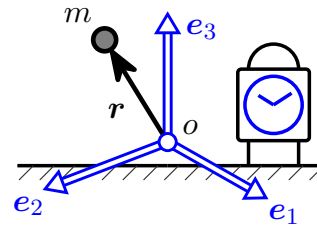


figure 9

Long time ago, the reference system was some "absolute space", empty at first, and then filled with the continuous elastic medium — the æther. Later, it

* The point mass (pointlike mass, material point) is the concept of an object, typically matter, that has the nonzero mass and is (or is being thought of as) infinitesimal in its volume (dimensions).

became clear that any frame of reference can be used for classical mechanics, but the preference is given to the so called "inertial" frames, where a particle in the absence of external interactions (or applied forces) moves "in free motion" — along a straight line with a constant velocity ($\dot{\boldsymbol{r}} = \textbf{constant}$), thence without acceleration ($\ddot{\boldsymbol{r}} = \boldsymbol{0}$)

$$\dot{\boldsymbol{r}} = \textbf{constant} = \dot{x}_i \boldsymbol{e}_i \quad \Rightarrow \quad \dot{x}_i = \text{constant} \quad \Leftarrow \quad \boldsymbol{e}_i = \textbf{constant}$$

The measure of interaction in mechanics is the vector of force $\boldsymbol{F}$. In the widely known* Newton's equation

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \tag{1.1}$$

the force $\boldsymbol{F}$ can depend only on position, velocity and explicitly on time, whereas acceleration $\ddot{\boldsymbol{r}}$ is directly proportional to force $\boldsymbol{F}$ with coefficient $1/m$.

Here're theses of the dynamics of a collection of particles.

The force $\boldsymbol{F}_k$, acting on the k-th particle (figure 10)

$$m_k \ddot{\boldsymbol{r}}_k = \boldsymbol{F}_k,$$
$$\boldsymbol{F}_k = \boldsymbol{F}_k^{(e)} + \sum_j \boldsymbol{F}_{kj}^{(i)}. \tag{1.2}$$

$\boldsymbol{F}_k^{(e)}$ is the external force — such forces emanate from objects outside the system being considered. The second addend is the sum of internal forces (force $\boldsymbol{F}_{kj}^{(i)}$ is the interaction induced by the j-th particle on the k-th particle). Internal interactions happen only between

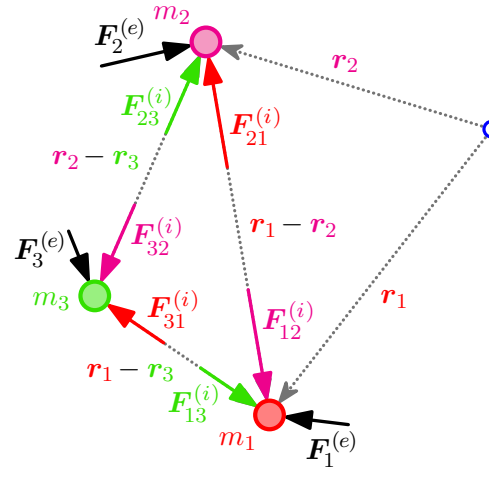


figure 10

*"Axiomata sive Leges Motus" ("Axioms or Laws of Motion") were written by Isaac Newton in his Philosophiæ Naturalis Principia Mathematica, first published in 1687. Reprint (en Latin), 1871. Translated into English by Andrew Motte, 1846.

elements of the system and don't affect (mechanically) anything other. Neither particle interacts with itself, $\boldsymbol{F}_{kk}^{(i)} = \mathbf{0}\ \forall k$.

AXIOMATA
SIVE
LEGES MOTUS

Lex. I.

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Projectilia perseverant in motibus suis nisi quatenus a resistentia aeris retardantur & vi gravitatis impelluntur deorsum. Trochus, cujus partes cohærendo perpetuo retrahunt sese a motibus rectilineis, non cessat rotari nisi quatenus ab aere retardatur. Majora autem Planetarum & Cometarum corpora motus suos & progressivos & circulares in spatiis minus resistentibus factos conservant diutius.

Lex. II.

Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur.

Si vis aliqua motum quemvis generet, dupla duplum, tripla triplum generabit, sive simul & semel, sive gradatim & successive impressa fuerit. Et hic motus quoniam in eandem semper plagam cum vi generatrice determinatur, si corpus antea movebatur, motui ejus vel conspiranti additur, vel contrario subducitur, vel obliquo oblique adjicitur, & cum eo secundum utriusq; determinationem componitur.

Lex. III.

Actioni contrariam semper & æqualem esse reactionem : sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.

Quicquid premit vel trahit alterum, tantundem ab eo premitur vel trahitur. Siquis lapidem digito premit, premitur & hujus digitus a lapide. Si equus lapidem funi allegatum trahit, retrahetur etiam & equus aequaliter in lapidem : nam funis utrinq; distentus eodem relaxandi se conatu urgebit Equum versus lapidem, ac lapidem versus equum, tantumq; impediet progressum unius quantum promovet progressum alterius. Si corpus aliquod in corpus

aliut impingens, motum ejus vi sua quomodocunq: mutaverit, idem quoque vicissim in motu proprio eandem mutationem in partem contrariam vi alterius (ob æqualitatem pressionis mutuæ) subibit. His actionibus æquales fiunt mutationes non velocitatum sed motuum, (scilicet in corporibus non aliunde impeditis:) Mutationes enim velocitatum, in contrarias itidem partes factæ, quia motus æqualiter mutantur, sunt corporibus reciproce proportionales.

..... Rene Descartes' mechanics .....

Measuring motions in mechanics is, however, more controversial than measuring interactions. The discord and the extensive polemic on this topic dates back to the times of Newton and Leibniz. In those days, exploring how objects of various masses change the speed and velocity* of their motion when various forces are applied to it, both Newton and Leibnitz were looking for a useful invariant that would fit the observations.

- "*mv, the product of mass and velocity, is a useful quantity that remains constant*" thought Newton.
- "mv^2*, the product of mass and velocity squared, is a useful quantity that remains constant*" thought Leibniz.

And each of them believed that the quantity he proposed is more useful, more fundamental and more "fruitful".

Newton named *mv* as "quantitas motus" ("quantity of motion")

momentum is a measure of mechanical motion of an object

momentum depends on the weight (i.e. quantity) and velocity of an object.

Momentum is the product of mass and velocity, so

when either an object's mass or its velocity changes , then the momentum will change

.....

what is momentum? The measure of motion in mechanics is called "momentum"...

* *Speed* is the time rate of motion, that is *how fast* a thing moves along some path, a scalar. *Velocity* is the movement's rate and direction, that's *how fast* ***and where*** a thing moves, a vector.

....hmmmmm https://hsm.stackexchange.com/questions/769/when-and-by-whom-was-the-term-momentum-introduced

why "mass by velocity" measures the amount of motion ? there are two momenta known, the linear (translational) one and the angular (rotational) one, why?

Why is momentum defined as mass times velocity?

https://physics.stackexchange.com/a/577486/377185

If you read the history, you'll find there was much discussion, rivalry, and even bad blood as each pushed the benefits of their particular view. Each thought that their quantity was more fundamental, or more important.

Now, we see that both are useful, just in different contexts.

I'm sure somebody briefly toyed with the expressions like mv^3 and maybe m^2v before quickly finding that they didn't stay constant under any reasonable set of constraints, so had no predictive power. That's why they're not named, or used for anything.

So why has the quantity mv been given a name? Because it's useful, it's conserved, and it allows us to make predictions about some parameters of a mechanical system as it undergoes interactions with other things.

from *Leibniz and the Vis Viva Controversy by Carolyn Iltis (1971)*

Roger Boscovich in 1745 and Jean d'Alembert in 1758 both pointed out that vis viva mv^2 and momentum mv were equally valid.

The momentum of a body is actually the Newtonian force F acting through a time, since $dv = adt$ and $mdv = madt = Fdt$.

The kinetic energy is the Newtonian force F acting over a space, since $d(v^2) = 2vdv = 2ads$ and $md(v^2) = 2mads$ or $\frac{1}{2}md(v^2) = Fds$.

$$v \equiv \dot{s}, \quad a \equiv \dot{v} \equiv \ddot{s}$$

$$\begin{aligned} ds &= \dot{s}dt & d\dot{s} &= \ddot{s}dt \\ &= vdt & dv &= adt \end{aligned}$$

$$\begin{aligned} vdv &= vadt = avdt = ads \\ \dot{s}d\dot{s} &= \dot{s}\ddot{s}dt = \ddot{s}\dot{s}dt = \ddot{s}ds \end{aligned}$$

The amount of movement of some object is the product of the mass and velocity of that object.

When the two objects collide, ...

the (linear, translational) momentum

$$\begin{aligned} m_k \dot{\boldsymbol{r}}_k &\quad \text{— for the } k\text{-th particle}, \\ \sum_k m_k \dot{\boldsymbol{r}}_k &\quad \text{— for the whole discrete system} \end{aligned} \tag{1.3}$$

and the angular (rotational) momentum

$$\begin{aligned} \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k &\quad \text{— for the } k\text{-th particle}, \\ \sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k &\quad \text{— for the whole discrete system}. \end{aligned} \tag{1.4}$$

From (1.2) together with the action–reaction principle

$$\boldsymbol{F}_{kj}^{(i)} = -\boldsymbol{F}_{jk}^{(i)} \ \forall k, j \ \Rightarrow \ \sum_{k,j} \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0} \tag{1.5}$$

ensues the balance of linear momentum

$$\Big(\sum_k m_k \dot{\boldsymbol{r}}_k \Big)^{\bullet} = \sum_k m_k \ddot{\boldsymbol{r}}_k = \sum_k \boldsymbol{F}_k^{(e)}. \tag{1.6}$$

And here's the balance of angular momentum*

$$\Big(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \Big)^{\bullet} = \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k \tag{1.7}$$

— is the sum $\sum \boldsymbol{M}_k$ of moments. The moment $\boldsymbol{M}_k$, acting on the k-th particle

$$\boldsymbol{M}_k = \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k^{(e)} + \boldsymbol{r}_k \times \sum_j \boldsymbol{F}_{kj}^{(i)}. \tag{1.8}$$

When in addition to the action–reaction principle, all internal interactions between particles are assumed to be central, that is

$$\boldsymbol{F}_{kj}^{(i)} \parallel \big(\boldsymbol{r}_k - \boldsymbol{r}_j \big) \ \Leftrightarrow \ \big(\boldsymbol{r}_k - \boldsymbol{r}_j \big) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0}, \tag{1.9}$$

* $\Big(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \Big)^{\bullet} = \sum_k \dot{\boldsymbol{r}}_k \times m_k \dot{\boldsymbol{r}}_k + \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k = \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k$
$\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0} \ \forall \boldsymbol{a} \ \Rightarrow \ \dot{\boldsymbol{r}}_k \times \dot{\boldsymbol{r}}_k = \boldsymbol{0}$

the balance of rotational (angular) momentum becomes*

$$\Big(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k\Big)^{\bullet} = \sum_k \boldsymbol{r}_k \times \boldsymbol{F}_k^{(e)}. \tag{1.10}$$

Thus, all changes in the linear and angular momenta are due only to external forces $\boldsymbol{F}_k^{(e)}$, not internal ones.

Unlike for momenta, the balance of kinetic energy $\mathrm{K} \equiv \frac{1}{2}\sum m_k \dot{\boldsymbol{r}}_k \cdot \dot{\boldsymbol{r}}_k$ (mv^2 is Leibniz's "vis viva") includes the power of internal forces as well

$$\begin{aligned}\dot{\mathrm{K}} = \Big(\tfrac{1}{2}\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \dot{\boldsymbol{r}}_k\Big)^{\bullet} &= \tfrac{1}{2}\sum_k \big(m_k \ddot{\boldsymbol{r}}_k \cdot \dot{\boldsymbol{r}}_k + m_k \dot{\boldsymbol{r}}_k \cdot \ddot{\boldsymbol{r}}_k\big) \\ = \sum_k m_k \ddot{\boldsymbol{r}}_k \cdot \dot{\boldsymbol{r}}_k = \sum_k \boldsymbol{F}_k \cdot \dot{\boldsymbol{r}}_k &= \sum_k \Big(\boldsymbol{F}_k^{(e)} + \sum_j \boldsymbol{F}_{kj}^{(i)}\Big) \cdot \dot{\boldsymbol{r}}_k \\ &= \sum_k \boldsymbol{F}_k^{(e)} \cdot \dot{\boldsymbol{r}}_k + \sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \dot{\boldsymbol{r}}_k \end{aligned} \tag{1.11}$$

or, using the action–reaction principle (1.5),

$$\dot{\mathrm{K}} - \sum_k \boldsymbol{F}_k^{(e)} \cdot \dot{\boldsymbol{r}}_k = \tfrac{1}{2}\sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \big(\dot{\boldsymbol{r}}_k + \dot{\boldsymbol{r}}_k\big) = \tfrac{1}{2}\sum_{k,j} \Big(\boldsymbol{F}_{kj}^{(i)} \cdot \dot{\boldsymbol{r}}_k - \boldsymbol{F}_{jk}^{(i)} \cdot \dot{\boldsymbol{r}}_k\Big),$$

and since $\sum_{k,j} \boldsymbol{F}_{jk}^{(i)} \cdot \dot{\boldsymbol{r}}_k = \sum_{j,k} \boldsymbol{F}_{kj}^{(i)} \cdot \dot{\boldsymbol{r}}_j = \sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \dot{\boldsymbol{r}}_j$

$$\dot{\mathrm{K}} = \sum_k \boldsymbol{F}_k^{(e)} \cdot \dot{\boldsymbol{r}}_k + \tfrac{1}{2}\sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \big(\dot{\boldsymbol{r}}_k - \dot{\boldsymbol{r}}_j\big). \tag{1.12}$$

.....

all bodies that are limited in free motion possess potential energy

.....

* $\forall k,j\ \ \boldsymbol{F}_{kj}^{(i)} = -\boldsymbol{F}_{jk}^{(i)}$ and $\big(\boldsymbol{r}_k - \boldsymbol{r}_j\big) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0}\ \Rightarrow$

$$\sum_k \boldsymbol{r}_k \times \sum_j \boldsymbol{F}_{kj}^{(i)} = \tfrac{1}{2}\sum_{k,j} \big(\boldsymbol{r}_k + \boldsymbol{r}_k\big) \times \boldsymbol{F}_{kj}^{(i)} = \tfrac{1}{2}\sum_{k,j} \big(\boldsymbol{r}_k - \boldsymbol{r}_j\big) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0}$$

$$\sum_{k,j} \boldsymbol{r}_k \times \boldsymbol{F}_{kj}^{(i)} = -\sum_{k,j} \boldsymbol{r}_k \times \boldsymbol{F}_{jk}^{(i)} = -\sum_{j,k} \boldsymbol{r}_j \times \boldsymbol{F}_{kj}^{(i)} = -\sum_{k,j} \boldsymbol{r}_j \times \boldsymbol{F}_{kj}^{(i)}$$

$$W(\boldsymbol{F}, \boldsymbol{u}) = \boldsymbol{F} \cdot \boldsymbol{u}$$

as the exact (full) differential

$$dW = \frac{\partial W}{\partial \boldsymbol{F}} \cdot d\boldsymbol{F} + \frac{\partial W}{\partial \boldsymbol{u}} \cdot d\boldsymbol{u}$$

by "product rule"

$$dW = d\big(\boldsymbol{F} \cdot \boldsymbol{u}\big) = d\boldsymbol{F} \cdot \boldsymbol{u} + \boldsymbol{F} \cdot d\boldsymbol{u}$$

$\frac{\partial W}{\partial \boldsymbol{F}} = \boldsymbol{u}$, $\frac{\partial W}{\partial \boldsymbol{u}} = \boldsymbol{F}$

...

Center of mass

- ✓ This is the unique point at any time where the weighted relative position of the distributed mass sums to zero.
- ✓ This is the point to which a force may be applied to cause a linear acceleration without an angular acceleration.
- ✓ This is a hypothetical point where the entire mass of an object may be assumed to be concentrated to visualise its motion.
- ✓ This is the particle equivalent of a given object for application of Newton's laws of motion.

When formulated for the center of mass, formulas in mechanics are often simplified.

...

— Are there any scenarios for which the center of mass is not almost exactly equivalent to the center of gravity?

— Non-uniform gravity field. In a uniform gravitational field, the center of mass is equal to the center of gravity.

...

Imposed on the positions and velocities of particles, there are restrictions of a geometrical or kinematical nature, called constraints.

Holonomic constraints are relations between position variables (and possibly time) which can be expressed as equality like

$$f(q^1, q^2, q^3, \ldots, q^n, t) = 0,$$

where $q^1, q^2, q^3, \ldots, q^n$ are n parameters (coordinates) that fully describe the system.

A constraint that cannot be expressed as such is nonholonomic.

Holonomic constraint depends only on coordinates and time. It does not depend on velocities or any higher time derivatives.

Velocity-dependent constraints like

$$f(q^1, q^2, \ldots, q^n, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n, t) = 0$$

are mostly not holonomic.

For example, the motion of a particle constrained to lie on a sphere's surface is subject to a holonomic constraint, but if the particle is able to fall off a sphere under the influence of gravity, the constraint becomes non-holonomic. For the first case the holonomic constraint may be given by the equation: $r^2 - a^2 = 0$, where r is the distance from the centre of a sphere of radius a. Whereas the second non-holonomic case may be given by: $r^2 - a^2 \geq 0$.

Three examples of nonholonomic constraints are: when the constraint equations are nonintegrable, when the constraints have inequalities, or with complicated non-conservative forces like friction.

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q^1, q^2, \ldots, q^n, t)$$

(assuming n independent parameters/coordinates)

.....

Newton's laws and d'Alembert's principle are non-variational principles of mechanics.

Mécanique analytique (1788–89) is a two volume French treatise on analytical mechanics, written by Joseph Louis Lagrange, and published 101 years following Isaac Newton's *Philosophiæ Naturalis Principia Mathematica*.

Joseph Louis Lagrange. Mécanique analytique. Nouvelle édition, revue et augmentée par l'auteur. Tome premier. Mme Ve Courcier, Paris, 1811. 490 pages.

Joseph Louis Lagrange. Mécanique analytique. Troisième édition, revue, corrigée et annotée par M. J. Bertrand. Tome second. Mallet-Bachelier, Paris, 1855. 416 pages.

The historical transition from geometrical methods, as presented in Newton's Principia, to methods of mathematical analysis.

Consider the exact differential of any set of location vectors $\boldsymbol{r}_i$, that are functions of other variable parameters (coordinates) $q^1, q^2, ..., q^n$ and time t.

The actual displacement is the differential

$$d\boldsymbol{r}_i = \frac{\partial \boldsymbol{r}_i}{\partial t}\, dt + \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q^j}\, dq^j$$

Now, imagine an arbitrary path through the configuration space/manifold. This means it has to satisfy the constraints of the system but not the actual applied forces

$$\delta\boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q^j}\, \delta q^j$$

A virtual infinitesimal displacement of a system of particles refers to a change in the configuration of a system as the result of any arbitrary infinitesimal change of location vectors (or coordinates) $\delta\boldsymbol{r}_k$, consistent with the forces and constraints imposed on the system at the current/given instant t. This displacement is called "virtual" to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the forces and constraints may be changing.

Assume the system is in equilibrium, that is the full force on each particle vanishes, $\boldsymbol{F}_i = \boldsymbol{0}\ \forall i$. Then clearly the term $\boldsymbol{F}_i \bullet \delta\boldsymbol{r}_i$, which is the virtual work of force $\boldsymbol{F}_i$ in displacement $\delta\boldsymbol{r}_i$, also vanishes for each particle, $\boldsymbol{F}_i \bullet \delta\boldsymbol{r}_i = 0\ \forall i$. The sum of these vanishing products over all particles is likewise equal to zero:

$$\sum_i \boldsymbol{F}_i \bullet \delta\boldsymbol{r}_i = 0.$$

Decompose the full force $\boldsymbol{F}_i$ into the applied (active) force $\boldsymbol{F}_i^{(a)}$ and the force of constraint $\boldsymbol{C}_i$,

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{(a)} + \boldsymbol{C}_i$$

We now restrict ourselves to systems for which the net virtual work of the force of every constraint is zero:

$$\sum_i \boldsymbol{C}_i \bullet \delta\boldsymbol{r}_i = 0.$$

We therefore have as the condition for equilibrium of a system that the virtual work of all applied forces vanishes:

$$\sum_i \boldsymbol{F}_i^{(a)} \bullet \delta\boldsymbol{r}_i = 0.$$

— the principle of virtual work.

Note that coefficients $\boldsymbol{F}_i^{(a)}$ can no longer be thought equal to zero: in common $\boldsymbol{F}_i^{(a)} \neq 0$, since $\delta\boldsymbol{r}_i$ are not independent but are bound by constraints.

A virtual displacement of a particle with vector radius $\boldsymbol{r}_k$ is variation $\delta\boldsymbol{r}_k$ — any infinitesimal change of vector $\boldsymbol{r}_k$, which is compatible with the constraints. If the system is free, that is there are no constraints, then virtual displacements $\delta\boldsymbol{r}_k$ are perfectly random.

A constraint can be holonomic, aka geometric, bounding only positions (displacements) — a function of only coordinates and probably time

$$c(\boldsymbol{r}, t) = 0. \tag{2.1}$$

A nonholonomic, or differential, constraint includes time derivatives of coordinates

$$c(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = 0 \tag{2.2}$$

and is not integrable into a geometric constraint. A system with nonholonomic constraints depends on the path — the intermediate values of trajectory, not only upon the initial and final locations, and thus cannot be represented by a potential function $\Pi(\boldsymbol{r}, t)$.

When all constraints are holonomic, then the system is completely independent of the trajectory of transition, of any intermediate values along the path. Here, the virtual displacements of the k-th particle satisfy the equation

$$\sum_{j=1}^{m} \frac{\partial c_j}{\partial \boldsymbol{r}_k} \cdot \delta \boldsymbol{r}_k = 0. \tag{2.3}$$

In constrained (non-free) systems, all forces can be divided into two groups : the active forces $\boldsymbol{F}_k^{(a)}$ and the constraint (or reaction) forces.

Reaction $\boldsymbol{C}_k$ acts on the k-th particle from all material limiters and changes according to equation (2.1) for each constraint.

The constraints are assumed to be ideal, that is

$$\sum_k \boldsymbol{C}_k \cdot \delta \boldsymbol{r}_k = 0 \tag{2.4}$$

— the work of constraint (reaction) forces is equal to zero on any virtual displacement.

The principle of virtual work is

$$\sum_k \left(\boldsymbol{F}_k^{(a)} - m_k \ddot{\boldsymbol{r}}_k \right) \cdot \delta \boldsymbol{r}_k = 0, \tag{2.5}$$

where $\boldsymbol{F}_k^{(a)}$ are only active forces, without reactions of constraints.

Differential variational equation (2.5) may seem like a trivial consequence of the Newton's law (1.1) and the ideality of constraints (2.4). Nevertheless, the contents of (2.5) is incomparably broader. In this book the reader will see that the principle (2.5) can be put into the foundation of mechanics [92]. The various models of media, being described in this book, are based on that principle.

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Проявилась замечательная особенность (2.5) : это уравнение эквивалентно системе такого порядка, каково число степеней свободы системы, то есть сколько независимых вариаций $\delta\boldsymbol{r}_k$ мы имеем. Если система N точек имеет m связей, то число степеней свободы $n = 3N - m$.

...

§ 3. Balance of momentum, angular momentum, and energy

These balance equations may be related to the properties of space and time [93].— Emmy Noether's "conserved quantities from symmetries" theorem

Symmetry in the Universe	⇔	*Conservation Law*
translational symmetry	⇔	conservation of linear momentum
rotational symmetry	⇔	conservation of angular momentum
temporal symmetry	⇔	conservation of energy

Conservation of (linear) momentum (amount of movement) in a closed (isolated)* system is derived from the homogeneity of space *(любой параллельный перенос — трансляция — закрытой системы как целого не меняет свойства этой системы)*. Conservation of angular (rotational) momentum — следствие изотропии пространства *(свойства закрытой системы не меняются с любым*

* *An isolated (closed) system* is a system whose elements interact only with each other, but not with other elements.

поворотом такой системы как целого). Энергия же изолированной системы сохраняется, так как время однородно* (energy $\mathrm{E} \equiv \mathrm{K}(q, \dot{q}) + \Pi(q)$ такой системы не зависит явно от времени).

The balance equations can be derived from the principle of virtual work (2.5). Перепишем его, выделив внешние силы $\boldsymbol{F}_k^{(e)}$ и виртуальную работу внутренних сил $\delta W^{(i)} = \sum\limits_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \delta \boldsymbol{r}_k$

$$\sum_k \left(\boldsymbol{F}_k^{(e)} - m_k \ddot{\boldsymbol{r}}_k \right) \cdot \delta \boldsymbol{r}_k + \delta W^{(i)} = 0. \tag{3.1}$$

It's assumed that internal forces don't do any work on virtual displacements of a system as a rigid whole ($\boldsymbol{\delta\rho}$ and $\boldsymbol{\delta o}$ are some constant vectors describing translation and rotation)

$$\begin{aligned} &\delta \boldsymbol{r}_k = \boldsymbol{\delta\rho} + \boldsymbol{\delta o} \times \boldsymbol{r}_k, \\ &\boldsymbol{\delta\rho} = \textbf{constant},\ \boldsymbol{\delta o} = \textbf{constant} \end{aligned} \quad \Rightarrow \ \delta W^{(i)} = 0. \tag{3.2}$$

Premises and considerations for this assumption are as follows. The first — for the case of potential, such as elastic, forces. A variation of the work of potential internal forces $W^{(i)}$ is a variation of the potential Π with the opposite sign,

$$\delta W^{(i)} = -\delta \Pi. \tag{3.3}$$

And it's quite obvious that Π alters only by deforming. The second consideration — the internal forces are balanced in the sense that the net vector (the resultant force) and the net moment (the resultant couple) are $\mathbf{0}$,

(1.5) & (1.9)

$$\sum \dots$$

...

* Характеристики "однородность" and "изотропность" пространства, "однородность" времени не фигурируют среди аксиом классической механики.

Принимая (3.2) и подставляя в (3.1) сначала $\delta\boldsymbol{r}_k = \boldsymbol{\delta\rho}$ (трансляция), а затем $\delta\boldsymbol{r}_k = \boldsymbol{\delta o} \times \boldsymbol{r}_k$ (поворот), получаем баланс импульса (...) and баланс момента импульса (...).

...

§ 4. Hamilton's principle and Lagrange's equations

The two branches of analytical mechanics are Lagrangian mechanics (operating with generalized coordinates and corresponding generalized velocities in configuration space) and Hamiltonian mechanics (operating with coordinates and corresponding momenta in phase space). Both formulations are equivalent by a Legendre transformation on the generalized coordinates, velocities and momenta, therefore both contain the same information for describing the dynamics of a system.

Variational equation (2.5) is satisfied at any moment of time. Проинтегрируем его (оттуда лишь равенство F = ma)* по какому-либо промежутку $[t_1, t_2]$

$$\int_{t_1}^{t_2} \left(\delta\mathrm{K} + \sum_k \boldsymbol{F}_k \cdot \delta\boldsymbol{r}_k \right) dt - \left[\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta\boldsymbol{r}_k \right]_{t_1}^{t_2} = 0. \qquad (4.1)$$

Без ущерба для общности можно принять $\delta\boldsymbol{r}_k(t_1) = \delta\boldsymbol{r}_k(t_2) = \boldsymbol{0}$, then the non-integral term vanishes.

Generalized coordinates q^i ($i = 1, \ldots, n$ — the number of degrees o'freedom) are introduced. Location vectors become functions like $\boldsymbol{r}_k(q^i, t)$, тождественно удовлетворяющими уравнениям связей (2.1). Если связи стационарны, то есть уравнения (2.1) не содержат t, то остаётся $\boldsymbol{r}_k(q^i)$. Kinetic energy превращается в функцию $\mathrm{K}(q^i, \dot{q}^i, t)$, где явно входящее t характерно лишь для нестационарных связей.

* $\delta\mathrm{K} = \sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta\dot{\boldsymbol{r}}_k$, $\left(\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta\boldsymbol{r}_k \right)^{\cdot} = \sum_k m_k \ddot{\boldsymbol{r}}_k \cdot \delta\boldsymbol{r}_k + \underbrace{\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta\dot{\boldsymbol{r}}_k}_{\delta\mathrm{K}}$

$$\Rightarrow \quad \left[\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta\boldsymbol{r}_k \right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \sum_k m_k \ddot{\boldsymbol{r}}_k \cdot \delta\boldsymbol{r}_k \, dt + \int_{t_1}^{t_2} \delta\mathrm{K} dt$$

Hello to the essential concept of generalized forces. They originate from the virtual work $\boldsymbol{F}_k \cdot \delta \boldsymbol{r}_k$. With variation $\delta \boldsymbol{r}_k$ of the k-th point's location vector, expanded for generalized coordinates q^i,

$$\delta \boldsymbol{r}_k = \sum_i \frac{\partial \boldsymbol{r}_k}{\partial q^i} \delta q^i , \tag{4.2}$$

the virtual work can be written as

$$\sum_k \boldsymbol{F}_k \cdot \delta \boldsymbol{r}_k = \sum_k \boldsymbol{F}_k \cdot \sum_i \frac{\partial \boldsymbol{r}_k}{\partial q^i} \delta q^i = \sum_{i,k} \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i} \delta q^i$$
$$= \sum_i \left(\sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i} \right) \delta q^i = \sum_i Q_i \delta q^i , \tag{4.3}$$

where

$$Q_i \equiv \sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i} . \tag{4.4}$$

It's worth to accentuate once more the origin of generalized forces from work. Having chosen the generalized coordinates q^i for the problem, the applied forces $\boldsymbol{F}_k$ are then grouped into the sets of generalized forces Q_i.

The particular case of potential forces is very relevant for this book. A force is *potential* when the work done by it depends only on locations of points, but not on paths between them. Then the potential energy Π can be introduced as a scalar field, also dubbed a "potential field" or just "a potential", because it is a function of only coordinates $\Pi = \Pi(q^i)$ (and possibly time, $\Pi(q^i, t)$ — the explicit dependence on time t may appear due to non-stationary constraints or because the physical fields themselves depend on time). $\delta\Pi = \sum \frac{\partial \Pi}{\partial q^i} \delta q^i$ is a variation of energy Π. The function Π is usually defined so that a positive work is a reduction in the potential. Thus, if generalized forces are potential, then

$$\sum_i Q_i \delta q^i = -\delta\Pi, \quad Q_i = -\frac{\partial \Pi}{\partial q^i} . \tag{4.5}$$

...

Since there are Lagrange's equations "of the second kind", the reader may guess that equations "of the first kind" exist too. Yes, they are known. And they are worth mentioning at least because the derivation method, founded on these equations, is used in this book many times.

When constraints (2.1) are imposed on a system, the equality $\boldsymbol{F}_k = m_k \ddot{\boldsymbol{r}}_k$ doesn't follow from the variational equation (2.5), since virtual displacements $\delta\boldsymbol{r}_k$ are then not independent. Having m constraints and therefore m conditions (2.3) for variations, each of these conditions is multiplied by some scalar λ_a $(a = 1, \dots, m)$ and added to (2.5), turning into

$$\sum_{k=1}^{N} \left(\boldsymbol{F}_k^{(a)} + \sum_{a=1}^{m} \lambda_a \frac{\partial c_a}{\partial \boldsymbol{r}_k} - m_k \ddot{\boldsymbol{r}}_k \right) \cdot \delta\boldsymbol{r}_k = 0. \tag{4.6}$$

Among $3N$ components of variations $\delta\boldsymbol{r}_k$, m are dependent. Aha, and the number of Lagrange multipliers λ_a is m too. Если выбрать λ_α такие, что coefficients(??какие?) for dependent variations обращаются в нуль, то тогда у остальных вариаций коэффициенты(??) также будут нулевые из-за независимости. Следовательно, все выражения в скобках $(\cdots)$ равны нулю — это и есть Lagrange's equations of the first kind.

Since for each particle

...

§ 5. Statics

Let there is a mechanical system with stationary (constant over time) constraints under the action of static (not changing with time) active forces $\boldsymbol{F}_k$. In equilibrium all $\boldsymbol{r}_k = \textsf{constant}$, hence $\delta\boldsymbol{r}_k = \boldsymbol{0}$, $\frac{\partial \boldsymbol{r}_k}{\partial q^i} = \boldsymbol{0}$, and the principle of virtual work is formulated as

$$\sum_k \boldsymbol{F}_k \cdot \delta\boldsymbol{r}_k = 0 \quad \Leftrightarrow \quad \sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i} = Q_i = 0. \tag{5.1}$$

Both pieces are essential : and the variational equation, and zeros in the generalized forces.

Relations (5.1) are the most generic and universal equations of statics. In literature, the narrow conception of the equilibrium equations as the balance of forces and moments is widespread. But in that case too, as in any other, the set of the equilibrium equations exactly matches with the generalized coordinates. "The resultant force" (also referred to as "the net (full) force" or "the net vector") and "the resultant couple" ("the net couple", "the net moment") figure in the equilibrium equations* just because the system has translational and rotational degrees o'freedom. The huge popularity of forces and moments (force couples) comes not as much from the prevalence of statics of a perfectly non-deformable (ideally rigid) solid body, but more from the fact that the virtual work of internal forces on all movements of the system as a rigid whole is always equal to zero for any medium.

Let two kinds o'forces act in the system : potential, with the coordinate-dependent energy $\Pi(q^i)$, and plus external ones $\overset{\circ}{Q}{}_j^{(e)} \equiv P_j$. From (5.1) follow the equilibrium equations

$$\frac{\partial \Pi}{\partial q^i} = P_i \tag{5.2}$$

and the exact differential of $\Pi(q^i)$ (time independent) is

$$d\Pi = \sum_i \frac{\partial \Pi}{\partial q^i} dq^i = \sum_i P_i \, dq^i. \tag{5.3}$$

Equations (5.2) formulate the problem of statics, non-linear in overall, about the relation of the equilibrium position $q^i_\circ$ with the external loads P_i.

* Since **Louis Poinsot** described the reduction of any set of forces, acting on the same absolutely rigid system, into the single force and the single couple in his book "Élémens de statique", first published in 1803 — Éléments de statique, chez Calixte-Volland, 1803 (an XII). 267 p. Onzième (11ème) édition : Gauthier-Villars, 1873.

A linear system with quadratic potential Π as a function of coordinates

$$\Pi = \frac{1}{2}\sum_{i,k} C_{ik}\, q^i q^k \tag{5.4}$$

$$\sum_k C_{ik} q^k = P_i. \tag{5.5}$$

Here figure elements C_{ik} of "the stiffness matrix", coordinates q^k and external loads P_i.

Structures (both human-made artificial and in the nature) most often have a positive-definite stiffness matrix C_{ik}. Then $\det C_{ik} > 0$, the solution of a linear algebraic system (5.5) is unique, and this solution can be substituted by minimization of the quadratic form

$$\mathscr{E}(q^j) \equiv \Pi - \sum_i P_i q^i = \frac{1}{2}\sum_{i,k} q^i C_{ik} q^k - \sum_i P_i q^i \;\to\; \min. \tag{5.6}$$

However, the design may be so unfortunate that the stiffness matrix becomes singular (noninvertible) with $\det C_{ik} = 0$ (or the determinant is very close to zero, $\det C_{ik} \approx 0$ — the nearly singular matrix). Then the solution of the linear problem of statics (5.5) exists only when external loads P_i are orthogonal to all linearly independent solutions of the homogeneous conjugate system

...

The famous theorems of statics for linear continua (§ 4.5) can be easily proved for a finite number of degrees o'freedom. The Clapeyron's theorem looks here like

...

The reciprocal work theorem ("the work W_{12} of the first set's forces on displacements from the forces of the second is equal to the work W_{21} of the second set's forces on displacements from the forces of the first") instantly derives from (5.5) :

(....)

Here the symmetry of the stiffness matrix $C_{ij} = C_{ji}$ is essential — that the system is conservative.

.....

Turning back to the problem (5.2), sometimes called the Lagrange's theorem. Inverted by Legendre transform(ation), it translates into

$$d\Big(\sum_i P_i q^i\Big) = \sum_i d\big(P_i q^i\big) = \sum_i \big(q^i\, dP_i + P_i\, dq^i\big),$$
$$\sum_i d\big(P_i q^i\big) - \underbrace{\sum_i P_i\, dq^i}_{d\Pi} = \sum_i q^i\, dP_i\,,$$
$$d\Big(\sum_i P_i q^i - \Pi\Big) = \sum_i q^i\, dP_i = \underbrace{\sum_i \frac{\partial \amalg}{\partial P_i}\, dP_i}_{d\amalg},$$

where appears the exact differential of the so-called "complementary energy" $\amalg$

$$q^i = \frac{\partial \amalg}{\partial P_i}, \quad \amalg(P_i) = \sum_i P_i q^i - \Pi. \tag{5.7}$$

This is known as the Castigliano theorem*. For a linear system (5.5) $\Rightarrow \amalg(P_i) = \Pi(q^i)$. Theorem (5.7) is sometimes very useful — when the complementary energy as the function of external loads $\amalg(P_i)$ is easy to find. Someone may come across the so-called "statically determinate" structures (systems), for which all internal forces can luckily be found just only from the balance (equilibrium) equations for forces and moments. For such structures, (5.7) is effective.

Unlike the linear problem (5.5), the nonlinear problem (5.2) may have no solutions at all or may have several of them.

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* **Carlo Alberto Castigliano**. Intorno ai sistemi elastici, Dissertazione presentata da Castigliano Alberto alla Commissione Esaminatrice della R. Scuola d'Applicazione degli Ingegneri in Torino per ottenere la Laurea di Ingegnere Civile. Torino, Vincenzo Bona, 1873.

The overview of statics in classical mechanics I am ending with the d'Alembert's principle*: the dynamic equations differ from the static ones only in additional "inertia forces" ("fictitious forces") $m_k \ddot{\boldsymbol{r}}_k$. The d'Alembert's principle is pretty obvious, but applying it everytime & everywhere is a mistake. As example, the equations of motion for a viscous fluid (Navier–Stokes equations) in statics and in dynamics differ not only in inertial adjunct. Nevertheless, for solid elastic bodies the d'Alembert's principle always apply.

§ 6. Small oscillations (vibrations)

If the statics of a linear system is described by equation (5.5), then in the dynamics we have

$$\sum_k \left(A_{ik} \ddot{q}_k + C_{ik} q^k \right) = P_i(t), \tag{6.1}$$

where A_{ik} is the symmetric and positive "matrix of kinetic energy".

Any description of oscillations almost always includes the term "mode". A mode of vibration can be defined as a way of vibrating or a pattern of vibration. A normal mode is a pattern of periodic motion, when all parts of an oscillating system move sinusoidally with the same frequency and with a fixed phase relation. The free motion described by the normal modes takes place at fixed frequencies — the natural resonant frequencies of an oscillating system.

* **Jean Le Rond d'Alembert**. Traité de Dynamique, dans lequel les Loix de l'Equilibre & du mouvement des Corps sont réduites au plus petit nombre possible, & démontrées d'une maniére nouvelle, & où l'on donne un Principe général pour trouver le Mouvement de plusieurs Corps qui agissent les uns sur les autres, d'une maniére quelconque. Paris : David l'aîné, MDCCXLIII (1743).

The most generic motion of an oscillating system is some superposition of normal modes of this system.*

A research of an oscillating system most often begins with orthogonal (normal) "modes" — harmonics, free (without any driving or damping force) sinusoidal oscillations

$$q^k(t) = \overset{*}{q}_k \sin \omega_k t.$$

Multipliers $\overset{*}{q}_k = \text{constant}$ are orthogonal (normal) "modes" of oscillation, ω_k are natural (resonant, eigen-) frequencies. This set, dependent on the structure of an oscillating object, the materials and the boundary conditions, is found from the eigenvalue problem

$$\begin{gathered} P_i = 0, \quad \ddot{\overset{*}{q}}_k = -\omega_k^2 \overset{*}{q}_k \sin \omega_k t, \quad (6.1) \quad \Rightarrow \\ \Rightarrow \quad \sum_k \left(C_{ik} - A_{ik} \omega_k^2 \right) \overset{*}{q}_k \sin \omega_k t = 0 \end{gathered} \tag{6.2}$$

...

The Duhamel's integral is a way of calculating the response of linear systems to an arbitrary time-varying external perturbation.

...

§ 7. Perfectly rigid undeformable solid body

> "Absolutely rigid", aka "absolutely solid" and "absolutely durable" — the pipe dream of any engineer.

One more concept, modeled in classical generic mechanics, is the (perfectly) rigid body. That is a solid** body, in which

* The modes are "normal" in the sense that they move independently, and an excitation of one mode will never cause a motion of another mode. In mathematical terms, normal modes are orthogonal to each other. In music, normal modes of vibrating instruments (strings, air pipes, percussion and others) are called "harmonics" or "overtones".

** "Rigid" is inelastic and not flexible, and "solid" is not fluid. A solid substance retains its size and shape without a container (as opposed to a fluid substance, a liquid or a gas).

deformation is zero (or is negligibly small — so small that it can be neglected). The distance between any two points of a non-deformable rigid body remains constant regardless of external forces exerted on it.

A non-deformable rigid body is modeled using the "continual approach" as a continuous distribution of mass (a material continuum, a continuous medium), rather than using the "discrete approach" (that is modeling a body as a discrete collection of particles, § 1).

The mass of a material continuum is distributed continuously throughout its volume,

$$dm \equiv \rho\, d\mathcal{V} \tag{7.1}$$

(ρ is a volume(tric) mass density and $d\mathcal{V}$ is an infinitesimal volume).

A formula with summation over discrete points becomes a formula for a continuous body by replacing the masses of particles with the mass (7.1) of an infinitesimal volume element $d\mathcal{V}$ with integration over the entire volume of a body. In particular, here are the formulas for the (linear) momentum

$$\sum_k m_k \dot{\boldsymbol{r}}_k \quad \text{becomes} \quad \int_{\mathcal{V}} \dot{\boldsymbol{r}}\, dm \tag{7.2}$$

and for the angular momentum

$$\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \quad \text{becomes} \quad \int_{\mathcal{V}} \boldsymbol{r} \times \dot{\boldsymbol{r}}\, dm. \tag{7.3}$$

To fully describe the location (position, place) of any non-deformable body with all its points, it's enough to choose some unique point as the "pole", to find or to set the location $\boldsymbol{p} = \boldsymbol{p}(t)$ of the chosen point, as well as the angular orientation of a body relative to the pole (figure 11). As a result, any motion of an undeformable rigid body is either a rotation around the chosen pole, or an equal displacement of the pole and all body's points — a translation (a linear motion)*, or a combination of them both.

* A translation can also be thought of as a rotation with the revolution center at infinity.

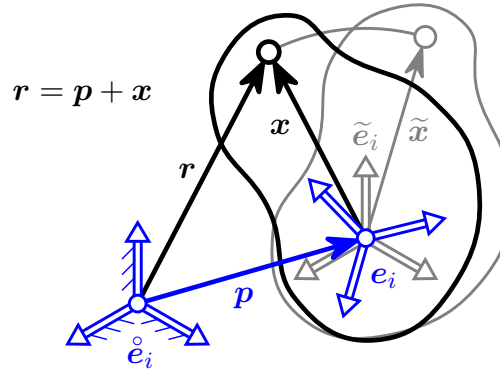

figure 11

$\mathring{\boldsymbol{e}}_i$ — the triplet of mutually perpendicular unit vectors, called the "basis vectors", immovable relatively to the absolute (or to any inertial) reference system

✓ $\mathring{\boldsymbol{e}}_i$ is the immovable (stationary) basis

✓ $\boldsymbol{e}_i$ is the basis which moves along with the body

By adding the basis $\boldsymbol{e}_i$ (it moves together with the body), the body's angular orientation can be determined by the rotation tensor $\boldsymbol{O} \equiv \boldsymbol{e}_i\widetilde{\boldsymbol{e}}_i$.

Then any motion of a body is completely described by two functions, $\boldsymbol{p}(t)$ and $\boldsymbol{O}(t)$.

The location vector of some body's point

$$\boldsymbol{r} = \boldsymbol{p} + \boldsymbol{x} \tag{7.4}$$

$\widetilde{\boldsymbol{x}} = x_i\widetilde{\boldsymbol{e}}_i$, $\boldsymbol{x} = x_i\boldsymbol{e}_i$

(11.3), § 1.11

$\boldsymbol{x} = \boldsymbol{O} \cdot \widetilde{\boldsymbol{x}}$

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{p}} + \dot{\boldsymbol{x}},$$

For a non-deformable rigid body, components x_i don't depend on time: $x_i = \mathsf{constant}(t)$ and $\dot{\boldsymbol{x}} = x_i\dot{\boldsymbol{e}}_i$

$\dot{\boldsymbol{x}} = \dot{\boldsymbol{O}} \cdot \mathring{\boldsymbol{x}}$

$x_i\dot{\boldsymbol{e}}_i = \dot{\boldsymbol{O}} \cdot x_i\mathring{\boldsymbol{e}}_i \;\Leftrightarrow\; \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{O}} \cdot \mathring{\boldsymbol{e}}_i$

...

The linear momentum and the rotational (angular) momentum of a non-deformable continuous body are described by the following integrals

...

...

$$\int_{\mathcal{V}} \boldsymbol{p}\, dm = \boldsymbol{p} \int_{\mathcal{V}} dm = \boldsymbol{p} m$$

$$\int_{\mathcal{V}} \boldsymbol{x}\, dm = \boldsymbol{\Xi} m, \;\; \boldsymbol{\Xi} \equiv m^{-1} \int_{\mathcal{V}} \boldsymbol{x}\, dm$$

Three inertial characteristics of the body:

- ✓ integral mass $m = \int_{\mathcal{V}} dm = \int_{\mathcal{V}} \rho\, d\mathcal{V}$ — the mass of the whole body,
- ✓ eccentricity vector $\boldsymbol{\Xi}$ — measures the offset of the chosen pole from the body's "center of mass",
- ✓ inertia tensor ${}^2\boldsymbol{\mathfrak{I}}$.

The eccentricity vector is equal to the null vector only when the chosen pole coincides with the "center of mass" — the unique point within a body with location vector $\boldsymbol{\imath}$, in short

$$\boldsymbol{\Xi} = \boldsymbol{0} \;\Leftrightarrow\; \boldsymbol{p} = \boldsymbol{\imath}.$$

$$\boldsymbol{x} = \boldsymbol{r} - \boldsymbol{p}, \;\; \boldsymbol{\Xi} m = \int_{\mathcal{V}} \big(\boldsymbol{r} - \boldsymbol{\imath}\big)\, dm = \boldsymbol{0},$$

$$\int_{\mathcal{V}} \boldsymbol{r}\, dm - \boldsymbol{\imath} \int_{\mathcal{V}} dm = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{\imath} = m^{-1} \int_{\mathcal{V}} \boldsymbol{r}\, dm$$

...

Introducing the (pseudo)vector of angular velocity $\boldsymbol{\omega}$, ...

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i$$

...

inertia tensor ${}^2\mathfrak{J}$

$$ {}^2\mathfrak{J} \equiv -\int_{\mathcal{V}} (\boldsymbol{x}\times\boldsymbol{E})\cdot(\boldsymbol{x}\times\boldsymbol{E})\,dm = \int_{\mathcal{V}} (\boldsymbol{x}\cdot\boldsymbol{x}\boldsymbol{E} - \boldsymbol{x}\boldsymbol{x})\,dm $$

It is assumed (can be proven?) that the inertia tensor changes only due to a rotation

$$ {}^2\mathfrak{J} = \boldsymbol{O}\cdot{}^2\mathring{\mathfrak{J}}\cdot\boldsymbol{O}^{\mathsf{T}} $$

and if some basis $\boldsymbol{e}_j$ is moving along with the body, the inertia components in that basis don't change over time

$$ {}^2\mathfrak{J} = \mathfrak{J}_{ab}\boldsymbol{e}_a\boldsymbol{e}_b,\ \ \mathfrak{J}_{ab} = \mathsf{constant}(t), $$

thus the time derivative is

$$ \begin{aligned} {}^2\dot{\mathfrak{J}} &= \mathfrak{J}_{ab}\left(\dot{\boldsymbol{e}}_a\boldsymbol{e}_b + \boldsymbol{e}_a\dot{\boldsymbol{e}}_b\right) = \mathfrak{J}_{ab}\left(\boldsymbol{\omega}\times\boldsymbol{e}_a\boldsymbol{e}_b + \boldsymbol{e}_a\boldsymbol{\omega}\times\boldsymbol{e}_b\right) \\ &= \boldsymbol{\omega}\times\mathfrak{J}_{ab}\boldsymbol{e}_a\boldsymbol{e}_b - \mathfrak{J}_{ab}\boldsymbol{e}_a\boldsymbol{e}_b\times\boldsymbol{\omega} = \boldsymbol{\omega}\times{}^2\mathfrak{J} - {}^2\mathfrak{J}\times\boldsymbol{\omega} \end{aligned} $$

Substitution of (....) into (1.6) and (??) gives equations of balance — the balance of linear momentum and the balance of rotational momentum — for the model of a continuous non-deformable rigid body

...

here $\boldsymbol{f}$ is the external force per mass unit, $\boldsymbol{F}$ is the resultant of external forces (also called the "equally acting force" or the "main vector"), $\boldsymbol{M}$ is the resultant of external couples (the "main couple", the "main moment").

...

Рассмотрим моделирование совершенно жёсткого (недеформируемого) твёрдого тела по принципу виртуальной работы (2.5)

.... (7.4) $\Rightarrow \delta\boldsymbol{r} = \delta\boldsymbol{p} + \delta\boldsymbol{x}$

(begin copied from § 1.13)

Варьируя тождество (11.5), получим $\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O}\cdot\delta\boldsymbol{O}^{\mathsf{T}}$. Этот тензор антисимметричен, и потому выражается через свой сопутствующий вектор $\boldsymbol{\delta o}$ как $\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\delta o}\times\boldsymbol{E}$. Приходим к соотношениям

$$\delta\boldsymbol{O} = \boldsymbol{\delta o}\times\boldsymbol{O},\ \ \boldsymbol{\delta o} = -\frac{1}{2}\Big(\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}}\Big)_{\times}, \tag{7.5}$$

(end of copied from § 1.13)

....

§ 8. Mechanics of relative motion

До этого не ставился вопрос о системе отсчёта, всё рассматривалось в некой “абсолютной” системе или одной из инерциальных систем (§ 1). Теперь представим себе две системы: “абсолютную” и “подвижную”

...

$$\mathring{\boldsymbol{r}} = \boldsymbol{r} + \boldsymbol{x}$$
$$\boldsymbol{r} = \rho_i\mathring{\boldsymbol{e}}_i,\ \ \boldsymbol{x} = x_i\boldsymbol{e}_i$$
$$\dot{\mathring{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}}$$
$$\dot{\boldsymbol{r}} = \dot{\rho}_i\mathring{\boldsymbol{e}}_i,\ \ \dot{\boldsymbol{x}} = \big(x_i\boldsymbol{e}_i\big)^{\bullet} = \dot{x}_i\boldsymbol{e}_i + x_i\dot{\boldsymbol{e}}_i$$

$x_i \neq \text{constant} \Rightarrow \dot{x}_i \neq 0$

By (11.11, § 1.11)

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega}\times\boldsymbol{e}_i \ \Rightarrow\ x_i\dot{\boldsymbol{e}}_i = \boldsymbol{\omega}\times x_i\boldsymbol{e}_i = \boldsymbol{\omega}\times\boldsymbol{x}$$

$\dot{\boldsymbol{x}} = \dot{x}_i\boldsymbol{e}_i + \boldsymbol{\omega}\times\boldsymbol{x}$

$$\boldsymbol{v} \equiv \dot{\mathring{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}} = \underbrace{\dot{\boldsymbol{r}} + \boldsymbol{\omega}\times\boldsymbol{x}}_{\boldsymbol{v}_e}\ \underbrace{-\,\boldsymbol{\omega}\times\boldsymbol{x} + \dot{\boldsymbol{x}}}_{\boldsymbol{v}_{rel}}$$

$\dot{\boldsymbol{x}} - \boldsymbol{\omega}\times\boldsymbol{x} = \dot{x}_i\boldsymbol{e}_i \equiv \boldsymbol{v}_{rel}$ — relative velocity, $\dot{\boldsymbol{r}} + \boldsymbol{\omega}\times\boldsymbol{x} \equiv \boldsymbol{v}_e$

$$\boldsymbol{v} = \boldsymbol{v}_e + \boldsymbol{v}_{rel} \tag{8.1}$$

...

$$\mathring{\dot{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}}$$
$$\mathring{\ddot{\boldsymbol{r}}} = \ddot{\boldsymbol{r}} + \ddot{\boldsymbol{x}}$$
$$\boldsymbol{w} \equiv \dot{\boldsymbol{v}} = \mathring{\dot{\boldsymbol{r}}} = \ddot{\boldsymbol{r}} + \ddot{\boldsymbol{x}}$$
$$\ddot{\boldsymbol{r}} = \ddot{\rho}_i \mathring{\boldsymbol{e}}_i, \;\; \ddot{\boldsymbol{x}} = \big(x_i \boldsymbol{e}_i\big)^{\bullet\bullet} = \big(\dot{x}_i \boldsymbol{e}_i + x_i \dot{\boldsymbol{e}}_i\big)^{\bullet} = \ddot{x}_i \boldsymbol{e}_i + \dot{x}_i \dot{\boldsymbol{e}}_i + \dot{x}_i \dot{\boldsymbol{e}}_i + x_i \ddot{\boldsymbol{e}}_i$$

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \;\Rightarrow\; \ddot{\boldsymbol{e}}_i = \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)^{\bullet} = \dot{\boldsymbol{\omega}} \times \boldsymbol{e}_i + \boldsymbol{\omega} \times \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{\omega}} \times \boldsymbol{e}_i + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)$$

$$x_i \ddot{\boldsymbol{e}}_i = x_i \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)^{\bullet} = \dot{\boldsymbol{\omega}} \times x_i \boldsymbol{e}_i + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times x_i \boldsymbol{e}_i\big) = \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{x}\big)$$

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \;\Rightarrow\; \dot{x}_i \dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \dot{x}_i \boldsymbol{e}_i = \boldsymbol{\omega} \times \boldsymbol{v}_{rel}$$

$\ddot{x}_i \boldsymbol{e}_i \equiv \boldsymbol{w}_{rel}$ — relative acceleration
$2 \dot{x}_i \dot{\boldsymbol{e}}_i = 2\,\boldsymbol{\omega} \times \boldsymbol{v}_{rel} \equiv \boldsymbol{w}_{Cor}$ — Coriolis acceleration

$$\ddot{\boldsymbol{x}} = \boldsymbol{w}_{rel} + \boldsymbol{w}_{Cor} + x_i \ddot{\boldsymbol{e}}_i$$

$$\big(x_i \dot{\boldsymbol{e}}_i\big)^{\bullet} = \dot{x}_i \dot{\boldsymbol{e}}_i + x_i \ddot{\boldsymbol{e}}_i = \frac{1}{2} \boldsymbol{w}_{Cor} + x_i \ddot{\boldsymbol{e}}_i$$
$$\big(x_i \dot{\boldsymbol{e}}_i\big)^{\bullet} = \big(\boldsymbol{\omega} \times \boldsymbol{x}\big)^{\bullet} = \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \dot{\boldsymbol{x}}$$

$$\boldsymbol{\omega} \times \dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \big(\dot{x}_i \boldsymbol{e}_i + \boldsymbol{\omega} \times \boldsymbol{x}\big) = \underbrace{\boldsymbol{\omega} \times \dot{x}_i \boldsymbol{e}_i}_{\dot{x}_i \dot{\boldsymbol{e}}_i = \frac{1}{2} \boldsymbol{w}_{Cor}} + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{x}\big)$$

...

Bibliography

In a long list of the books about the classical mechanics, the reader can find the works of both the specialists in mechanics [87, 88, 94, 95, 96] and the broadly oriented theoretical physicists [93, 89]. The book by Felix R. Gantmacher (Феликс Р. Гантмахер) [92] with the compact but complete narration of the fundamentals is pretty interesting.

chapter 3

NONLINEAR ELASTIC MOMENTLESS CONTINUUM

§ 1. Continuum and the two approaches to describe it

According to the atomic theory, a substance is composed of discrete particles — atoms. Therefore a model of a system of particles with masses m_k and location vectors $\boldsymbol{r}_k(t)$ may seem suitable yet despite an unimaginable number of degrees of freedom, because amounts of memory and the speed of modern computers are characterized also by astronomical numbers.

But anyway, maybe it's worth choosing a fundamentally and qualitatively different model — a model of the material continuum, or the continuous medium, where the mass is distributed continuously within a volume, and the finite volume $\mathcal{V}$ contains the mass

$$m = \int_{\mathcal{V}} \rho \, d\mathcal{V}, \quad dm = \rho \, d\mathcal{V}, \tag{1.1}$$

here ρ is the volume(tric) mass density and $d\mathcal{V}$ is the infinitesimal volume element.

A real matter is modeled as a continuum, which can be thought of as an infinite set of vanishingly small particles, joined together.

A space of material points is only the first and simple idea of a continuous distribution of mass. More complex models are possible too, where particles have more degrees of freedom: not only of translation, but also of rotation, of internal deformation, and others. Knowing that such models are attracting more and more interest, in this chapter we will consider the classical concept of a continuous medium as "made of simple points".

At every moment of time t, a deformable continuum occupies a certain volume $\mathcal{V}$ of the space. This volume moves and deforms, but the set of particles inside this volume is constant. It is the balance of mass ("matter is neither created nor annihilated")

$$dm = \rho d\mathcal{V} = \rho' d\mathcal{V}' = \mathring{\rho} d\mathring{\mathcal{V}}, \quad m = \int\limits_{\mathcal{V}} \rho d\mathcal{V} = \int\limits_{\mathcal{V}'} \rho' d\mathcal{V}' = \int\limits_{\mathring{\mathcal{V}}} \mathring{\rho} d\mathring{\mathcal{V}}. \qquad (1.2)$$

Introducing some variable parameters q^i — the curvilinear coordinates, we have a relation for locations of particles

$$\boldsymbol{r} = \boldsymbol{r}(q^i, t). \qquad (1.3)$$

...

Material description

at the initial moment, in the so-called initial (original, ~~reference~~, "material") configuration

at some initial moment $t=0$

"запоминается" начальная ("материальная") конфигурация — locations in space of particles at some arbitrarily chosen "initial" moment $t=0$

$$\mathring{\boldsymbol{r}}(q^i) \equiv \boldsymbol{r}(q^i, 0)$$

Morphism (function) $\mathring{\boldsymbol{r}} = \mathring{\boldsymbol{r}}(q^i)$

isomorphism (bijective mapping) (invertible one-to-one relation) (взаимно однозначное)

Subsequent locations in space of particles are then dependent variables — functions of time and of the initial (material, "Lagrangian") coordinates/location $\mathring{\boldsymbol{r}}$

$$\boldsymbol{r} = \boldsymbol{r}(\mathring{\boldsymbol{r}}, t).$$

Для пространственного дифференцирования (постоянных во времени) отношений like $\varphi = \varphi(\mathring{\boldsymbol{r}})$

вводится локальный касательный базис $\mathring{\boldsymbol{r}}_{\partial i}$ и взаимный базис $\mathring{\boldsymbol{r}}^i$

$$\mathring{\boldsymbol{r}}_{\partial i} \equiv \partial_i \mathring{\boldsymbol{r}} \ \left(\partial_i \equiv \frac{\partial}{\partial q^i}\right), \quad \mathring{\boldsymbol{r}}^i \bullet \mathring{\boldsymbol{r}}_{\partial j} = \delta^i_j$$

"материальный" оператор Hamilton'a $\mathring{\boldsymbol{\nabla}}$

$$\boldsymbol{E} = \mathring{\boldsymbol{r}}^i \mathring{\boldsymbol{r}}_{\partial i} = \mathring{\boldsymbol{r}}^i \partial_i \mathring{\boldsymbol{r}} = \mathring{\boldsymbol{\nabla}} \mathring{\boldsymbol{r}}, \quad \mathring{\boldsymbol{\nabla}} \equiv \mathring{\boldsymbol{r}}^i \partial_i, \tag{1.4}$$

тогда $d\varphi = d\mathring{\boldsymbol{r}} \bullet \mathring{\boldsymbol{\nabla}} \varphi$.

...

But yet another approach may be effective — *the spatial (or "Eulerian") description*, when instead of focusing on how particles of a continuum move from the initial configuration through space and time, processes are considered at fixed points in space as time progresses. With relations like $\rho = \rho(\boldsymbol{r}, t)$, we track what's happening exactly in this place. Various particles, continuously leaving and coming here, do not confuse us.

...

the balance of mass in spatial description (the continuity equation for mass)

....

Jaumann derivative ("corotational time derivative") was first introduced by Gustav Jaumann*

Es sei $\frac{\partial}{\partial t}$ der Operator der lokalen Fluxion, d. i. der partiellen Fluxion in einem gegen das Koordinatensystem ruhenden Punkte des Raumes. Ferner sei $\frac{d}{dt}$ der Operator der totalen Fluxion, welcher definiert wird durch

$$\frac{da}{dt} = \frac{\partial a}{\partial t} + \boldsymbol{v} \bullet \boldsymbol{\nabla} a,$$
$$\frac{d\boldsymbol{a}}{dt} \overset{3}{=} \frac{\partial \boldsymbol{a}}{\partial t} + \boldsymbol{v} \bullet \boldsymbol{\nabla}; \boldsymbol{a} - \tfrac{1}{2} \big(\operatorname{rot} \boldsymbol{v}\big) \times \boldsymbol{a},$$
$$\frac{d\boldsymbol{\alpha}}{dt} \overset{9}{=} \frac{\partial \boldsymbol{\alpha}}{\partial t} + \boldsymbol{v} \bullet \boldsymbol{\nabla}; \boldsymbol{\alpha} - \tfrac{1}{2} \big(\operatorname{rot} \boldsymbol{v} \times \boldsymbol{\alpha} - \boldsymbol{\alpha} \times \operatorname{rot} \boldsymbol{v}\big).$$

* **Gustav Jaumann**. Geschlossenes System physikalischer und chemischer Differentialgesetze (I. Mitteilung) // Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Klasse, Abteilung IIa, Band CXX, 1911. Seiten 385–530.

Endlich verwenden wir die körperliche Fluxion eines Skalars:

$$\frac{\delta}{\delta t}a = \frac{\partial}{\partial t}a + \operatorname{div} a\boldsymbol{v} = \frac{d}{dt}a + a \operatorname{div} \boldsymbol{v}.$$

körperliche — bodily/телесная, material/вещественная(материальная), physical/физическая

$\boldsymbol{\nabla}\cdot\big(a\boldsymbol{v}\big) = a\,\boldsymbol{\nabla}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}a$

...

Пусть $\upsilon(\mathring{\boldsymbol{r}}, t)$ — какое-либо поле (?? только в материальном описании от $\mathring{\boldsymbol{r}}$??). Найдём скорость изменения интеграла по объёму

$$\Upsilon \equiv \int\limits_{\mathcal{V}} \rho\upsilon d\mathcal{V}$$

("υ is Υ per mass unit"). Seemingly difficult calculation of $\dot{\Upsilon}$ (since $\mathcal{V}$ is deforming) is actually quite simple with the balance of mass (1.2) :

$$\Upsilon = \int\limits_{\mathring{\mathcal{V}}} \mathring{\rho}\upsilon d\mathring{\mathcal{V}} \;\Rightarrow\; \dot{\Upsilon} = \int\limits_{\mathring{\mathcal{V}}} \mathring{\rho}\dot{\upsilon} d\mathring{\mathcal{V}} = \int\limits_{\mathcal{V}} \rho\dot{\upsilon} d\mathcal{V}. \tag{1.5}$$

$$\Psi = \textstyle\int_{\mathcal{V}} \rho\psi d\mathcal{V} = \int_{\mathcal{V}'} \rho'\psi d\mathcal{V}' \;\Rightarrow\; \dot{\Psi} = \int_{\mathcal{V}} \rho\dot{\psi} d\mathcal{V} = \int_{\mathcal{V}'} \rho'\dot{\psi} d\mathcal{V}'$$

...

It is not worth it to contrapose the material and the spatial descriptions. In this book both are used, depending on the situation.

Having the motion function $\boldsymbol{r}=\boldsymbol{r}(q^i,t)$, $\overset{\circ}{\boldsymbol{r}}(q^i)\equiv\boldsymbol{r}(q^i,0)$, the "nabla" operators $\boldsymbol{\nabla}\equiv\boldsymbol{r}^i\partial_i$, $\overset{\circ}{\boldsymbol{\nabla}}\equiv\overset{\circ}{\boldsymbol{r}}{}^i\partial_i$ and looking at differential relations for a certain infinitesimal vector in two configurations, the current with $d\boldsymbol{r}$ and the initial with $d\overset{\circ}{\boldsymbol{r}}$

$$\begin{aligned} d\boldsymbol{r} &= d\overset{\circ}{\boldsymbol{r}}\bullet\overbrace{\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{r}}^{\substack{\boldsymbol{F}^{\mathsf{T}}\\ \overset{\circ}{\boldsymbol{r}}{}^i\boldsymbol{r}_{\partial i}}} = \overbrace{\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}}^{\substack{\boldsymbol{F}\\ \boldsymbol{r}_{\partial i}\overset{\circ}{\boldsymbol{r}}{}^i}}\bullet d\overset{\circ}{\boldsymbol{r}} \\ d\overset{\circ}{\boldsymbol{r}} &= d\boldsymbol{r}\bullet\underbrace{\boldsymbol{\nabla}\overset{\circ}{\boldsymbol{r}}}_{\substack{\boldsymbol{r}^i\overset{\circ}{\boldsymbol{r}}_{\partial i}\\ \boldsymbol{F}^{-\mathsf{T}}}} = \underbrace{\boldsymbol{\nabla}\overset{\circ}{\boldsymbol{r}}{}^{\mathsf{T}}}_{\substack{\overset{\circ}{\boldsymbol{r}}_{\partial i}\boldsymbol{r}^i\\ \boldsymbol{F}^{-1}}}\bullet d\boldsymbol{r} \end{aligned} \tag{2.1}$$

here comes to mind to introduce the "motion gradient"*, picking one of these tensor multipliers for it : $\boldsymbol{F}\equiv\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}=\boldsymbol{r}_{\partial i}\overset{\circ}{\boldsymbol{r}}{}^i$.

Why this one? The reason to choose $\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}$ is another expression for the differential

$$\begin{aligned} d\boldsymbol{r} &= \frac{\partial\boldsymbol{r}}{\partial\overset{\circ}{\boldsymbol{r}}}\bullet d\overset{\circ}{\boldsymbol{r}} & \boldsymbol{F} &= \frac{\partial\boldsymbol{r}}{\partial\overset{\circ}{\boldsymbol{r}}} \\ d\overset{\circ}{\boldsymbol{r}} &= \frac{\partial\overset{\circ}{\boldsymbol{r}}}{\partial\boldsymbol{r}}\bullet d\boldsymbol{r} & \boldsymbol{F}^{-1} &= \frac{\partial\overset{\circ}{\boldsymbol{r}}}{\partial\boldsymbol{r}} \end{aligned}$$

$$\frac{\partial\boldsymbol{\zeta}}{\partial\overset{\circ}{\boldsymbol{r}}}=\partial_i\boldsymbol{\zeta}\,\overset{\circ}{\boldsymbol{r}}{}^i \qquad \frac{\partial\boldsymbol{\zeta}}{\partial\boldsymbol{r}}=\partial_i\boldsymbol{\zeta}\boldsymbol{r}^i$$

....

$$\boldsymbol{E}=\underbrace{\overset{\circ}{\boldsymbol{\nabla}}\overset{\circ}{\boldsymbol{r}}}_{\frac{\partial\overset{\circ}{\boldsymbol{r}}}{\partial\overset{\circ}{\boldsymbol{r}}}}=\underbrace{\boldsymbol{\nabla}\boldsymbol{r}}_{\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{r}}}$$

...

For cartesian coordinates with orthonormal basis $\boldsymbol{e}_i=$ **constant**

$$\boldsymbol{r}=x_i(t)\,\boldsymbol{e}_i,\ \ \overset{\circ}{\boldsymbol{r}}=x_i(0)\,\boldsymbol{e}_i=\overset{\circ}{x}_i\boldsymbol{e}_i,\ \ \overset{\circ}{x}_i\equiv x_i(0),$$

* Tensor $\boldsymbol{F}$ doesn't well suit the more popular name "deformation gradient", because this tensor describes not only the deformation itself, but also the rotation of a body as a whole without deformation.

$$\mathring{\boldsymbol{\nabla}} = \boldsymbol{e}_i \frac{\partial}{\partial \mathring{x}_i} = \boldsymbol{e}_i \mathring{\partial}_i, \quad \boldsymbol{\nabla} = \boldsymbol{e}_i \frac{\partial}{\partial x_i} = \boldsymbol{e}_i \partial_i,$$

$$\mathring{\boldsymbol{\nabla}} \boldsymbol{r} = \boldsymbol{e}_i \frac{\partial \boldsymbol{r}}{\partial \mathring{x}_i} = \boldsymbol{e}_i \frac{\partial (x_j \boldsymbol{e}_j)}{\partial \mathring{x}_i} = \frac{\partial x_j}{\partial \mathring{x}_i} \boldsymbol{e}_i \boldsymbol{e}_j = \mathring{\partial}_i x_j \boldsymbol{e}_i \boldsymbol{e}_j,$$
$$\boldsymbol{\nabla} \mathring{\boldsymbol{r}} = \boldsymbol{e}_i \frac{\partial \mathring{\boldsymbol{r}}}{\partial x_i} = \frac{\partial \mathring{x}_j}{\partial x_i} \boldsymbol{e}_i \boldsymbol{e}_j = \partial_i \mathring{x}_j \boldsymbol{e}_i \boldsymbol{e}_j$$

...

By the polar decomposition theorem (§ 1.**??**), the motion gradient decomposes into the rotation tensor $\boldsymbol{O}$ and the symmetric positive stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$:

$$\boldsymbol{F} = \boldsymbol{O} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{O}$$

...

When there's no rotation ($\boldsymbol{O} = \boldsymbol{E}$), then $\boldsymbol{F} = \boldsymbol{U} = \boldsymbol{V}$.

....

§ 3. Measures (tensors) of deformation

And this is where the extra complexity arose. Although, multivariance is often seen as a big gift.

The motion gradient $\boldsymbol{F}$ characterizes both the deformation of a body and the rotation of a body as a whole. The deformation-only tensors are the stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ from the polar decomposition $\boldsymbol{F} = \boldsymbol{O} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{O}$, as well as another tensors, originating from $\boldsymbol{U}$ or (and) $\boldsymbol{V}$.

The widely used ones are the "squares" of $\boldsymbol{U}$ and $\boldsymbol{V}$

$$\begin{aligned} \left(\boldsymbol{U}^2 =\right)\ & \boldsymbol{U} \cdot \boldsymbol{U} = \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \equiv \boldsymbol{G}, \\ \left(\boldsymbol{V}^2 =\right)\ & \boldsymbol{V} \cdot \boldsymbol{V} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathsf{T}} \equiv \boldsymbol{\Phi}. \end{aligned} \tag{3.1}$$

These are the Green's deformation tensor (or the right Cauchy–Green tensor) $\boldsymbol{G}$ and the Finger's deformation tensor (or the left Cauchy–Green tensor) $\boldsymbol{\Phi}$. They have the convenient link with the motion gradient $\boldsymbol{F}$, without calculating square roots (as it's needed for $\boldsymbol{U}$

and $\boldsymbol{V}$). That's the big reason why tensors $\boldsymbol{G}$ and $\boldsymbol{\Phi}$ are so widely used.

Tensor $\boldsymbol{G}$ was first used by George Green*.

An inversion of $\boldsymbol{\Phi}$ and $\boldsymbol{G}$ gives the two more deformation tensors

$$\begin{aligned} \boldsymbol{V}^{-2} = \boldsymbol{\Phi}^{-1} = \left(\boldsymbol{F}\cdot\boldsymbol{F}^{\mathsf{T}}\right)^{-1} = \boldsymbol{F}^{-\mathsf{T}}\cdot\boldsymbol{F}^{-1} \equiv {}^{2}\boldsymbol{c}, \\ \boldsymbol{U}^{-2} = \boldsymbol{G}^{-1} = \left(\boldsymbol{F}^{\mathsf{T}}\cdot\boldsymbol{F}\right)^{-1} = \boldsymbol{F}^{-1}\cdot\boldsymbol{F}^{-\mathsf{T}} \equiv {}^{2}\boldsymbol{f}, \end{aligned} \tag{3.2}$$

each of which is sometimes called the Piola tensor or the Finger tensor. The inverse of the left Cauchy–Green tensor $\boldsymbol{\Phi}$ is known as the Cauchy deformation tensor ${}^{2}\boldsymbol{c}$.

The components of these tensors are

$$\begin{aligned} \boldsymbol{G} &= \mathring{\boldsymbol{r}}^{i}\boldsymbol{r}_{\partial i}\cdot\boldsymbol{r}_{\partial j}\mathring{\boldsymbol{r}}^{j} = G_{ij}\mathring{\boldsymbol{r}}^{i}\mathring{\boldsymbol{r}}^{j}, \quad G_{ij} \equiv \boldsymbol{r}_{\partial i}\cdot\boldsymbol{r}_{\partial j}, \\ {}^{2}\boldsymbol{f} &= \mathring{\boldsymbol{r}}_{\partial i}\boldsymbol{r}^{i}\cdot\boldsymbol{r}^{j}\mathring{\boldsymbol{r}}_{\partial j} = G^{ij}\mathring{\boldsymbol{r}}_{\partial i}\mathring{\boldsymbol{r}}_{\partial j}, \quad G^{ij} \equiv \boldsymbol{r}^{i}\cdot\boldsymbol{r}^{j}, \\ {}^{2}\boldsymbol{c} &= \boldsymbol{r}^{i}\mathring{\boldsymbol{r}}_{\partial i}\cdot\mathring{\boldsymbol{r}}_{\partial j}\boldsymbol{r}^{j} = g_{ij}\boldsymbol{r}^{i}\boldsymbol{r}^{j}, \quad g_{ij} \equiv \mathring{\boldsymbol{r}}_{\partial i}\cdot\mathring{\boldsymbol{r}}_{\partial j}, \\ \boldsymbol{\Phi} &= \boldsymbol{r}_{\partial i}\mathring{\boldsymbol{r}}^{i}\cdot\mathring{\boldsymbol{r}}^{j}\boldsymbol{r}_{\partial j} = g^{ij}\boldsymbol{r}_{\partial i}\boldsymbol{r}_{\partial j}, \quad g^{ij} \equiv \mathring{\boldsymbol{r}}^{i}\cdot\mathring{\boldsymbol{r}}^{j}, \end{aligned}$$

and they coincide with the components of the unit (metric) tensor

$$\begin{aligned} \boldsymbol{E} = \boldsymbol{r}_{\partial i}\boldsymbol{r}^{i} = G_{ij}\boldsymbol{r}^{i}\boldsymbol{r}^{j} = \boldsymbol{r}^{i}\boldsymbol{r}_{\partial i} = G^{ij}\boldsymbol{r}_{\partial i}\boldsymbol{r}_{\partial j} \\ = \mathring{\boldsymbol{r}}^{i}\mathring{\boldsymbol{r}}_{\partial i} = g^{ij}\mathring{\boldsymbol{r}}_{\partial i}\mathring{\boldsymbol{r}}_{\partial j} = \mathring{\boldsymbol{r}}_{\partial i}\mathring{\boldsymbol{r}}^{i} = g_{ij}\mathring{\boldsymbol{r}}^{i}\mathring{\boldsymbol{r}}^{j}, \end{aligned}$$

but the components' bases are different. Using only the index notation, it's easy to get confused due to the similarity between the unit tensor $\boldsymbol{E}$ and the strain tensors $\boldsymbol{G}$, $\boldsymbol{\Phi}$, ${}^{2}\boldsymbol{f}$, ${}^{2}\boldsymbol{c}$. The direct indexless notation has the obvious advantage here.

As was mentioned in § 1.**??**, the invariants of the stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ are the same. If w_i are the three eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{V}$, that

* **Green, George**. (1839) On the propagation of light in crystallized media. *Transactions of the Cambridge Philosophical Society.* 1842, vol. 7, part II, pages 121–140.

is the roots of the characteristic equations for these tensors, then here are their invariants:

$$\begin{gathered}\mathrm{I}(\boldsymbol{U}) = \mathrm{I}(\boldsymbol{V}) = \operatorname{trace}\boldsymbol{U} = \operatorname{trace}\boldsymbol{V} = \sum U_{jj} = \sum V_{jj} = \sum w_i,\\ \mathrm{II}(\boldsymbol{U}) = \mathrm{II}(\boldsymbol{V}) = w_1w_2 + w_1w_3 + w_2w_3,\\ \mathrm{III}(\boldsymbol{U}) = \mathrm{III}(\boldsymbol{V}) = w_1w_2w_3.\end{gathered}$$

The invariants of $\boldsymbol{G}$ and $\boldsymbol{\Phi}$ coincide too:

$$\mathrm{I}(\boldsymbol{G}) = \mathrm{I}(\boldsymbol{\Phi}), \dots$$

Without deformation

$$\boldsymbol{F} = \boldsymbol{U} = \boldsymbol{V} = \boldsymbol{G} = \boldsymbol{\Phi} = {}^2\boldsymbol{f} = {}^2\boldsymbol{c} = \boldsymbol{E},$$

thus as characteristics of deformation it's worth taking the differences like $\boldsymbol{U} - \boldsymbol{E}$, $\boldsymbol{U}\cdot\boldsymbol{U} - \boldsymbol{E}$, ...

...

The right Cauchy–Green deformation tensor

George Green discovered a deformation tensor known as the right Cauchy–Green deformation tensor or Green's deformation tensor

$$\boldsymbol{G} = \boldsymbol{F}^{\mathsf{T}}\cdot\boldsymbol{F} = \boldsymbol{U}^2 \quad \text{or} \quad G_{ij} = F_{k'i}F_{k'j} = \frac{\partial x_{k'}}{\partial \mathring{x}_i}\frac{\partial x_{k'}}{\partial \mathring{x}_j}.$$

This tensor gives the "square" of local change in distances due to deformation: $d\boldsymbol{r}\cdot d\boldsymbol{r} = d\mathring{\boldsymbol{r}}\cdot\boldsymbol{G}\cdot d\mathring{\boldsymbol{r}}$

The most popular invariants of $\boldsymbol{G}$ are

$$\begin{aligned}\mathrm{I}(\boldsymbol{G}) &\equiv \operatorname{trace}\boldsymbol{G} = G_{ii} = \gamma_1^2 + \gamma_2^2 + \gamma_3^2\\ \mathrm{II}(\boldsymbol{G}) &\equiv \tfrac{1}{2}\left(G_{jj}^2 - G_{ik}G_{ki}\right) = \gamma_1^2\gamma_2^2 + \gamma_2^2\gamma_3^2 + \gamma_3^2\gamma_1^2\\ \mathrm{III}(\boldsymbol{G}) &\equiv \det\boldsymbol{G} = \gamma_1^2\gamma_2^2\gamma_3^2\end{aligned}$$

where γ_i are stretch ratios for unit fibers that are initially oriented along directions of eigenvectors of the right stretch tensor $\boldsymbol{U}$.

The inverse of Green's deformation tensor

Sometimes called the Finger tensor or the Piola tensor, the inverse of the right Cauchy–Green deformation tensor

$$^2\boldsymbol{f} = \boldsymbol{G}^{-1} = \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-\mathsf{T}} \quad \text{or} \quad f_{ij} = \frac{\partial \mathring{x}_i}{\partial x_{k'}} \frac{\partial \mathring{x}_j}{\partial x_{k'}}$$

The left Cauchy–Green or Finger deformation tensor

Swapping multipliers in the formula for the right Green–Cauchy deformation tensor leads to the left Cauchy–Green deformation tensor, defined as

$$\boldsymbol{\Phi} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathsf{T}} = \boldsymbol{V}^2 \quad \text{or} \quad \Phi_{ij} = \frac{\partial x_i}{\partial \mathring{x}_k} \frac{\partial x_j}{\partial \mathring{x}_k}$$

The left Cauchy–Green deformation tensor is often called the Finger's deformation tensor, named after Josef Finger (1894).

Invariants of $\boldsymbol{\Phi}$ are also used in expressions for strain energy density functions. The conventional invariants are defined as

$$\begin{aligned} I_1 &\equiv \Phi_{ii} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &\equiv \tfrac{1}{2}\left(\Phi_{ii}^2 - \Phi_{jk}\Phi_{kj}\right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\ I_3 &\equiv \det \boldsymbol{\Phi} = \mathcal{J}^2 = \lambda_1^2\lambda_2^2\lambda_3^2 \end{aligned}$$

($\mathcal{J} \equiv \det \boldsymbol{F}$ is the Jacobian, determinant of the motion gradient)

The Cauchy deformation tensor

The Cauchy deformation tensor is defined as the inverse of the left Cauchy–Green deformation tensor

$$^2\boldsymbol{c} = \boldsymbol{\Phi}^{-1} = \boldsymbol{F}^{-\mathsf{T}} \cdot \boldsymbol{F}^{-1} \quad \text{or} \quad c_{ij} = \frac{\partial \mathring{x}_k}{\partial x_i} \frac{\partial \mathring{x}_k}{\partial x_j}$$

$d\mathring{\boldsymbol{r}} \cdot d\mathring{\boldsymbol{r}} = d\boldsymbol{r} \cdot {}^2\boldsymbol{c} \cdot d\boldsymbol{r}$

This tensor is also called the Piola tensor or the Finger tensor in rheology and fluid dynamics literature.

The concept of *strain* is used to evaluate how much a given displacement differs locally from a body displacement as a whole (a "rigid body displacement"). One of such strains for large (finite) deformations is the *Green strain tensor* (*Green–Lagrangian strain tensor*, *Green–Saint-Venant strain tensor*). It measures how much $\boldsymbol{G}$ differs from $\boldsymbol{E}$

$$\boldsymbol{C} = \tfrac{1}{2}\big(\boldsymbol{G} - \boldsymbol{E}\big) = \tfrac{1}{2}\big(\boldsymbol{F}^{\mathsf{T}}\boldsymbol{\cdot}\boldsymbol{F} - \boldsymbol{E}\big) \tag{3.3}$$

or as the function of the displacement gradient tensor

$$\boldsymbol{C} = \tfrac{1}{2}\Big(\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{u} + \overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}} + \overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{u}\boldsymbol{\cdot}\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}\Big),$$

in cartesian coordinates

$$C_{ij} = \tfrac{1}{2}\Big(\frac{\partial x_{k'}}{\partial \overset{\circ}{x}_i}\frac{\partial x_{k'}}{\partial \overset{\circ}{x}_j} - \delta_{ij}\Big) = \tfrac{1}{2}\Big(\frac{\partial u_j}{\partial \overset{\circ}{x}_i} + \frac{\partial u_i}{\partial \overset{\circ}{x}_j} + \frac{\partial u_k}{\partial \overset{\circ}{x}_i}\frac{\partial u_k}{\partial \overset{\circ}{x}_j}\Big).$$

The *Almansi–Hamel strain tensor*, referenced to the deformed configuration ("Eulerian description"), is defined as

$${}^{2}\boldsymbol{a} = \tfrac{1}{2}\big(\boldsymbol{E} - {}^{2}\boldsymbol{c}\big) = \tfrac{1}{2}\big(\boldsymbol{E} - \boldsymbol{\Phi}^{-1}\big) \quad \text{or} \quad a_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial \overset{\circ}{x}_k}{\partial x_i}\frac{\partial \overset{\circ}{x}_k}{\partial x_j}\right)$$

or as function of the displacement gradient

$${}^{2}\boldsymbol{a} = \tfrac{1}{2}\big(\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{T}} + \boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}\boldsymbol{u}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{T}}\big)$$

$$a_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

Seth–Hill family of abstract strain tensors

B. R. Seth was the first to show that the Green and Almansi strain tensors are special cases of a more abstract measure of deformation. The idea was further expanded upon by Rodney Hill

in 1968 (publication??). The Seth–Hill family of strain measures (also called Doyle–Ericksen tensors) is expressed as

$$\boldsymbol{D}_{(m)} = \frac{1}{2m}\left(\boldsymbol{U}^{2m} - \boldsymbol{E}\right) = \frac{1}{2m}\left(\boldsymbol{G}^{m} - \boldsymbol{E}\right)$$

For various m it gives

$\boldsymbol{D}_{(1)} = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{E}\right) = \frac{1}{2}(\boldsymbol{G} - \boldsymbol{E})$	Green strain tensor
$\boldsymbol{D}_{(1/2)} = \boldsymbol{U} - \boldsymbol{E} = \boldsymbol{G}^{1/2} - \boldsymbol{E}$	Biot strain tensor
$\boldsymbol{D}_{(0)} = \ln \boldsymbol{U} = \frac{1}{2}\ln \boldsymbol{G}$	logarithmic strain, Hencky strain
$\boldsymbol{D}_{(-1)} = \frac{1}{2}\left(\boldsymbol{E} - \boldsymbol{U}^{-2}\right)$	Almansi strain

The second-order approximation of these tensors is

$$\boldsymbol{D}_{(m)} = \boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\nabla u} \cdot \boldsymbol{\nabla u}^{\mathsf{T}} - (1 - m)\,\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}$$

where $\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla u}^{\mathsf{S}}$ is the infinitesimal deformation tensor.

Many other different definitions of measures $\boldsymbol{D}$ are possible, provided that they satisfy these conditions:

- ✓ $\boldsymbol{D}$ vanishes for any movement of a body as a rigid whole
- ✓ dependence of $\boldsymbol{D}$ on displacement gradient tensor $\nabla \boldsymbol{u}$ is continuous, continuously differentiable and monotonic
- ✓ it's desired that $\boldsymbol{D}$ reduces to the infinitesimal linear deformation tensor $\boldsymbol{\varepsilon}$ when $\boldsymbol{\nabla u} \to 0$

For example, tensors from the set

$$\boldsymbol{D}^{(n)} = \left(\boldsymbol{U}^{n} - \boldsymbol{U}^{-n}\right)/2n$$

aren't from the Seth–Hill family, but for any n they have the same 2nd-order approximation as Seth–Hill measures with $m = 0$.

Wikipedia, the free encyclopedia — Finite strain theory

...

Heinrich Hencky. Über die Form des Elastizitätsgesetzes bei ideal elastischen Stoffen. Zeitschrift für technische Physik, Vol. 9 (1928), Seiten 215–220.

....

§ 4. Velocity field

This topic is discussed in nearly any book about continuum mechanics, however for solid elastic continua it's not very vital. Among various models of a material continuum, an elastic solid body is distinguished by interesting possibility of deriving the complete set (system) of equations for it via the single logically flawless procedure. But now we follow the way, usual for fluid continuum mechanics.

So, there's velocity field in spatial description $\boldsymbol{v} \equiv \dot{\boldsymbol{r}} = \boldsymbol{v}(\boldsymbol{r}, t)$. Decomposition of tensor $\boldsymbol{\nabla v} = \boldsymbol{\nabla \dot{r}} = \boldsymbol{r}^i \partial_i \dot{\boldsymbol{r}} = \boldsymbol{r}^i \dot{\boldsymbol{r}}_{\partial i}$* into symmetric and skewsymmetric parts (§ 1.8)

$$\boldsymbol{\nabla \dot{r}} = \boldsymbol{\nabla \dot{r}}^{\mathsf{S}} - \tfrac{1}{2}\big(\boldsymbol{\nabla} \times \dot{\boldsymbol{r}}\big) \times \boldsymbol{E}$$

or, introducing the rate of deformation tensor (rate of stretching tensor, strain rate tensor) $\boldsymbol{\mathcal{D}}$ and the vorticity tensor (rate of rotation tensor, spin tensor) $\boldsymbol{\mathcal{W}}$

$$\begin{gathered} \boldsymbol{\nabla v} = \boldsymbol{\mathcal{D}} - \boldsymbol{\mathcal{W}}, \\ \boldsymbol{\mathcal{D}} \equiv \boldsymbol{\nabla v}^{\mathsf{S}} = \boldsymbol{\nabla \dot{r}}^{\mathsf{S}} = \tfrac{1}{2}\big(\boldsymbol{r}^i \dot{\boldsymbol{r}}_{\partial i} + \dot{\boldsymbol{r}}_{\partial i} \boldsymbol{r}^i\big), \\ -\boldsymbol{\mathcal{W}} \equiv \boldsymbol{\nabla v}^{\mathsf{A}} = -\boldsymbol{w} \times \boldsymbol{E}, \quad \boldsymbol{w} \equiv \tfrac{1}{2} \boldsymbol{\nabla} \times \boldsymbol{v} = \tfrac{1}{2} \boldsymbol{r}^i \times \dot{\boldsymbol{r}}_{\partial i}, \end{gathered} \tag{4.1}$$

where also figures the vorticity (pseudo)vector $\boldsymbol{w}$, the companion of $\boldsymbol{\mathcal{W}}$.

* For sufficiently smooth functions, partial derivatives always commute, space and time ones too. Thus

$$\frac{\partial}{\partial q^i} \frac{\partial \boldsymbol{r}}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \boldsymbol{r}}{\partial q^i} \quad \text{or} \quad \partial_i \dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_{\partial i}$$

Components of the rate of deformation tensor in the current configuration's basis

$$\boldsymbol{\mathcal{D}} = \mathcal{D}_{ij}\boldsymbol{r}^i\boldsymbol{r}^j, \quad \mathcal{D}_{ij} = \boldsymbol{r}_{\partial i}\bullet\boldsymbol{\mathcal{D}}\bullet\boldsymbol{r}_{\partial j} = \frac{1}{2}\,\boldsymbol{r}_{\partial i}\bullet\left(\boldsymbol{r}^k\dot{\boldsymbol{r}}_k + \dot{\boldsymbol{r}}_k\boldsymbol{r}^k\right)\bullet\boldsymbol{r}_{\partial j} = \\ = \frac{1}{2}\left(\dot{\boldsymbol{r}}_{\partial i}\bullet\boldsymbol{r}_{\partial j} + \boldsymbol{r}_{\partial i}\bullet\dot{\boldsymbol{r}}_{\partial j}\right) = \frac{1}{2}\left(\boldsymbol{r}_{\partial i}\bullet\boldsymbol{r}_{\partial j}\right)^{\bullet}$$

...

$$\dot{G}_{ij}$$

$$G_{ij} \equiv \boldsymbol{r}_{\partial i}\bullet\boldsymbol{r}_{\partial j}$$

...

For elastic solid media, there's no need to discuss about rotations : the true representation appears along the way of logically harmonious conclusions and without additional hypotheses.

§ 5. Area vector. Surface change

Take an infinitesimal surface. The area vector by length is equal to the surface's area and is directed along the normal to this surface.

In the initial (original, undeformed, "material", ~~reference~~) configuration, the area vector can be represented as $\mathring{\boldsymbol{n}}do$. Surface's area do is infinitely small, and $\mathring{\boldsymbol{n}}$ is unit normal vector.

In the present (current, actual, deformed, "spatial") configuration, the same surface has area vector $\boldsymbol{n}d\mathcal{O}$.

With differential precision, these infinitesimal surfaces are parallelograms, thus

$$\begin{aligned}\mathring{\boldsymbol{n}}do &= d\mathring{\boldsymbol{r}}'\times d\mathring{\boldsymbol{r}}'' = \frac{\partial\mathring{\boldsymbol{r}}}{\partial q^i}\,dq^i\times\frac{\partial\mathring{\boldsymbol{r}}}{\partial q^j}\,dq^j = \mathring{\boldsymbol{r}}_{\partial i}\times\mathring{\boldsymbol{r}}_{\partial j}\,dq^i dq^j,\\ \boldsymbol{n}d\mathcal{O} &= d\boldsymbol{r}'\times d\boldsymbol{r}'' = \frac{\partial\boldsymbol{r}}{\partial q^i}\,dq^i\times\frac{\partial\boldsymbol{r}}{\partial q^j}\,dq^j = \boldsymbol{r}_{\partial i}\times\boldsymbol{r}_{\partial j}\,dq^i dq^j.\end{aligned} \tag{5.1}$$

Applying the transformation of volume (**??**), we have

$$dV = \mathcal{J} d\mathring{V} \;\Rightarrow\; \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j} \bullet \boldsymbol{r}_{\partial k} = \mathcal{J} \mathring{\boldsymbol{r}}_{\partial i} \times \mathring{\boldsymbol{r}}_{\partial j} \bullet \mathring{\boldsymbol{r}}_{\partial k} \;\Rightarrow$$
$$\Rightarrow\; \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j} \bullet \boldsymbol{r}_{\partial k} \boldsymbol{r}^k = \mathcal{J} \mathring{\boldsymbol{r}}_{\partial i} \times \mathring{\boldsymbol{r}}_{\partial j} \bullet \mathring{\boldsymbol{r}}_{\partial k} \boldsymbol{r}^k \;\Rightarrow$$
$$\Rightarrow\; \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j} = \mathcal{J} \mathring{\boldsymbol{r}}_{\partial i} \times \mathring{\boldsymbol{r}}_{\partial j} \bullet \boldsymbol{F}^{-1}.$$

Hence with (5.1) we come to the relation

$$\boldsymbol{n} dO = \mathcal{J} \mathring{\boldsymbol{n}} d\mathring{o} \bullet \boldsymbol{F}^{-1}, \tag{5.2}$$

called the Nanson's formula.

§ 6. Forces in continuum. Existence of the Cauchy stress tensor

Augustin-Louis Cauchy founded the *continuum mechanics* with the idea that two adjoining parts of a body interact with each other by means of contact forces on a dividing surface.

Assuming that these contact forces depend only on the perpendicular to the dividing surface and that surface contact forces are balanced by some volumetric force density, including inertia, Cauchy played with tetrahedrons and proved the existænce of the stress tensor.

"De la pression ou tension dans un corps solide." dans (i) *Exercices de mathématiques*, par M. **Augustin-Louis Cauchy**. Seconde année: 1827. Paris, Chez de Bure frères. Pages 42 à 59. (ii) *Œuvres complètes* d'**Augustin Cauchy**. Série 2, tome 7. Pages 60 à 78.

The particles of a momentless model of a continuum are points that have only translational degrees of freedom*. Thus there're no moments among generalized forces, and there can't be any external force couples.

Force $\rho \boldsymbol{f} dV$ acts on infinitesimal volume dV. If $\boldsymbol{f}$ is a mass force (acting per unit of mass), then $\rho \boldsymbol{f}$ is a volume one. Such forces

* The translational degrees of freedom come from the particle's ability to move freely in space.

originate from force fields, for example: the gravitational forces ("weight"), the forces of inertia in a non-inertial reference system, the electromagnetic forces in a medium with charges and currents.

Surface force $\boldsymbol{p}d\mathcal{O}$ acts on infinitesimal surface $d\mathcal{O}$. It may be a contact pressure or/and a friction, an electrostatic force with charges concentrated on the surface.

In a material continuum, like in any mechanical system, the external and the internal forces are distinguished. The internal forces balance the action of the external forces, and they are transmitted continuously from point to point. Since the times of Euler and Cauchy, the internal forces are assumed to be the surface short-range contact forces: on an infinitesimal surface $\boldsymbol{n}d\mathcal{O}$ acts the force $\boldsymbol{t}_{(\boldsymbol{n})}d\mathcal{O}$. It acts from?? that side of the two where the unit normal $\boldsymbol{n}$ is directed.

By the action–reaction principle, a traction vector $\boldsymbol{t}_{(\boldsymbol{n})}$ is reversed (alters direction) together with a unit normal vector $\boldsymbol{n}$: $\boldsymbol{t}_{(-\boldsymbol{n})} = -\boldsymbol{t}_{(\boldsymbol{n})}$. Sometimes this thesis is called "the Cauchy pill-box argument" and is proved thru the balance of momentum for an infinitely short cylinder with bases $\boldsymbol{n}d\mathcal{O}$ and $-\boldsymbol{n}d\mathcal{O}$.

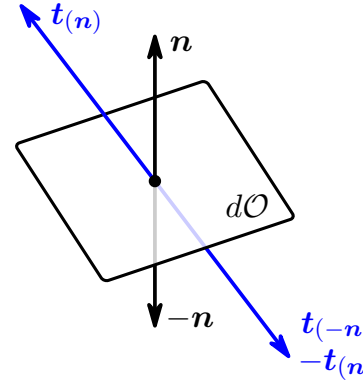

figure 12

Traction vector $\boldsymbol{t}_{(\boldsymbol{n})}$ on the surface with the unit normal $\boldsymbol{n}$ is called the surface traction vector or force-stress vector. However, $\boldsymbol{t}_{(\boldsymbol{n})}$ is not a vector field: traction $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{n}, \boldsymbol{r}, t)$ depends not only on location $\boldsymbol{r}$ of the point, but also on the local direction (defined by $\boldsymbol{n}$) of the surface element. An infinite number of surfaces of any direction contain the same point, and there are infinitely many traction vectors $\boldsymbol{t}_{(\boldsymbol{n})}$ at each point.

The stress at a point of continuum is *not a vector field*. Such a field is more complex, it is an infinite collection of all traction vectors for all infinitesimal surfaces of any direction, containing that point.

And in fact, an infinite collection of all traction vectorsat a point is completely defined by the only one single second complexity tensor — the Cauchy stress tensor $\boldsymbol{\tau}$.

The derivation of this thesis is described in many books. It is known as the theorem about existænce of the Cauchy stress tensor — the one with the impressive tetrahedron argument.

The Cauchy tetrahedron argument and the proof of the existænce of the Cauchy stress tensor.

On the surface of an infinitesimal material tetrahedron ...

.....

The traction vector $\boldsymbol{t}$ and its projections, $\boldsymbol{t}_\perp$ and $\boldsymbol{t}_\parallel$

✓ the projection of the traction vector on the unit normal vector

$$\boldsymbol{t}_\perp = \boldsymbol{t}_{\boldsymbol{n}} = \boldsymbol{t} \bullet \boldsymbol{n} \tag{6.1}$$

(is perpendicular to the cross-section area),

✓ the projection of the traction vector on the surface

$$\boldsymbol{t}_\parallel = \boldsymbol{t} - \boldsymbol{t}_\perp \tag{6.2}$$

....

§ 7. Balance of momentum and angular momentum

Consider some random finite volume $\mathcal{V}$ of an elastic medium, contained within surface $\mathcal{O}(\partial\mathcal{V})$. It is loaded with external forces, surface contact ones $\boldsymbol{p}d\mathcal{O}$ and body (mass or volume) ones $\boldsymbol{f}dm = \rho\boldsymbol{f}d\mathcal{V}$.

The integral formulation of the balance of momentum is as follows

$$\left(\int\limits_{\mathcal{V}} \rho\boldsymbol{v}d\mathcal{V}\right)^{\bullet} = \int\limits_{\mathcal{V}} \rho\boldsymbol{f}d\mathcal{V} + \oint\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{p}d\mathcal{O}. \tag{7.1}$$

... $\boldsymbol{p} = \boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \bullet \boldsymbol{\tau}$ ...

The derivative of the momentum on the left can be found as in (1.5), and the integral over the surface turns into the volume integral by the divergence theorem. This gives

$$\int\limits_{\mathcal{V}} \Big(\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\tau} + \rho \big(\boldsymbol{f} - \dot{\boldsymbol{v}} \big) \Big) d\mathcal{V} = \boldsymbol{0}.$$

But volume $\mathcal{V}$ is random, and therefore the integrand itself is also equal to the null vector — the equation of balance of momentum (forces) in local (differential) form

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\tau} + \rho \big(\boldsymbol{f} - \dot{\boldsymbol{v}} \big) = \boldsymbol{0}. \tag{7.2}$$

....

Now about the balance of the angular (rotational) momentum. Here is the integral formulation :

$$\left(\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \boldsymbol{v} d\mathcal{V} \right)^{\bullet} = \int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \boldsymbol{f} d\mathcal{V} + \oint\limits_{\mathcal{O}(\partial \mathcal{V})} \boldsymbol{r} \times \boldsymbol{p} d\mathcal{O}. \tag{7.3}$$

Дифференцируя левую часть ($\boldsymbol{v} \equiv \dot{\boldsymbol{r}}$)

$$\left(\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \dot{\boldsymbol{r}} \, d\mathcal{V} \right)^{\bullet} = \int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \ddot{\boldsymbol{r}} \, d\mathcal{V} + \int\limits_{\mathcal{V}} \underbrace{\dot{\boldsymbol{r}} \times \rho \dot{\boldsymbol{r}}}_{\boldsymbol{0}} d\mathcal{V},$$

применяя теорему о дивергенции к поверхностному интегралу ....

(... $\boldsymbol{p} = \boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \boldsymbol{\cdot} \boldsymbol{\tau}$ ...)

$$\boldsymbol{r} \times (\boldsymbol{n} \boldsymbol{\cdot} \boldsymbol{\tau}) = - (\boldsymbol{n} \boldsymbol{\cdot} \boldsymbol{\tau}) \times \boldsymbol{r} = - \boldsymbol{n} \boldsymbol{\cdot} (\boldsymbol{\tau} \times \boldsymbol{r}) \;\Rightarrow$$

$$\Rightarrow \oint\limits_{\mathcal{O}(\partial \mathcal{V})} \boldsymbol{r} \times (\boldsymbol{n} \boldsymbol{\cdot} \boldsymbol{\tau}) \, d\mathcal{O} = - \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \boldsymbol{\cdot} (\boldsymbol{\tau} \times \boldsymbol{r}) \, d\mathcal{V},$$

...

$$\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \ddot{\boldsymbol{r}} \, d\mathcal{V} = \int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \boldsymbol{f} d\mathcal{V} - \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \boldsymbol{\cdot} (\boldsymbol{\tau} \times \boldsymbol{r}) \, d\mathcal{V},$$

$$\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho\left(\boldsymbol{f} - \ddot{\boldsymbol{r}}\right) d\mathcal{V} - \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \times \boldsymbol{r})\, d\mathcal{V} = \boldsymbol{0},$$

...

$$\underbrace{\boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \times \boldsymbol{r})}_{\boldsymbol{r}^i \cdot \partial_i (\boldsymbol{\tau} \times \boldsymbol{r})} = \underbrace{(\boldsymbol{\nabla} \cdot \boldsymbol{\tau}) \times \boldsymbol{r}}_{\boldsymbol{r}^i \cdot (\partial_i \boldsymbol{\tau}) \times \boldsymbol{r}} + \boldsymbol{r}^i \cdot (\boldsymbol{\tau} \times \partial_i \boldsymbol{r})$$

$\boldsymbol{\tau} = \boldsymbol{e}_i \boldsymbol{t}_{(i)}$, $\boldsymbol{e}_i =$ **constant**

$$\begin{aligned} \boldsymbol{r}^i \cdot (\boldsymbol{\tau} \times \partial_i \boldsymbol{r}) = \boldsymbol{r}^i \cdot (\boldsymbol{e}_j \boldsymbol{t}_{(j)} \times \boldsymbol{r}_i) = \boldsymbol{r}^i \cdot \boldsymbol{e}_j \boldsymbol{t}_{(j)} \times \boldsymbol{r}_i = \\ = -\boldsymbol{e}_j \cdot \boldsymbol{r}^i \boldsymbol{r}_i \times \boldsymbol{t}_{(j)} = -\boldsymbol{e}_j \cdot \boldsymbol{E} \times \boldsymbol{t}_{(j)} = -\boldsymbol{e}_j \times \boldsymbol{t}_{(j)} = -\boldsymbol{\tau}_\times \end{aligned}$$

...

§ 8. Eigenvalues of the Cauchy stress tensor. Mohr's circles

Like any symmetric bivalent tensor, the Cauchy stress tensor $\boldsymbol{\tau}$ has три вещественных собственных числá σ_i, а также тройку взаимно перпендикулярных собственных векторов единичной длины (§ 1.10). Собственные числа тензора $\boldsymbol{\tau}$ называются главными напряжениями (principal stresses).

In the representation $\boldsymbol{\tau} = \sum \sigma_i \boldsymbol{e}_i \boldsymbol{e}_i$, the values of σ_i are most often sorted descending, $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and the triple $\boldsymbol{e}_i$ is oriented as "right".

Известна теорема о кругах Мора (Mohr's circles)*

...

Чтобы замкнуть набор (систему) уравнений модели сплошной среды, нужно добавить определяющие отношения (constitutive relations) — уравнения, связывающие напряжение с деформацией (и другие необходимые связи). However, for a solid elastic continuum такой длинный путь построения модели излишен, что читатель и увидит ниже.

* Mohr's circles, named after Christian Otto Mohr, is a two-dimensional graphical representation of transformation for the Cauchy stress tensor.

§ 9. Principle of virtual work (without Lagrange multipliers)

According to the principle of virtual work for some finite volume of a continuous medium

$$\int\limits_{\mathcal{V}}\Big(\rho\boldsymbol{f}\cdot\delta\boldsymbol{r}+\delta W^{(i)}\Big)d\mathcal{V}+\oint\limits_{\mathcal{O}(\partial\mathcal{V})}\boldsymbol{n}\cdot\boldsymbol{\tau}\cdot\delta\boldsymbol{r}\,d\mathcal{O}=0. \tag{9.1}$$

Here $\delta W^{(i)}$ is the work of internal forces per volume unit in the current configuration, $\boldsymbol{f}$ is the mass force (including dynamics, $\boldsymbol{f}\equiv\boldsymbol{f}_*-\ddot{\boldsymbol{r}}$), $\boldsymbol{p}=\boldsymbol{t}_{(\boldsymbol{n})}=\boldsymbol{n}\cdot\boldsymbol{\tau}$ is the surface force.

Applying the divergence theorem to the surface integral, using*

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\tau}\cdot\delta\boldsymbol{r})=\boldsymbol{\nabla}\cdot\boldsymbol{\tau}\cdot\delta\boldsymbol{r}+\boldsymbol{\tau}\cdot\cdot\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}}$$

and the randomness of $\mathcal{V}$, here comes the local differential version of (9.1)

$$\Big(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}+\rho\boldsymbol{f}\Big)\cdot\delta\boldsymbol{r}+\boldsymbol{\tau}\cdot\cdot\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}}+\delta W^{(i)}=0. \tag{9.2}$$

When a body virtually moves as a rigid whole, the work of internal forces nullifies

$$\delta\boldsymbol{r}=\boldsymbol{\delta\rho}+\boldsymbol{\delta o}\times\boldsymbol{r}\;\Rightarrow\;\delta W^{(i)}=0,$$
$$\big(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}+\rho\boldsymbol{f}\big)\cdot\big(\boldsymbol{\delta\rho}+\boldsymbol{\delta o}\times\boldsymbol{r}\big)+\boldsymbol{\tau}^{\mathsf{T}}\cdot\cdot\boldsymbol{\nabla}\big(\boldsymbol{\delta\rho}+\boldsymbol{\delta o}\times\boldsymbol{r}\big)=0,$$
$$\boldsymbol{\delta\rho}=\textbf{constant}\;\Rightarrow\;\boldsymbol{\nabla}\boldsymbol{\delta\rho}={}^2\boldsymbol{0},\quad\boldsymbol{\delta o}=\textbf{constant}\;\Rightarrow\;\boldsymbol{\nabla}\boldsymbol{\delta o}={}^2\boldsymbol{0},$$
$$\boldsymbol{\nabla}\big(\boldsymbol{\delta\rho}+\boldsymbol{\delta o}\times\boldsymbol{r}\big)=\boldsymbol{\nabla}\big(\boldsymbol{\delta o}\times\boldsymbol{r}\big)=\boldsymbol{\nabla}\boldsymbol{\delta o}\times\boldsymbol{r}-\boldsymbol{\nabla}\boldsymbol{r}\times\boldsymbol{\delta o}=$$
$$=-\boldsymbol{\nabla}\boldsymbol{r}\times\boldsymbol{\delta o}=-\boldsymbol{E}\times\boldsymbol{\delta o}$$
$$\dots$$

Assuming $\boldsymbol{\delta o}=\boldsymbol{0}$ (just a translation) $\Rightarrow\boldsymbol{\nabla}\delta\boldsymbol{r}=\boldsymbol{\nabla}\boldsymbol{\delta\rho}={}^2\boldsymbol{0}$, it turns into the balance of forces (of momentum)

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau}+\rho\boldsymbol{f}=\boldsymbol{0}. \tag{9.3}$$

* $\boldsymbol{r}^i\cdot\partial_i(\boldsymbol{\tau}\cdot\delta\boldsymbol{r})=\boldsymbol{r}^i\cdot(\partial_i\boldsymbol{\tau})\cdot\delta\boldsymbol{r}+\boldsymbol{r}^i\cdot\boldsymbol{\tau}\cdot\partial_i(\delta\boldsymbol{r}),$
$\boldsymbol{r}^i\cdot\boldsymbol{\tau}\cdot\partial_i(\delta\boldsymbol{r})=\boldsymbol{\tau}\cdot\cdot\partial_i(\delta\boldsymbol{r})\boldsymbol{r}^i=\boldsymbol{\tau}\cdot\cdot(\boldsymbol{r}^i\partial_i\delta\boldsymbol{r})^{\mathsf{T}}$

If $\delta\boldsymbol{r} = \delta\boldsymbol{o}\times\boldsymbol{r}$ (just a rotation) with $\delta\boldsymbol{o} = \textbf{constant}$, then

$$(17.12, \S\,1.17) \;\Rightarrow\; \boldsymbol{\nabla}\delta\boldsymbol{r} = \boldsymbol{\nabla}\delta\boldsymbol{o}\times\boldsymbol{r} - \boldsymbol{\nabla}\boldsymbol{r}\times\delta\boldsymbol{o} = -\boldsymbol{E}\times\delta\boldsymbol{o},$$
$$\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}} = \boldsymbol{E}\times\delta\boldsymbol{o}$$

With

$$(8.7, \S\,1.8) \;\Rightarrow\; \boldsymbol{\tau}_{\times} = -\boldsymbol{\tau}\bullet\!\bullet\,{}^{3}\boldsymbol{\epsilon},$$
$$\boldsymbol{\tau}\bullet\!\bullet\,(\boldsymbol{E}\times\delta\boldsymbol{o}) = \boldsymbol{\tau}\bullet\!\bullet\,\left(-{}^{3}\boldsymbol{\epsilon}\cdot\delta\boldsymbol{o}\right) = \left(-\boldsymbol{\tau}\bullet\!\bullet\,{}^{3}\boldsymbol{\epsilon}\right)\cdot\delta\boldsymbol{o} = \boldsymbol{\tau}_{\times}\cdot\delta\boldsymbol{o}$$

....

In an elastic continuum, the internal forces are potential

$$\delta W^{(i)} = -\rho\delta\widetilde{\Pi} \tag{9.4}$$

....

Variational equation (9.2) with the balance (9.3) of linear momentum (of forces) and the balance of angular momentum $\boldsymbol{\tau}_{\times} = 0 \;\Leftrightarrow\; \boldsymbol{\tau}^{\mathsf{T}} = \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{S}}$ for an elastic (9.4) continuum

$$\boldsymbol{\tau}\bullet\!\bullet\,\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = -\delta W^{(i)} = \rho\delta\widetilde{\Pi}. \tag{9.5}$$

What the potential energy $\widetilde{\Pi}$ per mass unit looks like is yet unknown, but it's obvious that $\widetilde{\Pi}$ is determined by deformation. The potential energy per unit volume $\mathring{\Pi}$ in the undeformed configuration can be presented as

$$\mathring{\Pi} \equiv \mathring{\rho}\widetilde{\Pi} \;\Rightarrow\; \delta\mathring{\Pi} = \mathring{\rho}\delta\widetilde{\Pi}. \tag{9.6}$$

With the balance of mass $\rho\mathcal{J} = \mathring{\rho} \;\Leftrightarrow\; \rho = \mathcal{J}^{-1}\mathring{\rho}$ ($\mathcal{J} \equiv \det\boldsymbol{F}$ is the Jacobian, determinant of the motion gradient)

$$\rho\delta\widetilde{\Pi} = \mathcal{J}^{-1}\delta\mathring{\Pi}.$$

> "The elastic potential energy density per volume unit", becomes when shorting "The elastic potential ...."
>
> Плотность упругой потенциальной энергии, запасённой|накопленной в единице объёма тела (среды).
>
> Дословный перевод с english на русский фразы "the elastic potential" даёт "упругий потенциал".

Полным аналогом (...) является равенство

...

§ 10. Constitutive relations of elasticity

The fundamental relation of elasticity (**??**)

...

$$\Pi(\boldsymbol{C}) = \int_0^{\boldsymbol{C}} \boldsymbol{\tau} \bullet\bullet\, d\boldsymbol{C}$$

If the strain energy density is path independent, then it acts as a potential for stress, that is

$$\boldsymbol{\tau} = \frac{\partial \Pi(\boldsymbol{C})}{\partial \boldsymbol{C}}$$

For adiabatic processes, Π is equal to the change in internal energy per unit of volume.

For isothermal processes, Π is equal to the Helmholtz free energy per unit of volume.

The natural configuration of a body is defined as the configuration in which the body is in stable thermal equilibrium with no external loads and zero stress and strain.

When we apply energy methods in elasticity, we implicitly assume that a body returns to its natural configuration after loads are removed. This implies that the Gibbs' condition is satisfied:

$$\Pi(\boldsymbol{C}) \geq 0 \quad \text{with} \quad \Pi(\boldsymbol{C}) = 0 \quad \text{iff} \quad \boldsymbol{C} = 0$$

...

Начальная конфигурация считается естественной (natural configuration) — недеформированной ненапряжённой : $\boldsymbol{C} = {}^2\mathbf{0} \Leftrightarrow \boldsymbol{\tau} = {}^2\mathbf{0}$, поэтому в Π нет линейных членов.

Тензор жёсткости ${}^4\boldsymbol{\mathcal{A}}$

...

A rubber-like material (an elastomer)

Для материала типа резины (эластомера) характерны больши́е деформации. Функция $\Pi(\mathrm{I}, \mathrm{II}, \mathrm{III})$ для такого материала бывает весьма сложной*.

Преимущества использования $\boldsymbol{u}$ и $\boldsymbol{C}$ исчезают, если деформации больши́е (коне́чные) — проще остаться с вектором-радиусом $\boldsymbol{r}$ ...

...

* **Harold Alexander**. A constitutive relation for rubber-like materials // International Journal of Engineering Science, volume 6 (September 1968), pages 549–563.

Соотношение Nanson'а $\boldsymbol{n}d\mathcal{O} = \mathcal{J}\mathring{\boldsymbol{n}}do \cdot \boldsymbol{F}^{-1}$ между векторами бесконечно малой площáдки в начальной ($\mathring{\boldsymbol{n}}do$) и в текущей ($\boldsymbol{n}d\mathcal{O}$) конфигурациях*

$$(5.2) \;\Rightarrow\; \boldsymbol{n}d\mathcal{O} \cdot \boldsymbol{\tau} = \mathcal{J}\mathring{\boldsymbol{n}}do \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \;\Rightarrow\; \boldsymbol{n} \cdot \boldsymbol{\tau}d\mathcal{O} = \mathring{\boldsymbol{n}} \cdot \mathcal{J}\boldsymbol{F}^{-1} \cdot \boldsymbol{\tau}do$$

gives the dual expression of a surface force

$$\boldsymbol{n} \cdot \boldsymbol{\tau}d\mathcal{O} = \mathring{\boldsymbol{n}} \cdot \boldsymbol{T}do, \;\; \boldsymbol{T} \equiv \mathcal{J}\boldsymbol{F}^{-1} \cdot \boldsymbol{\tau}. \tag{11.1}$$

Тензор $\boldsymbol{T}$ называется первым (несимметричным) тензором напряжения Piola–Kirchhoff, иногда — "номинальным напряжением" ("nominal stress") или "инженерным напряжением" ("engineering stress"). Бывает и когда какое-либо из этих (на)именований даётся транспонированному тензору

$$\boldsymbol{T}^{\mathsf{T}} = \mathcal{J}\boldsymbol{\tau}^{\mathsf{T}} \cdot \boldsymbol{F}^{-\mathsf{T}} = \mathcal{J}\boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathsf{T}}.$$

Обращение (11.1)

$$\mathcal{J}^{-1}\boldsymbol{F} \cdot \boldsymbol{T} = \mathcal{J}^{-1}\boldsymbol{F} \cdot \mathcal{J}\boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \;\Rightarrow\; \boldsymbol{\tau} = \mathcal{J}^{-1}\boldsymbol{F} \cdot \boldsymbol{T}$$

...

$$\delta\Pi = \boldsymbol{T} \cdot\cdot\, \delta\mathring{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}} \;\Rightarrow\; \Pi = \Pi(\mathring{\boldsymbol{\nabla}}\boldsymbol{r}) \tag{11.2}$$

— этот немного неожиданный результат получился благодаря коммутативности δ и $\mathring{\boldsymbol{\nabla}}$: $\mathring{\boldsymbol{\nabla}}\delta\boldsymbol{r}^{\mathsf{T}} = \delta\mathring{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}$ ($\boldsymbol{\nabla}$ and δ don't commute).

Тензор $\boldsymbol{T}$ оказался энергетически сопряжённым с $\boldsymbol{F} \equiv \mathring{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}$

$$\boldsymbol{T} = \frac{\partial\Pi}{\partial\mathring{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}} = \frac{\partial\Pi}{\partial\boldsymbol{F}}. \tag{11.3}$$

* Like before, $\boldsymbol{F} = \frac{\partial\boldsymbol{r}}{\partial\mathring{\boldsymbol{r}}} = \boldsymbol{r}_{\partial i}\mathring{\boldsymbol{r}}^{i} = \mathring{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}$ is the motion gradient, $\mathcal{J} \equiv \det\boldsymbol{F}$ is the Jacobian (the Jacobian determinant).

Второй (симметричный) тензор напряжения Piola–Kirchhoff $\boldsymbol{S}$ энергетически сопряжён с $\boldsymbol{G} \equiv \boldsymbol{F}^{\mathsf{T}} \boldsymbol{\cdot} \boldsymbol{F}$ и $\boldsymbol{C} \equiv \frac{1}{2}(\boldsymbol{G} - \boldsymbol{E})$

$$\begin{gathered} \delta\Pi(\boldsymbol{C}) = \boldsymbol{S} \boldsymbol{\cdot\cdot} \delta\boldsymbol{C} \;\Rightarrow\; \boldsymbol{S} = \frac{\partial \Pi}{\partial \boldsymbol{C}}, \\ d\boldsymbol{G} = 2\, d\boldsymbol{C} \;\Rightarrow\; \delta\Pi(\boldsymbol{G}) = \tfrac{1}{2}\boldsymbol{S} \boldsymbol{\cdot\cdot} \delta\boldsymbol{G},\;\; \boldsymbol{S} = 2\frac{\partial \Pi}{\partial \boldsymbol{G}}. \end{gathered} \tag{11.4}$$

Связь между первым и вторым тензорами

$$\boldsymbol{S} = \boldsymbol{T} \boldsymbol{\cdot} \boldsymbol{F}^{-\mathsf{T}} = \boldsymbol{F}^{-1} \boldsymbol{\cdot} \boldsymbol{T}^{\mathsf{T}} \;\Leftrightarrow\; \boldsymbol{T} = \boldsymbol{S} \boldsymbol{\cdot} \boldsymbol{F}^{\mathsf{T}},\;\; \boldsymbol{T}^{\mathsf{T}} = \boldsymbol{F} \boldsymbol{\cdot} \boldsymbol{S}$$

и между тензором $\boldsymbol{S}$ и тензором напряжения Cauchy $\boldsymbol{\tau}$

$$\boldsymbol{S} = \mathcal{J} \boldsymbol{F}^{-1} \boldsymbol{\cdot} \boldsymbol{\tau} \boldsymbol{\cdot} \boldsymbol{F}^{-\mathsf{T}} \;\Leftrightarrow\; \mathcal{J}^{-1} \boldsymbol{F} \boldsymbol{\cdot} \boldsymbol{S} \boldsymbol{\cdot} \boldsymbol{F}^{\mathsf{T}} = \boldsymbol{\tau}.$$

...

$$\boldsymbol{T} = \frac{\partial \Pi}{\partial \boldsymbol{C}} \boldsymbol{\cdot} \boldsymbol{F}^{\mathsf{T}} = 2\frac{\partial \Pi}{\partial \boldsymbol{G}} \boldsymbol{\cdot} \boldsymbol{F}^{\mathsf{T}}$$

$$\delta\boldsymbol{S} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} \boldsymbol{\cdot\cdot} \delta\boldsymbol{C} = \frac{\partial^2 \Pi}{\partial \boldsymbol{C} \partial \boldsymbol{C}} \boldsymbol{\cdot\cdot} \delta\boldsymbol{C}$$

$$\delta\boldsymbol{T} = \delta\boldsymbol{S} \boldsymbol{\cdot} \boldsymbol{F}^{\mathsf{T}} + \boldsymbol{S} \boldsymbol{\cdot} \delta\boldsymbol{F}^{\mathsf{T}}$$

...

The quantity $\boldsymbol{\kappa} = \mathcal{J}\boldsymbol{\tau}$ is called the *Kirchhoff stress tensor* and is used widely in numerical algorithms in metal plasticity (where there's no change in volume during plastic deformation). Another name for it is *weighted Cauchy stress tensor*.

...

Here's balance of forces (of momentum) with tensor $\boldsymbol{T}$ for any undeformed volume $\mathring{\mathcal{V}}$

$$\int\limits_{\mathcal{V}} \rho \boldsymbol{f}\, d\mathcal{V} + \int\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{n} \boldsymbol{\cdot} \boldsymbol{\tau}\, d\mathcal{O} = \int\limits_{\mathring{\mathcal{V}}} \mathring{\rho} \boldsymbol{f}\, d\mathring{\mathcal{V}} + \int\limits_{o(\partial\mathring{\mathcal{V}})} \mathring{\boldsymbol{n}} \boldsymbol{\cdot} \boldsymbol{T}\, do = \int\limits_{\mathring{\mathcal{V}}} \Big(\mathring{\rho} \boldsymbol{f} + \mathring{\boldsymbol{\nabla}} \boldsymbol{\cdot} \boldsymbol{T}\Big) d\mathring{\mathcal{V}} = \boldsymbol{0}$$

or in the local (differential) version

$$\mathring{\boldsymbol{\nabla}} \cdot \boldsymbol{T} + \mathring{\rho} \boldsymbol{f} = \boldsymbol{0}. \tag{11.5}$$

Advantages of this equation in comparison with (7.2) are : here figures the known mass density $\mathring{\rho}$ of an undeformed volume $\mathring{\mathcal{V}}$, and the operator $\mathring{\boldsymbol{\nabla}} \equiv \mathring{\boldsymbol{r}}^i \partial_i$ is defined through the known vectors $\mathring{\boldsymbol{r}}^i$. The appearance of $\boldsymbol{T}$ presents the specific property of an elastic solid body — "to retain" its initial configuration. Tensor $\boldsymbol{T}$ is unlikely useful in fluid mechanics.

The principle of virtual work for an arbitrary volume $\mathring{\mathcal{V}}$ of elastic ($\delta W^{(i)} = -\delta\Pi$) continuum :

$$\int\limits_{\mathring{\mathcal{V}}} \Big(\mathring{\rho} \boldsymbol{f} \cdot \delta \boldsymbol{r} - \delta\Pi \Big) d\mathring{\mathcal{V}} + \int\limits_{o(\mathring{\partial\mathcal{V}})} \mathring{\boldsymbol{n}} \cdot \boldsymbol{T} \cdot \delta \boldsymbol{r} \, do = 0,$$

$$\mathring{\boldsymbol{\nabla}} \cdot (\boldsymbol{T} \cdot \delta \boldsymbol{r}) = \mathring{\boldsymbol{\nabla}} \cdot \boldsymbol{T} \cdot \delta \boldsymbol{r} + \boldsymbol{T}^{\mathsf{T}} \cdot\cdot \mathring{\boldsymbol{\nabla}} \delta \boldsymbol{r}, \;\; \boldsymbol{T}^{\mathsf{T}} \cdot\cdot \mathring{\boldsymbol{\nabla}} \delta \boldsymbol{r} = \boldsymbol{T} \cdot\cdot \mathring{\boldsymbol{\nabla}} \delta \boldsymbol{r}^{\mathsf{T}}$$

$$\delta\Pi = \Big(\mathring{\rho} \boldsymbol{f} + \mathring{\boldsymbol{\nabla}} \cdot \boldsymbol{T} \Big) \cdot \delta \boldsymbol{r} + \boldsymbol{T} \cdot\cdot \mathring{\boldsymbol{\nabla}} \delta \boldsymbol{r}^{\mathsf{T}}$$

....

The first one is non-symmetric, it links forces in the deformed stressed configuration to the underfomed geometry and mass (volumes, areas, densities as they were initially), and it is energetically conjugate to the motion gradient (often mistakenly called the "deformation gradient", forgetting about rigid rotations). The first (or sometimes its transpose) is also known as "nominal stress" and "engineering stress".

The second one is symmetric, it links loads in the initial undeformed configuration to the initial mass and geometry, and it is conjugate to the right Cauchy–Green deformation tensor (and thus to the Cauchy–Green–Venant measure of deformation).

The first is simplier when you use just the motion gradient and is more universal, but the second is simplier when you prefer right Cauchy–Green deformation and its offsprings.

There's also popular Cauchy stress, which relates forces in the deformed configuration to the deformed geometry and mass.

"energetically conjugate" means that their product is kind of energy, here: elastic potential energy per unit of volume

.....

In the case of finite deformations, the Piola–Kirchhoff tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ describe the stress relative to the initial configuration. In contrast with them, the Cauchy stress tensor $\boldsymbol{\tau}$ describes the stress relative to the current configuration. For infinitesimal deformations, the Cauchy and Piola–Kirchhoff stress tensors are identical.

1st Piola–Kirchhoff stress tensor

The 1st Piola–Kirchhoff stress tensor $\boldsymbol{T}$ relates forces in the current (present, "spatial") configuration with areas in the initial ("material") configuration

$$\boldsymbol{T} = \mathcal{J}\, \boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathsf{T}}$$

where $\boldsymbol{F}$ is the motion gradient and $\mathcal{J} \equiv \det \boldsymbol{F}$ is the Jacobi determinant, Jacobian.

Because it relates different coordinate systems, the 1st Piola–Kirchhoff stress is a two-point tensor. Commonly, it's not symmetric.

The 1st Piola–Kirchhoff stress is the 3D generalization of the 1D concept of engineering stress.

If the material rotates without a change in stress (rigid rotation), the components of the 1st Piola–Kirchhoff stress tensor will vary with material orientation.

The 1st Piola–Kirchhoff stress is energy conjugate to the motion gradient.

2nd Piola–Kirchhoff stress tensor

The 2nd Piola–Kirchhoff stress tensor $\boldsymbol{S}$ relates forces in the initial configuration to areas in the initial configuration. The force in the initial configuration is obtained via mapping that preserves the relative relationship between the force direction and the area normal in the initial configuration.

$$\boldsymbol{S} = \mathcal{J}\, \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathsf{T}}$$

This tensor is a one-point tensor and it is symmetric.

If the material rotates without a change in stress (rigid rotation), the 2nd Piola–Kirchhoff stress tensor remain constant, irrespective of material orientation.

The 2nd Piola–Kirchhoff stress tensor is energy conjugate to the Green–Lagrange finite strain tensor.

....

§ 12. Variation of the present configuration

Usually the two configurations of a nonlinear elastic medium are considered : the initial one with location vectors $\mathring{\boldsymbol{r}}$ and the present (current) one with $\boldsymbol{r}$.

The following equations describe a small change of the current configuration with infinitesimal changes to the location vector $\delta\boldsymbol{r}$, to the vector of mass forces $\delta\boldsymbol{f}$, to the first Piola–Kirchhoff stress tensor $\delta\boldsymbol{T}$ and to the Green strain tensor $\delta\boldsymbol{C}$.

Varying (11.5), (....) and (3.3)* gives

$$\begin{gathered}
\mathring{\boldsymbol{\nabla}}\cdot\delta\boldsymbol{T}+\mathring{\rho}\delta\boldsymbol{f}=\boldsymbol{0},\\
\delta\boldsymbol{T}=\left(\frac{\partial^2\Pi}{\partial\boldsymbol{C}\partial\boldsymbol{C}}\cdot\cdot\,\delta\boldsymbol{C}\right)\cdot\boldsymbol{F}^{\mathsf{T}}+\frac{\partial\Pi}{\partial\boldsymbol{C}}\cdot\delta\boldsymbol{F}^{\mathsf{T}},\\
\delta\boldsymbol{F}^{\mathsf{T}}=\delta\mathring{\boldsymbol{\nabla}}\boldsymbol{r}=\mathring{\boldsymbol{\nabla}}\delta\boldsymbol{r}=\boldsymbol{F}^{\mathsf{T}}\cdot\boldsymbol{\nabla}\delta\boldsymbol{r},\ \ \delta\boldsymbol{F}=\delta\mathring{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}=\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}}\cdot\boldsymbol{F},\\
\delta\boldsymbol{C}=\tfrac{1}{2}\delta\big(\boldsymbol{F}^{\mathsf{T}}\cdot\boldsymbol{F}\big)=\tfrac{1}{2}\Big(\big(\delta\boldsymbol{F}^{\mathsf{T}}\big)\cdot\boldsymbol{F}+\boldsymbol{F}^{\mathsf{T}}\cdot\big(\delta\boldsymbol{F}\big)\Big),\\
\delta\boldsymbol{C}=\boldsymbol{F}^{\mathsf{T}}\cdot\boldsymbol{\delta\varepsilon}\cdot\boldsymbol{F},\ \ \boldsymbol{\delta\varepsilon}\equiv\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}}.
\end{gathered}\tag{12.1}$$

.....

$$(5.2)\ \Rightarrow\ \mathring{\boldsymbol{n}}d\mathring{o}=\mathcal{J}^{-1}\boldsymbol{n}d\mathcal{O}\cdot\boldsymbol{F}\ \Rightarrow\ \mathring{\boldsymbol{n}}\cdot\delta\boldsymbol{T}d\mathring{o}=\mathcal{J}^{-1}\boldsymbol{n}\cdot\boldsymbol{F}\cdot\delta\boldsymbol{T}d\mathcal{O}$$
$$\text{or}\ \ \mathring{\boldsymbol{n}}\cdot\delta\boldsymbol{T}d\mathring{o}=\boldsymbol{n}\cdot\boldsymbol{\delta\tau}d\mathcal{O},\ \ \boldsymbol{\delta\tau}\equiv\mathcal{J}^{-1}\boldsymbol{F}\cdot\delta\boldsymbol{T}$$

— tensor $\boldsymbol{\delta\tau}$ introduced here is related to variation $\delta\boldsymbol{T}$ just alike $\boldsymbol{\tau}$ is related to $\boldsymbol{T}$ ($\boldsymbol{\tau}=\mathcal{J}^{-1}\boldsymbol{F}\cdot\boldsymbol{T}$). From (12.1) and ...

.....

... and adjusting the coefficients of the linear function $\boldsymbol{\delta\tau}(\boldsymbol{\delta\varepsilon})$ (...)

§ 13. Internal constraints

До сих пор деформация считалась свободной, мера деформации $\boldsymbol{C}$ могла быть любой. Однако, существуют материалы со значи-

* $\boldsymbol{\nabla}=\boldsymbol{\nabla}\cdot\mathring{\boldsymbol{\nabla}}\mathring{\boldsymbol{r}}=\boldsymbol{r}^i\partial_i\cdot\mathring{\boldsymbol{r}}^j\partial_j\mathring{\boldsymbol{r}}\overset{?}{=}\boldsymbol{r}^i\partial_i\mathring{\boldsymbol{r}}\cdot\mathring{\boldsymbol{r}}^j\partial_j=\boldsymbol{\nabla}\mathring{\boldsymbol{r}}\cdot\mathring{\boldsymbol{\nabla}}=\boldsymbol{F}^{-\mathsf{T}}\cdot\mathring{\boldsymbol{\nabla}}$
$\mathring{\boldsymbol{\nabla}}=\mathring{\boldsymbol{\nabla}}\cdot\boldsymbol{\nabla}\boldsymbol{r}=\mathring{\boldsymbol{r}}^i\partial_i\cdot\boldsymbol{r}^j\partial_j\boldsymbol{r}\overset{?}{=}\mathring{\boldsymbol{r}}^i\partial_i\boldsymbol{r}\cdot\boldsymbol{r}^j\partial_j=\mathring{\boldsymbol{\nabla}}\boldsymbol{r}\cdot\boldsymbol{\nabla}=\boldsymbol{F}^{\mathsf{T}}\cdot\boldsymbol{\nabla}$

тельным сопротивлением некоторым видам деформации. Резина, например, изменению формы сопротивляется намного меньше, чем изменению объёма — некоторые виды резины можно считать несжимаемым материалом.

Понятие геометрической связи, развитое в общей механике ...

...

for incompressible materials $\Pi = \Pi(\mathrm{I}, \mathrm{II})$

Mooney–Rivlin model of incompressible material

$$\Pi = c_1\big(\mathrm{I} - 3\big) + c_2\big(\mathrm{II} - 3\big)$$

incompressible Treloar (neo-Hookean) material

$$c_2 = 0 \quad \Rightarrow \quad \Pi = c_1\big(\mathrm{I} - 3\big)$$

...

§ 14. Hollow sphere under pressure

Решение этой относительно простой задачи описано во многих книгах. В начальной (ненапряжённой) конфигурации имеем сферу с внутренним радиусом r_0 и наружным r_1. Давление равно p_0 внутри и p_1 снаружи.

Введём удобную для этой задачи сферическую систему координат в отсчётной конфигурации $q^1 = \theta$, $q^2 = \phi$, $q^3 = r$ (рисунок **??**). Эти же координаты будут и материальными. Имеем

...

§ 15. Stresses as Lagrange multipliers

The application of the principle of virtual work, described in § 9, was preceded by the introduction of the Cauchy stress tensor through the balance of forces for an infinitesimal tetrahedron (§ 6). But now the reader will see that this principle may be as well applied without any tetrahedrons.

Considering a continuum/body — not only elastic, with any virtual work of internal forces $\delta W^{(i)}$ (per unit mass) — loaded with external forces, mass ones $\boldsymbol{f}dm = \boldsymbol{f}\rho d\mathcal{V}$ (for brevity just $\boldsymbol{f}$, meaning $\boldsymbol{f} \equiv \boldsymbol{f}_* - \ddot{\boldsymbol{r}}$ in dynamics) and surface ones $\boldsymbol{p}d\mathcal{O}$. Then the variational equation of the principle of virtual work is

$$\int\limits_{\mathcal{V}} \rho\Big(\boldsymbol{f}\cdot\delta\boldsymbol{r} + \delta W^{(i)}\Big)d\mathcal{V} + \int\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{p}\cdot\delta\boldsymbol{r}\,d\mathcal{O} = 0. \tag{15.1}$$

Further, it's assumed that internal forces ("stresses") do not produce work when a continuum/body virtually moves (with $\delta\boldsymbol{r}$) as a whole without deformations (when $\boldsymbol{\delta\varepsilon} \equiv \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = {}^2\mathbf{0}$), that is

$$\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = {}^2\mathbf{0} \;\Rightarrow\; \delta W^{(i)} = 0. \tag{15.2}$$

(15.1) with condition (15.2) and without $\delta W^{(i)}$ becomes a variational equation with constraint.

The method of Lagrange multipliers makes $\delta\boldsymbol{r}$ random (independent) variations. Since at each point the constraint appears as a symmetric bivalent tensor, the Lagrange multiplier ${}^2\boldsymbol{\lambda}$ will likewise be such a tensor, bivalent and symmetric. The equation with this multiplier looks like

$$\int\limits_{\mathcal{V}} \Big(\rho\boldsymbol{f}\cdot\delta\boldsymbol{r} - {}^2\boldsymbol{\lambda}\cdot\cdot\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}}\Big)d\mathcal{V} + \int\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{p}\cdot\delta\boldsymbol{r}\,d\mathcal{O} = 0. \tag{15.3}$$

The symmetry of ${}^2\boldsymbol{\lambda}$ gives*

$${}^2\boldsymbol{\lambda} = {}^2\boldsymbol{\lambda}^{\mathsf{T}} \;\Rightarrow\; {}^2\boldsymbol{\lambda}\cdot\cdot\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = {}^2\boldsymbol{\lambda}\cdot\cdot\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}},$$

* $\boldsymbol{\Lambda}^{\mathsf{S}}\cdot\cdot\boldsymbol{X} = \boldsymbol{\Lambda}^{\mathsf{S}}\cdot\cdot\boldsymbol{X}^{\mathsf{T}} = \boldsymbol{\Lambda}^{\mathsf{S}}\cdot\cdot\boldsymbol{X}^{\mathsf{S}}$, $\quad \boldsymbol{\nabla}\cdot(\boldsymbol{B}\cdot\boldsymbol{a}) = (\boldsymbol{\nabla}\cdot\boldsymbol{B})\cdot\boldsymbol{a} + \boldsymbol{B}^{\mathsf{T}}\cdot\cdot\boldsymbol{\nabla}\boldsymbol{a}$

$$ {}^2\boldsymbol{\lambda} \bullet\bullet \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = \boldsymbol{\nabla}\bullet\big({}^2\boldsymbol{\lambda}\bullet\delta\boldsymbol{r}\big) - \boldsymbol{\nabla}\bullet{}^2\boldsymbol{\lambda}\bullet\delta\boldsymbol{r}. $$

Substituting this into (15.3) and applying the divergence theorem*, and the variational equation with multiplier ${}^2\boldsymbol{\lambda}$ becomes

$$ \int_{\mathcal{V}}\Big(\rho\boldsymbol{f} + \boldsymbol{\nabla}\bullet{}^2\boldsymbol{\lambda}\Big)\bullet\delta\boldsymbol{r}\,d\mathcal{V} + \int_{\mathcal{O}(\partial\mathcal{V})}\Big(\boldsymbol{p} - \boldsymbol{n}\bullet{}^2\boldsymbol{\lambda}\Big)\bullet\delta\boldsymbol{r}\,d\mathcal{O} = 0. \qquad (15.4) $$

But $\delta\boldsymbol{r}$ is random both on a surface and in a volume, thus

$$ \boldsymbol{p} = \boldsymbol{n}\bullet{}^2\boldsymbol{\lambda}, \quad \boldsymbol{\nabla}\bullet{}^2\boldsymbol{\lambda} + \rho\boldsymbol{f} = \boldsymbol{0} \qquad (15.5) $$

— the symmetric multiplier ${}^2\boldsymbol{\lambda}$, introduced formally, is in fact precisely the Cauchy stress tensor !

A similar introduction of stresses was presented in the book [47]. Here are no new results, but the very possibility of simultaneously deriving those equations of continuum mechanics, that were previously considered independent, is quite interesting. In subsequent chapters this technique is used for building new continuum models.

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* $\displaystyle\int_{\mathcal{V}}\boldsymbol{\nabla}\bullet\big({}^2\boldsymbol{\lambda}\bullet\delta\boldsymbol{r}\big)d\mathcal{V} = \int_{\mathcal{O}(\partial\mathcal{V})}\boldsymbol{n}\bullet\big({}^2\boldsymbol{\lambda}\bullet\delta\boldsymbol{r}\big)d\mathcal{O}, \quad \boldsymbol{n}\bullet\big({}^2\boldsymbol{\lambda}\bullet\delta\boldsymbol{r}\big) = \big(\boldsymbol{n}\bullet{}^2\boldsymbol{\lambda}\big)\bullet\delta\boldsymbol{r} = \boldsymbol{n}\bullet{}^2\boldsymbol{\lambda}\bullet\delta\boldsymbol{r}$

chapter 4

THE CLASSICAL LINEAR ELASTICITY

This chapter is about the geometrically linear model with infinitesimal displacements, where

- ✓ $\mathcal{V} = \mathring{\mathcal{V}}$, $\rho = \mathring{\rho}$ — "the equations can be written in the initial configuration" (sometimes it's called "the principle of initial dimensions"),
- ✓ operators $\mathring{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ are indistinguishable,
- ✓ operators δ and $\boldsymbol{\nabla}$ commute, thus for example $\delta\boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{\nabla}\delta\boldsymbol{u}$.

§1. The complete set of equations

Equations of the nonlinear elasticity, even in their simplest cases, lead to the mathematically complex problems. Therefore the linear theory of infinitesimal displacements is applied everywhere. This theory's equations were derived in the first half of the XIX$^{\text{th}}$ century by Cauchy, Navier, Lamé, Clapeyron, Poisson, Saint-Venant, George Green and the other scientists.

The complete closed set of equations of the classical linear theory in the direct invariant tensor notation, consisting of

- ✓ the balance of forces (of momentum),
- ✓ the stress–strain relations for a material,
- ✓ displacement $\boldsymbol{u} \mapsto \boldsymbol{\varepsilon}$ relative deformation,

is

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} = \boldsymbol{0}, \quad \boldsymbol{\sigma} = \frac{\partial \Pi}{\partial \boldsymbol{\varepsilon}} = {}^4\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}. \tag{1.1}$$

Here $\boldsymbol{\sigma}$ is the linear stress tensor, $\boldsymbol{g}$ is the resultant vector of volume loads*, $\boldsymbol{u}$ is the vector of displacement (the "absolute" displacement

* In linear theory, the "per volume unit" loads $\boldsymbol{g}d\mathcal{V}$ are used much more often than the "per mass unit" $\boldsymbol{f}dm$ ones ($\boldsymbol{g}d\mathcal{V} = \boldsymbol{f}dm$, $\boldsymbol{g} = \rho\boldsymbol{f}$), because $d\mathcal{V} = d\mathring{\mathcal{V}}$, $\rho d\mathcal{V} = \mathring{\rho} d\mathring{\mathcal{V}}$ and $\boldsymbol{f}dm = \boldsymbol{f}\mathring{\rho}d\mathring{\mathcal{V}} = \boldsymbol{f}\rho d\mathcal{V} = \boldsymbol{g}d\mathcal{V}$.

or absolute deformation), $\boldsymbol{\varepsilon}$ is the tensor of infinitesimal relative deformation (relative displacement, or strain*), $\Pi(\boldsymbol{\varepsilon})$ is the potential energy of deformation per volume unit and ${}^4\boldsymbol{\mathcal{A}}$ is the stiffness tensor. The latter is tetravalent with the following symmetry

$$ {}^4\boldsymbol{\mathcal{A}}_{12\rightleftarrows 34} = {}^4\boldsymbol{\mathcal{A}}, \quad {}^4\boldsymbol{\mathcal{A}}_{1\rightleftarrows 2} = {}^4\boldsymbol{\mathcal{A}}, \quad {}^4\boldsymbol{\mathcal{A}}_{3\rightleftarrows 4} = {}^4\boldsymbol{\mathcal{A}}. $$

But where does all this come from?

The equations (1.1) are exact, they can be derived by varying the equations of the nonlinear theory. Varying from an arbitrary configuration is described in § 3.12. The linear theory is the result of varying from the initial unstressed configuration, where

$$ \begin{gathered} \boldsymbol{F} = \boldsymbol{E}, \quad \boldsymbol{C} = {}^2\boldsymbol{0}, \quad \delta\boldsymbol{C} = \delta\boldsymbol{\varepsilon} = \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}}, \\ \boldsymbol{\tau} = {}^2\boldsymbol{0}, \quad \delta\boldsymbol{T} = \delta\boldsymbol{\tau} = \frac{\partial^2\Pi}{\partial\boldsymbol{C}\partial\boldsymbol{C}} \cdot\cdot\, \delta\boldsymbol{C}, \quad \boldsymbol{\nabla}\cdot\delta\boldsymbol{\tau} + \rho\delta\boldsymbol{f} = \boldsymbol{0}. \end{gathered} \tag{1.2} $$

It remains to change

- ✓ $\delta\boldsymbol{r}$ to $\boldsymbol{u}$,
- ✓ $\delta\boldsymbol{C} = \delta\boldsymbol{\varepsilon}$ to $\boldsymbol{\varepsilon}$,
- ✓ $\delta\boldsymbol{T} = \delta\boldsymbol{\tau}$ to $\boldsymbol{\sigma}$,
- ✓ $\partial^2\Pi/\partial\boldsymbol{C}\partial\boldsymbol{C}$ to ${}^4\boldsymbol{\mathcal{A}}$,
- ✓ $\rho\delta\boldsymbol{f}$ to $\boldsymbol{g}$.

If the derivation (1.2) via varying seems abstruse to the reader, it's possible to proceed from the following equations

$$ \begin{gathered} \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\boldsymbol{f} = \boldsymbol{0}, \quad \boldsymbol{\nabla} = \boldsymbol{F}^{-\mathsf{T}}\cdot\mathring{\boldsymbol{\nabla}}, \quad \boldsymbol{F} = \boldsymbol{E} + \mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}, \\ \boldsymbol{\tau} = J^{-1}\boldsymbol{F}\cdot\frac{\partial\Pi}{\partial\boldsymbol{C}}\cdot\boldsymbol{F}^{\mathsf{T}}, \quad \boldsymbol{C} = \mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{S}} + \tfrac{1}{2}\mathring{\boldsymbol{\nabla}}\boldsymbol{u}\cdot\mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}. \end{gathered} \tag{1.3} $$

Assuming the displacement $\boldsymbol{u}$ is small (infinitesimal), we'll move from (1.3) to (1.1).

* As written in the first sentence of "Historical introduction" in the Augustus Edward Hough Love's book [26].

Or so. Instead of $\boldsymbol{u}$ to take some small enough parameter $\chi\boldsymbol{u}$, $\chi \to 0$. And to represent thereafter the unknowns by the series in the integer exponents of parameter χ

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{(0)} + \chi\boldsymbol{\tau}^{(1)} + \dots, \;\; \boldsymbol{C} = \boldsymbol{C}^{(0)} + \chi\boldsymbol{C}^{(1)} + \dots,$$
$$\boldsymbol{\nabla} = \overset{\circ}{\boldsymbol{\nabla}} + \chi\boldsymbol{\nabla}^{(1)} + \dots, \;\; \boldsymbol{F} = \boldsymbol{E} + \chi\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}, \;\; J = 1 + \chi J^{(1)} + \dots$$

The complete set of equations (1.1) comes from the first (zeroth) terms of these series. In the book [55] this is called "formal approximation".

It is impossible to tell unambiguously how small the parameter χ should be — the answer depends on the situation and is determined by whether the linear model describes the effect we are interested in or not. When, as example, I'm interested in the relation between the frequency of a freely vibrating motion after the initial displacement, then a nonlinear model is needed.

A linear problem is posed in the initial volume $\mathcal{V} = \overset{\circ}{\mathcal{V}}$, bounded by the surface o with the area vector $\boldsymbol{n}do$ ("the principle of initial dimensions").

The boundary conditions (that is, conditions on the surface) most often are : on the part o_1 of the surface displacements are known, and on another part o_2 the forces are known.

$$\boldsymbol{u}\big|_{o_1} = \boldsymbol{u}_0, \quad \boldsymbol{n}\cdot\boldsymbol{\sigma}\big|_{o_2} = \boldsymbol{p}. \tag{1.4}$$

The more complex combinations happen too, if we know the certain components of the both $\boldsymbol{u}$ and $\boldsymbol{t}_{(n)} = \boldsymbol{n}\cdot\boldsymbol{\sigma}$ simultaneously. For example, on a flat face $x = \mathsf{constant}$ when pressing a stamp with a smooth surface $u_x = \nu(y,z)$, $\tau_{xy} = \tau_{xz} = 0$ (the function ν is determined by the stamp's shape).

In dynamics, vector $\boldsymbol{g}$ includes the inertial addend $-\rho\ddot{\boldsymbol{u}}$. The conditions on the surface (boundary conditions) here may depend on time. And the initial conditions for dynamic problems commonly are : positions $\boldsymbol{u}$ and velocities $\dot{\boldsymbol{u}}$, known at the specific point in time $t = 0$.

The linearity gives the principle of superposition (or independence) of the action of loads. When there are several loads, the problem can be solved for each load separately, and the complete solution is

then obtained by summation. For statics this means, for example, that if external loads $\boldsymbol{g}$ and $\boldsymbol{p}$ increase by m times (the body is fixed on o_1), then $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ will increase by m times too. The potential energy Π will increase by m^2 times. In reality, this is observed only when the loads are small.

The potential energy

With linearity, the potential energy density Π as a function of infinitesimal deformation $\boldsymbol{\varepsilon}$ is a quadratic form

$$\Pi(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\,\boldsymbol{\varepsilon} \bullet\bullet {}^4\boldsymbol{\mathcal{A}} \bullet\bullet \boldsymbol{\varepsilon}\,, \tag{1.5}$$

the variation of which is

$$\delta\Pi = \tfrac{1}{2}\,\delta\big(\boldsymbol{\varepsilon} \bullet\bullet {}^4\boldsymbol{\mathcal{A}} \bullet\bullet \boldsymbol{\varepsilon}\big) = \tfrac{1}{2}\big(\delta\boldsymbol{\varepsilon} \bullet\bullet {}^4\boldsymbol{\mathcal{A}} \bullet\bullet \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \bullet\bullet {}^4\boldsymbol{\mathcal{A}} \bullet\bullet \delta\boldsymbol{\varepsilon}\big) = \underbrace{\boldsymbol{\varepsilon} \bullet\bullet {}^4\boldsymbol{\mathcal{A}}}_{\frac{\partial\Pi}{\partial\boldsymbol{\varepsilon}} = \boldsymbol{\sigma}} \bullet\bullet \delta\boldsymbol{\varepsilon}$$

and the second variation

$$\delta^2\Pi = \delta\boldsymbol{\varepsilon} \bullet\bullet {}^4\boldsymbol{\mathcal{A}} \bullet\bullet \delta\boldsymbol{\varepsilon}\,.$$

The relation between the stiffness tensor and the potential energy of elastic deformation is now clear :

$${}^4\boldsymbol{\mathcal{A}} = \frac{\partial^2\Pi}{\partial\boldsymbol{\varepsilon}\,\partial\boldsymbol{\varepsilon}}\,,$$

because $\delta\Pi(\boldsymbol{\varepsilon}) = \dfrac{\partial\Pi}{\partial\boldsymbol{\varepsilon}} \bullet\bullet \delta\boldsymbol{\varepsilon}$ and $\delta^2\Pi(\boldsymbol{\varepsilon}) = \delta\boldsymbol{\varepsilon} \bullet\bullet \dfrac{\partial^2\Pi}{\partial\boldsymbol{\varepsilon}\,\partial\boldsymbol{\varepsilon}} \bullet\bullet \delta\boldsymbol{\varepsilon}$.

Adding that here, as well as for 1-variable calculus*, $\delta^2\Pi = 2\Pi(\delta\boldsymbol{\varepsilon})$, the first and second variations of Π may be written as

$$\begin{aligned} \delta\Pi(\boldsymbol{\varepsilon}) &= \frac{\partial\Pi}{\partial\boldsymbol{\varepsilon}} \bullet\bullet \delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \bullet\bullet {}^4\boldsymbol{\mathcal{A}} \bullet\bullet \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \bullet\bullet \delta\boldsymbol{\varepsilon}\,, \\ \delta^2\Pi(\boldsymbol{\varepsilon}) &= \delta\boldsymbol{\varepsilon} \bullet\bullet \frac{\partial^2\Pi}{\partial\boldsymbol{\varepsilon}\,\partial\boldsymbol{\varepsilon}} \bullet\bullet \delta\boldsymbol{\varepsilon} = \delta\boldsymbol{\varepsilon} \bullet\bullet {}^4\boldsymbol{\mathcal{A}} \bullet\bullet \delta\boldsymbol{\varepsilon} = 2\Pi(\delta\boldsymbol{\varepsilon}). \end{aligned} \tag{1.6}$$

* for $y(x) = \frac{1}{2}\alpha x^2 = \frac{1}{2} x\alpha x$ that is $d^2y = 2y(dx)$

$dy = \frac{1}{2}\big(dx\alpha x + x\alpha dx\big) = x\alpha dx$

$d^2y = dx\alpha dx = \alpha(dx)^2$

The d'Alembert–Lagrange principle of virtual work (2.5, § 2.2), which can be used as the foundation of mechanics, applies to the linear theory as well. Since internal forces in any elastic medium are potential ($\delta W^{(i)} = -\delta\Pi$), the principle is formulated as

$$\int_{\mathcal{V}} \Big((\boldsymbol{g} - \rho \ddot{\boldsymbol{u}}) \cdot \delta\boldsymbol{u} - \delta\Pi \Big) d\mathcal{V} + \int_{o_2} \boldsymbol{p} \cdot \delta\boldsymbol{u}\, do = 0, \quad \boldsymbol{u}\big|_{o_1} = \boldsymbol{0}. \tag{1.7}$$

In addition, if the medium is elastic *linearly*, then

$$\delta\boldsymbol{\varepsilon} = \delta\big(\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}\big) = \boldsymbol{\nabla}\big(\delta\boldsymbol{u}\big)^{\mathsf{S}} \text{ or just } \boldsymbol{\nabla}\delta\boldsymbol{u}^{\mathsf{S}},$$

$$\delta\Pi = \boldsymbol{\sigma} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla}\delta\boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \delta\boldsymbol{u}) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \cdot \delta\boldsymbol{u},$$

$$\int_{\mathcal{V}} \delta\Pi\, d\mathcal{V} = \oint_{o(\partial\mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \delta\boldsymbol{u}\, do - \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \cdot \delta\boldsymbol{u}\, d\mathcal{V},$$

and (1.7) becomes

$$\int_{\mathcal{V}} \Big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} - \rho\ddot{\boldsymbol{u}} \Big) \cdot \delta\boldsymbol{u}\, d\mathcal{V} + \int_{o_2} \Big(\boldsymbol{p} - \boldsymbol{n} \cdot \boldsymbol{\sigma} \Big) \cdot \delta\boldsymbol{u}\, do = 0.$$

Here, virtual displacements $\delta\boldsymbol{u}$ are compatible with the boundary condition for displacements in (1.7), $\delta\boldsymbol{u}\big|_{o_1} = \boldsymbol{0}$.

§ 2. The uniqueness of the solution in dynamics

Usually the uniqueness theorem is proven "by contradiction". Assume that there are two solutions : $\boldsymbol{u}_1(\boldsymbol{r}, t)$ and $\boldsymbol{u}_2(\boldsymbol{r}, t)$. If the difference $\boldsymbol{u}^* \equiv \boldsymbol{u}_1 - \boldsymbol{u}_2$ will be equal to $\boldsymbol{0}$, then these solutions coincide, that is the solution is unique.

But at first we'll make sure of the existænce of the energy integral — by deriving the balance of mechanical energy equation for the linear model of the small displacements theory

$$\int\limits_{\mathcal{V}}\Big(\mathrm{K}+\Pi\Big)^{\bullet}d\mathcal{V}=\int\limits_{\mathcal{V}}\boldsymbol{g}\cdot\dot{\boldsymbol{u}}\,d\mathcal{V}+\int\limits_{o_2}\boldsymbol{p}\cdot\dot{\boldsymbol{u}}\,do, \tag{2.1}$$

$$\boldsymbol{u}\big|_{o_1}=\boldsymbol{0},\quad \boldsymbol{n}\cdot\boldsymbol{\sigma}\big|_{o_2}=\boldsymbol{p},$$

$$\boldsymbol{u}\big|_{t=0}=\boldsymbol{u}^{\circ},\quad \dot{\boldsymbol{u}}\big|_{t=0}=\dot{\boldsymbol{u}}^{\circ}.$$

For the left-hand side we have

$$\dot{\mathrm{K}}=\tfrac{1}{2}\big(\rho\dot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}}\big)^{\bullet}=\tfrac{1}{2}\rho\big(\dot{\boldsymbol{u}}\cdot\ddot{\boldsymbol{u}}+\ddot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}}\big)=\rho\ddot{\boldsymbol{u}}\cdot\dot{\boldsymbol{u}},$$

$$\dot{\Pi}=\underbrace{\tfrac{1}{2}\big(\boldsymbol{\varepsilon}\cdot\cdot{}^{4}\boldsymbol{\mathcal{A}}\cdot\cdot\boldsymbol{\varepsilon}\big)^{\bullet}}_{2\boldsymbol{\varepsilon}\cdot\cdot{}^{4}\boldsymbol{\mathcal{A}}\cdot\cdot\dot{\boldsymbol{\varepsilon}}}=\boldsymbol{\sigma}\cdot\cdot\dot{\boldsymbol{\varepsilon}}=\boldsymbol{\sigma}\cdot\cdot\boldsymbol{\nabla}\dot{\boldsymbol{u}}^{\mathsf{S}}=\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}})-\underbrace{\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}}_{-(\boldsymbol{g}-\rho\ddot{\boldsymbol{u}})}\cdot\dot{\boldsymbol{u}}=$$
$$=\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}})+(\boldsymbol{g}-\rho\ddot{\boldsymbol{u}})\cdot\dot{\boldsymbol{u}}$$

(the balance of momentum $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\boldsymbol{g}-\rho\ddot{\boldsymbol{u}}=\boldsymbol{0}$ is used),

$$\dot{\mathrm{K}}+\dot{\Pi}=\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}})+\boldsymbol{g}\cdot\dot{\boldsymbol{u}}.$$

Applying the divergence theorem

$$\int\limits_{\mathcal{V}}\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}})\,d\mathcal{V}=\oint\limits_{o(\partial\mathcal{V})}\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}}\,do$$

and the boundary condition $\boldsymbol{n}\cdot\boldsymbol{\sigma}=\boldsymbol{p}$ on o_2, we get (2.1).

From (2.1) it follows that without loads (when there're no external forces, neither volume nor surface), and the full mechanical energy doesn't change :

$$\boldsymbol{g}=\boldsymbol{0}\ \text{ and }\ \boldsymbol{p}=\boldsymbol{0}\ \Rightarrow\ \int\limits_{\mathcal{V}}\big(\mathrm{K}+\Pi\big)\,d\mathcal{V}=\mathsf{constant}(t). \tag{2.2}$$

If at the moment $t=0$ there was unstressed ($\Pi=0$) rest ($\mathrm{K}=0$), then

$$\int\limits_{\mathcal{V}}\big(\mathrm{K}+\Pi\big)\,d\mathcal{V}=0. \tag{2.2'}$$

The kinetic energy is positive: $\mathrm{K} > 0$ if $\dot{\boldsymbol{u}} \neq \boldsymbol{0}$ and vanishes (nullifies) only when $\dot{\boldsymbol{u}} = \boldsymbol{0}$ — this ensues from its definition $\mathrm{K} \equiv \frac{1}{2}\rho\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}$. The potential energy, being a quadratic form $\Pi(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}$ (1.5), is positive too: $\Pi > 0$ if $\boldsymbol{\varepsilon} \neq {}^2\boldsymbol{0}$. Such is a priori requirement of the positive definiteness for stiffness tensor ${}^4\boldsymbol{\mathcal{A}}$. This is one of the "additional inequalities in the theory of elasticity" [27, 55].

Since K and Π are positive definite, (2.2') gives

$$\mathrm{K} = 0,\ \Pi = 0 \ \Rightarrow\ \dot{\boldsymbol{u}} = \boldsymbol{0},\ \boldsymbol{\varepsilon} = \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} = {}^2\boldsymbol{0} \ \Rightarrow\ \boldsymbol{u} = \boldsymbol{u}^{\circ} + \boldsymbol{\omega}^{\circ} \times \boldsymbol{r}$$

($\boldsymbol{u}^{\circ}$ and $\boldsymbol{\omega}^{\circ}$ are some constants of translation and rotation). With an immobile part of the surface

$$\boldsymbol{u}\big|_{o_1} = \boldsymbol{0} \ \Rightarrow\ \boldsymbol{u}^{\circ} = \boldsymbol{0} \text{ and } \boldsymbol{\omega}^{\circ} = \boldsymbol{0} \ \Rightarrow\ \boldsymbol{u} = \boldsymbol{0} \text{ everywhere.}$$

Now back to the two solutions, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Their difference $\boldsymbol{u}^* \equiv \boldsymbol{u}_1 - \boldsymbol{u}_2$ is a solution of the entirely "homogeneous" (with no constant terms at all) linear problem: in a volume $\boldsymbol{g} = \boldsymbol{0}$, in boundary and in initial conditions — zeroes. Therefore $\boldsymbol{u}^* = \boldsymbol{0}$, and the uniqueness is proven.

As for the existænce of a solution — it cannot be proven for the generic case by simple conclusions. I could only tell that a dynamic problem is evolutional, it describes the progress of a process in time.

The balance (the conservation) of momentum gives the acceleration $\ddot{\boldsymbol{u}}$. Then, moving to the "next time layer" $t + dt$:

$$\begin{gathered}
\dot{\boldsymbol{u}}(\boldsymbol{r}, t+dt) = \dot{\boldsymbol{u}}(\boldsymbol{r}, t) + \ddot{\boldsymbol{u}}dt, \\
\boldsymbol{u}(\boldsymbol{r}, t+dt) = \boldsymbol{u}(\boldsymbol{r}, t) + \dot{\boldsymbol{u}}dt, \\
\boldsymbol{\varepsilon}(\boldsymbol{r}, t+dt) = \big(\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{r}, t+dt)\big)^{\mathsf{S}} \ \Rightarrow\ \boldsymbol{\sigma}, \\
\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} = \rho\ddot{\boldsymbol{u}}(\boldsymbol{r}, t+dt)
\end{gathered}$$

and so forth. Surely, these considerations lack the "mathematical scrupulosity". If the reader is looking for such, there is, for example, the monograph by Philippe Ciarlet [22].

$$\boldsymbol{\sigma} = \frac{\partial \Pi}{\partial \boldsymbol{\varepsilon}} = {}^4\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot\cdot {}^4\boldsymbol{\mathcal{A}}$$

That relation between the stress and the deformation (strain), which in the XVII$^{\text{th}}$ century Robert Hooke could only phrase pretty vaguely*, is written as part of the complete set of equations (1.1) and is implemented via the stiffness tensor

$$ {}^4\boldsymbol{\mathcal{A}} = \frac{\partial^2 \Pi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}} = A^{ijkl} \boldsymbol{r}_{\partial i} \boldsymbol{r}_{\partial j} \boldsymbol{r}_{\partial k} \boldsymbol{r}_{\partial l}, \quad A^{ijkl} = \frac{\partial^2 \Pi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}. \qquad (3.1)$$

The stiffness tensor is the partial derivative of the scalar elastic potential energy density Π twice by the same bivalent infinitesimal deformation tensor $\boldsymbol{\varepsilon}$. It is symmetric in the pairs of indices: ${}^4\boldsymbol{\mathcal{A}}_{12 \rightleftarrows 34} = {}^4\boldsymbol{\mathcal{A}} \Leftrightarrow A^{ijkl} = A^{klij}$. Therefrom 36 constants out of $3^4 = 81$ "have a twin" and only 45 are independent. Furthermore, due to the symmetry of the infinitesimal deformation tensor $\boldsymbol{\varepsilon}$, the stiffness tensor ${}^4\boldsymbol{\mathcal{A}}$ is symmetric inside each pair of indices: $A^{ijkl} = A^{jikl} = A^{ijlk}$ $(= A^{jilk})$. This reduces the number of the independent constants (the "elastic moduli") to 21:

$$A^{abcd} = A^{cdab} = A^{bacd} = A^{abdc}$$

$$A^{1111}$$
$$A^{1112} = A^{1121} = A^{1211} = A^{2111}$$
$$A^{1113} = A^{1131} = A^{1311} = A^{3111}$$
$$A^{1122} = A^{2211}$$
$$A^{1123} = A^{1132} = A^{2311} = A^{3211}$$
$$A^{1133} = A^{3311}$$
$$A^{1212} = A^{1221} = A^{2112} = A^{2121}$$
$$A^{1213} = A^{1231} = A^{1312} = A^{1321} = A^{2113} = A^{2131} = A^{3112} = A^{3121}$$
$$A^{1222} = A^{2122} = A^{2212} = A^{2221}$$
$$A^{1223} = A^{1232} = A^{2123} = A^{2132} = A^{2312} = A^{2321} = A^{3212} = A^{3221}$$
$$A^{1233} = A^{2133} = A^{3312} = A^{3321}$$
$$A^{1313} = A^{1331} = A^{3113} = A^{3131}$$

* "*ceiiinosssttuu, id est, Ut tensio sic vis*" — **Robert Hooke**. Lectures de Potentia Restitutiva, Or of Spring Explaining the Power of Springing Bodies. London, 1678. 56 pages.

$$
\begin{aligned}
&A^{1322} = A^{2213} = A^{2231} = A^{3122}\\
&A^{1323} = A^{1332} = A^{2313} = A^{2331} = A^{3123} = A^{3132} = A^{3213} = A^{3231}\\
&A^{1333} = A^{3133} = A^{3313} = A^{3331}\\
&A^{2222}\\
&A^{2223} = A^{2232} = A^{2322} = A^{3222}\\
&A^{2233} = A^{3322}\\
&A^{2323} = A^{2332} = A^{3223} = A^{3232}\\
&A^{2333} = A^{3233} = A^{3323} = A^{3332}\\
&A^{3333}
\end{aligned}
$$

The moduli of the tetravalent stiffness tensor are often written as the symmetric 6×6 matrix

$$
\underset{6\times 6}{\big[\mathcal{A}\big]} = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16}\\
a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26}\\
a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36}\\
a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46}\\
a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56}\\
a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66}
\end{bmatrix} \equiv \begin{bmatrix}
A^{1111} & A^{1122} & A^{1133} & A^{1112} & A^{1113} & A^{1123}\\
A^{2211} & A^{2222} & A^{2233} & A^{1222} & A^{1322} & A^{2223}\\
A^{3311} & A^{3322} & A^{3333} & A^{1233} & A^{1333} & A^{2333}\\
A^{1211} & A^{2212} & A^{3312} & A^{1212} & A^{1213} & A^{1223}\\
A^{1311} & A^{2213} & A^{3313} & A^{1312} & A^{1313} & A^{1323}\\
A^{2311} & A^{2322} & A^{3323} & A^{2312} & A^{2313} & A^{2323}
\end{bmatrix}
$$

Even in Cartesian coordinates x, y, z, the quadratic form (1.5) looks pretty huge :

$$
\begin{aligned}
2\Pi = {} & a_{11}\varepsilon_x^2 + a_{22}\varepsilon_y^2 + a_{33}\varepsilon_z^2 + a_{44}\varepsilon_{xy}^2 + a_{55}\varepsilon_{xz}^2 + a_{66}\varepsilon_{yz}^2\\
& + 2\big[\varepsilon_x\big(a_{12}\varepsilon_y + a_{13}\varepsilon_z + a_{14}\varepsilon_{xy} + a_{15}\varepsilon_{xz} + a_{16}\varepsilon_{yz}\big)\\
& \quad + \varepsilon_y\big(a_{23}\varepsilon_z + a_{24}\varepsilon_{xy} + a_{25}\varepsilon_{xz} + a_{26}\varepsilon_{yz}\big)\\
& \quad\quad + \varepsilon_z\big(a_{34}\varepsilon_{xy} + a_{35}\varepsilon_{xz} + a_{36}\varepsilon_{yz}\big)\\
& \quad\quad\quad + \varepsilon_{xy}\big(a_{45}\varepsilon_{xz} + a_{46}\varepsilon_{yz}\big) + a_{56}\varepsilon_{xz}\varepsilon_{yz}\big].
\end{aligned}
\tag{3.2}
$$

When a material symmetry is added, then the number of the independent moduli of tensor ${}^4\boldsymbol{\mathcal{A}}$ decreases.

One plane of material symmetry, a monoclinic material

For a material with a symmetry plane of the elastic properties, for example $z = \mathsf{constant}$.

The change of signs of x and y coordinates does not change the potential energy density Π. And this is possible only when

$$\Pi\Big|_{\substack{\varepsilon_{xz}=-\varepsilon_{xz}\\ \varepsilon_{yz}=-\varepsilon_{yz}}} = \Pi \qquad \Leftrightarrow \qquad \begin{aligned} 0 = a_{15} = a_{16} = a_{25} = a_{26} \\ = a_{35} = a_{36} = a_{45} = a_{46} \end{aligned} \tag{3.3}$$

— the number of independent coefficients lowers to 13.

An orthotropic material

Let there be then the two planes of symmetry : $z = \mathsf{constant}$ and $y = \mathsf{constant}$. Because energy Π in such a case is not sensitive to the signs of ε_{yx} and ε_{yz}, in addition to (3.3) we have

$$a_{14} = a_{24} = a_{34} = a_{56} = 0 \tag{3.4}$$

— 9 constants remained.

A material with the three mutually orthogonal planes of symmetry — let these be the x and y, z planes — is called the orthotropic (orthogonally anisotropic). It's easy to see that (3.3) and (3.4) is the whole set of zero constants, in this case as well. So, an orthotropic material is characterized by the nine elastic moduli, and for orthotropy the two mutually perpendicular planes of symmetry are enough. The expression for the elastic energy density here can be simplified to

$$\begin{aligned} \Pi = \tfrac{1}{2}a_{11}\varepsilon_x^2 + \tfrac{1}{2}a_{22}\varepsilon_y^2 + \tfrac{1}{2}a_{33}\varepsilon_z^2 + \tfrac{1}{2}a_{44}\varepsilon_{xy}^2 + \tfrac{1}{2}a_{55}\varepsilon_{xz}^2 + \tfrac{1}{2}a_{66}\varepsilon_{yz}^2 \\ + a_{12}\varepsilon_x\varepsilon_y + a_{13}\varepsilon_x\varepsilon_z + a_{23}\varepsilon_y\varepsilon_z. \end{aligned}$$

For an orthotropic material, the shear (angular) deformations ε_{xy}, ε_{xz}, ε_{yz} are not linked to the normal stresses $\sigma_x = \partial\Pi/\partial\varepsilon_x$, $\sigma_y = \partial\Pi/\partial\varepsilon_y$, $\sigma_z = \partial\Pi/\partial\varepsilon_z$ (and vice versa).

The popular orthotropic material is wood. Elastic properties there differ along three mutually perpendicular lines : by the radius, along the circumference and along the trunk height.

A transversely isotropic material

One more case of anisotropy is a transversely isotropic material. It is characterized by an axis of anisotropy — let it be z. Then any plane

which is parallel* to z is a plane of material symmetry. It is clear that this material is orthotropic. But more than that, any rotation of the deformation tensor $\boldsymbol{\varepsilon}$ around the z axis doesn't change the elastic potential energy density Π. Thus

$$\frac{\partial \Pi}{\partial \boldsymbol{\varepsilon}} \bullet\bullet \big(\boldsymbol{k}\times\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\times\boldsymbol{k}\big) = 0, \tag{3.5}$$

because for any small rotation with vector $\boldsymbol{\delta o}$, the variation of the infinitesimal deformation tensor $\boldsymbol{\varepsilon}$ is $\boldsymbol{\delta o}\times\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\times\boldsymbol{\delta o}$, and $\boldsymbol{\delta o}$ goes along z with the unit vector $\boldsymbol{k} \equiv \boldsymbol{e}_z$. The equation (3.5) is true for any infinitesimal deformation $\boldsymbol{\varepsilon}$. In components

$$\begin{aligned}\big(a_{11}\varepsilon_x + a_{12}\varepsilon_y + a_{13}\varepsilon_z\big)\big(-2\varepsilon_{xy}\big) + \big(a_{12}\varepsilon_x + a_{22}\varepsilon_y + a_{23}\varepsilon_z\big)2\varepsilon_{xy} \\ + 2a_{44}\varepsilon_{xy}\big(\varepsilon_x - \varepsilon_y\big) + 2a_{55}\varepsilon_{xz}\big(-\varepsilon_{yz}\big) + 2a_{66}\varepsilon_{yz}\varepsilon_{xz} = 0\end{aligned}$$

$$\Rightarrow \quad a_{11} = a_{12} + a_{44} = a_{22}, \quad a_{13} = a_{23}, \quad a_{55} = a_{66}.$$

Writing the stress tensor like

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_\perp + \boldsymbol{s}\boldsymbol{k} + \boldsymbol{k}\boldsymbol{s} + \sigma_{zz}\boldsymbol{k}\boldsymbol{k}, \tag{3.6}$$

where

$$\begin{gathered}\boldsymbol{\sigma}_\perp \equiv \sigma_{\alpha\beta}\boldsymbol{e}_\alpha\boldsymbol{e}_\beta = \sigma_{xx}\boldsymbol{i}\boldsymbol{i} + \sigma_{xy}\big(\boldsymbol{i}\boldsymbol{j} + \boldsymbol{j}\boldsymbol{i}\big) + \sigma_{yy}\boldsymbol{j}\boldsymbol{j}, \\ \boldsymbol{s} \equiv \sigma_{\alpha z}\boldsymbol{e}_\alpha = \sigma_{xz}\boldsymbol{i} + \sigma_{yz}\boldsymbol{j} \\ \big(\ \alpha,\beta \text{ are } x \text{ or } y, \quad \boldsymbol{e}_x = \boldsymbol{i}, \quad \boldsymbol{e}_y = \boldsymbol{j}\ \big),\end{gathered}$$

the Hooke's law for a transversely isotropic material may be presented as

$$\begin{gathered}\boldsymbol{\sigma}_\perp = a_{44}\boldsymbol{\varepsilon}_\perp + \big(a_{12}\varepsilon_{\alpha\alpha} + a_{13}\varepsilon_z\big)\boldsymbol{E}_\perp, \quad \boldsymbol{s} = a_{55}\boldsymbol{\epsilon}, \quad \sigma_{zz} = a_{33}\varepsilon_z + a_{13}\varepsilon_{\alpha\alpha} \\ \big(\text{ here } \boldsymbol{\varepsilon}_\perp \equiv \varepsilon_{\alpha\beta}\boldsymbol{e}_\alpha\boldsymbol{e}_\beta, \quad \varepsilon_{\alpha\alpha} = \text{trace}\,\boldsymbol{\varepsilon}_\perp = \varepsilon_x + \varepsilon_y, \\ \boldsymbol{\epsilon} \equiv \varepsilon_{\alpha z}\boldsymbol{e}_\alpha, \quad \boldsymbol{E}_\perp \equiv \boldsymbol{e}_\alpha\boldsymbol{e}_\alpha = \boldsymbol{i}\boldsymbol{i} + \boldsymbol{j}\boldsymbol{j}\ \big)\end{gathered} \tag{3.7}$$

* If a plane is parallel to a line, this plane's normal vector is perpendicular to that line.

It comes that a transversely isotropic material is characterized by five non-null mutually independent components, the elastic moduli $a_{12} = A^{1122}$, $a_{13} = A^{1133}$, $a_{33} = A^{3333}$, $a_{44} = A^{1212}$, $a_{55} = A^{1313}$.

A crystal symmetry

There are only seven kinds of various primitive parallelepiped lattices (Bravais lattices) — the seven syngonies*, namely triclinic, monoclinic, orthorhombic (or just rhombic), rhombohedral (or trigonal), tetragonal, hexagonal and cubic.

Each case of crystal symmetry is characterized by the set of orthogonal** tensors $\boldsymbol{Q}$, for which the following equation

$$ {}^4\boldsymbol{\mathcal{A}} \bullet\bullet \left(\boldsymbol{Q} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{Q}^{\mathsf{T}} \right) = \boldsymbol{Q} \cdot \left({}^4\boldsymbol{\mathcal{A}} \bullet\bullet \boldsymbol{\varepsilon} \right) \cdot \boldsymbol{Q}^{\mathsf{T}} \quad \forall \boldsymbol{\varepsilon} \tag{3.8} $$

is true (for any infinitesimal deformation $\boldsymbol{\varepsilon}$).

Inverse relations

.....

$$ \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \frac{\partial \text{Ш}}{\partial \boldsymbol{\sigma}} = {}^4\boldsymbol{\mathcal{B}} \bullet\bullet \boldsymbol{\sigma}, \quad \text{Ш}(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \bullet\bullet \boldsymbol{\varepsilon} - \Pi(\boldsymbol{\varepsilon}) \tag{3.9} $$

For the linear model

$$ 2\Pi = \boldsymbol{\sigma} \bullet\bullet \boldsymbol{\varepsilon}, \quad \text{Ш} = \Pi = \tfrac{1}{2}\, \boldsymbol{\sigma} \bullet\bullet \boldsymbol{\varepsilon} \tag{3.10} $$

— the complementary energy density is numerically equal to the elastic potential energy density.

Material tensors

Many physical phenomena are described by tensors, including thermal, mechanical, electrical and magnetic properties.

* "syngony" = "lattice system" = "crystallographic system" = "crystal symmetry"

** Orthogonal tensors are those that satisfy the equality $\boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathsf{T}} = \boldsymbol{E}$ (11.5, § 1.11), describing rotations and mirror flippings.

The "material tensors" define the physical properties of bodies and media, kind of

- ✓ elasticity,
- ✓ thermal expansion,
- ✓ thermal conductivity,
- ✓ electrical conductivity,
- ✓ piezoelectric effect.

Piezoelectricity (the piezoelectric effect) is the coupling (for example, linear) between the mechanical strain and the electric charge in a material, it is the transduction of electrical and mechanical energy.

Certain materials generate an electrical charge when mechanical stress is applied to them. Piezoelectric materials directly transduce electrical and mechanical energy. The most famous piezoelectric material is quartz crystal. Certain ceramics are piezoelectric as well, piezoelectricity is often associated with ceramic materials. Piezoelectric behaviour is also observed in many polymers. And biomatter, such as bone and various proteins, too.

§ 4. Hooke's law for an isotropic medium

Besides anisotropic materials (crystals, composites, wood and others), there are isotropic ones — materials whose properties remain the same in any direction.

When a material is isotropic, (3.8) is satisfied for any orthogonal tensor $\boldsymbol{Q}$. And here, for an isotropic linear elastic medium, it's easier to introduce the potential energy density $\Pi(\boldsymbol{\varepsilon})$ as an isotropic function depending only on the invariants*

$$\Pi(\boldsymbol{\varepsilon}) = \Pi\big(\mathrm{I}(\boldsymbol{\varepsilon}), \mathrm{II}(\boldsymbol{\varepsilon})\big) = \alpha \mathrm{I}^2 + \beta \mathrm{II}. \tag{4.1}$$

(but II is not coefficient ${}_{\mathrm{cha}}\mathrm{II}$ from the solution of the characteristic equation (10.2, § 1.10))

$\Pi(\boldsymbol{\varepsilon})$ is a quadratic function (or a "quadratic form") with terms of only the second degree.

"quadratic form" = "homogeneous polynomial of the second degree" = "function with quadratic terms only"

* Any function whose arguments are only invariants is isotropic.

The isotropic function

$$\mathrm{I}(\boldsymbol{\varepsilon}), \mathrm{II}(\boldsymbol{\varepsilon}) \mapsto \Pi(\boldsymbol{\varepsilon}), \quad \Pi = \Pi(\mathrm{I}, \mathrm{II})$$

has terms of only the 2nd degree (I^2 and II), not of the 1st and not higher. And there's no third invariant $\mathrm{III}(\boldsymbol{\varepsilon})$ among the arguments of Π.

But why not add $\mathrm{III}^{\frac{2}{3}}$?

By excluding dependence on III, we simplify constitutive models while retaining accuracy for most practical applications.

For most isotropic elastic materials, it suffices to model their behavior using only I and II, which capture volumetric and distortional effects without explicitly including volume change through III.

In many cases of isotropic elastic materials, particularly those that are nearly incompressible (e.g., rubber-like materials), volume changes are negligible during deformation. There $\mathrm{III} = 1$.

For small deformations or incompressibility constraints, changes in $\Pi(\mathrm{I}, \mathrm{II})$ already account for all physically relevant behaviors. Including $\Pi(\mathrm{III})$ would unnecessarily complicate calculations without adding meaningful information about material response.

The third invariant III, which relates directly to volume changes, becomes redundant unless large compressive or expansive deformations occur. In linear elasticity (small strains), dependence on only I and II aligns with linear stress-strain relationships (the Hooke's law).

$$\mathrm{I}(\boldsymbol{\varepsilon}) = \operatorname{trace} \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\bullet} = \boldsymbol{\nabla} \cdot \boldsymbol{u}$$
$$\mathrm{II}(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} \cdot\cdot \boldsymbol{\varepsilon}$$

.....

$$2\Pi(\boldsymbol{\varepsilon}) = A \boldsymbol{\varepsilon}_{\bullet} \boldsymbol{\varepsilon}_{\bullet} + B \boldsymbol{\varepsilon} \cdot\cdot \boldsymbol{\varepsilon} \tag{4.2}$$

.....

Since $\boldsymbol{\varepsilon}_{\bullet} = \boldsymbol{\varepsilon} \cdot\cdot \boldsymbol{E} = \boldsymbol{E} \cdot\cdot \boldsymbol{\varepsilon}$ (4.12, § 1.4), the derivative of trace by the tensor itself is the unit dyad

$$\frac{\partial(\boldsymbol{\varepsilon} \cdot\cdot \boldsymbol{E})}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{\varepsilon}} \cdot\cdot \boldsymbol{E} + \boldsymbol{\varepsilon} \cdot\cdot \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{E} \cdot\cdot \boldsymbol{E} + \boldsymbol{\varepsilon} \cdot\cdot {}^2\mathbf{0} = \boldsymbol{E} + {}^2\mathbf{0}$$
$$\Rightarrow \frac{\partial \boldsymbol{\varepsilon}_{\bullet}}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{E} \tag{4.3}$$

An isotropic medium is characterized by the two non-zero elastic constants ("elastic moduli")

$$\Pi(\boldsymbol{\varepsilon}) = \alpha \mathrm{I}^2(\boldsymbol{\varepsilon}) + \beta \mathrm{II}(\boldsymbol{\varepsilon}) \tag{4.4}$$

$$\frac{\partial \Pi}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma}$$

$$\frac{\partial(\boldsymbol{\varepsilon}_\bullet \boldsymbol{\varepsilon}_\bullet)}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \boldsymbol{\varepsilon}_\bullet}{\partial \boldsymbol{\varepsilon}} \boldsymbol{\varepsilon}_\bullet + \boldsymbol{\varepsilon}_\bullet \frac{\partial \boldsymbol{\varepsilon}_\bullet}{\partial \boldsymbol{\varepsilon}} = 2\boldsymbol{\varepsilon}_\bullet \boldsymbol{E}$$

or

$$\frac{\partial(\mathrm{I}^2)}{\partial \boldsymbol{\varepsilon}} = 2\mathrm{I} \frac{\partial \mathrm{I}}{\partial \boldsymbol{\varepsilon}} = 2\mathrm{I}\boldsymbol{E} \quad \left(\text{where } \mathrm{I} = \boldsymbol{\varepsilon}_\bullet\right)$$

.....

$$\boldsymbol{\sigma} = \lambda \boldsymbol{\varepsilon} \bullet\bullet \boldsymbol{E}\boldsymbol{E} + 2\mu \boldsymbol{\varepsilon}$$

$$\boldsymbol{\sigma} = \lambda \boldsymbol{\varepsilon}_\bullet \boldsymbol{E} + 2\mu \boldsymbol{\varepsilon}$$

.....

In components for an isotropic medium

$$A_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu\left(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\right) \tag{4.5}$$

— these are components of an isotropic tensor of the fourth complexity, which don't change when the basis rotates.

.....

Pairs of elastic moduli

There are versions of the Hooke's law for the various pairs of elastic constants (elastic moduli). λ and μ are the Lamé parameters, μ (sometimes G) is the shear modulus, E — the Young's modulus (the modulus of tension or compression), ν is the Poisson's ratio, K — the bulk modulus.

.....

The a priori conditions for the values of elastic moduli are

$$\begin{aligned} E &> 0 \quad \text{— if something is stretched, it elongates,} \\ \mu &> 0 \quad \text{— the shear goes in the same way as the tangential (shear) stress component,} \\ K &> 0 \quad \text{— due to external pressure the volume decreases.} \end{aligned} \tag{4.6}$$

Inequalities for the elastic moduli (4.6) are sufficient for the positivity of Π.

When $\nu \to \frac{1}{2}$, the material becomes incompressible with an infinitely large bulk modulus $K \to \infty$. Negative values of ν are possible* too.

.....

Inverse relations, the complementary energy

$$2\Pi = \boldsymbol{\sigma} \bullet\bullet \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \frac{\partial \amalg}{\partial \boldsymbol{\sigma}} = {}^4\boldsymbol{\mathcal{B}} \bullet\bullet \boldsymbol{\sigma} = \boldsymbol{\sigma} \bullet\bullet {}^4\boldsymbol{\mathcal{B}}. \tag{4.7}$$

the Legendre transform

The complementary energy $\amalg$

$$\amalg(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \bullet\bullet \boldsymbol{\varepsilon} - \Pi(\boldsymbol{\varepsilon}). \tag{4.8}$$

In the linear theory, the "complementary energy" is numerically equal to the elastic potential energy

$$\underbrace{2\Pi}_{\boldsymbol{\sigma} \bullet\bullet \boldsymbol{\varepsilon}} - \underbrace{\Pi(\boldsymbol{\varepsilon})}_{\frac{1}{2}\boldsymbol{\sigma} \bullet\bullet \boldsymbol{\varepsilon}} = \amalg(\boldsymbol{\sigma}).$$

$$\amalg(\boldsymbol{\sigma}) = \Pi(\boldsymbol{\varepsilon})$$

.....

* Such a material, called an auxetic, becomes thicker when it stretches.

Clapeyron's theorem

In equilibrium with the external forces, the volume ones $\boldsymbol{g}$ and the surface ones $\boldsymbol{p}$, the work of these "statically frozen" (that is constant along time) forces on the actual displacements is equal to the double of* the energy of deformation

$$2\int\limits_{\mathcal{V}} \Pi d\mathcal{V} = \int\limits_{\mathcal{V}} \boldsymbol{g}\cdot\boldsymbol{u}\, d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p}\cdot\boldsymbol{u}\, do. \tag{5.1}$$

○ $2\Pi = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{u}) - \underbrace{\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}}_{-\boldsymbol{g}}\cdot\boldsymbol{u} \;\Rightarrow$

$$\Rightarrow\; 2\int\limits_{\mathcal{V}} \Pi d\mathcal{V} = \int\limits_{o_2} \underbrace{\boldsymbol{n}\cdot\boldsymbol{\sigma}}_{\boldsymbol{p}}\cdot\boldsymbol{u}\, do + \int\limits_{\mathcal{V}} \boldsymbol{g}\cdot\boldsymbol{u}\, d\mathcal{V} \quad ●$$

From (5.1) also follows, that without loading $\int_{\mathcal{V}} \Pi d\mathcal{V} = 0$. Because Π is positive, then the stress $\boldsymbol{\sigma}$, and deformation $\boldsymbol{\varepsilon}$ without a load are equal to zero.

$$\begin{aligned} 2\Pi &= \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\varepsilon} \\ \dot{\Pi} &= \boldsymbol{\sigma} \cdot\cdot \dot{\boldsymbol{\varepsilon}} \\ \delta\Pi &= \boldsymbol{\sigma} \cdot\cdot \delta\boldsymbol{\varepsilon} \end{aligned}$$

Π is equal to only the half of the work of the external forces.

The accumulated potential energy of deformation Π is equal to only the half of the work done by the external forces, acting from

* "Ce produit représentait d'ailleurs le double de la force vive que le ressort pouvait absorber par l'effet de sa flexion et qui était la mesure naturelle de sa puissance." —
Benoît Paul Émile Clapeyron. Mémoire sur le travail des forces élastiques dans un corps solide élastique déformé par l'action de forces extérieures. *Comptes rendus*, Tome XLVI, Janvier–Juin 1858. Pagine 208–212.

the unstressed configuration to the equilibrium with the external forces.

Clapeyron's theorem implies that the accumulated elastic energy accounts for only the half of the energy spent on the deformation. The remaining half of the work, done by the external forces, is lost somewhere before reaching the equilibrium.

Roger Fosdick and **Lev Truskinovsky**. About Clapeyron's Theorem in Linear Elasticity. *Journal of Elasticity*, Volume 72, July 2003. Pages 145–172.

In theory, the concept of the "static loading" is common. It's when the external load is applied infinitely slow (sounds like forever, yeah).

The work of the external forces on the actual displacements is equal to the double of the potential energy density 2Π.

Если снять внешние воздействия мгновенно (бесконечно быстро), то тело будет колебаться. Но из-за сопротивления среды (внутреннего трения) спустя некоторое время тело придёт в состояние равновесия.

Yes, only the half of the linear elastic energy is stored. The second half is the " additional energy ", which is lost before reaching of the equilibrium on the dynamics — on the internal energy of the particles (of the dissipation), on the vibrations and waves.

But any real loading would be neither a sudden loading nor an infinitely slow loading. These are the two extremes. The real dynamics of applying the loads will always be different from the theory.

Within infinitesimal variations, applied to an elastic medium real external forces work on virtual displacements $\delta\boldsymbol{\varepsilon}$ and produce work $\delta W^{(e)}$ that is exactly equal (for linear elastic only or for non-linear too??) to the variation of the elastic potential energy density

$$\delta W^{(e)} = \boldsymbol{\sigma} \bullet\bullet \delta\boldsymbol{\varepsilon} = \delta\Pi.$$

A linear elastic medium is medium, where the variation of work of the internal forces (that is stresses) is the variation of the potential energy density with the opposite sign $-\delta W^{(i)} = \delta\Pi = \delta W^{(e)}$, when only the displacements vary (while the stress loads do not).

It is necessary that the virtual work of the real external forces on variations of displacements be equal to the variation of internal energy with the opposite sign (for an elastic media — the variation of internal energy).

The uniqueness of the solution theorem

As in dynamics (§ 2), we suppose the existænce of the two solutions and are looking for their difference

..... (5.2)

The uniqueness of the solution, discovered by Gustav Kirchhoff for bodies with the simply connected contour*, is contrary to, as it seems, the everyday experience. Imagine a straight rod, clamped at the one end (the " cantilever ") and compressed at the second end with a longitudinal force (figure 13). When the load is large enough, the problem of statics has the two solutions, "straight" and "bent". Such a contradiction with the uniqueness theorem comes from the nonlinearity of this problem. If a load is small (infinitesimal), then the solution is described by the linear equations and is unique.

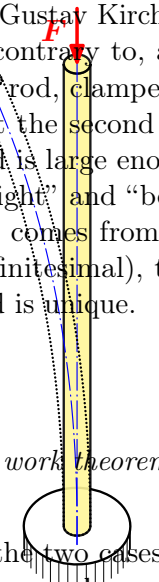

Figure 13

...

...

Reciprocal work theorem

Proposed by Enrico Betti**.

For a body with a fixed part o_1 of the surface, the two cases are considered : the first one with loads $\boldsymbol{g}_1$, $\boldsymbol{p}_1$ and the second one with loads $\boldsymbol{g}_2$, $\boldsymbol{p}_2$. The verbal formulation of the theorem is the same as in § 2.5. The mathematical notation

$$\overbrace{\int\limits_{\mathcal{V}} \boldsymbol{g}_1 \cdot \boldsymbol{u}_2 \, d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p}_1 \cdot \boldsymbol{u}_2 \, do}^{W_{12}} = \overbrace{\int\limits_{\mathcal{V}} \boldsymbol{g}_2 \cdot \boldsymbol{u}_1 \, d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p}_2 \cdot \boldsymbol{u}_1 \, do}^{W_{21}}. \qquad (5.3)$$

...

The reciprocal work theorem, also known as the Betti's theorem, claims that for a linear elastic structure subject to the two sets of forces, P and Q, the work done by set P through displacements produced by set Q is equal to the work done by set Q through displacements produced by set P. This

* **Gustav Robert Kirchhoff**. Über das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes. *Journal für die reine und angewandte Mathematik (Crelle's journal)*, 56. Band (1859). Seiten 285–313. (Seite 291)

** **Enrico Betti**. Teoria della elasticità. *Il Nuovo Cimento* (1869–1876), VII e VIII (1872). Pagina 69.

theorem has applications in structural engineering where it is used to define influence lines and derive the boundary element method.

...

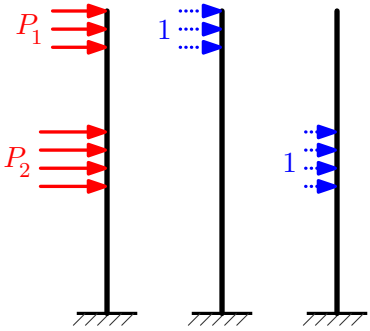

figure 14

The reciprocal work theorem finds unexpected and effective applications. For example, consider a rod-beam clamped at one end ("cantilever") and bent by the two forces with the integral values P_1 and P_2 (figure 14). While the linear theory is applied, displacements-deflections can be represented as

$$u_1 = \alpha_{11} P_1 + \alpha_{12} P_2,$$
$$u_2 = \alpha_{21} P_1 + \alpha_{22} P_2.$$

...

§ 6. Equations in displacements

The complete set of equations (1.1) contains unknowns $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{u}$. Excluding $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, we get the formulation in displacements (symmetrization of $\boldsymbol{\nabla u}$ is redundant due to the symmetry ${}^4\boldsymbol{\mathcal{A}}_{3\rightleftarrows 4} = {}^4\boldsymbol{\mathcal{A}}$).

$$\begin{gathered} \boldsymbol{\nabla} \cdot \left({}^4\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u} \right) + \boldsymbol{g} = \boldsymbol{0}, \\ \boldsymbol{u}\big|_{o_1} = \boldsymbol{u}_0, \quad \boldsymbol{n} \cdot {}^4\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u}\big|_{o_2} = \boldsymbol{p}. \end{gathered} \tag{6.1}$$

For an isotropic medium (6.1) becomes

....

The solution for the homogeneous part* of equation (...) was found by

Пётр Ф. Папкович (Pjotr F. Papkowitsch) Papkovich

* A homogeneous differential equation contains a differentiation and a homogeneous function with a set of variables.

Пётр Ф. Папкович. Выражение общего интеграла основных уравнений теории упругости через гармонические функции // Известия Академии наук СССР. Отделение математических и естественных наук. 1932, выпуск 10, страницы 1425–1435.

and Heinz Neuber

Heinz Neuber. Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie. Der Hohlkegel unter Einzellast als Beispiel // Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), 1934, Band 14, Nr. 4, Seiten 203–212.

Пётр Ф. Папкович (Pjotr F. Papkowitsch) in 1932 and Heinz Neuber in 1934 proposed to represent displacements as harmonic functions and, therefore, to make use of a wide catalogue of particular solutions to the Laplace's equation. And sometimes the problem of elasticity can be reduced, at least partially, to one of the classical problems of the theory of harmonic functions (potential theory).

Приведя полную совокупность уравнений (1.1) к дифференциальному уравнению второй степени (в компонентах — к трём таким уравнениям) for a twice-differentiable function (трёх функций в компонентах), описывающей перемещение точек тела, Пётр Папкович и Hanz Neuber смогли записать общее решение, однако, с существенным ограничением класса объёмных сил — рассматривались только потенциальные. Классические силы механического характера (силы тяжести, силы инерции при равномерном вращении тел, силы гравитационного взаимодействия) потенциальны.

Для консервативных объёмных сил возможно аналитическое решение. Подход Папковича–Neuber'а не применим, если воздействия механической и иной физической природы не потенциальны.

.....

Three-dimensional elasticity based on quaternion-valued potentials

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§ 7. Concentrated (point) force

A concentrated force acting on a point is a handy abstraction to simplify reality. However, it does not exist in the real world, where all forces either act in a volume — volume forces, or act over an area — surface (contact) forces.

Here is a rhetorical question : *why an elastic body withstands an applied load, "bears" it?* The book [12] by James Gordon gives the following answer : the body deforms, and thus the internal forces

appear, called "stresses", which can compensate (balance, equilibrate) an external load.

But a linear elastic body cannot take the load of a point force.

.... in the balance of forces (of momentum)

$$\int_{\mathcal{V}} \big(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \boldsymbol{g}\big)\, d\mathcal{V} = \boldsymbol{0}$$

$\mathcal{V} \sim r^3$, $\boldsymbol{g} \sim \frac{1}{r^3}$, $\boldsymbol{\sigma} \sim \frac{1}{r^2}$

$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\big(\text{trace}\ \boldsymbol{\varepsilon}\big)\boldsymbol{E} \Rightarrow \boldsymbol{\varepsilon} \sim \frac{1}{r^2}$, $\boldsymbol{u} \sim \frac{1}{r}$, thus $\boldsymbol{u} \to \boldsymbol{\infty}$ when $r \to 0$

...

the solution by Kelvin–Somigliana (William Thompson aka Lord Kelvin*, Carlo Somigliana)

... in an infinite medium

.....

the Saint-Venant's principle

Real things can have non-linearities at the places where the external loads are applied. But away from such places only the resulants are important.

As example for rods, the lengths of the non-linear loading regions are comparable with the sizes of the cross sections.

....

§ 8. Finding displacements by deformations

Like any bivalent tensor, the displacement gradient can be decomposed into the sum of the symmetric and antisymmetric parts

$$\boldsymbol{\nabla u} = \overbrace{\boldsymbol{\varepsilon}}^{\boldsymbol{\nabla u}^{\mathsf{S}}} - \overbrace{\boldsymbol{\omega}\times\boldsymbol{E}}^{-\boldsymbol{\nabla u}^{\mathsf{A}}},\quad \boldsymbol{\omega} \equiv \tfrac{1}{2}\,\boldsymbol{\nabla}\times\boldsymbol{u}, \tag{8.1}$$

The symmetric part $\boldsymbol{\nabla u}^{\mathsf{S}}$ is the linear deformation tensor $\boldsymbol{\varepsilon}$.

* William "Lord Kelvin" Thompson. Note on the Integration of the Equations of Equilibrium of an Elastic Solid. The Cambridge and Dublin Mathematical Journal, 1848 volume iii (vii), pages 87–89

The antisymmetric part $\boldsymbol{\nabla u}^{\mathsf{A}}$ can be denoted as $\boldsymbol{\Omega}$ and called the tensor of small rotations. Since any skew-symmetric bivalent tensor is uniquely representable by the vector (§ 1.8), one more field — the vector field of rotations $\boldsymbol{\omega}(\boldsymbol{r})$ — is needed to find displacements $\boldsymbol{u}$ by deformations $\boldsymbol{\varepsilon}$.

....

The deformation compatibility condition in the linear elasticity

The compatibility condition represent the integrability conditions for a symmetric bivalent tensor field. When such a tensor field is compatible, then it describes some deformation (strain).

In the displacement $\mapsto$ deformation relation $\boldsymbol{\varepsilon} = \boldsymbol{\nabla u}^{\mathsf{S}}$, the six components ε_{ij} of deformation $\boldsymbol{\varepsilon}$ originate from the only three components u_k of the displacement vector $\boldsymbol{u}$.

...

$$\operatorname{inc}\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\boldsymbol{\varepsilon}\right)^{\mathsf{T}} \quad \text{or} \quad \operatorname{inc}\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}\times\boldsymbol{\varepsilon}\times\boldsymbol{\nabla}$$

....

A contour here is arbitrary, therefore

$$\operatorname{inc}\boldsymbol{\varepsilon} = {}^{2}\mathbf{0}. \tag{8.2}$$

This relation is called the deformation compatibility condition or the deformation continuity equation(s).

...

Expression (8.2) constraints the possible types of the deformation (strain) field. The compatibility (continuity) condition ensures that no gaps and/or overlaps appear as the result of deformation.

(... add a picture here, the figure where the whole is cut into squares ...)

...

Tensor $\operatorname{inc}\boldsymbol{\varepsilon}$ is symmetric together with $\boldsymbol{\varepsilon}$

....

It was previously proven that

$$\operatorname{inc}\boldsymbol{\varepsilon} = {}^2\mathbf{0} \quad \Leftrightarrow \quad \triangle\boldsymbol{\varepsilon} + \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varepsilon}_{\bullet} = 2\big(\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\varepsilon}\big)^{\mathsf{S}}$$

in index notation for rectangular coordinates

$$\partial_m\partial_m\varepsilon_{ij} + \partial_i\partial_j\varepsilon_{mm} = \big(\partial_i\partial_m\varepsilon_{mj} + \partial_j\partial_m\varepsilon_{mi}\big)$$

with summations expanded

$$\left(\frac{\partial^2\varepsilon_{ij}}{\partial x_1^2} + \frac{\partial^2\varepsilon_{ij}}{\partial x_2^2} + \frac{\partial^2\varepsilon_{ij}}{\partial x_3^2}\right) + \left(\frac{\partial^2\varepsilon_{11}}{\partial x_i\partial x_j} + \frac{\partial^2\varepsilon_{22}}{\partial x_i\partial x_j} + \frac{\partial^2\varepsilon_{33}}{\partial x_i\partial x_j}\right)$$
$$= \left(\frac{\partial^2\varepsilon_{1j}}{\partial x_i\partial x_1} + \frac{\partial^2\varepsilon_{2j}}{\partial x_i\partial x_2} + \frac{\partial^2\varepsilon_{3j}}{\partial x_i\partial x_3}\right) + \left(\frac{\partial^2\varepsilon_{1i}}{\partial x_j\partial x_1} + \frac{\partial^2\varepsilon_{2i}}{\partial x_j\partial x_2} + \frac{\partial^2\varepsilon_{3i}}{\partial x_j\partial x_3}\right)$$

for $i=j\,(=a)$

$$\left(\frac{\partial^2\varepsilon_{aa}}{\partial x_1^2} + \frac{\partial^2\varepsilon_{aa}}{\partial x_2^2} + \frac{\partial^2\varepsilon_{aa}}{\partial x_3^2}\right) + \left(\frac{\partial^2\varepsilon_{11}}{\partial x_a^2} + \frac{\partial^2\varepsilon_{22}}{\partial x_a^2} + \frac{\partial^2\varepsilon_{33}}{\partial x_a^2}\right)$$
$$= 2\left(\frac{\partial^2\varepsilon_{1a}}{\partial x_a\partial x_1} + \frac{\partial^2\varepsilon_{2a}}{\partial x_a\partial x_2} + \frac{\partial^2\varepsilon_{3a}}{\partial x_a\partial x_3}\right) \quad \not{\sum}_a, \quad a=1,2,3$$

.....

$$\begin{cases} \dfrac{\partial^2\varepsilon_{22}}{\partial x_1^2} + \dfrac{\partial^2\varepsilon_{11}}{\partial x_2^2} = 2\dfrac{\partial^2\varepsilon_{21}}{\partial x_1\partial x_2} \\ \dfrac{\partial^2\varepsilon_{33}}{\partial x_2^2} + \dfrac{\partial^2\varepsilon_{22}}{\partial x_3^2} = 2\dfrac{\partial^2\varepsilon_{32}}{\partial x_2\partial x_3} \\ \dfrac{\partial^2\varepsilon_{11}}{\partial x_3^2} + \dfrac{\partial^2\varepsilon_{33}}{\partial x_1^2} = 2\dfrac{\partial^2\varepsilon_{13}}{\partial x_3\partial x_1} \\ \dfrac{\partial^2\varepsilon_{23}}{\partial x_1^2} + \dfrac{\partial^2\varepsilon_{11}}{\partial x_2\partial x_3} = \dfrac{\partial^2\varepsilon_{13}}{\partial x_2\partial x_1} + \dfrac{\partial^2\varepsilon_{12}}{\partial x_3\partial x_1} \\ \dfrac{\partial^2\varepsilon_{13}}{\partial x_2^2} + \dfrac{\partial^2\varepsilon_{22}}{\partial x_1\partial x_3} = \dfrac{\partial^2\varepsilon_{23}}{\partial x_1\partial x_2} + \dfrac{\partial^2\varepsilon_{21}}{\partial x_3\partial x_2} \\ \dfrac{\partial^2\varepsilon_{12}}{\partial x_3^2} + \dfrac{\partial^2\varepsilon_{33}}{\partial x_1\partial x_2} = \dfrac{\partial^2\varepsilon_{32}}{\partial x_1\partial x_3} + \dfrac{\partial^2\varepsilon_{31}}{\partial x_2\partial x_3} \end{cases}$$

— the deformation continuity (compatibility) equations in "classical" notation for rectangular coordinates (the six Saint-Venant's equations).

The last three can also be written as

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(\frac{\partial \varepsilon_{13}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} - \frac{\partial \varepsilon_{23}}{\partial x_1}\right)$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{21}}{\partial x_3} - \frac{\partial \varepsilon_{13}}{\partial x_2}\right)$$

$$\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{32}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right)$$

Isaac Todhunter. The Elastical Researches of Barré de Saint-Venant. Cambridge University Press, 1889.

[110.] *L'institut*, Vol. 26, 1858, pp. 178–9. Further results on Torsion communicated to the *Société Philomathique* (April 24 and May 15, 1858) and afterwards incorporated in the *Leçons de Navier* (pp. 305–6, 273–4). They relate to cross-sections in the form of doubly symmetrical quartic curves and to torsion about an external axis : see our Arts. 49 (c), 182 (b), 181 (d), and 182 (a).

[111.] Vol. 27, 1860, of same Journal, pp. 21–2. Saint-Venant presents to the *Société Philomathique* the model *de la surface décrite par une corde vibrante transportée d'un mouvement rapide perpendiculaire à son plan de vibration.* Copies of this as well as some other of Saint-Venant's models may still be obtained of M. Delagrave in Paris and are of considerable value for class-lectures on the vibration of elastic bodies.

[112.] Vol. 28, 1861, of same Journal, pp. 294–5. This gives an account of a paper of Saint-Venant's read before the *Société Philomathique* (July 28, 1860). In this he deduces the *conditions of compatibility*, or the six differential relations of the types :

$$2\frac{d^2 s_x}{dy\,dz} = \frac{d}{dx}\left(\frac{d\sigma_{xz}}{dy} + \frac{d\sigma_{xy}}{dz} - \frac{d\sigma_{yz}}{dx}\right)$$

$$\frac{d^2 \sigma_{yz}}{dy\,dz} = \frac{d^2 s_y}{dz^2} + \frac{d^2 s_z}{dy^2}$$

which must be satisfied by the strain-components. These conditions enable us in many cases to dispense with the consideration of the shifts. A proof of these conditions by Boussinesq will be found in the *Journal de Liouville*, Vol. 16, 1871, pp. 132–4. At the same meeting Saint-Venant

extended his results on torsion to : (1) prisms on any base with at each point only one plane of symmetry perpendicular to the sides, (2) prisms on an elliptic base with or without any plane of symmetry whatever ; see our Art. 190 (d).

What about nonlinear theory?

All equations of the linear theory have an analogue — the primary source — in the nonlinear theory. To find it for (8.2), remember the Cauchy–Green deformation tensor (§ 3.3) and the curvature tensors (§ 1.19)

.....

§ 9. Equations in stresses

The balance of forces (or of momentum)

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} = \boldsymbol{0} \tag{9.1}$$

does not quite yet determine the stresses. It's necessary as well that deformations (strains) $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ corresponding to stresses (3.9)

$$\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \frac{\partial \amalg}{\partial \boldsymbol{\sigma}} = {}^4\boldsymbol{\mathcal{B}} \cdot\cdot\, \boldsymbol{\sigma} \tag{9.2}$$

were compatible (§ 8)

$$\operatorname{inc} \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) \equiv \boldsymbol{\nabla} \times \Big(\boldsymbol{\nabla} \times \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) \Big)^{\mathsf{T}} = {}^2\boldsymbol{0}. \tag{9.3}$$

Gathered together, (9.1), (9.2) and (9.3) present the complete closed set (system) of equations in stresses.

...

§ 10. The principle of the minimum potential energy

When the existænce of the deformation energy function is assured, and the external forces are assumed to be constant during varying of

displacements, then the principle of virtual work leads to the principle of the minimum potential energy.

The formulation of the principle:

$$\mathscr{E}(\boldsymbol{u}) \equiv \int\limits_{\mathcal{V}} \Big(\Pi(\boldsymbol{u}) - \boldsymbol{g}\cdot\boldsymbol{u}\Big) d\mathcal{V} - \int\limits_{o_2} \boldsymbol{p}\cdot\boldsymbol{u}\, do \to \min, \quad \boldsymbol{u}\big|_{o_1} = \boldsymbol{u}_0. \quad (10.1)$$

The functional $\mathscr{E}(\boldsymbol{u})$, called the (full) potential energy enof a linear-elastic body, is minimal when displacements $\boldsymbol{u}$ are true — that is for the solution of a problem (6.1). The input functions $\boldsymbol{u}$ must satisfy the geometrical condition on o_1 (so they don't break the existing constraints and can be continuous or else $\Pi(\boldsymbol{u})$ will not be integrable) .

For the true field of displacements $\boldsymbol{u}$, the quadratic function

$$\Pi(\boldsymbol{u}) = \tfrac{1}{2}\boldsymbol{\nabla u} \cdot\cdot\, {}^4\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u}$$

becomes equal to the true potential energy of deformation. Then

$$\mathscr{E} = \mathscr{E}_{\min},$$

which according to the Clapeyron's theorem (5.1) is

$$\mathscr{E}_{\min} = \int\limits_{\mathcal{V}} \Pi(\boldsymbol{u})\, d\mathcal{V} - \left(\int\limits_{\mathcal{V}} \boldsymbol{g}\cdot\boldsymbol{u}\, d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p}\cdot\boldsymbol{u}\, do \right) = -\int\limits_{\mathcal{V}} \Pi(\boldsymbol{u})\, d\mathcal{V}.$$

Taking a some other satisfactory field of displacements $\boldsymbol{u}'$, look at the finite difference

$$\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) = \int\limits_{\mathcal{V}} \Big(\Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}) - \boldsymbol{g}\cdot(\boldsymbol{u}' - \boldsymbol{u})\Big) d\mathcal{V} - \int\limits_{o_2} \boldsymbol{p}\cdot(\boldsymbol{u}' - \boldsymbol{u})\, do,$$

seeking $\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) \geq 0$ or (ditto) $\mathscr{E}(\boldsymbol{u}') \geq \mathscr{E}(\boldsymbol{u})$.

$\boldsymbol{g} =$ **constant** and $\boldsymbol{p} =$ **constant**

$\Pi(\boldsymbol{a}) = \frac{1}{2}\boldsymbol{\nabla a} \cdot\cdot\, {}^4\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla a}$ (but *not* the linear $\frac{1}{2}\boldsymbol{\nabla u} \cdot\cdot\, {}^4\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla a}$ — this means $\Pi(\boldsymbol{a}) \neq \frac{1}{2}\boldsymbol{\sigma} \cdot\cdot \boldsymbol{\nabla a}$)

Constraints don't change: $(\boldsymbol{u}'-\boldsymbol{u})\big|_{o_1}=\boldsymbol{u}_0-\boldsymbol{u}_0=\boldsymbol{0}$. External surface force $\boldsymbol{p}\big|_{o_2}=\boldsymbol{t}_{(\boldsymbol{n})}=\boldsymbol{n}\cdot\boldsymbol{\sigma}$ on o_2 and $=\boldsymbol{0}$ elsewhere on $o(\partial\mathcal{V})$. $\boldsymbol{\sigma}=\boldsymbol{\nabla u}\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}=$ [2]**constant** along with constant $\boldsymbol{p}$ and $\boldsymbol{g}$. Therefore

$$\int\limits_{o_2}\boldsymbol{p}\cdot(\boldsymbol{u}'-\boldsymbol{u})\,do=\oint\limits_{o(\partial\mathcal{V})}\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot(\boldsymbol{u}'-\boldsymbol{u})\,do=\int\limits_{\mathcal{V}}\boldsymbol{\nabla}\cdot\Big(\boldsymbol{\sigma}\cdot(\boldsymbol{u}'-\boldsymbol{u})\Big)d\mathcal{V}=$$
$$=\int\limits_{\mathcal{V}}(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\cdot(\boldsymbol{u}'-\boldsymbol{u})\,d\mathcal{V}+\int\limits_{\mathcal{V}}\boldsymbol{\sigma}^{\mathsf{T}}\cdot\cdot\,\boldsymbol{\nabla}(\boldsymbol{u}'-\boldsymbol{u})\,d\mathcal{V}.$$

Due to symmetry $\boldsymbol{\sigma}^{\mathsf{T}}=\boldsymbol{\sigma}\;\Rightarrow\;\boldsymbol{\sigma}^{\mathsf{T}}\cdot\cdot\,\boldsymbol{\nabla a}=\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\nabla a}=\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\nabla a}^{\mathsf{S}}\;\forall\boldsymbol{a}$. Разность преобразуется до

$$\mathscr{E}(\boldsymbol{u}')-\mathscr{E}(\boldsymbol{u})=$$
$$=\int\limits_{\mathcal{V}}\Big(\Pi(\boldsymbol{u}')-\Pi(\boldsymbol{u})-\big(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\boldsymbol{g}\big)\cdot(\boldsymbol{u}'-\boldsymbol{u})-\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\nabla}(\boldsymbol{u}'-\boldsymbol{u})\Big)d\mathcal{V}.$$

And with the balance of momentum $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\boldsymbol{g}=\boldsymbol{0}$

$$\mathscr{E}(\boldsymbol{u}')-\mathscr{E}(\boldsymbol{u})=\int\limits_{\mathcal{V}}\Big(\Pi(\boldsymbol{u}')-\Pi(\boldsymbol{u})-\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\nabla}(\boldsymbol{u}'-\boldsymbol{u})\Big)d\mathcal{V}.$$

Here

$$\Pi(\boldsymbol{u}')=\tfrac{1}{2}\boldsymbol{\nabla u}'\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla u}',\quad\Pi(\boldsymbol{u})=\tfrac{1}{2}\boldsymbol{\nabla u}\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla u},$$

$$\Pi(\boldsymbol{u}')-\Pi(\boldsymbol{u})=\tfrac{1}{2}\Big(\boldsymbol{\nabla u}'\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla u}'-\boldsymbol{\nabla u}\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla u}\Big)$$

$${}^4\mathcal{A}_{12\rightleftarrows34}={}^4\boldsymbol{\mathcal{A}}\;\Rightarrow\;\boldsymbol{\nabla u}\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla u}'=\boldsymbol{\nabla u}'\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla u}$$

$$\tfrac{1}{2}\Big(\boldsymbol{\nabla u}'\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla u}'-\boldsymbol{\nabla u}\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla u}+\boldsymbol{\nabla u}\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla u}'-\boldsymbol{\nabla u}'\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla u}\Big)$$

$$\big(\boldsymbol{\nabla u}'-\boldsymbol{\nabla u}\big)=\boldsymbol{\nabla}(\boldsymbol{u}'-\boldsymbol{u})$$

for a finite difference of potentials

$$\tfrac{1}{2}\boldsymbol{\nabla}\big(\boldsymbol{u}'+\boldsymbol{u}\big)\cdot\cdot\,{}^4\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\big(\boldsymbol{u}'-\boldsymbol{u}\big)=\Pi(\boldsymbol{u}')-\Pi(\boldsymbol{u}),$$

adding to which

$$-\boldsymbol{\nabla u} \cdot\cdot\, {}^4\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u}) = -\boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u})$$

we get

$$\tfrac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u}) \cdot\cdot\, {}^4\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u}) = \Pi(\boldsymbol{u}' - \boldsymbol{u})$$

and finally*

$$\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) = \int_{\mathcal{V}} \Pi(\boldsymbol{u}' - \boldsymbol{u})\, d\mathcal{V}.$$

Since ${}^4\boldsymbol{\mathcal{A}}$ is positive definite (§ 2) $\Pi(\boldsymbol{w}) = \frac{1}{2}\boldsymbol{\nabla w} \cdot\cdot\, {}^4\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla w} \geq 0\ \forall \boldsymbol{w}$ (and $= 0$ only if $\boldsymbol{\nabla w} = \boldsymbol{0} \Leftrightarrow \boldsymbol{w} =$ **constant**: for a case of translation as a whole without deformation.

...

$\delta\boldsymbol{\nabla u} = \boldsymbol{\nabla}\delta\boldsymbol{u}$

...

the Ritz method

The minimum functional problem $\mathscr{E}(\boldsymbol{u})$ is approximately solved as

.....

the finite element method

.....

§ 11. The principle of the minimum complementary energy

When the stress–strain relations (the Hooke's law) assure the existænce of a complementary energy function and the geometrical boundary conditions are assumed constant during variation of stresses, then the principle of minimum complementary energy emerges.

* $b^2 - a^2 - 2a(b-a) = (b+a)(b-a) - 2a(b-a) = (b-a)^2$

"Complementary" work (energy) is named so as not to be confused with the "full" work by Clapeyron (5.1) $W = F(\int du) = 2\Pi$, where $F = \text{constant}$.

The complementary energy of a linearly elastic body is the following functional over the field of stresses :

$$\mathscr{D}(\boldsymbol{\sigma}) \equiv \int_{\mathcal{V}} \text{Ц}(\boldsymbol{\sigma})\, d\mathcal{V} - \int_{o_1} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{u}_0\, do, \quad \boldsymbol{u}_0 \equiv \boldsymbol{u}\big|_{o_1}, \tag{11.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \boldsymbol{\sigma}\big|_{o_2} = \boldsymbol{p}.$$

...

The variation of the balance of force equation

$$\delta\big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g}\big) = \boldsymbol{\nabla} \cdot \delta\boldsymbol{\sigma} = \boldsymbol{0}$$

...

The principle of the minimum complementary energy is very useful for estimating inexact (approximate) solutions. But for computations it isn't so essential as the (Lagrange) principle of minimum potential energy (10.1).

To derive the variational principles it is natural to use the principle of the virtual work (§ 2.2) as a foundation.

§ 12. Mixed principles of stationarity

Prange–Hellinger–Reissner Variational Principle,
named after *Ernst Hellinger*, *Georg Prange* and *Eric Reissner*.
Working independently of Hellinger and Prange, Eric Reissner published his famous six-page paper "On a variational theorem in elasticity" in 1950. In this paper he develops — without, however, considering Hamilton–Jacobi theory — a variational principle same to that of Prange and Hellinger.

Hu–Washizu Variational Principle,
named as *Hu Haichang* and *Kyuichiro Washizu*.

The following functional over the displacements and stresses

$$\mathcal{R}(\boldsymbol{u},\boldsymbol{\sigma}) = \int\limits_{\mathcal{V}} \Big[\boldsymbol{\sigma} \bullet\bullet \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} - \text{Ц}(\boldsymbol{\sigma}) - \boldsymbol{g} \bullet \boldsymbol{u}\Big] d\mathcal{V} - \\ - \int\limits_{o_1} \boldsymbol{n} \bullet \boldsymbol{\sigma} \bullet \big(\boldsymbol{u} - \boldsymbol{u}_0\big) do - \int\limits_{o_2} \boldsymbol{p} \bullet \boldsymbol{u}\, do \quad (12.1)$$

carries names of Reissner, Prange and Hellinger.

...

The advantage of the Reissner–Hellinger principle — freedom of variation. But it also has a drawback: on the true solution the functional has no extremum, but only stationarity.

Принцип можно использовать для построения приближённых решений методом Ritz (Ritz method). Задавая аппроксимации

...

Принцип Hu–Washizu [104] формулируется так:

$$\delta\mathcal{W}(\boldsymbol{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}) = 0,$$
$$\mathcal{W} \equiv \int\limits_{\mathcal{V}} \Big[\boldsymbol{\sigma} \bullet\bullet \Big(\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} - \boldsymbol{\varepsilon}\Big) + \Pi(\boldsymbol{\varepsilon}) - \boldsymbol{g} \bullet \boldsymbol{u}\Big] d\mathcal{V} - \\ - \int\limits_{o_1} \boldsymbol{n} \bullet \boldsymbol{\sigma} \bullet \big(\boldsymbol{u} - \boldsymbol{u}_0\big) do - \int\limits_{o_2} \boldsymbol{p} \bullet \boldsymbol{u}\, do. \quad (12.2)$$

Как и в принципе Reissner'a–Hellinger'a, здесь нет ограничений ни в объёме, ни на поверхности, но добавляется третий независимый аргумент $\boldsymbol{\varepsilon}$. Поскольку $\text{Ц} = \boldsymbol{\sigma} \bullet\bullet \boldsymbol{\varepsilon} - \Pi$, то (12.1) and (12.2) кажутся почти одним и тем же.

From the Hu–Washizu principle ensues the whole complete set of equations with boundary conditions, так как

.....

This is such a problem of the linear theory of elasticity where non-trivial results* are obtained by simple conclusions.

This problem is about an isotropic elastic continuum in the cartesian coordinates

$$x_\alpha, \qquad \alpha = 1, 2, \qquad x_1 \text{ and } x_2.$$

The plane $x_1,\ x_2$ is a cross-section of a rod, the third coordinate x_3 is perpendicular to the section. The basis vectors are

$$\boldsymbol{e}_i = \partial_i \boldsymbol{r}, \quad \boldsymbol{r} = x_i \boldsymbol{e}_i, \quad \boldsymbol{e}_i \boldsymbol{e}_i = \boldsymbol{E} \Leftrightarrow \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}.$$

In case of antiplane strain (antiplane shear), the field of displacements $\boldsymbol{u}(\boldsymbol{r})$ is parallel to the third coordinate x_3 :

$$\boldsymbol{u} = \upsilon \boldsymbol{e}_3,$$

and υ doesn't depend on x_3 :

$$\upsilon = \upsilon(x_1, x_2), \quad \partial_3 \upsilon = 0.$$

The deformation

$$\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla} \boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla} \big(\upsilon \boldsymbol{e}_3 \big)^{\mathsf{S}} = \boldsymbol{e}_3 \boldsymbol{\nabla} \upsilon^{\mathsf{S}} + \upsilon \underbrace{\boldsymbol{\nabla} \boldsymbol{e}_3^{\mathsf{S}}}_{^2\mathbf{0}} = \tfrac{1}{2} \big(\boldsymbol{\nabla} \upsilon \boldsymbol{e}_3 + \boldsymbol{e}_3 \boldsymbol{\nabla} \upsilon \big) \quad (13.1)$$

In the plane x_1, x_2 of the cross-section

$$\mu = \mu(x_1, x_2), \qquad \partial_3 \mu = 0$$

— a possible inhomogeneity of the medium.

* Non-trivial in the theory of elasticity is, for example, when dividing the force by the area gives an infinitely large error in calculating the stress.

M. de Saint-Venant. Memoire sur la torsion des prismes (1853)

Adhémar-Jean-Claude Barré de Saint-Venant. Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exerçant simultanément. 1856. 327 pages.

1. Memoire sur la torsion des prismes, avec des considerations sur leur flexion, etc. Memoires presentes par divers savants a l'Academie des scienees, t. 14, 1856.

2. Memoire sur la flexion des prismes, etc. Journal de mathematiques pures et appliquees, publie par J. Liouville, 2me serie, t. 1, 1856.

Перевод на русский язык: **Сен-Венан Б.** Мемуар о кручении призм. Мемуар об изгибе призм. М.: Физматгиз, 1961. 518 страниц.

стр. 379–494

This problem, which was studied in detail by Adhémar-Jean-Claude Barré de Saint-Venant, is contained in almost every book on the linear elasticity. It considers a cylinder of some section, loaded only by the surface forces at the ends (... add a figure ...)

$$z = \ell : \boldsymbol{k} \cdot \boldsymbol{\sigma} = \boldsymbol{p}(x_\alpha),$$
$$z = 0 : -\boldsymbol{k} \cdot \boldsymbol{\sigma} = \boldsymbol{p}_0(x_\alpha),$$

where $\boldsymbol{k} \equiv \boldsymbol{e}_3$, $\alpha = 1, 2$, $\boldsymbol{x} \equiv x_\alpha \boldsymbol{e}_\alpha$. Coordinates are x_1, x_2, z.

The resultant (the sum) of the external forces is equal to $\mathbf{0}$, and the resultant couple is directed along the z axis :

$$\int\limits_o \boldsymbol{p} do = \mathbf{0}, \quad \int\limits_o \boldsymbol{x} \times \boldsymbol{p} do = M\boldsymbol{k}.$$

On torsion, the tangential stress components $\tau_{z1} \equiv \boldsymbol{k} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{e}_1$ and $\tau_{z2} \equiv \boldsymbol{k} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{e}_2$ arise. Assuming that only these components of $\boldsymbol{\sigma}$ are non-zero

$$\boldsymbol{\sigma} = \boldsymbol{s}\boldsymbol{k} + \boldsymbol{k}\boldsymbol{s}, \quad \boldsymbol{s} \equiv \tau_{z\alpha} \boldsymbol{e}_\alpha.$$

The solution of this problem simplifies if the equations in stresses are used.

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=\mathbf{0}\ \Rightarrow\ \boldsymbol{\nabla}_\perp\cdot\boldsymbol{s}=\mathbf{0}\ (\boldsymbol{\nabla}_\perp\equiv\boldsymbol{e}_\alpha\partial_\alpha),\ \partial_z\boldsymbol{s}=\mathbf{0}, \tag{14.1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\boldsymbol{\sigma}+\frac{1}{1+\nu}\boldsymbol{\nabla}\boldsymbol{\nabla}\sigma=^2\mathbf{0}\ \Rightarrow\ \triangle_\perp\boldsymbol{s}=\mathbf{0}\ (\triangle_\perp\equiv\partial_\alpha\partial_\alpha). \tag{14.2}$$

The independence of $\boldsymbol{s}$ from z makes it possible to replace the three-dimensional operators with the two-dimensional ones.

...

§ 15. Plane deformation

Here the displacement vector $\boldsymbol{u}$ is parallel to the plane x_1, x_2 and does not depend on the third coordinate z.

For example рассмотрим полуплоскость с сосредоточенной нормальной силой Q на краю (?? рисунок ??)

...

Bibliography

There are several dozen books on the classical linear theory of elasticity that haven't lost their relevance over time. First of all, the fundamental monograph by Анатолий Лурье (Anatoliy Lurie) [28] and his earlier book [29] about solving spatial problems. Quite rich in content is the Witold Nowacki's book [38]. There the author spent many pages describing the problems of both statics and "elastokinetics" (that is dynamics), and the last chapter of this book describes the linear Cosserat continuum — that's what the next chapter is about. Being mathematically capacious and saturated, the theory of elasticity attracts mathematicians, as it happened with the monograph [22] by Philippe Ciarlet. The Augustus Love's book [26] cannot go unmentioned as well. Климентий Черны́х (Klimentiy Chernih) described in [58] how to model linear elastic media with anisotropy.

chapter 5

MICROPOLAR THREE-DIMENSIONAL CONTINUUM

§1. Introduction to the linear micropolar theory

The characteristic feature of a classical elastic media (chapter 3 and 4) is that they are made from the "simple points". A particle of a classical continuum has only the translational degrees of freedom, and the only single vector $\boldsymbol{r}(q^i, t)$ describes its movement. Therefore the external loads in such a model are only the forces, the volume and the surface ones. There are no moments.

But it's not so hard to build the more complex models of a continuous medium, where the particles have not only the translational degrees of freedom, but some additional ones as well. These new degrees o'freedom are correlated with the new loads. The most natural of the non-classical models of a three-dimensional medium was proposed by the Cosserat brothers in 1909 [23]. Every particle of the Cosserat's continuum is an infinitesimal absolutely rigid body with the six degrees of freedom, the three translational and the three rotational. The loads in such a medium are forces and moments. The work of the Cosserat brothers remained unnoticed for the half a century, but then the interest in this topic arose.

from ***Nowacki W.*** *The Linear Theory of Micropolar Elasticity.* In: *Micropolar Elasticity. International Centre for Mechanical Sciences (Courses and Lectures), vol. 151, 1974, pp. 1–43*

Woldemar Voigt tried to remove the shortcomings of the classical theory of elasticity [***W. Voigt****. Theoretische Studien über die Elasticitätsverhältnisse der Krystalle. Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen, 34: 3–51, 1887*] by the assumption

that the interaction of two parts of the body is transmitted through an area element do by means not only of the force vector $\boldsymbol{p}do$ but also by the moment vector $\boldsymbol{m}do$. Thus, besides the force stresses σ_{ji} also the moment stresses have been defined.

However, the complete theory of asymmetric elasticity was developed by the brothers ***François et Eugène Cosserat*** who published it in 1909 in the work *"Théorie des corps déformables"*.

They assumed that the bodies are composed from the connected particles. These particles are like a small rigid bodies.

During the deforming each particle is displaced by vector $\boldsymbol{u}(\boldsymbol{r},t)$ and rotated by vector $\boldsymbol{\varphi}(\boldsymbol{r},t)$, the functions of the location $\boldsymbol{r}$ and time t. Thus the points (particles) of the Cosserat's continuum possess the orientation (it is a "polar media"). So we can speak of the rotation of a point. The mutually independent vectors $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ define deformations of a body.

The introduction of $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ with the assumption that the transmission of forces through an area element do is happened by the force vector $\boldsymbol{p}$ and the moment vector $\boldsymbol{m}$ leads in the consequence to the asymmetric stress tensors σ_{qp} and μ_{qp}.

The theory of the brothers E. and F. Cosserat remained unnoticed during their lifetime. It was so because their theory was non-linear and thus included large deformations, because the frames of their theory were out of the frames of the classical linear elasticity. They tried to construct the unified field theory, containing mechanics, optics and electrodynamics, combined by a principle of the least action.

The research in the field of the general theories of continuous media conducted in the last fifteen years, drew the attention of the scientists to the Cosserats' work. Looking for the new models, describing the behaviour of the real elastic media more precisely, the models similar to, or identical with that of Cosserats' have been encountered. Here I want to mention, first of all, the papers by C. Truesdell and R. A. Toupin [***C. Truesdell*** *and* ***R. A. Toupin****. The classical field theories. Encyclopœdia of Physics, Chapter 1, Springer-Verlag, Berlin, 1960*], G. Grioli [***Grioli G.*** *Elasticité asymmetrique. Ann. di Mat. Pura et Appl. Ser. IV, 50 (1960)*], R. D. Mindlin and H. F. Tiersten [***Mindlin, R. D.; Tiersten, H. F.*** *Effects of couple-stresses in linear elasticity. Arch. Rational Mech. Anal. 11. 1962. 415–448*].

In the truly micropolar continuum, the vector field of displacements $\boldsymbol{u}(\boldsymbol{r},t)$ and the field of rotations $\boldsymbol{\varphi}(\boldsymbol{r},t)$ are mutually independent. This

is also called the model with free rotation. Also it is the geometrically linear model, that's the case of the very small, the infinitesimal displacements and the infinitesimall rotations. Here operators $\mathring{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ are indistinguishable. $\mathcal{V}=\mathring{\mathcal{V}}$, $\rho=\mathring{\rho}$ and therefore equations "can be written in the initial configuration". Also operators δ and $\boldsymbol{\nabla}$ commute ($\delta\boldsymbol{\nabla}\boldsymbol{u}=\boldsymbol{\nabla}\delta\boldsymbol{u}$, $\delta\boldsymbol{\nabla}\boldsymbol{\varphi}=\boldsymbol{\nabla}\delta\boldsymbol{\varphi}$).

To build this model, I use the principle of virtual work. This principle says that the variation of work of the real external forces on virtual displacements is equal with the opposite sign to the variation of work , the real stresses on virtual deformations

$$\int\limits_{\mathcal{V}}\Big(\boldsymbol{f}\cdot\delta\boldsymbol{u}+\boldsymbol{m}\cdot\delta\boldsymbol{\varphi}\Big)d\mathcal{V}+\int\limits_{o}\Big(\boldsymbol{p}\cdot\delta\boldsymbol{u}+\boldsymbol{M}\cdot\delta\boldsymbol{\varphi}\Big)do=-\int\limits_{\mathcal{V}}\delta W^{(i)}d\mathcal{V}.$$

Here $\boldsymbol{f}$ and $\boldsymbol{m}$ are the external forces and moments per "one volume", $\boldsymbol{p}$ and $\boldsymbol{M}$ are external forces too, but per surface unit (the surface loads, they act only on a certain part of o on the boundary surface.

$\delta W^{(i)}$ is the work of internal forces density per volume unit

As before, we suppose that $\delta W^{(i)}$ nullifies when the body moves as a rigid whole without deformation:

$$\delta\boldsymbol{u}=\delta\boldsymbol{\varphi}\times\boldsymbol{r}+\textbf{constant},\ \ \delta\boldsymbol{\varphi}=\textbf{constant}\ \Rightarrow\ \delta W^{(i)}=0,$$
$$\boldsymbol{\nabla}\delta\boldsymbol{u}=\boldsymbol{\nabla}\delta\boldsymbol{\varphi}\times\boldsymbol{r}-\boldsymbol{\nabla}\boldsymbol{r}\times\delta\boldsymbol{\varphi}=-\boldsymbol{E}\times\delta\boldsymbol{\varphi}=-\delta\boldsymbol{\varphi}\times\boldsymbol{E},\ \ \boldsymbol{\nabla}\delta\boldsymbol{\varphi}={}^2\boldsymbol{0}.$$

Introducing the deformation (strain) tensors — the tensor of the relative displacement between particles $\boldsymbol{\gamma}$ (the distortion tensor, the distortion is the relative displacement between particles) and the curvature-twist tensor $\boldsymbol{\kappa}$ (it also has other names: the torsion-flexure tensor, or the wryness tensor)

$$\boldsymbol{\gamma}\equiv\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\varphi}\times\boldsymbol{E},\ \ \boldsymbol{\kappa}\equiv\boldsymbol{\nabla}\boldsymbol{\varphi},\tag{1.1}$$

$$\boldsymbol{\gamma}_{\times}=\boldsymbol{\nabla}\times\boldsymbol{u}-2\boldsymbol{\varphi},\ \ \boldsymbol{\kappa}_{\times}=\boldsymbol{\nabla}\times\boldsymbol{\varphi},$$
$$\delta\boldsymbol{\gamma}=\boldsymbol{\nabla}\delta\boldsymbol{u}+\delta\boldsymbol{\varphi}\times\boldsymbol{E},\ \ \delta\boldsymbol{\kappa}=\boldsymbol{\nabla}\delta\boldsymbol{\varphi},$$

with the needed absence of any virtual deformations $\delta\boldsymbol{\gamma}={}^2\boldsymbol{0}$ and $\delta\boldsymbol{\kappa}={}^2\boldsymbol{0}$.

Before in § 3.15, stresses appeared as Lagrange's multipliers in the principle of virtual work, when variation of the internal work $\delta W^{(i)} = 0$. The same for the micropolar continuum :

$$\int\limits_{\mathcal{V}} \Big(\boldsymbol{f} \cdot \delta \boldsymbol{u} + \boldsymbol{m} \cdot \delta \boldsymbol{\varphi} - \boldsymbol{\tau} \cdot\cdot\, \delta \boldsymbol{\gamma}^{\mathsf{T}} - \boldsymbol{\mu} \cdot\cdot\, \delta \boldsymbol{\kappa}^{\mathsf{T}} \Big)\, d\mathcal{V} + \\ + \int\limits_{o} \Big(\boldsymbol{p} \cdot \delta \boldsymbol{u} + \boldsymbol{M} \cdot \delta \boldsymbol{\varphi} \Big)\, do = 0. \quad (1.2)$$

Lagrange's multipliers at each point are non-symmetric tensors of the second complexity, $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$.

Transforming $-\boldsymbol{\tau} \cdot\cdot\, \delta \boldsymbol{\gamma}^{\mathsf{T}}$ and $-\boldsymbol{\mu} \cdot\cdot\, \delta \boldsymbol{\kappa}^{\mathsf{T}}$

$$\delta \boldsymbol{\gamma}^{\mathsf{T}} = \boldsymbol{\nabla} \delta \boldsymbol{u}^{\mathsf{T}} - \delta \boldsymbol{\varphi} \times \boldsymbol{E}, \;\; \delta \boldsymbol{\kappa}^{\mathsf{T}} = \boldsymbol{\nabla} \delta \boldsymbol{\varphi}^{\mathsf{T}},$$
$$-\boldsymbol{\tau} \cdot\cdot\, \delta \boldsymbol{\gamma}^{\mathsf{T}} = -\boldsymbol{\tau} \cdot\cdot\, \boldsymbol{\nabla} \delta \boldsymbol{u}^{\mathsf{T}} + \boldsymbol{\tau} \cdot\cdot\, (\delta \boldsymbol{\varphi} \times \boldsymbol{E}), \;\; -\boldsymbol{\mu} \cdot\cdot\, \delta \boldsymbol{\kappa}^{\mathsf{T}} = -\boldsymbol{\mu} \cdot\cdot\, \boldsymbol{\nabla} \delta \boldsymbol{\varphi}^{\mathsf{T}}.$$

From

$$(8.7, \S\, 1.8) \;\Rightarrow\; \boldsymbol{A}_{\times} = -\boldsymbol{A} \cdot\cdot\, {}^{3}\boldsymbol{\epsilon},$$

$$\boldsymbol{A} \cdot\cdot\, (\boldsymbol{b} \times \boldsymbol{E}) = \boldsymbol{A} \cdot\cdot\, (\boldsymbol{E} \times \boldsymbol{b}) = \boldsymbol{A} \cdot\cdot\, \big(-{}^{3}\boldsymbol{\epsilon} \cdot \boldsymbol{b}\big) = \\ = \big(-\boldsymbol{A} \cdot\cdot\, {}^{3}\boldsymbol{\epsilon}\big) \cdot \boldsymbol{b} = \boldsymbol{A}_{\times} \cdot \boldsymbol{b} \;\Rightarrow$$

$$\Rightarrow\; \boldsymbol{\tau} \cdot\cdot\, (\delta \boldsymbol{\varphi} \times \boldsymbol{E}) = \boldsymbol{\tau}_{\times} \cdot \delta \boldsymbol{\varphi}$$

and the "product rule"

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \cdot \delta \boldsymbol{u}) = (\boldsymbol{\nabla} \cdot \boldsymbol{\tau}) \cdot \delta \boldsymbol{u} + \boldsymbol{\tau} \cdot\cdot\, \boldsymbol{\nabla} \delta \boldsymbol{u}^{\mathsf{T}},$$
$$\boldsymbol{\nabla} \cdot (\boldsymbol{\mu} \cdot \delta \boldsymbol{\varphi}) = (\boldsymbol{\nabla} \cdot \boldsymbol{\mu}) \cdot \delta \boldsymbol{\varphi} + \boldsymbol{\mu} \cdot\cdot\, \boldsymbol{\nabla} \delta \boldsymbol{\varphi}^{\mathsf{T}},$$

follows

$$-\boldsymbol{\tau} \cdot\cdot\, \delta \boldsymbol{\gamma}^{\mathsf{T}} = (\boldsymbol{\nabla} \cdot \boldsymbol{\tau}) \cdot \delta \boldsymbol{u} - \boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \cdot \delta \boldsymbol{u}) + \boldsymbol{\tau}_{\times} \cdot \delta \boldsymbol{\varphi},$$
$$-\boldsymbol{\mu} \cdot\cdot\, \delta \boldsymbol{\kappa}^{\mathsf{T}} = (\boldsymbol{\nabla} \cdot \boldsymbol{\mu}) \cdot \delta \boldsymbol{\varphi} - \boldsymbol{\nabla} \cdot (\boldsymbol{\mu} \cdot \delta \boldsymbol{\varphi}).$$

Because $\boldsymbol{a} \cdot ({}^{2}\boldsymbol{B} \cdot \boldsymbol{c}) = (\boldsymbol{a} \cdot {}^{2}\boldsymbol{B}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot {}^{2}\boldsymbol{B} \cdot \boldsymbol{c}$, after integration by the divergence theorem

$$\int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \cdot \delta \boldsymbol{u})\, d\mathcal{V} = \oint\limits_{\widehat{o}(\partial \mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\tau} \cdot \delta \boldsymbol{u}\, do, \quad \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\boldsymbol{\mu} \cdot \delta \boldsymbol{\varphi})\, d\mathcal{V} = \oint\limits_{\widehat{o}(\partial \mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\mu} \cdot \delta \boldsymbol{\varphi}\, do$$

(1.2) turns into

$$\int\limits_{\mathcal{V}}\Big((\boldsymbol{\nabla}\cdot\boldsymbol{\tau}+\boldsymbol{f})\cdot\delta\boldsymbol{u}+(\boldsymbol{\nabla}\cdot\boldsymbol{\mu}+\boldsymbol{\tau}_{\times}+\boldsymbol{m})\cdot\delta\boldsymbol{\varphi}\Big)d\mathcal{V}+ \\ +\int\limits_{o}\Big((\boldsymbol{p}-\boldsymbol{n}\cdot\boldsymbol{\tau})\cdot\delta\boldsymbol{u}+(\boldsymbol{M}-\boldsymbol{n}\cdot\boldsymbol{\mu})\cdot\delta\boldsymbol{\varphi}\Big)do=0.$$

From the randomness of variations $\delta\boldsymbol{u}$ and $\delta\boldsymbol{\varphi}$ inside a volume ensues the balance of forces and moments

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau}+\boldsymbol{f}=\boldsymbol{0},\ \ \boldsymbol{\nabla}\cdot\boldsymbol{\mu}+\boldsymbol{\tau}_{\times}+\boldsymbol{m}=\boldsymbol{0}, \tag{1.3}$$

and the randomness of a surface gives boundary conditions

$$\boldsymbol{n}\cdot\boldsymbol{\tau}=\boldsymbol{p},\ \ \boldsymbol{n}\cdot\boldsymbol{\mu}=\boldsymbol{M}. \tag{1.4}$$

The force stress tensor $\boldsymbol{\tau}$ satisfies the same differential "equilibrium equations"* and the same boundary conditions, as for the momentless continuum. But tensor $\boldsymbol{\tau}$ is yet non-symmetric : instead of $\boldsymbol{\tau}_{\times}=\boldsymbol{0}$ here is $\boldsymbol{\nabla}\cdot\boldsymbol{\mu}+\boldsymbol{\tau}_{\times}+\boldsymbol{m}=\boldsymbol{0}$ — the couple stresses $\boldsymbol{\mu}$ appear, and the volume moment load $\boldsymbol{m}$ is not zero.

The meaning of components of the couple stress tensor $\boldsymbol{\mu}$ is similar to the meaning of components of the force stress tensor $\boldsymbol{\tau}$. For an orthonormal basis, moment $\boldsymbol{M}_i=\boldsymbol{e}_i\cdot\boldsymbol{\mu}=\mu_{ik}\boldsymbol{e}_k$ acts on an area with the normal vector $\boldsymbol{e}_i$. The diagonal components μ_{11}, μ_{22}, μ_{33} are the twisting moments and the nondiagonal are the bending ones (?? figure??).

...

eccentricity vector $\boldsymbol{æ}$ and inertia tensor ${}^2\boldsymbol{\mathfrak{I}}$

For an isotropic medium $\boldsymbol{æ}=\boldsymbol{0}$, ${}^2\boldsymbol{\mathfrak{I}}=\mathfrak{I}\boldsymbol{E}$.

...

* In quotes, because *equilibrium equations* are quite *everything* that ensues from the principle of virtual work in statics.

In this book, if something is "elastic", then this implies the potentiality of internal forces : $\delta W^{(i)} = -\delta\Pi$, where Π is the energy of deformation per volume unit (continuing to model a geometrically linear material, $\mathcal{V} = \mathring{\mathcal{V}}$).

Having relations (...)

...

$$\delta\Pi = -\delta W^{(i)} = \boldsymbol{\tau} \bullet\bullet \delta\boldsymbol{\gamma}^{\mathsf{T}} + \boldsymbol{\mu} \bullet\bullet \delta\boldsymbol{\kappa}^{\mathsf{T}} \;\Rightarrow$$
$$\Rightarrow\;\; \boldsymbol{\tau} = \frac{\partial\Pi}{\partial\boldsymbol{\gamma}},\;\; \boldsymbol{\mu} = \frac{\partial\Pi}{\partial\boldsymbol{\kappa}}. \tag{2.1}$$

And yep, the last equalities are the relations of elasticity (or, in other words,— the constitutive equations).

Decomposing the strain (deformation) and stress tensors into the symmetric and the antisymmetric parts

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}^{\mathsf{S}} - \tfrac{1}{2}\,\boldsymbol{\gamma}_{\times} \times \boldsymbol{E},\;\; \boldsymbol{\kappa} = \boldsymbol{\kappa}^{\mathsf{S}} - \tfrac{1}{2}\,\boldsymbol{\kappa}_{\times} \times \boldsymbol{E},$$
$$\delta\boldsymbol{\gamma}^{\mathsf{T}} = \delta\boldsymbol{\gamma}^{\mathsf{S}} + \tfrac{1}{2}\,\delta\boldsymbol{\gamma}_{\times} \times \boldsymbol{E},\;\; \delta\boldsymbol{\kappa}^{\mathsf{T}} = \delta\boldsymbol{\kappa}^{\mathsf{S}} + \tfrac{1}{2}\,\delta\boldsymbol{\kappa}_{\times} \times \boldsymbol{E},$$
$$\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{S}} - \tfrac{1}{2}\,\boldsymbol{\tau}_{\times} \times \boldsymbol{E},\;\; \boldsymbol{\mu} = \boldsymbol{\mu}^{\mathsf{S}} - \tfrac{1}{2}\,\boldsymbol{\mu}_{\times} \times \boldsymbol{E},$$

the expression $\delta\Pi = \boldsymbol{\tau} \bullet\bullet \delta\boldsymbol{\gamma}^{\mathsf{T}} + \boldsymbol{\mu} \bullet\bullet \delta\boldsymbol{\kappa}^{\mathsf{T}}$ changes into

$$\delta\Pi = \dots \tag{2.2}$$

...

$$\boldsymbol{\gamma}_{\times} = \boldsymbol{\nabla} \times \boldsymbol{u} - 2\boldsymbol{\varphi},$$

$$\boldsymbol{\kappa}_{\times} = \boldsymbol{\nabla} \times \boldsymbol{\varphi},$$

$$\boldsymbol{\varepsilon} \equiv \boldsymbol{\gamma}^{\mathsf{S}} = \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}},$$

...

The classical isotropic linear elastic material behavior is described by two material parameters, for example, the Young's modulus and the Poisson's ratio, while the isotropic Cosserat continuum needs six material parameters

even when assumed to be linear, homogeneous and isotropic, it requires six independent material constants, in contrast to only two such constants for the classical continuum

...

Relations (2.1) are inverted by a Legendre transformation

$$\begin{gathered} \boldsymbol{\gamma} = \frac{\partial \text{Ц}}{\partial \boldsymbol{\tau}}, \ \boldsymbol{\kappa} = \frac{\partial \text{Ц}}{\partial \boldsymbol{\mu}}, \\ \text{Ц}(\boldsymbol{\tau}, \boldsymbol{\mu}) = \boldsymbol{\tau} \bullet\bullet \boldsymbol{\gamma}^{\mathsf{T}} + \boldsymbol{\mu} \bullet\bullet \boldsymbol{\kappa}^{\mathsf{T}} - \Pi(\boldsymbol{\gamma}, \boldsymbol{\kappa}). \end{gathered} \tag{2.3}$$

...

material's intrinsic (internal) length scale ℓ

Если устремить ℓ к нулю, то исчезает вклад $\boldsymbol{\kappa}$ в Π, а с ним и моментные напряжения $\boldsymbol{\mu}$. Когда вдобавок нет объёмной моментной нагрузки $\boldsymbol{m}$, тогда тензор $\boldsymbol{\tau}$ становится симметричным: $\boldsymbol{\nabla} \bullet \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \boldsymbol{m} = \boldsymbol{0}$, $\boldsymbol{\mu} = {}^2\boldsymbol{0}$, $\boldsymbol{m} = \boldsymbol{0} \ \Rightarrow \ \boldsymbol{\tau}_{\times} = \boldsymbol{0}$, и модель превращается в классическую безмоментную.

Yet using of the micropolar model is natural for a case when the real material has a certain smallest volume "which is impossible to penetrate". And such a situation occurs quite often : for composites with a "representative" volume, for polycrystalline materials, for polymers with large molecules (macromolecules).

§ 3. Compatibility equations

Having the identity $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \boldsymbol{a} = {}^2\boldsymbol{0} \ \ \forall \boldsymbol{a}$ and the definitions for the deformation tensors (1.1)

$$\begin{gathered} \boldsymbol{\kappa} \equiv \boldsymbol{\nabla} \boldsymbol{\varphi} \ \Rightarrow \ \boldsymbol{\nabla} \times \boldsymbol{\kappa} = {}^2\boldsymbol{0}, \\ \boldsymbol{\gamma} - \boldsymbol{\varphi} \times \boldsymbol{E} = \boldsymbol{\nabla} \boldsymbol{u} \ \Rightarrow \ \boldsymbol{\nabla} \times \big(\boldsymbol{\gamma} - \boldsymbol{\varphi} \times \boldsymbol{E} \big) = {}^2\boldsymbol{0} \ \ldots \end{gathered} \tag{3.1}$$

....

§ 4. Theorems of statics

Theorems of statics for linear conservative systems, easily derivable for a finite number of degrees o'freedom (§ 2.5), such as the minimality of energy, the uniqueness of the solution, the Clapeyron's theorem, the reciprocal work theorem, are equitable (5.2, § 4.5), (5.1, § 4.5), (5.3, § 4.5) for infinity degrees o'freedom as well, that is for any linear elastic continuum, including the micropolar model (a medium with force couples or moments).

...

§ 5. The Cosserat pseudocontinuum

Besides the model with free rotation ("the truly micropolar continuum"), there is the simplified model of a medium with the force couples, in which rotations are expressed via displacements, like for the linear momentless ("classical") continuum (8.1, § 4.8)

$$\boldsymbol{\varphi} = \tfrac{1}{2}\,\boldsymbol{\nabla}\times\boldsymbol{u} \;\Leftrightarrow\; \boldsymbol{\gamma}_{\times} = \mathbf{0} \;\Leftrightarrow\; \boldsymbol{\gamma} = \boldsymbol{\varepsilon} = \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} \tag{5.1}$$

— the model with constrained rotation.*

The symmetry of $\boldsymbol{\gamma}$ ($\boldsymbol{\gamma}_{\times} = \mathbf{0}$) can be thought of as an internal constraint (§ 3.13). Here $\boldsymbol{\gamma}_{\times}$ disappears from the energy Π, and a relation of elasticity for $\boldsymbol{\tau}_{\times}$ cannot be written. That place in the complete set of equations is taken by the equation of a constraint.

For the classical (linear momentless) model, the complete set (system) may be presented as the just one equation for the vector $\boldsymbol{u}$ (6.1, § 4.6). The model with moments and free rotation may be described by the two vector equations for $\boldsymbol{u}$ and $\boldsymbol{\varphi}$. And for the model with constrained rotation it's again the single equation for $\boldsymbol{u}$.

.....

* Cosserat brothers named it *cas de trièdre caché* (case of latent trihedron).

§ 6. Plane deformation

Все переменные в этой постановке проблемы не зависят от декартовой координаты $z \equiv x_3$ (единичный вектор оси $\boldsymbol{k} \equiv \boldsymbol{e}_z \equiv \boldsymbol{e}_3$). Смещения и силы перпендикулярны оси z, а повороты и моменты — параллельны ей:

...

Это краткое изложение плоской задачи относится к модели с независимыми поворотами. ПсевдоконтинуумCosserat (модель “со стеснённым вращением”) получается либо наложением внутренней связи $\boldsymbol{\gamma}_\times = \mathbf{0}$, либо предельным переходом ...

Подробнее о плоской моментной задаче написано в книгах Н. Ф. Морозова [**?**, 72].

§ 7. Nonlinear theory

Кажущееся на первый взгляд чрезвычайно трудным, построение теории конечных деформаций континуума Cosserat становится прозрачным, если опираться на общую механику, тензорное исчисление и нелинейную теорию безмоментной среды.

Построение модели упругого континуума проходит обычно четыре этапа:

- ✓ определение степеней свободы частиц,
- ✓ выявление нагрузок (“силовых факторов”, напряжений) и условий их баланса,
- ✓ подбор соответствующих мер деформации

и, наконец,

- ✓ вывод соотношений упругости между напряжениями и деформациями.

Этот путь очень сокращается, если опираться на принцип виртуальной работы.

Как и в гл. 3, среда состоит из частиц с материальными координатами q^i и вектором-радиусом $\boldsymbol{r}(q^i, t)$. В начальной (исходной, отсчётной) конфигурации $\boldsymbol{r}(q^i, 0) \equiv \mathring{\boldsymbol{r}}(q^i)$. Но кроме трансляции,

частицы имеют независимые степени свободы поворота, описываемого ортогональным тензором

$$\boldsymbol{O}(q^i,t) \equiv \boldsymbol{a}_j \mathring{\boldsymbol{a}}^j = \boldsymbol{a}^j \mathring{\boldsymbol{a}}_j = \boldsymbol{O}^{-\mathsf{T}},$$

где тройка векторов $\boldsymbol{a}_j(q^i,t)$ жёстко связана с каждой частицей, показывая угловую ориентацию относительно как-либо выбираемых* векторов $\mathring{\boldsymbol{a}}_j(q^i) \equiv \boldsymbol{a}_j(q^i,0)$, $\boldsymbol{a}_j = \boldsymbol{O}\bullet\mathring{\boldsymbol{a}}_j$; $\boldsymbol{a}^j$ — тройка взаимных векторов: $\boldsymbol{a}_j\boldsymbol{a}^j = \boldsymbol{a}^j\boldsymbol{a}_j = \boldsymbol{E}$ ($t{=}0$, $\mathring{\boldsymbol{a}}^j$: $\mathring{\boldsymbol{a}}_j\mathring{\boldsymbol{a}}^j = \mathring{\boldsymbol{a}}^j\mathring{\boldsymbol{a}}_j = \boldsymbol{E}$). Движение среды полностью определяется функциями $\boldsymbol{r}(q^i,t)$ and $\boldsymbol{O}(q^i,t)$.

Имея представления $\mathring{\boldsymbol{r}}(q^i)$ и $\boldsymbol{r}(q^i,t)$, вводим базис $\boldsymbol{r}_i \equiv \partial_i \boldsymbol{r}$, взаимный базис $\boldsymbol{r}^i$: $\boldsymbol{r}_j\bullet\boldsymbol{r}^i = \delta^i_j$, differential operators $\mathring{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$, а также motion gradient $\boldsymbol{F}$

$$\mathring{\boldsymbol{\nabla}} \equiv \mathring{\boldsymbol{r}}^i\partial_i, \quad \boldsymbol{\nabla} \equiv \boldsymbol{r}^i\partial_i, \quad \boldsymbol{\nabla} = \boldsymbol{F}^{-\mathsf{T}}\bullet\mathring{\boldsymbol{\nabla}}, \quad \boldsymbol{F} \equiv \mathring{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}} = \boldsymbol{r}_i\mathring{\boldsymbol{r}}^i. \tag{7.1}$$

Вариационное уравнение принципа виртуальной работы для континуума с нагрузками в объёме и на поверхности:

$$\int\limits_{\mathcal{V}} \Big(\rho\big(\boldsymbol{f}\bullet\delta\boldsymbol{r} + \boldsymbol{m}\bullet\delta\boldsymbol{\varphi}\big) + \delta W^{(i)}\Big)\,d\mathcal{V} + \\ + \int\limits_{\mathcal{O}} \Big(\boldsymbol{p}\bullet\delta\boldsymbol{r} + \boldsymbol{M}\bullet\delta\boldsymbol{\varphi}\Big)\,d\mathcal{O} = 0. \tag{7.2}$$

Here ρ is mass density; $\boldsymbol{f}$ and $\boldsymbol{m}$ are external force and moment per mass unit; $\boldsymbol{p}$ and $\boldsymbol{M}$ — они же per surface unit; $\delta W^{(i)}$ — работа внутренних сил per volume unit в текущей конфигурации. Вектор ма́лого поворота $\delta\boldsymbol{\varphi}$

$$\boldsymbol{O}\bullet\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{E} \;\Rightarrow\; \delta\boldsymbol{O}\bullet\boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O}\bullet\delta\boldsymbol{O}^{\mathsf{T}} \;\Rightarrow \\ \Rightarrow\; \delta\boldsymbol{O}\bullet\boldsymbol{O}^{\mathsf{T}} = \delta\boldsymbol{\varphi}\times\boldsymbol{E} = \delta\boldsymbol{\varphi}\times\boldsymbol{O}\bullet\boldsymbol{O}^{\mathsf{T}} \;\Rightarrow\; \delta\boldsymbol{O} = \delta\boldsymbol{\varphi}\times\boldsymbol{O},$$

* Один из вариантов: $\mathring{\boldsymbol{a}}_j = \mathring{\boldsymbol{r}}_j \equiv \partial_j\boldsymbol{r}$. Другое предложение: $\mathring{\boldsymbol{a}}_j$ это ортонормальная тройка собственных векторов тензора инерции частицы. Вообще, $\mathring{\boldsymbol{a}}_j$ могут быть любой тройкой линейно-независимых векторов.

$$\delta\boldsymbol{\varphi} = -\tfrac{1}{2}\left(\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}}\right)_{\times}$$

При движении среды как жёсткого целого нет деформаций, и работа $\delta W^{(i)}$ внутренних сил равна нулю:

$$\delta\boldsymbol{r} = \textsf{constant} + \delta\boldsymbol{\varphi}\times\boldsymbol{r},\ \ \delta\boldsymbol{\varphi} = \textsf{constant}\ \Rightarrow\ \delta W^{(i)} = 0,$$
$$\boldsymbol{\nabla}\delta\boldsymbol{\varphi} = {}^{2}\mathbf{0},\ \ \boldsymbol{\nabla}\delta\boldsymbol{r} = \boldsymbol{\nabla}\delta\boldsymbol{\varphi}\times\boldsymbol{r} - \boldsymbol{\nabla}\boldsymbol{r}\times\delta\boldsymbol{\varphi} = -\boldsymbol{E}\times\delta\boldsymbol{\varphi} = -\,\delta\boldsymbol{\varphi}\times\boldsymbol{E},$$
$$\boldsymbol{\nabla}\delta\boldsymbol{r} + \delta\boldsymbol{\varphi}\times\boldsymbol{E} = {}^{2}\mathbf{0}.$$

К нагрузкам. Несимметричные тензоры напряжения, силового $\boldsymbol{\tau}$ и моментного $\boldsymbol{\mu}$, введём как множители Lagrange'а:

$$\int\limits_{\mathcal{V}}\Big(\rho\left(\boldsymbol{f}\cdot\delta\boldsymbol{r} + \boldsymbol{m}\cdot\delta\boldsymbol{\varphi}\right) - \boldsymbol{\tau}\cdot\cdot\left(\boldsymbol{\nabla}\delta\boldsymbol{r} + \delta\boldsymbol{\varphi}\times\boldsymbol{E}\right)^{\mathsf{T}} - \boldsymbol{\mu}\cdot\cdot\boldsymbol{\nabla}\delta\boldsymbol{\varphi}^{\mathsf{T}}\Big)\,d\mathcal{V} + \\ + \int\limits_{\mathcal{O}}\Big(\boldsymbol{p}\cdot\delta\boldsymbol{r} + \boldsymbol{M}\cdot\delta\boldsymbol{\varphi}\Big)\,d\mathcal{O} = 0. \quad (7.3)$$

После тех же преобразований, что и в § 1, получаем

...

Отсюда вытекают уравнения баланса сил и моментов в объёме и краевые условия в виде формул типа Cauchy. Они по существу те же, что и в линейной теории.

Найдём теперь тензоры деформации. Их можно вводить по-разному, если требовать лишь одного — нечувствительности к движению среды как жёсткого целого. Читатель найдёт не один вариант таких тензоров. Однако, вид тензоров деформации "подсказывает" принцип виртуальной работы.

...

§ 8. Nonlinear model with constrained rotation

Вспомним переход к модели со стеснённым вращением в линейной теории (§ 5). Разделились соотношения упругости для симметричной части тензора силового напряжения $\boldsymbol{\tau}^{\mathsf{S}}$ и кососимметричной его части $\boldsymbol{\tau}_{\times}$. Возникла внутренняя связь $\boldsymbol{\gamma}_{\times} = \mathbf{0}$

...

Все работы по моментной теории упругости упоминают книгу братьев Eugène et François Cosserat [23], где трёхмерной среде посвящена одна глава из шести. Переведённая монография W. Nowacki [38] была одной из первых книг на русском языке с изложением линейной моментной теории. Ранее эта область представлялась статьями — например, R. D. Mindlin'a и H. F. Tiersten'a [35]. Краткое изложение моментной теории, но с подробным рассмотрением задач содержится в книгах Н. Ф. Морозова [**?**, 72].

chapter 6

THERMOELASTICITY

Hitherto, modeling in this book was limited to just only mechanics. Everyday experience shows, however, that changes in temperature (heating or cooling) deform bodies. Thermal deformations and stresses often play the primary role and can lead to a breakage.

(In this chapter, only the momentless model is considered.)

§ 1. The first law of thermodynamics

Very successful in mechanics, the principle of virtual work is not applicable to thermomechanics*. Considering thermal effects, it's possible to lean on the two laws of thermodynamics.

The first, discovered by Joule, Mayer, and Helmholtz — is the balance of energy : the rate of internal energy change $\dot{E}$ is equal to the sum of the power of external forces $P^{(e)}$ and the rate of heat supply $\dot{Q}$

$$\dot{E} = P^{(e)} + \dot{Q}. \tag{1.1}$$

Internal energy E is the sum of the kinetic and the potential energies of the particles. For any finite volume of a material continuum

$$E = \int_{\mathcal{V}} \rho \left(\tfrac{1}{2} \dot{\boldsymbol{r}} \bullet \dot{\boldsymbol{r}} + e \right) d\mathcal{V}. \tag{1.2}$$

With the balance of mass $dm = \rho d\mathcal{V} = \rho' d\mathcal{V}'$, $m = \int_{\mathcal{V}} \rho d\mathcal{V} = \int_{\mathcal{V}'} \rho' d\mathcal{V}'$ and

$$\Psi = \textstyle\int_{\mathcal{V}} \rho \psi d\mathcal{V} = \int_{\mathcal{V}'} \rho' \psi d\mathcal{V}' \Rightarrow \dot{\Psi} = \int_{\mathcal{V}} \rho \dot{\psi} d\mathcal{V} = \int_{\mathcal{V}'} \rho' \dot{\psi} d\mathcal{V}',$$

* Analogue of the principle of virtual work will be presented below in § 8.

it's easy to get the time derivative of the internal energy

$$\dot{E} = \int_{\mathcal{V}} \rho \left(\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + \dot{e} \right) d\mathcal{V}. \tag{1.3}$$

The power of the external forces for some finite volume of a momentless continuum

$$P^{(e)} = \int_{\mathcal{V}} \rho \boldsymbol{f} \cdot \dot{\boldsymbol{r}}\, d\mathcal{V} + \oint_{O(\partial\mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\tau} \cdot \dot{\boldsymbol{r}}\, dO = \int_{\mathcal{V}} \left(\rho \boldsymbol{f} \cdot \dot{\boldsymbol{r}} + \boldsymbol{\nabla} \cdot \left(\boldsymbol{\tau} \cdot \dot{\boldsymbol{r}} \right) \right) d\mathcal{V} =$$
$$= \int_{\mathcal{V}} \left(\left(\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \boldsymbol{f} \right) \cdot \dot{\boldsymbol{r}} + \boldsymbol{\tau} \cdot\cdot \boldsymbol{\nabla} \dot{\boldsymbol{r}}^{\mathsf{S}} \right) d\mathcal{V}. \tag{1.4}$$

As before (chapter 3), $\boldsymbol{\tau}$ is Cauchy stress tensor, $\boldsymbol{f}$ is mass force (without inertial part $-\ddot{\boldsymbol{r}}$, which is included in $\dot{E}$), $\boldsymbol{n} \cdot \boldsymbol{\tau}$ is surface force. The symmetry of $\boldsymbol{\tau}$ is used for expanding $\boldsymbol{\nabla} \cdot \left(\boldsymbol{\tau} \cdot \dot{\boldsymbol{r}} \right)$:

$$\boldsymbol{\tau}^{\mathsf{T}} = \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{S}} \;\Rightarrow\; \boldsymbol{\tau} \cdot\cdot \boldsymbol{\nabla} \dot{\boldsymbol{r}}^{\mathsf{T}} = \boldsymbol{\tau} \cdot\cdot \boldsymbol{\nabla} \dot{\boldsymbol{r}}^{\mathsf{S}},$$
$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{\tau} \cdot \dot{\boldsymbol{r}} \right) = \left(\boldsymbol{\nabla} \cdot \boldsymbol{\tau} \right) \cdot \dot{\boldsymbol{r}} + \boldsymbol{\tau} \cdot\cdot \boldsymbol{\nabla} \dot{\boldsymbol{r}}^{\mathsf{T}} = \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \cdot \dot{\boldsymbol{r}} + \boldsymbol{\tau} \cdot\cdot \boldsymbol{\nabla} \dot{\boldsymbol{r}}^{\mathsf{S}}.$$

Denominating the rate of deformation tensor as $\boldsymbol{\mathcal{D}} \equiv \boldsymbol{\nabla} \dot{\boldsymbol{r}}^{\mathsf{S}}$

$$P^{(e)} = \int_{\mathcal{V}} \left(\left(\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \boldsymbol{f} \right) \cdot \dot{\boldsymbol{r}} + \boldsymbol{\tau} \cdot\cdot \boldsymbol{\mathcal{D}} \right) d\mathcal{V}. \tag{1.4'}$$

Heat arrives in a volume of continuum by two ways. The first is a surface heat transfer (heat conduction, thermal conductivity, convection, diffusion), occurring via matter, upon a contact of two media. This can be described by heat flux vector $\boldsymbol{q}$. Through an infinitesimal area in the current configuration towards normal vector $\boldsymbol{n}$ per unit of time passes heat flux $\boldsymbol{q} \cdot \boldsymbol{n} dO$. For a surface with finite dimensions this expression needs to be integrated. It's usually assumed

$$\boldsymbol{q} = -{}^2\boldsymbol{k} \cdot \boldsymbol{\nabla} \Theta, \tag{1.5}$$

where Θ is the temperature (temperature field); ${}^2\boldsymbol{k}$ is the thermal conductivity tensor as the property of material, for an isotropic material ${}^2\boldsymbol{k} = k\boldsymbol{E}$ and $\boldsymbol{q} = -k\,\boldsymbol{\nabla}\Theta$.

The second way is a volume heat transfer (thermal radiation). Solar energy, flame of a campfire, a microwave oven are familiar examples of pervasive heating by radiation. Thermal radiation occurs via electromagnetic waves and doesn't need an interjacent medium. Heat is radiated (emitted) by any matter (with temperature above the absolute zero 0 K). The rate of heat transfer by radiation per mass unit b or per volume unit $B = \rho b$ is assumed as known.

Therefore, the rate of heat supply for a finite volume is

$$\dot{Q} = -\oint\limits_{O(\partial\mathcal{V})} \boldsymbol{n}\bullet\boldsymbol{q}\,dO + \int\limits_{\mathcal{V}} \rho b\,d\mathcal{V} = \int\limits_{\mathcal{V}} \Big(-\boldsymbol{\nabla}\bullet\boldsymbol{q} + \rho b\Big)\,d\mathcal{V}. \qquad (1.6)$$

Applying (1.3), (1.4') and (1.6) to formulation (1.1) gives the equality of integrals over a volume

$$\int\limits_{\mathcal{V}} \rho\Big(\ddot{\boldsymbol{r}}\bullet\dot{\boldsymbol{r}} + \dot{e}\Big)\,d\mathcal{V} = \int\limits_{\mathcal{V}} \Big(\big(\boldsymbol{\nabla}\bullet\boldsymbol{\tau} + \rho\boldsymbol{f}\big)\bullet\dot{\boldsymbol{r}} + \boldsymbol{\tau}\bullet\bullet\boldsymbol{\mathcal{D}} - \boldsymbol{\nabla}\bullet\boldsymbol{q} + \rho b\Big)\,d\mathcal{V}.$$

And since volume $\mathcal{V}$ is random, integrands are equal too

$$\rho\ddot{\boldsymbol{r}}\bullet\dot{\boldsymbol{r}} + \rho\dot{e} = \big(\boldsymbol{\nabla}\bullet\boldsymbol{\tau} + \rho\boldsymbol{f}\big)\bullet\dot{\boldsymbol{r}} + \boldsymbol{\tau}\bullet\bullet\boldsymbol{\mathcal{D}} - \boldsymbol{\nabla}\bullet\boldsymbol{q} + \rho b.$$

With the balance of momentum

$$\boldsymbol{\nabla}\bullet\boldsymbol{\tau} + \rho\big(\boldsymbol{f} - \ddot{\boldsymbol{r}}\big) = \boldsymbol{0} \quad (7.2, \S\,3.7)$$

this simplifies to

$$\rho\dot{e} = \boldsymbol{\tau}\bullet\bullet\boldsymbol{\mathcal{D}} - \boldsymbol{\nabla}\bullet\boldsymbol{q} + \rho b \qquad (1.7)$$

— the local (differential) version of the balance of energy.

...

§ 2. The second law

The following concept of laws of thermodynamics is widespread: change in internal energy dE is equal to the sum of work of external forces $\partial W^{(e)}$ and supplied heat ∂Q

$$dE = \partial W^{(e)} + \partial Q.$$

Work $\partial W^{(e)}$ and heat ∂Q are inexact differentials*, but the quotient $\partial Q/\Theta$ becomes the exact differential — the differential dS of the entropy.

Further, all processes are divided into the reversible ones, for which $dS = \partial Q/\Theta$, and the irreversible ones with the characteristic Clausius inequality $dS \geq \partial Q/\Theta$.

But how to adapt this for a continuum with an inhomogeneous temperature field?

Sometimes a process within an infinitesimal volume is thought of as reversible, then the equality like

$$\rho\Theta\dot{s} = -\boldsymbol{\nabla}\cdot\boldsymbol{q} + \rho b \tag{2.1}$$

is proposed (s is entropy per mass unit and $\dot{s}$ is the time derivative of it, that is the rate of entropy change).

However, there's always heat dissipation — an irreversible process, and therefore (2.1) looks disputable.

The most appropriate expression of the second law of thermodynamics for a material continuum seems to be the Clausius–Duhem inequality

$$\left(\int\limits_{\mathcal{V}} \rho s\, d\mathcal{V}\right)^{\bullet} \geq -\oint\limits_{O(\partial\mathcal{V})} \frac{\boldsymbol{q}}{\Theta}\cdot\boldsymbol{n}\, dO + \int\limits_{\mathcal{V}} \frac{\rho b}{\Theta}\, d\mathcal{V}. \tag{2.2}$$

This inequality as the imbalance of entropy defines the rate of entropy production.

$$-\oint\limits_{O(\partial\mathcal{V})} \boldsymbol{n}\cdot\boldsymbol{q}\Theta^{-1} dO = -\int\limits_{\mathcal{V}} \boldsymbol{\nabla}\cdot\left(\boldsymbol{q}\Theta^{-1}\right) d\mathcal{V}$$

$$-\boldsymbol{\nabla}\cdot\left(\boldsymbol{q}\Theta^{-1}\right) = -\left(\boldsymbol{\nabla}\cdot\boldsymbol{q}\right)\Theta^{-1} - \boldsymbol{q}\cdot\left(\boldsymbol{\nabla}\tfrac{1}{\Theta}\right)$$

* Because work and heat depend on the path of the process (they are path functions), they can't be the full (exact) differentials, contrasting with the idea of the exact differential, expressed via the gradient of another function and therefore path independent.

$$-\boldsymbol{\nabla}\frac{1}{\Theta} = \frac{1}{\Theta^2}\boldsymbol{\nabla}\Theta$$

$$-\boldsymbol{\nabla}\cdot\left(\boldsymbol{q}\Theta^{-1}\right) = -\left(\boldsymbol{\nabla}\cdot\boldsymbol{q}\right)\Theta^{-1} + \left(\boldsymbol{q}\cdot\boldsymbol{\nabla}\Theta\right)\Theta^{-2}$$

$$\rho\dot{s} \geq \left(-\boldsymbol{\nabla}\cdot\boldsymbol{q} + \rho b\right)\Theta^{-1} + \left(\boldsymbol{q}\cdot\boldsymbol{\nabla}\Theta\right)\Theta^{-2} \tag{2.3}$$

The Clausius–Duhem inequality is also called the dissipation inequality. For a real matter, the dissipation is always greater than zero, it can never be negative and can't be zero whenever irreversible processes are present.

...

Helmholtz free energy per mass unit

$$a \equiv e - \Theta s, \tag{2.4}$$

$$\dot{a} = \dot{e} - \Theta\dot{s} - \dot{\Theta}s$$

§ 3. Constitutive equations

К балансу импульса, балансу момента импульса и законам термодинамики нужно добавить определяющие уравнения, выражающие свойства среды. Эти уравнения

...

Термоупругим называется материал, в котором свободная энергия a и энтропия s — функции деформации $\boldsymbol{C}$ и температуры Θ

$$a = a(\boldsymbol{C}, \Theta)$$
$$\dot{a} = \frac{\partial a}{\partial \boldsymbol{C}} \cdot\cdot\, \dot{\boldsymbol{C}} + \frac{\partial a}{\partial \Theta}\dot{\Theta}$$

...

§ 4. Heat equation

In mathematical physics, a parabolic differential equation like

$$k\triangle\Theta + B = c\dot{\Theta} \tag{4.1}$$

is declared as the "heat equation". Here k is thermal conductivity, $B = \rho b$ is the rate of heat transfer by radiation per volume unit, c is the thermal capacity per volume unit. The boundary conditions most often are : the external temperature $\Theta_1^{(e)}$ on O_1 part of the surface

$$\Theta\big|_{O_1} = \Theta_1^{(e)} \quad \text{and} \quad k\,\partial_n\Theta\big|_{O_2} = q^{(e)}$$

— the heat flux $q^{(e)}$ from the outside of O_2 part of the surface.

Sometimes, the external flux $q^{(e)}$ is thought to be proportional with some coefficient $\hbar$ to the difference between the ambient temperature $\Theta^{(e)}$ and the body temperature Θ

$$k\,\partial_n\Theta + \hbar\left(\Theta - \Theta^{(e)}\right) = 0.$$

If the heat transfer coefficient $\hbar$ is infinitely large, it turns into the first condition $\Theta = \Theta^{(e)}$, and when $\hbar \to 0$ — into the condition $\partial_n\Theta = 0$ of thermal insulation.

But how is equation (4.1) related to the fundamental principles of balance? Since no special "thermal energy" exists, but there is the internal energy, and it changes according to the first law of thermodynamics (1.7).

....

$$e = a + \Theta s \;\Rightarrow\; \dot{e} = \dot{a} + \dot{\Theta}s + \Theta\dot{s}$$

$$\rho\dot{e} = \rho\left(\dot{a} + \dot{\Theta}s + \Theta\dot{s}\right) = \rho\Big(\overbrace{\frac{\partial a}{\partial \boldsymbol{C}} \bullet\bullet \dot{\boldsymbol{C}} + \underbrace{\frac{\partial a}{\partial \Theta}\dot{\Theta}}_{=0 \;\Leftarrow\; s = -\frac{\partial a}{\partial \Theta}} + \dot{\Theta}s}^{\dot{a}(\boldsymbol{C},\Theta)} + \Theta\dot{s}\Big)$$

...

§ 5. Linear thermoelasticity

Квадратичная аппроксимация свободной энергии наиболее естественна в линейной теории

...

Полагая поле температуры известным

...

§ 7. Thermal stress

Это напряжение сто́ит рассмотреть детально, хотя оно и определяется очевидным образом полями смещений и температуры. For an equilibrium свободного тела без внешних нагрузок

...

§ 8. Variational formulations

Когда температура постоянна, уравнения термоупругости выглядят так же, как в механике.

...

Для переноса вариационного метода на термоупругость достаточно заменить in the (Lagrange's) principle of minimum potential energy $\Pi(\boldsymbol{C})$ на Helmholtz free energy $A(\boldsymbol{C}, \Theta)$, а в принципе Reissner'a–Hellinger'a заменить $Ц(\boldsymbol{\tau})$ на Gibbs free enthalpy (Gibbs function) $G(\boldsymbol{\tau}, \Theta)$.

The *Gibbs free energy* (the *Gibbs energy* or the *Gibbs function* or the *free enthalpy**) is a thermodynamic potential which measures the maximum of the reversible work produced with a constant temperature and pressure.

...

...

Более сложные вариационные постановки для нестационарных задач можно найти, например, в книге [90].

* To distinguish it from the Helmholtz free energy.

Шириной и глубиной описания термоупругости выделяются книги W. Nowacki [37, 38], книга E. Melan'a и H. Parkus'a [32] и монография H. Parkus'a [43]. C. Truesdell [55] внёс большой вклад в создание и распространение новых взглядов на термодинамику сплошной среды. Чёткое изложение основных законов есть у C. Teodosiu [51]. Методы расчёта температурных полей представлены у Н. М. Беляева и А. А. Рядно [90].

chapter 7

MAGNETOELASTICITY

Much in the modern world is built upon the theory of electromagnetism. This theory was created in the XIX$^{\text{th}}$ century. Its creators — Gian Domenico Romagnosi, Hans Christian Ørsted, André-Marie Ampère, Michael Faraday, James Clerk Maxwell, Oliver Heaviside, Heinrich Hertz, Hendrik Lorentz and others — relied on the experiments with electric circuits and didn't imagine the existænce of electromagnetic waves. Nevertheless, the entities describing electricity and magnetism at each point were introduced as vectors, along with the differential equations featuring these vectors. This happened due to *the æther*, because the creators of the theory were convinced of its existænce and thus utilized the concept of it.

When electric currents flow in a body (a medium), the magnetic field produces a load, a body deforms, and this deformation alters the magnetic field itself. If the field is highly sensitive to deformations, then a joint problem of elasticity and magnetism emerges.

§ 1. Electromagnetic field

Here is the summary of the theory of electromagnetism.

The theory describes the couple of the closely intertwined together vector fields, the electric one $\mathbb{E}(\boldsymbol{r}, t)$ and the magnetic one $\mathbb{B}(\boldsymbol{r}, t)$. What are the vector $\mathbb{E}$ and the pseudovector $\mathbb{B}$ can be figured out from the expression for the electromagnetic force, or the Lorentz force. This force $\mathbf{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t, q)$ acts on a point-like charge — a vanishingly small (infinitesimal in size) particle that contains the electric charge q and moves with the velocity $\dot{\boldsymbol{r}}$

$$\mathbf{F} = q\big(\mathbb{E} + \dot{\boldsymbol{r}} \times \mathbb{B}\big). \tag{1.1}$$

In essence, the part of the electromagnetic force arising from an interaction with a moving charge — the magnetic force $q\dot{\boldsymbol{r}} \times \mathbb{B}$ —

reveals the magnetic field $\mathbb{B}$, while the other part — the electric force $q\mathbb{E}$ — reveals the electric field $\mathbb{E}$.

The acute question *"in which exactly frame of reference is velocity $\dot{\boldsymbol{r}}$ of a charged particle measured?"* leads to the special theory of relativity*. However, here I will follow the classical concept about the existænce of the absolute space and time as the most frequently chosen frame of reference.

Continual model

May the reader guess by the book's title, the model of what kind awaits him hereafter? Yep, ignoring the discreteness of charge, that any electric charge may be only an integer multiplier of a single lone eléctron's charge, there is the model of the continuous distribution of charge within a volume, when the finite volume $\mathcal{V}$ contains the electric charge**

$$q = \int_{\mathcal{V}} \varrho \, d\mathcal{V}, \quad dq = \varrho \, d\mathcal{V} \tag{1.2}$$

(the charge density $\varrho(\boldsymbol{r}, t)$ is the electric charge per volume unit).

On a continuum with charges and currents acts the "ponderomotive" force $\mathbf{f}$ — the electromagnetic Lorentz force per volume unit

$$\mathbf{f} = \varrho \left(\mathbb{E} + \dot{\boldsymbol{r}} \times \mathbb{B} \right) = \varrho \mathbb{E} + \boldsymbol{j} \times \mathbb{B} \tag{1.3}$$

— the differential (the local, the microscopic, the continual) version of (1.1). Here $\boldsymbol{j} \equiv \varrho \dot{\boldsymbol{r}}$ is the volume(tric) density of the electric current, in other words "the flux of electric charge".

A vacuum is a medium without a matter, a "free space". Within a vacuum there's no ϱ and $\boldsymbol{j}$, it is a region without charges, $\varrho = 0$, and without currents, $\boldsymbol{j} = \mathbf{0}$, thus there's no ponderomotive force, $\mathbf{f} = \mathbf{0}$.

...

* **Albert Einstein**. Zur Elektrodynamik bewegter Körper. // Annalen der Physik, IV. Folge, Band 17, 1905. Seiten 891–921.

** Doesn't it remind of something? Even (1.1, § 3.1)?

Electromagnetic phenomena are usually described by the Maxwell's equations. Differential versions of these equations are

$$\boldsymbol{\nabla}\cdot\mathbb{E}=\frac{\varrho}{\varepsilon_0} \qquad \text{Gauss's theorem for electricity} \qquad (1.4^{\alpha})$$

$$\boldsymbol{\nabla}\times\mathbb{E}=-\dot{\mathbb{B}} \qquad \text{Maxwell–Faraday equation (Faraday's law of induction)} \qquad (1.4^{\beta})$$

$$\boldsymbol{\nabla}\cdot\mathbb{B}=0 \qquad \text{Gauss's theorem for magnetism} \qquad (1.4^{\gamma})$$

$$c^2\,\boldsymbol{\nabla}\times\mathbb{B}=\frac{\boldsymbol{j}}{\varepsilon_0}+\dot{\mathbb{E}} \qquad \text{Ampère's circuital law with Maxwell's term } \dot{\mathbb{E}} \text{ for the balance of electric charge} \qquad (1.4^{\delta})$$

....

speed of light in vacuum $c = 299\,792\,458\ {}^{\mathrm{m}}/_{\mathrm{s}}$

"electric constant", vacuum permittivity ε_0

$\varepsilon_0 \approx 8.8541878\cdot 10^{-12}\ \mathrm{F}\cdot\mathrm{m}^{-1}$ (farads per metre)

"magnetic constant", vacuum permeability $\mu_0 = \dfrac{1}{\varepsilon_0 c^2}$

...

With μ_0, the equation (1.4^{δ}) is sometimes written as

$$\boldsymbol{\nabla}\times\mathbb{B}=\mu_0\boldsymbol{j}+\frac{1}{c^2}\dot{\mathbb{E}} \quad \text{or} \quad \boldsymbol{\nabla}\times\mathbb{B}=\mu_0\boldsymbol{j}+\mu_0\varepsilon_0\dot{\mathbb{E}}.$$

...

The balance of charge

The balance of charge — the continuity equation for electric charges — mathematically follows from the Maxwell's equations

$$\left.\begin{aligned} \boldsymbol{\nabla}\cdot(1.4^{\delta}) \;&\Rightarrow\; c^2\,\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\mathbb{B}\big)=\frac{\boldsymbol{\nabla}\cdot\boldsymbol{j}}{\varepsilon_0}+\boldsymbol{\nabla}\cdot\dot{\mathbb{E}} \\ (1.4^{\alpha})^{\bullet} \;&\Rightarrow\; \boldsymbol{\nabla}\cdot\dot{\mathbb{E}}=\frac{\dot{\varrho}}{\varepsilon_0} \\ &\quad\; \boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\boldsymbol{a}\big)=0\ \ \forall\boldsymbol{a},\ \ \boldsymbol{j}\equiv\varrho\dot{\boldsymbol{r}} \end{aligned}\right\}\ \Rightarrow$$

$$\Rightarrow\ \boldsymbol{\nabla}\cdot\big(\varrho\dot{\boldsymbol{r}}\big)+\dot{\varrho}=0. \qquad (1.5)$$

Upon a continuum, the electromagnetic field acts with the ponderomotive force (1.3). But there's also another expression of interaction, the bivalent "Maxwell" stress tensor

$$^2\boldsymbol{M} \equiv \varepsilon_0\Big(\mathbb{E}\mathbb{E} + c^2\,\mathbb{B}\mathbb{B} - \tfrac{1}{2}\big(\mathbb{E}\cdot\mathbb{E} + c^2\,\mathbb{B}\cdot\mathbb{B}\big)\boldsymbol{E}\Big). \tag{1.6}$$

It derives from (1.3) and Maxwell's equations

$$\begin{aligned}(1.4^{\alpha}) \;&\Rightarrow\; \varrho = \varepsilon_0\boldsymbol{\nabla}\cdot\mathbb{E}\\ (1.4^{\delta}) \;&\Rightarrow\; \boldsymbol{j} = \varepsilon_0 c^2\,\boldsymbol{\nabla}\times\mathbb{B} - \varepsilon_0\dot{\mathbb{E}}\end{aligned}$$

$$\begin{aligned}(1.3) \;\Rightarrow\; \boldsymbol{f} &= \varepsilon_0\boldsymbol{\nabla}\cdot\mathbb{E}\mathbb{E} + \big(\varepsilon_0 c^2\,\boldsymbol{\nabla}\times\mathbb{B} - \varepsilon_0\dot{\mathbb{E}}\big)\times\mathbb{B} =\\ &= \varepsilon_0\boldsymbol{\nabla}\cdot\mathbb{E}\mathbb{E} + \varepsilon_0 c^2\big(\boldsymbol{\nabla}\times\mathbb{B}\big)\times\mathbb{B} - \varepsilon_0\dot{\mathbb{E}}\times\mathbb{B}\end{aligned}$$

$$\left.\begin{aligned}\big(\mathbb{E}\times\mathbb{B}\big)^{\bullet} &= \dot{\mathbb{E}}\times\mathbb{B} + \mathbb{E}\times\dot{\mathbb{B}}\\ (1.4^{\beta}) \;&\Rightarrow\; \dot{\mathbb{B}} = -\boldsymbol{\nabla}\times\mathbb{E}\end{aligned}\right\} \Rightarrow\; \dot{\mathbb{E}}\times\mathbb{B} = \big(\mathbb{E}\times\mathbb{B}\big)^{\bullet} + \mathbb{E}\times\big(\boldsymbol{\nabla}\times\mathbb{E}\big)$$

Then

$$\boldsymbol{f} = \varepsilon_0\boldsymbol{\nabla}\cdot\mathbb{E}\mathbb{E} - \varepsilon_0 c^2\,\mathbb{B}\times\big(\boldsymbol{\nabla}\times\mathbb{B}\big) - \varepsilon_0\mathbb{E}\times\big(\boldsymbol{\nabla}\times\mathbb{E}\big) - \varepsilon_0\big(\mathbb{E}\times\mathbb{B}\big)^{\bullet}$$

For the symmetry with $\boldsymbol{\nabla}\cdot\mathbb{E}\mathbb{E}$, the null vector

$$(1.4^{\gamma}) \;\Rightarrow\; \boldsymbol{\nabla}\cdot\mathbb{B}\mathbb{B} = \boldsymbol{0},\;\; c^2\,\boldsymbol{\nabla}\cdot\mathbb{B}\mathbb{B} = \boldsymbol{0}$$

is added to $\boldsymbol{f}$.

...

$$\begin{aligned}(17.11, \S\,1.17) \;&\Rightarrow\; \boldsymbol{\nabla}\cdot\big(\boldsymbol{a}\boldsymbol{a}\big) = \big(\boldsymbol{\nabla}\cdot\boldsymbol{a}\big)\boldsymbol{a} + \boldsymbol{a}\cdot\boldsymbol{\nabla}\boldsymbol{a}\\ (17.13, \S\,1.17) \;&\Rightarrow\; \boldsymbol{\nabla}\big(\boldsymbol{a}\cdot\boldsymbol{a}\big) = 2\,\boldsymbol{\nabla}\boldsymbol{a}\cdot\boldsymbol{a}\end{aligned}$$

$$\boldsymbol{\nabla}\cdot\big(\boldsymbol{a}\cdot\boldsymbol{a}\boldsymbol{E}\big) = \boldsymbol{\nabla}\cdot\boldsymbol{E}\big(\boldsymbol{a}\cdot\boldsymbol{a}\big) = \boldsymbol{\nabla}\big(\boldsymbol{a}\cdot\boldsymbol{a}\big) = 2\,\boldsymbol{\nabla}\boldsymbol{a}\cdot\boldsymbol{a}$$

$$\boldsymbol{\nabla}\cdot{}^2\boldsymbol{M} = \varepsilon_0\Big(\boldsymbol{\nabla}\cdot\big(\mathbb{E}\mathbb{E}\big) + c^2\,\boldsymbol{\nabla}\cdot\big(\mathbb{B}\mathbb{B}\big) - \tfrac{1}{2}\,\boldsymbol{\nabla}\cdot\big(\mathbb{E}\cdot\mathbb{E}\boldsymbol{E} + c^2\,\mathbb{B}\cdot\mathbb{B}\boldsymbol{E}\big)\Big) =$$
$$= \varepsilon_0\Big(\boldsymbol{\nabla}\cdot\mathbb{E}\mathbb{E} + \mathbb{E}\cdot\boldsymbol{\nabla}\mathbb{E} + c^2\,\boldsymbol{\nabla}\cdot\mathbb{B}\mathbb{B} + c^2\,\mathbb{B}\cdot\boldsymbol{\nabla}\mathbb{B} -$$
$$- \boldsymbol{\nabla}\mathbb{E}\cdot\mathbb{E} - c^2\,\boldsymbol{\nabla}\mathbb{B}\cdot\mathbb{B}\Big)$$

...

§ 2. Electromagnetic waves

To derive wave equations

$$\boldsymbol{\nabla}\times(1.4^{\beta}) \;\Rightarrow\; \boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbb{E}\big) = -\boldsymbol{\nabla}\times\dot{\mathbb{B}}$$
$$\boldsymbol{\nabla}\times(1.4^{\delta}) \;\Rightarrow\; \boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbb{B}\big) = \boldsymbol{\nabla}\times\left(\frac{\boldsymbol{j}}{\varepsilon_0 c^2} + \frac{\dot{\mathbb{E}}}{c^2}\right)$$

$$\boldsymbol{\nabla}\times\dot{\mathbb{E}} = \big(\boldsymbol{\nabla}\times\mathbb{E}\big)^{\bullet},\ (1.4^{\beta}) \;\Rightarrow\; \boldsymbol{\nabla}\times\dot{\mathbb{E}} = -\ddot{\mathbb{B}}$$
$$\boldsymbol{\nabla}\times\dot{\mathbb{B}} = \big(\boldsymbol{\nabla}\times\mathbb{B}\big)^{\bullet},\ (1.4^{\delta}) \;\Rightarrow\; \boldsymbol{\nabla}\times\dot{\mathbb{B}} = \frac{\boldsymbol{j}^{\bullet}}{\varepsilon_0 c^2} + \frac{\ddot{\mathbb{E}}}{c^2}$$

$$\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbb{E}\big) = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbb{E} - \triangle\mathbb{E}$$
$$\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbb{B}\big) = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbb{B} - \triangle\mathbb{B}$$

$$\triangle\mathbb{E} - \boldsymbol{\nabla}\underbrace{\boldsymbol{\nabla}\cdot\mathbb{E}}_{\varrho/\varepsilon_0\,(1.4^{\alpha})} = \frac{\ddot{\mathbb{E}}}{c^2} + \frac{\boldsymbol{j}^{\bullet}}{\varepsilon_0 c^2}$$
$$\triangle\mathbb{B} - \boldsymbol{\nabla}\underbrace{\boldsymbol{\nabla}\cdot\mathbb{B}}_{0\,(1.4^{\gamma})} = \frac{\ddot{\mathbb{B}}}{c^2} - \frac{\boldsymbol{\nabla}\times\boldsymbol{j}}{\varepsilon_0 c^2}$$

...
$\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\times\boldsymbol{a}\big) = 0\ \ \forall\boldsymbol{a}$
$\boldsymbol{\nabla}\times\boldsymbol{\nabla} = \boldsymbol{0},\ \ \boldsymbol{\nabla}\times\boldsymbol{\nabla}\alpha = \boldsymbol{0}\ \ \forall\alpha$
vector potential $\mathbb{A}$

$$\boldsymbol{\nabla}\cdot\mathbb{B} = 0\ \ (1.4^{\gamma}) \;\Leftrightarrow\; \mathbb{B} = \boldsymbol{\nabla}\times\mathbb{A}$$

potential $\mathbb{A}$ is not unique and has gauge freedom $\mathbb{A} + \boldsymbol{\nabla} a$

$$\mathbb{B} = \boldsymbol{\nabla} \times \big(\mathbb{A} + \boldsymbol{\nabla} a\big) \;\Leftrightarrow\; \boldsymbol{\nabla} \cdot \mathbb{B} = 0 \;\; (1.4^{\gamma})$$

scalar potential ϕ

$$\begin{gathered}\boldsymbol{\nabla} \times \mathbb{E} = -\dot{\mathbb{B}} \;\; (1.4^{\beta}) \;\Rightarrow\; \boldsymbol{\nabla} \times \mathbb{E} = -\boldsymbol{\nabla} \times \big(\dot{\mathbb{A}} + \boldsymbol{\nabla} \dot{a}\big) \;\Rightarrow \\ \Rightarrow\; \boldsymbol{\nabla} \times \big(\mathbb{E} + \dot{\mathbb{A}} + \boldsymbol{\nabla} \dot{a}\big) = \mathbf{0} \;\Rightarrow\; -\boldsymbol{\nabla}\phi = \mathbb{E} + \dot{\mathbb{A}} + \boldsymbol{\nabla} \dot{a} \;\Rightarrow \\ \Rightarrow\; \mathbb{E} = -\boldsymbol{\nabla}\big(\phi + \dot{a}\big) - \dot{\mathbb{A}}. \quad (2.1)\end{gathered}$$

And

$$\boldsymbol{\nabla} \cdot \mathbb{E} = \frac{\varrho}{\varepsilon_0} \;\; (1.4^{\alpha}) \;\Rightarrow\; \begin{aligned} -\triangle\big(\phi + \dot{a}\big) - \boldsymbol{\nabla} \cdot \dot{\mathbb{A}} &= \frac{\varrho}{\varepsilon_0} \\ -\triangle\phi - \boldsymbol{\nabla} \cdot \big(\dot{\mathbb{A}} + \boldsymbol{\nabla} \dot{a}\big) &= \frac{\varrho}{\varepsilon_0} \end{aligned} \qquad (2.2)$$

$\boldsymbol{\nabla} \times \big(\boldsymbol{\nabla} \times \mathbb{A}\big) = \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbb{A} - \triangle \mathbb{A}$

$\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} a - \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\boldsymbol{\nabla} a = \mathbf{0}$ (partial derivatives of a smooth function commute)

$$\begin{gathered}c^2 \boldsymbol{\nabla} \times \mathbb{B} = \frac{\boldsymbol{j}}{\varepsilon_0} + \dot{\mathbb{E}} \;\; (1.4^{\delta}) \;\Rightarrow \\ \Rightarrow\; c^2 \boldsymbol{\nabla} \times \big(\boldsymbol{\nabla} \times \mathbb{A}\big) = \frac{\boldsymbol{j}}{\varepsilon_0} - \boldsymbol{\nabla}\big(\dot{\phi} + \ddot{a}\big) - \ddot{\mathbb{A}} \;\Rightarrow \\ \Rightarrow\; c^2 \big(\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbb{A} - \triangle \mathbb{A}\big) = \frac{\boldsymbol{j}}{\varepsilon_0} - \boldsymbol{\nabla}\dot{\phi} - \boldsymbol{\nabla}\ddot{a} - \ddot{\mathbb{A}}. \quad (2.3)\end{gathered}$$

With a gauge freedom it's possible to simplify the wave equations (2.3) and (2.2), assuming that

$$\left.\begin{aligned} -\boldsymbol{\nabla}\ddot{a} = \boldsymbol{\nabla}\dot{\phi} + c^2 \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \mathbb{A} \;&\Rightarrow\; \ddot{a} = -\dot{\phi} - c^2 \boldsymbol{\nabla} \cdot \mathbb{A} \\ -\boldsymbol{\nabla} \cdot \big(\dot{\mathbb{A}} + \boldsymbol{\nabla} \dot{a}\big) = \tfrac{1}{c^2} \ddot{\phi} \;&\Rightarrow\; \dot{\phi} = -c^2 \boldsymbol{\nabla} \cdot \big(\mathbb{A} + \boldsymbol{\nabla} a\big) \end{aligned}\right\} \Rightarrow$$
$$\Rightarrow\; c^2 \boldsymbol{\nabla} \cdot \mathbb{A} + c^2 \triangle a - c^2 \boldsymbol{\nabla} \cdot \mathbb{A} = \ddot{a},$$

finally presenting as the homogeneous wave equation for a

$$\ddot{a} = c^2 \triangle a. \qquad (2.4)$$

The more popular condition is even more stiff

$$\triangle a = 0 \;\Rightarrow\; \ddot{a} = 0,\;\; \dot{\phi} + c^2\,\boldsymbol{\nabla}\bullet\,\mathbb{A} = 0.$$

This Lorenz gauge condition gives the same effect, being just the particular — the harmonic — case of (2.4).

Following from (2.2) and (2.3) with condition (2.4), equations of electromagnetic waves in the potential formulation are

$$\begin{aligned} -\triangle\phi + \tfrac{1}{c^2}\ddot{\phi} &= \frac{\varrho}{\varepsilon_0}, \\ -c^2\triangle\mathbb{A} &= \frac{\dot{\boldsymbol{j}}}{\varepsilon_0} - \ddot{\mathbb{A}}. \end{aligned} \tag{2.5}$$

...

§ 3. Electrostatics

Рассмотрение этого вопроса полезно и для последующего описа́ния магнетизма. В статике

$$\dot{\boldsymbol{r}} = \mathbf{0} \;\Rightarrow\; \mathbb{B} = \mathbf{0}$$

...

The volume “ponderomotive” force, с которой электростатическое поле действует на среду ...

...

Maxwell stress tensor (1.6) in electrostatics is

$${}^2\boldsymbol{M} = \varepsilon_0\left(\mathbb{E}\mathbb{E} - \tfrac{1}{2}\,\mathbb{E}\bullet\mathbb{E}\boldsymbol{E}\right)$$

...

§ 4. Dielectrics

Начнём с рассмотрения электростатического поля

...

В диэлектриках нет свободных зарядов: charge density $\varrho = 0$. Здесь вводится плотность дипольного момента

...

§ 5. Magnetostatics

Если поле (а с ним ...)

...

§ 6. Magnetics, or magnetic materials

Having the relations of magnetostatics for the overall case, here I'll go to a material matter — and there's already some previous experience from electrostatics of dielectrics.

Начнём с рассмотрения

...

...

О том, насколько поведение реальных материалов соответствует представленным здесь формальным построениям — сей вопрос is out of scope этой книги.

§ 7. Magnetic rigidity

В электротехнике распространены обмотки всевозможной формы, в которых провод намотан так, что образуется некое массивное тело. Такие обмотки есть в статоре генератора автомобиля (да и в роторе), в больших промышленных электромагнитах и в магнитных системах установок "токама́к" (**то**роидальная **ка**мера с **ма**гнитными **к**атушками) для управляемого термоядерного синтеза — примеров много. Сочетание токопровода и изоляции образует периодический композит, и одной из главных нагрузок для него является пондеромоторная магнитная сила. Рассчитывая деформации и механические напряжения в обмотке, начинают с определения магнитных сил. Поскольку распределение токов

задано известной геометрией проводов, достаточно интегрирования по формуле Био-Савара (**??**). Термин “магнитоупругость” при этом неуместен, так как задачи магнитостатики и упругости решаются раздельно.

Однако при деформации обмотки меняются и поле $\boldsymbol{j}$, и вызываемое им поле $\mathbb{B}$. Объёмная сила становится равной

$$\boldsymbol{f} = (\boldsymbol{j} \times \mathbb{B})_0 + \dots \tag{7.1}$$

Подчёркнутое слагаемое соответствует недеформированному состоянию. Обусловленное деформацией изменение объёмной силы линейно связано с малым смещением $\boldsymbol{u}$, поэтому матричное (после дискретизации) уравнение в смещениях можно представить в виде

$$(C + C_m)\, u = \mathring{F}. \tag{7.2}$$

К “обычному” оператору линейной упругости C добавилась магнитная жёсткость C_m, $\mathring{F}$ — силы в недоформированном состоянии.

Добавка C_m пропорциональна квадрату тока и может стать весьма существенной в магнитных системах с мощным полем. Учёт её нужен и в случае небольшого значения C.

For plain conditions, the structure will sustain the load, but further loading is risky and may be unbearable.

Magnetic rigidity plays the big role in stability problems. Matrix C_m is symmetric since magnetic forces are potential. The Euler's static approach (§ 13.1) gives the “critical” parameter values.

Как иллюстрацию рассмотрим простую задачу о балке в продольном магнитном поле. Балка располагается на декартовой оси z, концы $z{=}0$ и $z{=}l$ закреплены, магнитная индукция $\mathbb{B} = B\boldsymbol{k} = \textbf{constant}$, по балке течёт постоянный (по величине) ток I. В классической модели балки при равных жёсткостях на изгиб для прогиба $\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2$ легко получить следующую постановку:

...

Вводя компле́ксную комбинацию $u \equiv u_1 + \mathrm{i}u_2$, будем иметь

...

с общим решением

...

Подстановка в граничные условия приводит к однородной системе для постоянных A_k. Приравняв нулю определитель, придём к характеристическому уравнению

...

Наименьший положительный корень $x = 3.666$, так что критическая комбинация параметров такова:

$$\left(IBl^3/a\right)_* = 394.2.$$

Поле $\mathbb{B}$ в этом решении считалось внешним и не варьировалось. Но если собственное поле тока в стержне сравнимо с $\mathbb{B}$, то решение изме́нится and усложни́тся.

Bibliography

Основы электродинамики хорошо изложены во многих книгах [91, 86], но для приложений в механике выделяется книга И. Е. Тамма [97]. Растёт список литературы по связанным задачам электромагнетизма и упругости [44, 45]. Как введение в эту область может быть полезна монография В. Новацкого [39].

chapter 8

PERTURBATION METHODS (ASYMPTOTIC METHODS)

Of approximate approaches to the analysis of nonlinear systems, perturbation methods are applied the most often.

§1. Asymptotic decomposition

Until now, the arguments of a function were only the coordinates and time. Dimensions of a body, elastic moduli, ranges of actions — all these parameters were thought to be known. All asymptotic methods are based on the study of how the solution depends on the parameters.

A decomposition of perturbation usually consists of the first two terms. When a problem with unknown u contains parameter a then, assuming

$$a = a_0 + \chi a_1,$$

the solution is sought as a series

$$u = u_0 + \chi u_1 + \chi^2 u_2 + \dots \tag{1.1}$$

Additional argument χ is called a formal small parameter.

Although a decomposition of perturbation may diverge, it may nevertheless describe the solution more useful than representations with the uniform and absolute convergence.

Decomposition (1.1) looks like a common power series. However, the approach of the perturbation methods is different. Here, series are considered as asymptotic with convergence $\chi \to 0$ and not $n \to \infty$ (where n defines how many terms to hold)

$$\phi = \sum_{k=0}^{\infty} \phi_k(\chi), \quad \lim_{\chi \to 0} \frac{\phi - \sum\limits_{k=0}^{n} \phi_k}{\phi_n} = 0. \tag{1.2}$$

In other words, the remainder of series is smaller — of a higher degree of infinitesimality — than the last withheld term. Obviously, a power series (1.1) also converges as an asymptotic series.

But the decomposable unknowns usually also depend on the "main" arguments — the coordinates and time. The convergence for $\chi \to 0$ must be uniform across these "main" arguments — this is a requirement for the effective use of asymptotic decompositions. For example

$$\sin(1+\chi)t = \sin t + \chi t \cos \chi t - \tfrac{1}{2}\chi^2 t^2 \sin \chi t + \dots$$

doesn't satisfy the requirement of uniformity, because the subsequent terms prevail over the previous ones when $t \to \infty$.

Why asymptotic methods are so attractive? As example take a solution of the equation

$$f(u, \chi) = 0.$$

Substituting decomposition (1.1) into (1.2) and equating coefficients before the same degrees of χ, we get

$$\begin{gathered} f(u_0, 0) = 0, \\ \big(\partial_u f\big)_0 + u_1 \big(\partial_\chi f\big)_0 = 0, \\ \big(\partial_u f\big)_0 u_2 + \tfrac{1}{2}\big(\partial_u^2 f\big)_0 u_1^2 + \tfrac{1}{2}\big(\partial_\chi^2 f\big)_0 + \big(\partial_u \partial_\chi f\big)_0 u_1 = 0, \\ \dots \end{gathered}$$

If the problem for u_0 is uniquely solvable at the first step, then subsequent steps will give corrections $u_1, u_2, \dots$ Small corrections are barely important and u_0 is enough, but then the asymptotic analysis disappears, since formally small terms in (1.2) are just discarded. However, it happens that these corrections contain in self some important information, absent in u_0 — then they play the main role. It's worth mentioning that all corrections follow from the linear problem with the same operator $\big(\partial_u f\big)_0$.

But all non-trivial and effective solutions are obtained in other ways, non-uniqueness of the solution at the first step characterizes them. About this — in the next sections.

And yes, asymptotic methods change the initially complex formulation of the problem (1.2) to a simpler one. Essential is that this does not happen by "simple" discarding of terms, but quite correctly — equalities remain equalities. However, the convergence is not proven, so there is no complete mathematical accuracy.

In previous chapters, asymptotic problems have already arisen. The linear theory follows from the nonlinear theory thru an asymptotic decomposition by the value of load (§ 4.1). The momentless theory derives (accurate to edge effects) from the moment (micropolar) theory (chapter 5) when the "moment" stiffnesses are approaching infinity. In the thermoelasticity (chapter 6), the use of the heat equation (4.1, § 6.4) instead of the whole entire balance of energy needs to be proved by asymptotic methods.

To a certain degree the introduction of a small parameter $\chi \to 0$ is the weakness of all asymptotic approaches. A protestation like "infinitesimal parameters do not exist, all quantities are finite" is barely constructive here. The more relevant question is: what is a small parameter? Usually a problem is reformulated in "dimensionless" quantities, then that "dimensionless" parameter is taken as a small parameter χ, which turns out to be small. But another way is possible too : if it's known that some parameter ω influences the solution only a little, then, redesignating it as $\chi\omega$, do asymptotic analysis for $\chi \to 0$.

Surely, these are not the "laws" of asymptotics, but only considerations. There's no common theory of asymptotic methods, and their application is to some extent an art.

The deeper description of asymptotic methods is written in the books by Ali Hasan Nayfeh [108, 109].

§ 2. Splitting in a linear algebraic system

This simple case well illustrates asymptotic methods.

A linear system is considered

$$C_{ij}\, u_j = f_i, \;\; C_{ij} = C_{ij}^{(0)} + \chi C_{ij}^{(1)} \tag{2.1}$$

with the matrix C_{ij} and the sets of unknowns u_j and loads f_i. The process of constructing an asymptotic solution is determined by whether the matrix $C_{ij}^{(0)}$ is singular (invertible) or not. Three cases are possible.

1° $\det C_{ij}^{(0)} \neq 0$. Однородная задача

$$C_{ij}^{(0)} u_j = 0 \tag{2.2}$$

имеет лишь тривиальное (нулевое) решение. Матрица $C_{ij}^{(0)}$ обратима, неоднородная задача всегда однозначно реши́ма. Решение строится как

$$u_j = \dots \tag{2.3}$$

...

2° $\det C_{ij}^{(0)} = 0$

...

3° $\det C_{ij}^{(0)} = 0$

...

§ 3. Poincaré method

This method, associated with the name Jules Henri Poincaré, is widely known in the theory of nonlinear oscillations. It is intended, in particular, to determine the periodic solutions of equation

$$\ddot{u} + u = \chi f(u, \dot{u}) \tag{3.1}$$

....

§ 4. Van der Pol averaging method

Here again figures equation (3.1), but the solutions being found are not only periodic anymore. Introducing the phase plane

...

Процедура осреднения применяется во многих темах, таких как тонкие тела и композиты. Осреднение вне асимптотических

методов ведёт обычно к не за́мкнутому набору уравнений. Для замыкания системы приходится добавлять некие гипотезы, убавляющие убедительность теории. Иная ситуация в асимптотике: условия разрешимости для поправочных членов с необходимостью приводят к соответствующим интегральным соотношениям.

§ 5. Coalescence of asymptotic decompositions

Основоположник метода сращивания внешних и внутренних асимптотических разложений — Ludwig Prandtl. Рассматривая течение вязкой жидкости, он заметил, что влияние малой вязкости локализовано у края — в тонком слое на краю. Вдали́ от края жидкость ведёт себя как идеальная. Одни и те же уравнения Navier–Stokes по-разному упрощаются вдали от края и около него [...]

Метод сращивания состоит из трёх процедур: построения внешнего разложения, построения внутренних разложений и сращивания внешнего разложения с внутренними. Метод предназначен для дифференциальных уравнений с малым параметром при старших производных. Вдали от края решение меняется плавно, формально малые члены можно отбросить, уравнение имеет пониженный порядок — всё это характерно для внешнего разложения. У края наоборот: решение меняется быстро, первостепенную роль играют старшие производные, хотя имеют малые коэффициенты. Но внешнее и внутреннее разложения — это разные формы одного решения, они должны быть состыкованы процедурой сращивания. Рассмотрим пример.

Задача о прогибе $u(x)$ натянутой струны с закреплёнными концами под действием равномерно распределённой нагрузки может быть поставлена так:

...

§ 6. Multiple-scale analysis (method of multiple scales)

Этот метод привлекателен, естественен и — как написано у Ali Hasan Nayfeh

...

§ 7. Equations with slowly varying parameters

Рассмотрим гармонический осциллятор, собственная частота которого медленно меняется во времени

...

§ 8. Thin bodies

Задачи теории упругости часто ставятся для тонких тел — стержней, пластин и оболочек. Таковы многие элементы конструкций, но и в природе вне человека тонкие тела встречаются довольно часто.

Решение задач упругости для тонких тел многие десятилетия основывалось на неких гипотезах о распределении решения по толщине и о порядках одних неизвестных относительно других. Построенные так теории сыграли большу́ю роль в практике инженерных расчётов. Однако, им не хватало логической стройности и убедительности, их хотелось обосновать, уточнить — а в последнее время и уничтожить (в связи́ с появлением великолепных компьютеров). Но открытие асимптотического расщепления прояснило картину: в тонком теле трёхмерная задача расщепляется на задачи меньшей размерности. Классические теории тонких тел получили и подтверждение, и развитие.

Рассмотрим задачу о кручении из ...

...

Ali Hasan Nayfeh's book [109] is an excellent introduction to perturbation methods (asymptotic methods).
Всё разнообразие асимптотических методов представлено в монографиях ...

chapter 9

RODS

§1. Initial concepts

Rod is a thin long body. It is thought of (and modeled) as a spatial curve — the axis of rod, coated with a material (a figure).

The axis of rod is described as a curve by parameterizing the location vector of points of a curve. This is a morphism (a function) of one variable coordinate s,

$$\boldsymbol{r} = \boldsymbol{r}(s). \tag{1.1}$$

Material coating gives at each rod's point a plane figure, perpendicular to the axis — normal section $\Omega(s)$.

...

$\boldsymbol{c} = \boldsymbol{c}(t)$ is a parametric curve parameterized by parameter t. If $dt \neq d\ell$, then a parameterization is not natural. For the natural parameterization $dt = d\ell$, where

$$d\ell = \sqrt{\left(dq^1\right)^2 + \left(dq^2\right)^2 + \left(dq^3\right)^2}.$$

Many different functions draw the same curve. But among various parametrizations of a curve, the parametrization by the arc length is special, it is also called the *natural parametrization*.

The length of an infinitesimal piece of a curve is described by the Πυθαγόρας-formula

$$d\ell = \sqrt{dx^2 + dy^2 + dz^2}$$

where $dx \equiv dq^1$, $dy \equiv dq^2$, $dz \equiv dq^3$ are infinitesimal changes of coordinates. $d\ell$ is called the differential length, that is the length of an almost straight very small piece of a curve.

$\boldsymbol{c}(s)$ is a parametric curve parameterized by the arc length (the natural parametrization), its derivative by the arc length parameter is denoted as $\boldsymbol{c}' \equiv \dfrac{d\boldsymbol{c}}{ds}$.

If the arc length (natural) parametrization $\boldsymbol{r}=\boldsymbol{r}(s)$ is used, then the length of derivative $\boldsymbol{r}'(s) \equiv \dfrac{d\boldsymbol{r}(s)}{ds}$ (of the tangent vector) is always equal to the one unit of length:

$$\boldsymbol{r}(s) = q_i(s)\,\boldsymbol{e}_i(s) = q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 + q_3\boldsymbol{e}_3,$$

$$\boldsymbol{r}'(s) = \frac{d\boldsymbol{r}(s)}{ds} = \frac{dq_i(s)}{ds}\,\boldsymbol{e}_i(s) = q_i'(s)\,\boldsymbol{e}_i(s),$$

$$\|\boldsymbol{r}'(s)\|^2 \equiv \boldsymbol{r}'(s)\bullet\boldsymbol{r}'(s) = q_i'(s)\,\boldsymbol{e}_i\bullet q_j'(s)\,\boldsymbol{e}_j = q_i'(s)\,q_j'(s)\,\delta_{ij} = \sum_{i=1}^{3}\big(q_i'(s)\big)^2,$$

$$ds = \sqrt{(dq_1)^2 + (dq_2)^2 + (dq_3)^2} \;\;\Rightarrow\;\; ds^2 = \sum_{i=1}^{3}\big(dq_i(s)\big)^2,$$

$$\|\boldsymbol{r}'(s)\|^2 = \frac{\sum\limits_{i=1}^{3}\big(dq_i(s)\big)^2}{(ds)^2} = 1 \;\;\Rightarrow\;\; \|\boldsymbol{r}'(s)\| = 1.$$

...

In each section we select two perpendicular axes x_α with co-directed unit vectors $\boldsymbol{e}_\alpha$ ($\alpha=1,2$). The reason of selection for all rod sections is the same, for example, the main axes of inertia of the section are chosen everywhere.

The actual axis and the initial axis are different

When vector $\mathring{\boldsymbol{e}}_3$ is directed along the tangent to the initial axis with location $\mathring{\boldsymbol{r}}$, it is written as $\mathring{\boldsymbol{r}}' \equiv \dfrac{\partial\mathring{\boldsymbol{r}}}{\partial s} \equiv \mathring{\boldsymbol{r}}_{\partial s} \equiv \mathring{\boldsymbol{e}}_3$.

Vector $\boldsymbol{e}_3 \equiv \boldsymbol{k}$ is directed along the tangent (tangentially) to the actual axis.

Together with the unit vector, tangent to the actual axis

$$\boldsymbol{r}' \equiv \frac{d\boldsymbol{r}}{ds} \equiv \boldsymbol{r}_{\partial s} \equiv \boldsymbol{e}_3 \equiv \boldsymbol{k},$$

we'll get for each s a triple of mutually perpendicular unit vectors.

The curvature and the torsion of the rod's axis can be described by vector $\boldsymbol{\psi} = \psi_j \boldsymbol{e}_j$:

$$\boldsymbol{e}_j' = \boldsymbol{\psi} \times \boldsymbol{e}_j, \quad \boldsymbol{\psi} = \frac{1}{2}\, \boldsymbol{e}_j \times \boldsymbol{e}_j'. \tag{1.2}$$

For a cylindrical (prismatic) rod $\boldsymbol{\psi} = \boldsymbol{0}$.

However, (1.2) is only an initial concept of vector $\boldsymbol{\psi}$ as of geometric features. Further in § 2, after adopting the material structure of a rod, a concept of $\boldsymbol{\psi}$ will change.

Moreover, at each point of the rod's axis, thought of as a curve, there's also another triple of mutually perpendicular unit vectors, the one with the normal and the binormal vectors.

Tangent $\boldsymbol{T}$, normal $\boldsymbol{N}$ and binormal $\boldsymbol{B}$ vectors, together called the *Frenet–Serret frame*, are defined as:

- ✓ $\boldsymbol{T}$ is the unit vector tangent to a curve. The length of the tangent vector is always one unit, if the natural (arc length) parametrization of a curve is used. The tangent vector points to where a curve continues further.
- ✓ $\boldsymbol{N}$ is the normal unit vector, the derivativeof $\boldsymbol{T}$ by the curve's parameter (for instance, the arc length of a curve). The normal vector is always perpendicular to the tangent vector and points towards the center of curvature. It is divided by its length $\|\boldsymbol{N}\|$ to be the one unit long.
- ✓ $\boldsymbol{B}$ is the binormal unit vector, the "×"-product ("cross product") of $\boldsymbol{T}$ and $\boldsymbol{N}$, $\boldsymbol{B} \equiv \boldsymbol{T} \times \boldsymbol{N}$.

The Frenet–Serret formulas describe the derivatives of tangent, normal and binormal unit vectors through relations with each other.

$$\begin{aligned} \frac{d\boldsymbol{T}}{ds} &= \kappa \boldsymbol{N}, \\ \frac{d\boldsymbol{N}}{ds} &= -\kappa \boldsymbol{T} + \tau \boldsymbol{B}, \\ \frac{d\boldsymbol{B}}{ds} &= -\tau \boldsymbol{N}, \end{aligned}$$

where $^{d}/_{ds}$ denotes the derivative by the arc length, κ is the curvature and τ is the curve's torsion. The associated collection — $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, κ, τ — is called the Frenet–Serret apparatus.

Two scalars κ and τ effectively define the curvature and the torsion of a space curve. Intuitively, the curvature measures the deviation of a curve from a straight line, while the torsion measures the deviation of a curve from being planar.

Two functions $\kappa(s)$ and $\tau(s)$ completely define the geometry of a curve, because they are the coefficients of a system of ordinary differential equations for $\boldsymbol{T}$, $\boldsymbol{N}$ and $\boldsymbol{B}$. Knowing $\boldsymbol{T}(s)$, we'll obtain $\boldsymbol{r}(s)$ via an integration of what? with an accuracy up to a constant rigid movement without deformations.

...

$$\boldsymbol{T} = \frac{d\boldsymbol{r}}{ds} \quad \text{or} \quad \boldsymbol{T} = \boldsymbol{r}'$$

The derivative of $\boldsymbol{T}$ consists of two multipliers — the curvature κ and the unit normal vector $\boldsymbol{N}$

$$\frac{d\boldsymbol{T}}{ds} = \kappa\boldsymbol{N} \quad \text{or} \quad \boldsymbol{T}' = \kappa\boldsymbol{N}$$

Curvature κ is equal to the magnitude (length) of vector $\boldsymbol{N}$ (the derivative of vector $\boldsymbol{T}$, the second derivative of location vector $\boldsymbol{r}$)

$$\kappa = \|\boldsymbol{N}\| = \|\boldsymbol{T}'\| = \left\|\frac{d\boldsymbol{T}}{ds}\right\| = \|\boldsymbol{r}''\| = \left\|\frac{d^2\boldsymbol{r}}{ds^2}\right\|$$

Vector $\boldsymbol{N}$ itself is divided by its length thus its length is equal to the one unit.

$$\boldsymbol{N} = \frac{\frac{d\boldsymbol{T}}{ds}}{\left\|\frac{d\boldsymbol{T}}{ds}\right\|} \quad \text{or} \quad \boldsymbol{N} = \frac{\boldsymbol{T}'}{\|\boldsymbol{T}'\|}$$

The radius of curvature is the reciprocal of curvature.

$$\frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds} = \boldsymbol{N} \quad \text{or} \quad \frac{1}{\kappa}\boldsymbol{T}' = \boldsymbol{N}$$

...

$$\kappa \geq 0$$

The Frenet–Serret frame is defined only if the curvature is nonzero ($\kappa > 0$), it is not defined if $\kappa = 0$.

A line with the nonzero curvature $\kappa \neq 0$ is considered a curve.

The zero curvature implies that a line is straight, and it lies in a plane, making the torsion equal to zero too ($\tau = 0$).

...

$\boldsymbol{T}$ always has the unit magnitude (length). Since the length of $\boldsymbol{T}$ is constant, then $\boldsymbol{N}$ — the derivative of $\boldsymbol{T}$ and the second derivative of location vector $\boldsymbol{r}$ — is always perpendicular to $\boldsymbol{T}$

$$\boldsymbol{T} \cdot \boldsymbol{T} = 1 \;\Rightarrow\; \boldsymbol{T}' \cdot \boldsymbol{T} = 0 \;\Rightarrow\; \boldsymbol{N} \cdot \boldsymbol{T} = 0.$$

Vectors of the Frenet–Serret frame make an orthonormal basis $\boldsymbol{f}_i$:

$$\boldsymbol{f}_1 = \boldsymbol{T},\;\; \boldsymbol{f}_2 = \boldsymbol{N},\;\; \boldsymbol{f}_3 = \boldsymbol{B}.$$

The location vector in the Frenet–Serret basis

$$\boldsymbol{r}(s) = q_j(s)\boldsymbol{f}_j(s) = q_1(s)\underbrace{\boldsymbol{f}_1(s)}_{\boldsymbol{T}} + q_2(s)\underbrace{\boldsymbol{f}_2(s)}_{\boldsymbol{N}} + q_3(s)\underbrace{\boldsymbol{f}_3(s)}_{\boldsymbol{B}}.$$

The tensor version of the Frenet–Serret formulas

$$\boldsymbol{f}_i{}' = {}^2\boldsymbol{d} \cdot \boldsymbol{f}_i. \tag{1.3}$$

The Frenet–Serret formulas written using the matrix notation

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}' \\ \boldsymbol{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix}.$$

Tensor ${}^2\boldsymbol{d}$ is skewsymmetric, so it can be represented via the companion pseudovector (§ 1.8). This pseudovector is known as the Darboux vector.

$$\boldsymbol{D} = \tau\boldsymbol{T} + \kappa\boldsymbol{B}$$
$$\boldsymbol{D} = \tau\boldsymbol{T} + 0\boldsymbol{N} + \kappa\boldsymbol{B}$$

With the Darboux vector, the Frenet–Serret formulas turn into the following:

$$\begin{aligned} \boldsymbol{T}' &= \boldsymbol{D} \times \boldsymbol{T}, \\ \boldsymbol{N}' &= \boldsymbol{D} \times \boldsymbol{N}, \\ \boldsymbol{B}' &= \boldsymbol{D} \times \boldsymbol{B} \end{aligned}$$

or as the vector version of (1.3)

$$\boldsymbol{f}_i' = \boldsymbol{D} \times \boldsymbol{f}_i \,. \tag{1.4}$$

The Darboux vector is the angular velocity vector of the Frenet–Serret frame.

...

Approximate applied theories of rods like the "strength of materials" use such concepts as internal force $\boldsymbol{Q}$ and internal moment $\boldsymbol{M}$. The following relations connect them with the stress tensor

$$\boldsymbol{Q}(s) = \int\limits_{\Omega} \boldsymbol{t}_{(\boldsymbol{k})} \, d\Omega = \int\limits_{\Omega} \boldsymbol{k} \bullet \boldsymbol{\tau} \, d\Omega \,, \tag{1.5}$$

$$\boldsymbol{M}(s) = \int\limits_{\Omega} \boldsymbol{x} \times \boldsymbol{t}_{(\boldsymbol{k})} \, d\Omega = \int\limits_{\Omega} \boldsymbol{x} \times \boldsymbol{k} \bullet \boldsymbol{\tau} \, d\Omega \,. \tag{1.6}$$

$$\boldsymbol{k} \equiv \boldsymbol{e}_3, \quad \boldsymbol{x} = x_\alpha \boldsymbol{e}_\alpha, \quad \alpha = 1, 2$$

...

These thoughts about geometry and about mechanics — in particular, about internal force (1.5) and internal moment (1.6) — concern only some unique single configuration of a rod. It's meaningless to continue these thoughts, because in reality plane and normal sections do not remain plane and normal after deforming.

synonyms for "deplanations"

warp, warping = деформация (deformation, strain), искривление, искажение (distort, distortion), перекóс (skew), перекоси́ть, перека́шиваться, коро́биться, покоро́бить, коробле́ние, becoming twisted or bent

And the addition of some assumptions-hypotheses to the model, like "there are no warping (deplanations)" and initially plane sections remain plane* introduces essential contradictions with reality. Enough to recall just one fact that without deplanations it's impossible to acceptably describe the torsion of a rod (and not only torsion).

The very reasonable approach to modeling deformations of an elastic rod consists in asymptotic splitting of the three-dimensional problem with a small thickness. But for a complex asymptotic procedure it would be much simpler to have whatever solution version aforehand. And the direct approach, when the one-dimensional model of a rod is a material line, gives such a version.

The primary question for building the one-dimensional model: what degrees of freedom — besides translation — do the particles of a material line possess?

It is known that rods are sensitive to the moment loads. And the presence of moments among generalized forces indicates the presence of rotational degrees of freedom. Therefore as the one-dimensional model of a rod it is reasonable to take the Cosserat line — it consists of infinitesimal absolutely rigid bodies. However, another new degrees of freedom may also appear — as for thin-walled rods, described in the dedicated chapter (chapter 10).

In the mechanics of a continuous elastic media, the place of rods is special. At first, the moments play here the main role, not the role of

* The two very popular beam models exist which postulate the hypothesis about the absence of deplanations. In the Euler–Bernoulli beam theory, shear deformations are neglected, plane sections remain plane and perpendicular to the axis. In the Timoshenko beam theory there's a constant transverse shear along the section, so plane sections still remain plane, but they are no longer perpendicular to the axis.

the small additions as in a three-dimensional Cosserat continuum. At second, the rods can be used to test models with the additional degrees of freedom, before the presence of these degrees will be researched for the three-dimensional models.

The following section presents and describes a simple one-dimensional Cosserat-like moment model.

§ 2. Kinematics of Cosserat lines

Model described further is a simplified version of chapter 5.

There's no more a triple of material coordinates q_i, but the only one — s. It may be the arc length parameter in the initial configuration. The motion of a particle over time is described by the location vector $\boldsymbol{r}(s,t)$ and the rotation tensor $\boldsymbol{O}(s,t)$. The linear and angular (11.8, § 1.11) velocities of a rod's particle are

$$\dot{\boldsymbol{r}} = \boldsymbol{v}, \tag{2.1}$$

$$\dot{\boldsymbol{O}} = \boldsymbol{\omega} \times \boldsymbol{O} \;\Leftrightarrow\; \boldsymbol{\omega} = -\tfrac{1}{2}\left(\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^{\mathsf{T}}\right)_{\times}. \tag{2.2}$$

Deformation of a rod as a Cosserat line is defined by two vectors

$$\boldsymbol{\Gamma} \equiv \boldsymbol{r}' - \boldsymbol{O} \cdot \mathring{\boldsymbol{r}}', \tag{2.3}$$

$$\boldsymbol{\kappa} \equiv -\tfrac{1}{2}\left(\boldsymbol{O}' \cdot \boldsymbol{O}^{\mathsf{T}}\right)_{\times} \;\Leftrightarrow\; \boldsymbol{O}' = \boldsymbol{\kappa} \times \boldsymbol{O}. \tag{2.4}$$

$$\left(\mathring{\boldsymbol{r}}(s) \equiv \boldsymbol{r}(s,0)\right)$$

$$\|\boldsymbol{e}_3\| = \|\mathring{\boldsymbol{e}}_3\| = 1 = \text{constant}$$

(2.3) and (2.4) are really the deformation vectors, it follows from their equality to zero on movements of a body as a rigid whole (.... add some equation(s) here describing movements as a rigid whole .....).

Further we will clarify the idea of the first deformation vector $\boldsymbol{\Gamma}$. Without a loss of universality, the parameter s is the initial arc length,

the third initial basis vector $\mathring{\boldsymbol{e}}_3$ is directed along the tangent in the initial configuration: $\mathring{\boldsymbol{e}}_3 = \mathring{\boldsymbol{r}}'$. And then

$$\boldsymbol{\Gamma} \equiv \boldsymbol{r}' - \boldsymbol{O}\bullet\mathring{\boldsymbol{r}}',\ \ \mathring{\boldsymbol{r}}' = \mathring{\boldsymbol{e}}_3,\ \ \boldsymbol{O}\bullet\mathring{\boldsymbol{r}}' = \boldsymbol{O}\bullet\mathring{\boldsymbol{e}}_3 = \boldsymbol{e}_3 \\ \Rightarrow\ \ \boldsymbol{\Gamma} = \boldsymbol{r}' - \boldsymbol{e}_3, \quad (2.5)$$

$$\boldsymbol{r}' = \boldsymbol{\Gamma} + \boldsymbol{e}_3,\ \ \|\boldsymbol{r}'\|^2 = \boldsymbol{r}'\bullet\boldsymbol{r}' = \big(\boldsymbol{\Gamma} + \boldsymbol{e}_3\big)\bullet\big(\boldsymbol{\Gamma} + \boldsymbol{e}_3\big) \\ = \boldsymbol{\Gamma}\bullet\boldsymbol{\Gamma} + 2\boldsymbol{\Gamma}\bullet\boldsymbol{e}_3 + \boldsymbol{e}_3\bullet\boldsymbol{e}_3,$$

$$\|\boldsymbol{r}'\| = \sqrt{\|\boldsymbol{\Gamma}\|^2 + 2\Gamma_3 + 1},$$

$$\|\boldsymbol{r}'\| - 1 = \sqrt{\|\boldsymbol{\Gamma}\|^2 + 2\Gamma_3 + 1} - 1 = \Gamma_3 + \infty^{-1}\big(\|\boldsymbol{\Gamma}\|^2\big). \quad (2.6)$$

Equality (2.6) describes a relative elongation. Roughly speaking, component $\Gamma_3 \equiv \boldsymbol{\Gamma}\bullet\boldsymbol{e}_3$ can be considered an elongation, and components $\Gamma_1 \equiv \boldsymbol{\Gamma}\bullet\boldsymbol{e}_1$, $\Gamma_2 \equiv \boldsymbol{\Gamma}\bullet\boldsymbol{e}_2$ present a transverse shear. It's more accurate to rely on formulas (2.5) and (2.6).

.....

The Cosserat-like model of a rod doesn't have a section as a plane figure.

....

§ 3. Balance of forces and moments

Possible loads acting on a rod as the Cosserat line are forces and moments: on infinitesimal element ds of a rod act external force $\boldsymbol{q}ds$ and external moment $\boldsymbol{m}ds$. The internal interactions will be force $\boldsymbol{Q}(s)$ and moment $\boldsymbol{M}(s)$ — this is the action of the particle with coordinate $s{+}0$ on the particle with $s{-}0$. The action–reaction principle gives that a reverse (a change of direction) of coordinate s changes signs of $\boldsymbol{Q}$ and $\boldsymbol{M}$.

...

§ 4. Principle of virtual work and consequences

For a piece of rod $s_0 \le s \le s_1$ formulation of the principle is as follows

.....

Conventionally $\boldsymbol{a}$ is the tensor of stiffness for bending and twisting, $\boldsymbol{b}$ is the tensor of stiffness for (ex)tension and shear, and $\boldsymbol{c}$ is the tensor of crosslinks.

Stiffness tensors rotate along with a particle:

...

§ 5. Classical Kirchhoff's model

It is also called the *Kirchhoff's rod theory.*

Until now functions $\boldsymbol{r}(s,t)$ and $\boldsymbol{O}(s,t)$ were independent. The Kirchhoff's classical theory postulates the internal con*strain*t

$$\boldsymbol{\Gamma} = \boldsymbol{0} \;\Leftrightarrow\; \boldsymbol{r}_{\partial s} = \boldsymbol{O} \bullet \mathring{\boldsymbol{r}}_{\partial s} \text{ or } \boldsymbol{r}' = \boldsymbol{O} \bullet \mathring{\boldsymbol{r}}'. \tag{5.1}$$

Having the idea of vector $\boldsymbol{\Gamma}$ (2.5), here we can tell that: (1) a rod is non-extensible, (2) there are no transverse shears.

If basis vector $\mathring{\boldsymbol{e}}_3$ was directed along the tangent to the axis in the initial configuration, then it will remain on the tangent also after deforming. The rod particles rotate only together with the tangent to the axis and around it.

Уравнения баланса сил и моментов (импульса и момента импульса) не меняются от введения связи (5.1). Но локальное вариационное соотношение (...) становится короче:

...

§ 6. Euler's problem about stability of rods

Рассматривается прямой стержень, защемлённый на одном конце и нагруженный силой $\boldsymbol{P}$ на другом (рисунок ?? 123 ??). Сила “мёртвая” (не меняется в процессе деформирования)

...

§ 7. Variational equations

In the nonlinear mechanics of elastic media variational equations are useful, which describe a small change in the current configuration (§ 3.12).

Variating equations of the complete system of the Cosserat model, we get

$$\delta \boldsymbol{Q}' + \delta \boldsymbol{q} = \rho \big(\boldsymbol{u} + \delta \boldsymbol{\tau} \times \boldsymbol{\varepsilon} \big)^{\bullet\bullet},$$
$$\delta \boldsymbol{M}' + \boldsymbol{u}' \times \boldsymbol{Q} + \boldsymbol{r}' \times \delta \boldsymbol{Q} + \delta \boldsymbol{m} = \dots$$

...

§ 8. Non-shear model with (ex)tension

Kirchhoff's model with an internal constraint $\boldsymbol{\Gamma} = \boldsymbol{0}$ (5.1) doesn't describe the simplest case of a straight rod extension/compression. This nuisance disappears with "softening" of the con*strain*t, for example by adding the possibility of (ex)tension and inhibiting only the transverse shear

$$\boldsymbol{\Gamma} = \Gamma \boldsymbol{e}_3 \;\Leftrightarrow\; \Gamma_\alpha = 0 \tag{8.1}$$

...

§ 9. Mechanics of flexible thread

A thread is a momentless rod.

A flexible thread (chain) is simpler than a rod, because its particles are "simple" material points with only translational degrees of freedom. Therefore среди нагрузок нет моментов, только "линейные" силы — внешние распределённые $\boldsymbol{q}$ и внутренние сосредоточенные $\boldsymbol{Q}$. Движение нити полностью определяется одним вектором-радиусом $\boldsymbol{r}(s,t)$, а инерционные свойства — линейной плотностью $\rho(s)$.

Вот принцип виртуальной работы для кускá нити $s_0 \le s \le s_1$

$$\int\limits_{s_0}^{s_1} \Big((\boldsymbol{q} - \rho \ddot{\boldsymbol{r}}) \cdot \delta \boldsymbol{r} - \delta \Pi \Big) ds + \Big[\boldsymbol{Q} \cdot \delta \boldsymbol{r} \Big]_{s_0}^{s_1} = 0. \tag{9.1}$$

...

Механика нити детально описана в книге [33].

§ 10. Linear theory

В линейной теории внешние воздействия считаются малыми, а отсчётная конфигурация — ненапряжённым состоянием покоя. Уравнения в вариациях в этом случае дают

...

§ 11. Rod of small thickness

When the rod's relative thickness is small, then for such a rod the Cosserat-like model can be replaced with the classical one.

The "smallness" of thickness is determined by the ratio of rigidities (stiffnesses). But since $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are measured in different units, taking some range of length $\hbar$ and denoting $\boldsymbol{a} = \hbar^2 \widehat{\boldsymbol{a}}$ and $\boldsymbol{c} = \hbar \widehat{\boldsymbol{c}}$, the units of tensors $\widehat{\boldsymbol{a}}$, $\boldsymbol{b}$ and $\widehat{\boldsymbol{c}}$ will be the same. By picking $\hbar$ so that the characteristic values of tensors $\widehat{\boldsymbol{a}}$, $\boldsymbol{b}$ and $\widehat{\boldsymbol{c}}$ become closer, the rod's "equivalent thickness" h can be found (for real three-dimensional rods, h is somewhere around the cross-section diameter).

Representing $\boldsymbol{Q}$ and $\boldsymbol{M}$ via vectors of infinitesimal linear deformation

....

The transition from the Cosserat-like model to the classical one seems more obvious if equations (...) are integrated immediately.

...

§ 12. Saint-Venant's problem

It's hard to overestimate the role that the Saint-Venant's classical solution plays in the mechanics of rods. This solution was already considered earlier in § 4.14.

In place of conditions ...

.....

§ 13. Finding stiffness by energy

To find the stiffness tensors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ for the one-dimensional model it's enough to have solutions of 3-dimensional problems for a rod. But here emerge the two questions : which specific problems to consider and what to take from solutions?

Saint-Venant's problem выделяется среди прочих, ведь оттуда берётся жёсткость на кручение.

Вдобавок есть много точных решений, получаемых таким путём : задаётся поле $\boldsymbol{u}(\boldsymbol{r})$, определяется $\breve{\boldsymbol{\tau}} = {}^4\boldsymbol{\mathcal{C}} \bullet\bullet \boldsymbol{\nabla}\boldsymbol{u}$, затем находятся объёмные $\boldsymbol{f} = -\boldsymbol{\nabla} \bullet \breve{\boldsymbol{\tau}}$ и поверхностные $\boldsymbol{p} = \boldsymbol{n} \bullet \breve{\boldsymbol{\tau}}$ нагрузки.

But what to do with the solution? It's clear that $\boldsymbol{Q}$ and $\boldsymbol{M}$ in a rod are integrals over the rod's section (...). However it's not clear at all, what to consider as displacement and what as rotation in the one-dimensional model. Taking, for example, the following version (the index of $\boldsymbol{u}$ shows the model's dimensions)

$$\boldsymbol{u}_1(z) = \Omega^{-1} \int\limits_{\Omega} \boldsymbol{u}_3(\boldsymbol{x}, z)\, d\Omega, \quad \boldsymbol{\theta}(z) = \tfrac{1}{2}\, \Omega^{-1} \int\limits_{\Omega} \boldsymbol{\nabla} \times \boldsymbol{u}_3\, d\Omega,$$

nevertheless, other possible representations may be proposed as well.

Besides $\boldsymbol{Q}$ and $\boldsymbol{M}$, есть ещё величина, не вызывающая сомнений — упругая энергия. Естественно потребовать, чтобы в одномерной и в трёхмерной моделях энергии на единицу длины совпали. And

to avoid differences in interpretations of $\boldsymbol{u}_1$ and $\boldsymbol{\theta}$, I propose to proceed from the complementary energy Ц$(\boldsymbol{M}, \boldsymbol{Q})$:

$$\text{Ц}(\boldsymbol{M}, \boldsymbol{Q}) = \int_{\Omega} \Pi_3 \, d\Omega$$

...

§ 14. Variational method of building one-dimensional model

Мы только что определили жёсткости стержня, полагая, что одномерная модель линии типа Cosserat адекватно описывает поведение трёхмерной модели. “Одномерные” представления ассоциируются со следующей картиной смещений в сечении:

$$\boldsymbol{u}(s, \boldsymbol{x}) = \boldsymbol{U}(s) + \boldsymbol{\theta}(s) \times \boldsymbol{x}. \tag{14.1}$$

Однако, такое поле $\boldsymbol{u}$ не удовлетворяет уравнениям трёхмерной теории(??добавить, каким именно). Невозможно пренебречь возникающими невязками в дифференциальных уравнениях и краевых условиях.

Формально “чистым” является вариационный метод сведения трёхмерной проблемы к одномерной, называемый иногда методом внутренних связей. Аппроксимация (14.1) подставляется в трёхмерную формулировку вариационного принципа минимума потенциальной энергии (10.1, § 4.10)

$$\mathscr{E}(\boldsymbol{u}) = \int_{\mathcal{V}} \Big(\Pi(\boldsymbol{u}) - \boldsymbol{f} \cdot \boldsymbol{u} \Big) d\mathcal{V} - \int_{o_2} \boldsymbol{p} \cdot \boldsymbol{u} \, do \to \min,$$

которая после интегрирования по сечению становится одномерной. Если $\boldsymbol{U}$ и $\boldsymbol{\theta}$ варьируются независимо, получаем модель типа Коссера. В случае $\boldsymbol{U}' = \boldsymbol{\theta} \times \boldsymbol{t}$ приходим к классической модели.

Метод внутренних связей привлекателен, его продолжают “переоткрывать”. С его помощью возможно моделировать тела

с неоднородностью и анизотропией, он легко обобщается на динамику, если $\boldsymbol{f}$ дополнить неварьируемой динамической добавкой до $\boldsymbol{f} - \rho\ddot{\boldsymbol{u}}$. Можно рассматривать и стержни переменного сечения, и даже нелинейно упругие, ведь вариационная постановка есть (гл. 3).

Аппроксимацию (14.1) можно дополнить слагаемыми с внутренними степенями свободы. Понимая необходимость учёта депланаций, некоторые авторы

...

...

For variational construction of one-dimensional models it's convenient to use the Reissner–Hellinger principle (§ 4.12) with independent approximation of stresses [17]. In this case, some consistency between $\boldsymbol{u}$ and $\boldsymbol{\tau}$ is needed.

Many advantages of the variational method are opposed by the one, but significant disadvantage. Introducing approximations within cross-sections, we impose our unreal simplifications on reality. The variational method is more suitable for applied calculations.

§ 15. Asymptotic splitting of three-dimensional problem

Asymptotic splitting can be considered fundamental for describing the mechanics of rods. One-dimensional models paint only the part of a picture, two-dimensional problems in cross-sections paint the other part, and together they present the solution of a three-dimensional problem for a small thickness.

How to introduce a small parameter χ into a three-dimensional problem? The easiest way to do it is through representation of the location vector (§ 1):

$$\boldsymbol{R}(x_\alpha, s) = \chi^{-1}\boldsymbol{r}(s) + \boldsymbol{x}, \quad \boldsymbol{x} \equiv x_\alpha \boldsymbol{e}_\alpha(s), \quad \alpha = 1, 2.$$

For an orthonormal basis, upper and lower indices do not differ

$$q_i = q^i, \ \boldsymbol{R}^i = \boldsymbol{R}_i = \boldsymbol{R}_{\partial i} = \frac{\partial \boldsymbol{R}}{\partial q^i}.$$

Three coordinates are

$$q^1 = x_1, \;\; q^2 = x_2, \;\; q^3 = s.$$

The basis vectors are

$$\boldsymbol{e}_1, \;\; \boldsymbol{e}_2, \;\; \boldsymbol{t} \equiv \boldsymbol{e}_3.$$

Representation of the Hamilton operator $\boldsymbol{\nabla}$

$$\boldsymbol{\nabla} = \boldsymbol{R}^i \frac{\partial}{\partial q^i} = \boldsymbol{\nabla}_\perp + \mathrm{v}^{-1}\boldsymbol{t}\big(\partial_s - \psi_t \mathrm{D}\big), \;\; \boldsymbol{\nabla}_\perp \equiv \boldsymbol{e}_\alpha \frac{\partial}{\partial x_\alpha},$$
$$\mathrm{v} \equiv \boldsymbol{R}_{\partial 1} \times \boldsymbol{R}_{\partial 2} \boldsymbol{\cdot} \boldsymbol{R}_{\partial 3} = \chi^{-1} + \boldsymbol{t} \boldsymbol{\cdot} \boldsymbol{\psi}_\perp \times \boldsymbol{x}, \;\; \mathrm{D} \equiv \boldsymbol{t} \boldsymbol{\cdot} \boldsymbol{x} \times \boldsymbol{\nabla}_\perp,$$
$$\boldsymbol{\psi}_\perp + \psi_t \boldsymbol{t} = \boldsymbol{\psi}$$

(the meaning of vector $\boldsymbol{\psi}$ is the same as in § 1).

The Cauchy stress tensor

$$\boldsymbol{\tau} = \boldsymbol{\tau}_\perp + \sigma_t \boldsymbol{t}\boldsymbol{t} + 2\boldsymbol{t}\boldsymbol{\tau}_\parallel^{\mathsf{S}}, \;\; \boldsymbol{\tau}_\perp \equiv \tau_{\alpha\beta}\boldsymbol{e}_\alpha\boldsymbol{e}_\beta, \;\; \boldsymbol{\tau}_\parallel \equiv \tau_{3\alpha}\boldsymbol{e}_\alpha.$$

...

§ 16. Thermal deformation and stress

The direct approach, very efficient for describing one-dimensional Cosserat and Kirchhoff models, isn't applicable for problems of thermoelasticity. The transition from the three-dimensional model to the one-dimensional can be realized either by the variational or the asymptotic way.

Описанный в § 14 вариационный метод целиком переносится на термоупругость — включая задачи с неоднородностью и анизотропией, переменным сечением, динамические и даже нелинейные. Для этого нужно (§ 6.8) in the Lagrange principle of minimum potential energy заменить потенциал $\Pi(\boldsymbol{\varepsilon})$ свободной энергией

...

Unlike other topics, rods are presented very modestly in books. The narration style of “the strength of materials” prevails there, and more exact and perfect approaches seem impossible or unnecessary to the majority of authors.

But many interesting articles have been published, their reviews are presented, for example, by S. Antman [1], В. В. Елисеев [17] and А. А. Илюхин [**?**].

chapter 10

THIN-WALLED RODS

§1. Variational approach

In chapter 9, rods with massive cross-sections were described. But other rods are also widely used — thin-walled ones, with sections of narrow bands of various shapes: corner, Z-beam, channel (C-beam), I-beam, ... If rods are like lines, then in a thin-walled rod the section itself also looks like line. The three dimensions — the thickness and the length of a section, as well as the length of a rod — have different decimal orders.

The applied theories of thin-walled rods are known, they are described, for example, in books ...

There is also the exact theory, based on asymptotic splitting of the three-dimensional problem [17]. In sum, the complex asymptotic analysis confirmed most of the hypotheses of applied theories.

As introduction to the mechanics of thin-walled rods, here is the following variational procedure with warping (deplanation) of cross-sections.

Consider the simplest case of a cylindrical rod with a thin, simply connected section (... рисунок ...). Площадь сечения do, боковая поверхность свободна, нагрузка в объёме — $\boldsymbol{f}$, на торце $z=z_1$ известны поверхностные силы $\boldsymbol{p}(\boldsymbol{x})$ ($\boldsymbol{x}=x_\alpha \boldsymbol{e}_\alpha$), торец $z=z_0$ закреплён.

...

§ 2. Equations with small parameter

Рассмотрим призматический стержень с односвязным сечением в виде тонкой криволинейной полоски постоянной толщины h. Радиус-вектор в объёме предста́вим следующим образом:

...

§ 3. First step of the asymptotic procedure

Outer decomposition

Из системы

...

Inner decomposition near s_0

Выпишем уравнения для

...

Coalescence

Стыковка внутреннего и внешнего разложений

...

§ 4. Second step

Outer decomposition

Из системы

...

Inner decomposition near $s = s_0$

Из общей системы

...

Coalescence

Поскольку рассматриваются поправочные члены асимптотических разложений

...

§ 5. Third step

Outer decomposition

Из системы

....

Inner decomposition around $s = s_0$

Как уже́ отмечалось, внутренние разложения нужны для постановки краевых условий на концах

...

Coalescence

В плоской задаче имеем следующее двучленное внешнее разложение:

....

§ 6. Fourth step

Здесь понадобится лишь внешнее разложение. Более того: в этом приближении мы не будем искать решения уравнений — будет достаточно лишь условий разрешимости. Напомним, что философия наша такова: разыскиваются лишь главные члены асимптотических разложений, но для полного их определения могут понадобиться

...

Расписывая тензорное соотношение

...

§ 8. Results of asymptotic analysis

Нахождение главных членов асимптотики of stresses and displacements для тонкостенных стержней оказалось намного сложнее, чем для массивных сечений. Дадим сводку полученных выше итоговых результатов.

Смещение:

$$\boldsymbol{u}_\perp = \lambda^{-4}\big(\boldsymbol{U}_\perp(z) + \theta(z)\boldsymbol{k}\times\boldsymbol{r}\big) + \infty^{-1}(\lambda^{-3}), \tag{8.1}$$

$$u_{[z]} = \lambda^{-3}\big(-\dot{\boldsymbol{U}}_\perp\cdot\boldsymbol{r} + \dot{\theta}\omega(s) + U_z(z)\big) + \infty^{-1}(\lambda^{-2}). \tag{8.2}$$

Напряжения:

...

Bibliography

Помимо книг В. З. Власов'а [8], Г. Ю. Джанелидзе и Я. Г. Пановко [15], отметим статью Я. Г. Пановко и Е. А. Бейлина [41].

Материал этой главы содержится в диссертации Владимира В. Елисеева [17], там также есть обширный список статей на тему тонкостенных стержней.

chapter 11

SHELLS AND PLATES

§1. Surface geometry

A surface is described by a function (a morphism)

$$\boldsymbol{r} = \boldsymbol{r}(q^\alpha),\ \ \alpha = 1,2 \tag{1.1}$$

of two mutually independent variable parameters (coordinates) q^α, then $\boldsymbol{r}$ is the location vector (the radius vector) of the surface's points.

Examples

- ✓ a linear mapping is a plane $\boldsymbol{r}(a,b) = a\boldsymbol{e}_1 + b\big(\boldsymbol{e}_2 + \boldsymbol{e}_3\big)$
- ✓ a helicoid $\boldsymbol{r}(u,v) = u\sin v\,\boldsymbol{e}_1 + u\cos v\,\boldsymbol{e}_2 + v\boldsymbol{e}_3$
- ✓ a cone $\boldsymbol{r}(u,v) = u\sin v\,\boldsymbol{e}_1 + u\cos v\,\boldsymbol{e}_2 + u\boldsymbol{e}_3$
- ✓ a cylinder of radius $r = \mathsf{constant}$
$$\boldsymbol{r}(u,v) = r\big(\cos u\,\boldsymbol{e}_1 + \sin u\,\boldsymbol{e}_2\big) + v\boldsymbol{e}_3$$
- ✓ a torus of revolution with radii r and R
$$\boldsymbol{r}(p,q) = \boldsymbol{e}_1\big(r\cos p + R\big)\cos q + \boldsymbol{e}_2\big(r\cos p + R\big)\sin q + \boldsymbol{e}_3 r\sin p$$
- ✓ a 2-sphere — a torus with $R = 0$
$$\boldsymbol{r}(p,q) = r\big(\cos p\cos q\,\boldsymbol{e}_1 + \cos p\sin q\,\boldsymbol{e}_2 + \sin p\,\boldsymbol{e}_3\big)$$
- ✓ a paraboloid $\boldsymbol{r}(u,v) = u\boldsymbol{e}_1 + v\boldsymbol{e}_2 + \big(u^2+v^2\big)\boldsymbol{e}_3$ or via a cylindrical parameterization $\boldsymbol{r}(\rho,\vartheta) = \rho\cos\vartheta\,\boldsymbol{e}_1 + \rho\sin\vartheta\,\boldsymbol{e}_2 + \rho^2\boldsymbol{e}_3$ for a paraboloid of revolution.

The continuous change of the first coordinate q^1, while the second one $q^2 = u^* = \mathsf{constant}$ is "frozen", gives the coordinate line $\overset{(1)}{\boldsymbol{r}}(q^1) = \boldsymbol{r}(q^1, u^*)$. The crossing of the two coordinate lines $q^1 = v^*$ and $q^2 = w^*$ uniquely identifies the point $\boldsymbol{r}(v^*, w^*)$ of the surface.

Vectors

$$\boldsymbol{r}_{\partial\alpha} \equiv \partial_\alpha \boldsymbol{r},\ \ \partial_\alpha \equiv \frac{\partial}{\partial q^\alpha} \tag{1.2}$$

are tangent to coordinate lines. If they are linearly independent (that is $\boldsymbol{r}_{\partial 1} \times \boldsymbol{r}_{\partial 2} \neq \boldsymbol{0}$)*, they compose the local basis for representing any vector $\overset{(2)}{\boldsymbol{v}}$ in the tangent plane as linear combination

$$\begin{gathered} \overset{(2)}{\boldsymbol{v}} = v^\alpha \boldsymbol{r}_{\partial\alpha} = v_\alpha \boldsymbol{r}^\alpha, \\ v^\alpha = \overset{(2)}{\boldsymbol{v}} \bullet \boldsymbol{r}^\alpha, \quad v_\alpha = \overset{(2)}{\boldsymbol{v}} \bullet \boldsymbol{r}_{\partial\alpha}, \quad \boldsymbol{r}^\alpha \bullet \boldsymbol{r}_{\partial\beta} = \delta^\alpha_\beta . \end{gathered} \tag{1.3}$$

Here $\boldsymbol{r}^\alpha$ is the local reciprocal basis in the (co)tangent plane.

The dield of unit normal vectors $\boldsymbol{n}(q^\alpha)$ adds at every point of the surface ($\forall \boldsymbol{r}(q^\alpha) \Leftrightarrow \forall q^\alpha$) the unit** normal

$$\boldsymbol{n} = \frac{\boldsymbol{r}_{\partial 1} \times \boldsymbol{r}_{\partial 2}}{\| \boldsymbol{r}_{\partial 1} \times \boldsymbol{r}_{\partial 2} \|} . \tag{1.4}$$

At non-singular points, three vectors $\boldsymbol{r}_{\partial 1}$, $\boldsymbol{r}_{\partial 2}$ and $\boldsymbol{n}$ can be taken as a basis for the entire three-dimensional space, giving decomposition for any vector and any tensor, for example $\overset{(3)}{\boldsymbol{u}} = u^\alpha \boldsymbol{r}_{\partial\alpha} + u^n \boldsymbol{n}$.

$u^n = u_n$

for a 2-sphere of unit radius

$$\boldsymbol{r}_{\partial p} \times \boldsymbol{r}_{\partial q} = -\det \begin{bmatrix} -\sin p \cos q & \boldsymbol{e}_1 & -\cos p \sin q \\ -\sin p \sin q & \boldsymbol{e}_2 & \cos p \cos q \\ \cos p & \boldsymbol{e}_3 & 0 \end{bmatrix} = \ldots$$

The bivalent unit ("metric") tensors, $\boldsymbol{E}$ in space and $\boldsymbol{I}$ in the tangent plane

$$\boldsymbol{E} = \boldsymbol{I} + \boldsymbol{n}\boldsymbol{n}, \quad \boldsymbol{I} \equiv \boldsymbol{E} - \boldsymbol{n}\boldsymbol{n} = \boldsymbol{r}_{\partial\alpha} \boldsymbol{r}^\alpha = \boldsymbol{r}^\alpha \boldsymbol{r}_{\partial\alpha} .$$

* Sometimes somewhere — at the so-called singular points — it's not so. As example, for a 2-sphere of unit radius

$$\begin{aligned} \boldsymbol{r}(p,q) &= \cos p \cos q\, \boldsymbol{e}_1 + \cos p \sin q\, \boldsymbol{e}_2 + \sin p\, \boldsymbol{e}_3 , \\ \boldsymbol{r}_{\partial p} = \partial_p \boldsymbol{r} &= -\sin p \cos q\, \boldsymbol{e}_1 - \sin p \sin q\, \boldsymbol{e}_2 + \cos p\, \boldsymbol{e}_3 , \\ \boldsymbol{r}_{\partial q} = \partial_q \boldsymbol{r} &= -\cos p \sin q\, \boldsymbol{e}_1 + \cos p \cos q\, \boldsymbol{e}_2 + 0\, \boldsymbol{e}_3 , \end{aligned}$$

a pole $p = \pm\frac{\pi}{2}$ is a singular point: $\boldsymbol{r}_{\partial q}\big|_{p=\pm\pi/2} = \boldsymbol{0}$.

** $\|\boldsymbol{a}\| \equiv \sqrt{\boldsymbol{a} \bullet \boldsymbol{a}}$ is the length of vector $\boldsymbol{a}$.

Representation of the location vector $\overset{(3)}{\boldsymbol{\mathcal{R}}}$ for any point in space at distance h from the surface ($\partial_\alpha h = 0$)

$$\overset{(3)}{\boldsymbol{\mathcal{R}}}(q^\alpha, h) = \boldsymbol{r}(q^\alpha) + h\boldsymbol{n}(q^\alpha) \tag{1.5}$$

gives the local basis in the tangent space

$$\overset{(3)}{\boldsymbol{\mathcal{R}}}_{\partial n} = \boldsymbol{n} = \overset{(3)}{\boldsymbol{\mathcal{R}}}{}^n,$$
$$\overset{(3)}{\boldsymbol{\mathcal{R}}}_{\partial\alpha} \equiv \partial_\alpha \overset{(3)}{\boldsymbol{\mathcal{R}}} = \partial_\alpha \boldsymbol{r} + h\partial_\alpha \boldsymbol{n} = \boldsymbol{r}_{\partial\alpha} + h\boldsymbol{r}_{\partial\alpha} \bullet \boldsymbol{r}^\beta \partial_\beta \boldsymbol{n}.$$

...

differential operator "nabla"

in space $\boldsymbol{\nabla} \equiv \overset{(3)}{\boldsymbol{\mathcal{R}}}{}^i \partial_i$

in the tangent plane $\overset{(2)}{\boldsymbol{\nabla}} \equiv \boldsymbol{r}^\alpha \partial_\alpha$

...

$$\overset{(3)}{\boldsymbol{\mathcal{R}}}_{\partial\alpha} = \boldsymbol{r}_{\partial\alpha} \bullet \left(\boldsymbol{r}^\beta \boldsymbol{r}_{\partial\beta} + h\boldsymbol{r}^\beta \partial_\beta \boldsymbol{n}\right) = \boldsymbol{r}_{\partial\alpha} \bullet \left(\boldsymbol{I} + h\overset{(2)}{\boldsymbol{\nabla}}\boldsymbol{n}\right) = \boldsymbol{r}_{\partial\alpha} \bullet \left(\boldsymbol{I} - {}^{2}\overset{(2)}{\boldsymbol{c}} h\right)$$

The two-coordinate bivalent tensor

$${}^{2}\overset{(2)}{\boldsymbol{c}} \equiv -\overset{(2)}{\boldsymbol{\nabla}}\boldsymbol{n} = -\boldsymbol{r}^\alpha \partial_\alpha \boldsymbol{n} \tag{1.6}$$

characterizes the surface's curvature.

....

cobasis $\overset{(3)}{\boldsymbol{\mathcal{R}}}{}^\alpha \bullet \overset{(3)}{\boldsymbol{\mathcal{R}}}_{\partial\beta} = \delta^\alpha_\beta$, $\overset{(3)}{\boldsymbol{\mathcal{R}}}{}^i \bullet \overset{(3)}{\boldsymbol{\mathcal{R}}}_{\partial j} = \delta^i_j$

$$\overset{(3)}{\boldsymbol{\mathcal{R}}}{}^\alpha = \left(\boldsymbol{I} + h\overset{(2)}{\boldsymbol{\nabla}}\boldsymbol{n}\right)^{-1} \bullet \boldsymbol{r}^\alpha, \quad \overset{(3)}{\boldsymbol{\mathcal{R}}}{}^n = \boldsymbol{n}$$

relation between $\boldsymbol{\nabla}$ and $\overset{(2)}{\boldsymbol{\nabla}}$

$$\boldsymbol{\nabla} = \left(\boldsymbol{I} + h\overset{(2)}{\boldsymbol{\nabla}}\boldsymbol{n}\right)^{-1} \bullet \overset{(2)}{\boldsymbol{\nabla}} + \boldsymbol{n}\partial_n$$

...

§ 2. The model of a shell

Having the models of three-dimensional micropolar continuum (chapter 5) and one-dimensional rods (chapter 9, chapter 10), the mechanics of two-dimensional shells is pretty easy to describe.

As a geometrical object, the shell is defined by its middle surface and thickness h, thus in (1.5)

$$-h/2 \le h \le h/2 .$$

...

§ 3. The balance of forces and moments for a shell

When $\delta \boldsymbol{u} =$ **constant** and $\delta \boldsymbol{\varphi} = \mathbf{0}$ (a translation) ...

....

§ 4. Shells: The relations of elasticity

Локальное соотношение (**??**) после вывода уравнений баланса ...

...

§ 5. The classical theory of shells

Вышеизложенная теория (напоминающая балку Тимошенко и континуумы Cosserat) рассматривает rotations $\boldsymbol{\varphi}$ независимо от displacements $\boldsymbol{u}$. Но опыт подсказывает, что материальный элемент, нормальный к срединной поверхности до деформации, остаётся таким и после деформации (кинематическая гипотеза Kirchhoff'a). В классической теории Kirchhoff'a, Арона и Love'a field $\boldsymbol{\varphi}$ выражается через $\boldsymbol{u}$, что в конце концов даёт свести всё к одному векторному уравнению для $\boldsymbol{u}$.

Предположим, что в основе классической теории лежит внутренняя связь

...

§ 6. Shells: A plate

Plate is the simplest kind of shell. Единичный перпендикуляр $\boldsymbol{n} = \boldsymbol{k}$ направлен по декартовой оси z, в качестве координат ...

...

§ 7. Shells: Approach with Lagrange multipliers

Уязвимым местом этого изложения теории оболочек являются формулы

...

§ 8. Cylindrical shell

Существуют разные уравнения цилиндрической оболочки. Приводятся громоздкие выкладки с отбрасыванием некоторых малых членов, и не всегда ясно, какие именно члены действительно можно отбросить.

Предлагаемая читателю теория оболочек иного свойства: лишних членов нет, все уравнения записаны в компактной тензорной форме — остаётся лишь грамотно действовать с компонентами тензоров. В качестве иллюстрации рассмотрим цилиндрическую оболочку.

В декартовой системе

...

§ 9. Shells: Common theorems

Пусть край закреплён

...

§ 10. Shells: Boundary conditions

В рамках рассматриваемого прямого подхода к оболочкам как материальным поверхностям наиболее надёжным способом вывода граничных условий представляется вариационный. Исходим из вариационного уравнения:

...

§ 11. Shells of revolution

Surface of revolution (reference surface of shell of revolution) is created by rotating a plane curve (the meridian, the generatrix) about a straight line in the plane of curve (an axis of rotation).

Разберёмся в геометрии поверхности вращения (рисунок). Меридиан можно задать зависимостью декартовых координат

...

§ 12. Momentless theory of shells

В отличие от пластины, оболочка способна выдерживать нормальную распределённую нагрузку без появления внутренних моментов. В безмоментном состоянии напряжения равномерно распределены по толщине оболочки, безмоментные оболочечные конструкции можно считать оптимально спроектированными.

Уравнения безмоментной теории

...

§ 13. Shells: Nonlinear momentless theory

Вышеизложенную безмоментную теорию оболочек возможно просто и корректно обобщить на нелинейную постановку. Материальная поверхность состоит из частиц

...

§ 14. Shells: Other variant of classical theory

Выше при изложении моментной теории оболочек частицы материальной поверхности считались твёрдыми телами с шестью степенями свободы

...

§ 15. Plates: Overall concepts

Пластиной называется тонкое трёхмерное тело, ограниченное двумя параллельными плоскостями и боковой цилиндрической поверхностью (?? рисунок ??). В декартовых координатах x_1, x_2, z поперечная координата ...

...

В теории пластин рассматриваются двумерные задачи. Переход от трёхмерной задачи наиболее достоверен на пути асимптотики. Но логическая стройность и эффективность присуща и вариационному подходу, основанному на аппроксимации по толщине решения трёхмерной вариационной задачи. Самое же простое корректное изложение теории пластин характерно для прямого подхода к ним как материальным плоскостям.

...

§ 16. Timoshenko-like model of a plate (direct approach)

Пластина рассматривается как материальная плоскость, частицы которой

...

§ 17. Kirchhoff's classical theory of plates

Принимается внутренняя связь

...

§ 18. Plates: Asymptotic matching of two-dimensional models

При малой толщине из теории типа Тимошенко следует классическая теория. Толщина h определяется отношением жёсткостей. Перепишем

...

§ 19. Plates: Variational transition from the three-dimensional model

Using the variational principles by Lagrange or by Hellinger and Reissner с аппроксимацией решения по толщине, можно получить двумерные вариационные формулировки проблем. Из этих вариационных принципов вытекают и соотношения внутри области, and the naturalboundary conditions.

For example, here is the model of the Timoshenko type with the approximation of displacements

...

...

The variational transitions can be easily generalized for the cases of the temperature deformations, the inhomogeneity (heterogeneity) and anisotropy материала, the dynamics. The advantage of the Hellinger–Reissner principle состоит в явном представлении напряжений. Зато принцип Лагранжа применим and к нелинейным задачам (in the chapter 3 описана трёхмерная постановка).

§ 20. Plates: Splitting of three-dimensional bending problem

Двумерная классическая теория изгиба пластин легко выводится из трёхмерной постановки с малым параметром. Представив радиус-вектор в объёме

...

§ 21. Circular plates

В качестве иллюстрации рассмотрим широко представленный в литературе вопрос об уравнениях теории Kirchhoff'a в полярных координатах.

...

§ 22. Plates: Plane stress

Это вторая из двух задач, о которых говорилось в § 15. Силы

...

Bibliography

Теория оболочек изложена в монографиях А. Л. Гольденвейзера [9], В. В. Новожилова [40], А. И. Лурье [30], В. С. Черниной [**?**] и ряде других. Достоинства этих книг перекрывают неразвитость формального аппарата. Переход от трёхмерной модели оболочки к двумерной рассмотрен у ...

...

Техническая теория изгиба пластин изложена ...

...

chapter 12

OSCILLATIONS AND WAVES

§1. Vibrations of three-dimensional bodies

Dynamic problem of the classical linear elasticity is

$$\begin{gathered}\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\boldsymbol{f}=\rho\ddot{\boldsymbol{u}},\ \ \boldsymbol{\sigma}={}^4\boldsymbol{\mathcal{A}}\cdot\cdot\boldsymbol{\nabla}\boldsymbol{u},\\ \boldsymbol{u}\big|_{o_1}=\boldsymbol{0},\ \ \boldsymbol{n}\cdot\boldsymbol{\sigma}\big|_{o_2}=\boldsymbol{p},\\ \boldsymbol{u}\big|_{t=0}=\boldsymbol{u}^\circ,\ \ \dot{\boldsymbol{u}}\big|_{t=0}=\dot{\boldsymbol{u}}^\circ.\end{gathered}\tag{1.1}$$

According to the common theory (§ 2.6), we begin with the analysis of harmonics (orthogonal oscillations):

$$\boldsymbol{f}=\boldsymbol{0},\ \ \boldsymbol{p}=\boldsymbol{0},\ \ \boldsymbol{u}(\boldsymbol{r},t)=\boldsymbol{U}(\boldsymbol{r})\sin\omega t,$$

$$\boldsymbol{\nabla}\cdot\big({}^4\boldsymbol{\mathcal{A}}\cdot\cdot\boldsymbol{\nabla}\boldsymbol{U}\big)+\rho\omega^2\boldsymbol{U}=\boldsymbol{0}.\tag{1.2}$$

If a homogeneous problem has a nontrivial solution, then the values of ω are natural resonant frequencies, and $\boldsymbol{U}(\boldsymbol{r})$ are orthogonal (normal) "modes".

The time independent equation (1.2) looks like equation of linear elastostatics (6.1, § 4.6) when the volume load is equal to $\omega^2\rho\boldsymbol{U}$. The surface load on o_2 is equal to zero. The Clapeyron's identity (5.1, § 4.5) gives

$$\omega^2\int_{\mathcal{V}}\rho\,\boldsymbol{U}\cdot\boldsymbol{U}d\mathcal{V}=2\int_{\mathcal{V}}\Pi\big(\boldsymbol{\nabla}\boldsymbol{U}^{\mathsf{S}}\big)d\mathcal{V}.\tag{1.3}$$

It also means that $\omega^2\geq 0$, and $=0$ only if a continuum moves (with $\boldsymbol{U}$) as a rigid whole. When even a small part of a surface is fixed, then all $\omega_i>0$.

And here we assume that ω^2 and $\boldsymbol{U}$ are real numbers. This can be proven "by contradiction". If $\Im\mathrm{m}\,\omega^2 \neq 0$, then conjugate frequency $\overline{\omega}^2$ is part of the oscillation spectrum too, and "mode" $\overline{\boldsymbol{U}}$ for this frequency has conjugate components. Using then the reciprocal work theorem (5.3, § 4.5) for $\boldsymbol{U}$ and $\overline{\boldsymbol{U}}$, we have

$$\omega^2 \int\limits_{\mathcal{V}} \rho \boldsymbol{U} \cdot \overline{\boldsymbol{U}} d\mathcal{V} = \overline{\omega}^2 \int\limits_{\mathcal{V}} \rho \overline{\boldsymbol{U}} \cdot \boldsymbol{U} d\mathcal{V} \;\Rightarrow\; \omega^2 = \overline{\omega}^2 \;\Rightarrow\; \Im\mathrm{m}\,\omega^2 = 0.$$

...

However, a bright picture with decomposition by modes is of little use for practical calculations of oscillations (vibrations) of a three-dimensional elastic body. The reason is density of the spectrum, driven oscillations excite many modes. When the natural frequency density is high, even a small friction qualitatively changes the resonance curve. Damping (decrease in amplitudes) in real bodies is also important. In addition, wave nature of non-stationary processes hinders to just transfer the theory of oscillations of discrete systems to the continuum: in case of sudden local excitation it's more correct to consider waves instead of superposing modes.

The way from a continuous dynamic model to a discrete one goes through the variational approach.

$$\int\limits_{\mathcal{V}} \Big((\boldsymbol{f} - \rho \ddot{\boldsymbol{u}}) \cdot \delta \boldsymbol{u} - \boldsymbol{\sigma} \cdot\cdot\, \delta \boldsymbol{\varepsilon} \Big) d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p} \cdot \delta \boldsymbol{u}\, do = 0. \qquad (1.4)$$

This is the principle of virtual work with forces of inertia. Looking for an approximate solution in series

$$\boldsymbol{u}(\boldsymbol{r}, t) = \sum_{k=1}^{N} \alpha_k(t) \varphi_k(\boldsymbol{r}),$$

where φ_k are given $\Big(\varphi_k\big|_{o_1} = \boldsymbol{0}\Big)$, and $\alpha_k(t)$ are varying. The solution???? is a/?????the system of ordinary equations

....

In place of the principle of virtual work (1.4), the Hellinger–Reissner's mixed formulation with an independent approximation of stresses can be applied.

In the dynamic theory of elasticity, the Laplace integral transform is often applied. For simple shaped bodies it is sometimes possible to find an analytical solution in transforms. The original can be found by the numerical inversion, but sometimes it's possible to take the Riemann–Mellin integral* using the "saddle-point method" (or the "method of steepest descents") with a deformation of contour in the complex plane [46, 68].

§ 2. Vibrations of rods

In the linear dynamics of rods we have the following system for forces $\boldsymbol{Q}$, force couples (moments) $\boldsymbol{M}$, displacements $\boldsymbol{u}$ and rotations $\boldsymbol{\theta}$ (§ 9.10):

$$\boldsymbol{Q}' + \boldsymbol{q} = \rho\big(\ddot{\boldsymbol{u}} + \ddot{\boldsymbol{\theta}} \times \boldsymbol{\varepsilon}\big), \;\; \boldsymbol{M}' + \boldsymbol{r}' \times \boldsymbol{Q} + \boldsymbol{m} = \boldsymbol{J} \cdot \ddot{\boldsymbol{\theta}} + \rho\boldsymbol{\varepsilon} \times \ddot{\boldsymbol{u}}, \tag{2.1}$$

$$\ldots \tag{2.2}$$

...

§ 3. Small perturbations of parameters

With small perturbations of mass and rigidity, рассмотрим задачу об определении собственных частот и форм :

$$\begin{gathered} \big(C_{ij} - \omega^2 A_{ij}\big) U_j = 0, \\ C_{ij} = C_{ij}^{(0)} + \chi C_{ij}^{(1)}, \;\; A_{ij} = A_{ij}^{(0)} + \chi A_{ij}^{(1)}, \;\; \chi \to 0. \end{gathered} \tag{3.1}$$

Находя́ решение в виде

$$\omega = \omega^{(0)} + \chi\omega^{(1)} + \ldots, \;\; U_j = U_j^{(0)} + \chi U_j^{(1)} + \ldots,$$

получаем последовательность задач

* The Riemann–Mellin integral does the inverse Laplace transform $F(s) \mapsto f(t)$.

...

§ 4. Vibrations of shells

Динамика оболочек рассматривалась многими

...

§ 5. Waves in an elastic continuum

Рассмотрим линейные уравнения динамики однородной изотропной среды без объёмных сил

...

§ 6. Waves in a rod

Рассмотрим прямой стержень. Продольная деформация описывается уравнениями

...

§ 7. Nonlinear oscillations

Рассмотрим простой пример: продольные колебания прямого стержня с ма́лой нелинейной добавкой в соотношениях упругости

...

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chapter 13

STABILITY

§ 1. Various approaches to the problem of stability

There's the well developed theory of stability by А. Ляпунов (Aleksandr Lyapunov). It looks at the dynamics of small deviations and says that if initial deviations (for example, from the equilibrium) were "quite close (small)", and they don't rise any further, remaining "quite close" forever, then the process is stable. This is the **dynamic approach** to the problem of stability, and this approach is the most reasonable.

However, for problems about the stability of the equilibrium, the other approach has found much popularity. It is called the **static approach**, and is widely known under the Leonhard Euler's name. When the equations of statics give a nontrivial solution for small disturbances-displacements, then the values of parameters are assumed to be "critical". In other words, if there's a non-isolated equilibrium*, then it's considered "critical". With this approach it's enough to solve the eigenvalue problem.

More approaches exist as well. One of them is the **method of imperfections** : if small random changes of the initial shape, the stiffnesses, the loads and other variables cause only a small change of a deformed configuration, then this equilibrium is stable. Or the **energy approach** : when a loss of stability becomes energetically beneficial, that is when it leads to a decrease in energy, then such a stability loss really happens.

* "Non-isolated" means that at the same time there appear many possible adjacent forms of the same one equilibrium.

The mentioned approaches draw a motley picture. Yet it is pretty easy to visualize for a model with a finite number of degrees o'freedom.

Pretty abstract and universal, the Lagrange's equations

$$A_{jk}\,\ddot{q}_k = Q_j\big(q_k, p\big). \tag{1.1}$$

Here q_k — generalized coordinates, Q_j — generalized forces, ...

...

.....

§ 2. Classical problems with rods

Состояние перед варьированием описывается уравнениями нелинейной теории стержней Kirchhoff'a

...

§ 3. "Tracking" loads

В проблемах устойчивости весьма весóмо поведение нагрузки в процессе деформирования. Ведь в уравнения входит вариация (of what?) $\delta\boldsymbol{q}$, она равна нулю лишь для "dead" loads. Распространены the "tracking" loads, which change in a certain way along with the deviations (displacements) of particles.

The Euler's static approach to the problem of stability

...

§ 4. The role of additional pliabilities

Для прямого консольного стержня, сжатого постоянной силой $\boldsymbol{F}$ на свободном конце, критическая нагрузка определяется формулой Euler'a

...

§ 5. Variational formulations

Во всех разделах линейной теории упругости большу́ю роль играют вариационные постановки. Среди прочего, они составляют основу метода конечных элементов как варианта метода Ritz'a.

Менее развиты вариационные постановки для проблем устойчивости. Здесь получил популярность метод

...

§ 6. Nonconservative problems

В уравнении динамики (...) матрица позиционных сил

...

§ 7. Case of multiple roots

Вернёмся к проблеме устойчивости (...) в случае циркуляционных сил. Как уже отмечалось(где??), критическая ситуация характеризуется

...

Bibliography

Увлекательные вопросы устойчивости упругих систем освещены в книгах ...

chapter 14

DEFECTS

§ 1. Volterra dislocations

Consider the classical linear three-dimensional elastic medium (chapter 4). As shown in § 4.8, the equation of compatibility of deformations

...

§ 2. Straight-line dislocations

A dislocation line may be any curve, closed inside a body or with ends on the surface. For a dislocation of a random shape in a limitless medium, it's not so difficult to obtain an appropriate solution [18]. Мы же ограничимся простейшим случаем прямолинейной дислокации. Ищется решение

...

§ 3. Action of stress field on dislocation

Рассмотрим среду, содержащую внутри себя дислокацию с замкнутой линией C. Тело нагружено объёмными $\boldsymbol{v}$ и поверхностными $\boldsymbol{p}$ силами. Обозначим

...

§ 4. About movement of dislocations

Рассмотрим это явление, следуя [24]. Ограничимся случаем прямолинейной винтовой дислокации, движущейся с постоянной скоростью

...

§ 5. Point defects

Речь пойдёт о континуальной модели таких явлений как вакансии, примесные частицы или междоузельные атомы в кристаллической решётке. В случае дислокации рассматривались

...

§ 6. Force acting on a point defect

Дефект находится в континууме, нагруженном объёмными $\boldsymbol{v}$ и поверхностными $\boldsymbol{p}$ силами. Суперпозиция

...

§ 7. Continuously distributed dislocations

Начнём со сложения векторов Бюргерса. При обходе сразу двух дислокаций (рис. ?? 40 ??) по контуру

...

§ 8. Stress during winding of coil

Не только дислокации и точечные дефекты, но и макроскопические факторы могут быть источниками собственных напряжений. При намотке катушки (рис. ?? 42 ??) в ней возникают напряжения от натяжения ленты. Расчёт этих напряжений очень сложен, если рассматривать детально процесс укладки ленты.

Но существует чёткий алгоритм Southwell'a [50] расчёта напряжений в катушке: укладка каждого нового витка вызывает внутри катушки приращения напряжений, определяемые соотношениями линейной упругости. There are two stages, and the first one consists in solving the Lamé problem about deformation of a hollow cylinder under external pressure (рис. ?? 43 ??)

...

Bibliography

Dislocations and point defects in linear elastic bodies have been considered by many authors: John Eshelby [18], Roland deWit [14], Cristian Teodosiu [51], Alan Cottrell [24]. The theory of eigen-stresses (eigen-spannungen) is explained by Ekkehart Kröner in [25]. Calculation of stresses when winding a coil is described by Richard Southwell in his book [50].

chapter 15

FRACTURES

§ 1. Traditional criteria of toughness

After determining the stresses in the body, how to estimate the toughness? When a sample of material is uniaxially extended during a test, there obviously is some limit — the "tensile strength" σ_*, above which the material is destroyed. So for uniaxial (ex)tension with stress σ, the toughness is thought to be sufficient if $\sigma \le \sigma_*/k$, where k is the so-called safety factor (coefficient of safety). But this approach isn't very convincing, because the values of σ_* obtained from uniaxial tension tests have a big scatter, and the choice of safety factor is sometimes tied to bureaucracy. However, such views on analysis of toughness are quite widespread. Without criticizing, I will mention the most popular of them.

Maximum principal stress criterion : a destruction occurs when the biggest eigenvalue ("principal") stress σ_1 reaches the limit, $\sigma_1 = \sigma_*$. But in the case of uniaxial compression with $\sigma_1 = 0$ this hypothesis is false.

Критерий максимального касательного напряжения (Tresca criterion): destruction happens if $\sigma_1 - \sigma_3 = \sigma_*$ (σ_3 — наименьшее из собственных напряжений). Это более соответствует началу пластического течения.

Критерий максимального удлинения : наибольшее из собственных значений тензора деформации $\varepsilon_1 = \varepsilon_*$. Это приемлемо и для сжатия с $\varepsilon_1 > 0$.

Критерий энергии деформации : $\Pi = \Pi_*$. Здесь учитывается, что разрушение требует энергии, а источником её может быть лишь само́ деформированное тело. Однако достаточный запас энергии — необходимое, но не единственное условие разрушения; должен

включиться некий механизм преобразования упругой энергии в работу разрушения.

Критерий энергии формоизменения (von Mises yield criterion) : $\boldsymbol{s} \bullet\bullet \boldsymbol{s} = 2\tau_*^2$, $\boldsymbol{s} \equiv \boldsymbol{\tau} - \frac{1}{3}\boldsymbol{E}\,\mathrm{trace}\,\boldsymbol{\tau}$ ("девиатор напряжений"). Здесь не играет роли энергия объёмной деформации. Richard von Mises предложил* этот критерий как гладкую аппроксимацию условия Henri Tresca.

Критерий Mohr'a. Представим себе множество предельных состояний ...

...

§ 2. Antiplane deformation of continuum with a crack

Любая регулярная функция компле́ксного переменного $z = x + iy$ содержит в себе решение какой-либо антиплоской задачи статики без ...

...

§ 3. Crack in plane deformation

Рассмотрим плоскую область произвольного очертания с трещиной внутри; нагрузка приложена и "в объёме", и на внешнем крае. Как и при антиплоской деформации, решение строится в два этапа

...

§ 4. Crack-driving force

Это едва ли не основное понятие механики трещин. Рассмотрим его, следуя

* **R. von Mises**. Mechanik der festen Körper im plastisch-deformablen Zustand. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse.* 1913, Seiten 582–592.

...

§ 5. Criterion of crack growth

Связанная с энергией $\mathscr{E}$ трещинодвижущая сила F — не единственное воздействие на передний край трещины. Должна быть ещё некая сила сопротивления F_*; рост трещины начинается, когда

...

§ 6. J-integral

Одно из самых известных понятий в механике трещин выражается интегралом

$$J = \dots \tag{6.1}$$

...

§ 7. Stress intensity factors

Расчёт прочности тела с трещиной сводится к определению коэффициентов интенсивности напряжений. Методы расчёта таких коэффициентов — как аналитические, так и численные — хорошо освещены в литературе.

Рассмотрим ещё один подход к задачам механики трещин, разработанный

...

§ 8. Barenblatt's model

An unlimited increase of stress at the edge of a crack seems indeed dubious. Singular solutions desire support by some additional reasonings or by using of another model. And such support was given in the work

...

§ 9. Deformational criterion

D. S. Dugdale*, а также М. Я. Леонов и В. В. Панасюк** предложили модель, напоминающую построения Баренблатта. Также есть силы сцепления q и равен нулю итоговый коэффициент интенсивности напряжений. Но, во-первых, q имеет иной вид:

...

Второе отличие рассматриваемой модели — в формулировке критерия прочности: трещина начинает расти, когда расхождение берегов в конце свободного участка достигает критического значения δ_* (этот параметр — константа материала), то есть при

...

§ 10. Growth of cracks

Пусть нагрузка на тело с трещиной выросла настолько, что выполняется условие

...

§ 11. Elastic field ahead of a moving crack

Considering this topic

...

* **Dugdale, D. S.** Yielding of steel sheets containing slits. *Journal of the Mechanics and Physics of Solids.* 1960, Volume 8, Issue 2, pages 100–104.

** **Леонов М. Я.**, **Панасюк В. В.** Развитие мельчайших трещин в твёрдом теле. *Прикладная механика.* 1959, Т. 5, № 4, с. 391–401.

§ 12. Balance of energy for a moving crack

The balance of energy equation in the linear theory ($\Pi = \frac{1}{2}\,\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}$, $\mathrm{K} = \frac{1}{2}\,\rho\,\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}$):

$$\int\limits_{\mathcal{V}} \Big(\mathrm{K} + \Pi\Big)^{\bullet} d\mathcal{V} = \int\limits_{\mathcal{V}} \boldsymbol{f} \cdot \dot{\boldsymbol{u}}\, d\mathcal{V} + \int\limits_{\mathcal{O}} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\, d\mathcal{O}. \tag{12.1}$$

...

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Список книг по механике трещин уже́ вели́к. В нём сто́ит отметить работы Л. М. Качанова [70], Н. Ф. Морозова [72], В. З. Партона и Е. М. Морозова [74], Г. П. Черепанова [76]. Обзор статей есть у ... Экспериментальные данные представлены, например, в [71].

chapter 16

COMPOSITES

Anisotropic materials are either natural (which appeared without human participation) — crystals, or human-made artificial — composites (composite materials, structurally anisotropic materials).

§ 1. Introductory thoughts

When the clay is used as a building material, a shredded straw is added to it. Working with epoxy gum, pretty practical is to blend it (before solidifying) with a filler, which can be a powder, fibers, pieces o' fabric. These were examples of composites (composite materials, composite mixtures). The new types of composites are used more and more widely, replacing the steel, the aluminum alloys and other popular homogeneous materials.

The composites can be defined as the micro-inhomogeneous materials, where happens some averaging together with the new properties. The "usual" mechanics of continua is applicable, surely, to the composite materials too. But it's barely possible and is absurd to model the every aspect of a composite material. The new approach is needed that will deal with the complexity of material structure. As for a gas, when instead of describing the dynamics of individual molecules we introduce the parameters, such as "the pressure", "the temperature" and others.

The mechanics of a homogeneous non-composite continuum considers just the single "macroscopic" length ℓ, assuming that any small volumes have the same properties as finite volumes. The characteristic size there is the volume divided by the surface area.

For a composite it's different, here we have the three scopes of length : $\ell \gg \ell' \gg \ell_0$. The largest ℓ presents the macroscopic dimensions of a body. The smallest ℓ_0 is near to the size of elements of the material structure, for example particles of a filler powder.

The intermediate scope ℓ' shows the size of a so-called "representative" volume, of "the unit cell" — that small enough volume, where the specific properties of this composite become sensible.

In composites, the complex problem with an inhomogeneous body is split into the two : for a body as a whole (the macrolevel) and for a "representative" volume (the microlevel). At the macrolevel (scope ℓ), a composite is modeled as a homogeneous medium with "the effective properties", and the "representative" volumes play there the role of particles. At the microlevel (scope ℓ_0) fields are very inhomogeneous, but some averaging by the "representative" volume leads to the macrolevel. The complex problem for a composite as an inhomogeneous body splits into the two : for a representative volume (the microlevel) and for a body as a whole (the macrolevel).

These (somewhat blurry) thoughts, however, may seem not very convincing. To convince more, there's the "random field theory" : elastic moduli of the original inhomogeneous medium are stationary random* functions ${}^4\boldsymbol{\mathcal{A}}(\boldsymbol{r})$ ....

.....

The mechanics of composites appeared not very long ago, but it is developing pretty intensively. Due to the high crack resistance of composite materials, the most widely researched topic is the fracture (destruction) mechanics of composites.

§ 2. Effective fields

Any field u in a composite is usually represented as the sum $u = u^* + u'$, where u^* is some smoothed "effective" field, and u' is a fast oscillating fluctuation. It's often assumed

$$u^*(A) = \langle u \rangle \equiv \mathcal{V}^{-1} \int\limits_{\mathcal{V}} u \, d\mathcal{V}, \tag{2.1}$$

* A stationary random function is a random function that doesn't change its statistical properties over time.

where $\langle u \rangle$ is an average within a representative volume centered at point A. Averaging (2.1)

...

§ 3. Boundary value problems for a representative volume

How the elastic moduli are determined for a homogeneous medium? Without the real possibility to get a relation between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ for a point of infinitesimal size, experiments with finite volume bodies are carried out under a so-called "homogeneous stress" — when the stress is the same at any point of a body. In composites, the role of a point plays a "representative" volume.

....

§ 4. Hill's fork

Using Voigt and Reuss theories, Hill derived upper and lower bounds on the effective properties of a composite material [**Hill, R. W.** The elastic behaviour of a crystalline aggregate. *Proceedings of the Physical Society*, Section A, Volume 65, Issue 5 (May 1952). Pages 349–354.]

The scale separation is motivated by the material properties, at both scales continuum mechanics models the underlying system. Such an approach uses energy equivalence at both scales as proposed in Hill (1972).

Hill R (1972) On constitutive macro-variables for heterogeneous solids at finite strain (pages 131–147)

For a composite material, at least two different materials with known material models and parameters, generate a homogenized material modeled with a predetermined constitutive equation. Determination of material parameters of the homogenized material is a challenging task.

Отметив, что

...

Итак, эффективные модули определяются энергией представительного объёма в первой или второй задачах:

...

§ 6. Effective moduli for material with spherical inclusions

В однородной матрице случайным образом, но достаточно равномерно, распределены сферические включения радиусом a. Получившийся композит на макроуровне будет изотропным, его упругие свойства полностью определяются ...

...

§ 7. Self-consistent method

Выше мы опирались на две задачи for a representative volume и определяли effective modules from the equality of energies. В основе метода самосогласования лежит новая идея: представительный объём помещается в безграничную среду с эффективными свойствами, на бесконечности состояние считается однородным, эффективные модули находятся из некоторых дополнительных условий самосогласования.

Обратимся снова к вопросу об объёмном модуле среды со сферическими включениями. Задача сферически симметрична; для включения по-прежнему

...

§ 8. Hashin–Shtrikman principle

Hashin and Shtrikman derived upper and lower bounds for the effective elastic properties of quasi-isotropic and quasi-homogeneous multiphase materials using a variational approach [**Hashin, Z.**; **Shtrikman, S.** A variational approach to the theory of the elastic behaviour of multiphase materials.

// Journal of the Mechanics and Physics of Solids. Volume 11, Issue 2 (March–April 1963). Pages 127–140.]

Hashin Z., Shtrikman S. (1962) On some variational principles in anisotropic and nonhomogeneous elasticity. Journal of the Mechanics and Physics of Solids 10(4): pages 335–342

Вилка Hill'а основана на обычных экстремальных принципах теории упругости. Специально для механики композитов Hashin и Shtrikman построили очень своеобразный функционал, который на некотором точном решении может иметь как максимум, так и минимум, давая возможность с двух сторон оценивать эффективные модули [83].

Рассмотрим первую из двух задач для представительного объёма

....

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The books by R. Christensen [77] and Б. Победря (B. Pobedrya) [79] contain both the fundamentals of the mechanics of composites, and the problems that remain relevant. The most demanding reader will appreciate the monograph by Т. Шермергор (T. Shermergor) [83]. Among all the variety of publications on the fracture (destruction) mechanics of composites, it's worth highlighting the book by Г. Черепанов (G. Cherepanov) [82].

chapter 17

PERIODIC COMPOSITES

A periodic composite is made of a repetition, finite or infinite, of a unit cell.

§ 1. One-dimensional problem

In one-dimensional problem of statics, the equation
...

§ 2. Three-dimensional continuum

From the equations in displacements (...),

....

§ 3. Fibrous structure

In this case tensor ${}^4\boldsymbol{\mathcal{A}}$ is constant along the axis

...

§ 4. Statics of a periodic rod

In the equations of linear statics of a rod (...)

.....

Bibliography

The asymptotic method, underlying this chapter, is presented with varying degrees of mathematical scrupulousness in the books [80, 81, 78, 79].

chapter 18

OUT OF ELASTICITY, OR PLASTICITY

The opposite of elasticity is plasticity, that is when something is deformed then it stays deformed after unstressing.

...

§ 1. When elasticity is over

When elasticity is over .....

....

§ 2. Where goes the energy

Mechanical energy is lost whenever an object undergoes plastic deformation.

The energy goes into changing the shape, and after all it stays changed.

....

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About some publications on plasticity .....

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