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PHYSICS *of* ELASTIC CONTINUA

CONTENTS

chapter 4

THE CLASSICAL LINEAR ELASTICITY

This chapter is about the geometrically linear model with infinitesimal displacements, where

- ✓ $\mathcal{V} = \mathring{\mathcal{V}}$, $\rho = \mathring{\rho}$ — "the equations can be written in the initial configuration" (sometimes it is called "the principle of initial dimensions"),
- ✓ operators $\mathring{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ are indistinguishable,
- ✓ operators δ and $\boldsymbol{\nabla}$ commute, thus for example $\delta \boldsymbol{\nabla} \boldsymbol{u} = \boldsymbol{\nabla} \delta \boldsymbol{u}$.

§ 1. The complete set of equations

Equations of the nonlinear elasticity, even in their simplest cases, lead to the mathematically complex problems. Therefore the linear theory of infinitesimal displacements is applied everywhere. This theory's equations were derived in the first half of the XIX$^{\text{th}}$ century by Cauchy, Navier, Lamé, Clapeyron, Poisson, Saint-Venant, George Green and the other scientists.

The complete closed set of equations of the classical linear theory in the direct invariant tensor notation, including

- ✓ the balance of forces (of momentum, of *vis viva*),
- ✓ the stress–strain relations for a material,
- ✓ $\boldsymbol{u} \mapsto \boldsymbol{\varepsilon}$,

is

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\sigma} + \boldsymbol{v} = \boldsymbol{0}, \quad \boldsymbol{\sigma} = \frac{\partial \Pi}{\partial \boldsymbol{\varepsilon}} = {}^{4}\!\boldsymbol{\mathcal{A}} \boldsymbol{\cdot\cdot}\, \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \boldsymbol{\nabla} \boldsymbol{u}^{\mathsf{S}}. \tag{1.1}$$

Here $\boldsymbol{\sigma}$ is the linear stress tensor, $\boldsymbol{v}$ is the resultant vector of volume loads, $\boldsymbol{\varepsilon}$ is the tensor of infinitesimal relative deformation (strain), $\Pi(\boldsymbol{\varepsilon})$ is the potential energy of deformation per volume unit and ${}^{4}\!\boldsymbol{\mathcal{A}}$ is the stiffness tensor. The latter is tetravalent with the following symmetry

$$^4\!\boldsymbol{\mathcal{A}}_{12\rightleftarrows 34} = {}^4\!\boldsymbol{\mathcal{A}}, \quad {}^4\!\boldsymbol{\mathcal{A}}_{1\rightleftarrows 2} = {}^4\!\boldsymbol{\mathcal{A}}, \quad {}^4\!\boldsymbol{\mathcal{A}}_{3\rightleftarrows 4} = {}^4\!\boldsymbol{\mathcal{A}}.$$

But where does this set (system) of equations follow from?

The equations (1.1) are exact, they can be derived by varying the equations of the nonlinear theory. Varying from an arbitrary configuration is described in § ??.??. The linear theory is the result of varying from the initial unstressed configuration, when

$$\begin{gathered}\boldsymbol{F} = \boldsymbol{E}, \;\; \boldsymbol{C} = {}^2\boldsymbol{0}, \;\; \delta\boldsymbol{C} = \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}}, \\ \boldsymbol{\tau} = {}^2\boldsymbol{0}, \;\; \delta\boldsymbol{\tau} = \delta\boldsymbol{T} = \frac{\partial^2\Pi}{\partial\boldsymbol{C}\partial\boldsymbol{C}} \cdot\cdot\, \delta\boldsymbol{C}, \;\; \boldsymbol{\nabla}\cdot\delta\boldsymbol{\tau} + \rho\,\delta\boldsymbol{f} = \boldsymbol{0}.\end{gathered} \tag{1.2}$$

It remains to change

- ✓ $\delta\boldsymbol{r}$ to $\boldsymbol{u}$,
- ✓ $\delta\boldsymbol{C}$ to $\boldsymbol{\varepsilon}$,
- ✓ $\delta\boldsymbol{\tau}$ to $\boldsymbol{\sigma}$,
- ✓ $\partial^2\Pi/_{\partial\boldsymbol{C}\partial\boldsymbol{C}}$ to ${}^4\!\boldsymbol{\mathcal{A}}$,
- ✓ $\rho\,\delta\boldsymbol{f}$ to $\boldsymbol{v}$.

If the derivation of (1.2) seems abstruse to the reader, it's possible to proceed from the following equations

$$\begin{gathered}\boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\boldsymbol{f} = \boldsymbol{0}, \;\; \boldsymbol{\nabla} = \boldsymbol{F}^{-\mathsf{T}}\cdot\mathring{\boldsymbol{\nabla}}, \;\; \boldsymbol{F} = \boldsymbol{E} + \mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}, \\ \boldsymbol{\tau} = J^{-1}\boldsymbol{F}\cdot\frac{\partial\Pi}{\partial\boldsymbol{C}}\cdot\boldsymbol{F}^{\mathsf{T}}, \;\; \boldsymbol{C} = \mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{S}} + \tfrac{1}{2}\mathring{\boldsymbol{\nabla}}\boldsymbol{u}\cdot\mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}.\end{gathered} \tag{1.3}$$

Assuming the displacement $\boldsymbol{u}$ is small (infinitesimal), we'll move from (1.3) to (1.1).

Or so. Instead of $\boldsymbol{u}$ to take some small enough parameter $\chi\boldsymbol{u}$, $\chi \to 0$. And to represent thereafter the unknowns by the series in the integer exponents of parameter χ

$$\begin{gathered}\boldsymbol{\tau} = \boldsymbol{\tau}^{(0)} + \chi\boldsymbol{\tau}^{(1)} + \ldots, \;\; \boldsymbol{C} = \boldsymbol{C}^{(0)} + \chi\boldsymbol{C}^{(1)} + \ldots, \\ \boldsymbol{\nabla} = \mathring{\boldsymbol{\nabla}} + \chi\boldsymbol{\nabla}^{(1)} + \ldots, \;\; \boldsymbol{F} = \boldsymbol{E} + \chi\mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}, \;\; J = 1 + \chi J^{(1)} + \ldots\end{gathered}$$

The complete set of equations (1.1) comes from the first (zeroth) terms of these series. In the book [61] this is called "formal approximation".

It is impossible to tell unambiguously how small the parameter χ should be — the answer depends on the situation and is determined by whether the linear model describes the effect we are interested in or not. When, as example, I'm interested in a relation between the frequency of a freely vibrating motion after the initial displacement, then a nonlinear model is needed.

A linear problem is posed in the initial volume $\mathcal{V} = \mathring{\mathcal{V}}$, bounded by the surface o with the area vector $\boldsymbol{n} do$ ("the principle of initial dimensions").

The boundary conditions most often are: on the part o_1 of the surface displacements are known, and on another part o_2 the forces are known.

$$\boldsymbol{u}\Big|_{o_1} = \boldsymbol{u}_0 \,, \quad \boldsymbol{n} \cdot \boldsymbol{\sigma}\Big|_{o_2} = \boldsymbol{p} . \tag{1.4}$$

The more complex combinations happen too, if we know the certain components of the both $\boldsymbol{u}$ and $\boldsymbol{t}_{(n)} = \boldsymbol{n} \cdot \boldsymbol{\sigma}$ simultaneously. For example, on a flat face $x = \mathsf{constant}$ when pressing a stamp with a smooth surface $u_x = \nu(y, z)$, $\tau_{xy} = \tau_{xz} = 0$ (the function ν is determined by the stamp's shape).

For the dynamic problems we have $\boldsymbol{f} - \rho \ddot{\boldsymbol{u}}$ instead of just $\boldsymbol{f}$. And the initial conditions for the dynamic problems are set as it's common in mechanics — on the positions and on the velocities: at the given moment of time $t = 0$ $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ are known. The linearity of the problems

The linearity gives the principle of superposition (or independence) of the action of loads. When there are several loads, the problem can be solved for the each load separately. And then the complete solution can be obtained by the summation. For statics this means, for example, the following: if external loads $\boldsymbol{f}$ and $\boldsymbol{p}$ increase by m times (body is fixed on o_1), then $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ will increase by m times too. Potential energy density Π will increase by m^2 times. In reality such is observed only when the loads are small.

The density of the potential energy of the elastic deformation Π

$$\Pi(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\, \boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}$$

and its variation

$$\delta\Pi = \tfrac{1}{2}\,\delta\big(\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}\big) = \tfrac{1}{2}\big(\delta\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \delta\boldsymbol{\varepsilon}\big) = \underbrace{\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}}}_{\boldsymbol{\sigma}} \cdot\cdot\, \delta\boldsymbol{\varepsilon}$$

$$\delta\Pi(\boldsymbol{\varepsilon}) = \frac{\partial\Pi}{\partial\boldsymbol{\varepsilon}} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \delta\boldsymbol{\varepsilon}$$

$$\delta^2\Pi(\boldsymbol{\varepsilon}) = \delta\boldsymbol{\varepsilon} \cdot\cdot\, \frac{\partial^2\Pi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = \delta\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = 2\Pi(\delta\boldsymbol{\varepsilon})$$

As was noted in chapter **??**, the principle of the virtual work (the d'Alembert–Lagrange principle) can be put into the foundation of mechanics. This principle is true for the linear theory too (the internal forces in an elastic medium are potential $\delta W^{(i)} = -\,\delta\Pi$)

$$\int\limits_{\mathcal{V}} \Big((\boldsymbol{f} - \rho\ddot{\boldsymbol{u}}) \cdot \delta\boldsymbol{u} - \delta\Pi \Big) d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p} \cdot \delta\boldsymbol{u}\, do = 0, \quad \boldsymbol{u}\big|_{o_1} = \boldsymbol{0}, \tag{1.5}$$

because

$$\delta\Pi = \boldsymbol{\sigma} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla}\delta\boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \delta\boldsymbol{u}) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \cdot \delta\boldsymbol{u},$$

$$\int\limits_{\mathcal{V}} \delta\Pi\, d\mathcal{V} = \oint\limits_{o(\partial\mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \delta\boldsymbol{u}\, do - \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \cdot \delta\boldsymbol{u}\, d\mathcal{V}$$

and the left part of (1.5) becomes

$$\int\limits_{\mathcal{V}} \Big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} - \rho\ddot{\boldsymbol{u}} \Big) \cdot \delta\boldsymbol{u}\, d\mathcal{V} + \int\limits_{o_2} \Big(\boldsymbol{p} - \boldsymbol{n} \cdot \boldsymbol{\sigma} \Big) \cdot \delta\boldsymbol{u}\, do,$$

that is equal to zero. Notice the boundary condition $\boldsymbol{u}\big|_{o_1} = \boldsymbol{0}$: the virtual displacements are compatible with this constraint $\delta\boldsymbol{u}\big|_{o_1} = \boldsymbol{0}$.

§ 2. The uniqueness of the solution in dynamics

As is typical for linear mathematical physics, the uniqueness theorem is proven "by contradiction". Assume that there are two solutions: $\boldsymbol{u}_1(\boldsymbol{r}, t)$ and $\boldsymbol{u}_2(\boldsymbol{r}, t)$. If the difference $\boldsymbol{u}^* \equiv \boldsymbol{u}_1 - \boldsymbol{u}_2$ will be equal to $\boldsymbol{0}$, then these solutions coincide, that is the solution is unique.

But at first we'll make sure of the existænce of the energy integral — by deriving the balance of mechanical energy equation for the linear model of the small displacements theory

$$\int\limits_{\mathcal{V}} \bigl(\mathrm{K} + \Pi\bigr)^{\bullet} d\mathcal{V} = \int\limits_{\mathcal{V}} \boldsymbol{f} \cdot \dot{\boldsymbol{u}} \, d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p} \cdot \dot{\boldsymbol{u}} \, do, \tag{2.1}$$

$$\boldsymbol{u}\Big|_{o_1} = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \boldsymbol{\sigma}\Big|_{o_2} = \boldsymbol{p},$$

$$\boldsymbol{u}\Big|_{t=0} = \boldsymbol{u}^{\circ}, \quad \dot{\boldsymbol{u}}\Big|_{t=0} = \dot{\boldsymbol{u}}^{\circ}.$$

For the left-hand side we have

$$\dot{\mathrm{K}} = \tfrac{1}{2} \bigl(\rho \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}\bigr)^{\bullet} = \tfrac{1}{2} \rho \bigl(\dot{\boldsymbol{u}} \cdot \ddot{\boldsymbol{u}} + \ddot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}\bigr) = \rho \ddot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}},$$

$$\begin{aligned} \dot{\Pi} = \tfrac{1}{2} \underbrace{\bigl(\boldsymbol{\varepsilon} \cdot\cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\varepsilon}\bigr)^{\bullet}}_{2\boldsymbol{\varepsilon} \cdot\cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \dot{\boldsymbol{\varepsilon}}} = \boldsymbol{\sigma} \cdot\cdot \dot{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\nabla} \dot{\boldsymbol{u}}^{\mathsf{S}} = \boldsymbol{\nabla} \cdot \bigl(\boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\bigr) - \underbrace{\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}}_{-(\boldsymbol{f} - \rho \ddot{\boldsymbol{u}})} \cdot \dot{\boldsymbol{u}} = \\ = \boldsymbol{\nabla} \cdot \bigl(\boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\bigr) + \bigl(\boldsymbol{f} - \rho \ddot{\boldsymbol{u}}\bigr) \cdot \dot{\boldsymbol{u}} \end{aligned}$$

(the balance of momentum $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} - \rho \ddot{\boldsymbol{u}} = \boldsymbol{0}$ is used),

$$\dot{\mathrm{K}} + \dot{\Pi} = \boldsymbol{\nabla} \cdot \bigl(\boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\bigr) + \boldsymbol{f} \cdot \dot{\boldsymbol{u}}.$$

Applying the divergence theorem

$$\int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot \bigl(\boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\bigr) \, d\mathcal{V} = \oint\limits_{o(\partial \mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}} \, do$$

and the boundary condition $\boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{p}$ on o_2, we get (2.1).

From (2.1) it follows that without loads (when there're no external forces, neither volume nor surface), and the full mechanical energy doesn't change:

$$\boldsymbol{f} = \boldsymbol{0} \ \text{ and } \ \boldsymbol{p} = \boldsymbol{0} \ \Rightarrow \int\limits_{\mathcal{V}} \bigl(\mathrm{K} + \Pi\bigr) \, d\mathcal{V} = \mathsf{constant}(t). \tag{2.2}$$

If at the moment $t = 0$ there was unstressed ($\Pi = 0$) rest ($\mathrm{K} = 0$), then

$$\int\limits_{\mathcal{V}} \bigl(\mathrm{K} + \Pi\bigr) \, d\mathcal{V} = 0. \tag{2.2'}$$

The kinetic energy is positive: $\mathrm{K} > 0$ if $\dot{\boldsymbol{u}} \neq \boldsymbol{0}$ and vanishes (nullifies) only when $\dot{\boldsymbol{u}} = \boldsymbol{0}$ — this ensues from its definition $\mathrm{K} \equiv \frac{1}{2}\rho\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}$. The potential energy, being a quadratic form $\Pi(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}$, is positive too: $\Pi > 0$ if $\boldsymbol{\varepsilon} \neq {}^2\boldsymbol{0}$. Such is a priori requirement of the positive definiteness for stiffness tensor ${}^4\!\boldsymbol{\mathcal{A}}$. This is one of "additional inequalities in the theory of elasticity" [29, 61].

Since K and Π are positive definite, (2.2') gives

$$\mathrm{K} = 0,\ \Pi = 0 \ \Rightarrow\ \dot{\boldsymbol{u}} = \boldsymbol{0},\ \boldsymbol{\varepsilon} = \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} = {}^2\boldsymbol{0} \ \Rightarrow\ \boldsymbol{u} = \boldsymbol{u}^{\circ} + \boldsymbol{\omega}^{\circ} \times \boldsymbol{r}$$

($\boldsymbol{u}^{\circ}$ and $\boldsymbol{\omega}^{\circ}$ are some constants of translation and rotation). With an immobile part of the surface

$$\boldsymbol{u}\big|_{o_1} = \boldsymbol{0} \ \Rightarrow\ \boldsymbol{u}^{\circ} = \boldsymbol{0} \text{ and } \boldsymbol{\omega}^{\circ} = \boldsymbol{0} \ \Rightarrow\ \boldsymbol{u} = \boldsymbol{0} \text{ everywhere.}$$

Now remember two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Their difference $\boldsymbol{u}^* \equiv \boldsymbol{u}_1 - \boldsymbol{u}_2$ is a solution of an entirely "homogeneous" (with no constant terms at all) linear problem: in a volume $\boldsymbol{f} = \boldsymbol{0}$, in boundary and in initial conditions — zeroes. Therefore $\boldsymbol{u}^* = \boldsymbol{0}$, and the uniqueness is proven.

As for the existænce of a solution — it cannot be proven for the generic case by simple conclusions. I could only tell that a dynamic problem is evolutional, it describes the progress of a process in time.

The balance (the conservation) of momentum gives the acceleration $\ddot{\boldsymbol{u}}$. Then, moving to the "next time layer" $t + dt$:

$$\begin{gathered}
\dot{\boldsymbol{u}}(\boldsymbol{r}, t+dt) = \dot{\boldsymbol{u}}(\boldsymbol{r}, t) + \ddot{\boldsymbol{u}}dt,\\
\boldsymbol{u}(\boldsymbol{r}, t+dt) = \boldsymbol{u}(\boldsymbol{r}, t) + \dot{\boldsymbol{u}}dt,\\
\boldsymbol{\varepsilon}(\boldsymbol{r}, t+dt) = \big(\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{r}, t+dt)\big)^{\mathsf{S}} \ \Rightarrow\ \boldsymbol{\sigma},\\
\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \rho\ddot{\boldsymbol{u}}(\boldsymbol{r}, t+dt)
\end{gathered}$$

and so forth. Surely, these considerations lack the mathematical scrupulosity. The latter can be found, for example, in the Philippe Ciarlet's monograph [54].

$$\boldsymbol{\sigma} = \frac{\partial\Pi}{\partial\boldsymbol{\varepsilon}} = {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}}$$

§ 3. Hooke's law

That relation between the stress and the deformation (strain), which in the XVII$^{\text{th}}$ century Robert Hooke could only phrase pretty vaguely*, is written as part of the complete set of equations (1.1) and is implemented via the stiffness tensor

$$^{4}\!\boldsymbol{\mathcal{A}} = \frac{\partial^2 \Pi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}} = A^{ijkl} \boldsymbol{r}_{\partial i} \boldsymbol{r}_{\partial j} \boldsymbol{r}_{\partial k} \boldsymbol{r}_{\partial l}\,, \quad A^{ijkl} = \frac{\partial^2 \Pi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}\,. \tag{3.1}$$

The stiffness tensor is the partial derivative of the scalar elastic potential energy density Π twice by the same bivalent tensor of infinitesimal deformation $\boldsymbol{\varepsilon}$. It is symmetric in the pairs of indices: $^{4}\!\boldsymbol{\mathcal{A}}_{12\rightleftarrows 34} = {}^{4}\!\boldsymbol{\mathcal{A}} \Leftrightarrow A^{ijkl} = A^{klij}$. Therefrom we have 36 constants out of $3^4 = 81$ "have a twin" and only 45 are independent. Furthermore, due to the symmetry of the infinitesimal deformation tensor $\boldsymbol{\varepsilon}$, the stiffness tensor $^{4}\!\boldsymbol{\mathcal{A}}$ is symmetric inside each pair of indices: $A^{ijkl} = A^{jikl} = A^{ijlk}$ $(= A^{jilk})$. This reduces the number of the independent constants (the "elastic moduli") to 21:

$$
\begin{array}{l}
A^{abcd} = A^{cdab} = A^{bacd} = A^{abdc} \\
A^{1111} \\
A^{1112} = A^{1121} = A^{1211} = A^{2111} \\
A^{1113} = A^{1131} = A^{1311} = A^{3111} \\
A^{1122} = A^{2211} \\
A^{1123} = A^{1132} = A^{2311} = A^{3211} \\
A^{1133} = A^{3311} \\
A^{1212} = A^{1221} = A^{2112} = A^{2121} \\
A^{1213} = A^{1231} = A^{1312} = A^{1321} = A^{2113} = A^{2131} = A^{3112} = A^{3121} \\
A^{1222} = A^{2122} = A^{2212} = A^{2221} \\
A^{1223} = A^{1232} = A^{2123} = A^{2132} = A^{2312} = A^{2321} = A^{3212} = A^{3221} \\
A^{1233} = A^{2133} = A^{3312} = A^{3321} \\
A^{1313} = A^{1331} = A^{3113} = A^{3131} \\
A^{1322} = A^{2213} = A^{2231} = A^{3122} \\
A^{1323} = A^{1332} = A^{2313} = A^{2331} = A^{3123} = A^{3132} = A^{3213} = A^{3231} \\
A^{1333} = A^{3133} = A^{3313} = A^{3331} \\
A^{2222} \\
A^{2223} = A^{2232} = A^{2322} = A^{3222} \\
A^{2233} = A^{3322} \\
A^{2323} = A^{2332} = A^{3223} = A^{3232} \\
A^{2333} = A^{3233} = A^{3323} = A^{3332} \\
A^{3333}
\end{array}
$$

* "*ceiiinosssttuu, id est, Ut tensio sic vis*" — **Robert Hooke**. Lectures de Potentia Restitutiva, Or of Spring Explaining the Power of Springing Bodies. London, 1678. 56 pages.

The moduli of the tetravalent stiffness tensor are often written as the symmetric 6×6 matrix

$$\underset{6\times6}{[\mathcal{A}]} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix} \equiv \begin{bmatrix} A^{1111} & A^{1122} & A^{1133} & A^{1112} & A^{1113} & A^{1123} \\ A^{2211} & A^{2222} & A^{2233} & A^{1222} & A^{1322} & A^{2223} \\ A^{3311} & A^{3322} & A^{3333} & A^{1233} & A^{1333} & A^{2333} \\ A^{1211} & A^{2212} & A^{3312} & A^{1212} & A^{1213} & A^{1223} \\ A^{1311} & A^{2213} & A^{3313} & A^{1312} & A^{1313} & A^{1323} \\ A^{2311} & A^{2322} & A^{3323} & A^{2312} & A^{2313} & A^{2323} \end{bmatrix} \tag{3.2}$$

Even in Cartesian coordinates x, y, z the quadratic form of elastic energy density $\Pi(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}$ looks pretty huge:

$$\begin{aligned} 2\Pi = a_{11}\varepsilon_x^2 + a_{22}\varepsilon_y^2 + a_{33}\varepsilon_z^2 + a_{44}\varepsilon_{xy}^2 + a_{55}\varepsilon_{xz}^2 + a_{66}\varepsilon_{yz}^2 \\ +2\big[\varepsilon_x\big(a_{12}\varepsilon_y + a_{13}\varepsilon_z + a_{14}\varepsilon_{xy} + a_{15}\varepsilon_{xz} + a_{16}\varepsilon_{yz}\big) \\ +\varepsilon_y\big(a_{23}\varepsilon_z + a_{24}\varepsilon_{xy} + a_{25}\varepsilon_{xz} + a_{26}\varepsilon_{yz}\big) \\ +\varepsilon_z\big(a_{34}\varepsilon_{xy} + a_{35}\varepsilon_{xz} + a_{36}\varepsilon_{yz}\big) \\ +\varepsilon_{xy}\big(a_{45}\varepsilon_{xz} + a_{46}\varepsilon_{yz}\big) + a_{56}\varepsilon_{xz}\varepsilon_{yz}\big]. \end{aligned} \tag{3.3}$$

When a material symmetry is added, then the number of the independent moduli of tensor ${}^4\!\boldsymbol{\mathcal{A}}$ decreases.

For a material with a symmetry plane of the elastic properties, for example $z = \mathsf{constant}$.

The change of signs of the coordinates x and y does not change the potential energy density Π. And it's possible only when

$$\Pi\Big|_{\substack{\varepsilon_{xz} = -\varepsilon_{xz} \\ \varepsilon_{yz} = -\varepsilon_{yz}}} = \Pi \qquad \Leftrightarrow \qquad \begin{aligned} 0 = a_{15} = a_{16} = a_{25} = a_{26} \\ = a_{35} = a_{36} = a_{45} = a_{46} \end{aligned} \tag{3.4}$$

— the number of the independent coefficients lowers to 13.

Let there be then the two planes of symmetry: $z = \mathsf{constant}$ and $y = \mathsf{constant}$. Because energy Π in such a case is not sensitive to the signsof ε_{yx} and ε_{yz}, in addition to (3.4) we have

$$a_{14} = a_{24} = a_{34} = a_{56} = 0 \tag{3.5}$$

— 9 constants are left.

A material with the three mutually orthogonal planes of symmetry — let these be the x and y, z planes — is called the orthotropic (orthogonally anisotropic) . It's easy to see that (3.4) and (3.5) — this is the whole set of null constants, in this case too. So, an orthotropic material is characterized by the nine constants, and for the orthotropy the two mutually perpendicular planes of symmetry are enough. The expression for the elastic energy density here can be simplified to

$$\Pi = \tfrac{1}{2}a_{11}\varepsilon_x^2 + \tfrac{1}{2}a_{22}\varepsilon_y^2 + \tfrac{1}{2}a_{33}\varepsilon_z^2 + \tfrac{1}{2}a_{44}\varepsilon_{xy}^2 + \tfrac{1}{2}a_{55}\varepsilon_{xz}^2 + \tfrac{1}{2}a_{66}\varepsilon_{yz}^2 \\ + a_{12}\varepsilon_x\varepsilon_y + a_{13}\varepsilon_x\varepsilon_z + a_{23}\varepsilon_y\varepsilon_z .$$

For an orthotropic material, the shear (angular) deformations ε_{xy}, ε_{xz}, ε_{yz} are not linked to the normal stresses $\sigma_x = {}^{\partial\Pi}\!/_{\partial\varepsilon_x}$, $\sigma_y = {}^{\partial\Pi}\!/_{\partial\varepsilon_y}$, $\sigma_z = {}^{\partial\Pi}\!/_{\partial\varepsilon_z}$ (and vice versa).

The popular orthotropic material is wood. Elastic properties there differ along three mutually perpendicular lines: by the radius, along the circumference and along the trunk height.

A transversely isotropic material

One more case of anisotropy is a transversely isotropic material. It is characterized by an axis of anisotropy — let it be z. Then any plane which is parallel* to z is a plane of material symmetry. It is clear that this material is orthotropic. But more than that, any rotation of the deformation tensor $\boldsymbol{\varepsilon}$ around the z axis doesn't change the elastic potential energy density Π. Thus

$$\frac{\partial\Pi}{\partial\boldsymbol{\varepsilon}} \cdot\cdot \big(\boldsymbol{k}\times\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\times\boldsymbol{k}\big) = 0, \tag{3.6}$$

because for any small rotation with vector $\delta\mathbf{o}$, the variation of the infinitesimal deformation tensor $\boldsymbol{\varepsilon}$ is $\delta\mathbf{o}\times\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}\times\delta\mathbf{o}$, and $\delta\mathbf{o}$ goes along z with the unit vector $\boldsymbol{k}\equiv\boldsymbol{e}_z$. The equation (3.6) is true for any infinitesimal deformation $\boldsymbol{\varepsilon}$. In components

$$\big(a_{11}\varepsilon_x + a_{12}\varepsilon_y + a_{13}\varepsilon_z\big)\big({-2}\varepsilon_{xy}\big) + \big(a_{12}\varepsilon_x + a_{22}\varepsilon_y + a_{23}\varepsilon_z\big)2\varepsilon_{xy} \\ + 2a_{44}\varepsilon_{xy}\big(\varepsilon_x - \varepsilon_y\big) + 2a_{55}\varepsilon_{xz}\big({-\varepsilon_{yz}}\big) + 2a_{66}\varepsilon_{yz}\varepsilon_{xz} = 0$$

$$\Rightarrow \quad a_{11} = a_{12} + a_{44} = a_{22}, \;\; a_{13} = a_{23}, \;\; a_{55} = a_{66}.$$

Writing the stress tensor like

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\perp} + \boldsymbol{s}\boldsymbol{k} + \boldsymbol{k}\boldsymbol{s} + \tau_{zz}\boldsymbol{k}\boldsymbol{k}, \tag{3.7}$$

* If a plane is parallel to a line, this plane's normal vector is perpendicular to that line.

where

$$\boldsymbol{\sigma}_{\perp} \equiv \tau_{\alpha\beta}\,\boldsymbol{e}_{\alpha}\boldsymbol{e}_{\beta} = \tau_{xx}\,\boldsymbol{ii} + \tau_{xy}\big(\boldsymbol{ij}+\boldsymbol{ji}\big) + \tau_{yy}\,\boldsymbol{jj}\,,$$
$$\boldsymbol{s} \equiv \tau_{\alpha z}\,\boldsymbol{e}_{\alpha} = \tau_{xz}\,\boldsymbol{i} + \tau_{yz}\,\boldsymbol{j}$$
$$\big(\,\alpha,\beta \text{ are } x \text{ or } y,\quad \boldsymbol{e}_x = \boldsymbol{i},\ \ \boldsymbol{e}_y = \boldsymbol{j}\,\big),$$

the Hooke's law for a transversely isotropic material may be presented as

$$\boldsymbol{\sigma}_{\perp} = a_{44}\boldsymbol{\varepsilon}_{\perp} + \big(a_{12}\varepsilon_{\alpha\alpha} + a_{13}\varepsilon_z\big)\boldsymbol{E}_{\perp}\,,\quad \boldsymbol{s} = a_{55}\,\boldsymbol{\epsilon},\quad \tau_{zz} = a_{33}\varepsilon_z + a_{13}\varepsilon_{\alpha\alpha} \tag{3.8}$$
$$\big(\,\text{here }\ \boldsymbol{\varepsilon}_{\perp} \equiv \varepsilon_{\alpha\beta}\,\boldsymbol{e}_{\alpha}\boldsymbol{e}_{\beta}\,,\quad \varepsilon_{\alpha\alpha} = \operatorname{trace}\boldsymbol{\varepsilon}_{\perp} = \varepsilon_x + \varepsilon_y\,,$$
$$\boldsymbol{\epsilon} \equiv \varepsilon_{\alpha z}\,\boldsymbol{e}_{\alpha}\,,\quad \boldsymbol{E}_{\perp} \equiv \boldsymbol{e}_{\alpha}\boldsymbol{e}_{\alpha} = \boldsymbol{ii} + \boldsymbol{ji} + \boldsymbol{ij} + \boldsymbol{jj}\,\big)$$

It comes that a transversely isotropic material is characterized by five non-null mutually independent components, the elastic moduli $a_{12} = A^{1122}$, $a_{13} = A^{1133}$, $a_{33} = A^{3333}$, $a_{44} = A^{1212}$, $a_{55} = A^{1313}$.

A crystal symmetry

There are many different kinds of a crystal symmetry : triclinic, monoclinic, rhombic, tetragonal and others [66]. Each case of symmetry is characterized by the set of orthogonal* tensors $\boldsymbol{Q}$, for which the following equation

$${}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \left(\boldsymbol{Q} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{Q}^{\mathsf{T}}\right) = \boldsymbol{Q} \cdot \left({}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}\right) \cdot \boldsymbol{Q}^{\mathsf{T}} \quad \forall \boldsymbol{\varepsilon} \tag{3.9}$$

is true (for any infinitesimal deformation $\boldsymbol{\varepsilon}$).

Inverse relations

....

$$2\Pi = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon},\quad \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \frac{\partial \widehat{\Pi}}{\partial \boldsymbol{\sigma}} = {}^4\boldsymbol{\mathcal{B}} \cdot\cdot\, \boldsymbol{\sigma},\quad \widehat{\Pi}(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon} - \Pi(\boldsymbol{\varepsilon}) \tag{3.10}$$

For the linear model, "complementary energy" $\widehat{\Pi} = \Pi = \frac{1}{2}\,\boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon}$

....

The theory, describing the linear properties of anisotropic materials — the crystals, the composites, the wood and others, ...

* Orthogonal tensors are those that satisfy the equality $\boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathsf{T}} = \boldsymbol{E}$ (**??**, § **??**.**??**), describing rotations and mirror flippings.

The “material tensors” define the physical properties of bodies and media, kind of

- ✓ the elasticity,
- ✓ the thermal expansion,
- ✓ the thermal conductivity,
- ✓ the electrical conductivity,
- ✓ piezoelectric effect.

The traction vector $\boldsymbol{t}$ and its projections, $\boldsymbol{t}_{\perp}$ and $\boldsymbol{t}_{\parallel}$.

- ✓ the projection of the traction vector on the unit normal vector

$$\boldsymbol{t}_{\perp} = \boldsymbol{t}_{\boldsymbol{n}} = \boldsymbol{t} \bullet \boldsymbol{n} \tag{3.11}$$

(is perpendicular to the cross-section area),

- ✓ the projection of the traction vector on the plane

$$\boldsymbol{t}_{\parallel} = \boldsymbol{t} - \boldsymbol{t}_{\perp} \tag{3.12}$$

§ 4. Hooke’s law for an isotropic material

....

$$= \boldsymbol{E} \tag{4.1}$$

In the components for an isotropic medium we have

$$A_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu \bigl(\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp} \bigr) \tag{4.2}$$

— these are components of an isotropic tensor of the fourth complexity. These components don’t change when a basis rotates.

....

§ 5. Theorems of statics

Clapeyron’s theorem

In equilibrium with the external forces, the volume ones $\boldsymbol{f}$ and the surface ones $\boldsymbol{p}$, the work of these “statically frozen” (that is constant along time) forces on the actual displacements is equal to the double of* the energy of deformation

$$2\int\limits_{\mathcal{V}} \Pi \, d\mathcal{V} = \int\limits_{\mathcal{V}} \boldsymbol{f} \cdot \boldsymbol{u} \, d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p} \cdot \boldsymbol{u} \, do. \tag{5.1}$$

○ $2\Pi = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla} \boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{u}) - \underbrace{\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}}_{-\boldsymbol{f}} \cdot \boldsymbol{u} \;\Rightarrow$

$$\Rightarrow\; 2\int\limits_{\mathcal{V}} \Pi \, d\mathcal{V} = \int\limits_{o_2} \underbrace{\boldsymbol{n} \cdot \boldsymbol{\sigma}}_{\boldsymbol{p}} \cdot \boldsymbol{u} \, do + \int\limits_{\mathcal{V}} \boldsymbol{f} \cdot \boldsymbol{u} \, d\mathcal{V} \quad \bullet$$

From (5.1) also follows, that without loading $\int_{\mathcal{V}} \Pi \, d\mathcal{V} = 0$. Because Π is positive, then the stress $\boldsymbol{\sigma}$, and deformation $\boldsymbol{\varepsilon}$ without a load are equal to zero.

$$2\Pi = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon}$$
$$\dot{\Pi} = \boldsymbol{\sigma} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}}$$
$$\delta\Pi = \boldsymbol{\sigma} \cdot\cdot\, \delta\boldsymbol{\varepsilon}$$

Π is equal to only the half of the work of the external forces.

The accumulated potential energy of deformation Π is equal to only the half of the work done by the external forces, acting from the unstressed configuration to the equilibrium with the external forces.

Clapeyron's theorem implies that the accumulated elastic energy accounts for only the half of the energy spent on the deformation. The remaining half of the work, done by the external forces, is lost somewhere before reaching the equilibrium.

Roger Fosdick and **Lev Truskinovsky**. About Clapeyron's Theorem in Linear Elasticity. *Journal of Elasticity*, Volume 72, July 2003. Pages 145–172.

* "Ce produit représentait d'ailleurs le double de la force vive que le ressort pouvait absorber par l'effet de sa flexion et qui était la mesure naturelle de sa puissance." —
Benoît Paul Émile Clapeyron. Mémoire sur le travail des forces élastiques dans un corps solide élastique déformé par l'action de forces extérieures. *Comptes rendus*, Tome XLVI, Janvier–Juin 1858. Pagine 208–212.

In theory, the concept of the "static loading" is common. It's when the external load is applied infinitely slow (sounds like forever, yeah).

The work of the external forces on the actual displacements is equal to the double of the potential energy density 2Π.

Если снять внешние воздействия мгновенно (бесконечно быстро), то тело будет колебаться. Но из-за сопротивления спустя некоторое время тело придёт в состояние равновесия.

Yes, only the half of the linear elastic energy is stored. The second half is the " additional energy ", which is lost before reaching of the equilibrium on the dynamics — on the internal energy of the particles (of the dissipation), on the vibrations and waves.

But any real loading would be neither a sudden loading nor an infinitely slow loading. These are the two extremes. The real dynamics of applying the loads will always be different from the theory.

In the area of infinitesimal variations the real external forces, applied to the elastic medium, work on virtual displacements and produce the work, which is exactly equal to the variation of the elastic potential energy density.

$$\boldsymbol{\sigma} \cdot\cdot\, \delta \boldsymbol{\varepsilon} = \delta \Pi.$$

A linear elastic medium is a medium, where a variation of work of the internal forces (that is stresses) is a variation of the potential energy density with the opposite sign $-\delta W^{(i)} = \delta \Pi = \delta W^{(e)}$, when the only displacements vary (the stress loads do not vary).

It is necessary that the virtual work of the real external forces on variations of displacements would be equal to the variation of the internal energy with the opposite sign (for an elastic media — the variation of the internal energy).

The uniqueness of the solution theorem

As in dynamics (§ 2), we suppose the existænce of the two solutions and are looking for their difference

....

The uniqueness of the solution, discovered by Gustav Kirchhoff for bodies with the simply connected contour*, is contrary to, as it seems, the everyday experience. Imagine a straight rod, clamped at the one end (the " cantilever ") and compressed at the second end with a longitudinal force (fig. 1). When the load is large enough, the problem of statics has the two solutions, "straight" and "bent". Such a contradiction with the uniqueness theorem comes from the nonlinearity of this problem. If a load is small (infinitesimal), then the solution is described by the linear equations and is unique.

§ 6. Equations in displacements

The complete set of equations (1.1) contains unknowns $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{u}$. Excluding $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, we get the formulation in displacements (symmetrization of $\boldsymbol{\nabla u}$ is redundant due to the symmetry ${}^4\!\boldsymbol{\mathcal{A}}_{3\rightleftarrows 4} = {}^4\!\boldsymbol{\mathcal{A}}$).

$$\begin{gathered} \boldsymbol{\nabla} \cdot \left({}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u} \right) + \boldsymbol{f} = \boldsymbol{0}, \\ \boldsymbol{u}\big|_{o_1} = \boldsymbol{u}_0, \;\; \boldsymbol{n} \cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u}\big|_{o_2} = \boldsymbol{p}. \end{gathered} \tag{6.1}$$

In an isotropic medium (6.1) takes the form

...

Общее решение однородного уравнения (...) нашёл Heinz Neuber

П. Ф. Папкович

...

§ 7. Concentrated force in an infinite medium

A concentrated force is a useful mathematical idealization, but it cannot be found in the real world, where all forces are either body forces acting over a volume or surface forces acting over an area.

Here is a rhetorical question: *why an elastic body withstands an applied load, "bears" it?* The book [12] by James Gordon gives the following answer: the body deforms, and thus the internal forces appear, called "the stresses", which can compensate an external load.

...

§ 8. Finding displacements by deformations

Like any bivalent tensor, the displacement gradient can be decomposed into the sum of the symmetric and antisymmetric parts

$$\boldsymbol{\nabla}\boldsymbol{u} = \overbrace{\boldsymbol{\varepsilon}}^{\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}} - \overbrace{\boldsymbol{\omega}\times\boldsymbol{E}}^{-\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{A}}},\;\; \boldsymbol{\omega} \equiv \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{u}, \tag{8.1}$$

The symmetric part $\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}$ is the linear deformation tensor $\boldsymbol{\varepsilon}$.

The antisymmetric part $\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{A}}$ we will denote as $\boldsymbol{\Omega}$ and will call it the tensor of small rotations. Any antisymmetric bivalent tensor can be represented by a vector (§ **??**.**??**). So, to find displacements $\boldsymbol{u}$ by deformations $\boldsymbol{\varepsilon}$, one more field is needed — the field of rotations $\boldsymbol{\omega}(\boldsymbol{r})$.

....

The compatibility conditions in the linear elasticity

The Saint-Venant's compatibility conditions represent the integrability conditions for a symmetric bivalent tensor field. When such a tensor field is compatible, then it describes some deformation (strain).

In the displacement $\mapsto$ deformation relation $\boldsymbol{\varepsilon} = \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}$, the six components ε_{ij} of deformation $\boldsymbol{\varepsilon}$ originate from only three components u_k of the displacement vector $\boldsymbol{u}$.

The compatibility conditions determine whether this deformation does not cause any gaps and/or overlaps.

(.... add a picture here .....)

...

$$\operatorname{inc}\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\boldsymbol{\varepsilon}\right)^{\mathsf{T}}$$

A contour here is arbitrary, so we have the relation

$$\operatorname{inc}\boldsymbol{\varepsilon} = {}^{2}\mathbf{0}, \tag{8.2}$$

called the compatibility of deformations equation.

...

Expression (8.2) provides constraints on possible variants of a deformation (strain) field.

(... the figure with cut squares ...)

...

Tensor $\operatorname{inc}\boldsymbol{\varepsilon}$ is symmetric together with the $\boldsymbol{\varepsilon}$

...

All equations of the linear theory have an analogue (primary source) in the nonlinear theory. To find it for (8.2), remember the Cauchy–Greendeformation tensor (§ ??.??) and curvature tensors (§ ??.??)

....

....

§ 9. Equations in stresses

The balance of forces (or of momentum)

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0} \tag{9.1}$$

does not quite yet determine the stresses. It's necessary as well that deformations (strains) $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ corresponding to stresses (3.10)

$$\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \frac{\partial \widehat{\Pi}}{\partial \boldsymbol{\sigma}} = {}^4\!\boldsymbol{\mathcal{B}} \cdot\cdot\, \boldsymbol{\sigma} \tag{9.2}$$

were compatible (§ 8)

$$\operatorname{inc} \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) \equiv \boldsymbol{\nabla} \times \Big(\boldsymbol{\nabla} \times \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) \Big)^{\mathsf{T}} = {}^2\mathbf{0}. \tag{9.3}$$

Gathered together, (9.1), (9.2) and (9.3) present the complete closed set (system) of equations in stresses.

...

§ 10. The principle of the minimum potential energy

When the existænce of the deformation energy function is assured, and the external forces are assumed to be constant during varying of displacements, then the principle of virtual work leads to the principle of the minimum potential energy.

The formulation of the principle:

$$\mathscr{E}(\boldsymbol{u}) \equiv \int\limits_{\mathcal{V}} \Big(\Pi(\boldsymbol{u}) - \boldsymbol{f} \cdot \boldsymbol{u} \Big) d\mathcal{V} - \int\limits_{o_2} \boldsymbol{p} \cdot \boldsymbol{u} \, do \to \min, \;\; \boldsymbol{u}\big|_{o_1} = \boldsymbol{u}_0 . \tag{10.1}$$

The functional $\mathscr{E}(\boldsymbol{u})$, called the (full) potential energy enof a linear-elastic body, is minimal when displacements $\boldsymbol{u}$ are true — that is for the solution of a problem (6.1). The input functions $\boldsymbol{u}$ must satisfy the geometrical condition on o_1 (so they don't break the existing constraints and can be continuous or else $\Pi(\boldsymbol{u})$ will not be integrable) .

For the true field of displacements $\boldsymbol{u}$, the quadratic function

$$\Pi(\boldsymbol{u}) = \tfrac{1}{2} \boldsymbol{\nabla} \boldsymbol{u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla} \boldsymbol{u}$$

becomes equal to the true potential energy of deformation. Then

$$\mathscr{E} = \mathscr{E}_{\min},$$

which according to the Clapeyron's theorem (5.1) is

$$\mathscr{E}_{\min} = \int\limits_{\mathcal{V}} \Pi(\boldsymbol{u})\, d\mathcal{V} - \Bigg(\int\limits_{\mathcal{V}} \boldsymbol{f} \cdot \boldsymbol{u}\, d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p} \cdot \boldsymbol{u}\, do \Bigg) = - \int\limits_{\mathcal{V}} \Pi(\boldsymbol{u})\, d\mathcal{V}.$$

Taking a some other satisfactory field of displacements $\boldsymbol{u}'$, look at the finite difference

$$\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) = \int\limits_{\mathcal{V}} \Big(\Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}) - \boldsymbol{f} \cdot (\boldsymbol{u}' - \boldsymbol{u}) \Big) d\mathcal{V} - \int\limits_{o_2} \boldsymbol{p} \cdot (\boldsymbol{u}' - \boldsymbol{u})\, do,$$

seeking $\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) \geq 0$ or (ditto) $\mathscr{E}(\boldsymbol{u}') \geq \mathscr{E}(\boldsymbol{u})$.

$\boldsymbol{f} = \textsf{constant}$ and $\boldsymbol{p} = \textsf{constant}$

$\Pi(\boldsymbol{a}) = \frac{1}{2} \boldsymbol{\nabla} \boldsymbol{a} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla} \boldsymbol{a}$ (but *not* the linear $\frac{1}{2} \boldsymbol{\nabla} \boldsymbol{u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla} \boldsymbol{a}$ — this means $\Pi(\boldsymbol{a}) \neq \frac{1}{2} \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla} \boldsymbol{a}$)

Constraints don't change: $(\boldsymbol{u}' - \boldsymbol{u})\big|_{o_1} = \boldsymbol{u}_0 - \boldsymbol{u}_0 = \boldsymbol{0}$. External surface force $\boldsymbol{p}\big|_{o_2} = \boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{\sigma}$ on o_2 and $= \boldsymbol{0}$ elsewhere on $o(\partial\mathcal{V})$. $\boldsymbol{\sigma} = \boldsymbol{\nabla} \boldsymbol{u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} = {}^2\textsf{constant}$ along with constant $\boldsymbol{p}$ and $\boldsymbol{f}$. Therefore

$$\int\limits_{o_2} \boldsymbol{p} \cdot (\boldsymbol{u}' - \boldsymbol{u})\, do = \oint\limits_{o(\partial\mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot (\boldsymbol{u}' - \boldsymbol{u})\, do = \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot \Big(\boldsymbol{\sigma} \cdot (\boldsymbol{u}' - \boldsymbol{u}) \Big) d\mathcal{V} =$$

$$= \int\limits_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot (\boldsymbol{u}' - \boldsymbol{u})\, d\mathcal{V} + \int\limits_{\mathcal{V}} \boldsymbol{\sigma}^{\mathsf{T}} \cdot\cdot\, \boldsymbol{\nabla} (\boldsymbol{u}' - \boldsymbol{u})\, d\mathcal{V}.$$

Due to symmetry $\boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma} \;\Rightarrow\; \boldsymbol{\sigma}^{\mathsf{T}} \cdot\cdot\, \boldsymbol{\nabla} \boldsymbol{a} = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla} \boldsymbol{a} = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla} \boldsymbol{a}^{\mathsf{S}} \;\forall \boldsymbol{a}$. Разность преобразуется до

$$\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) =$$

$$= \int\limits_{\mathcal{V}} \Big(\Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}) - \big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} \big) \cdot (\boldsymbol{u}' - \boldsymbol{u}) - \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla} (\boldsymbol{u}' - \boldsymbol{u}) \Big) d\mathcal{V}.$$

And with the balance of momentum $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}$

$$\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) = \int\limits_{\mathcal{V}} \Big(\Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}) - \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u}) \Big) d\mathcal{V}.$$

Here

$$\Pi(\boldsymbol{u}') = \tfrac{1}{2} \boldsymbol{\nabla}\boldsymbol{u}' \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u}', \quad \Pi(\boldsymbol{u}) = \tfrac{1}{2} \boldsymbol{\nabla}\boldsymbol{u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u},$$

$$\Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}) = \tfrac{1}{2} \Big(\boldsymbol{\nabla}\boldsymbol{u}' \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u}' - \boldsymbol{\nabla}\boldsymbol{u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u} \Big)$$

$${}^4\!\boldsymbol{\mathcal{A}}_{12\rightleftarrows 34} = {}^4\!\boldsymbol{\mathcal{A}} \;\;\Rightarrow\;\; \boldsymbol{\nabla}\boldsymbol{u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u}' = \boldsymbol{\nabla}\boldsymbol{u}' \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u}$$

$$\tfrac{1}{2} \Big(\boldsymbol{\nabla}\boldsymbol{u}' \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u}' - \boldsymbol{\nabla}\boldsymbol{u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u}' - \boldsymbol{\nabla}\boldsymbol{u}' \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u} \Big)$$

$$\big(\boldsymbol{\nabla}\boldsymbol{u}' - \boldsymbol{\nabla}\boldsymbol{u} \big) = \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u})$$

for a finite difference of potentials

$$\tfrac{1}{2} \boldsymbol{\nabla}\big(\boldsymbol{u}' + \boldsymbol{u}\big) \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\big(\boldsymbol{u}' - \boldsymbol{u}\big) = \Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}),$$

adding to which

$$-\boldsymbol{\nabla}\boldsymbol{u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u}) = -\boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u})$$

we get

$$\tfrac{1}{2} \boldsymbol{\nabla}\big(\boldsymbol{u}' - \boldsymbol{u}\big) \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\big(\boldsymbol{u}' - \boldsymbol{u}\big) = \Pi(\boldsymbol{u}' - \boldsymbol{u})$$

and finally*

$$\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) = \int\limits_{\mathcal{V}} \Pi(\boldsymbol{u}' - \boldsymbol{u}) \, d\mathcal{V}.$$

Since ${}^4\!\boldsymbol{\mathcal{A}}$ is positive definite (§ 2) $\Pi(\boldsymbol{w}) = \frac{1}{2} \boldsymbol{\nabla}\boldsymbol{w} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{w} \geq 0 \;\forall \boldsymbol{w}$ (and $= 0$ only if $\boldsymbol{\nabla}\boldsymbol{w} = \boldsymbol{0} \Leftrightarrow \boldsymbol{w} =$ **constant**: for a case of translation as a whole without deformation.

* $b^2 - a^2 - 2a(b - a) = (b + a)(b - a) - 2a(b - a) = (b - a)^2$

...

$\delta \boldsymbol{\nabla} \boldsymbol{u} = \boldsymbol{\nabla} \delta \boldsymbol{u}$

...

the Ritz method

The minimum functional problem $\mathscr{E}(\boldsymbol{u})$ is approximately solved as

....

the finite element method

....

§ 11. The principle of the minimum complementary energy

When the stress–strain relations (the Hooke's law) assure the existænce of a complementary energy function and the geometrical boundary conditions are assumed constant during variation of stresses, then the principle of minimum complementary energy emerges.

The complementary energy of a linear-elastic body is the following functional over the field of stresses:

$$\mathscr{D}(\boldsymbol{\sigma}) \equiv \int\limits_{\mathcal{V}} \widehat{\Pi}(\boldsymbol{\sigma})\, d\mathcal{V} - \int\limits_{o_1} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{u}_0 \, do, \;\; \boldsymbol{u}_0 \equiv \boldsymbol{u}\big|_{o_1}, \tag{11.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}, \;\; \boldsymbol{n} \cdot \boldsymbol{\sigma}\big|_{o_2} = \boldsymbol{p}.$$

...

The variation of the balance of force equation

$$\delta\big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f}\big) = \boldsymbol{\nabla} \cdot \delta\boldsymbol{\sigma} = \boldsymbol{0}$$

...

The principle of the minimum complementary energy is very useful for estimating inexact (approximate) solutions. But for computations it isn't so essential as the (Lagrange) principle of minimum potential energy (10.1).

To derive the variational principles it is natural to use the principle of the virtual work (§ **??.??**) as a foundation.

§ 12. Mixed principles of stationarity

Prange–Hellinger–Reissner Variational Principle,
named after *Ernst Hellinger*, *Georg Prange* and *Eric Reissner*.

Working independently of Hellinger and Prange, Eric Reissner published his famous six-page paper "On a variational theorem in elasticity" in 1950. In this paper he develops — without, however, considering Hamilton–Jacobi theory — a variational principle same to that of Prange and Hellinger.

Hu–Washizu Variational Principle,
named as *Hu Haichang* and *Kyuichiro Washizu*.

The following functional over the displacements and stresses

$$\mathcal{R}(\boldsymbol{u},\boldsymbol{\sigma}) = \int\limits_{\mathcal{V}} \Big[\boldsymbol{\sigma} \cdot\cdot \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} - \widehat{\Pi}(\boldsymbol{\sigma}) - \boldsymbol{f} \cdot \boldsymbol{u} \Big] d\mathcal{V} - \\ - \int\limits_{o_1} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot (\boldsymbol{u} - \boldsymbol{u}_0)\, do - \int\limits_{o_2} \boldsymbol{p} \cdot \boldsymbol{u}\, do \tag{12.1}$$

carries names of Reissner, Prange and Hellinger.

...

The advantage of the Reissner–Hellinger principle — freedom of variation. But it also has a drawback: on the true solution the functional has no extremum, but only stationarity.

Принцип можно использовать для построения приближённых решений методом Ritz (Ritz method). Задавая аппроксимации

...

Принцип Hu–Washizu [103] формулируется так:

$$\delta \mathcal{W}(\boldsymbol{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}) = 0,$$

$$\mathcal{W} \equiv \int\limits_{\mathcal{V}} \Big[\boldsymbol{\sigma} \cdot\cdot \Big(\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} - \boldsymbol{\varepsilon} \Big) + \Pi(\boldsymbol{\varepsilon}) - \boldsymbol{f} \cdot \boldsymbol{u} \Big] d\mathcal{V} - \\ - \int\limits_{o_1} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot (\boldsymbol{u} - \boldsymbol{u}_0)\, do - \int\limits_{o_2} \boldsymbol{p} \cdot \boldsymbol{u}\, do. \tag{12.2}$$

Как и в принципе Рейсснера–Хеллингера, здесь нет ограничений ни в объёме, ни на поверхности, но добавляется третий независимый аргумент $\boldsymbol{\varepsilon}$. Поскольку $\widehat{\Pi} = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\varepsilon} - \Pi$, то (12.1) and (12.2) кажутся почти одним и тем же.

From the Hu–Washizu principle ensues the complete system of equations with boundary conditions, так как

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§ 13. Antiplane shear

This is such a problem of the linear theory of elasticity, where the non-trivial results are obtained by the simple outputs* .

This problem is about an isotropic elastic continuum in the cartesian coordinates

$$x_\alpha, \qquad \alpha = 1, 2, \qquad x_1 \text{ and } x_2.$$

The plane $x_1,\ x_2$ is a cross-section of a rod, the third coordinate x_3 is perpendicular to the section. The basis vectorsare

$$\boldsymbol{e}_i = \partial_i \boldsymbol{r}, \quad \boldsymbol{r} = x_i \boldsymbol{e}_i, \quad \boldsymbol{e}_i \boldsymbol{e}_i = \boldsymbol{E} \Leftrightarrow \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}.$$

In a case of an antiplane strain (an antiplane shear), the field of displacements $\boldsymbol{u}(\boldsymbol{r})$ is parallel to the third coordinate x_3:

$$\boldsymbol{u} = \upsilon \boldsymbol{e}_3,$$

and υ doesn't depend on x_3:

$$\upsilon = \upsilon(x_1, x_2), \quad \partial_3 \upsilon = 0.$$

The deformation

$$\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla}\big(\upsilon \boldsymbol{e}_3\big)^{\mathsf{S}} = \boldsymbol{e}_3 \boldsymbol{\nabla}\upsilon^{\mathsf{S}} + \upsilon \underbrace{\boldsymbol{\nabla}\boldsymbol{e}_3}_{{}^2\mathbf{0}}{}^{\mathsf{S}} = \tfrac{1}{2}\big(\boldsymbol{\nabla}\upsilon\, \boldsymbol{e}_3 + \boldsymbol{e}_3 \boldsymbol{\nabla}\upsilon\big) \qquad (13.1)$$

In the plane x_1, x_2 of the section

$$\mu = \mu(x_1, x_2), \partial_3 \mu = 0$$

is a possible inhomogeneity of the medium.

* Non-trivial in the theory of elasticity is, for example, when the division of a force by an area gives an infinitely large error in the calculation of the stresses.

§ 14. The torsion of rods

M. de Saint-Venant. Memoire sur la torsion des prismes (1853)

Adhémar-Jean-Claude Barré de Saint-Venant. Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exerçant simultanément. 1856. 327 pages.

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стр. 379–494

This problem, which was studied in detail by Adhémar-Jean-Claude Barré de Saint-Venant, is contained in almost every book about the linear elasticity. It considers a cylinder of some section, loaded only by the surface forces at the ends (... add a figure ...)

$$\begin{aligned} z = \ell:\ \boldsymbol{k} \cdot \boldsymbol{\sigma} &= \boldsymbol{p}(x_\alpha), \\ z = 0:\ -\boldsymbol{k} \cdot \boldsymbol{\sigma} &= \boldsymbol{p}_0(x_\alpha), \end{aligned}$$

where $\boldsymbol{k} \equiv \boldsymbol{e}_3$, $\alpha = 1, 2$, $\boldsymbol{x} \equiv x_\alpha \boldsymbol{e}_\alpha$. Coordinates are x_1, x_2, z.

The resultant (the sum) of the external forces is equal to $\mathbf{0}$, and the resultant couple is directed along the z axis:

$$\int\limits_o \boldsymbol{p} do = \mathbf{0}, \ \ \int\limits_o \boldsymbol{x} \times \boldsymbol{p} do = M\boldsymbol{k}.$$

It is known that the torsion gives the tangential components of stress $\tau_{z1} \equiv \boldsymbol{k} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{e}_1$ and $\tau_{z2} \equiv \boldsymbol{k} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{e}_2$. Assuming that only these components of tensor $\boldsymbol{\sigma}$ are non-zero

$$\boldsymbol{\sigma} = \boldsymbol{s}\boldsymbol{k} + \boldsymbol{k}\boldsymbol{s}, \ \ \boldsymbol{s} \equiv \tau_{z\alpha} \boldsymbol{e}_\alpha.$$

The solution of this problem simplifies if the equations in stresses are used.

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mathbf{0} \;\Rightarrow\; \boldsymbol{\nabla}_{\perp} \cdot \boldsymbol{s} = \mathbf{0} (\boldsymbol{\nabla}_{\perp} \equiv \boldsymbol{e}_{\alpha} \partial_{\alpha}), \partial_z \boldsymbol{s} = \mathbf{0}, \tag{14.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \boldsymbol{\sigma} + \frac{1}{1+\nu} \boldsymbol{\nabla} \boldsymbol{\nabla} \sigma = {}^{2}\mathbf{0} \;\Rightarrow\; \triangle_{\perp} \boldsymbol{s} = \mathbf{0} (\triangle_{\perp} \equiv \partial_{\alpha} \partial_{\alpha}). \tag{14.2}$$

The independence of $\boldsymbol{s}$ from z makes it possible to replace the three-dimensional operators with the two-dimensional ones.

...

§ 15. Plane deformation

Here the displacement vector $\boldsymbol{u}$ is parallel to the plane x_1, x_2 and does not depend on the third coordinate z.

For example рассмотрим полуплоскость с сосредоточенной нормальной силой Q на краю (?? рисунок ??)

...

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There are several dozens of books on the classical linear theory of elasticity, which present some interest. Primarily this is the monograph by Anatoliy I. Lurie [30]. His earlier book [31] is dedicated to solving of the spatial problems. Witold Nowacki published his work [41], filled with the miscellaneous content. The author solved the dynamic problems. Also in his book is a description of the continuum of the Cosserat brothers. Being mathematically complex, the theory of elasticity attracts mathematicians, for example there is the monograph [54] of Philippe G. Ciarlet. Климентий Черны́х (Klimentiy Chernih) described the features of the anisotropy in the linear elastic media [66].

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