Vadique Myself

PHYSICS *of* ELASTIC CONTINUA

CONTENTS

chapter 2

CLASSICAL AND ANALYTICAL MECHANICS

When relativistic mechanics (for the very fast) and quantum mechanics (for the very small) emerged at the beginning of the XX$^{\text{th}}$ century, the equations of mechanics existed prior to that, still perfectly suitable for describing objects of everyday sizes and speeds, needed a new name. The "classical" was chosen back then to refer to the equations, describing reality without any quantum and relativistic effects influencing it.

§1. Discrete collection of particles

Classical mechanics models physical objects by discretizing them into a collection of particles ("pointlike masses", "material points"*).

In a collection of N particles, each k-th particle has its nonzero mass $m_k = \mathsf{constant} > 0$ and the motion function $\boldsymbol{r}_k(t)$. The function $\boldsymbol{r}_k(t)$ is measured relative to the chosen reference system.

The "reference system" (or "reference frame") consists of (figure 1)

- ✓ some "null" reference point o,
- ✓ a set of coordinates, which give the units of spatial measurements,
- ✓ a clock.

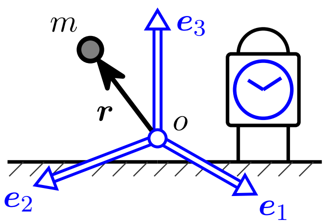

figure 1

Long time ago, the reference system was some "absolute space", empty at first, and then filled with the continuous elastic medium — the æther. Later, it

* The point mass (pointlike mass, material point) is the concept of an object, typically matter, that has the nonzero mass and is (or is being thought of as) infinitesimal in its volume (dimensions).

became clear that any frame of reference can be used for classical mechanics, but the preference is given to the so called "inertial" frames, where a particle in the absence of external interactions (or applied forces) moves "in free motion" — along a straight line with a constant velocity ($\dot{\boldsymbol{r}} = \mathbf{constant}$), thence without acceleration ($\ddot{\boldsymbol{r}} = \mathbf{0}$)

$$\dot{\boldsymbol{r}} = \mathbf{constant} = \dot{x}_i \boldsymbol{e}_i \;\Rightarrow\; \dot{x}_i = \text{constant} \;\Leftarrow\; \boldsymbol{e}_i = \mathbf{constant}$$

The measure of interaction in mechanics is the vector of force $\boldsymbol{F}$. In the widely known* Newton's equation

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \tag{1.1}$$

the force $\boldsymbol{F}$ can depend only on position, velocity and explicitly on time, whereas acceleration $\ddot{\boldsymbol{r}}$ is directly proportional to force $\boldsymbol{F}$ with coefficient $1/m$.

Here're theses of the dynamics of a collection of particles.

The force $\boldsymbol{F}_k$, acting on the k-th particle (figure 2)

$$m_k \ddot{\boldsymbol{r}}_k = \boldsymbol{F}_k,$$
$$\boldsymbol{F}_k = \boldsymbol{F}_k^{(e)} + \sum_j \boldsymbol{F}_{kj}^{(i)}. \tag{1.2}$$

$\boldsymbol{F}_k^{(e)}$ is the external force — such forces emanate from objects outside the system being considered. The second addend is the sum of internal forces (force $\boldsymbol{F}_{kj}^{(i)}$ is the interaction induced by the j-th particle on the k-th particle). Internal interactions happen only between

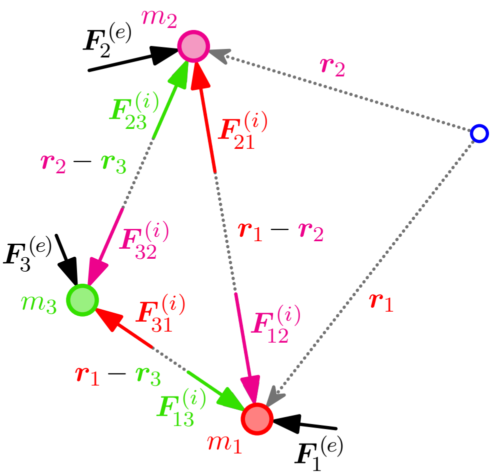

figure 2

* "Axiomata sive Leges Motus" ("Axioms or Laws of Motion") were written by Isaac Newton in his Philosophiæ Naturalis Principia Mathematica, first published in 1687. Reprint (en Latin), 1871. Translated into English by Andrew Motte, 1846.

elements of the system and don't affect (mechanically) anything other. Neither particle interacts with itself, $\boldsymbol{F}_{kk}^{(i)} = \mathbf{0}\ \forall k$.

.... Rene Descartes' mechanics ....

Measuring motions in mechanics is, however, more controversial than measuring interactions. The discord and the extensive polemic on this topic dates back to the times of Newton and Leibniz. In those days, exploring how objects of various masses change the speed and velocity* of their motion when various forces are applied to it, both Newton and Leibnitz were looking for a useful invariant that would fit the observations.

- " *mv, the product of mass and velocity, is a useful quantity that is conserved* " thought Newton.
- " *mv^2, the product of mass and velocity squared, is a useful quantity that is conserved* " thought Leibniz.

And each of them believed that the quantity he proposed is more useful, more fundamental and more "fruitful".

Newton named mv as "quantitas motus" ("quantity of motion")

momentum is a measure of mechanical motion of an object

momentum depends on the weight (i.e. quantity) and velocity of an object.

Momentum is the product of mass and velocity, so

when either an object's mass or its velocity changes , then the momentum will change

....

what is momentum? The measure of movement in mechanics is called "momentum"

BUT WHY?

why "mass by velocity" measures the amount of movement ? there are two momenta known, the linear (translational) one and the angular (rotational) one, why?

* *Speed* is the time rate of motion, that is *how fast* a thing moves along some path, a scalar. *Velocity* is the movement's rate and direction, that's *how fast* ***and where*** a thing moves, a vector.

Why is momentum defined as mass times velocity?

https://physics.stackexchange.com/a/577486/377185

Newton thought mv was a useful conserved quantity. Leibniz thought mv^2 was a useful conserved quantity.

If you read the history, you'll find there was much discussion, rivalry, and even bad blood as each pushed the benefits of their particular view. Each thought that their quantity was more fundamental, or more important.

Now, we see that both are useful, just in different contexts.

I'm sure somebody briefly toyed with the expressions like mv^3 and maybe m^2v before quickly finding that they didn't stay constant under any reasonable set of constraints, so had no predictive power. That's why they're not named, or used for anything.

So why has the quantity mv been given a name? Because it's useful, it's conserved, and it allows us to make predictions about some parameters of a mechanical system as it undergoes interactions with other things.

from *Leibniz and the Vis Viva Controversy by Carolyn Iltis (1971)*

Roger Boscovich in 1745 and Jean d'Alembert in 1758 both pointed out that vis viva mv^2 and momentum mv were equally valid.

The momentum of a body is actually the Newtonian force F acting through a time, since $v = at$ and $mv = mat = Ft$.

The kinetic energy is the Newtonian force acting over a space, since $v^2 = 2as$ and $mv^2 = 2mas$ or $\frac{1}{2}mv^2 = Fs$.

The amount of movement of some object is the product of the mass and velocity of that object.

When two objects collide, ...

the (linear, translational) momentum

$$\begin{aligned} &m_k \dot{\boldsymbol{r}}_k \quad \text{— for the } k\text{-th particle}, \\ &\sum_k m_k \dot{\boldsymbol{r}}_k \quad \text{— for the whole discrete system} \end{aligned} \tag{1.3}$$

and the angular (rotational) momentum

$$\begin{aligned} &\boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \quad \text{— for the } k\text{-th particle}, \\ &\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \quad \text{— for the whole discrete system}. \end{aligned} \tag{1.4}$$

From (1.2) together with the action–reaction principle

$$\boldsymbol{F}_{kj}^{(i)} = -\boldsymbol{F}_{jk}^{(i)} \;\forall k,j \;\;\Rightarrow\;\; \sum_{k,j} \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0} \tag{1.5}$$

ensues the balance of linear momentum

$$\Bigl(\sum_k m_k \dot{\boldsymbol{r}}_k\Bigr)^{\bullet} = \sum_k m_k \ddot{\boldsymbol{r}}_k = \sum_k \boldsymbol{F}_k^{(e)}. \tag{1.6}$$

And here's the balance of angular momentum*

$$\Bigl(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k\Bigr)^{\bullet} = \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k \tag{1.7}$$

— is the sum $\sum \boldsymbol{M}_k$ of moments. The moment $\boldsymbol{M}_k$, acting on the k-th particle

$$\boldsymbol{M}_k = \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k^{(e)} + \boldsymbol{r}_k \times \sum_j \boldsymbol{F}_{kj}^{(i)}. \tag{1.8}$$

When in addition to the action–reaction principle, all internal interactions between particles are assumed to be central, that is

$$\boldsymbol{F}_{kj}^{(i)} \parallel \bigl(\boldsymbol{r}_k - \boldsymbol{r}_j\bigr) \;\;\Leftrightarrow\;\; \bigl(\boldsymbol{r}_k - \boldsymbol{r}_j\bigr) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0}, \tag{1.9}$$

the balance of rotational (angular) momentum becomes**

$$\Bigl(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k\Bigr)^{\bullet} = \sum_k \boldsymbol{r}_k \times \boldsymbol{F}_k^{(e)}. \tag{1.10}$$

Thus, all changes in the linear and angular momenta are due only to external forces $\boldsymbol{F}_k^{(e)}$, not internal ones.

* $$\Bigl(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k\Bigr)^{\bullet} = \sum_k \dot{\boldsymbol{r}}_k \times m_k \dot{\boldsymbol{r}}_k + \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k = \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k$$

$$\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0} \;\; \forall \boldsymbol{a} \;\;\Rightarrow\;\; \dot{\boldsymbol{r}}_k \times \dot{\boldsymbol{r}}_k = \boldsymbol{0}$$

** $$\forall k,j \;\; \boldsymbol{F}_{kj}^{(i)} = -\boldsymbol{F}_{jk}^{(i)} \;\text{ and }\; \bigl(\boldsymbol{r}_k - \boldsymbol{r}_j\bigr) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0} \;\;\Rightarrow$$

$$\sum_k \boldsymbol{r}_k \times \sum_j \boldsymbol{F}_{kj}^{(i)} = \tfrac{1}{2} \sum_{k,j} \bigl(\boldsymbol{r}_k + \boldsymbol{r}_k\bigr) \times \boldsymbol{F}_{kj}^{(i)} = \tfrac{1}{2} \sum_{k,j} \bigl(\boldsymbol{r}_k - \boldsymbol{r}_j\bigr) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0}$$

$$\sum_{k,j} \boldsymbol{r}_k \times \boldsymbol{F}_{kj}^{(i)} = -\sum_{k,j} \boldsymbol{r}_k \times \boldsymbol{F}_{jk}^{(i)} = -\sum_{j,k} \boldsymbol{r}_j \times \boldsymbol{F}_{kj}^{(i)} = -\sum_{k,j} \boldsymbol{r}_j \times \boldsymbol{F}_{kj}^{(i)}$$

Unlike for momenta, the balance of kinetic energy $\mathrm{K} \equiv {}^1\!/_2 \sum m_k \dot{\boldsymbol{r}}_k \cdot \dot{\boldsymbol{r}}_k$ (mv^2 is Leibniz's “vis viva”) includes the power of internal forces as well

$$\begin{aligned}\dot{\mathrm{K}} = \Big(\frac{1}{2}\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \dot{\boldsymbol{r}}_k\Big)^{\bullet} &= \frac{1}{2}\sum_k \big(m_k \ddot{\boldsymbol{r}}_k \cdot \dot{\boldsymbol{r}}_k + m_k \dot{\boldsymbol{r}}_k \cdot \ddot{\boldsymbol{r}}_k\big) \\ &= \sum_k m_k \ddot{\boldsymbol{r}}_k \cdot \dot{\boldsymbol{r}}_k = \sum_k \boldsymbol{F}_k \cdot \dot{\boldsymbol{r}}_k = \sum_k \Big(\boldsymbol{F}_k^{(e)} + \sum_j \boldsymbol{F}_{kj}^{(i)}\Big) \cdot \dot{\boldsymbol{r}}_k \\ &= \sum_k \boldsymbol{F}_k^{(e)} \cdot \dot{\boldsymbol{r}}_k + \sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \dot{\boldsymbol{r}}_k \end{aligned} \qquad (1.11)$$

or, using the action–reaction principle (1.5),

$$\dot{\mathrm{K}} - \sum_k \boldsymbol{F}_k^{(e)} \cdot \dot{\boldsymbol{r}}_k = \frac{1}{2}\sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \big(\dot{\boldsymbol{r}}_k + \dot{\boldsymbol{r}}_k\big) = \frac{1}{2}\sum_{k,j} \Big(\boldsymbol{F}_{kj}^{(i)} \cdot \dot{\boldsymbol{r}}_k - \boldsymbol{F}_{jk}^{(i)} \cdot \dot{\boldsymbol{r}}_k\Big),$$

and since $\sum_{k,j} \boldsymbol{F}_{jk}^{(i)} \cdot \dot{\boldsymbol{r}}_k = \sum_{j,k} \boldsymbol{F}_{kj}^{(i)} \cdot \dot{\boldsymbol{r}}_j = \sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \dot{\boldsymbol{r}}_j$

$$\dot{\mathrm{K}} = \sum_k \boldsymbol{F}_k^{(e)} \cdot \dot{\boldsymbol{r}}_k + \frac{1}{2}\sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \big(\dot{\boldsymbol{r}}_k - \dot{\boldsymbol{r}}_j\big). \qquad (1.12)$$

....

all bodies that are limited in free motion possess potential energy

.....

Work

$$W(\boldsymbol{F}, \boldsymbol{u}) = \boldsymbol{F} \cdot \boldsymbol{u}$$

as the exact (full) differential

$$dW = \frac{\partial W}{\partial \boldsymbol{F}} \cdot d\boldsymbol{F} + \frac{\partial W}{\partial \boldsymbol{u}} \cdot d\boldsymbol{u}$$

by “product rule”

$$dW = d\big(\boldsymbol{F} \cdot \boldsymbol{u}\big) = d\boldsymbol{F} \cdot \boldsymbol{u} + \boldsymbol{F} \cdot d\boldsymbol{u}$$

$$\frac{\partial W}{\partial \boldsymbol{F}} = \boldsymbol{u},\ \frac{\partial W}{\partial \boldsymbol{u}} = \boldsymbol{F}$$

...

Center of mass

✓ This is the unique point at any time where the weighted relative position of the distributed mass sums to zero.

✓ This is the point to which a force may be applied to cause a linear acceleration without an angular acceleration.

✓ This is a hypothetical point where the entire mass of an object may be assumed to be concentrated to visualise its motion.

✓ This is the particle equivalent of a given object for application of Newton's laws of motion.

When formulated for the center of mass, formulas in mechanics are often simplified.

...

— Are there any scenarios for which the center of mass is not almost exactly equivalent to the center of gravity?

— Non-uniform gravity field. In a uniform gravitational field, the center of mass is equal to the center of gravity.

...

Constraints

Imposed on the positions and velocities of particles, there are restrictions of a geometrical or kinematical nature, called constraints.

Holonomic constraints are relations between position variables (and possibly time) which can be expressed as equality like

$$f(q^1, q^2, q^3, \ldots, q^n, t) = 0,$$

where $q^1, q^2, q^3, \ldots, q^n$ are n parameters (coordinates) that fully describe the system.

A constraint that cannot be expressed as such is nonholonomic.

Holonomic constraint depends only on coordinates and time. It does not depend on velocities or any higher time derivatives.

Velocity-dependent constraints like

$$f(q^1, q^2, \ldots, q^n, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n, t) = 0$$

are mostly not holonomic.

For example, the motion of a particle constrained to lie on a sphere's surface is subject to a holonomic constraint, but if the particle is able to fall off a sphere under the influence of gravity, the constraint becomes non-holonomic. For the first case the holonomic constraint may be given by the equation: $r^2 - a^2 = 0$, where r is the distance from the centre of a sphere of radius a. Whereas the second non-holonomic case may be given by: $r^2 - a^2 \geq 0$.

Three examples of nonholonomic constraints are: when the constraint equations are nonintegrable, when the constraints have inequalities, or with complicated non-conservative forces like friction.

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q^1, q^2, \ldots, q^n, t)$$

(assuming n independent parameters/coordinates)

.....

§ 2. The principle of virtual work

Mécanique analytique (1788–89) is a two volume French treatise on analytical mechanics, written by Joseph Louis Lagrange, and published 101 years following Isaac Newton's *Philosophiæ Naturalis Principia Mathematica*.

Joseph Louis Lagrange. Mécanique analytique. Nouvelle édition, revue et augmentée par l'auteur. Tome premier. Mme Ve Courcier, Paris, 1811. 490 pages.

Joseph Louis Lagrange. Mécanique analytique. Troisième édition, revue, corrigée et annotée par M. J. Bertrand. Tome second. Mallet-Bachelier, Paris, 1855. 416 pages.

The historical transition from geometrical methods, as presented in Newton's Principia, to methods of mathematical analysis.

Consider the exact differential of any set of location vectors $\boldsymbol{r}_i$, that are functions of other variable parameters (coordinates) $q^1, q^2, ..., q^n$ and time t.

The actual displacement is the differential

$$d\boldsymbol{r}_i = \frac{\partial \boldsymbol{r}_i}{\partial t} dt + \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q^j} dq^j$$

Now, imagine an arbitrary path through the configuration space/manifold. This means it has to satisfy the constraints of the system but not the actual applied forces

$$\delta \boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q^j} \delta q^j$$

A virtual infinitesimal displacement of a system of particles refers to a change in the configuration of a system as the result of any arbitrary infinitesimal change of location vectors (or coordinates) $\delta \boldsymbol{r}_k$, consistent with the forces and constraints imposed on the system at the current/given instant t. This displacement is called "virtual" to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the forces and constraints may be changing.

Assume the system is in equilibrium, that is the full force on each particle vanishes, $\boldsymbol{F}_i = \boldsymbol{0}\ \forall i$. Then clearly the term $\boldsymbol{F}_i \bullet \delta \boldsymbol{r}_i$, which is the virtual work of force $\boldsymbol{F}_i$ in displacement $\delta \boldsymbol{r}_i$, also vanishes for each particle, $\boldsymbol{F}_i \bullet \delta \boldsymbol{r}_i = 0\ \forall i$. The sum of these vanishing products over all particles is likewise equal to zero:

$$\sum_i \boldsymbol{F}_i \bullet \delta \boldsymbol{r}_i = 0.$$

Decompose the full force $\boldsymbol{F}_i$ into the applied (active) force $\boldsymbol{F}_i^{(a)}$ and the force of constraint $\boldsymbol{C}_i$,

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{(a)} + \boldsymbol{C}_i$$

We now restrict ourselves to systems for which the net virtual work of the force of every constraint is zero:

$$\sum_i \boldsymbol{C}_i \bullet \delta \boldsymbol{r}_i = 0.$$

We therefore have as the condition for equilibrium of a system that the virtual work of all applied forces vanishes:

$$\sum_i \boldsymbol{F}_i^{(a)} \bullet \delta \boldsymbol{r}_i = 0.$$

— the principle of virtual work.

Note that coefficients $\boldsymbol{F}_i^{(a)}$ can no longer be thought equal to zero: in common $\boldsymbol{F}_i^{(a)} \neq 0$, since $\delta \boldsymbol{r}_i$ are not independent but are bound by constraints.

A virtual displacement of a particle with vector radius $\boldsymbol{r}_k$ is variation $\delta \boldsymbol{r}_k$ — any infinitesimal change of vector $\boldsymbol{r}_k$, which is compatible with the constraints. If the system is free, that is there are no constraints, then virtual displacements $\delta \boldsymbol{r}_k$ are perfectly random.

Связи бывают голономные (holonomic, или геометрические), связывающие только положения (смещения) — they are functions of only the coordinates and probably time

$$c(\boldsymbol{r}, t) = 0 \tag{2.1}$$

— и неголономные (или дифференциальные), содержащие производные координат по времени: $c(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = 0$ and not интегрируемые till the geometrical constraints.

When all constraints are holonomic, then the virtual displacements of a particle “k” satisfy the equation

$$\sum_{j=1}^{m} \frac{\partial c_j}{\partial \boldsymbol{r}_k} \bullet \delta \boldsymbol{r}_k = 0. \tag{2.2}$$

In constrained (non-free) systems, all forces can be divided into two groups : the active forces $\boldsymbol{F}_k^{(a)}$ and the constraint (or reaction) forces.

Реакция $\boldsymbol{\Phi}_k$ действует со стороны всех материальных ограничителей на частицу “k” и меняется согласно уравнению (2.1) для каждой связи.

The constraints are assumed to be ideal, that is

$$\sum_k \boldsymbol{\Phi}_k \cdot \delta \boldsymbol{r}_k = 0 \tag{2.3}$$

— the work of constraint (reaction) forces is equal to zero on any virtual displacements.

The principle of virtual work is

$$\sum_k \Big(\boldsymbol{F}_k^{(a)} - m_k \ddot{\boldsymbol{r}}_k \Big) \cdot \delta \boldsymbol{r}_k = 0, \tag{2.4}$$

where $\boldsymbol{F}_k^{(a)}$ are only active forces, without reactions of constraints.

Differential variational equation (2.4) may seem like a trivial consequence of the Newton's law (1.1) and the ideality of constraints (2.3). Однако содержание (2.4) несравненно обширнее. Читатель вскоре увидит, что принцип (2.4) может быть положен в основу механики [92]. The various models of elastic media, being described in this book, are based on this principle.

Для примера рассмотрим совершенно жёсткое (недеформируемое) твёрдое тело.

.... (7.4) $\Rightarrow \delta \boldsymbol{r} = \delta \boldsymbol{p} + \delta \boldsymbol{x}$

(begin copied from § **??**.**??**)

Варьируя тождество (**??**), получим $\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = - \boldsymbol{O} \cdot \delta \boldsymbol{O}^{\mathsf{T}}$. Этот тензор антисимметричен, и потому выражается через свой сопутствующий вектор $\boldsymbol{\delta o}$ как $\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\delta o} \times \boldsymbol{E}$. Приходим к соотношениям

$$\delta \boldsymbol{O} = \boldsymbol{\delta o} \times \boldsymbol{O}, \;\; \boldsymbol{\delta o} = -\frac{1}{2} \Big(\delta \boldsymbol{O} \cdot \boldsymbol{O}^{\mathsf{T}} \Big)_{\times}, \tag{2.5}$$

(end of copied from § **??**.**??**)

...

Проявилась замечательная особенность (2.4) : это уравнение эквивалентно системе такого порядка, каково число степеней свободы системы, то есть сколько независимых вариаций $\delta \boldsymbol{r}_k$ мы имеем. Если система N точек имеет m связей, то число степеней свободы $n = 3N - m$.

...

§ 3. Balance of momentum, rotational momentum, and energy

Эти уравнения баланса могут быть связаны со свойствами пространства and времени [93]. Сохранение импульса (количества движения) в за́мкнутой (изолированной)* системе выводится из однородности пространства *(при любом параллельном переносе — трансляции — замкнутой системы как целого свойства этой системы не меняются)*. Сохранение момента импульса — следствие изотропии пространства *(свойства замкнутой системы не меняются при любом повороте этой системы как целого)*. Энергия же изолированной системы сохраняется, так как время однородно** (energy $\mathrm{E} \equiv \mathrm{K}(q, \dot{q}) + \Pi(q)$ такой системы не зависит явно от времени).

The balance equations can be derived from the principle of virtual work (2.4). Перепишем его, выделив внешние силы $\boldsymbol{F}_k^{(e)}$ и виртуальную работу внутренних сил $\delta W^{(i)} = \sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \delta \boldsymbol{r}_k$

$$\sum_k \Big(\boldsymbol{F}_k^{(e)} - m_k \ddot{\boldsymbol{r}}_k \Big) \cdot \delta \boldsymbol{r}_k + \delta W^{(i)} = 0. \tag{3.1}$$

It's assumed that internal forces don't do any work on virtual displacements of a system as a rigid whole ($\boldsymbol{\delta\rho}$ and $\boldsymbol{\delta o}$ are some constant vectors describing translation and rotation)

$$\begin{array}{l} \delta \boldsymbol{r}_k = \boldsymbol{\delta\rho} + \boldsymbol{\delta o} \times \boldsymbol{r}_k \,, \\ \boldsymbol{\delta\rho} = \textbf{constant}, \;\; \boldsymbol{\delta o} = \textbf{constant} \end{array} \;\Rightarrow\; \delta W^{(i)} = 0. \tag{3.2}$$

Premises and considerations for this assumption are as follows. The first — for the case of potential, such as elastic, forces.

* *За́мкнутая (изолированная) система* это такая система частиц, которые взаимодействуют only друг с другом, but ни с какими другими телами.

** Характеристики “однородность” and “изотропность” пространства, “однородность” времени не фигурируют среди аксиом классической механики.

A variation of the work of potential internal forces $W^{(i)}$ is a variation of the potential Π with the opposite sign,

$$\delta W^{(i)} = -\delta\Pi. \tag{3.3}$$

And it's quite obvious that Π alters only by deforming. The second consideration — the internal forces are balanced in the sense that the net vector (the resultant force) and the net moment (the resultant couple) are $\mathbf{0}$,

(1.5) & (1.9)

$$\sum \cdots$$

...

Принимая (3.2) и подставляя в (3.1) сначала $\delta \boldsymbol{r}_k = \boldsymbol{\delta\rho}$ (трансляция), а затем $\delta \boldsymbol{r}_k = \boldsymbol{\delta o} \times \boldsymbol{r}_k$ (поворот), получаем баланс импульса (...) and баланс момента импульса (...).

...

§ 4. Hamilton's principle and Lagrange's equations

The two branches of analytical mechanics are Lagrangian mechanics (using generalized coordinates and corresponding generalized velocities in configuration space) and Hamiltonian mechanics (using coordinates and corresponding momenta in phase space). Both formulations are equivalent by a Legendre transformation on the generalized coordinates, velocities and momenta, therefore both contain the same information for describing the dynamics of a system.

Variational equation (2.4) is satisfied at any moment of time. Проинтегрируем его (оттуда лишь равенство F = ma)* по какому-либо промежутку $[t_1, t_2]$

$$\int_{t_1}^{t_2} \Bigl(\delta \mathrm{K} + \sum_k \boldsymbol{F}_k \cdot \delta \boldsymbol{r}_k \Bigr) dt - \Bigl[\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k \Bigr]_{t_1}^{t_2} = 0. \tag{4.1}$$

Без ущерба для общности можно принять $\delta \boldsymbol{r}_k(t_1) = \delta \boldsymbol{r}_k(t_2) = \boldsymbol{0}$, then the non-integral term vanishes.

Generalized coordinates q^i ($i = 1, \ldots, n$ — the number of degrees o'freedom) are introduced. Location vectors become functions like $\boldsymbol{r}_k(q^i, t)$, тождественно удовлетворяющими уравнениям связей (2.1). Если связи стационарны, то есть уравнения (2.1) не содержат t, то остаётся $\boldsymbol{r}_k(q^i)$. Kinetic energy превращается в функцию $\mathrm{K}(q^i, \dot{q}^i, t)$, где явно входящее t характерно лишь для нестационарных связей.

Hello to the essential concept of generalized forces. They originate from the virtual work $\boldsymbol{F}_k \cdot \delta \boldsymbol{r}_k$. With variation $\delta \boldsymbol{r}_k$ of the k-th point's location vector, expanded for generalized coordinates q^i,

$$\delta \boldsymbol{r}_k = \sum_i \frac{\partial \boldsymbol{r}_k}{\partial q^i} \delta q^i , \tag{4.2}$$

the virtual work can be written as

$$\begin{aligned} \sum_k \boldsymbol{F}_k \cdot \delta \boldsymbol{r}_k &= \sum_k \boldsymbol{F}_k \cdot \sum_i \frac{\partial \boldsymbol{r}_k}{\partial q^i} \delta q^i = \sum_{i,k} \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i} \delta q^i \\ &= \sum_i \Bigl(\sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i} \Bigr) \delta q^i = \sum_i Q_i \delta q^i , \end{aligned} \tag{4.3}$$

where

$$Q_i \equiv \sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i} . \tag{4.4}$$

* $\delta \mathrm{K} = \sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta \dot{\boldsymbol{r}}_k$, $\Bigl(\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k \Bigr)^{\bullet} = \sum_k m_k \ddot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k + \underbrace{\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta \dot{\boldsymbol{r}}_k}_{\delta \mathrm{K}}$

$$\Rightarrow \quad \Bigl[\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k \Bigr]_{t_1}^{t_2} = \int_{t_1}^{t_2} \sum_k m_k \ddot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k \, dt + \int_{t_1}^{t_2} \delta \mathrm{K} dt$$

It's worth to accentuate once more the origin of generalized forces from work. Having chosen the generalized coordinates q^i for the problem, the applied forces $\boldsymbol{F}_k$ are then grouped into the sets of generalized forces Q_i.

The particular case of potential forces is very relevant for this book. A force is *potential* when the work done by it depends only on locations of points, but not on paths between them. Then the potential energy Π can be introduced as a scalar field, also dubbed a "potential field" or just "a potential", because it is a function of only coordinates $\Pi = \Pi(q^i)$ (and possibly time, $\Pi(q^i, t)$ — the explicit dependence on time t may appear due to non-stationary constraints or because the physical fields themselves depend on time). $\delta\Pi = \sum \frac{\partial\Pi}{\partial q^i}\delta q^i$ is a variation of energy Π. The function Π is usually defined so that a positive work is a reduction in the potential. Thus, if generalized forces are potential, then

$$\sum_i Q_i \delta q^i = -\delta\Pi, \quad Q_i = -\frac{\partial\Pi}{\partial q^i}. \tag{4.5}$$

...

Lagrange's equations of the first kind

Since there are Lagrange's equations "of the second kind", the reader may guess that equations "of the first kind" exist too. Yes, they are known. And they are worth mentioning at least because the derivation method, founded on these equations, is used in this book many times.

When constraints (2.1) are imposed on a system, the equality $\boldsymbol{F}_k = m_k \ddot{\boldsymbol{r}}_k$ doesn't follow from the variational equation (2.4), since virtual displacements $\delta\boldsymbol{r}_k$ are then not independent. Having m constraints and therefore m conditions (2.2) for variations, each of these conditions is multiplied by some scalar λ_a $(a = 1, \ldots, m)$ and added to (2.4), turning into

$$\sum_{k=1}^{N} \left(\boldsymbol{F}_k^{(a)} + \sum_{a=1}^{m} \lambda_a \frac{\partial c_a}{\partial \boldsymbol{r}_k} - m_k \ddot{\boldsymbol{r}}_k \right) \cdot \delta\boldsymbol{r}_k = 0. \tag{4.6}$$

Among $3N$ components of variations $\delta\boldsymbol{r}_k$, m are dependent. Aha, and the number of Lagrange multipliers λ_a is m too. Если выбрать λ_α

такие, что coefficients(??каки́е?) for dependent variations обращаются в нуль, то тогда у остальных вариаций коэффициенты(??) также будут нулевые из-за независимости. Следовательно, все выражения в скобках $(\cdots)$ равны нулю — это и есть Lagrange's equations of the first kind.

Since for each particle

...

§ 5. Statics

Let there is a mechanical system with stationary (constant over time) constraints under the action of static (not changing with time) active forces $\boldsymbol{F}_k$. In equilibrium all $\boldsymbol{r}_k = \textbf{constant}$, hence $\delta \boldsymbol{r}_k = \boldsymbol{0}$, $\dfrac{\partial \boldsymbol{r}_k}{\partial q^i} = \boldsymbol{0}$, and the principle of virtual work is formulated as

$$\sum_k \boldsymbol{F}_k \bullet \delta \boldsymbol{r}_k = 0 \quad \Leftrightarrow \quad \sum_k \boldsymbol{F}_k \bullet \frac{\partial \boldsymbol{r}_k}{\partial q^i} = Q_i = 0. \tag{5.1}$$

Both pieces are essential : and the variational equation, and zeros in the generalized forces.

Relations (5.1) are the most generic and universal equations of statics. In literature, the narrow conception of the equilibrium equations as the balance of forces and moments is widespread. But in that case too, as in any other, the set of the equilibrium equations exactly matches with the generalized coordinates. "The resultant force" (also referred to as "the net (full) force" or "the net vector") and "the resultant couple" ("the net couple", "the net moment") figure in the equilibrium equations* just because the system has translational and rotational degrees o'freedom. The huge popularity of forces and moments (force couples) comes not as much from the prevalence of statics of a perfectly non-deformable (ideally rigid) solid body, but more from the fact that the virtual work of internal forces on all

* Since describing the reduction of any set of forces, acting on the same absolutely rigid body, into the single force and the single couple in the book "Éléments de statique" (1873) by Louis Poinsot.

movements of the system as a rigid whole is always equal to zero for any medium.

Let two kinds o'forces act in the system : potential, with the coordinate-dependent energy $\Pi(q^i)$, and plus external ones $\overset{\circ}{Q}{}_j^{(e)} \equiv P_j$. From (5.1) follow the equilibrium equations

$$\frac{\partial \Pi}{\partial q^i} = P_i \tag{5.2}$$

and the exact differential of $\Pi(q^i)$ (time independent) is

$$d\Pi = \sum_i \frac{\partial \Pi}{\partial q^i} dq^i = \sum_i P_i \, dq^i. \tag{5.3}$$

Equations (5.2) formulate the problem of statics, non-linear in overall, about the relation of the equilibrium position $q^i_\circ$ with the external loads P_i.

A linear system with quadratic potential Π as a function of coordinates

$$\Pi = \frac{1}{2} \sum_{i,k} C_{ik} \, q^i q^k \tag{5.4}$$

$$\sum_k C_{ik} q^k = P_i . \tag{5.5}$$

Here figure elements C_{ik} of "the stiffness matrix", coordinates q^k and external loads P_i.

Structures (both human-made artificial and in the nature) most often have a positive-definite stiffness matrix C_{ik}. Then $\det C_{ik} > 0$, the solution of a linear algebraic system (5.5) is unique, and this solution can be substituted by minimization of the quadratic form

$$\mathscr{E}(q^j) \equiv \Pi - \sum_i P_i q^i = \frac{1}{2} \sum_{i,k} q^i C_{ik} \, q^k - \sum_i P_i q^i \to \text{ min.} \tag{5.6}$$

However, the design may be so unfortunate that the stiffness matrix becomes singular (noninvertible) with $\det C_{ik} = 0$ (or the determinant is very close to zero, $\det C_{ik} \approx 0$ — the nearly singular matrix). Then the solution of the linear problem of statics (5.5) exists only when external loads P_i are orthogonal to all linearly independent solutions of the homogeneous conjugate system

...

The famous theorems of statics for linear continua (§ **??.??**) can be easily proved for a finite number of degrees o'freedom. The Clapeyron's theorem looks here like

...

The reciprocal work theorem ("the work W_{12} of the first set's forces on displacements from the forces of the second is equal to the work W_{21} of the second set's forces on displacements from the forces of the first") instantly derives from (5.5) :

(....)

Here the symmetry of the stiffness matrix $C_{ij} = C_{ji}$ is essential — that the system is conservative.

.....

Turning back to the problem (5.2), sometimes called the Lagrange's theorem. Inverted by Legendre transform(ation), it translates into

$$d\Bigl(\sum_i P_i\, q^i\Bigr) = \sum_i d\left(P_i\, q^i\right) = \sum_i \left(q^i\, dP_i + P_i\, dq^i\right),$$

$$\sum_i d\left(P_i\, q^i\right) - \underbrace{\sum_i P_i\, dq^i}_{d\Pi} = \sum_i q^i\, dP_i\,,$$

$$d\Bigl(\sum_i P_i\, q^i - \Pi\Bigr) = \sum_i q^i\, dP_i = \underbrace{\sum_i \frac{\partial ⨿}{\partial P_i}\, dP_i}_{d⨿}\,,$$

where appears the exact differential of the so-called "complementary energy" ⨿

$$q^i = \frac{\partial ⨿}{\partial P_i}\,, \quad ⨿(P_i) = \sum_i P_i\, q^i - \Pi. \tag{5.7}$$

This is known as the Castigliano theorem*. For a linear system (5.5) $\Rightarrow$ $\amalg(P_i) = \Pi(q^i)$. Theorem (5.7) is sometimes very useful — when the complementary energy as the function of external loads $\amalg(P_i)$ is easy to find. Someone may come across the so-called "statically determinate" structures (systems), for which all internal forces can luckily be found just only from the balance (equilibrium) equations for forces and moments. For such structures, (5.7) is effective.

Unlike the linear problem (5.5), the nonlinear problem (5.2) may have no solutions at all or may have several of them.

....

The overview of statics in classical mechanics I am ending with the d'Alembert's principle** : the dynamic equations differ from the static ones only in additional "inertia forces" ("fictitious forces") $m_k \ddot{\boldsymbol{r}}_k$. The d'Alembert's principle is pretty obvious, but applying it everytime & everywhere is a mistake. As example, the equations of motion for a viscous fluid (Navier–Stokes equations) in statics and in dynamics differ not only in inertial adjunct. Nevertheless, for solid elastic bodies the d'Alembert's principle always apply.

§ 6. Small oscillations (vibrations)

If the statics of a linear system is described by equation (5.5), then in the dynamics we have

$$\sum_k \left(A_{ik} \ddot{q}_k + C_{ik} q^k \right) = P_i(t), \tag{6.1}$$

* **Carlo Alberto Castigliano**. Intorno ai sistemi elastici, Dissertazione presentata da Castigliano Alberto alla Commissione Esaminatrice della R. Scuola d'Applicazione degli Ingegneri in Torino per ottenere la Laurea di Ingegnere Civile. Torino, Vincenzo Bona, 1873.

** **Jean Le Rond d'Alembert**. Traité de Dynamique, dans lequel les Loix de l'Equilibre & du mouvement des Corps sont réduites au plus petit nombre possible, & démontrées d'une maniére nouvelle, & où l'on donne un Principe général pour trouver le Mouvement de plusieurs Corps qui agissent les uns sur les autres, d'une maniére quelconque. Paris : David l'aîné, MDCCXLIII (1743).

where A_{ik} is the symmetric and positive "matrix of kinetic energy".

Any description of oscillations almost always includes the term "mode". A mode of vibration can be defined as a way of vibrating or a pattern of vibration. A normal mode is a pattern of periodic motion, when all parts of an oscillating system move sinusoidally with the same frequency and with a fixed phase relation. The free motion described by the normal modes takes place at fixed frequencies — the natural resonant frequencies of an oscillating system.

The most generic motion of an oscillating system is some superposition of normal modes of this system.*

A research of an oscillating system most often begins with orthogonal (normal) "modes" — harmonics, free (without any driving or damping force) sinusoidal oscillations

$$q^k(t) = \overset{*}{q}_k \sin \omega_k t.$$

Multipliers $\overset{*}{q}_k = \mathsf{constant}$ are orthogonal (normal) "modes" of oscillation, ω_k are natural (resonant, eigen-) frequencies. This set, dependent on the structure of an oscillating object, the materials and the boundary conditions, is found from the eigenvalue problem

$$\begin{gathered} P_i = 0, \;\; \overset{\bullet\bullet}{q}_k = -\omega_k^2 \, \overset{*}{q}_k \sin \omega_k t, \;\; (6.1) \;\Rightarrow \\ \Rightarrow \;\; \sum_k \Bigl(C_{ik} - A_{ik} \, \omega_k^2 \Bigr) \, \overset{*}{q}_k \sin \omega_k t = 0 \end{gathered} \tag{6.2}$$

...

The Duhamel's integral is a way of calculating the response of linear systems to an arbitrary time-varying external perturbation.

...

§ 7. Perfectly rigid undeformable solid body

* The modes are "normal" in the sense that they move independently, and an excitation of one mode will never cause a motion of another mode. In mathematical terms, normal modes are orthogonal to each other. In music, normal modes of vibrating instruments (strings, air pipes, percussion and others) are called "harmonics" or "overtones".

"Absolutely rigid", aka "absolutely solid" and "absolutely durable" — the pipe dream of any engineer.

One more concept, modeled in classical generic mechanics, is the (perfectly) rigid body. That is a solid* body, in which deformation is zero (or is negligibly small — so small that it can be neglected). The distance between any two points of a non-deformable rigid body remains constant regardless of external forces exerted on it.

A non-deformable rigid body is modeled using the "continual approach" as a continuous distribution of mass (a material continuum, a continuous medium), rather than using the "discrete approach" (that is modeling a body as a discrete collection of particles, § 1).

The mass of a material continuum is distributed continuously throughout its volume,

$$dm \equiv \rho \, d\mathcal{V} \tag{7.1}$$

(ρ is a volume(tric) mass density and $d\mathcal{V}$ is an infinitesimal volume).

A formula with summation over discrete points becomes a formula for a continuous body by replacing the masses of particles with the mass (7.1) of an infinitesimal volume element $d\mathcal{V}$ with integration over the entire volume of a body. In particular, here are the formulas for the (linear) momentum

$$\sum_k m_k \dot{\boldsymbol{r}}_k \;\; \text{becomes} \;\; \int\limits_{\mathcal{V}} \dot{\boldsymbol{r}} \, dm \tag{7.2}$$

and for the angular momentum

$$\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \;\; \text{becomes} \;\; \int\limits_{\mathcal{V}} \boldsymbol{r} \times \dot{\boldsymbol{r}} \, dm \, . \tag{7.3}$$

To fully describe the location (position, place) of any non-deformable body with all its points, it's enough to choose some

* "Rigid" is inelastic and not flexible, and "solid" is not fluid. A solid substance retains its size and shape without a container (as opposed to a fluid substance, a liquid or a gas).

unique point as the “pole”, to find or to set the location $\boldsymbol{p} = \boldsymbol{p}(t)$ of the chosen point, as well as the angular orientation of a body relative to the pole (figure 3). As a result, any motion of an undeformable rigid body is either a rotation around the chosen pole, or an equal displacement of the pole and all body’s points — a translation (a linear motion)*, or a combination of them both.

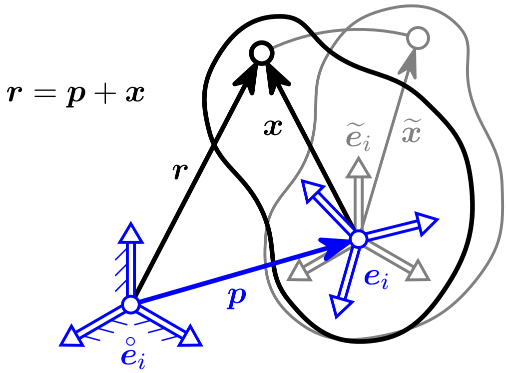

figure 3

$\mathring{\boldsymbol{e}}_i$ — the triplet of mutually perpendicular unit vectors, called the “basis vectors”, immovable relatively to the absolute (or to any inertial) reference system

✔ $\mathring{\boldsymbol{e}}_i$ is the immovable (stationary) basis

✔ $\boldsymbol{e}_i$ is the basis which moves along with the body

By adding the basis $\boldsymbol{e}_i$ (it moves together with the body), the body’s angular orientation can be determined by the rotation tensor $\boldsymbol{O} \equiv \boldsymbol{e}_i \widetilde{\boldsymbol{e}}_i$.

Then any motion of a body is completely described by two functions, $\boldsymbol{p}(t)$ and $\boldsymbol{O}(t)$.

The location vector of some body’s point

$$\boldsymbol{r} = \boldsymbol{p} + \boldsymbol{x} \tag{7.4}$$

$\widetilde{\boldsymbol{x}} = x_i \widetilde{\boldsymbol{e}}_i$, $\boldsymbol{x} = x_i \boldsymbol{e}_i$

* A translation can also be thought of as a rotation with the revolution center at infinity.

(??), § ??.??

$\boldsymbol{x} = \boldsymbol{O} \cdot \widetilde{\boldsymbol{x}}$

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{p}} + \dot{\boldsymbol{x}},$$

For a non-deformable rigid body, components x_i don't depend on time: $x_i = \mathsf{constant}(t)$ and $\dot{\boldsymbol{x}} = x_i \dot{\boldsymbol{e}}_i$

$\dot{\boldsymbol{x}} = \dot{\boldsymbol{O}} \cdot \mathring{\boldsymbol{x}}$

$x_i \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{O}} \cdot x_i \mathring{\boldsymbol{e}}_i \;\Leftrightarrow\; \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{O}} \cdot \mathring{\boldsymbol{e}}_i$

...

The linear momentum and the rotational (angular) momentum of a non-deformable continuous body are described by the following integrals

...

...

$$\int\limits_{\mathcal{V}} \boldsymbol{p} dm = \boldsymbol{p} \int\limits_{\mathcal{V}} dm = \boldsymbol{p} m$$

$$\int\limits_{\mathcal{V}} \boldsymbol{x} dm = \boldsymbol{\Xi} m, \;\; \boldsymbol{\Xi} \equiv m^{-1} \int\limits_{\mathcal{V}} \boldsymbol{x} dm$$

Three inertial characteristics of the body:

- ✓ integral mass $m = \int_{\mathcal{V}} dm = \int_{\mathcal{V}} \rho d\mathcal{V}$ — the mass of the whole body,
- ✓ eccentricity vector $\boldsymbol{\Xi}$ — measures the offset of the chosen pole from the body's "center of mass",
- ✓ inertia tensor ${}^{2}\boldsymbol{\mathfrak{I}}$.

The eccentricity vector is equal to the null vector only when the chosen pole coincides with the "center of mass" — the unique point within a body with location vector $\boldsymbol{\kappa}$, in short

$$\boldsymbol{\Xi} = \boldsymbol{0} \;\Leftrightarrow\; \boldsymbol{p} = \boldsymbol{\kappa}.$$

$$\boldsymbol{x} = \boldsymbol{r} - \boldsymbol{p}, \quad \boldsymbol{\Xi} m = \int\limits_{\mathcal{V}} (\boldsymbol{r} - \boldsymbol{\kappa}) dm = \boldsymbol{0},$$

$$\int\limits_{\mathcal{V}} \boldsymbol{r}\, dm - \boldsymbol{\kappa} \int\limits_{\mathcal{V}} dm = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{\kappa} = m^{-1} \int\limits_{\mathcal{V}} \boldsymbol{r}\, dm$$

...

Introducing the (pseudo)vector of angular velocity $\boldsymbol{\omega}$, ...

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i$$

...

inertia tensor ${}^2\boldsymbol{\mathfrak{I}}$

$${}^2\boldsymbol{\mathfrak{I}} \equiv - \int\limits_{\mathcal{V}} \big(\boldsymbol{x} \times \boldsymbol{E}\big) \cdot \big(\boldsymbol{x} \times \boldsymbol{E}\big)\, dm = \int\limits_{\mathcal{V}} \big(\boldsymbol{x} \cdot \boldsymbol{x} \boldsymbol{E} - \boldsymbol{x}\boldsymbol{x}\big)\, dm$$

It is assumed (can be proven?) that the inertia tensor changes only due to a rotation

$${}^2\boldsymbol{\mathfrak{I}} = \boldsymbol{O} \cdot {}^2\mathring{\boldsymbol{\mathfrak{I}}} \cdot \boldsymbol{O}^{\mathsf{T}}$$

and if some basis $\boldsymbol{e}_j$ is moving along with the body, the inertia components in that basis don't change over time

$${}^2\boldsymbol{\mathfrak{I}} = \mathfrak{I}_{ab}\, \boldsymbol{e}_a \boldsymbol{e}_b, \;\; \mathfrak{I}_{ab} = \mathsf{constant}(t),$$

thus the time derivative is

$$\begin{aligned} {}^2\dot{\boldsymbol{\mathfrak{I}}} &= \mathfrak{I}_{ab} \big(\dot{\boldsymbol{e}}_a \boldsymbol{e}_b + \boldsymbol{e}_a \dot{\boldsymbol{e}}_b\big) = \mathfrak{I}_{ab} \big(\boldsymbol{\omega} \times \boldsymbol{e}_a \boldsymbol{e}_b + \boldsymbol{e}_a \boldsymbol{\omega} \times \boldsymbol{e}_b\big) \\ &= \boldsymbol{\omega} \times \mathfrak{I}_{ab}\, \boldsymbol{e}_a \boldsymbol{e}_b - \mathfrak{I}_{ab}\, \boldsymbol{e}_a \boldsymbol{e}_b \times \boldsymbol{\omega} = \boldsymbol{\omega} \times {}^2\boldsymbol{\mathfrak{I}} - {}^2\boldsymbol{\mathfrak{I}} \times \boldsymbol{\omega} \end{aligned}$$

Substitution of (....) into (1.6) and (**??**) gives equations of balance — the balance of linear momentum and the balance of rotational momentum — for the model of a continuous non-deformable rigid body

...

here $\boldsymbol{f}$ is the external force per mass unit, $\boldsymbol{F}$ is the resultant of external forces (also called the "equally acting force" or the "main vector"), $\boldsymbol{M}$ is the resultant of external couples (the "main couple", the "main moment").

...

§ 8. Mechanics of relative motion

До этого не ставился вопрос о системе отсчёта, всё рассматривалось в некой "абсолютной" системе или одной из инерциальных систем (§ 1). Теперь представим себе две системы: "абсолютную" и "подвижную"

...

$$\mathring{\boldsymbol{r}} = \boldsymbol{r} + \boldsymbol{x}$$
$$\boldsymbol{r} = \rho_i \mathring{\boldsymbol{e}}_i, \;\; \boldsymbol{x} = x_i \boldsymbol{e}_i$$
$$\dot{\mathring{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}}$$
$$\dot{\boldsymbol{r}} = \dot{\rho}_i \mathring{\boldsymbol{e}}_i, \;\; \dot{\boldsymbol{x}} = \big(x_i \boldsymbol{e}_i\big)^{\bullet} = \dot{x}_i \boldsymbol{e}_i + x_i \dot{\boldsymbol{e}}_i$$

$x_i \neq \text{constant} \Rightarrow \dot{x}_i \neq 0$

By (??, § ??.??)

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \;\Rightarrow\; x_i \dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times x_i \boldsymbol{e}_i = \boldsymbol{\omega} \times \boldsymbol{x}$$

$\dot{\boldsymbol{x}} = \dot{x}_i \boldsymbol{e}_i + \boldsymbol{\omega} \times \boldsymbol{x}$

$$\boldsymbol{v} \equiv \dot{\mathring{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}} = \underbrace{\dot{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{x}}_{\boldsymbol{v}_e} \underbrace{- \boldsymbol{\omega} \times \boldsymbol{x} + \dot{\boldsymbol{x}}}_{\boldsymbol{v}_{rel}}$$

$\dot{\boldsymbol{x}} - \boldsymbol{\omega} \times \boldsymbol{x} = \dot{x}_i \boldsymbol{e}_i \equiv \boldsymbol{v}_{rel}$ — relative velocity, $\dot{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{x} \equiv \boldsymbol{v}_e$

$$\boldsymbol{v} = \boldsymbol{v}_e + \boldsymbol{v}_{rel} \tag{8.1}$$

...

$$\overset{\bullet}{\overset{\circ}{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}}$$
$$\overset{\bullet\bullet}{\overset{\circ}{\boldsymbol{r}}} = \ddot{\boldsymbol{r}} + \ddot{\boldsymbol{x}}$$
$$\boldsymbol{w} \equiv \dot{\boldsymbol{v}} = \overset{\bullet\bullet}{\overset{\circ}{\boldsymbol{r}}} = \ddot{\boldsymbol{r}} + \ddot{\boldsymbol{x}}$$
$$\ddot{\boldsymbol{r}} = \ddot{\rho}_i \overset{\circ}{\boldsymbol{e}}_i, \;\; \ddot{\boldsymbol{x}} = \big(x_i \boldsymbol{e}_i\big)^{\bullet\bullet} = \big(\dot{x}_i \boldsymbol{e}_i + x_i \dot{\boldsymbol{e}}_i\big)^{\bullet} = \ddot{x}_i \boldsymbol{e}_i + \dot{x}_i \dot{\boldsymbol{e}}_i + \dot{x}_i \dot{\boldsymbol{e}}_i + x_i \ddot{\boldsymbol{e}}_i$$

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \;\Rightarrow\; \ddot{\boldsymbol{e}}_i = \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)^{\bullet} = \dot{\boldsymbol{\omega}} \times \boldsymbol{e}_i + \boldsymbol{\omega} \times \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{\omega}} \times \boldsymbol{e}_i + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)$$

$$x_i \ddot{\boldsymbol{e}}_i = x_i \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)^{\bullet} = \dot{\boldsymbol{\omega}} \times x_i \boldsymbol{e}_i + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times x_i \boldsymbol{e}_i\big) = \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{x}\big)$$

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \;\Rightarrow\; \dot{x}_i \dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \dot{x}_i \boldsymbol{e}_i = \boldsymbol{\omega} \times \boldsymbol{v}_{rel}$$

$\ddot{x}_i \boldsymbol{e}_i \equiv \boldsymbol{w}_{rel}$ — relative acceleration
$2\dot{x}_i \dot{\boldsymbol{e}}_i = 2\,\boldsymbol{\omega} \times \boldsymbol{v}_{rel} \equiv \boldsymbol{w}_{Cor}$ — Coriolis acceleration

$$\ddot{\boldsymbol{x}} = \boldsymbol{w}_{rel} + \boldsymbol{w}_{Cor} + x_i \ddot{\boldsymbol{e}}_i$$

$$\big(x_i \dot{\boldsymbol{e}}_i\big)^{\bullet} = \dot{x}_i \dot{\boldsymbol{e}}_i + x_i \ddot{\boldsymbol{e}}_i = \tfrac{1}{2} \boldsymbol{w}_{Cor} + x_i \ddot{\boldsymbol{e}}_i$$
$$\big(x_i \dot{\boldsymbol{e}}_i\big)^{\bullet} = \big(\boldsymbol{\omega} \times \boldsymbol{x}\big)^{\bullet} = \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \dot{\boldsymbol{x}}$$

$$\boldsymbol{\omega} \times \dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \big(\dot{x}_i \boldsymbol{e}_i + \boldsymbol{\omega} \times \boldsymbol{x}\big) = \underbrace{\boldsymbol{\omega} \times \dot{x}_i \boldsymbol{e}_i}_{\dot{x}_i \dot{\boldsymbol{e}}_i = \frac{1}{2} \boldsymbol{w}_{Cor}} + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{x}\big)$$

...

Bibliography

In a long list of the books about the classical mechanics, the reader can find the works of both the specialists in mechanics [87, 88, 94, 95, 96] and the broadly oriented theoretical physicists [93, 89]. The book by Felix R. Gantmacher (Феликс Р. Гантмахер) [92] with the compact but complete narration of the fundamentals is pretty interesting.

LIST OF PUBLICATIONS

1. **Antman, Stuart S.** The theory of rods. In: Truesdell C. (editor) Mechanics of solids. Volume II. Linear theories of elasticity and thermoelasticity. Linear and nonlinear theories of rods, plates, and shells. Springer-Verlag, 1973. Pages 641–703.
2. **Алфутов Н. А.** Основы расчета на устойчивость упругих систем. Издание 2-е. М.: Машиностроение, 1991. 336 с.
3. **Артоболевский И. И.**, **Бобровницкий Ю. И.**, **Генкин М. Д.** Введение в акустическую динамику машин. «Наука», 1979. 296 с.
4. **Ахтырец Г. П.**, **Короткин В. И.** Использование МКЭ при решении контактной задачи теории упругости с переменной зоной контакта // Известия северо-кавказского научного центра высшей школы (СКНЦ ВШ). Серия естественные науки. Ростов-на-Дону: Издательство РГУ, 1984. № 1. С. 38–42.
5. **Ахтырец Г. П.**, **Короткин В. И.** К решению контактной задачи с помощью метода конечных элементов // Механика сплошной среды. Ростов-на-Дону: Издательство РГУ, 1988. С. 43–48.
6. **Бидерман В. Л.** Механика тонкостенных конструкций. М.: Машиностроение, 1977. 488 с.
7. **Вениамин И. Блох**. Теория упругости. Харьков: Издательство Харьковского Государственного Университета, 1964. 484 с.
8. **Власов В. З.** Тонкостенные упругие стержни. М.: Физматгиз, 1959. 568 с.
9. **Гольденвейзер А. Л.** Теория упругих тонких оболочек. «Наука», 1976. 512 с.
10. **Гольденвейзер А. Л.**, **Лидский В. Б.**, **Товстик П. Е.** Свободные колебания тонких упругих оболочек. «Наука», 1979. 383 с.
11. **Gordon, James E.** Structures, or Why things don't fall down. Penguin Books, 1978. 395 pages. *Перевод:* **Гордон Дж.** Конструкции, или почему не ломаются вещи. «Мир», 1980. 390 с.

12. **Gordon, James E.** The new science of strong materials, or Why you don't fall through the floor. Penguin Books, 1968. 269 pages. *Перевод:* **Гордон Дж.** Почему мы не проваливаемся сквозь пол. «Мир», 1971. 272 с.
13. **Александр Н. Гузь**. Устойчивость упругих тел при конечных деформациях. Киев: "Наукова думка", 1973. 271 с.
14. *Перевод:* **Де Вит Р.** Континуальная теория дисклинаций. «Мир», 1977. 208 с.
15. **Джанелидзе Г. Ю.**, **Пановко Я. Г.** Статика упругих тонкостенных стержней. Л., М.: Гостехиздат, 1948. 208 с.
16. **Димитриенко Ю. И.** Тензорное исчисление: Учебное пособие для вузов. М.: "Высшая школа", 2001. 575 с.
17. **Dorin Ieşan**. Classical and generalized models of elastic rods. 2nd edition. CRC Press, Taylor & Francis Group, 2009. 369 pages
18. **Владимир В. Елисеев**. Одномерные и трёхмерные модели в механике упругих стержней. Диссертация на соискание учёной степени доктора физико-математических наук. ЛГТУ, 1991. 300 с.
19. **Eshelby, John D.** The continuum theory of lattice defects // Solid State Physics, Academic Press, vol. 3, 1956, pp. 79–144. *Перевод:* **Эшелби Дж.** Континуальная теория дислокаций. М.: ИИЛ, 1963. 247 с.
20. **Журавлёв В. Ф.** Основы теоретической механики. 3-е издание, переработанное. М.: ФИЗМАТЛИТ, 2008. 304 с.
21. **Зубов Л. М.** Методы нелинейной теории упругости в теории оболочек. Изд-во Ростовского ун-та, 1982. 144 с.
22. **Кац, Арнольд М.** Теория упругости. 2-е издание, стереотипное. Санкт-Петербург: Издательство «Лань», 2002. 208 с.
23. **Качанов Л. М.** Основы механики разрушения. «Наука», 1974. 312 с.
24. **Керштейн И. М.**, **Клюшников В. Д.**, **Ломакин Е. В.**, **Шестериков С. А.** Основы экспериментальной механики разрушения. Изд-во МГУ, 1989. 140 с.
25. **Cosserat E.** et **Cosserat F.** Théorie des corps déformables. Paris: A. Hermann et Fils, 1909. 226 p.
26. **Cottrell, Alan**. Theory of crystal dislocations. Gordon and Breach (Documents on Modern Physics), 1964. 94 p. *Перевод:* **Коттрел А.** Теория дислокаций. «Мир», 1969. 96 с.

27. **Kröner, Ekkehart** *(i)* Kontinuumstheorie der Versetzungen und Eigenspannungen. Springer-Verlag, 1958. 180 pages. *(ii)* Allgemeine Kontinuumstheorie der Versetzungen und Eigenspannungen // Archive for Rational Mechanics and Analysis. Volume 4, Issue 1 (January 1959), pp. 273–334. *Перевод:* **Крёнер Э.** Общая континуальная теория дислокаций и собственных напряжений. «Мир», 1965. 104 с.

28. **Augustus Edward Hough Love**. A treatise on the mathematical theory of elasticity. Volume I. Cambridge, 1892. 354 p. Volume II. Cambridge, 1893. 327 p. 4th edition. Cambridge, 1927. Dover, 1944. 643 p. *Перевод:* **Аугустус Ляв** Математическая теория упругости. М.: ОНТИ, 1935. 674 с.

29. **Лурье А. И.** Нелинейная теория упругости. «Наука», 1980. 512 с. *Translation:* **Lurie, A. I.** Nonlinear Theory of Elasticity: translated from the Russian by K. A. Lurie. Elsevier Science Publishers B.V., 1990. 617 p.

30. **Лурье А. И.** Теория упругости. «Наука», 1970. 940 с. *Translation:* **Lurie, A. I.** Theory of Elasticity (translated by A. Belyaev). Springer-Verlag, 2005. 1050 p.

31. **Лурье А. И.** Пространственные задачи теории упругости. М.: Гостехиздат, 1955. 492 с.

32. **Лурье А. И.** Статика тонкостенных упругих оболочек. М., Л.: Гостехиздат, 1947. 252 с.

33. **George E. Mase**. Schaum's outline of theory and problems of continuum mechanics (Schaum's outline series). McGraw–Hill, 1970. 221 p. *Перевод:* **Джордж Мейз**. Теория и задачи механики сплошных сред. Издание 3-е. URSS, 2010. 320 с.

34. **Ernst Melan**, **Heinz Parkus**. Wärmespannungen infolge stationärer Temperaturfelder. Wein, Springer-Verlag, 1953. 114 Seiten. *Перевод:* **Мелан Э.**, **Паркус Г.** Термоупругие напряжения, вызываемые стационарными температурными полями. М.: Физматгиз, 1958. 167 с.

35. **Меркин Д. Р.** Введение в механику гибкой нити. «Наука», 1980. 240 с.

36. **Меркин Д. Р.** Введение в теорию устойчивости движения. 3-е издание. «Наука», 1987. 304 с.

37. **Mindlin, Raymond David** and **Tiersten, Harry F.** Effects of couple-stresses in linear elasticity // Archive for Rational Mechanics and Analysis. Volume 11, Issue 1 (January 1962), pp. 415–448. *Перевод:* **Миндлин Р. Д.**, **Тирстен Г. Ф.** Эффекты моментных напряжений в линейной теории упругости // Механика: Сборник переводов и обзоров иностранной периодической литературы. «Мир», 1964. № 4 (86). С. 80–114.
38. **Морозов Н. Ф.** Математические вопросы теории трещин. «Наука», 1984. 256 с.
39. **Naghdi P. M.** The theory of shells and plates. In: Truesdell C. (editor) Mechanics of solids. Volume II. Linear theories of elasticity and thermoelasticity. Linear and nonlinear theories of rods, plates, and shells. Springer-Verlag, 1973. Pages 425–640.
40. **Witold Nowacki**. Dynamiczne zagadnienia termosprężystości. Warszawa: Państwowe wydawnictwo naukowe, 1966. 366 stron. *Translation:* **Nowacki, Witold**. Dynamic problems of thermoelasticity. Leyden: Noordhoff international publishing, 1975. 436 pages. *Перевод:* **Витольд Новацкий**. Динамические задачи термоупругости. «Мир», 1970. 256 с.
41. **Witold Nowacki**. Teoria sprężystości. Warszawa: Państwowe wydawnictwo naukowe, 1970. 769 stron. *Перевод:* **Новацкий Витольд**. Теория упругости. «Мир», 1975. 872 с.
42. **Witold Nowacki**. Efekty elektromagnetyczne w stałych ciałach odkształcalnych. Państwowe wydawnictwo naukowe, 1983. 147 stron. *Перевод:* **Новацкий В.** Электромагнитные эффекты в твёрдых телах. «Мир», 1986. 160 с.
43. **Новожилов В. В.** Теория тонких оболочек. 2-е издание. Л.: Судпромгиз, 1962. 431 с.
44. **Пановко Я. Г.**, **Бейлин Е. А.** Тонкостенные стержни и системы, составленные из тонкостенных стержней. В сборнике: Рабинович И. М. (редактор) Строительная механика в СССР 1917–1967. М.: Стройиздат, 1969. С. 75–98.
45. **Пановко Я. Г.**, **Губанова И. И.** Устойчивость и колебания упругих систем. Современные концепции, парадоксы и ошибки. 4-е издание. «Наука», 1987. 352 с.
46. **Heinz Parkus**. Instationäre Wärmespannungen. Springer-Verlag, 1959. 176 Seiten. *Перевод:* **Паркус Г.** Неустановившиеся температурные напряжения. М.: Физматгиз, 1963. 252 с.

47. **Партон В. З.** Механика разрушения: от теории к практике. «Наука», 1990. 240 с.

48. **Партон В. З.**, **Кудрявцев Б. А.** Электромагнитоупругость пьезоэлектрических и электропроводных тел. «Наука», 1988. 472 с.

49. **Партон В. З.**, **Морозов Е. М.** Механика упругопластического разрушения. 2-е издание. «Наука», 1985. 504 с.

50. **Подстригач Я. С.**, **Бурак Я. И.**, **Кондрат В. Ф.** Магнитотермоупругость электропроводных тел. Киев: Наукова думка, 1982. 296 с.

51. **Поручиков В. Б.** Методы динамической теории упругости. «Наука», 1986. 328 с.

52. **Southwell, Richard V.** An introduction to the theory of elasticity for engineers and physicists. Dover Publications, 1970. 509 pages. *Перевод:* **Саусвелл Р. В.** Введение в теорию упругости для инженеров и физиков. М.: ИИЛ, 1948. 675 с.

53. **Седов Л. И.** Механика сплошной среды. Том 2. 6-е издание. «Лань», 2004. 560 с.

54. **Ciarlet, Philippe G.** Mathematical elasticity. Volume 1: Three-dimensional elasticity. Elsevier Science Publishers B. V., 1988. xlii + 452 pp. *Перевод:* **Филипп Сьярле** Математическая теория упругости. «Мир», 1992. 472 с.

55. **Adhémar-Jean-Claude Barré de Saint-Venant**. Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exerçant simultanément. Memoires presentes par divers savants a l'Academie des scienees, t. 14, année 1856. 327 pages. *Перевод на русский язык:* **Сен-Венан Б.** Мемуар о кручении призм. Мемуар об изгибе призм. М.: Физматгиз, 1961. 518 страниц.

56. **Adhémar-Jean-Claude Barré de Saint-Venant**. Mémoire sur la flexion des prismes .... Journal de mathematiques pures et appliquees, publie par J. Liouville. 2me serie, t. 1, année 1856. *Перевод на русский язык:* **Сен-Венан Б.** Мемуар о кручении призм. Мемуар об изгибе призм. М.: Физматгиз, 1961. 518 страниц.

57. **Cristian Teodosiu**. Elastic models of crystal defects. Springer-Verlag, 1982. 336 pages. *Перевод:* **Теодосиу К.** Упругие модели дефектов в кристаллах. «Мир», 1985. 352 с.

58. **Тимошенко Степан П.** Устойчивость стержней, пластин и оболочек. «Наука», 1971. 808 с.

59. **Тимошенко Степан П.**, **Войновский-Кригер С.** Пластинки и оболочки. «Наука», 1966. 635 с.

60. **Stephen P. Timoshenko** and **James N. Goodier**. Theory of Elasticity. 2nd edition. McGraw–Hill, 1951. 506 pages. 3rd edition. McGraw–Hill, 1970. 567 pages. *Перевод:* **Тимошенко Степан П.**, **Джеймс Гудьер**. Теория упругости. 2-е издание. «Наука», 1979. 560 с.

61. **Truesdell, Clifford A.** A first course in rational continuum mechanics. Volume 1: General concepts. 2nd edition. Academic Press, 1991. 391 pages. *Перевод:* **Трусделл К.** Первоначальный курс рациональной механики сплошных сред. «Мир», 1975. 592 с.

62. **Феодосьев В. И.** Десять лекций-бесед по сопротивлению материалов. 2-е издание. «Наука», 1975. 173 с.

63. *Перевод:* **Хеллан К.** Введение в механику разрушения. «Мир», 1988. 364 с.

64. *Перевод:* **Циглер Г.** Основы теории устойчивости конструкций. «Мир», 1971. 192 с.

65. **Черепанов Г. П.**. Механика хрупкого разрушения. «Наука», 1974. 640 с.

66. **Черны́х К. Ф.** Введение в анизотропную упругость. «Наука», 1988. 192 с.

67. **Шермергор Т. Д.** Теория упругости микронеоднородных сред. «Наука», 1977. 400 с.

Oscillations and waves

68. **Timoshenko, Stephen P.**; **Young, Donovan H.**; **William Weaver, jr.** Vibration problems in engineering. 5th edition. John Wiley & Sons, 1990. 624 pages. *Перевод:* **Тимошенко Степан П.**, **Янг Донован Х.**, **Уильям Уивер**. Колебания в инженерном деле. М.: Машиностроение, 1985. 472 с.

69. **Бабаков И. М.** Теория колебаний. 4-е издание. «Дрофа», 2004. 592 с.

70. **Бидерман В. Л.** Теория механических колебаний. М.: Высшая школа, 1980. 408 с.

71. **Болотин В. В.** Случайные колебания упругих систем. «Наука», 1979. 336 с.

72. **Гринченко В. Т.**, **Мелешко В. В.** Гармонические колебания и волны в упругих телах. Киев: Наукова думка, 1981. 284 с.

73. **Whitham, Gerald B.** Linear and nonlinear waves. John Wiley & Sons, 1974. 636 pages. *Перевод:* **Уизем Дж.** Линейные и нелинейные волны. «Мир», 1977. 624 с.

74. **Kolsky, Herbert**. Stress waves in solids. Oxford, Clarendon Press, 1953. 211 p. 2nd edition. Dover Publications, 2012. 224 p. *Перевод:* **Кольский Г.** Волны напряжения в твёрдых телах. М.: ИИЛ, 1955. 192 с.

75. **Энгельбрехт Ю. К.**, **Нигул У. К.** Нелинейные волны деформации. «Наука», 1981. 256 с.

76. **Слепян Л. И.** Нестационарные упругие волны. Л.: Судостроение, 1972. 376 с.

77. **Григолюк Э. И.**, **Селезов И. Т.** Неклассические теории колебаний стержней, пластин и оболочек. (Итоги науки и техники. Механика твёрдых деформируемых тел. Том 5.) М.: ВИНИТИ, 1973. 272 с.

Composites

78. **Christensen, Richard M.** Mechanics of composite materials. New York: Wiley, 1979. 348 p. *Перевод:* **Кристенсен Р.** Введение в механику композитов. «Мир», 1982. 336 с.

79. **Кравчук А. С.**, **Майборода В. П.**, **Уржумцев Ю. С.** Механика полимерных и композиционных материалов. Экспериментальные и численные методы. «Наука», 1985. 304 с.

80. **Победря Б. Е.** Механика композиционных материалов. Изд-во Моск. ун-та, 1984. 336 с.

81. **Черепанов Г. П.** Механика разрушения композиционных материалов. «Наука», 1983. 296 с.

82. **Бахвалов Н. С.**, **Панасенко Г. П.** Осреднение процессов в периодических средах. Математические задачи механики композиционных материалов. «Наука», 1984. 352 с.

83. **Bensoussan A.**, **Lions J.-L.**, **Papanicolaou G.** Asymptotic analysis for periodic structures. Amsterdam: North-Holland, 1978. 700 p.

The finite element method

84. **Зенкевич О.**, **Морган К.** Конечные элементы и аппроксимация. «Мир», 1986. 318 с.

85. **Шабров Н. Н.** Метод конечных элементов в расчётах деталей тепловых двигателей. Л.: Машиностроение, 1983. 212 с.

Mechanics, thermodynamics, electromagnetism

86. **Feynman, Richard Ph. • Leighton, Robert B. • Sands, Matthew**. The Feynman Lectures on Physics. New millennium edition. Volume II: Mainly electromagnetism and matter. Basic Books, 2011. 566 pages. *Online:* The Feynman Lectures on Physics. Online edition.
87. **Goldstein, Herbert**; **Poole, Charles P.**; **Safko, John L.** Classical Mechanics. 3rd edition. Addison–Wesley, 2001. 638 pages. *Перевод:* **Голдстейн Г.**, **Пул Ч.**, **Сафко Дж.** Классическая механика. URSS, 2012. 828 с.
88. **Pars, Leopold A.** A treatise on analytical dynamics. London: Heinemann, 1965. 641 pages. *Перевод:* **Парс Л. А.** Аналитическая динамика. «Наука», 1971. 636 с.
89. **Ter Haar, Dirk**. Elements of hamiltonian mechanics. 2nd edition. Pergamon Press, 1971. 201 pages. *Перевод:* **Тер Хаар Д.** Основы гамильтоновой механики. «Наука», 1974. 223 с.
90. **Беляев Н. М.**, **Рядно А. А.** Методы теории теплопроводности. М.: Высшая школа, 1982. В 2-х томах. Том 1, 328 с. Том 2, 304 с.
91. **Бредов М. М.**, **Румянцев В. В.**, **Топтыгин И. Н.** Классическая электродинамика. «Наука», 1985. 400 с.
92. **Феликс Р. Гантмахер** Лекции по аналитической механике. Издание 2-е. «Наука», 1966. 300 с.
93. **Ландау Л. Д.**, **Лифшиц Е. М.** Краткий курс теоретической физики. Книга 1. Механика. Электродинамика. «Наука», 1969. 271 с.
94. **Лойцянский Л. Г.**, **Лурье А. И.** Курс теоретической механики: В 2-х томах. «Дрофа», 2006. Том 1: Статика и кинематика. 9-е издание. 447 с. Том 2: Динамика. 7-е издание. 719 с.
95. **Лурье А. И.** Аналитическая механика. М.: Физматгиз, 1961. 824 с.
96. **Ольховский И. И.** Курс теоретической механики для физиков. 3-е издание. Изд-во МГУ, 1978. 575 с.
97. **Тамм И. Е.** Основы теории электричества. 11-е издание. М.: Физматлит, 2003. 616 с.

Tensors and tensor calculus

98. **McConnell, Albert Joseph**. Applications of tensor analysis. New York: Dover Publications, 1957. 318 pages. *Перевод:* **Мак-Коннел А. Дж.** Введение в тензорный анализ с приложениями к геометрии, механике и физике. М.: Физматгиз, 1963. 412 с.

99. **Schouten, Jan A.** Tensor analysis for physicists. 2nd edition. Dover Publications, 2011. 320 pages. *Перевод:* **Схоутен Я. А.** Тензорный анализ для физиков. «Наука», 1965. 456 с.
100. **Sokolnikoff, I. S.** Tensor analysis: Theory and applications to geometry and mechanics of continua. 2nd edition. John Wiley & Sons, 1965. 361 pages. *Перевод:* **Сокольников И. С.** Тензорный анализ (с приложениями к геометрии и механике сплошных сред). «Наука», 1971. 376 с.
101. **Рашевский П. К.** Риманова геометрия и тензорный анализ. Издание 3-е. «Наука», 1967. 664 с.

Variational methods

102. **Karel Rektorys**. Variační metody v inženýrských problémech a v problémech matematické fyziky. SNTL (Státní nakladatelství technické literatury), 1974. 593 s. *Translation:* **Rektorys, Karel**. Variational Methods in Mathematics, Science and Engineering. Second edition. D. Reidel Publishing Company, 1980. 571 p. *Перевод:* **Ректорис К.** Вариационные методы в математической физике. «Мир», 1985. 590 с.
103. **Washizu, Kyuichiro**. Variational methods in elasticity and plasticity. 3rd edition. Pergamon Press, Oxford, 1982. 630 pages. *Перевод:* **Васидзу К.** Вариационные методы в теории упругости и пластичности. «Мир», 1987. 542 с.
104. **Бердичевский В. Л.** Вариационные принципы механики сплошной среды. «Наука», 1983. 448 с.
105. **Михлин С. Г.** Вариационные методы в математической физике. Издание 2-е. «Наука», 1970. 512 с.

Perturbation methods (asymptotic methods)

106. **Cole, Julian D.** Perturbation methods in applied mathematics. Blaisdell Publishing Co., 1968. 260 pages. *Перевод:* **Коул Дж.** Методы возмущений в прикладной математике. «Мир», 1972. 274 с.
107. **Nayfeh, Ali H.** Introduction to perturbation techniques. Wiley, 1981. 536 pages. *Перевод:* **Найфэ Али Х.** Введение в методы возмущений. «Мир», 1984. 535 с.
108. **Nayfeh, Ali H.** Perturbation methods. Wiley-VCH, 2004. 425 pages.
109. **Боголюбов Н. Н.**, **Митропольский Ю. А.** Асимптотические методы в теории нелинейных колебаний. «Наука», 1974. 504 с.

110. **Васильева А. Б.**, **Бутузов В. Ф.** Асимптотические методы в теории сингулярных возмущений. М.: Высшая школа, 1990. 208 с.

111. **Зино И. Е.**, **Тропп Э. А.** Асимптотические методы в задачах теории теплопроводности и термоупругости. Изд-во ЛГУ, 1978. 224 с.

112. **Моисеев Н. Н.** Асимптотические методы нелинейной механики. 2-е издание. «Наука», 1981. 400 с.

113. **Товстик П. Е.** Устойчивость тонких оболочек: асимптотические методы. «Наука», 1995. 319 с.

Other topics of mathematics

114. **Collatz, Lothar**. Eigenwertaufgaben mit technischen Anwendungen. 2. Auflage. Akademische Verlagsgesellschaft Geest & Portig, Leipzig, 1963. 500 Seiten. *Перевод:* **Коллатц Л.** Задачи на собственные значения (с техническими приложениями). «Наука», 1968. 504 с.

115. **Dwight, Herbert Bristol**. Tables of integrals and other mathematical data. 4th edition. The Macmillan Co., 1961. 336 pages. *Перевод:* **Двайт Г. Б.** Таблицы интегралов и другие математические формулы. Издание 4-е. «Наука», 1973. 228 с.

116. **Kamke, Erich**. Differentialgleichungen, Lösungsmethoden und Lösungen. Bd. I. Gewöhnliche Differentialgleichungen. 10. Auflage. Teubner Verlag, 1977. 670 Seiten. *Перевод:* **Камке Э.** Справочник по обыкновенным дифференциальным уравнениям. 6-е издание. «Лань», 2003. 576 с.

117. **Korn, Granino A.** and **Korn, Theresa M.** Mathematical handbook for scientists and engineers: definitions, theorems, and formulas for reference and review. Revised edition. Dover Publications, 2013. 1152 pages. *Перевод:* **Корн Г.**, **Корн Т.** Справочник по математике для научных работников и инженеров. «Наука», 1974. 832 с.

118. **Лаврентьев М. А.**, **Шабат Б. В.** Методы теории функций комплексного переменного. 4-е издание. «Наука», 1973. 736 с.

119. **Погорелов А. В.** Дифференциальная геометрия. Издание 6-е. «Наука», 1974. 176 с.