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PHYSICS *of* ELASTIC CONTINUA

INTRODUCTION

I presented the following models of an elastic continuum in this book: the nonlinear and the linear ones, the micropolar and the classical momentless; the three-dimensional, the two-dimensional (shells and plates) and the one-dimensional (rods, including thin-walled ones). I also explained the fundamentals of dynamics — oscillations, waves and stability. For the thermoelasticity and the magnetoelasticity, I gave the summary of the classical theories of the thermodynamics and the electrodynamics. The dynamics of destruction is described via the theories of defects and fractures. The approaches to modeling of the human-made inhomogeneous materials (the "composites") are also shown.

The word "continua" in the title of the book says that a body (a medium) is modeled here not as a discrete collection of particles, but as a continuous field of location vectors, a continuous matter. It gives a large convenience, because the apparatus of calculus of infinitesimals can be used for such models.

The book is written using the compact and elegant direct indexless tensor notation. The mathematical apparatus for interpreting the direct tensor relations is located in the first chapter.

I am writing this book simultaneously in the two languages, English and Russian. The reader is free to pick any language of the two.

github.com/VadiqueMe/PhysicsOfElasticContinua

CONTENTS

chapter 1
MATHEMATICAL APPARATUS

0.1. What is abstract? What does math do?

Math is abstract.

Numbers are not real entities. They are purely imaginary concepts.

When we do math, we are playing a game in a world of imagination.

We cannot experience numbers. One can make up stories about them, such as $1 + 1 = 2$. But no one can ever experience such an operation since there's no such thing as *one*.

0.2. Points

....

> Euclid's Elements
> Book I
> Definition 1
> A point is that which has no part.

The description of a point, "that which has no part", indicates that Euclid will be treating a point as having no width, length, or breadth, but as an indivisible location.

0.3. Lines, curved and straight (linear)

> Euclid's Elements
> Book I
> Definition 2
> A line is breadthless length.

"Line" is the second primitive term in the Elements. "Breadthless length" says that a line will have one dimension, length, but it won't

have breadth. The terms "length" and "breadth" are not defined in the Elements.

0.4. *A relation between lines and points*

> Euclid's Elements
> Book I
> Definition 3
> The ends of a line are points.

This statement doesn't mention how many ends a line can have.

0.5. *Do straight lines exists?*

The hypothesis of the existence of straight lines.

The existence of Euclidean straight lines in space.

> Euclid's Elements
> Book I
> Definition 4
> A straight line is a line which lies evenly with the points on itself.

To draw a straight line by hand is absolutely impossible.

0.6. *The existence of vectors. Do vectors exists?*

0.7. *Continuity of line*

0.8. *A point of reference*

0.9. *Translation as the easiest kind of motion. Translations and vectors*

0.10. *Straight line and vector*

A (geometric) vector may be like a straight line with an arrow at one of its ends. Then it is fully described (characterized) by the magnitude and the direction.

Within the abstract algebra, the word *vector* is about any object which can be summed with similar objects and scaled (multiplied) by scalars, and vector space is a synonym of linear space. Therefore I

clarify that in this book *vector* is nothing else than three-dimensional geometric (Ευκλείδειος, Euclidean) vector.

Why are vectors always straight (linear)?

(a) Vectors are linear (straight), they cannot be curved.

(b) Vectors are neither straight nor curved. A vector has the magnitude and the direction. A vector is not a line or a curve, albeit it can be represented by a straight line.

Vector can't be thought of as a line.

0.11. *Line which figures real numbers*

often just "number line"

0.12. *What is a distance?*
0.13. *Plane and more dimensional space*
0.14. *Distance on plane or more dimensional space*
0.15. *What is an angle?*

angle ≡ inclination /slope, slant/ of two lines

two lines sharing a common point are usually called intersecting lines

angle ≡ the amount of rotation of line or plane within space

angle ≡ the result of the dot product of two unit vectors gives angle's cosine

0.16. *Differentiation of continuous into small differential chunks*

small differential chunks

infinitesimal (infinitely small)

A mention of tensors may scare away the reader, commonly avoiding needless complications. Don't be afraid: tensors are used just due to their wonderful property of the invariance — the independence from a coordinate system.

§1. Vector

I propose to begin familiarizing with tensors via memoirs about such a phenomenon as a vector.

✓ A *point* has position in space. The only characteristic that distinguishes one point from another is its position.

✓ A *vector* has both magnitude and direction, but no specific position in space.

1.1. *What is a vector?*

What is "linear"?

(1) straight

(2) relating to, resembling, or having a graph that is a straight line

All vectors are linear objects.

Examples of vectors:

✓ A force acts on an object.

✓ The velocity of an object describes what's happening with this object at an instant.

Multiplication of a vector by a scalar

Multiplication by the minus one

The Newton's action–reaction principle "действие равно противодействию по магнитуде и обратно ему по направлению".

Each mechanical interaction of two objects is characterized by two forces that act on both interacting objects. These forces can be represented as two vectors that are equal in magnitude and reverse in direction.

Multiplying a vector by the negative one -1 reverses the vector's direction but doesn't change its magnitude.

1.2. *The addition and subtraction*

The sum (combination) of two or more vectors is the new "resultant" vector. There are two similar methods to calculate the resultant vector geometrically.

The "head to tail method" involves lining up the head of one vector with the tail of another. Here the resultant goes from the initial point (the "tail") of the first addend to the end point (the "head") of the second addend when the tail (the initial point) of the second one coincides with the head (the end point) of the first one.

[.... figure here ....]

The "parallelogram method" ...

[.... figure here ....]

The vector addition is commutative

$$\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}.$$

....

$$\boldsymbol{p} + \boldsymbol{q},$$
$$\boldsymbol{p} - \boldsymbol{q} = \boldsymbol{p} + (-\boldsymbol{q}) = \boldsymbol{p} + (-1)\boldsymbol{q}.$$

For every action, there's an equal (in magnitude) and opposite (in direction) reaction force.

A vector may be also represented as the sum (combination) of some trio of other vectors, called "basis", when the each of the three is scaled by a number (coefficient). Such a representation is called a "linear combination" of basis vectors. A list (array, tuple) of coefficients alone, without basis vectors, is not enough and can't represent a vector.

....

To get the numerical relations from the vector ones, a coordinate system is introduced, and on its axes the vector relations are projected.

....

У сами́х по себе векторов как элементов векторного пространства компонент нет. Vector components appear only when a certain basis is chosen, then any vector can be decomposed into components. В разных базисах компоненты одного и того же вектора отличаются друг от друга.

Here it is — a vector, $\boldsymbol{v}$ looks like a suitable name for it.

Like all geometric vectors, $\boldsymbol{v}$ is pretty well characterized by the two mutually independent properties: its length (magnitude, norm, modulus) and its direction in space. This characterization is complete, so some two vectors with the same magnitude and the same direction are considered equal.

Every vector exists objectively by itself, independently of methods and units of measurement of both lengths and directions (including any abstractions of such units and methods).

Εὐκλείδης Εὐkleídēs

ευκλείδειος euclidean

plane geometry is the two-dimensional Euclidean geometry

The Elements (Στοιχεῖα) (Stoikheîa)

1.3. The method of coordinates

......

figure 1
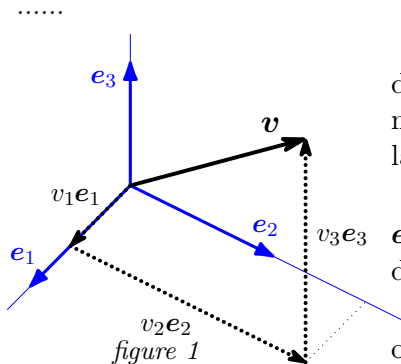
By choosing some mutually perpendicular unit vectors e_i as the basis for measurements, I introduce the rectangular ("cartesian") coordinates.

Three $(i = 1, 2, 3)$ basis vectors e_1, e_2, e_3 are needed for a three-dimensional — 3D — space.

Within such a system, "•"-products of the basis vectors are equal to the Kronecker delta

$$e_i \cdot e_j = \delta_{ij} = \begin{cases} 1, \ i = j \\ 0, \ i \neq j \end{cases}$$

for any orthonormal basis.

Decomposing vector v in some orthonormal basis e_i $(i = 1, 2, 3)$, we get coefficients v_i — the components of vector v in that basis (fig. 1)

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 \equiv \sum_{i=1}^{3} v_i e_i \equiv v_i e_i, \ \ v_i = v \cdot e_i. \tag{1.1}$$

Here and hereinafter, the Einstein's summation convention is accepted: an index repeated twice (and no more than twice) in a single term implies a summation over this index. And a non-repeating index is called "free", and it is identical in the both parts of the equation. These are examples:

$$\sigma = \tau_{ii}, \ \ p_j = n_i \tau_{ij}, \ \ m_i = e_{ijk} x_j f_k, \ \ a_i = \lambda b_i + \mu c_i.$$

(But equations $a = b_{kkk}$, $c = f_i + g_k$, $a_{ij} = k_i \gamma_{ij}$ are incorrect.)

Having components of a vector in an orthonormal basis, the length of this vector is retrieved by the "Πυθαγόρας' equation"

$$\boldsymbol{v}\bullet\boldsymbol{v} = v_i\boldsymbol{e}_i\bullet v_j\boldsymbol{e}_j = v_i\delta_{ij}v_j = v_iv_i, \quad \|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}\bullet\boldsymbol{v}} = \sqrt{v_iv_i}. \quad (1.2)$$

The magnitude represents the length independent of direction.
The direction of a vector in space is measured by the three angles (cosines of angles) between this vector and each of the basis ones:

$$\cos\angle(\boldsymbol{v}\,\hat{}\,\boldsymbol{e}_i) = \underbrace{\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\bullet\boldsymbol{e}_i}_{\boldsymbol{v}\bullet\boldsymbol{e}_i} = \frac{v_i}{\sqrt{v_jv_j}} \Leftrightarrow \underset{\sim}{v_i} = \|\boldsymbol{v}\|\cos\angle(\boldsymbol{v},\boldsymbol{e}_i). \quad (1.3)$$

Measurement of angles. The cosine of an angle between two vectors is the same as the dot product of these vectors if their magnitudes are equal to the one unit of length

When the magnitudes of two vectors are equal to the one unit of length, then the cosine of the least angle between them is the same as the dot product of these vectors. Any vector with the non-unit magnitude (but the null vector) can be "normalized" via dividing a vector by its magnitude.

$$\cos\angle(\boldsymbol{v},\boldsymbol{w}) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\bullet\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}.$$

To accompany the magnitude, which represents the length independent of direction, there's a way to represent the direction of a vector independent of its length. For this purpose, the unit vectors (the vectors with the magnitude of 1) are used.

A rotation matrix is just a transform that expresses the basis vectors of the input space in a different orientation. The length of the basis vectors will be the same, and the origin will not change. Also, the angle between the basis vectors will not change. All that changes is the relative direction of all of the basis vectors.

Therefore, a rotation matrix is not really just a "rotation" matrix; it is an orientation matrix.

There are also pseudovectors, waiting for the reader below in §6.

The angle between two random vectors. According to (1.3)

$$\cos\angle(\boldsymbol{v},\boldsymbol{e}_m) = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\bullet\boldsymbol{e}_m = \frac{v_m}{\sqrt{v_jv_j}},$$

$$\cos\angle(\boldsymbol{w},\boldsymbol{e}_n) = \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}\bullet\boldsymbol{e}_n = \frac{w_n}{\sqrt{w_kw_k}}.$$

The length (1.2) and the direction in space (1.3), that can be measured by the means of the trio of basic vectors, describe

a vector. And every vector possesses these properties[*]. However, this is not enough ("not sufficient" in jargon of the math books).
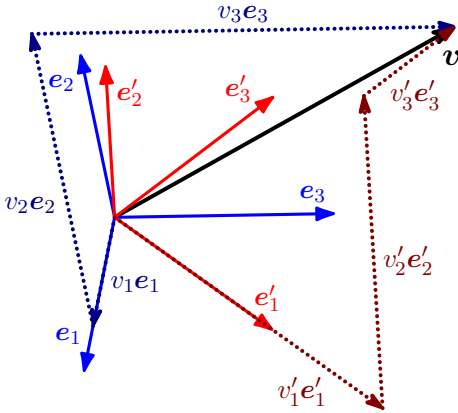
figure 2

A vector is not just a collection of components in some basis.

A triple of pairwise perpendicular unit vectors can only rotate and thereby it can characterize the angular orientation of other vectors.

The decomposition of the same vector $\boldsymbol{v}$ in the two cartesian systems with basis unit vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_i'$ (fig. 2) gives

$$\boldsymbol{v} = v_i \boldsymbol{e}_i = v_i' \boldsymbol{e}_i',$$

where

$$v_i = \boldsymbol{v} \cdot \boldsymbol{e}_i = v_k' \boldsymbol{e}_k' \cdot \boldsymbol{e}_i,$$
$$v_i' = \boldsymbol{v} \cdot \boldsymbol{e}_i' = v_k \boldsymbol{e}_k \cdot \boldsymbol{e}_i'.$$

Appeared here two-index objects (the two-dimensional arrays) $o_{k'i} \equiv \boldsymbol{e}_k' \cdot \boldsymbol{e}_i$ and $o_{ki'} \equiv \boldsymbol{e}_k \cdot \boldsymbol{e}_i'$ are used to shorten formulas.

The "$\cdot$"-product (dot product) of two vectors is commutative — that is, the swapping of multipliers doesn't change the result. Thus

$$o_{k'i} = \boldsymbol{e}_k' \cdot \boldsymbol{e}_i = \cos \angle(\boldsymbol{e}_k', \boldsymbol{e}_i) = \cos \angle(\boldsymbol{e}_i, \boldsymbol{e}_k') = \boldsymbol{e}_i \cdot \boldsymbol{e}_k' = o_{ik'}, \quad (1.3a)$$
$$o_{ki'} = \boldsymbol{e}_k \cdot \boldsymbol{e}_i' = \cos \angle(\boldsymbol{e}_k, \boldsymbol{e}_i') = \cos \angle(\boldsymbol{e}_i', \boldsymbol{e}_k) = \boldsymbol{e}_i' \cdot \boldsymbol{e}_k = o_{i'k}. \quad (1.3b)$$

The lines of equalities (1.3a) and (1.3b) are mutually reciprocal by multiplication

$$o_{k'i} o_{ki'} = 1, \quad o_{ki'} o_{k'i} = 1, \quad o_{k'i} o_{i'k} = 1, \quad o_{i'k} o_{k'i} = 1.$$

Multiplying of an orthogonal matrix by the components of any vector retains the length of this vector:

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = v_i' v_i' = o_{i'k} v_k \, o_{i'n} v_n = v_n v_n$$

— this conclusion leans on (??).

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[*] And what is the direction of the null vector ("(vanishing) vector") $\boldsymbol{0}$ with the zero length $\|\boldsymbol{0}\| = 0$? (The zero vector without a magnitude ends exactly where it begins and thus it is not directed anywhere, its direction is *undefined*.)

Orthogonal transformation of the vector components

$$\boldsymbol{v} \cdot \boldsymbol{e}'_i = v_k \boldsymbol{e}_k \cdot \boldsymbol{e}'_i = \boldsymbol{e}'_i \cdot \boldsymbol{e}_k v_k = o_{i'k} v_k = v'_i \qquad (1.4)$$

is sometimes used for defining a vector itself. If in each orthonormal basis $\boldsymbol{e}_i$ a triplet of numbers v_i is known, and with a rotation of the basis as a whole it is transformed according to (1.4). then this triplet of components represents an invariant object — vector $\boldsymbol{v}$.

§ 2. Tensor and its components

When in each orthonormal basis $\boldsymbol{e}_i$ we have a set of nine ($3^2 = 9$) numbers B_{ij} ($i, j = 1, 2, 3$), and this set is transformed during a transition to a new (rotated) orthonormal basis $\boldsymbol{e}'_i$ as

$$B'_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}_m B_{mn} \boldsymbol{e}_n \cdot \boldsymbol{e}'_j = \boldsymbol{e}'_i \cdot \boldsymbol{e}_m \boldsymbol{e}'_j \cdot \boldsymbol{e}_n B_{mn} = o_{i'm} o_{j'n} B_{mn}, \quad (2.1)$$

then this set of components presents an invariant object — a tensor of a second complexity (of a second valence, bivalent) $^2\boldsymbol{B}$.

In other words, tensor $^2\boldsymbol{B}$ reveals in every basis as a collection of its components B_{ij}, changing along with a basis according to (2.1).

The key example of a second complexity tensor is a dyad. Having two vectors $\boldsymbol{a} = a_i \boldsymbol{e}_i$ and $\boldsymbol{b} = b_i \boldsymbol{e}_i$, in each basis $\boldsymbol{e}_i$ assume $d_{ij} \equiv a_i b_j$. It's easy to see how components d_{ij} transform according to (2.1):

$$a'_i = o_{i'm} a_m, \ b'_j = o_{j'n} b_n \ \Rightarrow \ d'_{ij} = a'_i b'_j = o_{i'm} a_m o_{j'n} b_n = o_{i'm} o_{j'n} d_{mn}.$$

Resulting tensor $^2\boldsymbol{d}$ is called a dyadic product or just dyad and is written as $\boldsymbol{a} \otimes \boldsymbol{b}$ or $\boldsymbol{a}\boldsymbol{b}$. I choose the notation "$^2\boldsymbol{d} = \boldsymbol{a}\boldsymbol{b}$", without the $\otimes$ symbol.

Essential exemplar of a bivalent tensor is the unit tensor (other names are unit dyad, identity tensor and metric tensor). Let for any orthonormal (cartesian) basis $E_{ij} \equiv \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$. These are really components of tensor, (2.1) is actual: $E'_{mn} = o_{m'i} o_{n'j} E_{ij} = o_{m'i} o_{n'i} = \delta_{mn}$. I write this tensor as $\boldsymbol{E}$ (other popular choices are $\boldsymbol{I}$ and $^2\boldsymbol{1}$).

Invariableness of components upon any rotation makes the tensor $\boldsymbol{E}$ isotropic. There are no non-null (nonvanishing) isotropic vectors (all components of the null vector ("(vanishing) vector") $\boldsymbol{0}$ are equal to zero in any basis).

The next example is related to a linear transformation (a linear mapping) of vectors.

If $\boldsymbol{b} = b_i \boldsymbol{e}_i$ is linear (preserving addition and multiplication by number) function of $\boldsymbol{a} = a_j \boldsymbol{e}_j$, then $b_i = c_{ij} a_j$ in every basis. Transformation coefficients c_{ij} alter when a basis rotates:

$$b_i' = c_{ij}' a_j' = o_{i'k} b_k = o_{i'k} c_{kn} a_n, \quad a_n = o_{j'n} a_j' \quad \Rightarrow \quad c_{ij}' = o_{i'k} o_{j'n} c_{kn}.$$

It turns out that a set of two-index objects c_{ij}, c_{ij}', ..., describing the same linear mapping $\boldsymbol{a} \mapsto \boldsymbol{b}$, but in various bases, represents a single invariant object — a tensor of second complexity $^2\boldsymbol{c}$. And many book authors introduce tensors in that way, by means of linear mappings (linear transformations).

And the last example is a bilinear form $F(\boldsymbol{a}, \boldsymbol{b}) = f_{ij} a_i b_j$, where f_{ij} are coefficients, a_i and b_j are components of vector arguments $\boldsymbol{a} = a_i \boldsymbol{e}_i$ and $\boldsymbol{b} = b_j \boldsymbol{e}_j$. The result F is invariant (independent of basis) with the transformation (2.1) for coefficients f_{ij}:

$$F' = f_{ij}' a_i' b_j' = f_{mn} \underbrace{a_m b_n}_{o_{i'm} a_i' \, o_{j'n} b_j'} = F \quad \Leftrightarrow \quad f_{ij}' = o_{i'm} o_{j'n} f_{mn}.$$

If $f_{ij} = \delta_{ij}$, then $F = \delta_{ij} a_i b_j = a_i b_i$ — the "·"-product (dot product, scalar product) of two vectors. When both arguments are the same, such a homogeneous polynomial of second degree (quadratic) of one vector's components $F(\boldsymbol{a}, \boldsymbol{a}) = f_{ij} a_i a_j$ is called a quadratic form.

Now about more complex tensors (of valence larger than two). Tensor of third complexity $^3\boldsymbol{C}$ is represented by a collection of $3^3 = 27$ numbers C_{ijk}, changing with a rotation of basis as

$$C_{ijk}' = \boldsymbol{e}_i' \cdot \boldsymbol{e}_p \boldsymbol{e}_j' \cdot \boldsymbol{e}_q \boldsymbol{e}_k' \cdot \boldsymbol{e}_r C_{pqr} = o_{i'p} o_{j'q} o_{k'r} C_{pqr}. \qquad (2.2)$$

The primary example is a triad of three vectors $\boldsymbol{a} = a_i \boldsymbol{e}_i$, $\boldsymbol{b} = b_j \boldsymbol{e}_j$ and $\boldsymbol{c} = c_k \boldsymbol{e}_k$

$$t_{ijk} \equiv a_i b_j c_k \quad \Leftrightarrow \quad {}^3\boldsymbol{t} = \boldsymbol{abc}.$$

It is seen that orthogonal transformations (2.2) and (2.1) are results of "repeating" vector's (1.4). The reader will easily compose a transformation of components for tensor of any complexity and will write a corresponding polyad as an example.

Vectors with transformation (1.4) are tensors of the first complexity (monovalent tensors).

The least complex objects are scalars or tensors of the zeroth complexity. A scalar is a single ($3^0 = 1$) number, which doesn't depend on a basis: the energy, the mass, the temperature et al. But what are components, for example, of vector $\boldsymbol{v} = v_i\boldsymbol{e}_i$, $v_i = \boldsymbol{v} \cdot \boldsymbol{e}_i$? If not scalars, then what? Here could be no simple answer. In each particular basis, $\boldsymbol{e}_i$ are vectors and v_i are scalars.

§ 3. Tensor algebra, or operations with tensors

The whole tensor algebra can be built on the basis of the only five (four without the equality "=") operations. This paragraph is about them.

So, the tensor algebra consists of the five operations (actions).

Equality

This operation shows whether one tensor ("on the left") is equal to another tensor ("on the right"). Tensors can be equal only when their complexities (valencies) are the same. Tensors of different valencies cannot be equal or not equal.

$$..... \tag{3.1}$$

....

Linear combination

The first is the **linear combination**, it aggregates the addition and the multiplication by a number. Arguments of this operation and the result are of the same complexity. For a two tensors:

$$\lambda a_{ij...} + \mu b_{ij...} = c_{ij...} \Leftrightarrow \lambda\boldsymbol{a} + \mu\boldsymbol{b} = \boldsymbol{c}. \tag{3.2}$$

Here λ and μ are scalar coefficients; $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are tensors of the same complexity. It's easy to show that the components of the result $\boldsymbol{c}$ satisfy an orthogonal transformation like (2.1).

"Decomposition of vector by basis"— a representation of a vector as the sum $\boldsymbol{v} = v_i\boldsymbol{e}_i$ — is nothing else but the linear combination of basis vectors $\boldsymbol{e}_i$ with coefficients v_i.

The second operation is **the multiplication (tensor product, direct product)** . It takes arguments of any complexities, returning the result of a cumulative complexity. Examples:

$$v_i a_{jk} = C_{ijk} \iff \boldsymbol{v}\,{}^2\boldsymbol{a} = {}^3\boldsymbol{C},$$
$$a_{ij} B_{abc} = D_{ijabc} \iff {}^2\boldsymbol{a}\,{}^3\boldsymbol{B} = {}^5\boldsymbol{D}. \tag{3.3}$$

Transformation of a collection of result's components, such as $C_{ijk} = v_i a_{jk}$, during a rotation of basis is orthogonal, similar to (2.2), thus here's no doubt that this collection is a set of tensor components.

Primary and already known (from § 2) subtype of multiplication is the dyadic product of two vectors ${}^2\boldsymbol{A} = \boldsymbol{bc}$.

Contraction

The third operation is called **contraction**. It applies to bivalent and more complex tensors. This operation acts upon a single tensor, without other "participants". Roughly speaking, contracting a tensor is summing of its components over some pair of indices. As a result, tensor's complexity decreases by two.

For a trivalent tensor ${}^3\boldsymbol{D}$ there are the three possible contractions. They give vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ with components

$$a_i = D_{kki}, \quad b_i = D_{kik}, \quad c_i = D_{ikk}. \tag{3.4}$$

A rotation of basis

$$a'_i = D'_{kki} = \underbrace{o_{k'p} o_{k'q}}_{\delta_{pq}} o_{i'r} D_{pqr} = o_{i'r} D_{ppr} = o_{i'r} a_r$$

shows "the tensorial nature" as the result of contraction.

For a tensor of second complexity, the only one variant of contraction is possible. It gives a scalar, known as the trace

$$\boldsymbol{B_\bullet} \equiv \text{trace } \boldsymbol{B} \equiv \mathrm{I}(\boldsymbol{B}) = B_{kk}.$$

The trace of the unit tensor ("contraction of the Kronecker delta") is equal to the dimension of space

$$\text{trace } \boldsymbol{E} = \boldsymbol{E_\bullet} = \delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

Index juggling, transposing

The fourth operation is also applicable to a single tensor of second and bigger complexities. It is named as **index swap, index juggling, transposing**. From components of a tensor, the new collection is emerged with another sequence of indices, the result's complexity stays the same. For example, trivalent tensor ${}^3\boldsymbol{D}$ can give tensors ${}^3\boldsymbol{A}$, ${}^3\boldsymbol{B}$, ${}^3\boldsymbol{C}$ with components

$$
\begin{aligned}
{}^3\boldsymbol{A} &= {}^3\boldsymbol{D}_{1\rightleftarrows2} \Leftrightarrow A_{ijk} = D_{jik}, \\
{}^3\boldsymbol{B} &= {}^3\boldsymbol{D}_{1\rightleftarrows3} \Leftrightarrow B_{ijk} = D_{kji}, \\
{}^3\boldsymbol{C} &= {}^3\boldsymbol{D}_{2\rightleftarrows3} \Leftrightarrow C_{ijk} = D_{ikj}.
\end{aligned}
\tag{3.5}
$$

For a bivalent tensor, the only one transposition is possible: $\boldsymbol{A}^{\mathsf{T}} \equiv \boldsymbol{A}_{1\rightleftarrows2} = \boldsymbol{B} \Leftrightarrow B_{ij} = A_{ji}$. Obviously, $\left(\boldsymbol{A}^{\mathsf{T}}\right)^{\mathsf{T}} = \boldsymbol{A}$.

For the dyadic multiplication of two vectors, $\boldsymbol{ab} = \boldsymbol{ba}^{\mathsf{T}}$.

Combining operations

The four presented operations (actions) can be combined in various sequences.

The combination of multiplication (3.3) and contraction (3.4) — the "•"-product (dot product) — is the most frequently used. In the direct indexless notation this is denoted by large dot "•", which shows the contraction by adjacent indices:

$$
\boldsymbol{a} = \boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{c} \Leftrightarrow a_i = B_{ij}c_j, \quad \boldsymbol{A} = \boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{C} \Leftrightarrow A_{ij} = B_{ik}C_{kj}.
\tag{3.6}
$$

The defining property of the unit tensor — it is the neutral element for the dyadic product with the subsequent contraction by adjacent indices ("•"-product)

$$
{}^{\mathrm{n}}\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{E} = \boldsymbol{E}\boldsymbol{\cdot}{}^{\mathrm{n}}\boldsymbol{a} = {}^{\mathrm{n}}\boldsymbol{a} \quad \forall\, {}^{\mathrm{n}}\boldsymbol{a} \;\; \forall \mathrm{n} > 0.
\tag{3.7}
$$

In the commutative scalar product of two vectors, the dot represents the same: the dyadic product and the subsequent contraction

$$
\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{b} = (\boldsymbol{ab})_{\bullet} = a_i b_i = b_i a_i = (\boldsymbol{ba})_{\bullet} = \boldsymbol{b}\boldsymbol{\cdot}\boldsymbol{a}.
\tag{3.8}
$$

The following identity describes how to swap multipliers for the "•"-product (dot product) of two second complexity tensors

$$
\begin{aligned}
\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{Q} &= \left(\boldsymbol{Q}^{\mathsf{T}}\boldsymbol{\cdot}\boldsymbol{B}^{\mathsf{T}}\right)^{\mathsf{T}} \\
\left(\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{Q}\right)^{\mathsf{T}} &= \boldsymbol{Q}^{\mathsf{T}}\boldsymbol{\cdot}\boldsymbol{B}^{\mathsf{T}}.
\end{aligned}
\tag{3.9}
$$

For two dyads $\boldsymbol{B} = \boldsymbol{bd}$ and $\boldsymbol{Q} = \boldsymbol{pq}$

$$(\boldsymbol{bd} \cdot \boldsymbol{pq})^{\mathsf{T}} = \boldsymbol{pq}^{\mathsf{T}} \cdot \boldsymbol{bd}^{\mathsf{T}}$$
$$d_i p_i \, \boldsymbol{bq}^{\mathsf{T}} = \boldsymbol{qp} \cdot \boldsymbol{db}$$
$$d_i p_i \, \boldsymbol{qb} = p_i d_i \, \boldsymbol{qb}.$$

For a vector and a bivalent tensor

$$\boldsymbol{c} \cdot \boldsymbol{B} = \boldsymbol{B}^{\mathsf{T}} \cdot \boldsymbol{c}, \quad \boldsymbol{B} \cdot \boldsymbol{c} = \boldsymbol{c} \cdot \boldsymbol{B}^{\mathsf{T}}. \tag{3.10}$$

Contraction can be repeated two times or more: $\left(\boldsymbol{A} \cdot \boldsymbol{B}\right)_{\bullet} = \boldsymbol{A} \cdot\cdot \boldsymbol{B} = A_{ij} B_{ji}$, and here are useful equations for second complexity tensors

$$\boldsymbol{A} \cdot\cdot \boldsymbol{B} = \boldsymbol{B} \cdot\cdot \boldsymbol{A}, \quad \boldsymbol{d} \cdot \boldsymbol{A} \cdot \boldsymbol{b} = \boldsymbol{A} \cdot\cdot \boldsymbol{bd} = \boldsymbol{bd} \cdot\cdot \boldsymbol{A} = b_j d_i A_{ij},$$
$$\boldsymbol{A} \cdot\cdot \boldsymbol{B} = \boldsymbol{A}^{\mathsf{T}} \cdot\cdot \boldsymbol{B}^{\mathsf{T}} = A_{ij} B_{ji}, \quad \boldsymbol{A} \cdot\cdot \boldsymbol{B}^{\mathsf{T}} = \boldsymbol{A}^{\mathsf{T}} \cdot\cdot \boldsymbol{B} = A_{ij} B_{ij},$$
$$\boldsymbol{A} \cdot\cdot \boldsymbol{E} = \boldsymbol{E} \cdot\cdot \boldsymbol{A} = \boldsymbol{A}_{\bullet} = A_{jj},$$
$$\boldsymbol{A} \cdot \boldsymbol{B} \cdot\cdot \boldsymbol{E} = A_{ij} B_{jk} \delta_{ki} = \boldsymbol{A} \cdot\cdot \boldsymbol{B}, \quad \boldsymbol{A} \cdot \boldsymbol{A} \cdot\cdot \boldsymbol{E} = \boldsymbol{A} \cdot\cdot \boldsymbol{A}, \tag{3.11}$$
$$\boldsymbol{A} \cdot\cdot \boldsymbol{B} \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B} \cdot\cdot \boldsymbol{C} = \boldsymbol{C} \cdot\cdot \boldsymbol{A} \cdot \boldsymbol{B} = A_{ij} B_{jk} C_{ki},$$
$$\boldsymbol{A} \cdot\cdot \boldsymbol{B} \cdot \boldsymbol{C} \cdot \boldsymbol{D} = \boldsymbol{A} \cdot \boldsymbol{B} \cdot\cdot \boldsymbol{C} \cdot \boldsymbol{D} = \boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{C} \cdot\cdot \boldsymbol{D} =$$
$$= \boldsymbol{D} \cdot\cdot \boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{C} = A_{ij} B_{jk} C_{kh} D_{hi}.$$

§4. Polyadic representation (decomposition)

Before in §2, a tensor was presented as some invariant object, showing itself in every basis as a collection of numbers (components). Such a presentation is typical for majority of books about tensors. Index notation can be convenient, especially when only rectangular coordinates are used, but very often it is not. And the relevant case is physics of elastic continua: it needs more elegant, more powerful and perfect apparatus of the direct tensor calculus, operating with indexless invariant objects.

Linear combination $\boldsymbol{v} = v_i \boldsymbol{e}_i$ from decomposition (1.1) connects vector $\boldsymbol{v}$ with basis $\boldsymbol{e}_i$ and vector's components v_i in that basis. Soon we will get a similar relation for a tensor of any complexity.

Any bivalent tensor $^2\boldsymbol{B}$ has nine components B_{ij} in each basis. The number of various dyads $\boldsymbol{e}_i \boldsymbol{e}_j$ for the same basis is nine (3^2) too. Linear combining these dyads with coefficients B_{ij} gives the sum

$B_{ij}e_ie_j$. This is tensor, but what are its components, and how this representation changes or doesn't change with a rotation of basis?

Components of the constructed sum
$$\left(B_{ij}e_ie_j\right)_{pq} = B_{ij}\delta_{ip}\delta_{jq} = B_{pq}$$
are components of tensor ^2B. And with a rotation of basis
$$B_{ij}'e_i'e_j' = o_{i'p}o_{j'q}B_{pq}o_{i'n}e_no_{j'm}e_m = \delta_{pn}\delta_{qm}B_{pq}e_ne_m = B_{pq}e_pe_q.$$

Doubts are dropped: a tensor of second complexity can be (re)presented as the linear combination
$$^2B = B_{ij}e_ie_j \tag{4.1}$$
— the dyadic decomposition of a bivalent tensor.

For the unit tensor
$$E = E_{ij}e_ie_j = \delta_{ij}e_ie_j = e_ie_i = e_1e_1 + e_2e_2 + e_3e_3,$$
that's why E is called the unit dyad.

Polyadic representations like (4.1) help to operate with the tensors much easier:
$$v \cdot {}^2B = v_ie_i \cdot e_jB_{jk}e_k = v_i\delta_{ij}B_{jk}e_k = v_iB_{ik}e_k,$$
$$e_i \cdot {}^2B \cdot e_j = e_i \cdot B_{pq}e_pe_q \cdot e_j = B_{pq}\delta_{ip}\delta_{qj} = B_{ij} = {}^2B \cdot\cdot\, e_je_i. \tag{4.2}$$

§5. Matrices, permutations and determinants

The matrices are the convenient tool for solving the systems of linear equations and for arranging of elements. Any matrix has the same number of elements in each row and the same number of elements in each column.

Do you need the two-dimensional arrays? The matrices can be presented as tables full of rows and columns.

Do you want a rectangular arrangement of your elements? Matrices are full of numbers and expressions in the rows and columns.

Do you know that matrices are sometimes called arrays?

Matrix dimensions

Matrices come in all sizes that are dimensions .

The dimension of a matrix consists of the number of rows, then a multiplication sign ("×" is used the most often), and then the number of columns.

Examples.

$$[\underset{3\times 3}{\mathcal{A}}] = \text{......}$$

Matrix [A] is a 3×3 matrix, because it has 3 rows and 3 columns. Matrix [B] has 2 rows and 4 columns, so its dimension is 2×4. Matrix [C] is a column matrix (that is a matrix with just one column), and its dimension is 3×1. And [D] is a row matrix with dimension 1×6.

The matrix algebra

The matrix algebra includes the linear operations — the addition of matrices and the multiplication by scalar.

The dimension of a matrix is essential for the binary operations, that is for operations involving the two matrices.

An addition or subtraction of the two matrices is possible only if they have the same sizes.

5.1. *The multiplication of matrices*

....

$$[\underset{m\times n}{\mathcal{A}}] = \ldots$$

The matrix of the result, known as "the matrix product", has the number of rows of the first multiplier matrix and the number of columns of the second matrix.

....

Square matrices

....

Matrices and the one-dimensional arrays

The two indices of a table is more than the single index of a one-dimensional array. Due to this, an one-dimensional array could be presented as a table of rows or as a table of columns.

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix},$$

or the vertical tables

$$\begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix}.$$

$$\det_{i,j} \delta_{ij} = 1$$

...

the permutation parity symbols via the determinant

$$e_{pqr} = e_{ijk}\delta_{pi}\delta_{qj}\delta_{rk} = e_{ijk}\delta_{ip}\delta_{jq}\delta_{kr},$$

$$e_{pqr} = \det \begin{bmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{bmatrix} = \det \begin{bmatrix} \delta_{p1} & \delta_{p2} & \delta_{p3} \\ \delta_{q1} & \delta_{q2} & \delta_{q3} \\ \delta_{r1} & \delta_{r2} & \delta_{r3} \end{bmatrix}. \tag{5.1}$$

...

A determinant is not sensitive to transposing:

$$\det_{i,j} A_{ij} = \det_{i,j} A_{ji} = \det_{j,i} A_{ij}.$$

...

"The determinant of the matrix product of the two matrices is equal to the product of determinants of these matrices"

$$\det_{i,k} B_{ik} \det_{k,j} C_{kj} = \det_{i,j} B_{ik}C_{kj} \tag{5.2}$$

$$e_{fgh} \det_{m,n} B_{ms}C_{sn} = e_{pqr} B_{fi}C_{ip}B_{gj}C_{jq}B_{hk}C_{kr}$$

$$e_{fgh} \det_{m,s} B_{ms} = e_{ijk} B_{fi}B_{gj}B_{hk}$$

$$e_{ijk} \det_{s,n} C_{sn} = e_{pqr} C_{ip}C_{jq}C_{kr}$$

$$e_{fgh}\, e_{ijk} \det_{m,s} B_{ms} \det_{s,n} C_{sn} = e_{ijk}e_{pqr} B_{fi}B_{gj}B_{hk}C_{ip}C_{jq}C_{kr}$$

...

Определитель компонент of a bivalent tensor is invariant, он не меняется с поворотом базиса

$$A'_{ij} = o_{i'm}o_{j'n}A_{mn}$$

§ 6. The cross product

By common notions, the "×"-product (the "cross product", the "vector product", sometimes the "oriented area product") of the two vectors is the vector, heading perpendicular to the plane of multipliers, whose length is equal to the area of the parallelogram, spanned by the multipliers

$$\| \boldsymbol{a} \times \boldsymbol{b} \| = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \sin \angle(\boldsymbol{a}, \boldsymbol{b}).$$

However, a "×"-product isn't quite a vector, since it is not completely invariant.

The multipliers of the "×"-product $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ determine the result's direction in space, with an accuracy up to the sign fig. 3.

Once you pick as the positive the " right-chiral " ("right-handed") or the " left-chiral " ("left-handed") orientation of space, the one direction from the possible two, then the results of the "× "-products become completely determined.
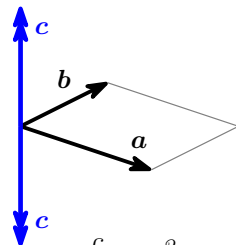
figure 3

"The chiral" means asymmetric in such a way that. the thing and its mirror image are not superimposable, a picture cannot be superposed on its mirror image by any combination of rotations and translations.

An object is chiral if it is distinguishable from its mirror image.

Vectors are usually measured via some a basis $\boldsymbol{e}_i$. They are decomposed into linear combinations like $\boldsymbol{a} = a_i \boldsymbol{e}_i$. So the orientation of space is equivalent to the orientation of the sequential triple of basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. It means that the sequence of basis vectors becomes significant (for linear combinations, the sequence of addends doesn't affect anything).

If two bases consist of different sequences of the same vectors within an oriented space, then their orientations differ by some permutation.

The orientation of the space is a (kind of) asymmetry. This asymmetry makes it impossible to replicate a reflection by the means of any rotations[*]
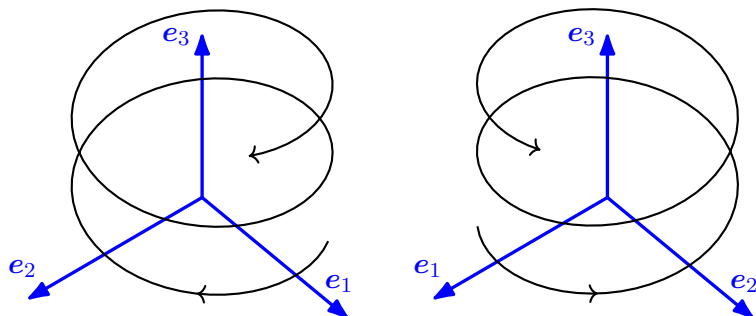
figure 4

A pseudovector is a vector-like object, invariant under any rotation.[**]

... put the figure here ...

Except in the rare cases, the direction of a fully invariant (polar) vector will change with a reflection.

A pseudovector (an axial vector), unlike a polar vector, doesn't change the component orthogonal to the plane of reflection, and turns out to be flipped relatively to the polar vectors and the geometry of the entire space. This happens because the sign (and, accordingly, the direction) of each axial vector changes along with changing the sign of the "×"-product — which corresponds to reflection.

The otherness of pseudovectors narrows the variety of formulas: a pseudovector is not additive with a vector. Formula $v = v_0 + \omega \times r$ is correct, because ω is pseudovector there, and with the cross product two "pseudo" give $(-1)^2 = 1$ ("mutually compensate each other").

---

[*] Applying only rotations, it's impossible to replace the left hand of a human figure into the right hand. But it is possible by reflection of a figure in a mirror.

[**] Rotations cannot change the orientation of a triple of basis vectors, only a reflection can.

The tensor of the permutations parity, the volumetric tensor of third complexity

$$^3\boldsymbol{\epsilon} = \in_{[ijk]} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k, \ \ \in_{[ijk]} \equiv \boldsymbol{e}_i \times \boldsymbol{e}_j \boldsymbol{\cdot} \boldsymbol{e}_k \tag{6.1}$$

with components $\in_{[ijk]}$ equal to "triple" ("mixed", "cross-dot") products of basis vectors.

The absolute value (modulus) of each nonzero component of tensor $^3\boldsymbol{\epsilon}$ is equal to the volume $\sqrt{g}$ of a parallelopiped, drew upon a basis. For basis $\boldsymbol{e}_i$ of pairwise perpendicular one-unit long vectors $\sqrt{g} = 1$.

Tensor $^3\boldsymbol{\epsilon}$ is isotropic, its components are constant enand independent of rotations of a basis. But a reflection — a change in the orientation of a triple of basis vectors (a change in "the direction of a screw") — changes the sign of $^3\boldsymbol{\epsilon}$, so this is pseudotensor (axial tensor).

If $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$, then $\boldsymbol{e}_i$ is ориентированная положительно тройка, произвольно выбираемая из двух вариантов (рис. 3). В таком случае компоненты $^3\boldsymbol{\epsilon}$ равны символу чётности перестановок $\in_{[ijk]} = e_{ijk}$. Когда же $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = -\boldsymbol{e}_3$, then triple $\boldsymbol{e}_i$ is oriented negatively (or "mirrored"). For mirrored triples $\in_{[ijk]} = -e_{ijk}$ (and $e_{ijk} = -\boldsymbol{e}_i \times \boldsymbol{e}_j \boldsymbol{\cdot} \boldsymbol{e}_k$).

With the Levi-Civita tensor $^3\boldsymbol{\epsilon}$ it is possible to take a fresh look at the cross product:

$$\in_{[ijk]} = \boldsymbol{e}_i \times \boldsymbol{e}_j \boldsymbol{\cdot} \boldsymbol{e}_k \ \Leftrightarrow \ \boldsymbol{e}_i \times \boldsymbol{e}_j = \in_{[ijk]} \boldsymbol{e}_k,$$

$$\boldsymbol{a} \times \boldsymbol{b} = a_i \boldsymbol{e}_i \times b_j \boldsymbol{e}_j = a_i b_j \boldsymbol{e}_i \times \boldsymbol{e}_j = a_i b_j \in_{[ijk]} \boldsymbol{e}_k =$$

$$= b_j a_i \boldsymbol{e}_j \boldsymbol{e}_i \boldsymbol{\cdot\cdot} \in_{[mnk]} \boldsymbol{e}_m \boldsymbol{e}_n \boldsymbol{e}_k = \boldsymbol{b}\boldsymbol{a} \boldsymbol{\cdot\cdot} \, ^3\boldsymbol{\epsilon},$$

$$= a_i \in_{[ijk]} \boldsymbol{e}_k b_j = -a_i \in_{[ikj]} \boldsymbol{e}_k b_j = -\boldsymbol{a} \boldsymbol{\cdot} \, ^3\boldsymbol{\epsilon} \boldsymbol{\cdot} \boldsymbol{b}. \tag{6.2}$$

So that, the cross product is not another new, entirely distinct operation. With the Levi-Civita tensor it reduces to the four already described (§ 3) and is applicable to tensors of any complexity.

"The cross product" is just the dot product — the combination of multiplication and contraction ($\S\,3$) — involving tensor $^3\boldsymbol{\epsilon}$. Such combinations are possible with any tensors:

$$\boldsymbol{a} \times {}^2\boldsymbol{B} = a_i \boldsymbol{e}_i \times B_{jk} \boldsymbol{e}_j \boldsymbol{e}_k = \underbrace{a_i B_{jk} \in_{[ijn]}}_{-a_i \in_{[inj]} B_{jk}} \boldsymbol{e}_n \boldsymbol{e}_k = -\boldsymbol{a} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^2\boldsymbol{B},$$

$$
\begin{aligned}
{}^2\boldsymbol{C} \times \boldsymbol{db} &= C_{ij} \boldsymbol{e}_i \boldsymbol{e}_j \times d_p b_q \boldsymbol{e}_p \boldsymbol{e}_q = \boldsymbol{e}_i C_{ij} d_p \underbrace{\in_{[jpk]}}_{-\in_{[pjk]}\,=\,-\in_{[jkp]}} \boldsymbol{e}_k b_q \boldsymbol{e}_q = \\
&= -{}^2\boldsymbol{C}\boldsymbol{d} \cdot\!\cdot {}^3\boldsymbol{\epsilon}\,\boldsymbol{b} = -{}^2\boldsymbol{C} \cdot {}^3\boldsymbol{\epsilon} \cdot \boldsymbol{db},
\end{aligned}
$$

$$\boldsymbol{E} \times \boldsymbol{E} = \boldsymbol{e}_i \boldsymbol{e}_i \times \boldsymbol{e}_j \boldsymbol{e}_j = \underbrace{-\in_{[ijk]} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k}_{+\in_{[ijk]} \boldsymbol{e}_i \boldsymbol{e}_k \boldsymbol{e}_j} = -{}^3\boldsymbol{\epsilon}. \qquad (6.3)$$

The last equation connects the isotropic tensors of second and third complexities.

Generalizing to all tensors of nonzero complexity

$$^{\mathrm{n}}\boldsymbol{\xi} \times {}^{\mathrm{m}}\boldsymbol{\zeta} = -{}^{\mathrm{n}}\boldsymbol{\xi} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^{\mathrm{m}}\boldsymbol{\zeta} \quad \forall\,{}^{\mathrm{n}}\boldsymbol{\xi}, {}^{\mathrm{m}}\boldsymbol{\zeta} \ \ \forall n > 0,\, m > 0. \qquad (6.4)$$

When one of the operands is the unit (metric) tensor, from (6.4) and (3.7) $\forall\,{}^{\mathrm{n}}\boldsymbol{\Upsilon} \ \forall\,\mathrm{n} > 0$

$$\boldsymbol{E} \times {}^{\mathrm{n}}\boldsymbol{\Upsilon} = -\boldsymbol{E} \cdot {}^3\boldsymbol{\epsilon} \cdot {}^{\mathrm{n}}\boldsymbol{\Upsilon} = -{}^3\boldsymbol{\epsilon} \cdot {}^{\mathrm{n}}\boldsymbol{\Upsilon},$$

$$^{\mathrm{n}}\boldsymbol{\Upsilon} \times \boldsymbol{E} = -{}^{\mathrm{n}}\boldsymbol{\Upsilon} \cdot {}^3\boldsymbol{\epsilon} \cdot \boldsymbol{E} = -{}^{\mathrm{n}}\boldsymbol{\Upsilon} \cdot {}^3\boldsymbol{\epsilon}.$$

The cross product of the two vectors is not commutative, but is anticommutative:

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{E}) = (\boldsymbol{a} \times \boldsymbol{E}) \cdot \boldsymbol{b} = -\boldsymbol{ab} \cdot\!\cdot {}^3\boldsymbol{\epsilon} = -{}^3\boldsymbol{\epsilon} \cdot\!\cdot \boldsymbol{ab},$$

$$\boldsymbol{b} \times \boldsymbol{a} = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{E}) = (\boldsymbol{b} \times \boldsymbol{E}) \cdot \boldsymbol{a} = -\boldsymbol{ba} \cdot\!\cdot {}^3\boldsymbol{\epsilon} = -{}^3\boldsymbol{\epsilon} \cdot\!\cdot \boldsymbol{ba}, \quad (6.5)$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{ab} \cdot\!\cdot {}^3\boldsymbol{\epsilon} = \boldsymbol{ba} \cdot\!\cdot {}^3\boldsymbol{\epsilon} \ \Rightarrow \ \boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}.$$

For any bivalent tensor $^2\boldsymbol{B}$ and a tensor of first complexity (vector) $\boldsymbol{a}$

$$^2\boldsymbol{B} \times \boldsymbol{a} = \boldsymbol{e}_i B_{ij} \boldsymbol{e}_j \times a_k \boldsymbol{e}_k = \left(-a_k \boldsymbol{e}_k \times \boldsymbol{e}_j B_{ij} \boldsymbol{e}_i\right)^{\!\top} = -\left(\boldsymbol{a} \times {}^2\boldsymbol{B}^{\top}\right)^{\!\top}.$$

However, in the particular case of the unit tensor $\boldsymbol{E}$ and a vector

$$\begin{aligned}
\boldsymbol{E} \times \boldsymbol{a} &= -\left(\boldsymbol{a} \times \boldsymbol{E}^{\top}\right)^{\!\top} = -\left(\boldsymbol{a} \times \boldsymbol{E}\right)^{\!\top} = \boldsymbol{a} \times \boldsymbol{E}, \\
\boldsymbol{E} \times \boldsymbol{a} &= \boldsymbol{a} \times \boldsymbol{E} = -\boldsymbol{a} \cdot {}^3\boldsymbol{\epsilon} = -{}^3\boldsymbol{\epsilon} \cdot \boldsymbol{a}.
\end{aligned} \qquad (6.6)$$

Справедливо такое соотношение

$$e_{ijk} e_{pqr} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{bmatrix} \qquad (6.7)$$

○ Доказательство начнём с представления символов чётности перестановки как определителей (5.1). $e_{ijk} = \pm\, \boldsymbol{e}_i \times \boldsymbol{e}_j \cdot \boldsymbol{e}_k$ по строкам, $e_{pqr} = \pm\, \boldsymbol{e}_p \times \boldsymbol{e}_q \cdot \boldsymbol{e}_r$ по столбцам, с "−" для "левой" тройки

$$e_{ijk} = \det \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}, \quad e_{pqr} = \det \begin{bmatrix} \delta_{p1} & \delta_{q1} & \delta_{r1} \\ \delta_{p2} & \delta_{q2} & \delta_{r2} \\ \delta_{p3} & \delta_{q3} & \delta_{r3} \end{bmatrix}.$$

Левая часть (6.7) есть произведение $e_{ijk}e_{pqr}$ этих определителей. Но $\det(AB) = (\det A)(\det B)$ — определитель произведения матриц равен произведению определителей (5.2). В матрице-произведении элемент $[\cdots]_{11}$ равен $\delta_{is}\delta_{ps} = \delta_{ip}$, как и в (6.7); легко проверить и другие фрагменты. ●

The contraction of (6.7) приводит к полезным формулам

$$e_{ijk}e_{pqk} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & \delta_{kk} \end{bmatrix} = \det \begin{bmatrix} \delta_{ip} & \delta_{iq} & \delta_{ik} \\ \delta_{jp} & \delta_{jq} & \delta_{jk} \\ \delta_{kp} & \delta_{kq} & 3 \end{bmatrix} =$$

$$= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jk}\delta_{kp} + \delta_{ik}\delta_{jp}\delta_{kq} - \delta_{ik}\delta_{jq}\delta_{kp} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jk}\delta_{kq} =$$

$$= 3\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp} + \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} - 3\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq} =$$

$$= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp},$$

$$e_{ijk}e_{pjk} = \delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp} = 3\delta_{ip} - \delta_{ip} = 2\delta_{ip},$$

$$e_{ijk}e_{ijk} = 2\delta_{ii} = 6.$$

Or in short

$$e_{ijk}e_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \quad e_{ijk}e_{pjk} = 2\delta_{ip}, \quad e_{ijk}e_{ijk} = 6. \tag{6.8}$$

The first of these formulas даёт представление двойного векторного произведения

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = a_i\boldsymbol{e}_i \times \in_{[pqj]} b_p c_q \boldsymbol{e}_j = \in_{[kij]}\in_{[pqj]} a_i b_p c_q \boldsymbol{e}_k =$$

$$= (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip})\, a_i b_p c_q \boldsymbol{e}_k = a_i b_k c_i \boldsymbol{e}_k - a_i b_i c_k \boldsymbol{e}_k =$$

$$= \boldsymbol{a} \cdot \boldsymbol{c}\boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{b}\boldsymbol{c} = \boldsymbol{a} \cdot (\boldsymbol{c}\boldsymbol{b} - \boldsymbol{b}\boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{c}\boldsymbol{b} - \boldsymbol{c}\boldsymbol{b} \cdot \boldsymbol{a}. \tag{6.9}$$

By another interpretation, the dot product of a dyad and a vector is not commutative: $\boldsymbol{b}\boldsymbol{d} \cdot \boldsymbol{c} \neq \boldsymbol{c} \cdot \boldsymbol{b}\boldsymbol{d}$, and this difference can be expressed as

$$\boldsymbol{b}\boldsymbol{d} \cdot \boldsymbol{c} - \boldsymbol{c} \cdot \boldsymbol{b}\boldsymbol{d} = \boldsymbol{c} \times (\boldsymbol{b} \times \boldsymbol{d}). \tag{6.10}$$

$\boldsymbol{a} \cdot \boldsymbol{b}\boldsymbol{c} = \boldsymbol{c}\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{c}\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}\boldsymbol{c}$

$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = -\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{c} \times (\boldsymbol{b} \times \boldsymbol{a})$

The same way it may be derived that

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{ba} - \boldsymbol{ab}) \boldsymbol{\cdot} \boldsymbol{c} = \boldsymbol{ba} \boldsymbol{\cdot} \boldsymbol{c} - \boldsymbol{ab} \boldsymbol{\cdot} \boldsymbol{c}. \qquad (6.11)$$

And following identities for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\begin{aligned}(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{E} &= \in_{[ijk]} a_i b_j \boldsymbol{e}_k \times \boldsymbol{e}_n \boldsymbol{e}_n = a_i b_j \in_{[ijk]} \in_{[knq]} \boldsymbol{e}_q \boldsymbol{e}_n = \\ &= a_i b_j (\delta_{in}\delta_{jq} - \delta_{iq}\delta_{jn}) \boldsymbol{e}_q \boldsymbol{e}_n = a_i b_j \boldsymbol{e}_j \boldsymbol{e}_i - a_i b_j \boldsymbol{e}_i \boldsymbol{e}_j = \\ &\hspace{5cm} = \boldsymbol{ba} - \boldsymbol{ab}, \quad (6.12)\end{aligned}$$

$$\begin{aligned}(\boldsymbol{a} \times \boldsymbol{E}) \boldsymbol{\cdot} (\boldsymbol{b} \times \boldsymbol{E}) &= (\boldsymbol{a} \boldsymbol{\cdot} {}^3\boldsymbol{\epsilon}) \boldsymbol{\cdot} (\boldsymbol{b} \boldsymbol{\cdot} {}^3\boldsymbol{\epsilon}) = \\ &= a_i \in_{[ipn]} \boldsymbol{e}_p \boldsymbol{e}_n \boldsymbol{\cdot} b_j \in_{[jsk]} \boldsymbol{e}_s \boldsymbol{e}_k = a_i b_j \in_{[ipn]} \in_{[nkj]} \boldsymbol{e}_p \boldsymbol{e}_k = \\ &= a_i b_j (\delta_{ik}\delta_{pj} - \delta_{ij}\delta_{pk}) \boldsymbol{e}_p \boldsymbol{e}_k = a_i b_j \boldsymbol{e}_j \boldsymbol{e}_i - a_i b_i \boldsymbol{e}_k \boldsymbol{e}_k = \\ &\hspace{5cm} = \boldsymbol{ba} - \boldsymbol{a} \boldsymbol{\cdot} \boldsymbol{b} \boldsymbol{E}. \quad (6.13)\end{aligned}$$

Finally, the one more correlation between the isotropic tensors of the second and third complexities:

$$^3\boldsymbol{\epsilon} \boldsymbol{\cdot\cdot} {}^3\boldsymbol{\epsilon} = \in_{[ijk]} \boldsymbol{e}_i \in_{[kjn]} \boldsymbol{e}_n = -2\delta_{in} \boldsymbol{e}_i \boldsymbol{e}_n = -2\boldsymbol{E}. \qquad (6.14)$$

§7. Symmetric and antisymmetric tensors

A tensor that does not change upon a permutation of some pair of its indices is called symmetric for that pair of indices. And if a tensor alternates the sign $(+/-)^*$ upon a permutation of some pair of indices, then it is called antisymmetric or skew-symmetric for that pair of indices.

The tensor of the parity of permutations $^3\boldsymbol{\epsilon}$ is antisymmetric by any pair of indices, it is completely (absolutely) antisymmetric (skew-symmetric).

Tensor of the second complexity $\boldsymbol{B}$ is symmetric if $\boldsymbol{B} = \boldsymbol{B}^\mathsf{T}$. When the transposing changes the sign of a tensor $\boldsymbol{A}^\mathsf{T} = -\boldsymbol{A}$, then it is antisymmetric (skew-symmetric).

$$\begin{aligned}\boldsymbol{C} = \boldsymbol{C}^\mathsf{S} + \boldsymbol{C}^\mathsf{A}, \ \boldsymbol{C}^\mathsf{T} = \boldsymbol{C}^\mathsf{S} - \boldsymbol{C}^\mathsf{A}; \\ \boldsymbol{C}^\mathsf{S} \equiv \frac{1}{2}(\boldsymbol{C} + \boldsymbol{C}^\mathsf{T}), \ \boldsymbol{C}^\mathsf{A} \equiv \frac{1}{2}(\boldsymbol{C} - \boldsymbol{C}^\mathsf{T}).\end{aligned} \qquad (7.1)$$

* $\cdot (-1)$

For a dyad $\boldsymbol{cd} = \boldsymbol{cd}^{\mathsf{S}} + \boldsymbol{cd}^{\mathsf{A}} = \frac{1}{2}\left(\boldsymbol{cd} + \boldsymbol{dc}\right) + \frac{1}{2}\left(\boldsymbol{cd} - \boldsymbol{dc}\right)$.

Произведение двух симметричных тензоров $\boldsymbol{C}^{\mathsf{S}} \boldsymbol{\cdot} \boldsymbol{D}^{\mathsf{S}}$ симметрично далеко не всегда, а лишь когда $\boldsymbol{D}^{\mathsf{S}} \boldsymbol{\cdot} \boldsymbol{C}^{\mathsf{S}} = \boldsymbol{C}^{\mathsf{S}} \boldsymbol{\cdot} \boldsymbol{D}^{\mathsf{S}}$, ведь по (??) $\left(\boldsymbol{C}^{\mathsf{S}} \boldsymbol{\cdot} \boldsymbol{D}^{\mathsf{S}}\right)^{\mathsf{T}} = \boldsymbol{D}^{\mathsf{S}} \boldsymbol{\cdot} \boldsymbol{C}^{\mathsf{S}}$.

В нечётномерных пространствах любой антисимметричный тензор второй сложности необрати́м, определитель матрицы компонент для него — нулевой.

Существует взаимно-однозначное соответствие между антисимметричными тензорами второй сложности и (псевдо)векторами. Компоненты кососимметричного тензора полностью определяются тройкой чисел (диагональные элементы матрицы компонент — нули, недиагональные — попарно противоположны). Dot product кососимметричного $\boldsymbol{A}$ и какого-нибудь тензора $^{\mathrm{n}}\boldsymbol{\xi}$ однозначно соответствует cross product'у псевдовектора $\boldsymbol{a}$ и того же тензора $^{\mathrm{n}}\boldsymbol{\xi}$

$$\begin{aligned}
\boldsymbol{b} = \boldsymbol{A} \boldsymbol{\cdot} {}^{\mathrm{n}}\boldsymbol{\xi} &\Leftrightarrow \boldsymbol{a} \times {}^{\mathrm{n}}\boldsymbol{\xi} = \boldsymbol{b} \quad \forall \boldsymbol{A} = \boldsymbol{A}^{\mathsf{A}} \ \ \forall {}^{\mathrm{n}}\boldsymbol{\xi} \ \ \forall \mathrm{n} > 0, \\
\boldsymbol{d} = {}^{\mathrm{n}}\boldsymbol{\xi} \boldsymbol{\cdot} \boldsymbol{A} &\Leftrightarrow {}^{\mathrm{n}}\boldsymbol{\xi} \times \boldsymbol{a} = \boldsymbol{d} \quad \forall \boldsymbol{A} = \boldsymbol{A}^{\mathsf{A}} \ \ \forall {}^{\mathrm{n}}\boldsymbol{\xi} \ \ \forall \mathrm{n} > 0.
\end{aligned} \tag{7.2}$$

Раскроем это соответствие $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{a})$:

$$\boldsymbol{A} \boldsymbol{\cdot} {}^{\mathrm{n}}\boldsymbol{\xi} = \boldsymbol{a} \times {}^{\mathrm{n}}\boldsymbol{\xi}$$
$$A_{hi}\boldsymbol{e}_h \boldsymbol{e}_i \boldsymbol{\cdot} \xi_{jk\ldots q}\boldsymbol{e}_j \boldsymbol{e}_k \ldots \boldsymbol{e}_q = a_i \boldsymbol{e}_i \times \xi_{jk\ldots q}\boldsymbol{e}_j \boldsymbol{e}_k \ldots \boldsymbol{e}_q$$
$$A_{hj}\xi_{jk\ldots q}\boldsymbol{e}_h \boldsymbol{e}_k \ldots \boldsymbol{e}_q = a_i \in_{[ijh]} \xi_{jk\ldots q}\boldsymbol{e}_h \boldsymbol{e}_k \ldots \boldsymbol{e}_q$$
$$A_{hj} = a_i \in_{[ijh]}$$
$$A_{hj} = -a_i \in_{[ihj]}$$
$$\boldsymbol{A} = -\boldsymbol{a} \boldsymbol{\cdot} {}^{3}\boldsymbol{\epsilon}$$

Так же из $^{\mathrm{n}}\boldsymbol{\xi} \boldsymbol{\cdot} \boldsymbol{A} = {}^{\mathrm{n}}\boldsymbol{\xi} \times \boldsymbol{a}$ получается $\boldsymbol{A} = -{}^{3}\boldsymbol{\epsilon} \boldsymbol{\cdot} \boldsymbol{a}$.

Или проще, согласно (6.4)

$$\begin{aligned}
\boldsymbol{A} = \boldsymbol{A} \boldsymbol{\cdot} \boldsymbol{E} = \boldsymbol{a} \times \boldsymbol{E} = -\boldsymbol{a} \boldsymbol{\cdot} {}^{3}\boldsymbol{\epsilon}, \\
\boldsymbol{A} = \boldsymbol{E} \boldsymbol{\cdot} \boldsymbol{A} = \boldsymbol{E} \times \boldsymbol{a} = -{}^{3}\boldsymbol{\epsilon} \boldsymbol{\cdot} \boldsymbol{a}.
\end{aligned}$$

(Псевдо)вектор $\boldsymbol{a}$ называется сопутствующим для тензора $\boldsymbol{A}$.

В общем, для взаимно-однозначного соответствия между $\boldsymbol{A}$ and $\boldsymbol{a}$ имеем

$$\begin{aligned}
\boldsymbol{A} &= -\boldsymbol{a} \boldsymbol{\cdot} {}^{3}\boldsymbol{\epsilon} = \boldsymbol{a} \times \boldsymbol{E} = -{}^{3}\boldsymbol{\epsilon} \boldsymbol{\cdot} \boldsymbol{a} = \boldsymbol{E} \times \boldsymbol{a}, \\
\boldsymbol{a} &= \boldsymbol{a} \boldsymbol{\cdot} \boldsymbol{E} = \boldsymbol{a} \boldsymbol{\cdot} \left(-\frac{1}{2}\,{}^{3}\boldsymbol{\epsilon} \boldsymbol{\cdot\cdot}\, {}^{3}\boldsymbol{\epsilon}\right) = \frac{1}{2}\,\boldsymbol{A} \boldsymbol{\cdot\cdot}\, {}^{3}\boldsymbol{\epsilon}.
\end{aligned} \tag{7.3}$$

The components of a skew-symmetric tensor $\boldsymbol{A}$ thru the components of the accompanying pseudovector $\boldsymbol{a}$

$$\boldsymbol{A} = -\,^3\boldsymbol{\epsilon}\boldsymbol{\cdot}\boldsymbol{a} = -\in_{[ijk]}\boldsymbol{e}_i\boldsymbol{e}_j a_k,$$

$$A_{ij} = -\in_{[ijk]}a_k = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

and vice versa

$$\boldsymbol{a} = \tfrac{1}{2}\,\boldsymbol{A}\boldsymbol{\cdot\cdot}\,^3\boldsymbol{\epsilon} = \tfrac{1}{2}\,A_{jk}\in_{[kji]}\boldsymbol{e}_i,$$

$$a_i = \tfrac{1}{2}\in_{[ikj]}A_{jk} = \frac{1}{2}\begin{bmatrix} \in_{[123]}A_{32} + \in_{[132]}A_{23} \\ \in_{[213]}A_{31} + \in_{[231]}A_{13} \\ \in_{[312]}A_{21} + \in_{[321]}A_{12} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}.$$

The easy to memorize "pseudovector invariant" $\boldsymbol{A}_\times$ comes from the original tensor $\boldsymbol{A}$ via replacing the dyadic product by the cross product

$$\boldsymbol{A}_\times \equiv A_{ij}\,\boldsymbol{e}_i \times \boldsymbol{e}_j = -\boldsymbol{A}\boldsymbol{\cdot\cdot}\,^3\boldsymbol{\epsilon},$$

$$\boldsymbol{A}_\times = \left(\boldsymbol{a}\times\boldsymbol{E}\right)_\times = -2\boldsymbol{a},\quad \boldsymbol{a} = -\tfrac{1}{2}\,\boldsymbol{A}_\times = -\tfrac{1}{2}\left(\boldsymbol{a}\times\boldsymbol{E}\right)_\times. \tag{7.4}$$

Explaination:

$$\boldsymbol{a}\times\boldsymbol{E} = -\tfrac{1}{2}\,\boldsymbol{A}_\times\times\boldsymbol{E} = -\tfrac{1}{2}\,A_{ij}\underbrace{\left(\boldsymbol{e}_i\times\boldsymbol{e}_j\right)}_{\in_{[ijn]}\boldsymbol{e}_n}\times\boldsymbol{e}_k\boldsymbol{e}_k$$

$$= -\tfrac{1}{2}\,A_{ij}\underbrace{\in_{[nij]}\in_{[nkp]}}_{\delta_{jp}\delta_{ik}\,-\,\delta_{ip}\delta_{jk}}\boldsymbol{e}_p\boldsymbol{e}_k = -\tfrac{1}{2}\,A_{ij}\left(\boldsymbol{e}_j\boldsymbol{e}_i - \boldsymbol{e}_i\boldsymbol{e}_j\right)$$

$$= -\tfrac{1}{2}\left(\boldsymbol{A}^\mathsf{T} - \boldsymbol{A}\right) = \boldsymbol{A}^\mathsf{A} = \boldsymbol{A}.$$

The accompanying vector can be introduced for any bivalent tensor. But only the asymmetric part contributes here: $\boldsymbol{C}^\mathsf{A} = -\tfrac{1}{2}\,\boldsymbol{C}_\times\times\boldsymbol{E}$.

For a symmetric tensor, the accompanying vector is zero:

$$\boldsymbol{B}_\times = \boldsymbol{0} \Leftrightarrow \boldsymbol{B} = \boldsymbol{B}^\mathsf{T} = \boldsymbol{B}.$$

With (7.4) the decomposition of some tensor $\boldsymbol{C}$ on the symmetric and the antisymmetric parts looks like

$$\boldsymbol{C} = \boldsymbol{C}^\mathsf{S} - \tfrac{1}{2}\,\boldsymbol{C}_\times\times\boldsymbol{E}. \tag{7.5}$$

For a dyad
$$(6.12) \Rightarrow (\boldsymbol{c} \times \boldsymbol{d}) \times \boldsymbol{E} = \boldsymbol{dc} - \boldsymbol{cd} = -2\boldsymbol{cd}^{\mathsf{A}}, \ (\boldsymbol{cd})_{\times} = \boldsymbol{c} \times \boldsymbol{d},$$

and its decomposition
$$\boldsymbol{cd} = \tfrac{1}{2}(\boldsymbol{cd} + \boldsymbol{dc}) - \tfrac{1}{2}(\boldsymbol{c} \times \boldsymbol{d}) \times \boldsymbol{E}. \tag{7.6}$$

§8. Polar decomposition

Any tensor of the second complexity $\boldsymbol{F}$ with $\det F_{ij} \neq 0$ (not singular) can be decomposed as

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Example. Polar decompose tensor $\boldsymbol{C} = C_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$, where $\boldsymbol{e}_k$ are pairwise perpendicular unit vectors and C_{ij} are the tensor's components.

$$C_{ij} = \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix}$$

$$\boldsymbol{O} = O_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = \boldsymbol{O}_1 \cdot \boldsymbol{O}_2$$

$$O_{ij} = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$\boldsymbol{C} = \boldsymbol{O} \cdot \boldsymbol{S}_{\mathsf{R}}, \ \boldsymbol{O}^{\mathsf{T}} \cdot \boldsymbol{C} = \boldsymbol{S}_{\mathsf{R}}$$

$$\boldsymbol{C} = \boldsymbol{S}_{\mathsf{L}} \cdot \boldsymbol{O}, \ \boldsymbol{C} \cdot \boldsymbol{O}^{\mathsf{T}} = \boldsymbol{S}_{\mathsf{L}}$$

$$S_{\mathsf{R}ij} = \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix}$$

$$S_{\mathsf{L}ij} = \begin{bmatrix} 104/5 & 47/5 & 5 \\ 47/5 & -129/5 & -10 \\ 5 & -10 & 3 \end{bmatrix}$$

...

§9. Eigenvectors and eigenvalues

If for some tensor $^2\boldsymbol{B}$ and the nonzero vector $\boldsymbol{a}$
$$^2\boldsymbol{B} \cdot \boldsymbol{a} = \eta\boldsymbol{a}, \ \boldsymbol{a} \neq \boldsymbol{0} \tag{9.1}$$

$$^2\boldsymbol{B} \cdot \boldsymbol{a} = \eta \boldsymbol{E} \cdot \boldsymbol{a}, \quad \left(^2\boldsymbol{B} - \eta \boldsymbol{E}\right) \cdot \boldsymbol{a} = \boldsymbol{0},$$

then η is called the eigenvalue (or the characteristic value) of tensor $^2\boldsymbol{B}$, and the axis (direction) of eigenvector $\boldsymbol{a}$ is called its characteristic axis (direction).

In components, this is an eigenvalue problem for a matrix. A homogeneous system of linear equations $(B_{ij} - \eta \delta_{ij}) a_j = 0$ has a non-zero solution if the determinant of a matrix of components

$$\det_{i,\, j} (B_{ij} - \eta \delta_{ij})$$

is equal to zero:

$$\det \begin{bmatrix} B_{11} - \eta & B_{12} & B_{13} \\ B_{21} & B_{22} - \eta & B_{23} \\ B_{31} & B_{32} & B_{33} - \eta \end{bmatrix} = -\eta^3 + \text{chaI}\,\eta^2 - \text{chaII}\,\eta + \text{chaIII} = 0;$$

(9.2)

$$\begin{aligned} \text{chaI} &= \text{trace}\,^2\boldsymbol{B} = B_{kk} = B_{11} + B_{22} + B_{33}, \\ \text{chaII} &= B_{11}B_{22} - B_{12}B_{21} + B_{11}B_{33} - B_{13}B_{31} + B_{22}B_{33} - B_{23}B_{32}, \\ \text{chaIII} &= \det\,^2\boldsymbol{B} = \det_{i,j} B_{ij} = e_{ijk} B_{1i} B_{2j} B_{3k} = e_{ijk} B_{i1} B_{j2} B_{k3}. \end{aligned}$$

(9.3)

The roots of the characteristic equation (9.2) — the eigenvalues — don't depend on the basis and therefore are invariants.

The coefficients of (9.3) also don't depend on the basis; they are called the first, the second and the third characteristic invariants of a tensor. The first invariant chaI is the trace. It was described earlier in §3. The second invariant chaII is the trace of the adjugate matrix — the transpose of the cofactor matrix (of the matrix of algebraic complements)

$$\text{chaII}(^2\boldsymbol{B}) \equiv \text{trace}\,(\text{adj}\, B_{ij})$$

(it's hard, yeah). Or

$$\text{chaII}(^2\boldsymbol{B}) \equiv \tfrac{1}{2}\left[(^2\boldsymbol{B}_{\bullet})^2 - {}^2\boldsymbol{B} \boldsymbol{\cdot\cdot}\,^2\boldsymbol{B}\right] = \tfrac{1}{2}\left[(B_{kk})^2 - B_{ij} B_{ji}\right].$$

And the third invariant chaIII is the determinant of a matrix of tensor components: $\text{chaIII}(^2\boldsymbol{B}) \equiv \det\,^2\boldsymbol{B}$.

This applies to all second complexity tensors. Besides that, in case of a symmetric tensor, the following is true:

1°. The eigenvalues of a symmetric bivalent tensor are real numbers.

2°. The characteristic axes (directions) for different eigenvalues are orthogonal to each other.

○ The first statement is proved by contradiction. If η is a complex root of (9.2) corresponding to eigenvector $\boldsymbol{a}$, then conjugate number $\overline{\eta}$ will also be the root of (9.2). Eigenvector $\overline{\boldsymbol{a}}$ with the conjugate components corresponds to it. And then

$$(9.1) \;\Rightarrow\; (\overline{\boldsymbol{a}}\boldsymbol{\cdot})\,{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a} = \eta\boldsymbol{a}, \;\; (\boldsymbol{a}\boldsymbol{\cdot})\,{}^2\boldsymbol{B}\boldsymbol{\cdot}\overline{\boldsymbol{a}} = \overline{\eta}\,\overline{\boldsymbol{a}} \;\Rightarrow$$
$$\Rightarrow\; \overline{\boldsymbol{a}}\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\overline{\boldsymbol{a}} = (\eta - \overline{\eta}\,)\boldsymbol{a}\boldsymbol{\cdot}\overline{\boldsymbol{a}}.$$

Here on the left is zero, because $\boldsymbol{a}\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{c} = \boldsymbol{c}\boldsymbol{\cdot}{}^2\boldsymbol{B}^{\mathsf{T}}\boldsymbol{\cdot}\boldsymbol{a}$ and ${}^2\boldsymbol{B} = {}^2\boldsymbol{B}^{\mathsf{T}}$. Thence $\eta = \overline{\eta}$, that is a real number.

Just as simple looks the proof of 2°:

$$\underbrace{\boldsymbol{a}_2\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a}_1 - \boldsymbol{a}_1\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{a}_2}_{=\,0} = (\eta_1 - \eta_2)\,\boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{a}_2, \; \eta_1 \neq \eta_2 \;\Rightarrow$$
$$\Rightarrow\; \boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{a}_2 = 0. \qquad\bullet$$

If the roots of the characteristic equation (the eigenvalues) are different, then one unit long eigenvectors $\boldsymbol{æ}_i$ compose an orthonormal basis. What are tensor components in such a basis?

$$\,^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{æ}_k = \cancel{\sum_k}\,\eta_k\,\boldsymbol{æ}_k, \;\; k = 1, 2, 3$$
$$\,^2\boldsymbol{B}\boldsymbol{\cdot}\underbrace{\boldsymbol{æ}_k\boldsymbol{æ}_k}_{E} = \sum_k \eta_k\,\boldsymbol{æ}_k\boldsymbol{æ}_k$$

In a common case $B_{ij} = \boldsymbol{e}_i\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{e}_j$. In the basis $\boldsymbol{æ}_1$, $\boldsymbol{æ}_2$, $\boldsymbol{æ}_3$ of mutually perpendicular one unit long $\boldsymbol{æ}_i\boldsymbol{\cdot}\boldsymbol{æ}_j = \delta_{ij}$ eigenvectors of a symmetric tensor

$$B_{11} = \boldsymbol{æ}_1\boldsymbol{\cdot}(\eta_1\boldsymbol{æ}_1\boldsymbol{æ}_1 + \eta_2\boldsymbol{æ}_2\boldsymbol{æ}_2 + \eta_3\boldsymbol{æ}_3\boldsymbol{æ}_3)\boldsymbol{\cdot}\boldsymbol{æ}_1 = \eta_1,$$
$$B_{12} = \boldsymbol{æ}_1\boldsymbol{\cdot}(\eta_1\boldsymbol{æ}_1\boldsymbol{æ}_1 + \eta_2\boldsymbol{æ}_2\boldsymbol{æ}_2 + \eta_3\boldsymbol{æ}_3\boldsymbol{æ}_3)\boldsymbol{\cdot}\boldsymbol{æ}_2 = 0,$$
$$\ldots$$

The matrix of components is diagonal and ${}^2\boldsymbol{B} = \sum \eta_i\,\boldsymbol{æ}_i\boldsymbol{æ}_i$.

Here goes a summation over the three repeating indices, because the special basis is used.

The case of multiplicity of the eigenvalues is considered in the limit.

If simpler $\eta_2 \to \eta_1$, then any linear combination of vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in the limit satisfies the equation (9.1).

Then any axis in the plane $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ becomes characteristic.

When the three eigenvalues coincide, then any axis in the space is characteristic.

Then $^2\boldsymbol{B} = \eta\boldsymbol{E}$, such tensors are called isotropic or "spherical".

§ 10. Collections of invariants of a symmetric bivalent tensor

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"The algebraic" invariants

...

"The characteristic" invariants

These are coefficients of a characteristic equation (9.1).

...

"The research" invariants

...

"The harmonic" invariants

...

§ 11. The rotation tensor

The relation between two "right" (or two "left") orthonormal bases $\boldsymbol{e}_i$ and $\mathring{\boldsymbol{e}}_i$ can be described by a two-index array represented as a matrix (§ 1, § 5)

$$\boldsymbol{e}_i = \boldsymbol{e}_i \cdot \underbrace{\mathring{\boldsymbol{e}}_j \mathring{\boldsymbol{e}}_j}_{\boldsymbol{E}} = o_{ij}^{\circ} \mathring{\boldsymbol{e}}_j, \ \ o_{ij}^{\circ} \equiv \boldsymbol{e}_i \cdot \mathring{\boldsymbol{e}}_j$$

("a matrix of cosines").

Also, a rotation of a tensor can be described by another tensor, called rotation tensor $\boldsymbol{O}$

$$\boldsymbol{e}_i = \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j \cdot \mathring{\boldsymbol{e}}_i = \boldsymbol{O} \cdot \mathring{\boldsymbol{e}}_i, \ \boldsymbol{O} \equiv \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j = \boldsymbol{e}_1 \mathring{\boldsymbol{e}}_1 + \boldsymbol{e}_2 \mathring{\boldsymbol{e}}_2 + \boldsymbol{e}_3 \mathring{\boldsymbol{e}}_3. \quad (11.1)$$

Components of $\boldsymbol{O}$ both in an initial $\mathring{\boldsymbol{e}}_i$ and in a rotated $\boldsymbol{e}_i$ bases are the same

$$\boldsymbol{e}_i \boldsymbol{\cdot} \boldsymbol{O} \boldsymbol{\cdot} \boldsymbol{e}_j = \underbrace{\boldsymbol{e}_i \boldsymbol{\cdot} \boldsymbol{e}_k}_{\delta_{ik}} \mathring{\boldsymbol{e}}_k \boldsymbol{\cdot} \boldsymbol{e}_j = \mathring{\boldsymbol{e}}_i \boldsymbol{\cdot} \boldsymbol{e}_j,$$

$$\mathring{\boldsymbol{e}}_i \boldsymbol{\cdot} \boldsymbol{O} \boldsymbol{\cdot} \mathring{\boldsymbol{e}}_j = \mathring{\boldsymbol{e}}_i \boldsymbol{\cdot} \boldsymbol{e}_k \underbrace{\mathring{\boldsymbol{e}}_k \boldsymbol{\cdot} \mathring{\boldsymbol{e}}_j}_{\delta_{kj}} = \mathring{\boldsymbol{e}}_i \boldsymbol{\cdot} \boldsymbol{e}_j. \tag{11.2}$$

In matrix notation, these components present the ???transposed???? matrix of cosines $o_{\underset{ji}{\circ}} = \mathring{\boldsymbol{e}}_i \boldsymbol{\cdot} \boldsymbol{e}_j$:

$$\boldsymbol{O} = o_{\underset{ji}{\circ}} \boldsymbol{e}_i \boldsymbol{e}_j = o_{\underset{ji}{\circ}} \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j.$$

Spatial transformations in the 3-dimensional Euclidean space $\mathbb{R}^3$ are distinguished into active or alibi transformations, and passive or alias transformations. An active transformation is a transformation which actually changes the physical position (alibi, elsewhere) of objects, which can be defined in the absence of a coordinate system; whereas a passive transformation is merely a change in the coordinate system in which the object is described (alias, other name) (change of coordinates, or change of basis). By transformation, math texts usually refer to active transformations.

Tensor $\boldsymbol{O}$ relates the two vectors — "before rotation" $\mathring{\boldsymbol{r}} = \rho_i \mathring{\boldsymbol{e}}_i$ and "after rotation" $\boldsymbol{r} = \rho_i \boldsymbol{e}_i$. Components $\rho_i = \mathsf{constant}$ of $\boldsymbol{r}$ in rotated basis $\boldsymbol{e}_i$ are the same as of $\mathring{\boldsymbol{r}}$ in immobile basis $\mathring{\boldsymbol{e}}_i$. So that the rotation tensor describes the rotation of the vector together with the basis. And since $\boldsymbol{e}_i = \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j \boldsymbol{\cdot} \mathring{\boldsymbol{e}}_i \Leftrightarrow \rho_i \boldsymbol{e}_i = \boldsymbol{e}_j \mathring{\boldsymbol{e}}_j \boldsymbol{\cdot} \rho_i \mathring{\boldsymbol{e}}_i$, then

$$\boldsymbol{r} = \boldsymbol{O} \boldsymbol{\cdot} \mathring{\boldsymbol{r}} \tag{11.3}$$

(this is the Rodrigues rotation formula).

Olinde Rodrigues. Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendants des causes qui peuvent les produire. *Journal de mathématiques pures et appliquées*, tome 5 (1840), pages 380–440.

For a second complexity tensor $\mathring{\boldsymbol{C}} = C_{ij} \mathring{\boldsymbol{e}}_i \mathring{\boldsymbol{e}}_j$, a rotation into the current position $\boldsymbol{C} = C_{ij} \boldsymbol{e}_i \boldsymbol{e}_j$ looks like

$$\boldsymbol{e}_i C_{ij} \boldsymbol{e}_j = \boldsymbol{e}_i \underbrace{\mathring{\boldsymbol{e}}_i \boldsymbol{\cdot} \mathring{\boldsymbol{e}}_p}_{} C_{pq} \mathring{\boldsymbol{e}}_q \boldsymbol{\cdot} \underbrace{\mathring{\boldsymbol{e}}_j}_{\delta_{ji}} \boldsymbol{e}_j \Leftrightarrow \boldsymbol{C} = \boldsymbol{O} \boldsymbol{\cdot} \mathring{\boldsymbol{C}} \boldsymbol{\cdot} \boldsymbol{O}^{\mathsf{T}}. \tag{11.4}$$

Essential property of a rotation tensor — orthogonality — is expressed as

$$\underset{\substack{e_i \overset{\circ}{e}_i}}{\boldsymbol{O}} \cdot \underset{\substack{\overset{\circ}{e}_j e_j}}{\boldsymbol{O}^\mathsf{T}} = \underset{\substack{\overset{\circ}{e}_i e_i}}{\boldsymbol{O}^\mathsf{T}} \cdot \underset{\substack{e_j \overset{\circ}{e}_j}}{\boldsymbol{O}} = \underset{\substack{\overset{\circ}{e}_i \overset{\circ}{e}_i \\ e_i e_i}}{\boldsymbol{E}}, \tag{11.5}$$

that is the transposed tensor coincides with the reciprocal tensor: $\boldsymbol{O}^\mathsf{T} = \boldsymbol{O}^{-1} \Leftrightarrow \boldsymbol{O} = \boldsymbol{O}^{-\mathsf{T}}$.

An orthogonal tensor retains lengths and angles (the metric) because it does not change the "•"-product of vectors

$$(\boldsymbol{O} \cdot \boldsymbol{a}) \cdot (\boldsymbol{O} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{O}^\mathsf{T} \cdot \boldsymbol{O} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{E} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}. \tag{11.6}$$

For all orthogonal tensors $(\det \boldsymbol{Q})^2 = 1$:

$$1 = \det \boldsymbol{E} = \det \left(\boldsymbol{Q} \cdot \boldsymbol{Q}^\mathsf{T} \right) = \left(\det \boldsymbol{Q} \right) \left(\det \boldsymbol{Q}^\mathsf{T} \right) = \left(\det \boldsymbol{Q} \right)^2.$$

A rotation tensor is an orthogonal tensor with $\det \boldsymbol{O} = 1$.

But not only rotation tensors possess the property of orthogonality. When in (11.1) the first basis is "left", and the second one is "right", then there's a combination of a rotation and a reflection ("rotoreflection") $\boldsymbol{O} = -\boldsymbol{E} \cdot \boldsymbol{O}$ with $\det \left(-\boldsymbol{E} \cdot \boldsymbol{O} \right) = -1$.

У любого бивалентного тензора в трёхмерном пространстве как минимум одно собственное число — the root of (9.2) is non-complex (real). For a rotation tensor, it is equal to one

$$\boldsymbol{O} \cdot \boldsymbol{a} = \eta \boldsymbol{a} \Rightarrow \overset{\boldsymbol{O} \cdot \boldsymbol{a}}{\overbrace{\boldsymbol{a} \cdot \underset{\boldsymbol{E}}{\underbrace{\boldsymbol{O}^\mathsf{T} \cdot \boldsymbol{O}}} \cdot \boldsymbol{a}}} = \eta \boldsymbol{a} \cdot \eta \boldsymbol{a} \Rightarrow \eta^2 = 1.$$

Соответствующая собственная ось называется осью поворота. Теорема Euler'a о конечном повороте в том и состоит, что такая ось существует. Если $\boldsymbol{k}$ — орт этой оси, а ϑ — конечная величина угла поворота, то тензор поворота представим как

$$\boldsymbol{O}(\boldsymbol{k}, \vartheta) = \boldsymbol{E} \cos \vartheta + \boldsymbol{k} \times \boldsymbol{E} \sin \vartheta + \boldsymbol{k}\boldsymbol{k} \left(1 - \cos \vartheta \right). \tag{11.7}$$

Доказывается эта формула так. Направление $\boldsymbol{k}$ при повороте не меняется $(\boldsymbol{O} \cdot \boldsymbol{k} = \boldsymbol{k})$, поэтому на оси поворота $\overset{\circ}{\boldsymbol{e}}_3 = \boldsymbol{e}_3 = \boldsymbol{k}$. В перпендикулярной плоскости (рис. 5) $\overset{\circ}{\boldsymbol{e}}_1 = \boldsymbol{e}_1 \cos \vartheta - \boldsymbol{e}_2 \sin \vartheta$, $\overset{\circ}{\boldsymbol{e}}_2 = \boldsymbol{e}_1 \sin \vartheta + \boldsymbol{e}_2 \cos \vartheta$, $\boldsymbol{O} = \boldsymbol{e}_i \overset{\circ}{\boldsymbol{e}}_i \Rightarrow$ (11.7).

Из (11.7) и (11.3) получаем формулу поворота Родрига в параметрах $\boldsymbol{k}$ и ϑ:

$$\boldsymbol{r} = \overset{\circ}{\boldsymbol{r}} \cos \vartheta + \boldsymbol{k} \times \overset{\circ}{\boldsymbol{r}} \sin \vartheta + \boldsymbol{k}\boldsymbol{k} \cdot \overset{\circ}{\boldsymbol{r}} \left(1 - \cos \vartheta \right).$$

$$\mathring{\boldsymbol{e}}_i = \mathring{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j \boldsymbol{e}_j$$

$$\begin{bmatrix} \mathring{\boldsymbol{e}}_1 \\ \mathring{\boldsymbol{e}}_2 \\ \mathring{\boldsymbol{e}}_3 \end{bmatrix} = \begin{bmatrix} \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{bmatrix}$$
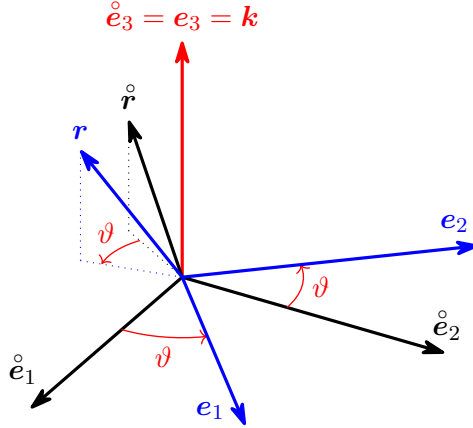
$$\begin{bmatrix} \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_2 & \mathring{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & \cos(90°+\vartheta) & \cos 90° \\ \cos(90°-\vartheta) & \cos\vartheta & \cos 90° \\ \cos 90° & \cos 90° & \cos 0° \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathring{\boldsymbol{e}}_1 = \boldsymbol{e}_1 \cos\vartheta - \boldsymbol{e}_2 \sin\vartheta$$
$$\mathring{\boldsymbol{e}}_2 = \boldsymbol{e}_1 \sin\vartheta + \boldsymbol{e}_2 \cos\vartheta$$
$$\mathring{\boldsymbol{e}}_3 = \boldsymbol{e}_3 = \boldsymbol{k}$$

$$\boldsymbol{O} = \boldsymbol{e}_1 \mathring{\boldsymbol{e}}_1 + \boldsymbol{e}_2 \mathring{\boldsymbol{e}}_2 + \boldsymbol{e}_3 \mathring{\boldsymbol{e}}_3 =$$

$$= \overbrace{\boldsymbol{e}_1 \boldsymbol{e}_1 \cos\vartheta - \boldsymbol{e}_1 \boldsymbol{e}_2 \sin\vartheta}^{\boldsymbol{e}_1 \mathring{\boldsymbol{e}}_1} + \overbrace{\boldsymbol{e}_2 \boldsymbol{e}_1 \sin\vartheta + \boldsymbol{e}_2 \boldsymbol{e}_2 \cos\vartheta}^{\boldsymbol{e}_2 \mathring{\boldsymbol{e}}_2} + \overbrace{\boldsymbol{k}\boldsymbol{k}}^{\boldsymbol{e}_3 \mathring{\boldsymbol{e}}_3} =$$

$$= \boldsymbol{E}\cos\vartheta - \underbrace{\boldsymbol{e}_3 \boldsymbol{e}_3}_{\boldsymbol{k}\boldsymbol{k}}\cos\vartheta + \underbrace{(\boldsymbol{e}_2 \boldsymbol{e}_1 - \boldsymbol{e}_1 \boldsymbol{e}_2)}_{\boldsymbol{e}_3 \times \boldsymbol{e}_i \boldsymbol{e}_i = {}^3\epsilon_{3ij}\boldsymbol{e}_j \boldsymbol{e}_i}\sin\vartheta + \boldsymbol{k}\boldsymbol{k} =$$

$$= \boldsymbol{E}\cos\vartheta + \boldsymbol{k} \times \boldsymbol{E}\sin\vartheta + \boldsymbol{k}\boldsymbol{k}(1 - \cos\vartheta)$$

рисунок 5
"Finite rotation"

В параметрах конечного поворота транспонирование, оно же обращение, тензора $\boldsymbol{O}$ эквивалентно перемене направления поворота — знака угла ϑ

$$\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{O}\big|_{\vartheta\,=\,-\vartheta} = \boldsymbol{E}\cos\vartheta - \boldsymbol{k}\times\boldsymbol{E}\sin\vartheta + \boldsymbol{kk}\,(1-\cos\vartheta).$$

Пусть теперь тензор поворота меняется со временем: $\boldsymbol{O}=\boldsymbol{O}(t)$. Псевдовектор угловой скорости $\boldsymbol{\omega}$ вводится через тензор поворота $\boldsymbol{O}$ таким путём. Дифференцируем тождество ортогональности (11.5) по времени*

$$\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}} + \boldsymbol{O}\cdot\dot{\boldsymbol{O}}^{\mathsf{T}} = {}^2\boldsymbol{0}.$$

Тензор $\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}}$ (по (??) $\big(\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}}\big)^{\mathsf{T}} = \boldsymbol{O}\cdot\dot{\boldsymbol{O}}^{\mathsf{T}}$) оказался антисимметричным. Поэтому согласно (7.3) он представи́м сопутствующим вектором как $\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\omega}\times\boldsymbol{E} = \boldsymbol{\omega}\times\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}}$. То есть

$$\dot{\boldsymbol{O}} = \boldsymbol{\omega}\times\boldsymbol{O}, \quad \boldsymbol{\omega} \equiv -\frac{1}{2}\big(\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}}\big)_{\times} \tag{11.8}$$

Помимо этого общего представления вектора $\boldsymbol{\omega}$, для него есть и другие. Например, через параметры конечного поворота.

Производная $\dot{\boldsymbol{O}}$ в параметрах конечного поворота в общем случае (оба параметра — и единичный вектор $\boldsymbol{k}$, и угол ϑ — переменны во времени):

$$\dot{\boldsymbol{O}} = \big(\boldsymbol{O}^{\mathsf{S}} + \boldsymbol{O}^{\mathsf{A}}\big)^{\bullet} = \big(\overbrace{\boldsymbol{E}\cos\vartheta + \boldsymbol{kk}\,(1-\cos\vartheta)}^{\boldsymbol{O}^{\mathsf{S}}} + \overbrace{\boldsymbol{k}\times\boldsymbol{E}\sin\vartheta}^{\boldsymbol{O}^{\mathsf{A}}}\big)^{\bullet} =$$

$$= \underbrace{(\boldsymbol{kk} - \boldsymbol{E})\,\dot{\vartheta}\sin\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta)}_{\dot{\boldsymbol{O}}^{\mathsf{S}}} + $$

$$+ \underbrace{\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\cos\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta}_{\dot{\boldsymbol{O}}^{\mathsf{A}}}.$$

Находим

$$\dot{\boldsymbol{O}}\cdot\boldsymbol{O}^{\mathsf{T}} = (\dot{\boldsymbol{O}}^{\mathsf{S}} + \dot{\boldsymbol{O}}^{\mathsf{A}})\cdot(\boldsymbol{O}^{\mathsf{S}} - \boldsymbol{O}^{\mathsf{A}}) =$$

$$= \dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{S}} + \dot{\boldsymbol{O}}^{\mathsf{A}}\cdot\boldsymbol{O}^{\mathsf{S}} - \dot{\boldsymbol{O}}^{\mathsf{S}}\cdot\boldsymbol{O}^{\mathsf{A}} - \dot{\boldsymbol{O}}^{\mathsf{A}}\cdot\boldsymbol{O}^{\mathsf{A}},$$

---

* Various notations are used to designate the time derivative. In addition to the Leibniz's notation dx/dt, the very popular one is the "dot-above" Newton's notation $\dot{x}$.

using

$$\boldsymbol{k}\cdot\boldsymbol{k} = 1 = \text{constant} \;\Rightarrow\; \boldsymbol{k}\cdot\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\cdot\boldsymbol{k} = 0 \;\Leftrightarrow\; \dot{\boldsymbol{k}}\cdot\boldsymbol{k} = \boldsymbol{k}\cdot\dot{\boldsymbol{k}} = 0,$$

$$\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{k} = \boldsymbol{k}\boldsymbol{k}, \;\; \dot{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{k} = \dot{\boldsymbol{k}}\boldsymbol{k}, \;\; \boldsymbol{k}\dot{\boldsymbol{k}}\cdot\boldsymbol{k}\boldsymbol{k} = {}^2\boldsymbol{0},$$

$$(\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\cdot\boldsymbol{k} = \boldsymbol{k} - \boldsymbol{k} = \boldsymbol{0}, \;\; (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k} = \boldsymbol{k}\boldsymbol{k} - \boldsymbol{k}\boldsymbol{k} = {}^2\boldsymbol{0},$$

$$\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{E}) = (\boldsymbol{k}\times\boldsymbol{E})\cdot\boldsymbol{k} = \boldsymbol{k}\times\boldsymbol{k} = \boldsymbol{0}, \;\; \boldsymbol{k}\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{E}) = (\boldsymbol{k}\times\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k} = {}^2\boldsymbol{0},$$

$$(\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E}) = -\boldsymbol{k}\times\boldsymbol{E},$$

$$(\boldsymbol{a}\times\boldsymbol{E})\cdot\boldsymbol{b} = \boldsymbol{a}\times(\boldsymbol{E}\cdot\boldsymbol{b}) = \boldsymbol{a}\times\boldsymbol{b} \;\Rightarrow\; (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k} = \dot{\boldsymbol{k}}\times\boldsymbol{k}\boldsymbol{k},$$

$$(6.13) \;\Rightarrow\; (\boldsymbol{k}\times\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E}) = \boldsymbol{k}\boldsymbol{k} - \boldsymbol{E}, \;\; (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E}) = \boldsymbol{k}\dot{\boldsymbol{k}} - \dot{\boldsymbol{k}}\cdot\boldsymbol{k}\,\boldsymbol{E},$$

$$(6.12) \;\Rightarrow\; \dot{\boldsymbol{k}}\boldsymbol{k} - \boldsymbol{k}\dot{\boldsymbol{k}} = (\boldsymbol{k}\times\dot{\boldsymbol{k}})\times\boldsymbol{E}, \;\; (\dot{\boldsymbol{k}}\times\boldsymbol{k})\boldsymbol{k} - \boldsymbol{k}(\dot{\boldsymbol{k}}\times\boldsymbol{k}) = \boldsymbol{k}\times(\dot{\boldsymbol{k}}\times\boldsymbol{k})\times\boldsymbol{E}$$

$$\dot{\boldsymbol{P}}^{\mathsf{S}}\cdot\boldsymbol{P}^{\mathsf{S}} =$$
$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\dot{\vartheta}\sin\vartheta\cdot\boldsymbol{E}\cos\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta)\cdot\boldsymbol{E}\cos\vartheta +$$
$$+ \cancel{(\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\dot{\vartheta}\sin\vartheta\cdot\boldsymbol{k}\boldsymbol{k}(1-\cos\vartheta)} + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta)\cdot\boldsymbol{k}\boldsymbol{k}(1-\cos\vartheta) =$$
$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\dot{\vartheta}\sin\vartheta\cos\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})\cos\vartheta\,(1-\cos\vartheta) + (\boldsymbol{k}\dot{\boldsymbol{k}}\cdot\boldsymbol{k}\boldsymbol{k} + \dot{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{k}\boldsymbol{k})(1-\cos\vartheta)^2 =$$
$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\dot{\vartheta}\sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta\,(1-\cos\vartheta) +$$
$$+ \dot{\boldsymbol{k}}\boldsymbol{k}\cos\vartheta - \dot{\boldsymbol{k}}\boldsymbol{k}\cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k} - 2\,\dot{\boldsymbol{k}}\boldsymbol{k}\cos\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\cos^2\vartheta =$$
$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\dot{\vartheta}\sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}}\cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\,(1-\cos\vartheta),$$

$$\dot{\boldsymbol{P}}^{\mathsf{A}}\cdot\boldsymbol{P}^{\mathsf{S}} =$$
$$= (\boldsymbol{k}\times\boldsymbol{E})\cdot\boldsymbol{E}\,\dot{\vartheta}\cos^2\vartheta + (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot\boldsymbol{E}\sin\vartheta\cos\vartheta +$$
$$+ \cancel{(\boldsymbol{k}\times\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k}\,\dot{\vartheta}\cos\vartheta\,(1-\cos\vartheta)} + (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot\boldsymbol{k}\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta) =$$
$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\cos^2\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{k}\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta),$$

$$\dot{\boldsymbol{P}}^{\mathsf{S}}\cdot\boldsymbol{P}^{\mathsf{A}} =$$
$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\dot{\vartheta}\sin\vartheta\cdot(\boldsymbol{k}\times\boldsymbol{E})\sin\vartheta + (\boldsymbol{k}\dot{\boldsymbol{k}} + \dot{\boldsymbol{k}}\boldsymbol{k})(1-\cos\vartheta)\cdot(\boldsymbol{k}\times\boldsymbol{E})\sin\vartheta =$$
$$= \cancel{\boldsymbol{k}\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{E})\dot{\vartheta}\sin^2\vartheta} - \boldsymbol{E}\cdot(\boldsymbol{k}\times\boldsymbol{E})\dot{\vartheta}\sin^2\vartheta + \Big(\boldsymbol{k}\dot{\boldsymbol{k}}\cdot(\boldsymbol{k}\times\boldsymbol{E}) + \cancel{\dot{\boldsymbol{k}}\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{E})}\Big)\sin\vartheta\,(1-\cos\vartheta) =$$
$$= -\boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\sin^2\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\times\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta),$$

$$\dot{\boldsymbol{P}}^{\mathsf{A}}\cdot\boldsymbol{P}^{\mathsf{A}} = (\boldsymbol{k}\times\boldsymbol{E})\dot{\vartheta}\cos\vartheta\cdot(\boldsymbol{k}\times\boldsymbol{E})\sin\vartheta + (\dot{\boldsymbol{k}}\times\boldsymbol{E})\cdot(\boldsymbol{k}\times\boldsymbol{E})\sin^2\vartheta =$$
$$= (\boldsymbol{k}\boldsymbol{k} - \boldsymbol{E})\dot{\vartheta}\sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\sin^2\vartheta;$$

$$\dot{\boldsymbol{P}} \cdot \boldsymbol{P}^{\mathsf{T}} = \dot{\boldsymbol{P}}^{\mathsf{S}} \cdot \boldsymbol{P}^{\mathsf{S}} + \dot{\boldsymbol{P}}^{\mathsf{A}} \cdot \boldsymbol{P}^{\mathsf{S}} - \dot{\boldsymbol{P}}^{\mathsf{S}} \cdot \boldsymbol{P}^{\mathsf{A}} - \dot{\boldsymbol{P}}^{\mathsf{A}} \cdot \boldsymbol{P}^{\mathsf{A}} =$$
$$= (\boldsymbol{kk} - \boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cos\vartheta + \boldsymbol{k}\dot{\boldsymbol{k}}\cos\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}}\cos^2\vartheta + \dot{\boldsymbol{k}}\boldsymbol{k}\,(1-\cos\vartheta) +$$
$$+ \boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\cos^2\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + \dot{\boldsymbol{k}}\times\boldsymbol{kk}\sin\vartheta\,(1-\cos\vartheta) +$$
$$+ \boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta}\sin^2\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}}\times\boldsymbol{k}\sin\vartheta\,(1-\cos\vartheta) - (\boldsymbol{kk}-\boldsymbol{E})\,\dot{\vartheta}\sin\vartheta\cos\vartheta - \boldsymbol{k}\dot{\boldsymbol{k}}\sin^2\vartheta =$$
$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta} + (\dot{\boldsymbol{k}}\boldsymbol{k} - \boldsymbol{k}\dot{\boldsymbol{k}})(1-\cos\vartheta) + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + (\dot{\boldsymbol{k}}\times\boldsymbol{kk} - \boldsymbol{k}\dot{\boldsymbol{k}}\times\boldsymbol{k})\sin\vartheta\,(1-\cos\vartheta) =$$
$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta} + \boldsymbol{k}\times\dot{\boldsymbol{k}}\times\boldsymbol{E}\,(1-\cos\vartheta) + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + \boldsymbol{k}\times(\dot{\boldsymbol{k}}\times\boldsymbol{k})\times\boldsymbol{E}\sin\vartheta\,(1-\cos\vartheta) =$$
$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta} + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta\cos\vartheta + (\dot{\boldsymbol{k}}\boldsymbol{k}\cdot\boldsymbol{k} - \boldsymbol{k}\dot{\boldsymbol{k}}\cdot\boldsymbol{k})\times\boldsymbol{E}\sin\vartheta\,(1-\cos\vartheta) + \boldsymbol{k}\times\dot{\boldsymbol{k}}\times\boldsymbol{E}\,(1-\cos\vartheta) =$$
$$= \boldsymbol{k}\times\boldsymbol{E}\,\dot{\vartheta} + \dot{\boldsymbol{k}}\times\boldsymbol{E}\sin\vartheta + \boldsymbol{k}\times\dot{\boldsymbol{k}}\times\boldsymbol{E}\,(1-\cos\vartheta).$$

Этот результат, подставленный в определение (11.8) псевдо-вектора $\boldsymbol{\omega}$, даёт
$$\boldsymbol{\omega} = \boldsymbol{k}\dot{\vartheta} + \dot{\boldsymbol{k}}\sin\vartheta + \boldsymbol{k}\times\dot{\boldsymbol{k}}\,(1-\cos\vartheta). \tag{11.9}$$
Вектор $\boldsymbol{\omega}$ получился разложенным по трём взаимно ортогональным направлениям — $\boldsymbol{k}$, $\dot{\boldsymbol{k}}$ и $\boldsymbol{k}\times\dot{\boldsymbol{k}}$. При неподвижной оси поворота $\dot{\boldsymbol{k}} = \boldsymbol{0} \Rightarrow \boldsymbol{\omega} = \boldsymbol{k}\dot{\vartheta}$.

Ещё одно представление $\boldsymbol{\omega}$ связано с компонентами тензора поворота (11.2). Поскольку $\boldsymbol{P} = o_{ji}\,\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j$, $\boldsymbol{P}^{\mathsf{T}} = o_{ij}\,\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j$, а векторы начального базиса $\mathring{\boldsymbol{e}}_i$ неподвижны (со временем не меняются), то
$$\dot{\boldsymbol{P}} = \dot{o}_{ji}\,\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j, \quad \dot{\boldsymbol{P}} \cdot \boldsymbol{P}^{\mathsf{T}} = \dot{o}_{ni}\,o_{nj}\,\mathring{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_j,$$
$$\boldsymbol{\omega} = -\tfrac{1}{2}\,\dot{o}_{ni}\,o_{nj}\,\mathring{\boldsymbol{e}}_i\times\mathring{\boldsymbol{e}}_j = \tfrac{1}{2}\in_{[jik]} o_{nj}\,\dot{o}_{ni}\,\mathring{\boldsymbol{e}}_k. \tag{11.10}$$

Отметим и формулы
$$(11.8) \Rightarrow \dot{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_i = \boldsymbol{\omega}\times\boldsymbol{e}_i\mathring{\boldsymbol{e}}_i \Rightarrow \dot{\boldsymbol{e}}_i = \boldsymbol{\omega}\times\boldsymbol{e}_i,$$
$$(11.8) \Rightarrow \boldsymbol{\omega} = -\tfrac{1}{2}\left(\dot{\boldsymbol{e}}_i\mathring{\boldsymbol{e}}_i\cdot\mathring{\boldsymbol{e}}_j\boldsymbol{e}_j\right)_{\times} = -\tfrac{1}{2}\left(\dot{\boldsymbol{e}}_i\boldsymbol{e}_i\right)_{\times} = \tfrac{1}{2}\,\boldsymbol{e}_i\times\dot{\boldsymbol{e}}_i. \tag{11.11}$$

Не всё то вектор, что имеет величину и направление. Поворот тела вокруг оси представляет, казалось бы, вектор: его численное значение равно углу поворота, а направление совпадает с направлением оси вращения. Однако, повороты не складываются как векторы[*].

На са́мом же деле последовательные повороты не складываются, а умножаются.

* Когда углы поворота не бесконечно-ма́лые.

Можно ли складывать угловые скорости? — Да, ведь угол поворота в $\dot{\vartheta}$ бесконечномалый. — Но только при вращении вокруг неподвижной оси?

...

Варьируя тождество (11.5), получим $\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O}\cdot\delta\boldsymbol{O}^{\mathsf{T}}$. Этот тензор антисимметричен, и потому выражается через свой сопутствующий вектор $\boldsymbol{\delta o}$ как $\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\delta o}\times\boldsymbol{E}$. Приходим к соотношениям

$$\delta\boldsymbol{O} = \boldsymbol{\delta o}\times\boldsymbol{O}, \quad \boldsymbol{\delta o} = -\frac{1}{2}\Big(\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}}\Big)_{\times}, \qquad (11.12)$$

аналогичным (11.8). Вектор бесконечно малого поворота $\boldsymbol{\delta o}$ это не "вариация $\mathbf{o}$", но единый символ (в отличие от $\delta\boldsymbol{O}$).

Малый поворот определяется вектором $\boldsymbol{\delta o}$, но конечный поворот тоже возможно представить как вектор.

...

§12. Variations

Further we will pretty often use the operation of varying. It is similar to the differentiation.

The variations are seen as the infinitesimal displacements, compatible with the constraints. If there are no restrictions for the variable x, then the variations δx are completely random. But when

$$x = x(y)$$

is the function of the independent argument y, then

$$\delta x = x'(y)\delta y.$$

Variations are similar to differentials. As example, if δx and δy are variations of x and y, u and v are the finite values, then we write $u\delta x + v\delta y = \delta w$ even if δw is not a variation of w.

In this case δw is a single symbol. Surely if $u = u(x,y)$, $v = v(x,y)$ and $\partial_x v = \partial_y u$ ($\frac{\partial}{\partial x}v = \frac{\partial}{\partial y}u$), then the sum $\delta w = u\delta x + v\delta y$ will be a variation of some w.

Varying the identity (11.5), we get

$$\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O}\cdot\delta\boldsymbol{O}^{\mathsf{T}}.$$

This tensor is antisymmetric, and thus is representable via its companion pseudovector $\delta\mathbf{o}$ as

$$\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = \delta\mathbf{o}\times\boldsymbol{E}.$$

We have the following relations

$$\delta\boldsymbol{O} = \delta\mathbf{o}\times\boldsymbol{O}, \quad \delta\mathbf{o} = -\frac{1}{2}\left(\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}}\right)_{\times}, \tag{12.1}$$

similar to (11.8). Vector $\delta\mathbf{o}$ of an infinitesimal rotation is not "a variation of $\mathbf{o}$", but a single symbol.

An infinitesimal rotation is defined by vector $\delta\mathbf{o}$, but a finite rotation is also possible to represent as a vector

...

§ 13. Polar decomposition

Any tensor of the second complexity $\boldsymbol{F}$ with $\det F_{ij} \neq 0$, that is a not singular tensor, can be decomposed as

...

Example. Polar decompose tensor $\boldsymbol{C} = C_{ij}\boldsymbol{e}_i\boldsymbol{e}_j$, where $\boldsymbol{e}_k$ are mutually perpendicular unit vectors of basis, and C_{ij} are tensor's components

$$C_{ij} = \begin{bmatrix} -5 & 20 & 11 \\ 10 & -15 & 23 \\ -3 & -5 & 10 \end{bmatrix}$$

$$\boldsymbol{O} = O_{ij}\boldsymbol{e}_i\boldsymbol{e}_j = \boldsymbol{O}_1\cdot\boldsymbol{O}_2$$

$$O_{ij} = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 0 & 4/5 & -3/5 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

$$\boldsymbol{C} = \boldsymbol{O}\cdot\boldsymbol{S}_{\mathsf{R}}, \quad \boldsymbol{O}^{\mathsf{T}}\cdot\boldsymbol{C} = \boldsymbol{S}_{\mathsf{R}}$$

$$\boldsymbol{C} = \boldsymbol{S}_{\mathsf{L}}\cdot\boldsymbol{O}, \quad \boldsymbol{C}\cdot\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{S}_{\mathsf{L}}$$

$$S_{\mathsf{R}ij} = \begin{bmatrix} 3 & 5 & -10 \\ 5 & 0 & 25 \\ -10 & 25 & -5 \end{bmatrix}$$

$$S_{\mathsf{L}ij} = \begin{bmatrix} 104/5 & 47/5 & 5 \\ 47/5 & -129/5 & -10 \\ 5 & -10 & 3 \end{bmatrix}$$

...

§14. In the oblique basis

Until now a basis of the three mutually perpendicular unit vectors $\boldsymbol{e}_i$ was used. Now we will take a basis of any three linearly independent (non-coplanar) vectors $\boldsymbol{a}_i$.

The decomposition of vector $\boldsymbol{v}$ in the basis $\boldsymbol{a}_i$ (fig. 6) is the linear combination

$$\boldsymbol{v} = v^i \boldsymbol{a}_i. \qquad (14.1)$$

The summation convention gains the new conditions: a summation index is repeated at different levels of the same monomial, and a free index stays at the equal height in every part of the expression ($a_i = b_{ij}c^j$ is correct, $a_i = b^i_{kk}$ is wrong twice).

figure 6

В таком базисе ужé $\boldsymbol{v} \cdot \boldsymbol{a}_i = v^k \boldsymbol{a}_k \cdot \boldsymbol{a}_i \neq v^i$, ведь тут $\boldsymbol{a}_i \cdot \boldsymbol{a}_k \neq \delta_{ik}$.

Дополним же базис $\boldsymbol{a}_i$ ещё другой тройкой векторов $\boldsymbol{a}^i$, называемых кобазисом или взаимным базисом, чтобы

$$\boldsymbol{a}_i \cdot \boldsymbol{a}^j = \delta_i^j, \ \ \boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta_j^i,$$
$$\boldsymbol{E} = \boldsymbol{a}^i \boldsymbol{a}_i = \boldsymbol{a}_i \boldsymbol{a}^i. \qquad (14.2)$$

Это — основное свойство кобазиса. Ортонормированный (ортонормальный) базис может быть определён как совпадáющий со своим кобазисом: $\boldsymbol{e}^i = \boldsymbol{e}_i$.

Для, к примеру, первого вектора кобазиса $\boldsymbol{a}^1$

$$\begin{cases} \boldsymbol{a}^1 \cdot \boldsymbol{a}_1 = 1 \\ \boldsymbol{a}^1 \cdot \boldsymbol{a}_2 = 0 \\ \boldsymbol{a}^1 \cdot \boldsymbol{a}_3 = 0 \end{cases} \Rightarrow \begin{cases} \boldsymbol{a}^1 \cdot \boldsymbol{a}_1 = 1 \\ \gamma \boldsymbol{a}^1 = \boldsymbol{a}_2 \times \boldsymbol{a}_3 \end{cases} \Rightarrow \begin{cases} \boldsymbol{a}^1 = 1/\gamma\ \boldsymbol{a}_2 \times \boldsymbol{a}_3 \\ \gamma = \boldsymbol{a}_2 \times \boldsymbol{a}_3 \cdot \boldsymbol{a}_1 \end{cases}$$

Коэффициент γ получился равным (с точностью до знака для "левой" тройки $\boldsymbol{a}_i$) объёму параллелепипеда, построенного на векторах $\boldsymbol{a}_i$. In §6 the same volume was presented as $\sqrt{g}$, and this is not without reason, because it coincides with the square root
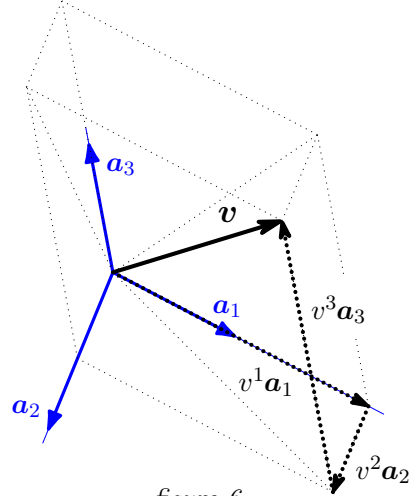
of gramian $g \equiv \det g_{ij}$ — determinant of the symmetric J. P. Gram matrix $g_{ij} \equiv \boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{a}_j$.

○ The proof resembles the derivation of (6.7). The "triple product" $\boldsymbol{a}_i \times \boldsymbol{a}_j \boldsymbol{\cdot} \boldsymbol{a}_k$ in some orthonormal basis $\boldsymbol{e}_i$ вычисли́мо как детерминант (c "$-$" для "левой" тройки $\boldsymbol{a}_i$) по строкам

$$\in_{[ijk]} \equiv \boldsymbol{a}_i \times \boldsymbol{a}_j \boldsymbol{\cdot} \boldsymbol{a}_k = \pm \det \begin{bmatrix} \boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{e}_1 & \boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{e}_2 & \boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{e}_3 \\ \boldsymbol{a}_j \boldsymbol{\cdot} \boldsymbol{e}_1 & \boldsymbol{a}_j \boldsymbol{\cdot} \boldsymbol{e}_2 & \boldsymbol{a}_j \boldsymbol{\cdot} \boldsymbol{e}_3 \\ \boldsymbol{a}_k \boldsymbol{\cdot} \boldsymbol{e}_1 & \boldsymbol{a}_k \boldsymbol{\cdot} \boldsymbol{e}_2 & \boldsymbol{a}_k \boldsymbol{\cdot} \boldsymbol{e}_3 \end{bmatrix}$$

или по столбцам

$$\in_{[pqr]} \equiv \boldsymbol{a}_p \times \boldsymbol{a}_q \boldsymbol{\cdot} \boldsymbol{a}_r = \pm \det \begin{bmatrix} \boldsymbol{a}_p \boldsymbol{\cdot} \boldsymbol{e}_1 & \boldsymbol{a}_q \boldsymbol{\cdot} \boldsymbol{e}_1 & \boldsymbol{a}_r \boldsymbol{\cdot} \boldsymbol{e}_1 \\ \boldsymbol{a}_p \boldsymbol{\cdot} \boldsymbol{e}_2 & \boldsymbol{a}_q \boldsymbol{\cdot} \boldsymbol{e}_2 & \boldsymbol{a}_r \boldsymbol{\cdot} \boldsymbol{e}_2 \\ \boldsymbol{a}_p \boldsymbol{\cdot} \boldsymbol{e}_3 & \boldsymbol{a}_q \boldsymbol{\cdot} \boldsymbol{e}_3 & \boldsymbol{a}_r \boldsymbol{\cdot} \boldsymbol{e}_3 \end{bmatrix}.$$

Произведение определителей $\in_{[ijk]} \in_{[pqr]}$ равно определителю произведения матриц, and elements of the latter are sums like $\boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{e}_s \boldsymbol{a}_p \boldsymbol{\cdot} \boldsymbol{e}_s = \boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{e}_s \boldsymbol{e}_s \boldsymbol{\cdot} \boldsymbol{a}_p = \boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{E} \boldsymbol{\cdot} \boldsymbol{a}_p = \boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{a}_p$, в результате

$$\in_{[ijk]} \in_{[pqr]} = \det \begin{bmatrix} \boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{a}_p & \boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{a}_q & \boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{a}_r \\ \boldsymbol{a}_j \boldsymbol{\cdot} \boldsymbol{a}_p & \boldsymbol{a}_j \boldsymbol{\cdot} \boldsymbol{a}_q & \boldsymbol{a}_j \boldsymbol{\cdot} \boldsymbol{a}_r \\ \boldsymbol{a}_k \boldsymbol{\cdot} \boldsymbol{a}_p & \boldsymbol{a}_k \boldsymbol{\cdot} \boldsymbol{a}_q & \boldsymbol{a}_k \boldsymbol{\cdot} \boldsymbol{a}_r \end{bmatrix};$$

$i{=}p{=}1,\ j{=}q{=}2,\ k{=}r{=}3 \ \Rightarrow \ \in_{[123]} \in_{[123]} = \det_{i,j} (\boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{a}_j) = \det_{i,j} g_{ij}$. ●

Representing $\boldsymbol{a}^1$ and other cobasis vectors as the sum

$$\pm 2\sqrt{g}\, \boldsymbol{a}^1 = \boldsymbol{a}_2 \times \boldsymbol{a}_3 \overbrace{- \boldsymbol{a}_3 \times \boldsymbol{a}_2}^{+\, \boldsymbol{a}_2 \times \boldsymbol{a}_3},$$

приходим к общей формуле (c "$-$" для "левой" тройки $\boldsymbol{a}_i$)

$$\boldsymbol{a}^i = \pm \frac{1}{2\sqrt{g}}\, e^{ijk} \boldsymbol{a}_j \times \boldsymbol{a}_k, \ \ \sqrt{g} \equiv \pm \boldsymbol{a}_1 \times \boldsymbol{a}_2 \boldsymbol{\cdot} \boldsymbol{a}_3 > 0. \qquad (14.3)$$

Здесь e^{ijk} по-прежнему символ перестановки Veblen'a (± 1 или 0): $e^{ijk} \equiv e_{ijk}$. Произведение $\boldsymbol{a}_j \times \boldsymbol{a}_k = \in_{[jkn]} \boldsymbol{a}^n$, компоненты тензора Ле́ви-Чиви́ты $\in_{[jkn]} = \pm e_{jkn}\sqrt{g}$, and by (6.8) $e^{ijk} e_{jkn} = 2\delta_n^i$. Thus

$$\boldsymbol{a}^1 = \pm 1\big/\sqrt{g}\, (\boldsymbol{a}_2 \times \boldsymbol{a}_3), \ \ \boldsymbol{a}^2 = \pm 1\big/\sqrt{g}\, (\boldsymbol{a}_3 \times \boldsymbol{a}_1), \ \ \boldsymbol{a}^3 = \pm 1\big/\sqrt{g}\, (\boldsymbol{a}_1 \times \boldsymbol{a}_2).$$

Example. Get cobasis for basis $\boldsymbol{a}_i$ when

$$\boldsymbol{a}_1 = \boldsymbol{e}_1 + \boldsymbol{e}_2,$$
$$\boldsymbol{a}_2 = \boldsymbol{e}_1 + \boldsymbol{e}_3,$$
$$\boldsymbol{a}_3 = \boldsymbol{e}_2 + \boldsymbol{e}_3.$$

$$\sqrt{g} = -\boldsymbol{a}_1 \times \boldsymbol{a}_2 \cdot \boldsymbol{a}_3 = -\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 2 \,;$$

$$-\boldsymbol{a}_2 \times \boldsymbol{a}_3 = \det \begin{bmatrix} 1 & \boldsymbol{e}_1 & 0 \\ 0 & \boldsymbol{e}_2 & 1 \\ 1 & \boldsymbol{e}_3 & 1 \end{bmatrix} = \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3,$$

$$-\boldsymbol{a}_3 \times \boldsymbol{a}_1 = \det \begin{bmatrix} 0 & \boldsymbol{e}_1 & 1 \\ 1 & \boldsymbol{e}_2 & 1 \\ 1 & \boldsymbol{e}_3 & 0 \end{bmatrix} = \boldsymbol{e}_1 + \boldsymbol{e}_3 - \boldsymbol{e}_2,$$

$$-\boldsymbol{a}_1 \times \boldsymbol{a}_2 = \det \begin{bmatrix} 1 & \boldsymbol{e}_1 & 1 \\ 1 & \boldsymbol{e}_2 & 0 \\ 0 & \boldsymbol{e}_3 & 1 \end{bmatrix} = \boldsymbol{e}_2 + \boldsymbol{e}_3 - \boldsymbol{e}_1$$

and finally

$$\boldsymbol{a}^1 = \tfrac{1}{2}\left(\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\right),$$
$$\boldsymbol{a}^2 = \tfrac{1}{2}\left(\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3\right),$$
$$\boldsymbol{a}^3 = \tfrac{1}{2}\left(-\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3\right).$$

Имея кобазис, возможно не только разложить по нему любой вектор (рис. 7), но и найти коэффициенты разложения (14.1):

$$\boldsymbol{v} = v^i \boldsymbol{a}_i = v_i \boldsymbol{a}^i,$$
$$\boldsymbol{v} \cdot \boldsymbol{a}^i = v^k \boldsymbol{a}_k \cdot \boldsymbol{a}^i = v^i, \ \ v_i = \boldsymbol{v} \cdot \boldsymbol{a}_i. \tag{14.4}$$

Коэффициенты v_i называются ковариантными компонентами вектора $\boldsymbol{v}$, а v^i — его контравариантными* компонентами.

Есть литература о тензорах, где introducing existence и различают ковариантные и контравариантные... векторы (and "covectors", "dual vectors"). Не сто́ит вводить читателя в заблуждение: вектор-то один и тот же, просто разложение по двум разным базисам даёт два набора компонент.

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* Потому что они меняются обратно (contra) изменению длин базисных векторов $\boldsymbol{a}_i$.
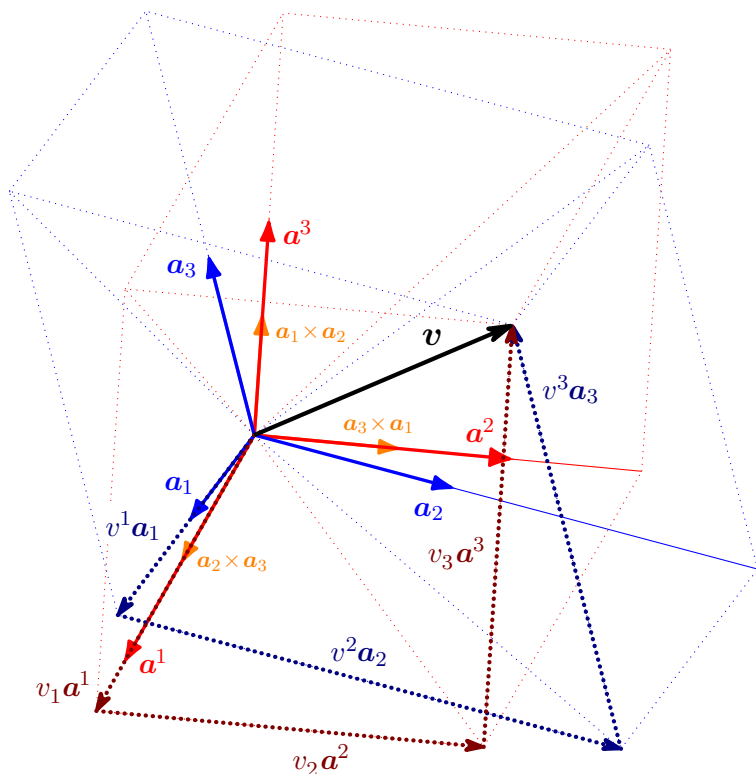
$$\boldsymbol{a}_1 \times \boldsymbol{a}_2 \boldsymbol{\cdot} \boldsymbol{a}_3 = \sqrt{g} = 0.56274$$
$$1/\sqrt{g} = 1.77703$$

$$\boldsymbol{a}_i \boldsymbol{\cdot} \boldsymbol{a}^j = \begin{bmatrix} \boldsymbol{a}_1 \boldsymbol{\cdot} \boldsymbol{a}^1 & \boldsymbol{a}_1 \boldsymbol{\cdot} \boldsymbol{a}^2 & \boldsymbol{a}_1 \boldsymbol{\cdot} \boldsymbol{a}^3 \\ \boldsymbol{a}_2 \boldsymbol{\cdot} \boldsymbol{a}^1 & \boldsymbol{a}_2 \boldsymbol{\cdot} \boldsymbol{a}^2 & \boldsymbol{a}_2 \boldsymbol{\cdot} \boldsymbol{a}^3 \\ \boldsymbol{a}_3 \boldsymbol{\cdot} \boldsymbol{a}^1 & \boldsymbol{a}_3 \boldsymbol{\cdot} \boldsymbol{a}^2 & \boldsymbol{a}_3 \boldsymbol{\cdot} \boldsymbol{a}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \delta_i^j$$

рисунок 7
"Decomposition of vector in oblique basis"

От векторов перейдём к тензорам второй сложности. Имеем четыре комплекта диад: $\boldsymbol{a}_i\boldsymbol{a}_j$, $\boldsymbol{a}^i\boldsymbol{a}^j$, $\boldsymbol{a}_i\boldsymbol{a}^j$, $\boldsymbol{a}^i\boldsymbol{a}_j$. Согласующиеся коэффициенты в декомпозиции тензора называются его контравариантными, ковариантными и смешанными компонентами:

$$^2\boldsymbol{B} = B^{ij}\boldsymbol{a}_i\boldsymbol{a}_j = B_{ij}\boldsymbol{a}^i\boldsymbol{a}^j = B^i_{\cdot j}\boldsymbol{a}_i\boldsymbol{a}^j = B^{\cdot j}_i\boldsymbol{a}^i\boldsymbol{a}_j,$$

$$B^{ij} = \boldsymbol{a}^i\boldsymbol\cdot{}^2\boldsymbol{B}\boldsymbol\cdot\boldsymbol{a}^j, \;\; B_{ij} = \boldsymbol{a}_i\boldsymbol\cdot{}^2\boldsymbol{B}\boldsymbol\cdot\boldsymbol{a}_j, \tag{14.5}$$

$$B^i_{\cdot j} = \boldsymbol{a}^i\boldsymbol\cdot{}^2\boldsymbol{B}\boldsymbol\cdot\boldsymbol{a}_j, \;\; B^{\cdot j}_i = \boldsymbol{a}_i\boldsymbol\cdot{}^2\boldsymbol{B}\boldsymbol\cdot\boldsymbol{a}^j.$$

Для двух видов смешанных компонент точка в индексе это просто свободное место: у $B^i_{\cdot j}$ верхний индекс "i" — первый, а нижний "j" — второй.

Компоненты единичного ("метрического") тензора $\boldsymbol{E}$

$$\boldsymbol{E} = \boldsymbol{a}^k\boldsymbol{a}_k = \boldsymbol{a}_k\boldsymbol{a}^k = g_{jk}\boldsymbol{a}^j\boldsymbol{a}^k = g^{jk}\boldsymbol{a}_j\boldsymbol{a}_k:$$

$$\boldsymbol{a}_i\boldsymbol\cdot\boldsymbol{E}\boldsymbol\cdot\boldsymbol{a}^j = \boldsymbol{a}_i\boldsymbol\cdot\boldsymbol{a}^j = \delta^j_i, \;\; \boldsymbol{a}^i\boldsymbol\cdot\boldsymbol{E}\boldsymbol\cdot\boldsymbol{a}_j = \boldsymbol{a}^i\boldsymbol\cdot\boldsymbol{a}_j = \delta^i_j,$$

$$\boldsymbol{a}_i\boldsymbol\cdot\boldsymbol{E}\boldsymbol\cdot\boldsymbol{a}_j = \boldsymbol{a}_i\boldsymbol\cdot\boldsymbol{a}_j \equiv g_{ij}, \;\; \boldsymbol{a}^i\boldsymbol\cdot\boldsymbol{E}\boldsymbol\cdot\boldsymbol{a}^j = \boldsymbol{a}^i\boldsymbol\cdot\boldsymbol{a}^j \equiv g^{ij}; \tag{14.6}$$

$$\boldsymbol{E}\boldsymbol\cdot\boldsymbol{E} = g_{ij}\boldsymbol{a}^i\boldsymbol{a}^j\boldsymbol\cdot g^{nk}\boldsymbol{a}_n\boldsymbol{a}_k = g_{ij}g^{jk}\boldsymbol{a}^i\boldsymbol{a}_k = \boldsymbol{E} \;\Rightarrow\; g_{ij}g^{jk} = \delta^k_i.$$

Вдобавок к (14.2) и (14.3) открылся ещё один способ найти векторы кобазиса — через матрицу g^{ij}, обратную матрице Грама g_{ij}. И наоборот:

$$\boldsymbol{a}^i = \boldsymbol{E}\boldsymbol\cdot\boldsymbol{a}^i = g^{jk}\boldsymbol{a}_j\boldsymbol{a}_k\boldsymbol\cdot\boldsymbol{a}^i = g^{jk}\boldsymbol{a}_j\delta^i_k = g^{ji}\boldsymbol{a}_j,$$

$$\boldsymbol{a}_i = \boldsymbol{E}\boldsymbol\cdot\boldsymbol{a}_i = g_{jk}\boldsymbol{a}^j\boldsymbol{a}^k\boldsymbol\cdot\boldsymbol{a}_i = g_{jk}\boldsymbol{a}^j\delta^k_i = g_{ji}\boldsymbol{a}^j. \tag{14.7}$$

Example. Using reversed Gram matrix, get cobasis for basis $\boldsymbol{a}_i$ when

$$\boldsymbol{a}_1 = \boldsymbol{e}_1 + \boldsymbol{e}_2,$$
$$\boldsymbol{a}_2 = \boldsymbol{e}_1 + \boldsymbol{e}_3,$$
$$\boldsymbol{a}_3 = \boldsymbol{e}_2 + \boldsymbol{e}_3.$$

$$g_{ij} = \boldsymbol{a}_i \cdot \boldsymbol{a}_j = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \ \det g_{ij} = 4,$$

$$\mathrm{adj}\, g_{ij} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}^{\mathsf{T}},$$

$$g^{ij} = g_{ij}^{-1} = \frac{\mathrm{adj}\, g_{ij}}{\det g_{ij}} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

Using $\boldsymbol{a}^i = g^{ij}\boldsymbol{a}_j$

$$\boldsymbol{a}^1 = g^{11}\boldsymbol{a}_1 + g^{12}\boldsymbol{a}_2 + g^{13}\boldsymbol{a}_3 = \tfrac{1}{2}\boldsymbol{e}_1 + \tfrac{1}{2}\boldsymbol{e}_2 - \tfrac{1}{2}\boldsymbol{e}_3,$$
$$\boldsymbol{a}^2 = g^{21}\boldsymbol{a}_1 + g^{22}\boldsymbol{a}_2 + g^{23}\boldsymbol{a}_3 = \tfrac{1}{2}\boldsymbol{e}_1 - \tfrac{1}{2}\boldsymbol{e}_2 + \tfrac{1}{2}\boldsymbol{e}_3,$$
$$\boldsymbol{a}^3 = g^{31}\boldsymbol{a}_1 + g^{32}\boldsymbol{a}_2 + g^{33}\boldsymbol{a}_3 = -\tfrac{1}{2}\boldsymbol{e}_1 + \tfrac{1}{2}\boldsymbol{e}_2 + \tfrac{1}{2}\boldsymbol{e}_3.$$

...

Единичный тензор (unit tensor, identity tensor, metric tensor)

$$\boldsymbol{E} \cdot \boldsymbol{\xi} = \boldsymbol{\xi} \cdot \boldsymbol{E} = \boldsymbol{\xi} \quad \forall \boldsymbol{\xi}$$

$$\boldsymbol{E} \cdot\cdot \boldsymbol{ab} = \boldsymbol{ab} \cdot\cdot \boldsymbol{E} = \boldsymbol{a} \cdot \boldsymbol{E} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}$$

$$\boldsymbol{E} \cdot\cdot \boldsymbol{A} = \boldsymbol{A} \cdot\cdot \boldsymbol{E} = \mathrm{trace}\, \boldsymbol{A}$$

$$\boldsymbol{E} \cdot\cdot \boldsymbol{A} = \boldsymbol{A} \cdot\cdot \boldsymbol{E} = \mathrm{trace}\, \boldsymbol{A} \neq \mathrm{notanymore}\, A_{jj}$$

Thus for, say, trace of some tensor $\boldsymbol{A} = A_{ij}\boldsymbol{r}^i\boldsymbol{r}^j$: $\boldsymbol{A} \cdot\cdot \boldsymbol{E} = \mathrm{trace}\, \boldsymbol{A}$, you have

$$\boldsymbol{A} \cdot\cdot \boldsymbol{E} = A_{ij}\boldsymbol{r}^i\boldsymbol{r}^j \cdot\cdot \boldsymbol{r}_{\partial k}\boldsymbol{r}^k = A_{ij}\boldsymbol{r}^i \cdot \boldsymbol{r}^j = A_{ij}g^{ij}$$

...

Тензор поворота (the rotation tensor)

$$\boldsymbol{P} = \boldsymbol{a}_i\mathring{\boldsymbol{a}}^i = \boldsymbol{a}^i\mathring{\boldsymbol{a}}_i = \boldsymbol{P}^{-\mathsf{T}}$$
$$\boldsymbol{P}^{-1} = \mathring{\boldsymbol{a}}_i\boldsymbol{a}^i = \mathring{\boldsymbol{a}}^i\boldsymbol{a}_i = \boldsymbol{P}^{\mathsf{T}}$$
$$\boldsymbol{P}^{\mathsf{T}} = \mathring{\boldsymbol{a}}^i\boldsymbol{a}_i = \mathring{\boldsymbol{a}}_i\boldsymbol{a}^i = \boldsymbol{P}^{-1}$$

...

... Характеристическое уравнение (9.2) быстро приводит к тождеству Кэ́ли–Га́мильтона (Cayley–Hamilton)

$$-\boldsymbol{B}\cdot\boldsymbol{B}\cdot\boldsymbol{B} + \mathrm{I}\,\boldsymbol{B}\cdot\boldsymbol{B} - \mathrm{II}\,\boldsymbol{B} + \mathrm{III}\,\boldsymbol{E} = {}^2\boldsymbol{0},$$
$$-\boldsymbol{B}^3 + \mathrm{I}\,\boldsymbol{B}^2 - \mathrm{II}\,\boldsymbol{B} + \mathrm{III}\,\boldsymbol{E} = {}^2\boldsymbol{0}. \tag{14.8}$$

§15. Tensor functions

In the concept of function $y = f(x)$ as of mapping (morphism) $f\colon x \mapsto y$, an input (argument) x and an output (result) y may be tensors of any complexities.

Consider at least a scalar function of a bivalent tensor $\varphi = \varphi(\boldsymbol{B})$. Examples are $\boldsymbol{B} \cdot\cdot \boldsymbol{\Phi}$ (or $\boldsymbol{p}\cdot\boldsymbol{B}\cdot\boldsymbol{q}$) and $\boldsymbol{B}\cdot\cdot\boldsymbol{B}$. Then in each basis $\boldsymbol{a}_i$ paired with cobasis $\boldsymbol{a}^i$ we have function $\varphi(B_{ij})$ of nine numeric arguments — components B_{ij} of tensor $\boldsymbol{B}$. For example

$$\varphi(\boldsymbol{B}) = \boldsymbol{B} \cdot\cdot \boldsymbol{\Phi} = B_{ij}\boldsymbol{a}^i\boldsymbol{a}^j \cdot\cdot \boldsymbol{a}_m\boldsymbol{a}_n \Phi^{mn} = B_{ij}\Phi^{ji} = \varphi(B_{ij}).$$

With any transition to a new basis, the result doesn't change: $\varphi(B_{ij}) = \varphi(B'_{ij}) = \varphi(\boldsymbol{B})$.

Differentiation of $\varphi(\boldsymbol{B})$ looks like

$$d\varphi = \frac{\partial\varphi}{\partial B_{ij}}\,dB_{ij} = \frac{\partial\varphi}{\partial\boldsymbol{B}} \cdot\cdot d\boldsymbol{B}^\mathsf{T}. \tag{15.1}$$

Tensor $\partial\varphi/\partial\boldsymbol{B}$ is called the derivative of function φ by argument $\boldsymbol{B}$; $d\boldsymbol{B}$ is the differential of tensor $\boldsymbol{B}$, $d\boldsymbol{B} = dB_{ij}\boldsymbol{a}^i\boldsymbol{a}^j$; $\partial\varphi/\partial B_{ij}$ are components (contravariant ones) of $\partial\varphi/\partial\boldsymbol{B}$

$$\boldsymbol{a}^i\cdot\frac{\partial\varphi}{\partial\boldsymbol{B}}\cdot\boldsymbol{a}^j = \frac{\partial\varphi}{\partial\boldsymbol{B}} \cdot\cdot \boldsymbol{a}^j\boldsymbol{a}^i = \frac{\partial\varphi}{\partial B_{ij}} \;\Leftrightarrow\; \frac{\partial\varphi}{\partial\boldsymbol{B}} = \frac{\partial\varphi}{\partial B_{ij}}\,\boldsymbol{a}_i\boldsymbol{a}_j.$$

...

$$\varphi(\boldsymbol{B}) = \boldsymbol{B}\cdot\cdot\boldsymbol{\Phi}$$
$$d\varphi = d(\boldsymbol{B}\cdot\cdot\boldsymbol{\Phi}) = d\boldsymbol{B}\cdot\cdot\boldsymbol{\Phi} = \boldsymbol{\Phi}\cdot\cdot d\boldsymbol{B} = \boldsymbol{\Phi}^\mathsf{T}\cdot\cdot d\boldsymbol{B}^\mathsf{T}$$
$$d\varphi = \frac{\partial\varphi}{\partial\boldsymbol{B}}\cdot\cdot d\boldsymbol{B}^\mathsf{T}, \quad \frac{\partial(\boldsymbol{B}\cdot\cdot\boldsymbol{\Phi})}{\partial\boldsymbol{B}} = \boldsymbol{\Phi}^\mathsf{T}$$
$$\boldsymbol{p}\cdot\boldsymbol{B}\cdot\boldsymbol{q} = \boldsymbol{B}\cdot\cdot\boldsymbol{q}\boldsymbol{p}$$
$$\frac{\partial(\boldsymbol{p}\cdot\boldsymbol{B}\cdot\boldsymbol{q})}{\partial\boldsymbol{B}} = \boldsymbol{p}\boldsymbol{q}$$

...

$$\varphi(\boldsymbol{B}) = \boldsymbol{B} \boldsymbol{\cdot\cdot} \boldsymbol{B}$$
$$d\varphi = d(\boldsymbol{B} \boldsymbol{\cdot\cdot} \boldsymbol{B}) = d...$$

...

Но согласно опять-таки (14.8) $-\boldsymbol{B}^2 + \mathrm{I}\boldsymbol{B} - \mathrm{II}\boldsymbol{E} + \mathrm{III}\boldsymbol{B}^{-1} = {}^2\boldsymbol{0}$, поэтому

....

Скалярная функция $\varphi(\boldsymbol{B})$ называется изотропной, если она не чувствительна к повороту аргумента:

$$\varphi(\boldsymbol{B}) = \varphi(\boldsymbol{O} \boldsymbol{\cdot} \overset{\circ}{\boldsymbol{B}} \boldsymbol{\cdot} \boldsymbol{O}^{\mathsf{T}}) = \varphi(\overset{\circ}{\boldsymbol{B}}) \quad \forall \boldsymbol{O} = \boldsymbol{a}_i \overset{\circ}{\boldsymbol{a}}{}^i = \boldsymbol{a}^i \overset{\circ}{\boldsymbol{a}}_i = \boldsymbol{O}^{-\mathsf{T}}$$

для любого ортогонального тензора $\boldsymbol{O}$ (тензора поворота, § 11).

Симметричный тензор $\boldsymbol{B}^{\mathsf{S}}$ полностью определяется тройкой инвариантов и угловой ориентацией собственных осей (они же взаимно ортогональны, § 9). Ясно, что изотропная функция $\varphi(\boldsymbol{B}^{\mathsf{S}})$ симметричного аргумента является функцией, входы-аргументы которой — только инварианты $\mathrm{I}(\boldsymbol{B}^{\mathsf{S}})$, $\mathrm{II}(\boldsymbol{B}^{\mathsf{S}})$, $\mathrm{III}(\boldsymbol{B}^{\mathsf{S}})$. Дифференцируется такая функция согласно (??), где транспонирование излишне.

§16. Spatial differentiation

««« rename: remove fields

> *Tensor field* is a tensor varying from point to point (variable in space, coordinate dependent).

Пусть at each point of some region of a three-dimensional space определена величина ς. Тогда говорят, что есть тензорное поле $\varsigma = \varsigma(\boldsymbol{r})$, where $\boldsymbol{r}$ is location vector (radius vector) of a point in space.

Величина ς может быть тензором любой сложности. Пример скалярного поля — поле температуры в среде, векторного поля — скорости частиц жидкости.

Концепт тензорного поля никак не связан с концептом поля с операциями $+$ и $*$ с 11 свойствами этих операций.

Не только для решения прикладных задач, но нередко и в "чистой теории" вместо аргумента $\boldsymbol{r}$ используется набор (какая-

либо тройка) криволинейных координат q^i. Если непрерывно менять лишь одну координату из трёх, получается координатная линия. Каждая точка трёхмерного пространства лежит на пересечении трёх координатных линий (рис. 8). Вектор положения точки выражается через набор координат as relation $\boldsymbol{r} = \boldsymbol{r}(q^i)$.

Commonly used sets of coordinates«« Rectangular ("cartesian"), spherical and cylindrical coordinates are

Curvilinear coordinates may be derived from a set of rectangular ("cartesian") coordinates by using a transformation that is locally invertible (a one-to-one map) at each point. Therefore rectangular coordinates of any point of space can be converted to some curvilinear coordinates and vice versa.
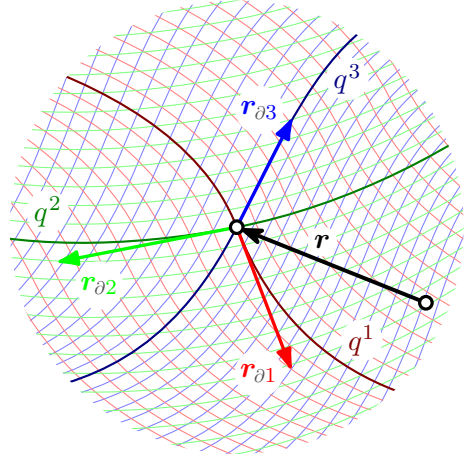

figure 8

...

The differential of a function presents a change in the linearization of this function.

...

partial derivative

$$\partial_i \equiv \frac{\partial}{\partial q^i}$$

...

differential of $\varsigma(q^i)$

$$d\varsigma = \frac{\partial \varsigma}{\partial q^i} dq^i = \partial_i \varsigma \, dq^i \tag{16.1}$$

...

Linearity

$$\partial_i\big(\lambda p + \mu q\big) = \lambda\big(\partial_i p\big) + \mu\big(\partial_i q\big) \tag{16.2}$$

"Product rule"
$$\partial_i(p \circ q) = (\partial_i p) \circ q + p \circ (\partial_i q) \tag{16.3}$$

...

Local basis $\boldsymbol{r}_{\partial i}$

The differential of location vector $\boldsymbol{r}(q^i)$ is
$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial q^i} dq^i = dq^i \boldsymbol{r}_{\partial i}, \quad \boldsymbol{r}_{\partial i} \equiv \frac{\partial \boldsymbol{r}}{\partial q^i} \equiv \partial_i \boldsymbol{r} \tag{16.4}$$

...

Local cobasis $\boldsymbol{r}^i$, $\boldsymbol{r}^i \cdot \boldsymbol{r}_{\partial j} = \delta_j^i$

...

$$\frac{\partial \varsigma}{\partial \boldsymbol{r}} = \frac{\partial \varsigma}{\partial q^i} \boldsymbol{r}^i = \partial_i \varsigma \, \boldsymbol{r}^i$$
$$d\varsigma = \frac{\partial \varsigma}{\partial \boldsymbol{r}} \cdot d\boldsymbol{r} = \partial_i \varsigma \, \boldsymbol{r}^i \cdot dq^j \boldsymbol{r}_{\partial j} = \partial_i \varsigma \, dq^i \tag{16.5}$$

...

The bivalent unit tensor (metric tensor) $\boldsymbol{E}$, which is neutral (3.7) to the "•"-product (dot product), can be represented as
$$\boldsymbol{E} = \boldsymbol{r}^i \boldsymbol{r}_{\partial i} = \underbrace{\boldsymbol{r}^i \partial_i}_{\nabla} \boldsymbol{r} = \boldsymbol{\nabla} \boldsymbol{r}, \tag{16.6}$$

where appears the differential "nabla" operator
$$\boldsymbol{\nabla} \equiv \boldsymbol{r}^i \partial_i. \tag{16.7}$$

...

$$d\varsigma = \frac{\partial \varsigma}{\partial \boldsymbol{r}} \cdot d\boldsymbol{r} = d\boldsymbol{r} \cdot \boldsymbol{\nabla}\varsigma = \partial_i \varsigma \, dq^i \tag{16.8}$$

$$d\boldsymbol{r} = d\boldsymbol{r} \cdot \overbrace{\boldsymbol{\nabla} \boldsymbol{r}}^{\boldsymbol{E}}$$

...

Divergence of the dyadic product of two vectors
$$\boldsymbol{\nabla} \cdot (\boldsymbol{ab}) = \boldsymbol{r}^i \partial_i \cdot (\boldsymbol{ab}) = \boldsymbol{r}^i \cdot \partial_i(\boldsymbol{ab}) = \boldsymbol{r}^i \cdot (\partial_i \boldsymbol{a})\boldsymbol{b} + \boldsymbol{r}^i \cdot \boldsymbol{a}(\partial_i \boldsymbol{b}) =$$
$$= (\boldsymbol{r}^i \cdot \partial_i \boldsymbol{a})\boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{r}^i(\partial_i \boldsymbol{b}) = (\boldsymbol{r}^i \partial_i \cdot \boldsymbol{a})\boldsymbol{b} + \boldsymbol{a} \cdot (\boldsymbol{r}^i \partial_i \boldsymbol{b}) =$$
$$= (\boldsymbol{\nabla} \cdot \boldsymbol{a})\boldsymbol{b} + \boldsymbol{a} \cdot (\boldsymbol{\nabla} \boldsymbol{b}) \tag{16.9}$$

— here's no need to expand vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, expanding just differential operator $\boldsymbol{\nabla}$.

...

Gradient of cross product of two vectors, applying "product rule" (16.3) and relation (6.5) for any two vectors (partial derivative ∂_i of some vector by scalar coordinate q^i is a vector too)

$$\boldsymbol{\nabla}(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{r}^i \partial_i (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{r}^i (\partial_i \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \partial_i \boldsymbol{b}) =$$
$$= \boldsymbol{r}^i (\partial_i \boldsymbol{a} \times \boldsymbol{b} - \partial_i \boldsymbol{b} \times \boldsymbol{a}) = \boldsymbol{r}^i \partial_i \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{r}^i \partial_i \boldsymbol{b} \times \boldsymbol{a} =$$
$$= \boldsymbol{\nabla} \boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{\nabla} \boldsymbol{b} \times \boldsymbol{a}. \quad (16.10)$$

...

Gradient of dot product of two vectors

$$\boldsymbol{\nabla}(\boldsymbol{a} \boldsymbol{\cdot} \boldsymbol{b}) = \boldsymbol{r}^i \partial_i (\boldsymbol{a} \boldsymbol{\cdot} \boldsymbol{b}) = \boldsymbol{r}^i (\partial_i \boldsymbol{a}) \boldsymbol{\cdot} \boldsymbol{b} + \boldsymbol{r}^i \boldsymbol{a} \boldsymbol{\cdot} (\partial_i \boldsymbol{b}) =$$
$$= (\boldsymbol{r}^i \partial_i \boldsymbol{a}) \boldsymbol{\cdot} \boldsymbol{b} + \boldsymbol{r}^i (\partial_i \boldsymbol{b}) \boldsymbol{\cdot} \boldsymbol{a} = (\boldsymbol{\nabla} \boldsymbol{a}) \boldsymbol{\cdot} \boldsymbol{b} + (\boldsymbol{\nabla} \boldsymbol{b}) \boldsymbol{\cdot} \boldsymbol{a}. \quad (16.11)$$

§17. The integral theorems

Для векторных полей известны интегральные теоремы Gauss'a и Stokes'a.

Gauss' theorem (divergence theorem) enables an integral taken over a volume to be replaced by one taken over the closed surface bounding that volume, and vice versa.

Stokes' theorem enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve. Stokes' theorem relates a line integral around a closed path to a surface integral over what is called a *capping surface* of the path.

Теорема Gauss'a о дивергенции

This theorem is about how to replace a volume integral with a surface one (and vice versa). В этой теореме рассматривается поток (ef)flux вектора через ограничивающую объём V за́мкнутую поверхность $\mathcal{O}(\partial V)$. Единичный вектор внешней нормали $\boldsymbol{n}$ к поверхности $\mathcal{O}(\partial V)$

$$\oint_{\mathcal{O}(\partial V)} \boldsymbol{n} \boldsymbol{\cdot} \boldsymbol{a} \, d\mathcal{O} = \int_{V} \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{a} \, dV. \quad (17.1)$$

Объём V нарезается тремя семействами координатных поверхностей на множество бесконечно малых элементов. Поток через

поверхность $\mathcal{O}(\partial V)$ равен сумме потоков через края получившихся элементов. В бесконечной малости каждый такой элемент — маленький локальный дифференциальный кубик (параллелепипед). ... Поток вектора $\boldsymbol{a}$ через грани малого кубика с объёмом dV равен $\sum_{i=1}^{6} \boldsymbol{n}_i \boldsymbol{\cdot} \boldsymbol{a} \mathcal{O}_i$, а поток через сам этот объём равен $\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{a}\, dV$.

Похожая трактовка этой теоремы есть, для примера, в курсе Richard'a Feynman'a [85].

(рисунок с кубиками)

to dice — нареза́ть кубиками

small cube, little cube

локально ортонормальные координаты $\boldsymbol{\xi} = \xi_i \boldsymbol{n}_i,\ d\boldsymbol{\xi} = d\xi_i \boldsymbol{n}_i,$ $\boldsymbol{\nabla} = \boldsymbol{n}_i \partial_i$

разложение вектора $\boldsymbol{a} = a_i \boldsymbol{n}_i$

Теорема Stokes'a о циркуляции

Эта теорема выражается равенством

...

§ 18. Curvature tensors

The *Riemann curvature tensor* or *Riemann–Christoffel tensor* (after **Bernhard Riemann** and **Elwin Bruno Christoffel**) is the most common method used to express the curvature of Riemannian manifolds. It's a tensor field, it assigns a tensor to each point of a Riemannian manifold, that measures the extent to which the metric tensor is not locally isometric to that of "flat" space. The curvature tensor measures noncommutativity of the covariant derivative, and as such is the integrability obstruction for the existence of an isometry with "flat" space.

Рассматривая тензорные поля в криволинейных координатах (§ 16), мы исходили из представления вектора-радиуса (вектора положения) точки функцией этих координат: $\boldsymbol{r} = \boldsymbol{r}(q^i)$. Этим отношением порождаются выражения

✔ векторов локального касательного базиса $\boldsymbol{r}_{\partial i} \equiv \partial \boldsymbol{r}/\partial q^i \equiv \partial_i \boldsymbol{r}$,

✔ компонент $g_{ij} \equiv \boldsymbol{r}_{\partial i} \boldsymbol{\cdot} \boldsymbol{r}_{\partial j}$ и $g^{ij} \equiv \boldsymbol{r}^i \boldsymbol{\cdot} \boldsymbol{r}^j = g_{ij}^{-1}$ единичного "метрического" тензора $\boldsymbol{E} = \boldsymbol{r}_{\partial i}\boldsymbol{r}^i = \boldsymbol{r}^i \boldsymbol{r}_{\partial i} = g_{jk}\boldsymbol{r}^j\boldsymbol{r}^k = g^{jk}\boldsymbol{r}_{\partial j}\boldsymbol{r}_{\partial k}$,

✔ векторов локального взаимного кокасательного базиса $\boldsymbol{r}^i \boldsymbol{\cdot} \boldsymbol{r}_{\partial j} = \delta^i_j$, $\boldsymbol{r}^i = g^{ij}\boldsymbol{r}_{\partial j}$,

✔ дифференциального набла-оператора Hamilton'a $\boldsymbol{\nabla} \equiv \boldsymbol{r}^i\partial_i$, $\boldsymbol{E} = \boldsymbol{\nabla}\boldsymbol{r}$,

✔ полного дифференциала $d\boldsymbol{\xi} = d\boldsymbol{r}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\xi}$,

✔ частных производных касательного базиса (вторых частных производных $\boldsymbol{r}$) $\boldsymbol{r}_{\partial i\partial j} \equiv \partial_i\partial_j\boldsymbol{r} = \partial_i\boldsymbol{r}_{\partial j}$,

✔ символов "связности" Христоффеля (Christoffel symbols) $\Gamma^k_{ij} \equiv \boldsymbol{r}_{\partial i\partial j}\boldsymbol{\cdot}\boldsymbol{r}^k$ и $\Gamma_{ij\overset{\bullet}{k}} \equiv \boldsymbol{r}_{\partial i\partial j}\boldsymbol{\cdot}\boldsymbol{r}_{\partial k}$.

Представим теперь, что функция $\boldsymbol{r}(q^k)$ не известна, но зато́ в каждой точке пространства известны шесть независимых компонент положительно определённой (all Gram matrices are non-negative definite) симметричной метрической матрицы Gram $g_{ij}(q^k)$.

the Gram matrix (or Gramian)

Билинейная форма ...

...

Поскольку шесть функций $g_{ij}(q^k)$ происходят от векторной функции $\boldsymbol{r}(q^k)$, то между элементами g_{ij} существуют некие соотношения.

Differential $d\boldsymbol{r}$ (16.4) is exact. This is true if and only if second partial derivatives commute:

$$d\boldsymbol{r} = \boldsymbol{r}_{\partial k}\,dq^k \;\Leftrightarrow\; \partial_i\boldsymbol{r}_{\partial j} = \partial_j\boldsymbol{r}_{\partial i} \;\text{ or }\; \boldsymbol{r}_{\partial i\partial j} = \boldsymbol{r}_{\partial j\partial i}.$$

Но это условие уже́ обеспечено симметрией g_{ij}

...

metric ("affine") connection ∇_i, её же называют "covariant derivative"

$$\boldsymbol{r}_{\partial i\partial j} = \underbrace{\boldsymbol{r}_{\partial i\partial j}\boldsymbol{\cdot}\overset{\boldsymbol{E}}{\overbrace{\boldsymbol{r}^k\boldsymbol{r}_{\partial k}}}}_{\Gamma^k_{ij}} = \underbrace{\boldsymbol{r}_{\partial i\partial j}\boldsymbol{\cdot}\overset{\boldsymbol{E}}{\overbrace{\boldsymbol{r}_{\partial k}\boldsymbol{r}^k}}}_{\Gamma_{ij\overset{\bullet}{k}}}$$

$$\Gamma^k_{ij}\boldsymbol{r}_{\partial k} = \boldsymbol{r}_{\partial i\partial j}\boldsymbol{\cdot}\boldsymbol{r}^k\boldsymbol{r}_{\partial k} = \boldsymbol{r}_{\partial i\partial j}$$

covariant derivative (affine connection) is only defined for vector fields

$$\boldsymbol{\nabla v} = \boldsymbol{r}^i \partial_i \left(v^j \boldsymbol{r}_{\partial j} \right) = \boldsymbol{r}^i \left(\partial_i v^j \boldsymbol{r}_{\partial j} + v^j \boldsymbol{r}_{\partial i \partial j} \right)$$

$$\boldsymbol{\nabla v} = \boldsymbol{r}^i \boldsymbol{r}_{\partial j} \nabla_i v^j, \ \ \nabla_i v^j \equiv \partial_i v^j + \Gamma_{in}^j v^n$$

$$\boldsymbol{\nabla r}_{\partial i} = \boldsymbol{r}^k \partial_k \boldsymbol{r}_{\partial i} = \boldsymbol{r}^k \boldsymbol{r}_{\partial k \partial i} = \boldsymbol{r}^k \boldsymbol{r}_{\partial n} \Gamma_{ki}^n, \ \ \nabla_i \boldsymbol{r}_{\partial n} = \Gamma_{in}^k \boldsymbol{r}_{\partial k}$$

Christoffel symbols describe a metric ("affine") connection, that is how the basis changes from point to point.

символы Christoffel'я это "components of connection" in local coordinates

...

torsion tensor $^3\boldsymbol{\mathfrak{T}}$ with components

$$\mathfrak{T}_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$$

determines the antisymmetric part of a connection

...

симметрия $\Gamma_{ij\underset{\bullet}{k}} = \Gamma_{ji\underset{\bullet}{k}}$, поэтому $3^3 - 3 \cdot 3 = 18$ разных (независимых) $\Gamma_{ij\underset{\bullet}{k}}$

$$\Gamma_{ij}^n g_{nk} = \Gamma_{ij\underset{\bullet}{k}} = \boldsymbol{r}_{\partial i \partial j} \boldsymbol{\cdot} \boldsymbol{r}_{\partial k} =$$
$$= \tfrac{1}{2} \left(\boldsymbol{r}_{\partial i \partial j} + \boldsymbol{r}_{\partial j \partial i} \right) \boldsymbol{\cdot} \boldsymbol{r}_{\partial k} + \tfrac{1}{2} \left(\boldsymbol{r}_{\partial j \partial k} - \boldsymbol{r}_{\partial k \partial j} \right) \boldsymbol{\cdot} \boldsymbol{r}_{\partial i} + \tfrac{1}{2} \left(\boldsymbol{r}_{\partial i \partial k} - \boldsymbol{r}_{\partial k \partial i} \right) \boldsymbol{\cdot} \boldsymbol{r}_{\partial j} =$$
$$= \tfrac{1}{2} \left(\boldsymbol{r}_{\partial i \partial j} \boldsymbol{\cdot} \boldsymbol{r}_{\partial k} + \boldsymbol{r}_{\partial i \partial k} \boldsymbol{\cdot} \boldsymbol{r}_{\partial j} \right) + \tfrac{1}{2} \left(\boldsymbol{r}_{\partial j \partial i} \boldsymbol{\cdot} \boldsymbol{r}_{\partial k} + \boldsymbol{r}_{\partial j \partial k} \boldsymbol{\cdot} \boldsymbol{r}_{\partial i} \right) - \tfrac{1}{2} \left(\boldsymbol{r}_{\partial k \partial i} \boldsymbol{\cdot} \boldsymbol{r}_{\partial j} + \boldsymbol{r}_{\partial k \partial j} \boldsymbol{\cdot} \boldsymbol{r}_{\partial i} \right) =$$
$$= \tfrac{1}{2} \Big(\partial_i (\boldsymbol{r}_{\partial j} \boldsymbol{\cdot} \boldsymbol{r}_{\partial k}) + \partial_j (\boldsymbol{r}_{\partial i} \boldsymbol{\cdot} \boldsymbol{r}_{\partial k}) - \partial_k (\boldsymbol{r}_{\partial i} \boldsymbol{\cdot} \boldsymbol{r}_{\partial j}) \Big) =$$
$$= \tfrac{1}{2} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right). \quad (18.1)$$

Все символы Christoffel'я тождественно равны нулю лишь в ортонормальной (декартовой) системе. (А какие они для косоугольной?)

Дальше: $d\boldsymbol{r}_{\partial i} = d\boldsymbol{r} \boldsymbol{\cdot} \boldsymbol{\nabla r}_{\partial i} = dq^k \partial_k \boldsymbol{r}_{\partial i} = \boldsymbol{r}_{\partial k \partial i} dq^k$ — тоже полные дифференциалы.

$$d\boldsymbol{r}_{\partial k} = \partial_i \boldsymbol{r}_{\partial k} dq^i = \frac{\partial \boldsymbol{r}_{\partial k}}{\partial q^1} dq^1 + \frac{\partial \boldsymbol{r}_{\partial k}}{\partial q^2} dq^2 + \frac{\partial \boldsymbol{r}_{\partial k}}{\partial q^3} dq^3$$

Поэтому $\partial_i \partial_j \boldsymbol{r}_{\partial k} = \partial_j \partial_i \boldsymbol{r}_{\partial k}$, $\partial_i \boldsymbol{r}_{\partial j \partial k} = \partial_j \boldsymbol{r}_{\partial i \partial k}$, и трёхиндексный объект из векторов третьих частных производных

$$\boldsymbol{r}_{\partial i \partial j \partial k} \equiv \partial_i \partial_j \partial_k \boldsymbol{r} = \partial_i \boldsymbol{r}_{\partial j \partial k} \quad (18.2)$$

симметричен по первому и второму индексам (а не только по второму и третьему). И тогда равен нулю $^4\mathbf{0}$ следующий тензор четвёртой сложности — *Riemann curvature tensor* (or *Riemann–Christoffel tensor*)

$$^4\mathfrak{R} = \mathfrak{R}_{hijk}\boldsymbol{r}^h\boldsymbol{r}^i\boldsymbol{r}^j\boldsymbol{r}^k, \;\; \mathfrak{R}_{hijk} \equiv \boldsymbol{r}_{\partial h}\bullet(\boldsymbol{r}_{\partial j\partial i\partial k} - \boldsymbol{r}_{\partial i\partial j\partial k}). \qquad (18.3)$$

Выразим компоненты $\mathfrak{R}_{ijkn}$ через метрическую матрицу g_{ij}. Начнём с дифференцирования локального кобазиса:

$$\boldsymbol{r}^i\bullet\boldsymbol{r}_{\partial k} = \delta^i_k \;\Rightarrow\; \partial_j\boldsymbol{r}^i\bullet\boldsymbol{r}_{\partial k} + \boldsymbol{r}^i\bullet\boldsymbol{r}_{\partial j\partial k} = 0 \;\Rightarrow\; \partial_j\boldsymbol{r}^i = -\Gamma^i_{jk}\boldsymbol{r}^k.$$

...

Шесть независимых компонент: $\mathfrak{R}_{1212}$, $\mathfrak{R}_{1213}$, $\mathfrak{R}_{1223}$, $\mathfrak{R}_{1313}$, $\mathfrak{R}_{1323}$, $\mathfrak{R}_{2323}$.

...

Symmetric bivalent *Ricci curvature tensor*

$$\boldsymbol{\mathscr{R}} \equiv \tfrac{1}{4}\,\mathfrak{R}_{abij}\,\boldsymbol{r}^a\times\boldsymbol{r}^b\boldsymbol{r}^i\times\boldsymbol{r}^j = \tfrac{1}{4}\in^{[abp]}\in^{[ijq]}\mathfrak{R}_{abij}\,\boldsymbol{r}_{\partial p}\boldsymbol{r}_{\partial q} = \mathscr{R}^{pq}\boldsymbol{r}_{\partial p}\boldsymbol{r}_{\partial q}$$

(coefficient $\tfrac{1}{4}$ is used here for convenience) with components

$$\mathscr{R}^{11} = \tfrac{1}{g}\,\mathfrak{R}_{2323},$$
$$\mathscr{R}^{21} = \tfrac{1}{g}\,\mathfrak{R}_{1323}, \;\; \mathscr{R}^{22} = \tfrac{1}{g}\,\mathfrak{R}_{1313},$$
$$\mathscr{R}^{31} = \tfrac{1}{g}\,\mathfrak{R}_{1223}, \;\; \mathscr{R}^{32} = \tfrac{1}{g}\,\mathfrak{R}_{1213}, \;\; \mathscr{R}^{33} = \tfrac{1}{g}\,\mathfrak{R}_{1212}.$$

Равенство тензора Риччи нулю $\boldsymbol{\mathscr{R}} = {}^2\mathbf{0}$ (в компонентах это шесть уравнений $\mathscr{R}^{ij} = \mathscr{R}^{ji} = 0$) is the necessary condition of integrability ("compatibility") для нахождения вектора-радиуса $\boldsymbol{r}(q^k)$ по полю $g_{ij}(q^k)$.

Bibliography

There are many books which describe only the apparatus of the tensor calculus [97, 98, 99, 16, **?**, 100].

However, the index notation (it's when the tensors are presented as the sets of components) is still more popular than the direct indexless notation.

The direct notation is widely used, for example, in the appendices to the books by Anatoliy I. Lurie (Анатолий И. Лурье) [28, 29].

In "Теории упругости" ("Theory of elasticity") by Вениамин Блох (Veniamin Blokh) [7] the direct indexless notation is used too.

The R. Feynman's lectures [85] contain the bright description of the vector fields theory.

Also, the information about the tensor calculus is the part of the unusual and interesting book by C. Truesdell [60].

CLASSICAL GENERIC MECHANICS

## §1. Discrete collection of particles

C lassical generic mechanics models physical objects by discretizing them into a collection of particles ("pointlike masses", "material points"[*]).

In a collection of N particles, each k-th particle has its nonzero mass $m_k = \mathsf{constant} > 0$ and motion function $\boldsymbol{r}_k(t)$. Function $\boldsymbol{r}_k(t)$ is measured relative to the chosen reference system.

A reference system (also "reference frame") consists of (fig. 9)
- ✓ some "null" reference point o,
- ✓ a set of coordinates,
- ✓ a clock.
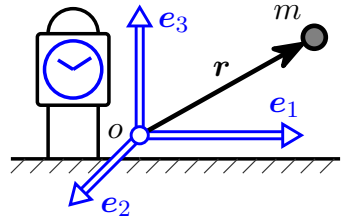
Long time ago, the reference system was some absolute space: empty at first, and then filled with the continuous elastic medium — the æther. Later it became clear that for classical mechanics any reference systems can be used, but the preference is given to the so called "inertial" systems, where a point moves with a constant velocity without acceleration ($\ddot{\boldsymbol{r}} = \boldsymbol{0}$, $\dot{\boldsymbol{r}} = \mathsf{constant}$) in the absence of external interactions.

figure 9

---

[*] Point mass (pointlike mass, material point) is the concept of an object, typically matter, that has nonzero mass and is (or is being thought of as) infinitesimal in its volume (dimensions).

A motion along a straight line with the constant velocity, also known as a free motion, supposes absence of external interactions (or applied forces):

$$\dot{\boldsymbol{r}} = \textcolor{green}{\mathbf{constant}} = \alpha_i \boldsymbol{e}_i \tag{1.1}$$

$$\alpha_i = \mathsf{constant} \tag{1.2}$$

The measure of interaction in mechanics is the vector of force $\boldsymbol{F}$. In the widely known[*] Newton's equation

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \tag{1.3}$$

the right-hand side can depend only on position, velocity and explicitly presented time, whereas acceleration $\ddot{\boldsymbol{r}}$ is directly proportional to force $\boldsymbol{F}$ with coefficient $1/m$.

---

[*] "Axiomata sive Leges Motus" ("Axioms or Laws of Motion") were written by Isaac Newton in his Philosophiæ Naturalis Principia Mathematica, first published in 1687. Reprint (en Latin), 1871. Translated into English by Andrew Motte, 1846.

Here're theses of the dynamics of a collection of particles.

The force acting on the k-th particle (fig. 10)

$$m_k \ddot{\boldsymbol{r}}_k = \boldsymbol{F}_k, \quad \boldsymbol{F}_k = \boldsymbol{F}_k^{(e)} + \sum_j \boldsymbol{F}_{kj}^{(i)}. \tag{1.4}$$

$\boldsymbol{F}_k^{(e)}$ is the external force — such forces emanate from objects outside the considered system. The second addend is the sum of internal forces (force $\boldsymbol{F}_{kj}^{(i)}$ is the interaction induced by the j-th particle on the k-th particle). Internal interactions happen only between elements of the observed system and don't affect (mechanically) anything other. Neither particle interacts with itself, $\boldsymbol{F}_{kk}^{(i)} = \boldsymbol{0} \ \forall k$.

figure 10

From (1.4) together with the action–reaction principle

$$\boldsymbol{F}_{kj}^{(i)} = -\boldsymbol{F}_{jk}^{(i)} \ \forall k, j \ \Rightarrow \ \sum_k \sum_j \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0},$$

ensues the balance of momentum

$$\left(\sum_k m_k \dot{\boldsymbol{r}}_k \right)^{\bullet} = \sum_k m_k \ddot{\boldsymbol{r}}_k = \sum_k \boldsymbol{F}_k^{(e)}. \tag{1.5}$$

The moment acting on the k-th particle

$$\boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k^{(e)} + \boldsymbol{r}_k \times \sum_j \boldsymbol{F}_{kj}^{(i)}. \tag{1.6}$$

It is relative to the reference point.

When in addition to the action–reaction principle, internal interactions between particles are central, that is

$$\boldsymbol{F}_{kj}^{(i)} \parallel (\boldsymbol{r}_k - \boldsymbol{r}_j) \ \Leftrightarrow \ (\boldsymbol{r}_k - \boldsymbol{r}_j) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0},$$
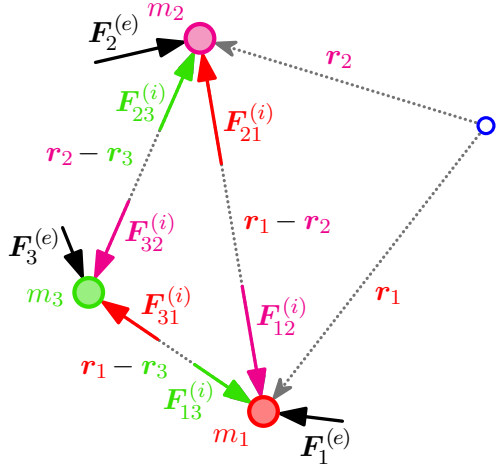
the balance of angular (rotational) momentum comes out*

$$\left(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \right)^{\bullet} = \sum_k \boldsymbol{r}_k \times \boldsymbol{F}_k^{(e)}. \tag{1.7}$$

Changes in momentum and angular momentum are determined only by external forces $\boldsymbol{F}_k^{(e)}$.

...

$$* \left(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \right)^{\bullet} = \sum_k \dot{\boldsymbol{r}}_k \times m_k \dot{\boldsymbol{r}}_k + \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k = \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k \,,$$

$$\boldsymbol{F}_{kj}^{(i)} = -\boldsymbol{F}_{jk}^{(i)} \ \text{ and } \ \left(\boldsymbol{r}_k - \boldsymbol{r}_j \right) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0} \ \Rightarrow$$

$$\Rightarrow \ \sum_{k=1}^{N} \boldsymbol{r}_k \times \sum_{j=1}^{N} \boldsymbol{F}_{kj}^{(i)} = \sum_{k=1}^{N} \sum_{j=k+1}^{N} \left(\boldsymbol{r}_k - \boldsymbol{r}_j \right) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0}$$

§ 2. Absolutely rigid undeformable solid body

An absolutely rigid undeformable body is a solid* body, in which deformation is zero (or negligibly small — so small that it can be neglected). The distance between any two points of a non-deformable rigid body remains constant regardless of external forces exerted on it.

A non-deformable rigid body is modeled using the "continual approach" as a continuous distribution of mass (a material continuum, a continuous medium), rather than using the "discrete approach" (that is modeling a body as a discrete collection of particles).

The mass of a material continuum is continuously distributed in its volume

$$dm \equiv \rho d\mathcal{V} \tag{2.1}$$

(ρ is a volume(tric) mass density and $d\mathcal{V}$ is an infinitesimal volume).

A formula with a summation over discrete points becomes a formula for a continuous body by replacing the masses of particles with the mass (2.1) of an infinitesimal volume element $d\mathcal{V}$, and with the following integration over the whole volume of a body. In particular, here are the formulas for the (linear) momentum

$$\sum_k m_k \dot{\boldsymbol{r}}_k \ \text{ becomes } \ \int_{\mathcal{V}} \dot{\boldsymbol{r}} \, dm \tag{2.2}$$

and the angular (rotational) momentum

$$\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \ \text{ becomes } \ \int_{\mathcal{V}} \boldsymbol{r} \times \dot{\boldsymbol{r}} \, dm. \tag{2.3}$$

To fully describe the location (position, place) of any non-deformable body with all its points, it's enough to choose some unique point as the "pole", to find or to set the location $\boldsymbol{p} = \boldsymbol{p}(t)$ of the chosen point, as well as the angular orientation of a body relative to the pole (fig. 11). As a result, any motion of an undeformable rigid body is either a rotation around the chosen pole, or an equal

---

* "Rigid" is inelastic and not flexible, and "solid" is not fluid. A solid substance retains its size and shape without a container (as opposed to a fluid substance — liquid or gas).

displacement of the pole and all body's points — translation (linear motion)[*], or a combination of them both.

$$\boldsymbol{r} = \boldsymbol{p} + \boldsymbol{x}$$
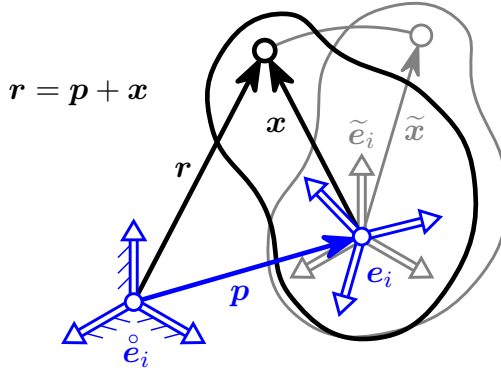
figure 11

$\mathring{\boldsymbol{e}}_i$ — triplet of orthonormal basis vectors, immovable relatively to the absolute (or any inertial) reference system

Имея неподвижный базис $\mathring{\boldsymbol{e}}_i$ and движущийся вместе с телом базис $\boldsymbol{e}_i$, ...

Если добавить базис $\boldsymbol{e}_i$ (этот базис движется вместе с телом), то угловая ориентация тела может быть определена тензором поворота $\boldsymbol{O} \equiv \boldsymbol{e}_i \widetilde{\boldsymbol{e}}_i$.

Then any motion of a body is completely described by two functions $\boldsymbol{p}(t)$ and $\boldsymbol{O}(t)$.

Вектор положения некоторой точки тела

$$\boldsymbol{r} = \boldsymbol{p} + \boldsymbol{x} \tag{2.4}$$

$\widetilde{\boldsymbol{x}} = x_i \widetilde{\boldsymbol{e}}_i,\ \boldsymbol{x} = x_i \boldsymbol{e}_i$

(11.3), § 1.11

$\boldsymbol{x} = \boldsymbol{O} \boldsymbol{\cdot} \widetilde{\boldsymbol{x}}$

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{p}} + \dot{\boldsymbol{x}},$$

For a non-deformable rigid body, components x_i don't depend on time: $x_i = \mathsf{constant}(t)$ and $\dot{\boldsymbol{x}} = x_i \dot{\boldsymbol{e}}_i$

---

[*] A translation can also be thought of as a rotation with the revolution center at the infinity

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{O}} \cdot \overset{\circ}{\boldsymbol{x}}$$
$$x_i \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{O}} \cdot x_i \overset{\circ}{\boldsymbol{e}}_i \;\Leftrightarrow\; \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{O}} \cdot \overset{\circ}{\boldsymbol{e}}_i$$

...

The linear momentum and the rotational (angular) momentum of a non-deformable continuous body are described by the following integrals

...

...

$$\int_{\mathcal{V}} \boldsymbol{p}\, dm = \boldsymbol{p} \int_{\mathcal{V}} dm = \boldsymbol{p} m$$

$$\int_{\mathcal{V}} \boldsymbol{x}\, dm = \boldsymbol{\Xi} m, \;\; \boldsymbol{\Xi} \equiv m^{-1} \int_{\mathcal{V}} \boldsymbol{x}\, dm$$

Three inertial characteristics of the body:

✓ integral mass $m = \int_{\mathcal{V}} dm = \int_{\mathcal{V}} \rho\, d\mathcal{V}$ — the mass of the whole body,

✓ eccentricity vector $\boldsymbol{\Xi}$ — measures the offset of the chosen pole from the body's "center of mass",

✓ inertia tensor ${}^2\boldsymbol{\mathfrak{J}}$.

The eccentricity vector is equal to the null vector only when the chosen pole coincides with the "center of mass" — the unique point within a body with location vector $\boldsymbol{\hbar}$, in short

$$\boldsymbol{\Xi} = \boldsymbol{0} \;\Leftrightarrow\; \boldsymbol{p} = \boldsymbol{\hbar}.$$

$$\boldsymbol{x} = \boldsymbol{r} - \boldsymbol{p}, \;\; \boldsymbol{\Xi} m = \int_{\mathcal{V}} (\boldsymbol{r} - \boldsymbol{\hbar})\, dm = \boldsymbol{0},$$

$$\int_{\mathcal{V}} \boldsymbol{r}\, dm - \boldsymbol{\hbar} \int_{\mathcal{V}} dm = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{\hbar} = m^{-1} \int_{\mathcal{V}} \boldsymbol{r}\, dm$$

...

Introducing (pseudo)vector of angular velocity $\boldsymbol{\omega}$, ...

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i$$

...

inertia tensor $^2\boldsymbol{\mathfrak{J}}$

$$^2\boldsymbol{\mathfrak{J}} \equiv -\int_{\mathcal{V}} (\boldsymbol{x} \times \boldsymbol{E}) \boldsymbol{\cdot} (\boldsymbol{x} \times \boldsymbol{E}) \, dm = \int_{\mathcal{V}} (\boldsymbol{x} \boldsymbol{\cdot} \boldsymbol{x} \boldsymbol{E} - \boldsymbol{x}\boldsymbol{x}) \, dm$$

It is assumed (can be proven?) that the inertia tensor changes only due to a rotation

$$^2\boldsymbol{\mathfrak{J}} = \boldsymbol{O} \boldsymbol{\cdot} {}^2\overset{\circ}{\boldsymbol{\mathfrak{J}}} \boldsymbol{\cdot} \boldsymbol{O}^{\mathsf{T}}$$

and its components in basis $\boldsymbol{e}_i$ (moving together with a body) don't change over time

$$^2\boldsymbol{\mathfrak{J}} = \mathfrak{J}_{ab} \boldsymbol{e}_a \boldsymbol{e}_b, \ \mathfrak{J}_{ab} = \mathsf{constant}(t)$$

thus the time derivative is

$$^2\overset{\bullet}{\boldsymbol{\mathfrak{J}}} = \mathfrak{J}_{ab} \big(\overset{\bullet}{\boldsymbol{e}}_a \boldsymbol{e}_b + \boldsymbol{e}_a \overset{\bullet}{\boldsymbol{e}}_b \big) = \mathfrak{J}_{ab} \big(\boldsymbol{\omega} \times \boldsymbol{e}_a \boldsymbol{e}_b + \boldsymbol{e}_a \boldsymbol{\omega} \times \boldsymbol{e}_b \big) =$$
$$= \mathfrak{J}_{ab} \big(\boldsymbol{\omega} \times \boldsymbol{e}_a \boldsymbol{e}_b - \boldsymbol{e}_a \boldsymbol{e}_b \times \boldsymbol{\omega} \big) = \boldsymbol{\omega} \times {}^2\boldsymbol{\mathfrak{J}} - {}^2\boldsymbol{\mathfrak{J}} \times \boldsymbol{\omega}$$

Substitution of (....) into (1.5) and (1.7) gives equations of balance — the balance of linear momentum and the balance of rotational momentum — for a model of continuous non-deformable rigid body

...

here $\boldsymbol{f}$ is the external force per mass unit, $\boldsymbol{F}$ is the resultant of external forces (also called the "equally acting force" or the "main vector"), $\boldsymbol{M}$ is the resultant of external couples (the "main couple", the "main moment").

...

— Are there any scenarios for which the center of mass is not almost exactly equivalent to the center of gravity?

— Non-uniform gravity field. In a uniform gravitational field, the center of mass is equal to the center of gravity.

...

Work

$$W(\boldsymbol{F}, \boldsymbol{u}) = \boldsymbol{F} \boldsymbol{\cdot} \boldsymbol{u}$$

as the exact (full) differential

$$dW = \frac{\partial W}{\partial \boldsymbol{F}} \boldsymbol{\cdot} d\boldsymbol{F} + \frac{\partial W}{\partial \boldsymbol{u}} \boldsymbol{\cdot} d\boldsymbol{u}$$

by "product rule"

$$dW = d\left(\boldsymbol{F} \cdot \boldsymbol{u}\right) = d\boldsymbol{F} \cdot \boldsymbol{u} + \boldsymbol{F} \cdot d\boldsymbol{u}$$

$$\frac{\partial W}{\partial \boldsymbol{F}} = \boldsymbol{u}, \ \frac{\partial W}{\partial \boldsymbol{u}} = \boldsymbol{F}$$

...

Constraints

Imposed on the positions and velocities of particles, there are restrictions of a geometrical or kinematical nature, called constraints.

Holonomic constraints are relations between position variables (and possibly time) which can be expressed as equality like

$$f(q^1, q^2, q^3, \ldots, q^n, t) = 0,$$

where $q^1, q^2, q^3, \ldots, q^n$ are n parameters (coordinates) that fully describe the system.

A constraint that cannot be expressed as such is nonholonomic.

Holonomic constraint depends only on coordinates and time. It does not depend on velocities or any higher time derivatives.

Velocity-dependent constraints like

$$f(q^1, q^2, \ldots, q^n, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n, t) = 0$$

are mostly not holonomic.

For example, the motion of a particle constrained to lie on a sphere's surface is subject to a holonomic constraint, but if the particle is able to fall off a sphere under the influence of gravity, the constraint becomes non-holonomic. For the first case the holonomic constraint may be given by the equation: $r^2 - a^2 = 0$, where r is the distance from the centre of a sphere of radius a. Whereas the second non-holonomic case may be given by: $r^2 - a^2 \geq 0$.

Three examples of nonholonomic constraints are: when the constraint equations are nonintegrable, when the constraints have inequalities, or with complicated non-conservative forces like friction.

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q^1, q^2, \ldots, q^n, t)$$

(assuming n independent parameters/coordinates)

§3. Principle of virtual work

Mécanique analytique (1788–89) is a two volume French treatise on analytical mechanics, written by Joseph Louis Lagrange, and published 101 years following Isaac Newton's *Philosophiæ Naturalis Principia Mathematica.*

Joseph Louis Lagrange. Mécanique analytique. Nouvelle édition, revue et augmentée par l'auteur. Tome premier. Mme Ve Courcier, Paris, 1811. 490 pages.

Joseph Louis Lagrange. Mécanique analytique. Troisième édition, revue, corrigée et annotée par M. J. Bertrand. Tome second. Mallet-Bachelier, Paris, 1855. 416 pages.

The historical transition from geometrical methods, as presented in Newton's Principia, to methods of mathematical analysis.

Consider the exact differential of any set of location vectors $\boldsymbol{r}_i$, that are functions of other variable parameters (coordinates) $q^1, q^2, ..., q^n$ and time t.

The actual displacement is the differential

$$d\boldsymbol{r}_i = \frac{\partial \boldsymbol{r}_i}{\partial t}\,dt + \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q^j}\,dq^j$$

Now, imagine an arbitrary path through the configuration space/manifold. This means it has to satisfy the constraints of the system but not the actual applied forces

$$\delta\boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q^j}\,\delta q^j$$

A virtual infinitesimal displacement of a system of particles refers to a change in the configuration of a system as the result of any arbitrary infinitesimal change of location vectors (or coordinates) $\delta\boldsymbol{r}_k$, consistent with the forces and constraints imposed on the system at the current/given instant t. This displacement is called "virtual" to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the forces and constraints may be changing.

Assume the system is in equilibrium, that is the full force on each particle vanishes, $\boldsymbol{F}_i = \boldsymbol{0}\ \forall i$. Then clearly the term $\boldsymbol{F}_i \cdot \delta\boldsymbol{r}_i$, which is the virtual work of force $\boldsymbol{F}_i$ in displacement $\delta\boldsymbol{r}_i$, also vanishes for each

particle, $\boldsymbol{F}_i \cdot \delta\boldsymbol{r}_i = 0 \; \forall i$. The sum of these vanishing products over all particles is likewise equal to zero:

$$\sum_i \boldsymbol{F}_i \cdot \delta\boldsymbol{r}_i = 0.$$

Decompose the full force $\boldsymbol{F}_i$ into the applied (active) force $\boldsymbol{F}_i^{(a)}$ and the force of constraint $\boldsymbol{\Phi}_i$,

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{(a)} + \boldsymbol{\Phi}_i$$

We now restrict ourselves to systems for which the net virtual work of the force of every constraint is zero:

$$\sum_i \boldsymbol{\Phi}_i \cdot \delta\boldsymbol{r}_i = 0.$$

We therefore have as the condition for equilibrium of a system that the virtual work of all applied forces vanishes:

$$\sum_i \boldsymbol{F}_i^{(a)} \cdot \delta\boldsymbol{r}_i = 0.$$

— the principle of virtual work.

Note that coefficients $\boldsymbol{F}_i^{(a)}$ can no longer be thought equal to zero: in common $\boldsymbol{F}_i^{(a)} \neq 0$, since $\delta\boldsymbol{r}_i$ are not independent but are bound by constraints.

A virtual displacement of a particle with vector radius $\boldsymbol{r}_k$ is variation $\delta\boldsymbol{r}_k$ — any infinitesimal change of vector $\boldsymbol{r}_k$, which is compatible with the constraints. If the system is free, that is there are no constraints, then virtual displacements $\delta\boldsymbol{r}_k$ are perfectly random.

Связи бывают голономные (holonomic, или геометрические), связывающие только положения (смещения) — they are functions of only the coordinates and probably time

$$c(\boldsymbol{r}, t) = 0 \tag{3.1}$$

— и неголономные (или дифференциальные), содержащие производные координат по времени: $c(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = 0$ and not интегрируемые till the geometrical constraints.

When all constraints are holonomic, then the virtual displacements of a particle "k" satisfy the equation

$$\sum_{j=1}^{m} \frac{\partial c_j}{\partial \boldsymbol{r}_k} \cdot \delta\boldsymbol{r}_k = 0. \tag{3.2}$$

В несвободных системах все силы делятся на две группы: активные и реакции связей. Реакция $\boldsymbol{\Phi}_k$ действует со стороны всех материальных ограничителей на частицу "k" и меняется согласно уравнению (3.1) для каждой связи. Связи предполагаются идеальными:

$$\sum_k \boldsymbol{\Phi}_k \boldsymbol{\cdot} \delta \boldsymbol{r}_k = 0 \quad - \tag{3.3}$$

работа реакций на любых виртуальных смещениях равна нулю.

Принцип виртуальной работы is

$$\sum_k \Big(\boldsymbol{F}_k^{(a)} - m_k \ddot{\boldsymbol{r}}_k \Big) \boldsymbol{\cdot} \delta \boldsymbol{r}_k = 0, \tag{3.4}$$

где $\boldsymbol{F}_k^{(a)}$ — лишь активные силы, без реакций связей.

Дифференциальное вариационное уравнение (3.4) может показаться тривиальным следствием закона Ньютона (1.3) и условия идеальности связей (3.3). Однако содержание (3.4) несравненно обширнее. Читатель вскоре увидит, что принцип (3.4) может быть положен в основу механики [91]. The various models of elastic bodies that I describe in this book are based on this principle.

Для примера рассмотрим совершенно жёсткое (недеформируемое) твёрдое тело.

.... (2.4) $\Rightarrow \delta \boldsymbol{r} = \delta \boldsymbol{p} + \delta \boldsymbol{x}$

(begin copied from § 1.12)

Варьируя тождество (11.5), получим $\delta \boldsymbol{O} \boldsymbol{\cdot} \boldsymbol{O}^\mathsf{T} = - \boldsymbol{O} \boldsymbol{\cdot} \delta \boldsymbol{O}^\mathsf{T}$. Этот тензор антисимметричен, и потому выражается через свой сопутствующий вектор $\boldsymbol{\delta o}$ как $\delta \boldsymbol{O} \boldsymbol{\cdot} \boldsymbol{O}^\mathsf{T} = \boldsymbol{\delta o} \times \boldsymbol{E}$. Приходим к соотношениям

$$\delta \boldsymbol{O} = \boldsymbol{\delta o} \times \boldsymbol{O}, \ \ \boldsymbol{\delta o} = -\frac{1}{2} \Big(\delta \boldsymbol{O} \boldsymbol{\cdot} \boldsymbol{O}^\mathsf{T} \Big)_\times, \tag{3.5}$$

(end of copied from § 1.12)

...

Проявилась замечательная особенность (3.4): это уравнение эквивалентно системе такого порядка, каково число степеней свободы системы, то есть сколько независимых вариаций $\delta \boldsymbol{r}_k$ мы имеем. Если в системе N точек есть m связей, то число степеней свободы $n = 3N - m$.

...

§4. Balance of momentum, rotational momentum, and energy

Эти уравнения баланса могут быть связаны со свойствами пространства and времени [92]. Сохранение импульса (количества движения) в за́мкнутой (изолированной)[*] системе выводится из однородности пространства *(при любом параллельном переносе — трансляции — замкнутой системы как целого свойства этой системы не меняются)*. Сохранение момента импульса — следствие изотропии пространства *(свойства замкнутой системы не меняются при любом повороте этой системы как целого)*. Энергия же изолированной системы сохраняется, так как время однородно[**] (энергия $\mathrm{E} \equiv \mathrm{K}(q, \dot{q}) + \Pi(q)$ такой системы не зависит явно от времени).

Уравнения баланса могут быть выведены из принципа виртуальной работы (3.4). Перепишем его в виде

$$\sum_k \left(\boldsymbol{F}_k^{(e)} - m_k \ddot{\boldsymbol{r}}_k \right) \cdot \delta \boldsymbol{r}_k + \delta W^{(i)} = 0, \tag{4.1}$$

где выделены внешние силы $\boldsymbol{F}_k^{(e)}$ и виртуальная работа внутренних сил $\delta W^{(i)} = \sum_k \sum_j \boldsymbol{F}_{kj}^{(i)} \cdot \delta \boldsymbol{r}_k$.

Предполагается, что внутренние силы не совершают работы на виртуальных смещениях тела как жёсткого целого ($\delta\boldsymbol{\rho}$ и $\delta\mathsf{o}$ — произвольные постоянные векторы, определяющие трансляцию и поворот)

$$\begin{aligned} \delta \boldsymbol{r}_k &= \delta\boldsymbol{\rho} + \delta\mathsf{o} \times \boldsymbol{r}_k, \\ \delta\boldsymbol{\rho} &= \mathsf{constant}, \ \delta\mathsf{o} = \mathsf{constant} \end{aligned} \quad \Rightarrow \ \delta W^{(i)} = 0. \tag{4.2}$$

---

[*] *За́мкнутая (изолированная) система* это такая система частиц, которые взаимодействуют only друг с другом, but ни с какими другими телами.

[**] Характеристики "однородность" and "изотропность" пространства, "однородность" времени не фигурируют среди аксиом классической механики.

Предпосылки-соображения для этого предположения таковы.

Первое — для случая упругих (потенциальных) внутренних сил. Тогда $\delta W^{(i)} = -\delta\Pi$ — вариация потенциала с противоположным знаком. Достаточно очевидно, что только лишь деформирование меняет Π.

Второе соображение — в том, что суммарный вектор и суммарный момент внутренних сил равен нулю

$$\sum \cdots$$

...

Принимая (4.2) и подставляя в (4.1) сначала $\delta r_k = \boldsymbol{\delta\rho}$ (трансляция), а затем $\boldsymbol{\delta r}_k = \boldsymbol{\delta o} \times \boldsymbol{r}_k$ (поворот), получаем баланс импульса (...) и баланс момента импульса (...).

...

§5. Hamilton's principle and Lagrange's equations

Two branches of analytical mechanics are Lagrangian mechanics (using generalized coordinates and corresponding generalized velocities in configuration space) and Hamiltonian mechanics (using coordinates and corresponding momenta in phase space). Both formulations are equivalent by a Legendre transformation on the generalized coordinates, velocities and momenta, therefore both contain the same information for describing the dynamics of a system.

Вариационное уравнение (3.4) удовлетворяется в любой момент времени. Проинтегрируем его[*] по какому-либо промежутку $[t_1, t_2]$

$$\int\limits_{t_1}^{t_2}\left(\delta\mathrm{K} + \sum_k \boldsymbol{F}_k \boldsymbol{\cdot} \delta\boldsymbol{r}_k\right)dt - \left[\sum_k m_k \dot{\boldsymbol{r}}_k \boldsymbol{\cdot} \delta\boldsymbol{r}_k\right]_{t_1}^{t_2} = 0. \qquad (5.1)$$

[*] $\delta\mathrm{K} = \sum\limits_k m_k\dot{\boldsymbol{r}}_k \boldsymbol{\cdot} \delta\dot{\boldsymbol{r}}_k, \quad \left(\sum_k m_k\dot{\boldsymbol{r}}_k \boldsymbol{\cdot} \delta\boldsymbol{r}_k\right)^{\boldsymbol{\cdot}} = \sum_k m_k\ddot{\boldsymbol{r}}_k \boldsymbol{\cdot} \delta\boldsymbol{r}_k + \underbrace{\sum_k m_k\dot{\boldsymbol{r}}_k \boldsymbol{\cdot} \delta\dot{\boldsymbol{r}}_k}_{\delta\mathrm{K}}$

$-\int\limits_{t_1}^{t_2}\sum_k m_k\ddot{\boldsymbol{r}}_k \boldsymbol{\cdot} \delta\boldsymbol{r}_k\,dt = \int\limits_{t_1}^{t_2}\delta\mathrm{K}dt - \left[\sum_k m_k\dot{\boldsymbol{r}}_k \boldsymbol{\cdot} \delta\boldsymbol{r}_k\right]_{t_1}^{t_2}$

Без ущерба для общности можно принять $\delta \boldsymbol{r}_k(t_1) = \delta \boldsymbol{r}_k(t_2) = \boldsymbol{0}$, тогда внеинтегральный член исчезает.

Вводятся обобщённые координаты q^i ($i = 1, \ldots, n$ — число степеней свободы). Векторы-радиусы становятся функциями вида $\boldsymbol{r}_k(q^i, t)$, тождественно удовлетворяющими уравнениям связей (3.1). Если связи стационарны, то есть (3.1) не содержат t, то остаётся $\boldsymbol{r}_k(q^i)$. Кинетическая энергия превращается в функцию $\mathrm{K}(q^i, \dot{q}^i, t)$, где явно входящее t характерно лишь для нестационарных связей.

Весьма существенно понятие обобщённых сил Q_i. Они вводятся через выражение виртуальной работы

$$\sum_k \boldsymbol{F}_k \boldsymbol{\cdot} \delta \boldsymbol{r}_k = \sum_i Q_i \delta q^i, \ \ Q_i \equiv \sum_k \boldsymbol{F}_k \boldsymbol{\cdot} \frac{\partial \boldsymbol{r}_k}{\partial q^i}. \tag{5.2}$$

Сто́ит акцентировать происхождение обобщённых сил через работу. Установив набор обобщённых координат системы, следует сгруппировать приложенные силы $\boldsymbol{F}_k$ в комплексы Q_i.

Если силы потенциальны с энергией $\Pi = \Pi(q^i, t)$, то

$$\sum_i Q_i \delta q^i = -\delta \Pi, \ \ Q_i = -\frac{\partial \Pi}{\partial q^i}. \tag{5.3}$$

Явное присутствие t может быть при нестационарности связей или зависимости физических полей от времени.

...

Известны уравнения Lagrange'a не только второго, но и первого рода. Рассмотрим их ради методики вывода, много раз применяемой в этой книге.

При наличии связей (3.1) равенство $\boldsymbol{F}_k = m_k \ddot{\boldsymbol{r}}_k$ не следует из вариационного уравнения (3.4), ведь тогда виртуальные смещения $\delta \boldsymbol{r}_k$ не независимы. Каждое из m (m — число связей) условий для вариаций (3.2) умножим на некий скаляр λ_α ($\alpha = 1, \ldots, m$) и добавим к (3.4):

$$\sum_{k=1}^N \left(\boldsymbol{F}_k + \sum_{\alpha=1}^m \lambda_\alpha \frac{\partial c_\alpha}{\partial \boldsymbol{r}_k} - m_k \ddot{\boldsymbol{r}}_k \right) \boldsymbol{\cdot} \delta \boldsymbol{r}_k = 0. \tag{5.4}$$

Среди $3N$ компонент вариаций $\delta \boldsymbol{r}_k$ зависимых m. Но столько же и множителей Лагранжа: подберём λ_α так, чтобы коэффици-

енты(??какие?) при зависимых вариациях обратились в нуль. Но при остальных вариациях коэффициенты(??) также должны быть нулями из-за независимости. Следовательно, все выражения в скобках $(\cdots)$ равны нулю — это и есть уравнения Lagrange'a первого рода.

Поскольку для каждой частицы

...

§6. Statics

Рассмотрим систему со стационарными (постоянными во времени) связями при статических (не меняющихся со временем) активных силах $\boldsymbol{F}_k$. В равновесии $\boldsymbol{r}_k = $ **constant**, и формулировка принципа виртуальной работы следующая:

$$\sum_k \boldsymbol{F}_k \cdot \delta\boldsymbol{r}_k = 0 \ \Leftrightarrow \ \sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i} = Q_i = 0. \qquad (6.1)$$

Существенны обе стороны этого положения: и вариационное уравнение, и равенство нулю обобщённых сил.

Соотношения (6.1) — это самые общие уравнения статики. В литературе распространено узкое представление об уравнениях равновесия как балансе сил и моментов. Но при этом нужно понимать, что набор уравнений равновесия точно соответствует обобщённым координатам. The resultant force (также называемая "равнодействующей силой" или "главным вектором") and the resultant couple ("главный момент") в уравнениях равновесия фигурируют*, поскольку у системы есть степени свободы трансляции и поворота. Огромная популярность сил и моментов связана не столько с известностью статики совершенно недеформируемого твёрдого тела, но с тем, что виртуальная работа внутренних сил на смещениях системы как жёсткого целого равна нулю в любой среде.

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* Since describing a composition of any system of forces, acting on the same absolutely rigid body, into a single force and a single couple (about a chosen point) in the book "Éléments de statique" by Louis Poinsot.

Пусть в системе действуют два вида сил: потенциальные с энергией от обобщённых координат $\Pi(q^i)$ и дополнительные внешние $\overset{\circ}{Q}_i$. Из (6.1) следуют уравнения равновесия

$$\frac{\partial \Pi}{\partial q^i} = \overset{\circ}{Q}_i, \tag{6.2}$$

$$d\Pi = \sum_i \frac{\partial \Pi}{\partial q^i}\, dq^i = \sum_i \overset{\circ}{Q}_i\, dq^i.$$

Здесь содержится нелинейная в общем случае задача статики о связи положения равновесия q^i с нагрузками $\overset{\circ}{Q}_i$.

В линейной системе с квадратичным потенциалом вида $\Pi = \frac{1}{2}\, C_{ik} q^k q^i$

$$\sum_k C_{ik} q^k = \overset{\circ}{Q}_i. \tag{6.3}$$

Here figure elements C_{ik} of "the stiffness matrix", coordinates q^k and loads $\overset{\circ}{Q}_i$.

Сказанное возможно обобщить и на континуальные линейные упругие среды.

Матрица жёсткости C_{ik} обычно бывает положительной (таково́ свойство конструкций и в природе, и в технике). Тогда $\det C_{ik} > 0$, линейная алгебраическая система (6.3) однозначно разрешима, а решение её можно заменить минимизацией квадратичной формы

$$\mathscr{E}(q^j) \equiv \Pi - \sum_i \overset{\circ}{Q}_i q^i = \frac{1}{2} \sum_{i,k} q^i C_{ik} q^k - \sum_i \overset{\circ}{Q}_i q^i \to \min. \tag{6.4}$$

Бывает однако, что конструкция неудачно спроектирована, тогда матрица жёсткости сингулярна (необратима) и $\det C_{ik} = 0$ (или же весьма близок к нулю — nearly singular матрица с $\det C_{ik} \approx 0$). Тогда решение линейной проблемы статики (6.3) существует лишь при ортогональности столбца нагрузок $\overset{\circ}{Q}_i$ всем линейно независимым решениям однородной сопряжённой системы

...

Известные теоремы статики линейно упругих систем легко доказываются в случае конечного числа степеней свободы. Теорема Clapeyron'a выражается равенством

...

Reciprocal work theorem ("работа W_{12} сил первого варианта на смещениях от сил второго равна работе W_{21} сил второго варианта на смещениях от сил первого") мгновенно выводится из (6.3):

(...)

Тут существенна симметрия матрицы жёсткости C_{ij}, то есть консервативность системы.

...

Но вернёмся к проблеме (6.2), иногда называемой теоремой Lagrange'а. Её можно обратить преобразованием Лежандра Legendre (involution) transform(ation):

$$d\left(\sum_i \mathring{Q}_i q^i\right) = \sum_i d\left(\mathring{Q}_i q^i\right) = \sum_i \left(q^i\, d\mathring{Q}_i + \mathring{Q}_i\, dq^i\right),$$

$$d\left(\sum_i \mathring{Q}_i q^i\right) - \overbrace{\sum_i \mathring{Q}_i\, dq^i}^{d\Pi} = \sum_i q^i\, d\mathring{Q}_i,$$

$$d\left(\sum_i \mathring{Q}_i q^i - \Pi\right) = \sum_i q^i\, d\mathring{Q}_i = \sum_i \frac{\partial\widehat{\Pi}}{\partial\mathring{Q}_i}\, d\mathring{Q}_i;$$

$$q^i = \frac{\partial\widehat{\Pi}}{\partial\mathring{Q}_i}, \quad \widehat{\Pi}(\mathring{Q}_i) = \sum_i \mathring{Q}_i q^i - \Pi. \tag{6.5}$$

Это теорема Castigliano, $\widehat{\Pi}$ называется дополнительной энергией. В линейной системе (6.3) $\Rightarrow \widehat{\Pi} = \Pi$. Теорема (6.5) бывает очень полезна — когда легко находится $\widehat{\Pi}(\mathring{Q}_i)$. Встречаются так называемые статически определимые системы, в которых все внутренние силы удаётся найти лишь из баланса сил и моментов. Для них (6.5) эффективна.

В отличие от линейной задачи (6.3), нелинейная задача (6.2) может не иметь решений вовсе или же иметь их несколько.

....

Рассказ о статике в общей механике закончим the d'Alembert's principle: уравнения динамики отличаются от статических лишь наличием дополнительных "сил инерции" $m_k\ddot{\boldsymbol{r}}_k$. Принцип d'Alembert'а достаточно очевиден, но бездумное применение может

привести к ошибкам. Например, уравнения вязкой жидкости в статике и в динамике отличаются не только лишь инерционными добавками. Но для твёрдых упругих тел принцип d'Alembert'а полностью справедлив.

§7. Mechanics of relative motion

До этого не ставился вопрос о системе отсчёта, всё рассматривалось в некой "абсолютной" системе или одной из инерциальных систем (§1). Теперь представим себе две системы: "абсолютную" и "подвижную"

...

$$\overset{\circ}{\boldsymbol{r}} = \boldsymbol{r} + \boldsymbol{x}$$

$$\boldsymbol{r} = \rho_i \overset{\circ}{\boldsymbol{e}}_i, \ \ \boldsymbol{x} = x_i \boldsymbol{e}_i$$

$$\overset{\circ}{\dot{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}}$$

$$\dot{\boldsymbol{r}} = \dot{\rho}_i \overset{\circ}{\boldsymbol{e}}_i, \ \ \dot{\boldsymbol{x}} = \big(x_i \boldsymbol{e}_i\big)^{\bullet} = \dot{x}_i \boldsymbol{e}_i + x_i \dot{\boldsymbol{e}}_i$$

$x_i \neq \text{constant} \Rightarrow \dot{x}_i \neq 0$

By (11.11, §1.11)

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \Rightarrow x_i \dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times x_i \boldsymbol{e}_i = \boldsymbol{\omega} \times \boldsymbol{x}$$

$$\dot{\boldsymbol{x}} = \dot{x}_i \boldsymbol{e}_i + \boldsymbol{\omega} \times \boldsymbol{x}$$

$$\boldsymbol{v} \equiv \overset{\circ}{\dot{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}} = \underbrace{\dot{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{x}}_{\boldsymbol{v}_e} \underbrace{- \boldsymbol{\omega} \times \boldsymbol{x} + \dot{\boldsymbol{x}}}_{\boldsymbol{v}_{rel}}$$

$\dot{\boldsymbol{x}} - \boldsymbol{\omega} \times \boldsymbol{x} = \dot{x}_i \boldsymbol{e}_i \equiv \boldsymbol{v}_{rel}$ — relative velocity, $\dot{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{x} \equiv \boldsymbol{v}_e$

$$\boldsymbol{v} = \boldsymbol{v}_e + \boldsymbol{v}_{rel} \tag{7.1}$$

...

$$\overset{\circ}{\dot{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}}$$

$$\overset{\circ}{\ddot{\boldsymbol{r}}} = \ddot{\boldsymbol{r}} + \ddot{\boldsymbol{x}}$$

$$\boldsymbol{w} \equiv \dot{\boldsymbol{v}} = \overset{\circ}{\ddot{\boldsymbol{r}}} = \ddot{\boldsymbol{r}} + \ddot{\boldsymbol{x}}$$

$$\ddot{\boldsymbol{r}} = \ddot{\rho}_i \overset{\circ}{\boldsymbol{e}}_i, \ \ \ddot{\boldsymbol{x}} = \big(x_i \boldsymbol{e}_i\big)^{\bullet\bullet} = \big(\dot{x}_i \boldsymbol{e}_i + x_i \dot{\boldsymbol{e}}_i\big)^{\bullet} = \ddot{x}_i \boldsymbol{e}_i + \dot{x}_i \dot{\boldsymbol{e}}_i + \dot{x}_i \dot{\boldsymbol{e}}_i + x_i \ddot{\boldsymbol{e}}_i$$

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \Rightarrow \ddot{\boldsymbol{e}}_i = \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)^{\bullet} = \dot{\boldsymbol{\omega}} \times \boldsymbol{e}_i + \boldsymbol{\omega} \times \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{\omega}} \times \boldsymbol{e}_i + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)$$

$$x_i \ddot{\boldsymbol{e}}_i = x_i \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)^{\bullet} = \dot{\boldsymbol{\omega}} \times x_i \boldsymbol{e}_i + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times x_i \boldsymbol{e}_i\big) = \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{x}\big)$$

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \Rightarrow \dot{x}_i \dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \dot{x}_i \boldsymbol{e}_i = \boldsymbol{\omega} \times \boldsymbol{v}_{rel}$$

$\ddot{x}_i \boldsymbol{e}_i \equiv \boldsymbol{w}_{rel}$ — relative acceleration

$2\dot{x}_i \dot{\boldsymbol{e}}_i = 2\boldsymbol{\omega} \times \boldsymbol{v}_{rel} \equiv \boldsymbol{w}_{Cor}$ — Coriolis acceleration

$$\ddot{\boldsymbol{x}} = \boldsymbol{w}_{rel} + \boldsymbol{w}_{Cor} + x_i \ddot{\boldsymbol{e}}_i$$

$$\left(x_i \dot{\boldsymbol{e}}_i\right)^{\bullet} = \dot{x}_i \dot{\boldsymbol{e}}_i + x_i \ddot{\boldsymbol{e}}_i = \tfrac{1}{2} \boldsymbol{w}_{Cor} + x_i \ddot{\boldsymbol{e}}_i$$

$$\left(x_i \dot{\boldsymbol{e}}_i\right)^{\bullet} = \left(\boldsymbol{\omega} \times \boldsymbol{x}\right)^{\bullet} = \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \dot{\boldsymbol{x}}$$

$$\boldsymbol{\omega} \times \dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \left(\dot{x}_i \boldsymbol{e}_i + \boldsymbol{\omega} \times \boldsymbol{x}\right) = \underbrace{\boldsymbol{\omega} \times \dot{x}_i \boldsymbol{e}_i}_{\dot{x}_i \dot{\boldsymbol{e}}_i = \frac{1}{2} \boldsymbol{w}_{Cor}} + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{x}\right)$$

...

§ 8. Small oscillations (vibrations)

If the statics of a linear system is described by equation (6.3), then in the dynamics we have

$$\sum_k \left(A_{ik}\ddot{q}_k + C_{ik}q^k\right) = \mathring{Q}_i(t), \tag{8.1}$$

where A_{ik} is the symmetric and positive "matrix of kinetic energy".

Any description of oscillations almost always includes the term "mode". A mode of vibration can be defined as a way of vibrating or a pattern of vibration. A normal mode is a pattern of periodic motion, when all parts of an oscillating system move sinusoidally with the same frequency and with a fixed phase relation. The free motion described by the normal modes takes place at fixed frequencies — the natural resonant frequencies of an oscillating system.

The most generic motion of an oscillating system is some superposition of normal modes of this system.*

---

* The modes are "normal" in the sense that they move independently, and an excitation of one mode will never cause a motion of another mode. In mathematical terms, normal modes are orthogonal to each other. In music, normal modes of vibrating instruments (strings, air pipes, percussion and others) are called "harmonics" or "overtones".

A research of an oscillating system most often begins with orthogonal (normal) "modes" — harmonics, free (without any driving or damping force) sinusoidal oscillations

$$q^k(t) = \overset{*}{q}_k \sin \omega_k t.$$

Multipliers $\overset{*}{q}_k = \mathsf{constant}$ are orthogonal (normal) "modes" of oscillation, ω_k are natural (resonant, eigen-) frequencies. This set, dependent on the structure of an oscillating object, the materials and the boundary conditions, is found from the eigenvalue problem

$$\overset{\circ}{Q}_i = 0, \;\; \ddot{\overset{*}{q}}_k = -\omega_k^2 \overset{*}{q}_k \sin \omega_k t, \;\; (8.1) \;\; \Rightarrow$$
$$\Rightarrow \sum_k \left(C_{ik} - A_{ik}\, \omega_k^2 \right) \overset{*}{q}_k \sin \omega_k t = 0 \tag{8.2}$$

...

The Duhamel's integral is a way of calculating the response of linear systems to an arbitrary time-varying external perturbation.

...

Bibliography

In a long list of the books about the classical mechanics, the reader can find the works of both the specialists in mechanics [86, 87, 93, 94, 95] and the broadly oriented theoretical physicists [92, 88]. The book by Felix R. Gantmacher (Феликс Р. Гантмахер) [91] with the compact but complete narration of the fundamentals is pretty interesting.

chapter 3
NONLINEAR ELASTIC MOMENTLESS CONTINUUM

§ 1. Continuum. The two descriptions of processes

A ccording to the atomic theory, a substance is composed of discrete particles — atoms. Therefore a model of a system of particles with masses m_k and location vectors $\boldsymbol{r}_k(t)$ may seem suitable yet despite an unimaginable number of degrees of freedom, because amounts of memory and speed of computers are characterized also by astronomical numbers.

But anyway, it's worth choosing a fundamentally and qualitatively different model — a model of material continuum, where the mass is continuously distributed in a volume: volume $\mathcal{V}$ contains mass

$$m = \int_{\mathcal{V}} \rho \, d\mathcal{V}, \ \ dm = \rho \, d\mathcal{V}, \tag{1.1}$$

here ρ is the volume(tric) mass density and $d\mathcal{V}$ is the infinitesimal volume element.

Real matter is modelled as a continuum, which can be thought of as an infinite set of vanishingly small particles, joined together.

A space of material points is only the first and simple idea of a continuous distribution of mass. More complex models are possible too, where particles have more degrees of freedom: not only of translation, but also of rotation, of internal deformation, and others. Knowing that such models are attracting more and more interest, in this chapter we will consider the classical concept of a continuous medium as "made of simple points".

At every moment of time t, a deformable continuum occupies a certain volume $\mathcal{V}$ of the space. This volume moves and deforms, but

the set of particles inside this volume is constant. It is the balance of mass ("matter is neither created nor annihilated")

$$dm = \rho d\mathcal{V} = \rho' d\mathcal{V}' = \mathring{\rho} d\mathring{\mathcal{V}}, \ \ m = \int\limits_{\mathcal{V}} \rho d\mathcal{V} = \int\limits_{\mathcal{V}'} \rho' d\mathcal{V}' = \int\limits_{\mathring{\mathcal{V}}} \mathring{\rho} d\mathring{\mathcal{V}}. \quad (1.2)$$

Introducing some variable parameters q^i — the curvilinear coordinates, we have a relation for locations of particles

$$\boldsymbol{r} = \boldsymbol{r}(q^i, t). \tag{1.3}$$

...

Material description

at the initial moment, in the so-called initial (original, ~~reference,~~ "material") configuration

at some initial moment $t = 0$

"запоминается" начальная ("материальная") конфигурация — locations in space of particles at some arbitrarily chosen "initial" moment $t = 0$

$$\mathring{\boldsymbol{r}}(q^i) \equiv \boldsymbol{r}(q^i, 0)$$

Morphism (function) $\mathring{\boldsymbol{r}} = \mathring{\boldsymbol{r}}(q^i)$

isomorphism (bijective mapping) (invertible one-to-one relation) (взаимно однозначное)

Subsequent locations in space of particles are then dependent variables — functions of time and of the initial (material, "Lagrangian") coordinates/location $\mathring{\boldsymbol{r}}$

$$\boldsymbol{r} = \boldsymbol{r}(\mathring{\boldsymbol{r}}, t).$$

Для пространственного дифференцирования (постоянных во времени) отношений like $\varphi = \varphi(\mathring{\boldsymbol{r}})$

вводится локальный касательный базис $\mathring{\boldsymbol{r}}_{\partial i}$ и взаимный базис $\mathring{\boldsymbol{r}}^i$

$$\mathring{\boldsymbol{r}}_{\partial i} \equiv \partial_i \mathring{\boldsymbol{r}} \ \left(\partial_i \equiv \frac{\partial}{\partial q^i}\right), \ \ \mathring{\boldsymbol{r}}^i \boldsymbol{\cdot} \mathring{\boldsymbol{r}}_{\partial j} = \delta^i_j$$

"материальный" оператор Hamilton'a $\mathring{\boldsymbol{\nabla}}$

$$\boldsymbol{E} = \mathring{\boldsymbol{r}}^i \mathring{\boldsymbol{r}}_{\partial i} = \mathring{\boldsymbol{r}}^i \partial_i \mathring{\boldsymbol{r}} = \mathring{\boldsymbol{\nabla}} \mathring{\boldsymbol{r}}, \ \ \mathring{\boldsymbol{\nabla}} \equiv \mathring{\boldsymbol{r}}^i \partial_i, \tag{1.4}$$

тогда $d\varphi = d\mathring{\boldsymbol{r}} \boldsymbol{\cdot} \mathring{\boldsymbol{\nabla}}\varphi$.

...

But yet another approach may be effective — *the spatial (or "Eulerian") description*, when instead of focusing on how particles of a continuum move from the initial configuration through space and time, processes are considered at fixed points in space as time progresses. With relations like $\rho = \rho(\boldsymbol{r}, t)$, we track what's happening exactly in this place. Various particles, continuously leaving and coming here, do not confuse us.

...

the balance of mass in spatial description (the continuity equation for mass)

...

Jaumann derivative ("corotational time derivative") was first introduced by Gustav Jaumann*

Es sei $\frac{\partial}{\partial t}$ der Operator der lokalen Fluxion, d. i. der partiellen Fluxion in einem gegen das Koordinatensystem ruhenden Punkte des Raumes. Ferner sei $\frac{d}{dt}$ der Operator der totalen Fluxion, welcher definiert wird durch

$$\frac{da}{dt} = \frac{\partial a}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} a,$$

$$\frac{d\boldsymbol{a}}{dt} \overset{3}{=} \frac{\partial \boldsymbol{a}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}; \boldsymbol{a} - \tfrac{1}{2}\left(\operatorname{rot} \boldsymbol{v}\right) \times \boldsymbol{a},$$

$$\frac{d\boldsymbol{\alpha}}{dt} \overset{9}{=} \frac{\partial \boldsymbol{\alpha}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}; \boldsymbol{\alpha} - \tfrac{1}{2}\left(\operatorname{rot} \boldsymbol{v} \times \boldsymbol{\alpha} - \boldsymbol{\alpha} \times \operatorname{rot} \boldsymbol{v}\right).$$

Endlich verwenden wir die körperliche Fluxion eines Skalars:

$$\frac{\delta}{\delta t} a = \frac{\partial}{\partial t} a + \operatorname{div} a\boldsymbol{v} = \frac{d}{dt} a + a \operatorname{div} \boldsymbol{v}.$$

körperliche — bodily/телесная, material/вещественная(материальная), physical/физическая

$$\boldsymbol{\nabla} \cdot \left(a\boldsymbol{v}\right) = a \boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} a$$

* **Gustav Jaumann**. Geschlossenes System physikalischer und chemischer Differentialgesetze (I. Mitteilung) // Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Klasse, Abteilung IIa, Band CXX, 1911. Seiten 385–530.

...

Пусть $\upsilon(\overset{\circ}{\boldsymbol{r}}, t)$ — какое-либо поле (?? только в материальном описании от $\overset{\circ}{\boldsymbol{r}}$??). Найдём скорость изменения интеграла по объёму

$$\Upsilon \equiv \int\limits_{\mathcal{V}} \rho\upsilon\,d\mathcal{V}$$

("υ is Υ per mass unit"). Seemingly difficult calculation of $\dot{\Upsilon}$ (since $\mathcal{V}$ is deforming) is actually quite simple together with (1.2):

$$\Upsilon = \int\limits_{\overset{\circ}{\mathcal{V}}} \overset{\circ}{\rho}\upsilon\,d\overset{\circ}{\mathcal{V}} \;\Rightarrow\; \dot{\Upsilon} = \int\limits_{\overset{\circ}{\mathcal{V}}} \overset{\circ}{\rho}\dot{\mathfrak{v}}\,d\overset{\circ}{\mathcal{V}} = \int\limits_{\mathcal{V}} \rho\dot{\mathfrak{v}}\,d\mathcal{V}. \qquad (1.5)$$

$$\Psi = \int_{\mathcal{V}} \rho\psi\,d\mathcal{V} = \int_{\mathcal{V}'} \rho'\psi\,d\mathcal{V}' \Rightarrow \dot{\Psi} = \int_{\mathcal{V}} \rho\dot{\psi}\,d\mathcal{V} = \int_{\mathcal{V}'} \rho'\dot{\psi}\,d\mathcal{V}'$$

...

It is not worth it to contrapose the material and the spatial descriptions. In this book both are used, depending on the situation.

§ 2. Motion gradient

Having motion function $\boldsymbol{r} = \boldsymbol{r}(q^i, t)$, $\mathring{\boldsymbol{r}}(q^i) \equiv \boldsymbol{r}(q^i, 0)$, "nabla" operators $\boldsymbol{\nabla} \equiv \boldsymbol{r}^i \partial_i$, $\mathring{\boldsymbol{\nabla}} \equiv \mathring{\boldsymbol{r}}^i \partial_i$ and looking at differential relations for a certain infinitesimal vector in two configurations, current with $d\boldsymbol{r}$ and initial with $d\mathring{\boldsymbol{r}}$

$$\begin{aligned}
d\boldsymbol{r} &= d\mathring{\boldsymbol{r}} \cdot \overbrace{\mathring{\boldsymbol{\nabla}} \boldsymbol{r}}^{\substack{\boldsymbol{F}^\mathsf{T} \\ \mathring{\boldsymbol{r}}^i \boldsymbol{r}_{\partial i}}} = \overbrace{\mathring{\boldsymbol{\nabla}} \boldsymbol{r}^\mathsf{T}}^{\substack{\boldsymbol{F} \\ \boldsymbol{r}_{\partial i} \mathring{\boldsymbol{r}}^i}} \cdot d\mathring{\boldsymbol{r}} \\
d\mathring{\boldsymbol{r}} &= \underbrace{d\boldsymbol{r} \cdot \boldsymbol{\nabla} \mathring{\boldsymbol{r}}}_{\substack{\boldsymbol{r}^i \mathring{\boldsymbol{r}}_{\partial i} \\ \boldsymbol{F}^{-\mathsf{T}}}} = \underbrace{\boldsymbol{\nabla} \mathring{\boldsymbol{r}}^\mathsf{T}}_{\substack{\mathring{\boldsymbol{r}}_{\partial i} \boldsymbol{r}^i \\ \boldsymbol{F}^{-1}}} \cdot d\boldsymbol{r}
\end{aligned} \qquad (2.1)$$

here comes to mind to introduce the "motion gradient"[*], picking one of these tensor multipliers for it: $\boldsymbol{F} \equiv \mathring{\boldsymbol{\nabla}} \boldsymbol{r}^\mathsf{T} = \boldsymbol{r}_{\partial i} \mathring{\boldsymbol{r}}^i$.

Why this one? The reason to choose $\mathring{\boldsymbol{\nabla}} \boldsymbol{r}^\mathsf{T}$ is another expression for the differential

$$dr = \frac{\partial \boldsymbol{r}}{\partial \mathring{\boldsymbol{r}}} \cdot d\mathring{\boldsymbol{r}}$$

$$\boldsymbol{F} = \frac{\partial \boldsymbol{r}}{\partial \mathring{\boldsymbol{r}}}$$

$$d\mathring{\boldsymbol{r}} = \frac{\partial \mathring{\boldsymbol{r}}}{\partial \boldsymbol{r}} \cdot d\boldsymbol{r}$$

$$\boldsymbol{F}^{-1} = \frac{\partial \mathring{\boldsymbol{r}}}{\partial \boldsymbol{r}}$$

$$\frac{\partial \boldsymbol{\zeta}}{\partial \mathring{\boldsymbol{r}}} = \partial_i \boldsymbol{\zeta} \mathring{\boldsymbol{r}}^i \qquad \frac{\partial \boldsymbol{\zeta}}{\partial \boldsymbol{r}} = \partial_i \boldsymbol{\zeta} \boldsymbol{r}^i$$

...

$$\boldsymbol{E} = \underbrace{\mathring{\boldsymbol{\nabla}} \mathring{\boldsymbol{r}}}_{\frac{\partial \mathring{\boldsymbol{r}}}{\partial \mathring{\boldsymbol{r}}}} = \underbrace{\boldsymbol{\nabla} \boldsymbol{r}}_{\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}}}$$

...

For cartesian coordinates with orthonormal basis $\boldsymbol{e}_i = \mathbf{constant}$

$$\boldsymbol{r} = x_i(t) \boldsymbol{e}_i, \quad \mathring{\boldsymbol{r}} = x_i(0) \boldsymbol{e}_i = \mathring{x}_i \boldsymbol{e}_i, \quad \mathring{x}_i \equiv x_i(0),$$

---

[*] Tensor $\boldsymbol{F}$ doesn't well suit its more popular name "deformation gradient", because this tensor describes not only the deformation itself, but also the rotation of a body as a whole without deformation.

$$\overset{\circ}{\boldsymbol{\nabla}} = \boldsymbol{e}_i \frac{\partial}{\partial \overset{\circ}{x}_i} = \boldsymbol{e}_i \overset{\circ}{\partial}_i, \quad \boldsymbol{\nabla} = \boldsymbol{e}_i \frac{\partial}{\partial x_i} = \boldsymbol{e}_i \partial_i,$$

$$\overset{\circ}{\boldsymbol{\nabla}} \boldsymbol{r} = \boldsymbol{e}_i \frac{\partial \boldsymbol{r}}{\partial \overset{\circ}{x}_i} = \boldsymbol{e}_i \frac{\partial (x_j \boldsymbol{e}_j)}{\partial \overset{\circ}{x}_i} = \frac{\partial x_j}{\partial \overset{\circ}{x}_i} \boldsymbol{e}_i \boldsymbol{e}_j = \overset{\circ}{\partial}_i x_j \boldsymbol{e}_i \boldsymbol{e}_j,$$

$$\boldsymbol{\nabla} \overset{\circ}{\boldsymbol{r}} = \boldsymbol{e}_i \frac{\partial \overset{\circ}{\boldsymbol{r}}}{\partial x_i} = \frac{\partial \overset{\circ}{x}_j}{\partial x_i} \boldsymbol{e}_i \boldsymbol{e}_j = \partial_i \overset{\circ}{x}_j \boldsymbol{e}_i \boldsymbol{e}_j$$

...

By the polar decomposition theorem (§ 1.13), the motion gradient decomposes into the rotation tensor $\boldsymbol{O}$ and the symmetric positive stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$:

$$\boldsymbol{F} = \boldsymbol{O} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{O}$$

...

When there's no rotation ($\boldsymbol{O} = \boldsymbol{E}$), then $\boldsymbol{F} = \boldsymbol{U} = \boldsymbol{V}$.

...

§ 3. Measures (tensors) of deformation

And this is where the extra complexity arose. Although, the multivariance is often seen as a big gift.

Motion gradient $\boldsymbol{F}$ characterizes both the deformation of a body and the rotation of a body as a whole. The deformation-only tensors are the stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ from the polar decomposition $\boldsymbol{F} = \boldsymbol{O} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{O}$, as well as another tensors, originating from $\boldsymbol{U}$ or (and) $\boldsymbol{V}$.

The widely used ones are the "squares" of $\boldsymbol{U}$ and $\boldsymbol{V}$

$$\begin{aligned} (\boldsymbol{U}^2 =) \; \boldsymbol{U} \cdot \boldsymbol{U} = \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \equiv \boldsymbol{G}, \\ (\boldsymbol{V}^2 =) \; \boldsymbol{V} \cdot \boldsymbol{V} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathsf{T}} \equiv \boldsymbol{\Phi}. \end{aligned} \tag{3.1}$$

These are the Green's deformation tensor (or the right Cauchy–Green tensor) $\boldsymbol{G}$ and the Finger's deformation tensor (or the left Cauchy–Green tensor) $\boldsymbol{\Phi}$. They have the convenient link with the motion gradient $\boldsymbol{F}$, without calculating square roots (as it's needed for $\boldsymbol{U}$ and $\boldsymbol{V}$). That's the big reason why tensors $\boldsymbol{G}$ and $\boldsymbol{\Phi}$ are so widely used.

Tensor $\boldsymbol{G}$ was first used by George Green[*].

An inversion of $\boldsymbol{\Phi}$ and $\boldsymbol{G}$ gives the two more deformation tensors

$$\boldsymbol{V}^{-2} = \boldsymbol{\Phi}^{-1} = \left(\boldsymbol{F} \cdot \boldsymbol{F}^\mathsf{T}\right)^{-1} = \boldsymbol{F}^{-\mathsf{T}} \cdot \boldsymbol{F}^{-1} \equiv {}^2\boldsymbol{c},$$
$$\boldsymbol{U}^{-2} = \boldsymbol{G}^{-1} = \left(\boldsymbol{F}^\mathsf{T} \cdot \boldsymbol{F}\right)^{-1} = \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-\mathsf{T}} \equiv {}^2\boldsymbol{f},$$

(3.2)

each of which is sometimes called the Piola tensor or the Finger tensor. The inverse of the left Cauchy–Green tensor $\boldsymbol{\Phi}$ is the Cauchy deformation tensor ${}^2\boldsymbol{c}$.

The components of these tensors are

$$\boldsymbol{G} = \mathring{r}^i r_{\partial i} \cdot r_{\partial j} \mathring{r}^j = G_{ij} \mathring{r}^i \mathring{r}^j, \; G_{ij} \equiv r_{\partial i} \cdot r_{\partial j},$$
$${}^2\boldsymbol{f} = \mathring{r}_{\partial i} r^i \cdot r^j \mathring{r}_{\partial j} = G^{ij} \mathring{r}_{\partial i} \mathring{r}_{\partial j}, \; G^{ij} \equiv r^i \cdot r^j,$$
$${}^2\boldsymbol{c} = r^i \mathring{r}_{\partial i} \cdot \mathring{r}_{\partial j} r^j = g_{ij} r^i r^j, \; g_{ij} \equiv \mathring{r}_{\partial i} \cdot \mathring{r}_{\partial j},$$
$$\boldsymbol{\Phi} = r_{\partial i} \mathring{r}^i \cdot \mathring{r}^j r_{\partial j} = g^{ij} r_{\partial i} r_{\partial j}, \; g^{ij} \equiv \mathring{r}^i \cdot \mathring{r}^j$$

coincide with the components of the unit (metric) tensor

$$\boldsymbol{E} = r_{\partial i} r^i = G_{ij} r^i r^j = r^i r_{\partial i} = G^{ij} r_{\partial i} r_{\partial j}$$
$$= \mathring{r}^i \mathring{r}_{\partial i} = g^{ij} \mathring{r}_{\partial i} \mathring{r}_{\partial j} = \mathring{r}_{\partial i} \mathring{r}^i = g_{ij} \mathring{r}^i \mathring{r}^j,$$

but the components' bases are different. Using the index notation, it's easy to get confused in the differences between the unit tensor $\boldsymbol{E}$ and the strain tensors $\boldsymbol{G}$, $\boldsymbol{\Phi}$, ${}^2\boldsymbol{f}$, ${}^2\boldsymbol{c}$. The direct indexless notation has the obvious advantage here.

As was mentioned in §1.13, the invariants of the stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ are the same. If w_i are the three eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{V}$, that is the roots of the characteristic equations for these tensors, then here are their invariants:

$$\mathrm{I}(\boldsymbol{U}) = \mathrm{I}(\boldsymbol{V}) = \operatorname{trace}\boldsymbol{U} = \operatorname{trace}\boldsymbol{V} = \sum U_{jj} = \sum V_{jj} = \sum w_i,$$
$$\mathrm{II}(\boldsymbol{U}) = \mathrm{II}(\boldsymbol{V}) = w_1 w_2 + w_1 w_3 + w_2 w_3,$$
$$\mathrm{III}(\boldsymbol{U}) = \mathrm{III}(\boldsymbol{V}) = w_1 w_2 w_3.$$

The invariants of $\boldsymbol{G}$ and $\boldsymbol{\Phi}$ coincide too:

$$\mathrm{I}(\boldsymbol{G}) = \mathrm{I}(\boldsymbol{\Phi}), \dots$$

[*] **Green, George**. (1839) On the propagation of light in crystallized media. *Transactions of the Cambridge Philosophical Society.* 1842, vol. 7, part II, pages 121–140.

Without a deformation

$$\boldsymbol{F} = \boldsymbol{U} = \boldsymbol{V} = \boldsymbol{G} = \boldsymbol{\Phi} = {}^2\!\boldsymbol{f} = {}^2\!\boldsymbol{c} = \boldsymbol{E},$$

thus as characteristics of deformation it's worth taking the differences like $\boldsymbol{U} - \boldsymbol{E}$, $\boldsymbol{U} \cdot \boldsymbol{U} - \boldsymbol{E}$, ...

...

The right Cauchy–Green deformation tensor

George Green discovered a deformation tensor known as the right Cauchy–Green deformation tensor or Green's deformation tensor

$$\boldsymbol{G} = \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} = \boldsymbol{U}^2 \quad \text{or} \quad G_{ij} = F_{k'i} F_{k'j} = \frac{\partial x_{k'}}{\partial \mathring{x}_i} \frac{\partial x_{k'}}{\partial \mathring{x}_j}.$$

This tensor gives the "square" of local change in distances due to deformation: $d\boldsymbol{r} \cdot d\boldsymbol{r} = d\mathring{\boldsymbol{r}} \cdot \boldsymbol{G} \cdot d\mathring{\boldsymbol{r}}$

The most popular invariants of $\boldsymbol{G}$ are

$$\mathrm{I}(\boldsymbol{G}) \equiv \operatorname{trace} \boldsymbol{G} = G_{ii} = \gamma_1^2 + \gamma_2^2 + \gamma_3^2$$
$$\mathrm{II}(\boldsymbol{G}) \equiv \tfrac{1}{2}\big(G_{jj}^2 - G_{ik}G_{ki}\big) = \gamma_1^2\gamma_2^2 + \gamma_2^2\gamma_3^2 + \gamma_3^2\gamma_1^2$$
$$\mathrm{III}(\boldsymbol{G}) \equiv \det \boldsymbol{G} = \gamma_1^2\gamma_2^2\gamma_3^2$$

where γ_i are stretch ratios for unit fibers that are initially oriented along directions of eigenvectors of the right stretch tensor $\boldsymbol{U}$.

The inverse of Green's deformation tensor

Sometimes called the Finger tensor or the Piola tensor, the inverse of the right Cauchy–Green deformation tensor

$$^2\!\boldsymbol{f} = \boldsymbol{G}^{-1} = \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-\mathsf{T}} \quad \text{or} \quad f_{ij} = \frac{\partial \mathring{x}_i}{\partial x_{k'}} \frac{\partial \mathring{x}_j}{\partial x_{k'}}$$

The left Cauchy–Green or Finger deformation tensor

Swapping multipliers in the formula for the right Green–Cauchy deformation tensor leads to the left Cauchy–Green deformation tensor, defined as

$$\boldsymbol{\Phi} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathsf{T}} = \boldsymbol{V}^2 \quad \text{or} \quad \Phi_{ij} = \frac{\partial x_i}{\partial \mathring{x}_k} \frac{\partial x_j}{\partial \mathring{x}_k}$$

The left Cauchy–Green deformation tensor is often called the Finger's deformation tensor, named after Josef Finger (1894).

Invariants of $\boldsymbol{\Phi}$ are also used in expressions for strain energy density functions. The conventional invariants are defined as

$$I_1 \equiv \Phi_{ii} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 \equiv \tfrac{1}{2}\left(\Phi_{ii}^2 - \Phi_{jk}\Phi_{kj}\right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$
$$I_3 \equiv \det \boldsymbol{\Phi} = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2$$

($J \equiv \det \boldsymbol{F}$ is the Jacobian, determinant of the motion gradient)

The Cauchy deformation tensor

The Cauchy deformation tensor is defined as the inverse of the left Cauchy–Green deformation tensor

$$^2\boldsymbol{c} = \boldsymbol{\Phi}^{-1} = \boldsymbol{F}^{-\mathsf{T}}\boldsymbol{\cdot}\boldsymbol{F}^{-1} \quad\text{or}\quad c_{ij} = \frac{\partial \mathring{x}_k}{\partial x_i}\frac{\partial \mathring{x}_k}{\partial x_j}$$

$$d\mathring{\boldsymbol{r}}\boldsymbol{\cdot}d\mathring{\boldsymbol{r}} = d\boldsymbol{r}\boldsymbol{\cdot}{}^2\boldsymbol{c}\boldsymbol{\cdot}d\boldsymbol{r}$$

This tensor is also called the Piola tensor or the Finger tensor in rheology and fluid dynamics literature.

Finite strain tensors

The concept of *strain* is used to evaluate how much a given displacement differs locally from a body displacement as a whole (a "rigid body displacement"). One of such strains for large deformations is the *Green strain tensor* (*Green–Lagrangian strain tensor*, *Green–Saint-Venant strain tensor*), defined as

$$\boldsymbol{C} = \tfrac{1}{2}\left(\boldsymbol{G} - \boldsymbol{E}\right) \quad\text{or}\quad C_{ij} = \frac{1}{2}\left(\frac{\partial x_{k'}}{\partial \mathring{x}_i}\frac{\partial x_{k'}}{\partial \mathring{x}_j} - \delta_{ij}\right)$$

or as the function of the displacement gradient tensor

$$\boldsymbol{C} = \tfrac{1}{2}\left(\mathring{\boldsymbol{\nabla}}\boldsymbol{u} + \mathring{\boldsymbol{\nabla}}\boldsymbol{u}^\mathsf{T} + \mathring{\boldsymbol{\nabla}}\boldsymbol{u}\boldsymbol{\cdot}\mathring{\boldsymbol{\nabla}}\boldsymbol{u}^\mathsf{T}\right)$$

in cartesian coordinates

$$C_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial \mathring{x}_i} + \frac{\partial u_i}{\partial \mathring{x}_j} + \frac{\partial u_k}{\partial \mathring{x}_i}\frac{\partial u_k}{\partial \mathring{x}_j}\right)$$

The Green strain tensor measures how much $\boldsymbol{G}$ differs from $\boldsymbol{E}$.

The *Almansi–Hamel strain tensor*, referenced to the deformed configuration ("Eulerian description"), is defined as

$$^2\boldsymbol{a} = \tfrac{1}{2}\left(\boldsymbol{E} - {}^2\boldsymbol{c}\right) = \tfrac{1}{2}\left(\boldsymbol{E} - \boldsymbol{\Phi}^{-1}\right) \quad \text{or} \quad a_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial \mathring{x}_k}{\partial x_i}\frac{\partial \mathring{x}_k}{\partial x_j}\right)$$

or as function of the displacement gradient

$$^2\boldsymbol{a} = \tfrac{1}{2}\left(\boldsymbol{\nabla u}^{\mathsf{T}} + \boldsymbol{\nabla u} - \boldsymbol{\nabla u} \boldsymbol{\cdot} \boldsymbol{\nabla u}^{\mathsf{T}}\right)$$

$$a_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

Seth–Hill family of abstract strain tensors

B. R. Seth was the first to show that the Green and Almansi strain tensors are special cases of a more abstract measure of deformation. The idea was further expanded upon by Rodney Hill in 1968 (publication??). The Seth–Hill family of strain measures (also called Doyle–Ericksen tensors) is expressed as

$$\boldsymbol{D}_{(m)} = \frac{1}{2m}\left(\boldsymbol{U}^{2m} - \boldsymbol{E}\right) = \frac{1}{2m}\left(\boldsymbol{G}^m - \boldsymbol{E}\right)$$

For various m it gives

$$\boldsymbol{D}_{(1)} = \tfrac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{E}\right) = \tfrac{1}{2}(\boldsymbol{G} - \boldsymbol{E}) \quad \text{Green strain tensor}$$

$$\boldsymbol{D}_{(1/2)} = \boldsymbol{U} - \boldsymbol{E} = \boldsymbol{G}^{1/2} - \boldsymbol{E} \qquad \text{Biot strain tensor}$$

$$\boldsymbol{D}_{(0)} = \ln \boldsymbol{U} = \tfrac{1}{2}\ln \boldsymbol{G} \qquad \text{logarithmic strain, Hencky strain}$$

$$\boldsymbol{D}_{(-1)} = \tfrac{1}{2}\left(\boldsymbol{E} - \boldsymbol{U}^{-2}\right) \qquad \text{Almansi strain}$$

The second-order approximation of these tensors is

$$\boldsymbol{D}_{(m)} = \boldsymbol{\varepsilon} + \tfrac{1}{2}\boldsymbol{\nabla u} \boldsymbol{\cdot} \boldsymbol{\nabla u}^{\mathsf{T}} - (1 - m)\boldsymbol{\varepsilon} \boldsymbol{\cdot} \boldsymbol{\varepsilon}$$

where $\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla u}^{\mathsf{S}}$ is the infinitesimal deformation tensor.

Many other different definitions of measures $\boldsymbol{D}$ are possible, provided that they satisfy these conditions:
- ✓ $\boldsymbol{D}$ vanishes for any movement of a body as a rigid whole
- ✓ dependence of $\boldsymbol{D}$ on displacement gradient tensor $\boldsymbol{\nabla u}$ is continuous, continuously differentiable and monotonic
- ✓ it's desired that $\boldsymbol{D}$ reduces to the infinitesimal linear deformation tensor $\boldsymbol{\varepsilon}$ when $\boldsymbol{\nabla u} \to 0$

For example, tensors from the set

$$D^{(n)} = (U^n - U^{-n})/2n$$

aren't from the Seth–Hill family, but for any n they have the same 2nd-order approximation as Seth–Hill measures with $m = 0$.

Wikipedia, the free encyclopedia — Finite strain theory

...

Logarithmic strain, Hencky's strain

Heinrich Hencky. Über die Form des Elastizitätsgesetzes bei ideal elastischen Stoffen. Zeitschrift für technische Physik, Vol. 9 (1928), Seiten 215–220.

....

§4. Velocity field

This topic is discussed in nearly any book about continuum mechanics, however for solid elastic continua it's not very vital. Among various models of a material continuum, an elastic solid body is distinguished by interesting possibility of deriving the complete set (system) of equations for it via the single logically flawless procedure. But now we follow the way, usual for fluid continuum mechanics.

So, there's velocity field in spatial description $v \equiv \dot{r} = v(r,t)$. Decomposition of tensor $\nabla v = \nabla \dot{r} = r^i \partial_i \dot{r} = r^i \dot{r}_{\partial i}{}^*$ into symmetric and skew-symmetric parts (§ 1.7)

$$\nabla \dot{r} = \nabla \dot{r}^{\,\mathsf{S}} - \tfrac{1}{2}(\nabla \times \dot{r}) \times E$$

$*$ For sufficiently smooth functions, partial derivatives always commute, space and time ones too. Thus

$$\frac{\partial}{\partial q^i}\frac{\partial r}{\partial t} = \frac{\partial}{\partial t}\frac{\partial r}{\partial q^i} \quad \text{or} \quad \partial_i \dot{r} = \dot{r}_{\partial i}$$

or, introducing the rate of deformation tensor (rate of stretching tensor, strain rate tensor) $\boldsymbol{D}$ and the vorticity tensor (rate of rotation tensor, spin tensor) $\boldsymbol{W}$

$$\boldsymbol{\nabla v} = \boldsymbol{D} - \boldsymbol{W},$$
$$\boldsymbol{D} \equiv \boldsymbol{\nabla v}^{\mathsf{S}} = \boldsymbol{\nabla \dot{r}}^{\,\mathsf{S}} = \tfrac{1}{2}\left(\boldsymbol{r}^i \dot{\boldsymbol{r}}_{\partial i} + \dot{\boldsymbol{r}}_{\partial i} \boldsymbol{r}^i\right), \qquad (4.1)$$
$$-\boldsymbol{W} \equiv \boldsymbol{\nabla v}^{\mathsf{A}} = -\boldsymbol{w} \times \boldsymbol{E}, \quad \boldsymbol{w} \equiv \tfrac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{v} = \tfrac{1}{2}\boldsymbol{r}^i \times \dot{\boldsymbol{r}}_{\partial i},$$

where also figures vorticity (pseudo)vector $\boldsymbol{w}$, the companion of vorticity tensor $\boldsymbol{W}$.

Components of the rate of deformation tensor in the current configuration's basis

$$\boldsymbol{D} = \mathcal{D}_{ij}\boldsymbol{r}^i\boldsymbol{r}^j, \quad \mathcal{D}_{ij} = \boldsymbol{r}_{\partial i} \cdot \boldsymbol{D} \cdot \boldsymbol{r}_{\partial j} = \tfrac{1}{2}\boldsymbol{r}_{\partial i} \cdot \left(\boldsymbol{r}^k \dot{\boldsymbol{r}}_k + \dot{\boldsymbol{r}}_k \boldsymbol{r}^k\right) \cdot \boldsymbol{r}_{\partial j} =$$
$$= \tfrac{1}{2}\left(\dot{\boldsymbol{r}}_{\partial i} \cdot \boldsymbol{r}_{\partial j} + \boldsymbol{r}_{\partial i} \cdot \dot{\boldsymbol{r}}_{\partial j}\right) = \tfrac{1}{2}\left(\boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j}\right)^{\bullet}$$

...

$$\dot{G}_{ij}$$
$$G_{ij} \equiv \boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j}$$

...

For elastic solid media, there's no need for discussion about rotations: the true representation appears along the way of logically harmonious conclusions without additional hypotheses.

§5. Area vector. Surface change

Take an infinitesimal surface. The area vector by length is equal to the surface's area and is directed along the normal to this surface.

In the initial (original, undeformed, "material", ~~reference~~) configuration, the area vector can be represented as $\mathring{\boldsymbol{n}}do$. Surface's area do is infinitely small, and $\mathring{\boldsymbol{n}}$ is unit normal vector.

In the present (current, actual, deformed, "spatial") configuration, the same surface has area vector $\boldsymbol{n}d\mathcal{O}$.

With differential precision, these infinitesimal surfaces are parallel-ograms, thus

$$\hat{\boldsymbol{n}}do = d\overset{\circ}{\boldsymbol{r}}' \times d\overset{\circ}{\boldsymbol{r}}'' = \frac{\partial \overset{\circ}{\boldsymbol{r}}}{\partial q^i}\, dq^i \times \frac{\partial \overset{\circ}{\boldsymbol{r}}}{\partial q^j}\, dq^j = \overset{\circ}{\boldsymbol{r}}_{\partial i} \times \overset{\circ}{\boldsymbol{r}}_{\partial j}\, dq^i dq^j,$$

$$\boldsymbol{n}d\mathcal{O} = d\boldsymbol{r}' \times d\boldsymbol{r}'' = \frac{\partial \boldsymbol{r}}{\partial q^i}\, dq^i \times \frac{\partial \boldsymbol{r}}{\partial q^j}\, dq^j = \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j}\, dq^i dq^j. \tag{5.1}$$

Applying the transformation of volume (**??**), we have

$$d\mathcal{V} = J d\overset{\circ}{\mathcal{V}} \;\Rightarrow\; \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j} \cdot \boldsymbol{r}_{\partial k} = J \overset{\circ}{\boldsymbol{r}}_{\partial i} \times \overset{\circ}{\boldsymbol{r}}_{\partial j} \cdot \overset{\circ}{\boldsymbol{r}}_{\partial k} \;\Rightarrow$$

$$\Rightarrow\; \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j} \cdot \boldsymbol{r}_{\partial k} \boldsymbol{r}^k = J \overset{\circ}{\boldsymbol{r}}_{\partial i} \times \overset{\circ}{\boldsymbol{r}}_{\partial j} \cdot \overset{\circ}{\boldsymbol{r}}_{\partial k} \boldsymbol{r}^k \;\Rightarrow$$

$$\Rightarrow\; \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j} = J \overset{\circ}{\boldsymbol{r}}_{\partial i} \times \overset{\circ}{\boldsymbol{r}}_{\partial j} \cdot \boldsymbol{F}^{-1}.$$

Hence with (5.1) we come to the relation

$$\boldsymbol{n}d\mathcal{O} = J\hat{\boldsymbol{n}}do \cdot \boldsymbol{F}^{-1}, \tag{5.2}$$

called the Nanson's formula.

§6. Forces in continuum. Existence of the Cauchy stress tensor

Augustin-Louis Cauchy founded the *continuum mechanics* by assuming that two parts of a continuum interact by means of surface densities of the contact forces concentrated on the dividing surface.

Assuming that these contact forces depend only on the perpendicular (normal) to the dividing surface and that the contact forces are balanced by some volume density of force (including inertia), Cauchy played with tetrahedrons and proved the existence of the stress tensor.

De la pression ou tension dans un corps solide. *Exercices de mathématiques*, par M. **Augustin-Louis Cauchy**. Seconde année: 1827. Paris, Chez de Bure frères. Pages 42 à 59.

The particles of a momentless model of a continuum are points with only the translational degrees of freedom[*]. Thus there're no moments among generalized forces as well as no external couples.

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[*] The translational degrees of freedom appear from the particle's ability to move free in the space.

Force $\rho \boldsymbol{f} d\mathcal{V}$ acts on infinitesimal volume $d\mathcal{V}$. If $\boldsymbol{f}$ is a mass force (acting per unit of mass), then $\rho \boldsymbol{f}$ is a volume one. Such forces originate from force fields, for example: the gravitational forces ("weight"), the forces of inertia in a non-inertial reference system, the electromagnetic forces in a medium with charges and currents.

Surface force $\boldsymbol{p} d\mathcal{O}$ acts on infinitesimal surface $d\mathcal{O}$. It may be a contact pressure or/and a friction, an electrostatic force with charges concentrated on the surface.

In a material continuum, like in any mechanical system, the external and the internal forces are distinguished. The internal forces balance the action of the external forces, and they are transmitted continuously from point to point. Since the times of Euler and Cauchy, the internal forces are assumed to be the surface short-range contact forces: on an infinitesimal surface $\boldsymbol{n} d\mathcal{O}$ acts a force $\boldsymbol{t}_{(\boldsymbol{n})} d\mathcal{O}$. It acts from?? that side of the two where the unit normal $\boldsymbol{n}$ is directed.
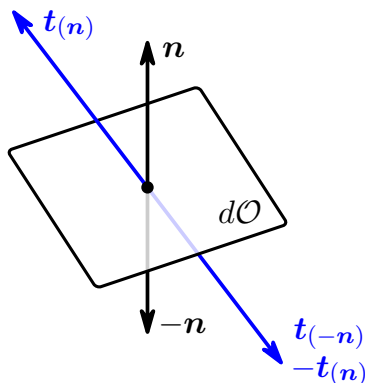

figure 12

By the action–reaction principle, the traction vector $\boldsymbol{t}_{(\boldsymbol{n})}$ changes its sign with the reversal of the direction of the normal $\boldsymbol{n}$: $\boldsymbol{t}_{(-\boldsymbol{n})} = -\boldsymbol{t}_{(\boldsymbol{n})}$. (Sometimes this thesis is proved via the balance of momentum for an infinitely short cylinder with bases $\boldsymbol{n} d\mathcal{O}$ and $-\boldsymbol{n} d\mathcal{O}$.)

Traction vector $\boldsymbol{t}_{(\boldsymbol{n})}$ on the surface with the unit normal $\boldsymbol{n}$ is called the surface traction vector or force-stress vector. However, $\boldsymbol{t}_{(\boldsymbol{n})}$ is not a vector field: traction $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{n}, \boldsymbol{r}, t)$ depends not only on location $\boldsymbol{r}$ of the point, but also on the local direction (defined by $\boldsymbol{n}$) of the surface element. An infinite number of surfaces of any direction contain the same point, and there are an infinitely many traction vectors $\boldsymbol{t}_{(\boldsymbol{n})}$ at each point.

The stress at a point of a continuum is *not a vector field.* It is a more complex field, an infinite collection of all traction vectors for any directions of the infinitesimal surfaces, containing this point in self.

Actually, an infinite set (a collection) of all traction vectors $\boldsymbol{t}_{(\boldsymbol{n})}$ at a point is completely defined by the only one single second complexity tensor — the Cauchy stress tensor $\boldsymbol{\tau}$.

The derivation of this thesis is described in many books. It is known as the theorem about existence of the Cauchy stress tensor — the one with the impressive tetrahedron argument.

The Cauchy tetrahedron argument and the proof of the existence of the Cauchy stress tensor.

На поверхности бесконечномалого материального тетраэдра ...

...

§7. Balance of momentum and angular momentum

Consider some random finite volume $\mathcal{V}$ of an elastic medium, contained within surface $\mathcal{O}(\partial\mathcal{V})$. It is loaded with external forces, surface contact ones $\boldsymbol{p}d\mathcal{O}$ and body (mass or volume) ones $\boldsymbol{f}dm = \rho\boldsymbol{f}d\mathcal{V}$.

The integral formulation of the balance of momentum is as follows

$$\left(\int_{\mathcal{V}} \rho\boldsymbol{v}d\mathcal{V}\right)^{\bullet} = \int_{\mathcal{V}} \rho\boldsymbol{f}d\mathcal{V} + \oint_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{p}d\mathcal{O}. \tag{7.1}$$

... $\boldsymbol{p} = \boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{\tau}$ ...

Импульс слева найдём по (1.5), а поверхностный интеграл превратим в объёмный по теореме о дивергенции. Получим

$$\int_{\mathcal{V}} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho\left(\boldsymbol{f} - \dot{\boldsymbol{v}}\right)\right)d\mathcal{V} = \boldsymbol{0}.$$

Но объём $\mathcal{V}$ случаен, поэтому равно нулю подынтегральное выражение. Приходим к уравнению баланса импульса в локальной (дифференциальной) форме

$$\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho\left(\boldsymbol{f} - \dot{\boldsymbol{v}}\right) = \boldsymbol{0}. \tag{7.2}$$

...

Now about the balance of the angular (rotational) momentum. Here is the integral formulation:

$$\left(\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \boldsymbol{v} d\mathcal{V}\right)^{\bullet} = \int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \boldsymbol{f} d\mathcal{V} + \oint\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{r} \times \boldsymbol{p} d\mathcal{O}. \qquad (7.3)$$

Дифференцируя левую часть ($\boldsymbol{v} \equiv \dot{\boldsymbol{r}}$)

$$\left(\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \dot{\boldsymbol{r}} \, d\mathcal{V}\right)^{\bullet} = \int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \ddot{\boldsymbol{r}} \, d\mathcal{V} + \int\limits_{\mathcal{V}} \underbrace{\dot{\boldsymbol{r}} \times \rho \dot{\boldsymbol{r}}}_{\boldsymbol{0}} \, d\mathcal{V},$$

применяя теорему о дивергенции к поверхностному интегралу ....

$$(\dots \; \boldsymbol{p} = \boldsymbol{t}_{(n)} = \boldsymbol{n} \cdot \boldsymbol{\tau} \dots)$$

$$\boldsymbol{r} \times (\boldsymbol{n} \cdot \boldsymbol{\tau}) = -(\boldsymbol{n} \cdot \boldsymbol{\tau}) \times \boldsymbol{r} = -\boldsymbol{n} \cdot (\boldsymbol{\tau} \times \boldsymbol{r}) \; \Rightarrow$$

$$\Rightarrow \oint\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{r} \times (\boldsymbol{n} \cdot \boldsymbol{\tau}) \, d\mathcal{O} = -\int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \times \boldsymbol{r}) \, d\mathcal{V},$$

...

$$\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \ddot{\boldsymbol{r}} \, d\mathcal{V} = \int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \boldsymbol{f} \, d\mathcal{V} - \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \times \boldsymbol{r}) \, d\mathcal{V},$$

$$\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \big(\boldsymbol{f} - \ddot{\boldsymbol{r}}\big) d\mathcal{V} - \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \times \boldsymbol{r}) \, d\mathcal{V} = \boldsymbol{0},$$

...

$$\underbrace{\boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \times \boldsymbol{r})}_{r^i \cdot \partial_i (\boldsymbol{\tau} \times \boldsymbol{r})} = \underbrace{(\boldsymbol{\nabla} \cdot \boldsymbol{\tau}) \times \boldsymbol{r}}_{r^i \cdot (\partial_i \boldsymbol{\tau}) \times \boldsymbol{r}} + r^i \cdot (\boldsymbol{\tau} \times \partial_i \boldsymbol{r})$$

$$\boldsymbol{\tau} = \boldsymbol{e}_i \boldsymbol{t}_{(i)}, \; \boldsymbol{e}_i = \text{constant}$$

$$r^i \cdot (\boldsymbol{\tau} \times \partial_i \boldsymbol{r}) = r^i \cdot (\boldsymbol{e}_j \boldsymbol{t}_{(j)} \times \boldsymbol{r}_i) = r^i \cdot \boldsymbol{e}_j \boldsymbol{t}_{(j)} \times \boldsymbol{r}_i =$$

$$= -\boldsymbol{e}_j \cdot r^i \boldsymbol{r}_i \times \boldsymbol{t}_{(j)} = -\boldsymbol{e}_j \cdot \boldsymbol{E} \times \boldsymbol{t}_{(j)} = -\boldsymbol{e}_j \times \boldsymbol{t}_{(j)} = -\boldsymbol{\tau}_{\times}$$

...

§8. Eigenvalues of the Cauchy stress tensor. Mohr's circles

Like any symmetric bivalent tensor, the Cauchy stress tensor $\boldsymbol{\tau}$ has три вещественных собственных числá σ_i, а также тройку

взаимно перпендикулярных собственных векторов единичной длины (§ 1.9). Собственные числа тензора $\boldsymbol{\tau}$ называются главными напряжениями (principal stresses).

In representation $\boldsymbol{\tau} = \sum \sigma_i \boldsymbol{e}_i \boldsymbol{e}_i$ most often indices are sorted descending $\sigma_1 \geq \sigma_2 \geq \sigma_3$, а тройка $\boldsymbol{e}_i$ ориентирована как "правая".

Известна теорема о кругах Мора (Mohr's circles)[*]

...

Чтобы замкнуть набор (систему) уравнений модели сплошной среды, нужно добавить определяющие отношения (constitutive relations) — уравнения, связывающие напряжение с деформацией (и другие необходимые связи). However, for a solid elastic continuum такой длинный путь построения модели излишен, что читатель и увидит ниже.

§9. Principle of virtual work (without Lagrange multipliers)

According to the principle of virtual work for some finite volume of continuum

$$\int_{\mathcal{V}} \Big(\rho \boldsymbol{f} \cdot \delta r + \delta W^{(i)} \Big) d\mathcal{V} + \oint_{\mathcal{O}(\partial \mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\tau} \cdot \delta r \, d\mathcal{O} = 0. \qquad (9.1)$$

Here $\delta W^{(i)}$ is work of internal forces per volume unit in the current configuration; $\boldsymbol{f}$ is mass force, with dynamics $\big(\boldsymbol{f} - \ddot{\boldsymbol{r}} \big)$; $\boldsymbol{p} = \boldsymbol{t}_{(n)} = \boldsymbol{n} \cdot \boldsymbol{\tau}$ is surface force.

Applying the divergence theorem to the surface integral, using[**]

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \cdot \delta r) = \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \cdot \delta r + \boldsymbol{\tau} \cdot \cdot \boldsymbol{\nabla} \delta r^{\mathsf{T}}$$

---

[*] Mohr's circles, named after Christian Otto Mohr, is a two-dimensional graphical representation of transformation for the Cauchy stress tensor.

[**] $\boldsymbol{r}^i \cdot \partial_i \big(\boldsymbol{\tau} \cdot \delta r \big) = \boldsymbol{r}^i \cdot \big(\partial_i \boldsymbol{\tau} \big) \cdot \delta r + \boldsymbol{r}^i \cdot \boldsymbol{\tau} \cdot \partial_i \big(\delta r \big),$

$\boldsymbol{r}^i \cdot \boldsymbol{\tau} \cdot \partial_i \big(\delta r \big) = \boldsymbol{\tau} \cdot \cdot \partial_i \big(\delta r \big) \boldsymbol{r}^i = \boldsymbol{\tau} \cdot \cdot \big(\boldsymbol{r}^i \partial_i \delta r \big)^{\mathsf{T}}$

and randomness of $\mathcal{V}$, we get the local differential version of (9.1)

$$\left(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}+\rho\boldsymbol{f}\right)\cdot\delta\boldsymbol{r}+\boldsymbol{\tau}\cdot\cdot\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}}+\delta W^{(i)}=0. \qquad (9.2)$$

When a body virtually moves as a rigid whole, the work of internal forces nullifies

$$\delta\boldsymbol{r}=\delta\boldsymbol{\rho}+\delta\mathbf{o}\times\boldsymbol{r} \ \Rightarrow\ \delta W^{(i)}=0,$$

$$\left(\boldsymbol{\nabla}\cdot\boldsymbol{\tau}+\rho\boldsymbol{f}\right)\cdot\left(\delta\boldsymbol{\rho}+\delta\mathbf{o}\times\boldsymbol{r}\right)+\boldsymbol{\tau}^{\mathsf{T}}\cdot\cdot\boldsymbol{\nabla}\left(\delta\boldsymbol{\rho}+\delta\mathbf{o}\times\boldsymbol{r}\right)=0,$$

$$\delta\boldsymbol{\rho}=\textsf{constant} \ \Rightarrow\ \boldsymbol{\nabla}\delta\boldsymbol{\rho}={}^{2}\mathbf{0}, \ \ \delta\mathbf{o}=\textsf{constant} \ \Rightarrow\ \boldsymbol{\nabla}\delta\mathbf{o}={}^{2}\mathbf{0},$$

$$\boldsymbol{\nabla}\left(\delta\boldsymbol{\rho}+\delta\mathbf{o}\times\boldsymbol{r}\right)=\boldsymbol{\nabla}\left(\delta\mathbf{o}\times\boldsymbol{r}\right)=\boldsymbol{\nabla}\delta\mathbf{o}\times\boldsymbol{r}-\boldsymbol{\nabla}\boldsymbol{r}\times\delta\mathbf{o}=$$
$$=-\boldsymbol{\nabla}\boldsymbol{r}\times\delta\mathbf{o}=-\boldsymbol{E}\times\delta\mathbf{o}$$

$$\cdots$$

Assuming $\delta\mathbf{o}=\mathbf{0}$ (just translation) $\Rightarrow \boldsymbol{\nabla}\delta\boldsymbol{r}=\boldsymbol{\nabla}\delta\boldsymbol{\rho}={}^{2}\mathbf{0}$, it turns into the balance of forces (of momentum)

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau}+\rho\boldsymbol{f}=\mathbf{0}.$$

If $\delta\boldsymbol{r}=\delta\mathbf{o}\times\boldsymbol{r}$ (just rotation) with $\delta\mathbf{o}=\textsf{constant}$, then

$$(16.10, \S\,1.16) \ \Rightarrow\ \boldsymbol{\nabla}\delta\boldsymbol{r}=\boldsymbol{\nabla}\delta\mathbf{o}\times\boldsymbol{r}-\boldsymbol{\nabla}\boldsymbol{r}\times\delta\mathbf{o}=-\boldsymbol{E}\times\delta\mathbf{o},$$
$$\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}}=\boldsymbol{E}\times\delta\mathbf{o}$$

With
$$(7.4, \S\,1.7) \ \Rightarrow\ \boldsymbol{\tau}_{\times}=-\boldsymbol{\tau}\cdot\cdot{}^{3}\boldsymbol{\epsilon},$$
$$\boldsymbol{\tau}\cdot\cdot\left(\boldsymbol{E}\times\delta\mathbf{o}\right)=\boldsymbol{\tau}\cdot\cdot\left(-{}^{3}\boldsymbol{\epsilon}\cdot\delta\mathbf{o}\right)=\left(-\boldsymbol{\tau}\cdot\cdot{}^{3}\boldsymbol{\epsilon}\right)\cdot\delta\mathbf{o}=\boldsymbol{\tau}_{\times}\cdot\delta\mathbf{o}$$
$$\cdots$$

In an elastic continuum, the internal forces are potential
$$\delta W^{(i)}=-\rho\,\delta\widetilde{\Pi}$$

...

"The elastic potential energy density per volume unit", becomes when shorting "The elastic potential ...."

Плотность упругой потенциальной энергии, запасённой|накопленной в единице объёма тела (среды).

Дословный перевод с english на русский фразы "the elastic potential" даёт "упругий потенциал".

....

$$\boldsymbol{\tau}\cdot\cdot\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}}=-\delta W^{(i)}=\rho\,\delta\widetilde{\Pi} \qquad (9.3)$$

...

Вид потенциала $\widetilde{\Pi}$ per mass unit пока неизвестен, но очевидно что $\widetilde{\Pi}$ определяется деформацией.

With the balance of mass $\rho J = \overset{\circ}{\rho} \Leftrightarrow \rho = J^{-1}\overset{\circ}{\rho}$ ($J \equiv \det \boldsymbol{F}$ is the Jacobian, determinant of the motion gradient), потенциал на единицу объёма в недеформированной конфигурации $\overset{\circ}{\Pi}$ имеет вид

$$\overset{\circ}{\Pi} \equiv \overset{\circ}{\rho}\widetilde{\Pi} \Rightarrow \delta\overset{\circ}{\Pi} = \overset{\circ}{\rho}\delta\widetilde{\Pi},$$
$$\rho\delta\widetilde{\Pi} = J^{-1}\delta\overset{\circ}{\Pi}. \tag{9.4}$$

Полным аналогом (...) является равенство

...

§ 10. Constitutive relations of elasticity

Fundamental relation of elasticity (**??**)

...

$$\Pi(\boldsymbol{C}) = \int\limits_{0}^{\boldsymbol{C}} \boldsymbol{\tau} \cdot\cdot \, d\boldsymbol{C}$$

If the strain energy density is path independent, then it acts as a potential for stress, that is

$$\boldsymbol{\tau} = \frac{\partial\Pi(\boldsymbol{C})}{\partial\boldsymbol{C}}$$

For adiabatic processes, Π is equal to the change in internal energy per unit of volume.

For isothermal processes, Π is equal to the Helmholtz free energy per unit of volume.

The natural configuration of a body is defined as the configuration in which the body is in stable thermal equilibrium with no external loads and zero stress and strain.

When we apply energy methods in elasticity, we implicitly assume that a body returns to its natural configuration after loads are removed. This implies that the Gibbs' condition is satisfied:

$$\Pi(\boldsymbol{C}) \geq 0 \quad \text{with} \quad \Pi(\boldsymbol{C}) = 0 \quad \text{iff} \quad \boldsymbol{C} = 0$$

...

Начальная конфигурация считается естественной (natural configuration) — недеформированной ненапряжённой: $C = {}^2\mathbf{0} \Leftrightarrow \boldsymbol{\tau} = {}^2\mathbf{0}$, поэтому в П нет линейных членов.

Тензор жёсткости ${}^4\boldsymbol{\mathcal{A}}$

...

A rubber-like material (an elastomer)

Для материала типа резины (эластомера) характерны больши́е деформации. Функция $\Pi(\mathrm{I}, \mathrm{II}, \mathrm{III})$ для такого материала бывает весьма сложной[*].

Преимущества использования $\boldsymbol{u}$ и $\boldsymbol{C}$ исчезают, если деформации больши́е (коне́чные) — проще остаться с вектором-радиусом $\boldsymbol{r}$ ...

...

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[*] **Harold Alexander**. A constitutive relation for rubber-like materials // International Journal of Engineering Science, volume 6 (September 1968), pages 549–563.

§11. Piola–Kirchhoff stress tensors and other measures of stress

Соотношение Nanson'a $\boldsymbol{n}d\mathcal{O} = J\overset{\circ}{\boldsymbol{n}}do \cdot \boldsymbol{F}^{-1}$ между векторами бесконечно малой площа́дки в начальной ($\overset{\circ}{\boldsymbol{n}}do$) и в текущей ($\boldsymbol{n}d\mathcal{O}$) конфигурациях*

$(5.2) \Rightarrow \boldsymbol{n}d\mathcal{O} \cdot \boldsymbol{\tau} = J\overset{\circ}{\boldsymbol{n}}do \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \Rightarrow \boldsymbol{n} \cdot \boldsymbol{\tau}d\mathcal{O} = \overset{\circ}{\boldsymbol{n}} \cdot J\boldsymbol{F}^{-1} \cdot \boldsymbol{\tau}do$

gives the dual expression of a surface force

$$\boldsymbol{n} \cdot \boldsymbol{\tau}d\mathcal{O} = \overset{\circ}{\boldsymbol{n}} \cdot \boldsymbol{T}do, \ \ \boldsymbol{T} \equiv J\boldsymbol{F}^{-1} \cdot \boldsymbol{\tau}. \tag{11.1}$$

Тензор $\boldsymbol{T}$ называется первым (несимметричным) тензором напряжения Piola–Kirchhoff, иногда — "номинальным напряжением" ("nominal stress") или "инженерным напряжением" ("engineering stress"). Бывает и когда какое-либо из этих (на)именований даётся транспонированному тензору

$$\boldsymbol{T}^{\mathsf{T}} = J\boldsymbol{\tau}^{\mathsf{T}} \cdot \boldsymbol{F}^{-\mathsf{T}} = J\boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathsf{T}}.$$

Обращение (11.1)

$$J^{-1}\boldsymbol{F} \cdot \boldsymbol{T} = J^{-1}\boldsymbol{F} \cdot J\boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \Rightarrow \boldsymbol{\tau} = J^{-1}\boldsymbol{F} \cdot \boldsymbol{T}$$

...

$$\delta\Pi = \boldsymbol{T} \cdot\cdot \delta\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}} \Rightarrow \Pi = \Pi(\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{r}) \tag{11.2}$$

— этот немного неожиданный результат получился благодаря коммутативности δ и $\overset{\circ}{\boldsymbol{\nabla}}$: $\overset{\circ}{\boldsymbol{\nabla}}\delta\boldsymbol{r}^{\mathsf{T}} = \delta\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}$ ($\boldsymbol{\nabla}$ and δ don't commute).

Тензор $\boldsymbol{T}$ оказался энергетически сопряжённым с $\boldsymbol{F} \equiv \overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}$

$$\boldsymbol{T} = \frac{\partial\Pi}{\partial\overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}} = \frac{\partial\Pi}{\partial\boldsymbol{F}}. \tag{11.3}$$

Второй (симметричный) тензор напряжения Piola–Kirchhoff $\boldsymbol{S}$ энергетически сопряжён с $\boldsymbol{G} \equiv \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F}$ и $\boldsymbol{C} \equiv \frac{1}{2}(\boldsymbol{G} - \boldsymbol{E})$

$$\delta\Pi(\boldsymbol{C}) = \boldsymbol{S} \cdot\cdot \delta\boldsymbol{C} \Rightarrow \boldsymbol{S} = \frac{\partial\Pi}{\partial\boldsymbol{C}},$$
$$d\boldsymbol{G} = 2d\boldsymbol{C} \Rightarrow \delta\Pi(\boldsymbol{G}) = \frac{1}{2}\boldsymbol{S} \cdot\cdot \delta\boldsymbol{G}, \ \ \boldsymbol{S} = 2\frac{\partial\Pi}{\partial\boldsymbol{G}}. \tag{11.4}$$

---

* Like before, $\boldsymbol{F} = \frac{\partial\boldsymbol{r}}{\partial\overset{\circ}{\boldsymbol{r}}} = \boldsymbol{r}_{\partial i}\overset{\circ}{\boldsymbol{r}}^i = \overset{\circ}{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}}$ is motion gradient, $J \equiv \det\boldsymbol{F}$ is the Jacobian (the Jacobian determinant).

Связь между первым и вторым тензорами

$$\boldsymbol{S} = \boldsymbol{T} \cdot \boldsymbol{F}^{-\mathsf{T}} = \boldsymbol{F}^{-1} \cdot \boldsymbol{T}^{\mathsf{T}} \Leftrightarrow \boldsymbol{T} = \boldsymbol{S} \cdot \boldsymbol{F}^{\mathsf{T}}, \ \boldsymbol{T}^{\mathsf{T}} = \boldsymbol{F} \cdot \boldsymbol{S}$$

и между тензором $\boldsymbol{S}$ и тензором напряжения Cauchy $\boldsymbol{\tau}$

$$\boldsymbol{S} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathsf{T}} \Leftrightarrow J^{-1}\boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^{\mathsf{T}} = \boldsymbol{\tau}.$$

...

$$\boldsymbol{T} = \frac{\partial \Pi}{\partial \boldsymbol{C}} \cdot \boldsymbol{F}^{\mathsf{T}} = 2\frac{\partial \Pi}{\partial \boldsymbol{G}} \cdot \boldsymbol{F}^{\mathsf{T}}$$

$$\delta \boldsymbol{S} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{C}} \cdot\cdot \,\delta \boldsymbol{C} = \frac{\partial^2 \Pi}{\partial \boldsymbol{C} \partial \boldsymbol{C}} \cdot\cdot \,\delta \boldsymbol{C}$$

$$\delta \boldsymbol{T} = \delta \boldsymbol{S} \cdot \boldsymbol{F}^{\mathsf{T}} + \boldsymbol{S} \cdot \delta \boldsymbol{F}^{\mathsf{T}}$$

...

The quantity $\boldsymbol{\kappa} = J\boldsymbol{\tau}$ is called the *Kirchhoff stress tensor* and is used widely in numerical algorithms in metal plasticity (where there's no change in volume during plastic deformation). Another name for it is *weighted Cauchy stress tensor*.

...

Here's balance of forces (of momentum) with tensor $\boldsymbol{T}$ for any undeformed volume $\overset{\circ}{\mathcal{V}}$

$$\int\limits_{\mathcal{V}} \rho \boldsymbol{f} d\mathcal{V} + \int\limits_{\mathcal{O}(\partial \mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\tau} d\mathcal{O} = \int\limits_{\overset{\circ}{\mathcal{V}}} \overset{\circ}{\rho} \boldsymbol{f} d\overset{\circ}{\mathcal{V}} + \int\limits_{o(\partial\overset{\circ}{\mathcal{V}})} \overset{\circ}{\boldsymbol{n}} \cdot \boldsymbol{T} do = \int\limits_{\overset{\circ}{\mathcal{V}}} \left(\overset{\circ}{\rho} \boldsymbol{f} + \overset{\circ}{\boldsymbol{\nabla}} \cdot \boldsymbol{T} \right) d\overset{\circ}{\mathcal{V}} = \boldsymbol{0}$$

or in the local (differential) version

$$\overset{\circ}{\boldsymbol{\nabla}} \cdot \boldsymbol{T} + \overset{\circ}{\rho} \boldsymbol{f} = \boldsymbol{0}. \tag{11.5}$$

Advantages of this equation in comparison with (7.2) are: here figures the known mass density $\overset{\circ}{\rho}$ of undeformed volume $\overset{\circ}{\mathcal{V}}$, and the operator $\overset{\circ}{\boldsymbol{\nabla}} \equiv \overset{\circ}{\boldsymbol{r}}{}^i \partial_i$ is defined through known vectors $\overset{\circ}{\boldsymbol{r}}{}^i$. Appearance of $\boldsymbol{T}$ reflects the specific property of an elastic solid body — "to remember" its initial configuration. Tensor $\boldsymbol{T}$ is unlikely useful in fluid mechanics.

Principle of virtual work for an arbitrary volume $\overset{\circ}{\mathcal{V}}$ of the elastic $(\delta W^{(i)} = -\delta\Pi)$ continuum:

$$\int\limits_{\overset{\circ}{\mathcal{V}}} \left(\overset{\circ}{\rho}\boldsymbol{f}\cdot\delta r - \delta\Pi\right)d\overset{\circ}{\mathcal{V}} + \int\limits_{o(\partial\overset{\circ}{\mathcal{V}})} \overset{\circ}{\boldsymbol{n}}\cdot\boldsymbol{T}\cdot\delta r\, do = 0,$$

$$\overset{\circ}{\boldsymbol{\nabla}}\cdot(\boldsymbol{T}\cdot\delta r) = \overset{\circ}{\boldsymbol{\nabla}}\cdot\boldsymbol{T}\cdot\delta r + \boldsymbol{T}^{\mathsf{T}}\cdot\cdot\overset{\circ}{\boldsymbol{\nabla}}\delta r,\ \ \boldsymbol{T}^{\mathsf{T}}\cdot\cdot\overset{\circ}{\boldsymbol{\nabla}}\delta r = \boldsymbol{T}\cdot\cdot\overset{\circ}{\boldsymbol{\nabla}}\delta r^{\mathsf{T}}$$

$$\delta\Pi = \left(\overset{\circ}{\rho}\boldsymbol{f} + \overset{\circ}{\boldsymbol{\nabla}}\cdot\boldsymbol{T}\right)\cdot\delta r + \boldsymbol{T}\cdot\cdot\overset{\circ}{\boldsymbol{\nabla}}\delta r^{\mathsf{T}}$$

....

The first one is non-symmetric, it links forces in the deformed stressed configuration to the underfomed geometry and mass (volumes, areas, densities as they were initially), and it is energetically conjugate to the motion gradient (often mistakenly called the "deformation gradient", forgetting about rigid rotations). The first (or sometimes its transpose) is also known as "nominal stress" and "engineering stress".

The second one is symmetric, it links loads in the initial undeformed configuration to the initial mass and geometry, and it is conjugate to the right Cauchy–Green deformation tensor (and thus to the Cauchy–Green–Venant measure of deformation).

The first is simpler when you use just the motion gradient and is more universal, but the second is simpler when you prefer right Cauchy–Green deformation and its offsprings.

There's also popular Cauchy stress, which relates forces in the deformed configuration to the deformed geometry and mass.

"energetically conjugate" means that their product is kind of energy, here: elastic potential energy per unit of volume

......

In the case of finite deformations, the Piola–Kirchhoff tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ describe the stress relative to the initial configuration. In contrast with them, the Cauchy stress tensor $\boldsymbol{\tau}$ describes the stress relative to the current configuration. For infinitesimal deformations, the Cauchy and Piola–Kirchhoff stress tensors are identical.

1st Piola–Kirchhoff stress tensor

The 1st Piola–Kirchhoff stress tensor $\boldsymbol{T}$ relates forces in the current (present, "spatial") configuration with areas in the initial ("material") configuration

$$\boldsymbol{T} = J\,\boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathsf{T}}$$

where $\boldsymbol{F}$ is the motion gradient and $J \equiv \det \boldsymbol{F}$ is the Jacobi determinant, Jacobian.

Because it relates different coordinate systems, the 1st Piola–Kirchhoff stress is a two-point tensor. Commonly, it's not symmetric.

The 1st Piola–Kirchhoff stress is the 3D generalization of the 1D concept of engineering stress.

If the material rotates without a change in stress (rigid rotation), the components of the 1st Piola–Kirchhoff stress tensor will vary with material orientation.

The 1st Piola–Kirchhoff stress is energy conjugate to the motion gradient.

2nd Piola–Kirchhoff stress tensor

The 2nd Piola–Kirchhoff stress tensor $\boldsymbol{S}$ relates forces in the initial configuration to areas in the initial configuration. The force in the initial configuration is obtained via mapping that preserves the relative relationship between the force direction and the area normal in the initial configuration.

$$\boldsymbol{S} = J\,\boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathsf{T}}$$

This tensor is a one-point tensor and it is symmetric.

If the material rotates without a change in stress (rigid rotation), the 2nd Piola–Kirchhoff stress tensor remain constant, irrespective of material orientation.

The 2nd Piola–Kirchhoff stress tensor is energy conjugate to the Green–Lagrange finite strain tensor.

...

§12. Variation of present configuration

Usually the two configurations of a nonlinear elastic medium are considered: the initial one with location vectors $\mathring{\boldsymbol{r}}$ and the current (present) one with $\boldsymbol{r}$.

And the following equations describe a small change of the current configuration with infinitesimal changes to the location vector $\delta\boldsymbol{r}$, to the vector of mass forces $\delta\boldsymbol{f}$, to the first Piola–Kirchhoff stress tensor $\delta\boldsymbol{T}$ and to the deformation tensor $\delta\boldsymbol{C}$.

By varying (11.5), (......)[*] and (......), we get

$$\mathring{\rho}\delta\boldsymbol{f} + \mathring{\boldsymbol{\nabla}}\boldsymbol{\cdot}\delta\boldsymbol{T} = \boldsymbol{0}, \ \ \delta\boldsymbol{T} = \left(\frac{\partial^2\Pi}{\partial\boldsymbol{C}\partial\boldsymbol{C}}\boldsymbol{\cdot\cdot}\delta\boldsymbol{C}\right)\boldsymbol{\cdot}\boldsymbol{F}^{\mathsf{T}} + \frac{\partial\Pi}{\partial\boldsymbol{C}}\boldsymbol{\cdot}\delta\boldsymbol{F}^{\mathsf{T}},$$

$$\delta\boldsymbol{F}^{\mathsf{T}} = \delta\mathring{\boldsymbol{\nabla}}\boldsymbol{r} = \mathring{\boldsymbol{\nabla}}\delta\boldsymbol{r} = \boldsymbol{F}^{\mathsf{T}}\boldsymbol{\cdot}\boldsymbol{\nabla}\delta\boldsymbol{r}, \ \ \delta\boldsymbol{F} = \delta\mathring{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}} = \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}}\boldsymbol{\cdot}\boldsymbol{F}, \quad (12.1)$$

$$\delta\boldsymbol{C} = \boldsymbol{F}^{\mathsf{T}}\boldsymbol{\cdot}\delta\boldsymbol{\varepsilon}\boldsymbol{\cdot}\boldsymbol{F}, \ \ \delta\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}}.$$

...

$$(5.2) \ \Rightarrow \ \mathring{\boldsymbol{n}}do = J^{-1}\boldsymbol{n}d\mathcal{O}\boldsymbol{\cdot}\boldsymbol{F} \ \Rightarrow \ \mathring{\boldsymbol{n}}\boldsymbol{\cdot}\delta\boldsymbol{T}do = J^{-1}\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{F}\boldsymbol{\cdot}\delta\boldsymbol{T}d\mathcal{O} \ \Rightarrow$$
$$\Rightarrow \ \mathring{\boldsymbol{n}}\boldsymbol{\cdot}\delta\boldsymbol{T}do = \boldsymbol{n}\boldsymbol{\cdot}\delta\boldsymbol{\tau}d\mathcal{O}, \ \ \delta\boldsymbol{\tau} \equiv J^{-1}\boldsymbol{F}\boldsymbol{\cdot}\delta\boldsymbol{T}$$

— введённый так tensor $\delta\boldsymbol{\tau}$ связан с вариацией $\delta\boldsymbol{T}$ как $\boldsymbol{\tau}$ связан с $\boldsymbol{T}$ ($\boldsymbol{\tau} = J^{-1}\boldsymbol{F}\boldsymbol{\cdot}\boldsymbol{T}$). Из (12.1) и ...

...

... корректируя коэффициенты линейной функции $\delta\boldsymbol{\tau}(\boldsymbol{\nabla}\delta\boldsymbol{r})$.

§13. Internal constraints

До сих пор деформация считалась свободной, мера деформации $\boldsymbol{C}$ могла быть любой. Однако, существуют материалы со значительным сопротивлением некоторым видам деформации. Резина, например, изменению формы сопротивляется намного меньше, чем изменению объёма — некоторые виды резины можно считать несжимаемым материалом.

Понятие геометрической связи, развитое в общей механике ...

---

[*] $\boldsymbol{\nabla} = \boldsymbol{\nabla}\boldsymbol{\cdot}\mathring{\boldsymbol{\nabla}}\mathring{\boldsymbol{r}} = \boldsymbol{r}^i\partial_i\boldsymbol{\cdot}\mathring{\boldsymbol{r}}^j\partial_j\mathring{\boldsymbol{r}} \stackrel{?}{=} \boldsymbol{r}^i\partial_i\mathring{\boldsymbol{r}}\boldsymbol{\cdot}\mathring{\boldsymbol{r}}^j\partial_j = \boldsymbol{\nabla}\mathring{\boldsymbol{r}}\boldsymbol{\cdot}\mathring{\boldsymbol{\nabla}} = \boldsymbol{F}^{-\mathsf{T}}\boldsymbol{\cdot}\mathring{\boldsymbol{\nabla}}$

$\mathring{\boldsymbol{\nabla}} = \mathring{\boldsymbol{\nabla}}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{r} = \mathring{\boldsymbol{r}}^i\partial_i\boldsymbol{\cdot}\boldsymbol{r}^j\partial_j\boldsymbol{r} \stackrel{?}{=} \mathring{\boldsymbol{r}}^i\partial_i\boldsymbol{r}\boldsymbol{\cdot}\boldsymbol{r}^j\partial_j = \mathring{\boldsymbol{\nabla}}\boldsymbol{r}\boldsymbol{\cdot}\boldsymbol{\nabla} = \boldsymbol{F}^{\mathsf{T}}\boldsymbol{\cdot}\boldsymbol{\nabla}$

...

for incompressible materials $\Pi = \Pi(\mathrm{I}, \mathrm{II})$

Mooney–Rivlin model of incompressible material
$$\Pi = c_1(\mathrm{I} - 3) + c_2(\mathrm{II} - 3)$$

incompressible Treloar (neo-Hookean) material
$$c_2 = 0 \quad \Rightarrow \quad \Pi = c_1(\mathrm{I} - 3)$$

...

§14. Hollow sphere under pressure

Решение этой относительно простой задачи описано во многих книгах. В начальной (ненапряжённой) конфигурации имеем сферу с внутренним радиусом r_0 и наружным r_1. Давление равно p_0 внутри и p_1 снаружи.

Введём удобную для этой задачи сферическую систему координат в отсчётной конфигурации $q^1 = \theta$, $q^2 = \phi$, $q^3 = r$ (рис. **??**). Эти же координаты будут и материальными. Имеем

...

§15. Stresses as Lagrange multipliers

Описанному ранее в §9 использованию принципа виртуальной работы предшествовало введение тензора напряжения Cauchy через баланс сил для бесконечномалого тетраэдра (§6). Но теперь мы увидим, что сей принцип примени́м и без рассуждений с тетраэдром.

Рассмотрим тело — не только лишь упругое, с любой виртуальной работой внутренних сил $\delta W^{(i)}$ per mass unit — нагруженное массовыми $\boldsymbol{f}dm$ (для краткости пишем $\boldsymbol{f}$ вместо $\boldsymbol{f} - \ddot{\boldsymbol{r}}$, так что динамика присутствует) и поверхностными $\boldsymbol{p}\,d\mathcal{O}$ внешними силами. Имеем вариационное уравнение

$$\int\limits_{\mathcal{V}} \rho\left(\boldsymbol{f}\cdot\delta\boldsymbol{r} + \delta W^{(i)}\right)d\mathcal{V} + \int\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{p}\cdot\delta\boldsymbol{r}\,d\mathcal{O} = 0. \qquad (15.1)$$

It is assumed that the internal forces don't produce a work, when a continuum (a body) virtually moves $\delta\boldsymbol{r}$ as a whole without deformations ($\delta\boldsymbol{\varepsilon} = {}^2\boldsymbol{0}$). Then

$$\delta\boldsymbol{\varepsilon} = \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = {}^2\boldsymbol{0} \;\Rightarrow\; \delta W^{(i)} = 0. \qquad (15.2)$$

Отбросив $\delta W^{(i)}$ в (15.1) при условии (15.2), получим вариационное уравнение со связью. Приём с множителями Лагранжа даёт возможность считать вариации $\delta\boldsymbol{r}$ независимыми. Поскольку в каждой точке связь представлена симметричным тензором второй сложности, то таким же тензором будут и множители Лагранжа ${}^2\boldsymbol{\lambda}$. Приходим к уравнению

$$\int\limits_{\mathcal{V}} \left(\rho\boldsymbol{f}\cdot\delta\boldsymbol{r} - {}^2\boldsymbol{\lambda}\mathbin{\boldsymbol{\cdot\cdot}}\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}}\right)d\mathcal{V} + \int\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{p}\cdot\delta\boldsymbol{r}\,d\mathcal{O} = 0. \qquad (15.3)$$

Благодаря симметрии ${}^2\boldsymbol{\lambda}$ имеем*

$$^2\boldsymbol{\lambda} = {}^2\boldsymbol{\lambda}^{\mathsf{T}} \;\Rightarrow\; {}^2\boldsymbol{\lambda}\mathbin{\boldsymbol{\cdot\cdot}}\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = {}^2\boldsymbol{\lambda}\mathbin{\boldsymbol{\cdot\cdot}}\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}},$$

$$^2\boldsymbol{\lambda}\mathbin{\boldsymbol{\cdot\cdot}}\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = \boldsymbol{\nabla}\boldsymbol{\cdot}\left({}^2\boldsymbol{\lambda}\cdot\delta\boldsymbol{r}\right) - \boldsymbol{\nabla}\boldsymbol{\cdot}{}^2\boldsymbol{\lambda}\cdot\delta\boldsymbol{r}.$$

$$^*\,\boldsymbol{\Lambda}^{\mathsf{S}}\mathbin{\boldsymbol{\cdot\cdot}}\boldsymbol{X} = \boldsymbol{\Lambda}^{\mathsf{S}}\mathbin{\boldsymbol{\cdot\cdot}}\boldsymbol{X}^{\mathsf{T}} = \boldsymbol{\Lambda}^{\mathsf{S}}\mathbin{\boldsymbol{\cdot\cdot}}\boldsymbol{X}^{\mathsf{S}}, \quad \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{B}\cdot\boldsymbol{a}) = (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{B})\cdot\boldsymbol{a} + \boldsymbol{B}^{\mathsf{T}}\mathbin{\boldsymbol{\cdot\cdot}}\boldsymbol{\nabla}\boldsymbol{a}$$

Подставив это в (15.3) и применив теорему о дивергенции, получаем

$$\int\limits_{\mathcal{V}} \Big(\rho \boldsymbol{f} + \boldsymbol{\nabla} \boldsymbol{\cdot}{}^{2}\boldsymbol{\lambda} \Big) \boldsymbol{\cdot} \delta \boldsymbol{r} \, d\mathcal{V} + \int\limits_{\mathcal{O}(\partial\mathcal{V})} \Big(\boldsymbol{p} - \boldsymbol{n} \boldsymbol{\cdot}{}^{2}\boldsymbol{\lambda} \Big) \boldsymbol{\cdot} \delta \boldsymbol{r} \, d\mathcal{O} = 0.$$

But $\delta \boldsymbol{r}$ is random on a surface and inside a volume, thus

$$\boldsymbol{p} = \boldsymbol{n} \boldsymbol{\cdot}{}^{2}\boldsymbol{\lambda}, \ \ \boldsymbol{\nabla} \boldsymbol{\cdot}{}^{2}\boldsymbol{\lambda} + \rho \boldsymbol{f} = \boldsymbol{0}$$

— formally introduced symmetric multiplier ${}^{2}\boldsymbol{\lambda}$ happened to be the Cauchy stress tensor.

Similar introduction of stresses was presented in book [**?**]. Here are no new results, но интересна самá возможность одновременного вывода тех уравнений механики сплошной среды, которые раньше считались независимыми. В следующих главах эта техника используется для построения новых континуальных моделей.

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chapter 4
THE CLASSICAL LINEAR ELASTICITY

This chapter is about the geometrically linear model with infinitesimal displacements, where

- ✓ $\mathcal{V} = \mathring{\mathcal{V}}$, $\rho = \mathring{\rho}$ — "the equations can be written in the initial configuration" (sometimes it is called "the principle of initial dimensions"),
- ✓ operators $\mathring{\nabla}$ and ∇ are indistinguishable,
- ✓ operators δ and ∇ commute, thus for example $\delta \nabla u = \nabla \delta u$.

§ 1. The complete set of equations

Equations of the nonlinear elasticity, even in their simplest cases, lead to the mathematically complex problems. Therefore the linear theory of infinitesimal displacements is applied everywhere. This theory's equations were derived in the first half of the XIXth century by Cauchy, Navier, Lamé, Clapeyron, Poisson, Saint-Venant, George Green and the other scientists.

The complete closed set of equations of the classical linear theory in the direct invariant tensor notation, including

- ✓ the balance of forces (of momentum, of *vis viva*),
- ✓ the stress–strain relations for a material,
- ✓ $u \mapsto \varepsilon$,

is

$$\nabla \cdot \sigma + v = 0, \quad \sigma = \frac{\partial \Pi}{\partial \varepsilon} = {}^4\!\mathcal{A} \cdot\!\cdot\, \varepsilon, \quad \varepsilon = \nabla u^{\mathsf{S}}. \tag{1.1}$$

Here σ is the linear stress tensor, v is the vector of volume loads, ε is the tensor of infinitesimal linear deformation, $\Pi(\varepsilon)$ is the potential energy of deformation per volume unit and ${}^4\!\mathcal{A}$ is the stiffness tensor. It is tetravalent with the following symmetry

$$ {}^4\!\mathcal{A}_{12 \rightleftarrows 34} = {}^4\!\mathcal{A}, \quad {}^4\!\mathcal{A}_{1 \rightleftarrows 2} = {}^4\!\mathcal{A}, \quad {}^4\!\mathcal{A}_{3 \rightleftarrows 4} = {}^4\!\mathcal{A}. $$

But where does this set (system) of equations follow from?

The equations (1.1) are exact, they can be derived by varying the equations of the nonlinear theory. Varying from an arbitrary configuration is described in § 3.12. The linear theory is the result of varying from the initial unstressed configuration, when

$$\boldsymbol{F} = \boldsymbol{E}, \quad \boldsymbol{C} = {}^2\boldsymbol{0}, \quad \delta\boldsymbol{C} = \nabla\delta r^{\mathsf{S}},$$

$$\boldsymbol{\tau} = {}^2\boldsymbol{0}, \quad \delta\boldsymbol{\tau} = \delta\boldsymbol{T} = \frac{\partial^2\Pi}{\partial\boldsymbol{C}\partial\boldsymbol{C}} \cdot\!\cdot\, \delta\boldsymbol{C}, \quad \nabla\cdot\delta\boldsymbol{\tau} + \rho\delta\boldsymbol{f} = \boldsymbol{0}. \tag{1.2}$$

It remains to change

- ✔ δr to $\boldsymbol{u}$,
- ✔ $\delta\boldsymbol{C}$ to $\boldsymbol{\varepsilon}$,
- ✔ $\delta\boldsymbol{\tau}$ to $\boldsymbol{\sigma}$,
- ✔ $\partial^2\Pi/\partial\boldsymbol{C}\partial\boldsymbol{C}$ to ${}^4\!\mathcal{A}$,
- ✔ $\rho\delta\boldsymbol{f}$ to $\boldsymbol{v}$.

If the derivation of (1.2) seems abstruse to the reader, it's possible to proceed from the following equations

$$\nabla\cdot\boldsymbol{\tau} + \rho\boldsymbol{f} = \boldsymbol{0}, \quad \nabla = \boldsymbol{F}^{-\mathsf{T}}\cdot\overset{\circ}{\nabla}, \quad \boldsymbol{F} = \boldsymbol{E} + \overset{\circ}{\nabla}\boldsymbol{u}^{\mathsf{T}},$$

$$\boldsymbol{\tau} = J^{-1}\boldsymbol{F}\cdot\frac{\partial\Pi}{\partial\boldsymbol{C}}\cdot\boldsymbol{F}^{\mathsf{T}}, \quad \boldsymbol{C} = \overset{\circ}{\nabla}\boldsymbol{u}^{\mathsf{S}} + \tfrac{1}{2}\overset{\circ}{\nabla}\boldsymbol{u}\cdot\overset{\circ}{\nabla}\boldsymbol{u}^{\mathsf{T}}. \tag{1.3}$$

Assuming the displacement $\boldsymbol{u}$ is small (infinitesimal), we'll move from (1.3) to (1.1).

Or so. Instead of $\boldsymbol{u}$ to take some small enough parameter $\chi\boldsymbol{u}$, $\chi \to 0$. And to represent thereafter the unknowns by the series in the integer exponents of parameter χ

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{(0)} + \chi\boldsymbol{\tau}^{(1)} + \dots, \quad \boldsymbol{C} = \boldsymbol{C}^{(0)} + \chi\boldsymbol{C}^{(1)} + \dots,$$

$$\nabla = \overset{\circ}{\nabla} + \chi\nabla^{(1)} + \dots, \quad \boldsymbol{F} = \boldsymbol{E} + \chi\overset{\circ}{\nabla}\boldsymbol{u}^{\mathsf{T}}, \quad J = 1 + \chi J^{(1)} + \dots$$

The complete set of equations (1.1) comes from the first (zeroth) terms of these series. In the book [60] this is called "formal approximation".

It is impossible to tell unambiguously how small the parameter χ should be — the answer depends on the situation and is determined by whether the linear model describes the effect we are interested in or not. When, as example, I'm interested in a relation between the frequency of a freely vibrating motion after the initial displacement, then a nonlinear model is needed.

A linear problem is posed in the initial volume $\mathcal{V} = \overset{\circ}{\mathcal{V}}$, bounded by the surface o with the area vector $\boldsymbol{n}do$ ("the principle of initial dimensions").

The boundary conditions most often are: on the part o_1 of the surface displacements are known, and on another part o_2 the forces are known.

$$\boldsymbol{u}\big|_{o_1} = \boldsymbol{u}_0, \quad \boldsymbol{n}\cdot\boldsymbol{\sigma}\big|_{o_2} = \boldsymbol{p}. \tag{1.4}$$

The more complex combinations happen too, if we know the certain components of the both $\boldsymbol{u}$ and $\boldsymbol{t}_{(n)} = \boldsymbol{n}\cdot\boldsymbol{\sigma}$ simultaneously. For example, on a flat face $x = \mathsf{constant}$ when pressing a stamp with a smooth surface $u_x = \nu(y, z)$, $\tau_{xy} = \tau_{xz} = 0$ (the function ν is determined by the stamp's shape).

For the dynamic problems we have $\boldsymbol{f} - \rho\ddot{\boldsymbol{u}}$ instead of just $\boldsymbol{f}$. And the initial conditions for the dynamic problems are set as it's common in mechanics — on the positions and on the velocities: at the given moment of time $t = 0$ $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ are known. The linearity of the problems

The linearity gives the principle of superposition (or independence) of the action of loads. When there are several loads, the problem can be solved for the each load separately. And then the complete solution can be obtained by the summation. For statics this means, for example, the following: if external loads $\boldsymbol{f}$ and $\boldsymbol{p}$ increase by m times (body is fixed on o_1), then $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ will increase by m times too. Potential energy density Π will increase by m^2 times. In reality such is observed only when the loads are small.

The density of the potential energy of the elastic deformation Π

$$\Pi(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\,\boldsymbol{\varepsilon}\cdot\!\cdot{}^4\!\boldsymbol{\mathcal{A}}\cdot\!\cdot\boldsymbol{\varepsilon}$$

and its variation

$$\delta\Pi = \tfrac{1}{2}\,\delta\big(\boldsymbol{\varepsilon}\cdot\!\cdot{}^4\!\boldsymbol{\mathcal{A}}\cdot\!\cdot\boldsymbol{\varepsilon}\big) = \tfrac{1}{2}\big(\delta\boldsymbol{\varepsilon}\cdot\!\cdot{}^4\!\boldsymbol{\mathcal{A}}\cdot\!\cdot\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}\cdot\!\cdot{}^4\!\boldsymbol{\mathcal{A}}\cdot\!\cdot\delta\boldsymbol{\varepsilon}\big) = \underbrace{\boldsymbol{\varepsilon}\cdot\!\cdot{}^4\!\boldsymbol{\mathcal{A}}}_{\boldsymbol{\sigma}}\cdot\!\cdot\delta\boldsymbol{\varepsilon}$$

$$\delta\Pi(\boldsymbol{\varepsilon}) = \frac{\partial\Pi}{\partial\boldsymbol{\varepsilon}}\cdot\!\cdot\delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma}\cdot\!\cdot\delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}\cdot\!\cdot{}^4\!\boldsymbol{\mathcal{A}}\cdot\!\cdot\delta\boldsymbol{\varepsilon}$$

$$\delta^2\Pi(\boldsymbol{\varepsilon}) = \delta\boldsymbol{\varepsilon}\cdot\!\cdot\frac{\partial^2\Pi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}}\cdot\!\cdot\delta\boldsymbol{\varepsilon} = \delta\boldsymbol{\varepsilon}\cdot\!\cdot{}^4\!\boldsymbol{\mathcal{A}}\cdot\!\cdot\delta\boldsymbol{\varepsilon} = 2\Pi(\delta\boldsymbol{\varepsilon})$$

As was noted in chapter 2, the principle of the virtual work (the d'Alembert–Lagrange principle) can be put into the foundation of

mechanics. This principle is true for the linear theory too (the internal forces in an elastic medium are potential $\delta W^{(i)} = -\delta\Pi$)

$$\int_{\mathcal{V}} \Big((\boldsymbol{f} - \rho\ddot{\boldsymbol{u}}) \cdot \delta\boldsymbol{u} - \delta\Pi \Big) d\mathcal{V} + \int_{o_2} \boldsymbol{p} \cdot \delta\boldsymbol{u} \, do = 0, \quad \boldsymbol{u}\big|_{o_1} = \boldsymbol{0}, \qquad (1.5)$$

because

$$\delta\Pi = \boldsymbol{\sigma} \cdot\!\cdot\, \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot\!\cdot\, \boldsymbol{\nabla}\delta\boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla}\!\cdot(\boldsymbol{\sigma} \cdot \delta\boldsymbol{u}) - \boldsymbol{\nabla}\!\cdot\boldsymbol{\sigma} \cdot \delta\boldsymbol{u},$$

$$\int_{\mathcal{V}} \delta\Pi \, d\mathcal{V} = \oint_{o(\partial\mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \delta\boldsymbol{u} \, do - \int_{\mathcal{V}} \boldsymbol{\nabla}\!\cdot\boldsymbol{\sigma} \cdot \delta\boldsymbol{u} \, d\mathcal{V}$$

and the left part of (1.5) becomes

$$\int_{\mathcal{V}} \Big(\boldsymbol{\nabla}\!\cdot\boldsymbol{\sigma} + \boldsymbol{f} - \rho\ddot{\boldsymbol{u}}\Big) \cdot \delta\boldsymbol{u} \, d\mathcal{V} + \int_{o_2} \Big(\boldsymbol{p} - \boldsymbol{n}\cdot\boldsymbol{\sigma}\Big) \cdot \delta\boldsymbol{u} \, do,$$

that is equal to zero. Notice the boundary condition $\boldsymbol{u}\big|_{o_1} = \boldsymbol{0}$: the virtual displacements are compatible with this constraint $\delta\boldsymbol{u}\big|_{o_1} = \boldsymbol{0}$.

§2. The uniqueness of the solution in dynamics

As is typical for linear mathematical physics, the uniqueness theorem is proven "by contradiction". Assume that there are two solutions: $\boldsymbol{u}_1(\boldsymbol{r}, t)$ and $\boldsymbol{u}_2(\boldsymbol{r}, t)$. If the difference $\boldsymbol{u}^* \equiv \boldsymbol{u}_1 - \boldsymbol{u}_2$ will be equal to $\boldsymbol{0}$, then these solutions coincide, that is the solution is unique.

But at first we'll make sure of the existence of the energy integral — by deriving the balance of mechanical energy equation for the linear model of the small displacements theory

$$\int_{\mathcal{V}} \Big(\mathrm{K} + \Pi\Big)^{\!\boldsymbol{\cdot}} d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{f} \cdot \dot{\boldsymbol{u}} \, d\mathcal{V} + \int_{o_2} \boldsymbol{p} \cdot \dot{\boldsymbol{u}} \, do, \qquad (2.1)$$

$$\boldsymbol{u}\big|_{o_1} = \boldsymbol{0}, \quad \boldsymbol{n}\cdot\boldsymbol{\sigma}\big|_{o_2} = \boldsymbol{p},$$

$$\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}^\circ, \quad \dot{\boldsymbol{u}}\big|_{t=0} = \dot{\boldsymbol{u}}^\circ.$$

For the left-hand side we have

$$\dot{\mathrm{K}} = \tfrac{1}{2}\left(\rho\dot{\boldsymbol{u}}\boldsymbol{\cdot}\dot{\boldsymbol{u}}\right)^{\bullet} = \tfrac{1}{2}\rho\left(\dot{\boldsymbol{u}}\boldsymbol{\cdot}\ddot{\boldsymbol{u}} + \ddot{\boldsymbol{u}}\boldsymbol{\cdot}\dot{\boldsymbol{u}}\right) = \rho\ddot{\boldsymbol{u}}\boldsymbol{\cdot}\dot{\boldsymbol{u}},$$

$$\dot{\Pi} = \tfrac{1}{2}\underbrace{\left(\boldsymbol{\varepsilon}\boldsymbol{\cdot\cdot}{}^{4}\!\boldsymbol{\mathcal{A}}\boldsymbol{\cdot\cdot}\boldsymbol{\varepsilon}\right)^{\bullet}}_{2\boldsymbol{\varepsilon}\boldsymbol{\cdot\cdot}{}^{4}\!\boldsymbol{\mathcal{A}}\boldsymbol{\cdot\cdot}\dot{\boldsymbol{\varepsilon}}} = \boldsymbol{\sigma}\boldsymbol{\cdot\cdot}\dot{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma}\boldsymbol{\cdot\cdot}\boldsymbol{\nabla}\dot{\boldsymbol{u}}^{\mathsf{S}} = \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\sigma}\boldsymbol{\cdot}\dot{\boldsymbol{u}}) - \underbrace{\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma}}_{-(\boldsymbol{f}-\rho\ddot{\boldsymbol{u}})}\boldsymbol{\cdot}\dot{\boldsymbol{u}} =$$

$$= \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\sigma}\boldsymbol{\cdot}\dot{\boldsymbol{u}}) + (\boldsymbol{f} - \rho\ddot{\boldsymbol{u}})\boldsymbol{\cdot}\dot{\boldsymbol{u}}$$

(the balance of momentum $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma} + \boldsymbol{f} - \rho\ddot{\boldsymbol{u}} = \boldsymbol{0}$ is used),

$$\dot{\mathrm{K}} + \dot{\Pi} = \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\sigma}\boldsymbol{\cdot}\dot{\boldsymbol{u}}) + \boldsymbol{f}\boldsymbol{\cdot}\dot{\boldsymbol{u}}.$$

Applying the divergence theorem

$$\int_{\mathcal{V}}\boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\sigma}\boldsymbol{\cdot}\dot{\boldsymbol{u}})\,d\mathcal{V} = \oint_{o(\partial\mathcal{V})}\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\sigma}\boldsymbol{\cdot}\dot{\boldsymbol{u}}\,do$$

and the boundary condition $\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\sigma} = \boldsymbol{p}$ on o_2, we get (2.1).

From (2.1) it follows that without loads (when there're no external forces, neither volume nor surface), and the full mechanical energy doesn't change:

$$\boldsymbol{f} = \boldsymbol{0} \text{ and } \boldsymbol{p} = \boldsymbol{0} \;\Rightarrow\; \int_{\mathcal{V}}(\mathrm{K} + \Pi)\,d\mathcal{V} = \mathsf{constant}(t). \qquad (2.2)$$

If at the moment $t=0$ there was unstressed ($\Pi{=}0$) rest (K$=$0), then

$$\int_{\mathcal{V}}(\mathrm{K} + \Pi)\,d\mathcal{V} = 0. \qquad (2.2')$$

The kinetic energy is positive: $\mathrm{K} > 0$ if $\dot{\boldsymbol{u}} \neq \boldsymbol{0}$ and vanishes (nullifies) only when $\dot{\boldsymbol{u}} = \boldsymbol{0}$ — this ensues from its definition $\mathrm{K} \equiv \tfrac{1}{2}\rho\dot{\boldsymbol{u}}\boldsymbol{\cdot}\dot{\boldsymbol{u}}$. The potential energy, being a quadratic form $\Pi(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\boldsymbol{\varepsilon}\boldsymbol{\cdot\cdot}{}^{4}\!\boldsymbol{\mathcal{A}}\boldsymbol{\cdot\cdot}\boldsymbol{\varepsilon}$, is positive too: $\Pi > 0$ if $\boldsymbol{\varepsilon} \neq {}^{2}\boldsymbol{0}$. Such is a priori requirement of the positive definiteness for stiffness tensor ${}^{4}\!\boldsymbol{\mathcal{A}}$. This is one of "additional inequalities in the theory of elasticity" [28, 60].

Since K and Π are positive definite, (2.2') gives

$$\mathrm{K} = 0,\; \Pi = 0 \;\Rightarrow\; \dot{\boldsymbol{u}} = \boldsymbol{0},\; \boldsymbol{\varepsilon} = \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} = {}^{2}\boldsymbol{0} \;\Rightarrow\; \boldsymbol{u} = \boldsymbol{u}^{\circ} + \boldsymbol{\omega}^{\circ}\times\boldsymbol{r}$$

($\boldsymbol{u}^{\circ}$ and $\boldsymbol{\omega}^{\circ}$ are some constants of translation and rotation). With an immobile part of the surface

$$\boldsymbol{u}|_{o_1} = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{u}^{\circ} = \boldsymbol{0} \text{ and } \boldsymbol{\omega}^{\circ} = \boldsymbol{0} \;\Rightarrow\; \boldsymbol{u} = \boldsymbol{0} \text{ everywhere.}$$

Now remember two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Their difference $\boldsymbol{u}^{*} \equiv \boldsymbol{u}_1 - \boldsymbol{u}_2$ is a solution of an entirely "homogeneous" (with no constant terms at all) linear problem: in a volume $\boldsymbol{f} = \boldsymbol{0}$, in boundary

and in initial conditions — zeroes. Therefore $\boldsymbol{u}^* = \boldsymbol{0}$, and the uniqueness is proven.

As for the existence of a solution — it cannot be proven for the generic case by simple conclusions. I could only tell that a dynamic problem is evolutional, it describes the progress of a process in time.

The balance (the conservation) of momentum gives the acceleration $\ddot{\boldsymbol{u}}$. Then, moving to the "next time layer" $t + dt$:

$$\dot{\boldsymbol{u}}(\boldsymbol{r}, t+dt) = \dot{\boldsymbol{u}}(\boldsymbol{r}, t) + \ddot{\boldsymbol{u}}dt,$$
$$\boldsymbol{u}(\boldsymbol{r}, t+dt) = \boldsymbol{u}(\boldsymbol{r}, t) + \dot{\boldsymbol{u}}dt,$$
$$\boldsymbol{\varepsilon}(\boldsymbol{r}, t+dt) = \big(\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{r}, t+dt)\big)^{\mathsf{S}} \Rightarrow \boldsymbol{\sigma},$$
$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma} + \boldsymbol{f} = \rho\ddot{\boldsymbol{u}}(\boldsymbol{r}, t+dt)$$

and so forth. Surely, these considerations lack the mathematical scrupulosity. The latter can be found, for example, in the Philippe Ciarlet's monograph [53].

§ 3. Hooke's law for an isotropic material

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$$= \boldsymbol{E} \tag{3.1}$$

In the components for an isotropic medium we have

$$A_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu\big(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\big) \tag{3.2}$$

— these are components of an isotropic tensor of the fourth complexity. These components don't change when a basis rotates.

....

§ 4. Theorems of statics

Clapeyron's theorem

In equilibrium with the external forces, the volume ones $\boldsymbol{f}$ and the surface ones $\boldsymbol{p}$, the work of these "statically frozen" (that is constant

along time) forces on the actual displacements is equal to the double of[*] the energy of deformation

$$2\int_{\mathcal{V}} \Pi d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{f} \cdot \boldsymbol{u}\, d\mathcal{V} + \int_{o_2} \boldsymbol{p} \cdot \boldsymbol{u}\, do. \tag{4.1}$$

$$\circ \quad 2\Pi = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{u}) - \underbrace{\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}}_{-\boldsymbol{f}} \cdot \boldsymbol{u} \ \Rightarrow$$

$$\Rightarrow \ 2\int_{\mathcal{V}} \Pi d\mathcal{V} = \int_{o_2} \underbrace{\boldsymbol{n} \cdot \boldsymbol{\sigma}}_{\boldsymbol{p}} \cdot \boldsymbol{u}\, do + \int_{\mathcal{V}} \boldsymbol{f} \cdot \boldsymbol{u}\, d\mathcal{V} \quad \bullet$$

From (4.1) also follows, that without loading $\int_{\mathcal{V}} \Pi d\mathcal{V} = 0$. Because Π is positive, then the stress $\boldsymbol{\sigma}$, and deformation $\boldsymbol{\varepsilon}$ without a load are equal to zero.

$$2\Pi = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\varepsilon}$$
$$\dot{\Pi} = \boldsymbol{\sigma} \cdot\cdot \dot{\boldsymbol{\varepsilon}}$$
$$\delta\Pi = \boldsymbol{\sigma} \cdot\cdot \delta\boldsymbol{\varepsilon}$$

Π is equal to only the half of the work of the external forces.

The accumulated potential energy of deformation Π is equal to only the half of the work done by the external forces, acting from the unstressed configuration to the equilibrium with the external forces.

Clapeyron's theorem implies that the accumulated elastic energy accounts for only the half of the energy spent on the deformation. The remaining half of the work, done by the external forces, is lost somewhere before reaching the equilibrium.

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[*] "Ce produit représentait d'ailleurs le double de la force vive que le ressort pouvait absorber par l'effet de sa flexion et qui était la mesure naturelle de sa puissance." —
Benoît Paul Émile Clapeyron. Mémoire sur le travail des forces élastiques dans un corps solide élastique déformé par l'action de forces extérieures. *Comptes rendus*, Tome XLVI, Janvier–Juin 1858. Pagine 208–212.

Roger Fosdick and **Lev Truskinovsky**. About Clapeyron's Theorem in Linear Elasticity. *Journal of Elasticity*, Volume 72, July 2003. Pages 145–172.

In theory, the concept of the "static loading" is common. It's when the external load is applied infinitely slow (sounds like forever, yeah).

The work of the external forces on the actual displacements is equal to the double of the potential energy density 2Π.

Если снять внешние воздействия мгновенно (бесконечно быстро), то тело будет колебаться. Но из-за сопротивления спустя некоторое время тело придёт в состояние равновесия.

Yes, only the half of the linear elastic energy is stored. The second half is the " additional energy ", which is lost before reaching of the equilibrium on the dynamics — on the internal energy of the particles (of the dissipation), on the vibrations and waves.

But any real loading would be neither a sudden loading nor an infinitely slow loading. These are the two extremes. The real dynamics of applying the loads will always be different from the theory.

In the area of infinitesimal variations the real external forces, applied to the elastic medium, work on virtual displacements and produce the work, which is exactly equal to the variation of the elastic potential energy density.

$$\boldsymbol{\sigma} \boldsymbol{\cdot\cdot} \delta\boldsymbol{\varepsilon} = \delta\Pi.$$

A linear elastic medium is a medium, where a variation of work of the internal forces (that is stresses) is a variation of the potential energy density with the opposite sign $-\delta W^{(i)} = \delta\Pi = \delta W^{(e)}$, when the only displacements vary (the stress loads do not vary).

It is necessary that the virtual work of the real external forces on variations of displacements would be equal to the variation of the internal energy with the opposite sign (for an elastic media — the variation of the internal energy).

The uniqueness of the solution theorem

As in dynamics (§ 2), we suppose the existence of the two solutions and are looking for their difference

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The uniqueness of the solution, discovered by Gustav Kirchhoff for bodies with the simply connected contour *, is contrary to, as it seems, the everyday experience. Imagine a straight rod, clamped at the one end (the " cantilever ") and compressed at the second end with a longitudinal force (fig. 13). When the load is large enough, the problem of statics has the two solutions, "straight" and "bent". Such a contradiction with the uniqueness theorem comes from the nonlinearity of this problem. If a load is small (infinitesimal), then the solution is described by the linear equations and is unique.

§5. Equations in displacements

The complete set of equations (1.1) contains unknowns $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{u}$. Excluding $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, we get the formulation in displacements (symmetrization of $\boldsymbol{\nabla u}$ is redundant due to the symmetry $^4\boldsymbol{A}_{3\rightleftarrows4} = {}^4\boldsymbol{A}$).

$$\boldsymbol{\nabla}\cdot\left({}^4\boldsymbol{A}\cdot\cdot\boldsymbol{\nabla u}\right) + \boldsymbol{f} = \boldsymbol{0},$$

$$\boldsymbol{u}\big|_{o_1} = \boldsymbol{u}_0, \ \ \boldsymbol{n}\cdot{}^4\boldsymbol{A}\cdot\cdot\boldsymbol{\nabla u}\big|_{o_2} = \boldsymbol{p}. \tag{5.1}$$

In an isotropic medium (5.1) takes the form

...

Общее решение однородного уравнения (...) нашёл Heinz Neuber

П. Ф. Папкович

...

§ 6. Concentrated
force in an infinite medium

A concentrated force is a useful mathematical idealization, but it cannot be found in the real world, where all forces are either body forces acting over a volume or surface forces acting over an area.

Here is a rhetorical question: *why an elastic body withstands an applied load, "bears" it?* The book [12] by James Gordon gives the following answer: the body deforms, and thus the internal forces appear, called "the stresses", which can compensate an external load.

...

§ 7. Finding displacements
by deformations

Like any bivalent tensor, the displacement gradient can be decomposed into the sum of the symmetric and antisymmetric parts

$$\boldsymbol{\nabla u} = \overbrace{\boldsymbol{\varepsilon}}^{\boldsymbol{\nabla u}^{\mathsf{S}}} - \overbrace{\boldsymbol{\omega} \times \boldsymbol{E}}^{-\boldsymbol{\nabla u}^{\mathsf{A}}}, \ \boldsymbol{\omega} \equiv \frac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{u}, \ (7.1)$$

The symmetric part $\boldsymbol{\nabla u}^{\mathsf{S}}$ is the linear deformation tensor $\boldsymbol{\varepsilon}$.

The antisymmetric part $\boldsymbol{\nabla u}^{\mathsf{A}}$ we will denote as $\boldsymbol{\Omega}$ and will call it the tensor of small rotations. Any antisymmetric bivalent tensor can be represented by a vector (§ 1.7). So, to find displacements $\boldsymbol{u}$ by deformations $\boldsymbol{\varepsilon}$, one more field is needed — the field of rotations $\boldsymbol{\omega}(\boldsymbol{r})$.

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The compatibility conditions in the linear elasticity

The Saint-Venant's compatibility conditions represent the integrability conditions for a symmetric bivalent tensor field. When such a tensor field is compatible, then it describes some deformation (strain).

In the displacement $\mapsto$ deformation relation $\boldsymbol{\varepsilon} = \boldsymbol{\nabla u}^{\mathsf{S}}$, the six components ε_{ij} of deformation $\boldsymbol{\varepsilon}$ originate from only three components u_k of the displacement vector $\boldsymbol{u}$.

The compatibility conditions determine whether this deformation does not cause any gaps and/or overlaps.

(.... add a picture here .....)

...

$$\operatorname{inc}\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\boldsymbol{\varepsilon}\right)^{\mathsf{T}}$$

A contour here is arbitrary, so we have the relation

$$\operatorname{inc}\boldsymbol{\varepsilon} = {}^{2}\mathbf{0}, \qquad (7.2)$$

called the compatibility of deformations equation.

...

Expression (7.2) provides constraints on possible variants of a deformation (strain) field.

(... the figure with cut squares ...)

...

Tensor $\operatorname{inc}\boldsymbol{\varepsilon}$ is symmetric together with the $\boldsymbol{\varepsilon}$

...

All equations of the linear theory have an analogue (primary source) in the nonlinear theory. To find it for (7.2), remember the Cauchy–Greendeformation tensor (§ 3.3) and curvature tensors (§ 1.18)

....

....

§ 8. Equations in stresses

The balance of forces (or of momentum)

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \qquad (8.1)$$

does not quite yet determine the stresses. It's necessary as well that deformations (strains) $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ corresponding to stresses (**??**)

$$\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \frac{\partial\widehat{\Pi}}{\partial\boldsymbol{\sigma}} = {}^4\boldsymbol{\mathcal{B}}\boldsymbol{\cdot\cdot}\boldsymbol{\sigma} \qquad (8.2)$$

were compatible (§ 7)

$$\operatorname{inc}\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) \equiv \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\boldsymbol{\varepsilon}(\boldsymbol{\sigma})\right)^{\mathsf{T}} = {}^2\boldsymbol{0}. \ (8.3)$$

Gathered together, (8.1), (8.2) and (8.3) present the complete closed set (system) of equations in stresses.

...

§ 9. The principle of the minimum potential energy

When the existence of the deformation energy function is assured, and the external forces are assumed to be constant during varying of displacements, then the principle of virtual work leads to the principle of the minimum potential energy.

The formulation of the principle:

$$\mathscr{E}(\boldsymbol{u}) \equiv \int_{\mathcal{V}}\Big(\Pi(\boldsymbol{u})-\boldsymbol{f}\cdot\boldsymbol{u}\Big)d\mathcal{V}-\int_{o_2}\boldsymbol{p}\cdot\boldsymbol{u}\,do \to \min, \quad \boldsymbol{u}\Big|$$

(9.1)

The functional $\mathscr{E}(\boldsymbol{u})$, called the (full) potential energy enof a linear-elastic body, is minimal when displacements $\boldsymbol{u}$ are true — that is for the solution of a problem (5.1). The input functions $\boldsymbol{u}$ must satisfy the geometrical condition on o_1 (so they don't break the existing constraints and can be continuous or else $\Pi(\boldsymbol{u})$ will not be integrable) .

For the true field of displacements $\boldsymbol{u}$, the quadratic function

$$\Pi(\boldsymbol{u}) = \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{u}\cdot\cdot\,{}^{4}\!\boldsymbol{\mathcal{A}}\cdot\cdot\boldsymbol{\nabla}\boldsymbol{u}$$

becomes equal to the true potential energy of deformation. Then

$$\mathscr{E} = \mathscr{E}_{\min},$$

which according to the Clapeyron's theorem (4.1) is

$$\mathscr{E}_{\min} = \int_{\mathcal{V}}\Pi(\boldsymbol{u})\,d\mathcal{V}-\left(\int_{\mathcal{V}}\boldsymbol{f}\cdot\boldsymbol{u}\,d\mathcal{V}+\int_{o_2}\boldsymbol{p}\cdot\boldsymbol{u}\,do\right)= -\int_{\mathcal{V}}$$

Taking a some other satisfactory field of displacements $\boldsymbol{u}'$, look at the finite difference

$$\mathscr{E}(\boldsymbol{u}')-\mathscr{E}(\boldsymbol{u}) = \int_{\mathcal{V}}\Big(\Pi(\boldsymbol{u}')-\Pi(\boldsymbol{u})-\boldsymbol{f}\cdot(\boldsymbol{u}'-\boldsymbol{u})\Big)d\mathcal{V}-\int_{o_2}$$

seeking $\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) \geq 0$ or (ditto) $\mathscr{E}(\boldsymbol{u}') \geq \mathscr{E}(\boldsymbol{u})$.

$\boldsymbol{f} = $ **constant** and $\boldsymbol{p} = $ **constant**

$\Pi(\boldsymbol{a}) = \frac{1}{2}\boldsymbol{\nabla}\boldsymbol{a}\cdot\cdot\,{}^{4}\!\boldsymbol{\mathcal{A}}\cdot\cdot\boldsymbol{\nabla}\boldsymbol{a}$ (but *not* the linear $\frac{1}{2}\boldsymbol{\nabla}\boldsymbol{u}\cdot\cdot\,{}^{4}\!\boldsymbol{\mathcal{A}}\cdot\cdot\boldsymbol{\nabla}\boldsymbol{a}$ — this means $\Pi(\boldsymbol{a}) \neq \frac{1}{2}\boldsymbol{\sigma}\cdot\cdot\boldsymbol{\nabla}\boldsymbol{a}$)

Constraints don't change:
$(\boldsymbol{u}' - \boldsymbol{u})\big|_{o_1} = \boldsymbol{u}_0 - \boldsymbol{u}_0 = \boldsymbol{0}$. External surface force $\boldsymbol{p}\big|_{o_2} = \boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{\sigma}$ on o_2 and $= \boldsymbol{0}$ elsewhere on $o(\partial\mathcal{V})$. $\boldsymbol{\sigma} = \boldsymbol{\nabla u} \cdot\cdot {}^4\!\boldsymbol{\mathcal{A}} = {}^2\textcolor{green}{\textbf{constant}}$ along with constant $\boldsymbol{p}$ and $\boldsymbol{f}$. Therefore

$$\int_{o_2} \boldsymbol{p} \cdot (\boldsymbol{u}' - \boldsymbol{u})\, do = \oint_{o(\partial\mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot (\boldsymbol{u}' - \boldsymbol{u})\, do = \int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \big(\boldsymbol{\sigma}$$

$$= \int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot (\boldsymbol{u}' - \boldsymbol{u})\, d\mathcal{V} + \int_{\mathcal{V}} \boldsymbol{\sigma}^{\mathsf{T}} \cdot\cdot \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u})\, d\mathcal{V}.$$

Due to symmetry $\boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma} \Rightarrow$ $\boldsymbol{\sigma}^{\mathsf{T}} \cdot\cdot \boldsymbol{\nabla a} = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\nabla a} = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\nabla a}^{\mathsf{S}} \; \forall \boldsymbol{a}$.

Разность преобразуется до

$$\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) =$$
$$= \int_{\mathcal{V}} \Big(\Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}) - \big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f}\big) \cdot (\boldsymbol{u}' - \boldsymbol{u}) - \boldsymbol{\sigma}$$

And with the balance of momentum $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}$

$$\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) = \int_{\mathcal{V}} \Big(\Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}) - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u}) \Big) d\mathcal{V}.$$

Here
$$\Pi(\boldsymbol{u}') = \tfrac{1}{2} \boldsymbol{\nabla u}' \cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot \boldsymbol{\nabla u}', \quad \Pi(\boldsymbol{u}) = \tfrac{1}{2} \boldsymbol{\nabla u} \cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot \boldsymbol{\nabla u},$$

$$\Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}) = \tfrac{1}{2} \Big(\boldsymbol{\nabla u}' \cdot\cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u}' - \boldsymbol{\nabla u} \cdot\cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u}$$

$${}^4\!\boldsymbol{\mathcal{A}}_{12 \rightleftarrows 34} = {}^4\!\boldsymbol{\mathcal{A}} \Rightarrow \boldsymbol{\nabla u} \cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot \boldsymbol{\nabla u}' = \boldsymbol{\nabla u}' \cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot \boldsymbol{\nabla u}$$

$$\tfrac{1}{2} \Big(\boldsymbol{\nabla u}' \cdot\cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u}' - \boldsymbol{\nabla u} \cdot\cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u} + \boldsymbol{\nabla u} \cdot\cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u}$$

$$\big(\boldsymbol{\nabla u}' - \boldsymbol{\nabla u} \big) = \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u})$$

for a finite difference of potentials
$$\tfrac{1}{2} \boldsymbol{\nabla}\big(\boldsymbol{u}' + \boldsymbol{u}\big) \cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot \boldsymbol{\nabla}\big(\boldsymbol{u}' - \boldsymbol{u}\big) = \Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}),$$
adding to which
$$-\boldsymbol{\nabla u} \cdot\cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u}) = -\boldsymbol{\sigma} \cdot\cdot \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u})$$

we get

$$\frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}'-\boldsymbol{u})\boldsymbol{\cdot\cdot}\,{}^{4}\!\boldsymbol{A}\boldsymbol{\cdot\cdot}\boldsymbol{\nabla}(\boldsymbol{u}'-\boldsymbol{u})=\Pi(\boldsymbol{u}'-\boldsymbol{u})$$

and finally[*]

$$\mathscr{E}(\boldsymbol{u}')-\mathscr{E}(\boldsymbol{u})=\int\limits_{\mathcal{V}}\Pi(\boldsymbol{u}'-\boldsymbol{u})d\mathcal{V}.$$

Since ${}^{4}\!\boldsymbol{A}$ is positive definite (§ 2) $\Pi(\boldsymbol{w})=\frac{1}{2}\boldsymbol{\nabla w}\boldsymbol{\cdot\cdot}\,{}^{4}\!\boldsymbol{A}\boldsymbol{\cdot\cdot}\boldsymbol{\nabla w}\geq 0 \;\forall\boldsymbol{w}$ (and $=0$ only if $\boldsymbol{\nabla w}=\boldsymbol{0}\Leftrightarrow\boldsymbol{w}=\mathsf{constant}$: for a case of translation as a whole without deformation.

...

$$\delta\boldsymbol{\nabla u}=\boldsymbol{\nabla}\delta\boldsymbol{u}$$

...

the Ritz method

The minimum functional problem $\mathscr{E}(\boldsymbol{u})$ is approximately solved as

....

the finite element method

....

§ 10. The principle of the minimum complementary energy

When the stress–strain relations (the Hooke's law) assure the existence of a complementary energy function and the geometrical boundary conditions are assumed constant during variation of stresses, then the principle of minimum complementary energy emerges.

---

[*] $b^2 - a^2 - 2a(b-a) = (b+a)(b-a) - 2a(b-a) = (b-a)^2$

The complementary energy of a linear-elastic body is the following functional over the field of stresses:

$$\mathscr{D}(\boldsymbol{\sigma}) \equiv \int\limits_{\mathcal{V}} \widehat{\Pi}(\boldsymbol{\sigma})\, d\mathcal{V} - \int\limits_{o_1} \boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\sigma}\boldsymbol{\cdot}\boldsymbol{u}_0\, do, \ \ \boldsymbol{u}_0 \equiv \boldsymbol{u}\big|_{o_1},$$

$$(10.1)$$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}, \ \ \boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\sigma}\big|_{o_2} = \boldsymbol{p}.$$

...

The variation of the balance of force equation

$$\delta\big(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma} + \boldsymbol{f}\big) = \boldsymbol{\nabla}\boldsymbol{\cdot}\delta\boldsymbol{\sigma} = \boldsymbol{0}$$

...

The principle of the minimum complementary energy is very useful for estimating inexact (approximate) solutions. But for computations it isn't so essential as the (Lagrange) principle of minimum potential energy (9.1).

To derive the variational principles it is natural to use the principle of the virtual work (§ 2.3) as a foundation.

§ 11. Mixed principles of stationarity

Prange–Hellinger–Reissner Variational Principle, named after *Ernst Hellinger*, *Georg Prange* and *Eric Reissner*.

> Working independently of Hellinger and Prange, Eric Reissner published his famous six-page paper "On a variational theorem in elasticity" in 1950. In this paper he develops — without, however,

considering Hamilton–Jacobi theory — a variational principle same to that of Prange and Hellinger.

Hu–Washizu Variational Principle,

named as *Hu Haichang* and *Kyuichiro Washizu*.

The following functional over the displacements and stresses

$$\mathcal{R}(\boldsymbol{u},\boldsymbol{\sigma}) = \int\limits_{\mathcal{V}}\Big[\boldsymbol{\sigma}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}-\widehat{\Pi}(\boldsymbol{\sigma})-\boldsymbol{f}\boldsymbol{\cdot}\boldsymbol{u}\Big]d\mathcal{V} - \\ -\int\limits_{o_1}\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\sigma}\boldsymbol{\cdot}(\boldsymbol{u}-\boldsymbol{u}_0)\,do-\int\limits_{o_2}\boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{u}\,do$$

(11.1

carries names of Reissner, Prange and Hellinger.

...

The advantage of the Reissner–Hellinger principle — freedom of variation. But it also has a drawback: on the true solution the functional has no extremum, but only stationarity.

Принцип можно использовать для построения приближённых решений методом Ritz (Ritz method). Задавая аппроксимации

...

Принцип Hu–Washizu [102] формулируется так:

$$\delta\mathcal{W}(\boldsymbol{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}) = 0,$$
$$\mathcal{W} \equiv \int\limits_{\mathcal{V}}\Big[\boldsymbol{\sigma}\boldsymbol{\cdot}\Big(\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}-\boldsymbol{\varepsilon}\Big)+\Pi(\boldsymbol{\varepsilon})-\boldsymbol{f}\boldsymbol{\cdot}\boldsymbol{u}\Big]d\mathcal{V} - \\ -\int\limits_{o_1}\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\sigma}\boldsymbol{\cdot}(\boldsymbol{u}-\boldsymbol{u}_0)\,do-\int\limits_{o_2}\boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{u}\,do.$$

(11.

Как и в принципе Рейсснера–Хеллингера, здесь нет ограничений ни в объёме, ни на поверхности, но добавляется третий независимый аргумент $\boldsymbol{\varepsilon}$. Поскольку $\widehat{\Pi} = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon} - \Pi$, то (11.1) and (11.2) кажутся почти одним и тем же.

From the Hu–Washizu principle ensues the complete system of equations with boundary conditions, так как

.....

§ 12. Antiplane shear

This is such a problem of the linear theory of elasticity, where the non-trivial results are obtained by the simple outputs[*] .

This problem is about an isotropic elastic continuum in the cartesian coordinates

$$x_\alpha, \quad \alpha = 1,2, \quad x_1 \text{ and } x_2.$$

The plane x_1, x_2 is a cross-section of a rod, the third coordinate x_3 is perpendicular to the section. The basis vectorsare

$$\boldsymbol{e}_i = \partial_i \boldsymbol{r}, \quad \boldsymbol{r} = x_i \boldsymbol{e}_i, \quad \boldsymbol{e}_i \boldsymbol{e}_i = \boldsymbol{E} \Leftrightarrow \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}.$$

In a case of an antiplane strain (an antiplane shear), the field of dis-

---

[*] Non-trivial in the theory of elasticity is, for example, when the division of a force by an area gives an infinitely large error in the calculation of the stresses.

placements $\boldsymbol{u}(\boldsymbol{r})$ is parallel to the third coordinate x_3:

$$\boldsymbol{u} = \upsilon\boldsymbol{e}_3,$$

and υ doesn't depend on x_3:

$$\upsilon = \upsilon(x_1, x_2), \quad \partial_3\upsilon = 0.$$

The deformation

$$\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla}\big(\upsilon\boldsymbol{e}_3\big)^{\mathsf{S}} = \boldsymbol{e}_3\boldsymbol{\nabla}\upsilon^{\mathsf{S}} + \upsilon\underbrace{\boldsymbol{\nabla}\boldsymbol{e}_3}_{(13.1)}{}^{\mathsf{S}} = \tfrac{1}{2}\big(\boldsymbol{\nabla}\upsilon\boldsymbol{e}$$

In the plane x_1, x_2 of the section

$$\mu = \mu(x_1, x_2), \partial_3\mu = 0$$

is a possible inhomogeneity of the medium.

§ 13. The torsion of rods

M. de Saint-Venant. Memoire sur la torsion des prismes (1853)

Adhémar-Jean-Claude Barré de Saint-Venant. Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exerçant simultanément. 1856. 327 pages.

1. Memoire sur la torsion des prismes, avec des considerations sur leur flexion, etc. Memoires presentes par divers savants a l'Academie des sciences, t. 14, 1856.

2. Memoire sur la flexion des prismes, etc. Journal de mathematiques pures et appliquees, publie par J. Liouville, 2me serie, t. 1, 1856.

стр. 379–494

This problem, which was studied in detail by Adhémar-Jean-Claude Barré de Saint-Venant, is contained in almost every book about the linear elasticity. It considers a cylinder of some section, loaded only by the surface forces at the ends (... add a figure ...)

$$z = \ell : \boldsymbol{k} \cdot \boldsymbol{\sigma} = \boldsymbol{p}(x_\alpha),$$
$$z = 0 : -\boldsymbol{k} \cdot \boldsymbol{\sigma} = \boldsymbol{p}_0(x_\alpha),$$

where $\boldsymbol{k} \equiv \boldsymbol{e}_3$, $\alpha = 1, 2$, $\boldsymbol{x} \equiv x_\alpha \boldsymbol{e}_\alpha$. Coordinates are x_1, x_2, z.

The resultant (the sum) of the external forces is equal to $\boldsymbol{0}$, and the resultant couple is directed along the z axis:

$$\int\limits_o \boldsymbol{p}\, do = \boldsymbol{0}, \quad \int\limits_o \boldsymbol{x} \times \boldsymbol{p}\, do = M\boldsymbol{k}.$$

It is known that the torsion gives the tangential components of stress $\tau_{z1} \equiv \boldsymbol{k} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{e}_1$ and $\tau_{z2} \equiv \boldsymbol{k} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{e}_2$. Assuming that only these components of tensor $\boldsymbol{\sigma}$ are non-zero

$$\boldsymbol{\sigma} = \boldsymbol{s}\boldsymbol{k} + \boldsymbol{k}\boldsymbol{s}, \ \ \boldsymbol{s} \equiv \tau_{z\alpha}\boldsymbol{e}_\alpha.$$

The solution of this problem simplifies if the equations in stresses are used.

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}=\mathbf{0} \Rightarrow \boldsymbol{\nabla}_\perp\cdot\boldsymbol{s}=\mathbf{0}(\boldsymbol{\nabla}_\perp \equiv \boldsymbol{e}_\alpha\partial_\alpha),\partial_z$$

(13.1)

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\boldsymbol{\sigma}+\frac{1}{1+\nu}\boldsymbol{\nabla}\boldsymbol{\nabla}\sigma={}^2\mathbf{0} \Rightarrow \triangle_\perp\boldsymbol{s}=\mathbf{0}(\triangle_\perp \equiv$$

(13.2)

The independence of $\boldsymbol{s}$ from z makes it possible to replace the three-dimensional operators with the two-dimensional ones.

...

§14. Plane deformation

Here the displacement vector $\boldsymbol{u}$ is parallel to the plane x_1, x_2 and does not depend on the third coordinate z.

For example рассмотрим полуплоскость с сосредоточенной нормальной силой Q на краю (?? рисунок ??)

...

There are several dozens of books on the c of elasticity, which present some interest. Primarily by Anatoliy I. Lurie [29]. His earlier book [30] is the spatial problems. Witold Nowacki published h the miscellaneous content. The author solved t Also in his book is a description of the contin brothers. Being mathematically complex, the theo mathematicians, for example there is the monogr Ciarlet. Климентий Черны́х (Klimentiy Chernih) of the anisotropy in the linear elastic media [65].

MICROPOLAR THREE-DIMENSIONAL CONTINUUM

§ 1. Introduction to the linear micropolar theory

The characteristic feature of a classical elastic media (chapter 3 and 4) is that they are made from the "simple points". A particle of a classical continuum has only the translational degrees of freedom, and the only single vector $\boldsymbol{r}(q^i, t)$ describes its movement. Therefore the external loads in such a model are only the forces, the volume and the surface ones. There are no moments.

But it is not so hard to build the more complex models of a continuous medium, where the particles have not only the degrees of the translation freedom, but also the some additional ones. These new degrees of freedom are related to the new loads. The most natural of the non-classical models of a three-dimensional medium was proposed by the Cosserat brothers in 1909 [24]. Every particle of the Cosserat's continuum is an infinitesimal absolutely rigid body with the six degrees of freedom, the three translational and the three rotational. The loads in such a medium are forces and moments. The work of the Cosserat brothers remained unnoticed for the half

a century, but then the interest to this topic arose.

Woldemar Voigt tried to remove the shortcomings of the classical theory of elasticity [**W. Voigt**. *Theoretische Studien über die Elasticitätsverhältnisse der Krystalle. Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen, 34: 3–51, 1887*] by the assumption that the interaction of two parts of the body is transmitted through an area element *do* by means not only of the force vector $\boldsymbol{p}do$ but also by the moment vector $\boldsymbol{m}do$. Thus, besides the force stresses σ_{ji} also the moment stresses have been defined.

However, the complete theory of asymmetric elasticity was developed by the brothers **François et Eugène Cosserat** who published it in 1909 in the work *"Théorie des corps déformables"*.

They assumed that the bodies are composed from the connected particles. These particles are like a small rigid bodies.

During the deforming each particle is displaced by vector $\boldsymbol{u}(\boldsymbol{r}, t)$ and rotated by vector $\boldsymbol{\varphi}(\boldsymbol{r}, t)$, the functions of the location $\boldsymbol{r}$ and time t. Thus the points (particles) of the Cosserat's continuum possess the orientation (it is a "polar media"). So we can speak of the rotation of a point. The mutually independent

vectors $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ define deformations of a body.

The introduction of $\boldsymbol{u}$ and $\boldsymbol{\varphi}$ with the assumption that the transmission of forces through an area element *do* is happened by the force vector $\boldsymbol{p}$ and the moment vector $\boldsymbol{m}$ leads in the consequence to the asymmetric stress tensors σ_{qp} and μ_{qp}.

The theory of the brothers E. and F. Cosserat remained unnoticed during their lifetime. It was so because their theory was non-linear and thus included large deformations, because the frames of their theory were out of the frames of the classical linear elasticity. They tried to construct the unified field theory, containing mechanics, optics and electrodynamics, combined by a principle of the least action.

The research in the field of the general theories of continuous media conducted in the last fifteen years, drew the attention of the scientists to the Cosserats' work. Looking for the new models, describing the behaviour of the real elastic media more precisely, the models similar to, or identical with that of Cosserats' have been encountered. Here I want to mention, first of all, the papers by C. Truesdell and R. A. Toupin [*C. Truesdell and R. A. Toupin. The classical field theories. Encyclopædia of Physics, Chap-*

ter 1, Springer-Verlag, Berlin, 1960], G. Grioli [**Grioli G.** *Elasticité asymmetrique. Ann. di Mat. Pura et Appl. Ser. IV, 50 (1960)*], R. D. Mindlin and H. F. Tiersten [**Mindlin, R. D.; Tiersten, H. F.** *Effects of couple-stresses in linear elasticity. Arch. Rational Mech. Anal. 11. 1962. 415–448*].

In the truly micropolar continuum, the vector field of displacements $\boldsymbol{u}(\boldsymbol{r}, t)$ and the field of rotations $\boldsymbol{\varphi}(\boldsymbol{r}, t)$ are mutually independent. This is also called the model with free rotation. Also it is the geometrically linear model, that's the case of the very small, the infinitesimal displacements and the infinitesimall rotations. Here operators $\overset{\circ}{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ are indistinguishable. $\mathcal{V} = \overset{\circ}{\mathcal{V}}, \; \rho = \overset{\circ}{\rho}$ and therefore equations "can be written in the initial configuration". Also operators δ and $\boldsymbol{\nabla}$ commute ($\delta\boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{\nabla}\delta\boldsymbol{u}$, $\delta\boldsymbol{\nabla}\boldsymbol{\varphi} = \boldsymbol{\nabla}\delta\boldsymbol{\varphi}$).

To build this model, I use the principle of the virtual work. This principle says that the variation of work of the real external forces on the virtual displacements is equal with the opposite sign to the variation of work , the real stresses on virtual deformations

$$\int_{\mathcal{V}} \Big(\boldsymbol{f} \cdot \delta\boldsymbol{u} + \boldsymbol{m} \cdot \delta\boldsymbol{\varphi} \Big) d\mathcal{V} + \int_{o} \Big(\boldsymbol{p} \cdot \delta\boldsymbol{u} + \boldsymbol{M} \cdot \delta\boldsymbol{\varphi} \Big) do$$

Here $\boldsymbol{f}$ and $\boldsymbol{m}$ are the external forces and moments per "one volume", $\boldsymbol{p}$ and $\boldsymbol{M}$ are external forces too, but per surface unit (the surface loads, they act only

on a certain part of o on the boundary surface.

$\delta W^{(i)}$ is the work of internal forces density per volume unit

As before, we suppose that $\delta W^{(i)}$ nullifies when the body moves as a rigid whole without deformation:

$$\delta \boldsymbol{u} = \delta \boldsymbol{\varphi} \times \boldsymbol{r} + \text{constant}, \ \delta \boldsymbol{\varphi} = \text{constant} \ \Rightarrow$$

$$\boldsymbol{\nabla}\delta \boldsymbol{u} = \boldsymbol{\nabla}\delta \boldsymbol{\varphi} \times \boldsymbol{r} - \boldsymbol{\nabla}\boldsymbol{r} \times \delta \boldsymbol{\varphi} = -\boldsymbol{E} \times \delta \boldsymbol{\varphi} = -\delta \boldsymbol{\varphi}$$

Introducing the deformation (strain) tensors — the tensor of the relative displacement between particles $\boldsymbol{\gamma}$ (the distortion tensor, the distortion is the relative displacement between particles) and the curvature-twist tensor $\boldsymbol{\kappa}$ (it also has other names: the torsion-flexure tensor, or the wryness tensor)

$$\boldsymbol{\gamma} \equiv \boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\varphi} \times \boldsymbol{E}, \ \boldsymbol{\kappa} \equiv \boldsymbol{\nabla}\boldsymbol{\varphi}, \quad (1.1)$$

$$\boldsymbol{\gamma}_\times = \boldsymbol{\nabla} \times \boldsymbol{u} - 2\boldsymbol{\varphi}, \ \boldsymbol{\kappa}_\times = \boldsymbol{\nabla} \times \boldsymbol{\varphi},$$

$$\delta\boldsymbol{\gamma} = \boldsymbol{\nabla}\delta\boldsymbol{u} + \delta\boldsymbol{\varphi} \times \boldsymbol{E}, \ \delta\boldsymbol{\kappa} = \boldsymbol{\nabla}\delta\boldsymbol{\varphi},$$

with the needed absence of any virtual deformations $\delta\boldsymbol{\gamma} = {}^2\boldsymbol{0}$ and $\delta\boldsymbol{\kappa} = {}^2\boldsymbol{0}$.

In § 3.15 for the momentless continuum, stresses appear as Lagrange multipliers in the principle of virtual work, when $\delta W^{(i)} = 0$. The same here:

$$\int_{\mathcal{V}} \left(\boldsymbol{f} \cdot \delta \boldsymbol{u} + \boldsymbol{m} \cdot \delta \boldsymbol{\varphi} - \boldsymbol{\tau} \cdot\!\!\cdot \delta\boldsymbol{\gamma}^\mathsf{T} - \boldsymbol{\mu} \cdot\!\!\cdot \delta\boldsymbol{\kappa}^\mathsf{T} \right) d\mathcal{V} +$$
$$+ \int_{o} \left(\boldsymbol{p} \cdot \delta \boldsymbol{u} + \boldsymbol{M} \cdot \delta \boldsymbol{\varphi} \right) do = 0. \quad (1.2)$$

Множители Лагранжа в каждой точке — это несимметричные тензоры второй сложности $\boldsymbol{\tau}$ и $\boldsymbol{\mu}$.

Преобразуем $-\boldsymbol{\tau}\boldsymbol{\cdot\!\cdot}\,\delta\boldsymbol{\gamma}^\mathsf{T}$ и $-\boldsymbol{\mu}\boldsymbol{\cdot\!\cdot}\,\delta\boldsymbol{\kappa}^\mathsf{T}$

$$\delta\boldsymbol{\gamma}^\mathsf{T} = \boldsymbol{\nabla}\delta\boldsymbol{u}^\mathsf{T} - \delta\boldsymbol{\varphi}\times\boldsymbol{E}, \;\; \delta\boldsymbol{\kappa}^\mathsf{T} =$$

$$-\boldsymbol{\tau}\boldsymbol{\cdot\!\cdot}\,\delta\boldsymbol{\gamma}^\mathsf{T} = -\boldsymbol{\tau}\boldsymbol{\cdot\!\cdot}\,\boldsymbol{\nabla}\delta\boldsymbol{u}^\mathsf{T} + \boldsymbol{\tau}\boldsymbol{\cdot\!\cdot}\,(\delta\boldsymbol{\varphi}\times\boldsymbol{E}), \;\; -\boldsymbol{\mu}\boldsymbol{\cdot\!\cdot}\,$$

Using (7.4, § 1.7) $\Rightarrow \boldsymbol{A}_\times = -\boldsymbol{A}\boldsymbol{\cdot\!\cdot}\,{}^3\boldsymbol{\epsilon},$

$$\boldsymbol{A}\boldsymbol{\cdot}(\boldsymbol{b}\times\boldsymbol{E}) = \boldsymbol{A}\boldsymbol{\cdot}(\boldsymbol{E}\times\boldsymbol{b}) = \boldsymbol{A}\boldsymbol{\cdot}\left(-{}^3\boldsymbol{\epsilon}\boldsymbol{\cdot}\boldsymbol{b}\right) =$$

$$= \left(-\boldsymbol{A}\boldsymbol{\cdot\!\cdot}\,{}^3\boldsymbol{\epsilon}\right)\boldsymbol{\cdot}\boldsymbol{b} = \boldsymbol{A}_\times\boldsymbol{\cdot}\boldsymbol{b} \Rightarrow$$

$$\Rightarrow \boldsymbol{\tau}\boldsymbol{\cdot\!\cdot}\,(\delta\boldsymbol{\varphi}\times\boldsymbol{E}) = \boldsymbol{\tau}_\times\boldsymbol{\cdot}\delta\boldsymbol{\varphi}$$

and "product rule"

$$\boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\tau}\boldsymbol{\cdot}\delta\boldsymbol{u}) = (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau})\boldsymbol{\cdot}\delta\boldsymbol{u} + \boldsymbol{\tau}\boldsymbol{\cdot\!\cdot}\,\boldsymbol{\nabla}\delta\boldsymbol{u}^\mathsf{T},$$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\mu}\boldsymbol{\cdot}\delta\boldsymbol{\varphi}) = (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mu})\boldsymbol{\cdot}\delta\boldsymbol{\varphi} + \boldsymbol{\mu}\boldsymbol{\cdot\!\cdot}\,\boldsymbol{\nabla}\delta\boldsymbol{\varphi}^\mathsf{T},$$

получаем

$$-\boldsymbol{\tau}\boldsymbol{\cdot\!\cdot}\,\delta\boldsymbol{\gamma}^\mathsf{T} = (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau})\boldsymbol{\cdot}\delta\boldsymbol{u} - \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\tau}\boldsymbol{\cdot}\delta\boldsymbol{u}) + \boldsymbol{\tau}_\times\boldsymbol{\cdot}\delta\boldsymbol{\varphi}$$

$$-\boldsymbol{\mu}\boldsymbol{\cdot\!\cdot}\,\delta\boldsymbol{\kappa}^\mathsf{T} = (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mu})\boldsymbol{\cdot}\delta\boldsymbol{\varphi} - \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\mu}\boldsymbol{\cdot}\delta\boldsymbol{\varphi}).$$

После интегрирования с применением теоремы о дивергенции[*]

$$\int\limits_{\mathcal{V}} \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\tau}\boldsymbol{\cdot}\delta\boldsymbol{u})\,d\mathcal{V} = \oint\limits_{\widehat{o}(\partial\mathcal{V})} \boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\tau}\boldsymbol{\cdot}\delta\boldsymbol{u}\,do, \int\limits_{\mathcal{V}} \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{\mu}\boldsymbol{\cdot}\delta\boldsymbol{\varphi})\,d\mathcal{V}$$

(1.2) приобретает вид

$$\int\limits_{\mathcal{V}}\Big((\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau}+\boldsymbol{f})\boldsymbol{\cdot}\delta\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mu}+\boldsymbol{\tau}_\times+\boldsymbol{m})\boldsymbol{\cdot}\delta\boldsymbol{\varphi}\Big)$$
$$+\int\limits_{o}\Big((\boldsymbol{p}-\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\tau})\boldsymbol{\cdot}\delta\boldsymbol{u} + (\boldsymbol{M}-\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\mu})\boldsymbol{\cdot}\delta\boldsymbol{\varphi}\Big)do = ($$

From randomness of variations $\delta\boldsymbol{u}$ and $\delta\boldsymbol{\varphi}$ inside a volume ensues the balance of forces and moments

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau}+\boldsymbol{f} = \boldsymbol{0}, \;\; \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mu}+\boldsymbol{\tau}_\times+\boldsymbol{m} = \boldsymbol{0}, \;\; (1.3)$$

and from randomness on a surface — boundary conditions

$$\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\tau} = \boldsymbol{p}, \;\; \boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\mu} = \boldsymbol{M}. \qquad (1.4)$$

---

[*] $\boldsymbol{a}\boldsymbol{\cdot}({}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{c}) = (\boldsymbol{a}\boldsymbol{\cdot}{}^2\boldsymbol{B})\boldsymbol{\cdot}\boldsymbol{c} = \boldsymbol{a}\boldsymbol{\cdot}{}^2\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{c}$

Force stress tensor $\boldsymbol{\tau}$ удовлетворяет тем же дифференциальным "уравнениям равновесия"[*] и краевым условиям, что и в безмоментной среде. But tensor $\boldsymbol{\tau}$ is asymmetric: instead of $\boldsymbol{\tau}_\times = \mathbf{0}$ here is $\boldsymbol{\nabla} \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \boldsymbol{m} = \mathbf{0}$ — couple stresses $\boldsymbol{\mu}$ appear, and volume moment load $\boldsymbol{m}$ is non-zero.

Meaning of components of the couple stress tensor $\boldsymbol{\mu}$ is revealed similarly as for $\boldsymbol{\tau}$. For an orthonormal basis, moment $\boldsymbol{M}_i = \boldsymbol{e}_i \cdot \boldsymbol{\mu} = \mu_{ik}\boldsymbol{e}_k$ acts on an area with normal $\boldsymbol{e}_i$. Diagonal components μ_{11}, μ_{22}, μ_{33} are twisting moments, nondiagonal are bending ones (?? figure??).

...

eccentricity vector $\boldsymbol{\textit{æ}}$ and inertia tensor ${}^2\mathfrak{J}$

For an isotropic medium $\boldsymbol{\textit{æ}} = \mathbf{0}$, ${}^2\mathfrak{J} = \mathfrak{J}\boldsymbol{E}$.

...

§ 2. Relations of elasticity

В этой книге упругой называется среда с потенциальными внутренними силами: $\delta W^{(i)} = -\delta\Pi$, где Π — энергия деформации на единицу объёма (continuing to model the geometrically linear material, $\mathcal{V} = \mathring{\mathcal{V}}$).

Имея соотношения (...)

---

[*] Кавычки здесь оттого, что *уравнения равновесия* это вообще всё, что вытекает из принципа виртуальной работы в статике.

...

$$\delta\Pi = -\delta W^{(i)} = \boldsymbol{\tau} \bullet \delta\boldsymbol{\gamma}^{\mathsf{T}} + \boldsymbol{\mu} \bullet \delta\boldsymbol{\kappa}^{\mathsf{T}} \Rightarrow$$

$$\Rightarrow \ \boldsymbol{\tau} = \frac{\partial\Pi}{\partial\boldsymbol{\gamma}}, \ \boldsymbol{\mu} = \frac{\partial\Pi}{\partial\boldsymbol{\kappa}}. \qquad (2.1)$$

Последние равенства — соотношения упругости (определяющие уравнения, constitutive equations).

Разлагая тензоры деформации и напряжения на симметричные и антисимметричные части

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}^{\mathsf{S}} - \tfrac{1}{2}\,\boldsymbol{\gamma}_{\times} \times \boldsymbol{E}, \ \ \boldsymbol{\kappa} = \boldsymbol{\kappa}^{\mathsf{S}} - \tfrac{1}{2}\,\boldsymbol{\kappa}_{\times} \times \boldsymbol{E},$$

$$\delta\boldsymbol{\gamma}^{\mathsf{T}} = \delta\boldsymbol{\gamma}^{\mathsf{S}} + \tfrac{1}{2}\,\delta\boldsymbol{\gamma}_{\times} \times \boldsymbol{E}, \ \ \delta\boldsymbol{\kappa}^{\mathsf{T}} = \delta\boldsymbol{\kappa}^{\mathsf{S}} + \tfrac{1}{2}\,\delta\boldsymbol{\kappa}_{\times} \times \boldsymbol{E}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{S}} - \tfrac{1}{2}\,\boldsymbol{\tau}_{\times} \times \boldsymbol{E}, \ \ \boldsymbol{\mu} = \boldsymbol{\mu}^{\mathsf{S}} - \tfrac{1}{2}\,\boldsymbol{\mu}_{\times} \times \boldsymbol{E},$$

преобразуем выражение $\delta\Pi = \boldsymbol{\tau} \bullet\bullet \delta\boldsymbol{\gamma}^{\mathsf{T}} + \boldsymbol{\mu} \bullet\bullet \delta\boldsymbol{\kappa}^{\mathsf{T}}$ как

$$\delta\Pi = \ldots \qquad (2.2)$$

...

$$\boldsymbol{\gamma}_{\times} = \boldsymbol{\nabla} \times \boldsymbol{u} - 2\boldsymbol{\varphi},$$

$$\boldsymbol{\kappa}_{\times} = \boldsymbol{\nabla} \times \boldsymbol{\varphi},$$

$$\boldsymbol{\varepsilon} \equiv \boldsymbol{\gamma}^{\mathsf{S}} = \boldsymbol{\nabla u}^{\mathsf{S}},$$

...

The classical isotropic linear elastic material behavior is described by two material parameters, for example, the Young's modulus and the Poisson's ratio, while the isotropic Cosserat continuum needs six material parameters

even when assumed to be linear, homogeneous and isotropic, it requires six independent material constants, in contrast to only two such constants for the classical continuum

...

Соотношения (2.1) обращаются преобразованием Лежандра

$$\boldsymbol{\gamma} = \frac{\partial \widehat{\Pi}}{\partial \boldsymbol{\tau}}, \ \boldsymbol{\kappa} = \frac{\partial \widehat{\Pi}}{\partial \boldsymbol{\mu}},$$

$$\widehat{\Pi}(\boldsymbol{\tau}, \boldsymbol{\mu}) = \boldsymbol{\tau} \boldsymbol{\cdot\cdot} \boldsymbol{\gamma}^{\mathsf{T}} + \boldsymbol{\mu} \boldsymbol{\cdot\cdot} \boldsymbol{\kappa}^{\mathsf{T}} - \Pi(\boldsymbol{\gamma}, \boldsymbol{\kappa}). \tag{2.3}$$

...

material's intrinsic (internal) length scale ℓ

Если устремить ℓ к нулю, то исчезает вклад $\boldsymbol{\kappa}$ в Π, а с ним и моментные напряжения $\boldsymbol{\mu}$. Когда вдобавок нет объёмной моментной нагрузки $\boldsymbol{m}$, тогда тензор $\boldsymbol{\tau}$ становится симметричным: $\boldsymbol{\nabla} \boldsymbol{\cdot} \mu + \boldsymbol{\tau}_{\times} + m = \mathbf{0}$, $\mu = {}^2\mathbf{0}$, $m = \mathbf{0} \Rightarrow \boldsymbol{\tau}_{\times} = \mathbf{0}$, и модель превращается в классическую безмоментную.

Yet using of the micropolar model is natural in a case when the real material has a certain smallest volume "which is impossible to enter into". And such a situation occurs quite often: composites with "representative" volume, polycrystalline materials, polymers with large molecules (macromolecules).

§ 3. Compatibility equations

Having identity $\boldsymbol{\nabla} \times \boldsymbol{\nabla} a = {}^2\mathbf{0} \quad \forall a$ and the definitions of the deformation tensors (1.1),

$$\boldsymbol{\kappa} \equiv \boldsymbol{\nabla}\boldsymbol{\varphi} \ \Rightarrow \ \boldsymbol{\nabla} \times \boldsymbol{\kappa} = {}^2\mathbf{0},$$

$$\boldsymbol{\gamma} - \boldsymbol{\varphi} \times \boldsymbol{E} = \boldsymbol{\nabla} u \ \Rightarrow \ \boldsymbol{\nabla} \times \left(\boldsymbol{\gamma} - \boldsymbol{\varphi} \times \boldsymbol{E} \right) = {}^2\mathbf{0} \ ... \tag{3.1}$$

...

§ 4. Theorems of statics

Теоремы статики линейных консервативных систем, легко выводимые для конечного числа степеней свободы (минимальность энергии, the Clapeyron's theorem, the reciprocal work theorem et al. — § 2.6), справедливые for the continual linear elastic media (гл. 4), including the micropolar model of continuum (a medium with the force couples or the moments).

...

§ 5. The Cosserat pseudocontinuum

Paul van Dyk - We Are Alive

Besides the model with free rotation ("the truly micropolar continuum"), there is the simplified model of a medium with the force couples, in which rotations are expressed via displacements, like in the classical momentless continuum:

$$\boldsymbol{\varphi} = \tfrac{1}{2}\,\boldsymbol{\nabla}\times\boldsymbol{u} \;\Leftrightarrow\; \boldsymbol{\gamma}_{\times} = \boldsymbol{0} \;\Leftrightarrow\; \boldsymbol{\gamma} = \boldsymbol{\varepsilon} = \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} \tag{5.1}$$

— the model with constrained rotation[*].

Равенство $\boldsymbol{\gamma}_{\times} = \boldsymbol{0}$ (симметрию $\boldsymbol{\gamma}$) возможно понимать как внутреннюю

---

[*] Cosserat brothers called it cas de trièdre caché (case of latent trihedron).

связь (§ 3.13). Аргумент $\boldsymbol{\gamma}_\times$ исчезает из энергии П, соотношение упругости для $\boldsymbol{\tau}_\times$ не может быть написано. Его место в полной системе занимает уравнение связи.

В классической (линейной безмоментной) теории упругости полная система сводится к одному уравнению для вектора $\boldsymbol{u}$ (§ 4.**??**). В моментной теории

...

§ 6. Plane deformation

Все переменные в этой постановке проблемы не зависят от декартовой координаты $z \equiv x_3$ (орт оси — $\boldsymbol{k}$). Смещения и силы перпендикулярны оси z, а повороты и моменты — параллельны ей:

...

Это краткое изложение плоской задачи относится к модели с независимыми поворотами. Псевдоконтинуум Cosserat (модель со стеснённым вращением) получается либо при наложении внутренней связи $\boldsymbol{\gamma}_\times = \boldsymbol{0}$, либо при предельном переходе ...

Подробнее о плоской моментной задаче написано в книгах Н. Ф. Морозова [**?**, 37].

§7. Nonlinear theory

Кажущееся на первый взгляд чрезвычайно трудным, построение теории конечных деформаций континуума Cosserat становится прозрачным, если опираться на общую механику, тензорное исчисление и нелинейную теорию безмоментной среды.

Построение модели упругого континуума проходит обычно четыре этапа:

- ✓ определение степеней свободы частиц,
- ✓ выявление нагрузок ("силовых факторов", напряжений) и условий их баланса,
- ✓ подбор соответствующих мер деформации

и, наконец,

- ✓ вывод соотношений упругости между напряжениями и деформациями.

Этот путь очень сокращается, если опираться на принцип виртуальной работы.

Как и в гл. 3, среда состоит из частиц с материальными координатами q^i и вектором-радиусом $\boldsymbol{r}(q^i, t)$. В начальной (исходной, отсчётной) конфигурации $\boldsymbol{r}(q^i, 0) \equiv \overset{\circ}{\boldsymbol{r}}(q^i)$. Но кроме трансляции, частицы имеют независимые степени свободы поворота, описываемого ортогональным тензором

$$\boldsymbol{O}(q^i, t) \equiv \boldsymbol{a}_j \overset{\circ}{\boldsymbol{a}}{}^j = \boldsymbol{a}^j \overset{\circ}{\boldsymbol{a}}_j = \boldsymbol{O}^{-\mathsf{T}},$$

где тройка векторов $\boldsymbol{a}_j(q^i, t)$ жёстко связана с каждой частицей,

показывая угловую ориентацию относительно как-либо выбираемых векторов $\mathring{\boldsymbol{a}}_j(q^i) \equiv \boldsymbol{a}_j(q^i, 0)$, $\boldsymbol{a}_j = \boldsymbol{O} \cdot \mathring{\boldsymbol{a}}_j$; $\boldsymbol{a}^j$ — тройка взаимных векторов: $\boldsymbol{a}_j \boldsymbol{a}^j = \boldsymbol{a}^j \boldsymbol{a}_j = \boldsymbol{E}$ ($t=0$, $\mathring{\boldsymbol{a}}^j$: $\mathring{\boldsymbol{a}}_j \mathring{\boldsymbol{a}}^j = \mathring{\boldsymbol{a}}^j \mathring{\boldsymbol{a}}_j = \boldsymbol{E}$). Движение среды полностью определяется функциями $\boldsymbol{r}(q^i, t)$ and $\boldsymbol{O}(q^i, t)$.

Имея представления $\mathring{\boldsymbol{r}}(q^i)$ и $\boldsymbol{r}(q^i, t)$, вводим базис $\boldsymbol{r}_i \equiv \partial_i \boldsymbol{r}$, взаимный базис $\boldsymbol{r}^i$: $\boldsymbol{r}_j \cdot \boldsymbol{r}^i = \delta_j^i$, differential operators $\mathring{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$, а также motion gradient $\boldsymbol{F}$

$$\mathring{\boldsymbol{\nabla}} \equiv \mathring{\boldsymbol{r}}^i \partial_i, \ \boldsymbol{\nabla} \equiv \boldsymbol{r}^i \partial_i, \ \boldsymbol{\nabla} = \boldsymbol{F}^{-\mathsf{T}} \cdot \mathring{\boldsymbol{\nabla}}, \ \boldsymbol{F} \equiv \mathring{\boldsymbol{\nabla}} \boldsymbol{r}^\mathsf{T} = \boldsymbol{r}_i \tag{7.1}$$

Вариационное уравнение принципа виртуальной работы для континуума с нагрузками в объёме и на поверхности:

$$\int_{\mathcal{V}} \Big(\rho \left(\boldsymbol{f} \cdot \delta \boldsymbol{r} + \boldsymbol{m} \cdot \delta\boldsymbol{\varphi} \right) + \delta W^{(i)} \Big) d\mathcal{V} +$$
$$+ \int_{\mathcal{O}} \Big(\boldsymbol{p} \cdot \delta \boldsymbol{r} + \boldsymbol{M} \cdot \delta\boldsymbol{\varphi} \Big) d\mathcal{O} = 0. \tag{7.2}$$

Here ρ is mass density; $\boldsymbol{f}$ and $\boldsymbol{m}$ are external force and moment per mass unit; $\boldsymbol{p}$ and $\boldsymbol{M}$ — они же per surface unit; $\delta W^{(i)}$ — работа внутренних сил per

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Один из вариантов: $\mathring{\boldsymbol{a}}_j = \mathring{\boldsymbol{r}}_j \equiv \partial_j \boldsymbol{r}$. Другое предложение: $\mathring{\boldsymbol{a}}_j$ это ортонормальная тройка собственных векторов тензора инерции частицы. Вообще, $\mathring{\boldsymbol{a}}_j$ могут быть любой тройкой линейно-независимых векторов.

volume unit в текущей конфигурации. Вектор ма́лого поворота $\delta\boldsymbol{\varphi}$

$$\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{E} \Rightarrow \delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O}\cdot\delta\boldsymbol{O}^{\mathsf{T}} \Rightarrow$$
$$\Rightarrow \delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = \delta\boldsymbol{\varphi}\times\boldsymbol{E} = \delta\boldsymbol{\varphi}\times\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} \Rightarrow \delta\boldsymbol{O} = \delta\boldsymbol{\varphi}$$
$$\delta\boldsymbol{\varphi} = -\tfrac{1}{2}\big(\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}}\big)_{\times}$$

При движении среды как жёсткого целого нет деформаций, и работа $\delta W^{(i)}$ внутренних сил равна нулю:

$$\delta\boldsymbol{r} = \textbf{constant} + \delta\boldsymbol{\varphi}\times\boldsymbol{r}, \ \ \delta\boldsymbol{\varphi} = \textbf{constant} =$$
$$\boldsymbol{\nabla}\delta\boldsymbol{\varphi} = {}^2\boldsymbol{0}, \ \ \boldsymbol{\nabla}\delta\boldsymbol{r} = \boldsymbol{\nabla}\delta\boldsymbol{\varphi}\times\boldsymbol{r} - \boldsymbol{\nabla}\boldsymbol{r}\times\delta\boldsymbol{\varphi} = -\boldsymbol{E}\times$$
$$\boldsymbol{\nabla}\delta\boldsymbol{r} + \delta\boldsymbol{\varphi}\times\boldsymbol{E} = {}^2\boldsymbol{0}.$$

К нагрузкам. Несимметричные тензоры напряжения, силового $\boldsymbol{\tau}$ и моментного $\boldsymbol{\mu}$, введём как множители Lagrange'а:

$$\int_{\mathcal{V}}\Big(\rho\big(\boldsymbol{f}\cdot\delta\boldsymbol{r} + \boldsymbol{m}\cdot\delta\boldsymbol{\varphi}\big) - \boldsymbol{\tau}\cdot\cdot\big(\boldsymbol{\nabla}\delta\boldsymbol{r} + \delta\boldsymbol{\varphi}\times\boldsymbol{E}\big)$$
$$+ \int_{\mathcal{O}}\Big(\boldsymbol{p}\cdot\delta\boldsymbol{r} + \boldsymbol{M}\cdot\delta\boldsymbol{\varphi}\Big)d\mathcal{O} = 0. \quad (7.3)$$

После тех же преобразований, что и в § 1, получаем

...

Отсюда вытекают уравнения баланса сил и моментов в объёме и краевые условия в виде формул типа Cauchy. Они по существу те же, что и в линейной теории.

Найдём теперь тензоры деформации. Их можно вводить по-разному, если требовать лишь одного — нечувствительности к движению среды как жёсткого целого. Читатель найдёт не один вариант таких тензоров. Одна-

ко, вид тензоров деформации "подсказывает" принцип виртуальной работы.

...

§ 8. Nonlinear model with constrained rotation

Вспомним переход к модели со стеснённым вращением в линейной теории ($\S\,5$). Разделились соотношения упругости для симметричной части тензора силового напряжения $\boldsymbol{\tau}^{\mathsf{S}}$ и кососимметричной его части $\boldsymbol{\tau}_{\times}$. Возникла внутренняя связь $\boldsymbol{\gamma}_{\times} = \mathbf{0}$

...

Bibliography

Все работы по моментной теории упругости упоминают книгу братьев Eugène et François Cosserat [24], где трёхмерной среде посвящена одна глава из шести. Переведённая монография W. Nowacki [40] была одной из первых книг на русском языке с изложением линейной моментной теории. Ранее эта область представлялась статьями — например, R. D. Mindlin'a и H. F. Tiersten'a [36]. Краткое изложение моментной теории, но с подробным рассмотрением задач содержится в книгах Н. Ф. Морозова [?, 37].

chapter 6
THERMOELASTICITY

In this chapter
only the momentless model is considered.

§1. The first
law of thermodynamics

Hitherto modeling was limited to mechanics only. It is widely known, however, that a change in temperature causes a deformation. Temperature deformation and stress often play the primary role and can lead to a breakage.

So effective in mechanics, the principle of virtual work is not applicable to thermomechanics[*]. Considering thermal effects, it's possible to lean on the two laws of thermodynamics.

The first, discovered by Joule, Mayer, and Helmholtz — this is the adapted version of the balance of energy: rate of internal energy change $\dot{E}$ is equal to the sum of power of external forces $P^{(e)}$ and rate of heat supply $\dot{Q}$

$$\dot{E} = P^{(e)} + \dot{Q}. \tag{1.1}$$

Internal energy E is the sum of the kinetic and the potential energies

[*] Analogue of the principle of virtual work will be presented below in §8.

of the particles. For any finite volume of a material continuum

$$E = \int_{\mathcal{V}} \rho\left(\tfrac{1}{2}\dot{\boldsymbol{r}}\boldsymbol{\cdot}\dot{\boldsymbol{r}} + e\right)d\mathcal{V}. \qquad (1.2)$$

With the balance of mass $dm = \rho\,d\mathcal{V} = \rho' d\mathcal{V}'$, $m = \int_{\mathcal{V}} \rho\,d\mathcal{V} = \int_{\mathcal{V}'} \rho' d\mathcal{V}'$ and
$\Psi = \int_{\mathcal{V}} \rho\psi\,d\mathcal{V} = \int_{\mathcal{V}'} \rho'\psi\,d\mathcal{V}' \Rightarrow \dot{\Psi} = \int_{\mathcal{V}} \rho\dot{\psi}\,d\mathcal{V} = \int$
it's easy to get the time derivative of internal energy

$$\dot{E} = \int_{\mathcal{V}} \rho\left(\ddot{\boldsymbol{r}}\boldsymbol{\cdot}\dot{\boldsymbol{r}} + \dot{e}\right)d\mathcal{V}. \qquad (1.3)$$

The power of the external forces for some finite volume of a momentless continuum

$$P^{(e)} = \int_{\mathcal{V}} \rho\boldsymbol{f}\boldsymbol{\cdot}\dot{\boldsymbol{r}}\,d\mathcal{V} + \oint_{O(\partial\mathcal{V})} \boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\tau}\boldsymbol{\cdot}\dot{\boldsymbol{r}}\,dO = \int_{\mathcal{V}}\left(\rho\boldsymbol{f}\boldsymbol{\cdot}\dot{\boldsymbol{r}} + \boldsymbol{\nabla}\boldsymbol{\cdot}\right.$$

$$= \int_{\mathcal{V}}\left(\left(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau} + \rho\boldsymbol{f}\right)\boldsymbol{\cdot}\dot{\boldsymbol{r}} + \boldsymbol{\tau}\boldsymbol{\cdot\cdot}\boldsymbol{\nabla}\dot{\boldsymbol{r}}^{\mathsf{S}}\right)d\mathcal{V}.$$

$$(1.4)$$

As before (chapter 3), $\boldsymbol{\tau}$ is Cauchy stress tensor, $\boldsymbol{f}$ is mass force (without inertial part $-\ddot{\boldsymbol{r}}$, which is included in $\dot{E}$), $\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\tau}$ is surface force. The symmetry of $\boldsymbol{\tau}$ is used for expanding $\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\boldsymbol{\tau}\boldsymbol{\cdot}\dot{\boldsymbol{r}}\right)$:

$$\boldsymbol{\tau}^{\mathsf{T}} = \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{S}} \Rightarrow \boldsymbol{\tau}\boldsymbol{\cdot\cdot}\boldsymbol{\nabla}\dot{\boldsymbol{r}}^{\mathsf{T}} = \boldsymbol{\tau}\boldsymbol{\cdot\cdot}\boldsymbol{\nabla}\dot{\boldsymbol{r}}^{\mathsf{S}},$$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\boldsymbol{\tau}\boldsymbol{\cdot}\dot{\boldsymbol{r}}\right) = \left(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau}\right)\boldsymbol{\cdot}\dot{\boldsymbol{r}} + \boldsymbol{\tau}\boldsymbol{\cdot\cdot}\boldsymbol{\nabla}\dot{\boldsymbol{r}}^{\mathsf{T}} = \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau}\boldsymbol{\cdot}\dot{\boldsymbol{r}} +$$

Denominating the rate of deformation tensor as $\boldsymbol{\mathcal{D}} \equiv \boldsymbol{\nabla}\dot{\boldsymbol{r}}^{\mathsf{S}}$

$$P^{(e)} = \int_{\mathcal{V}}\left(\left(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\tau} + \rho\boldsymbol{f}\right)\boldsymbol{\cdot}\dot{\boldsymbol{r}} + \boldsymbol{\tau}\boldsymbol{\cdot\cdot}\boldsymbol{\mathcal{D}}\right)d\mathcal{V}.$$

$$(1.4')$$

Heat arrives in a volume of continuum by two ways. The first is a surface heat transfer (heat conduction, thermal conductivity, convection, diffusion), occurring via matter, upon a contact of two media. This can be described by heat flux vector $\boldsymbol{q}$. Through an infinitesimal area in the current configuration towards normal vector $\boldsymbol{n}$ per unit of time passes heat flux $\boldsymbol{q} \cdot \boldsymbol{n} dO$. For a surface with finite dimensions this expression needs to be integrated. It's usually assumed

$$\boldsymbol{q} = -{}^2\boldsymbol{k} \cdot \boldsymbol{\nabla}\Theta, \qquad (1.5)$$

where Θ is temperature (temperature field); ${}^2\boldsymbol{k}$ is thermal conductivity tensor as property of the material, for isotropic material ${}^2\boldsymbol{k} = k\boldsymbol{E}$ and $\boldsymbol{q} = -k\boldsymbol{\nabla}\Theta$.

The second way is a volume heat transfer (thermal radiation). Solar energy, flame of a campfire, a microwave oven are familiar examples of pervasive heating by radiation. Thermal radiation occurs via electromagnetic waves and doesn't need an interjacent medium. Heat is radiated (emitted) by any matter (with temperature above the absolute zero $0\,\mathrm{K}$). The rate of heat transfer by radiation per mass unit b or per volume unit $B = \rho b$ is assumed as known.

Therefore, the rate of heat supply for a finite volume is

$$\dot{Q} = -\oint_{O(\partial \mathcal{V})} \boldsymbol{n} \boldsymbol{\cdot} \boldsymbol{q}\, dO + \int_{\mathcal{V}} \rho b\, d\mathcal{V} = \int_{\mathcal{V}} \left(-\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{q} + \rho b\right) d\mathcal{V}.$$

$$(1.6)$$

Applying (1.3), (1.4') and (1.6) to formulation (1.1) gives the equality of integrals over a volume

$$\int_{\mathcal{V}} \rho \left(\ddot{\boldsymbol{r}} \boldsymbol{\cdot} \dot{\boldsymbol{r}} + \dot{e}\right) d\mathcal{V} = \int_{\mathcal{V}} \left((\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\tau} + \rho \boldsymbol{f}) \boldsymbol{\cdot} \dot{\boldsymbol{r}} + \boldsymbol{\tau} \boldsymbol{\cdot\cdot} \boldsymbol{D} - $$

And since volume $\mathcal{V}$ is random, integrands are equal too

$$\rho \ddot{\boldsymbol{r}} \boldsymbol{\cdot} \dot{\boldsymbol{r}} + \rho \dot{e} = \left(\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\tau} + \rho \boldsymbol{f}\right) \boldsymbol{\cdot} \dot{\boldsymbol{r}} + \boldsymbol{\tau} \boldsymbol{\cdot} \boldsymbol{D} - \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{q} + \rho b.$$

With the balance of momentum

$$\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\tau} + \rho\left(\boldsymbol{f} - \ddot{\boldsymbol{r}}\right) = \boldsymbol{0} \quad (7.2, \S\, 3.7)$$

this simplifies to

$$\rho \dot{e} = \boldsymbol{\tau} \boldsymbol{\cdot\cdot} \boldsymbol{D} - \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{q} + \rho b \qquad (1.7)$$

— the local (differential) version of the balance of energy.

...

§ 2. The second law

The following concept of laws of thermodynamics is widespread: change in internal energy dE is equal to the sum of work of external forces $\partial W^{(e)}$ and supplied heat ∂Q

$$dE = \partial W^{(e)} + \partial Q.$$

Work $\partial W^{(e)}$ and heat ∂Q are inexact differentials*, but quotient $\partial Q/_\Theta$ becomes the exact differential — differential dS of the entropy.

Further, processes are divided into reversible ones, for which $dS = \partial Q/_\Theta$, and irreversible ones with characteristic Clausius inequality $dS \geq \partial Q/_\Theta$.

But how to adapt this for a continuum with an inhomogeneous temperature field?

Sometimes a process in an infinitesimal volume is thought of as reversible, then the equality like

$$\rho\Theta\dot{s} = -\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q} + \rho b \qquad (2.1)$$

is proposed (s is entropy per mass unit and $\dot{s}$ is the time derivative of it, that is the rate of entropy change).

However, there's always heat dissipation — an irreversible process, and therefore (2.1) looks disputable.

The most appropriate expression of the second law of thermodynamics for a material continuum seems to be the Clausius–Duhem inequality

$$\left(\int_{\mathcal{V}} \rho s\, d\mathcal{V}\right)^{\bullet} \geq -\oint_{O(\partial\mathcal{V})} \frac{\boldsymbol{q}}{\Theta}\boldsymbol{\cdot}\boldsymbol{n}\, dO + \int_{\mathcal{V}} \frac{\rho b}{\Theta}\, d\mathcal{V}.$$

$$(2.2)$$

---

* Because work and heat depend on the path of the process (are path functions), they can't be full (exact) differentials, contrasting with the idea of the exact differential, expressed via the gradient of another function and therefore path independent.

This inequality as the imbalance of entropy defines the rate of entropy production.

$$-\oint\limits_{O(\partial\mathcal{V})}\boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{q}\Theta^{-1}dO = -\int\limits_{\mathcal{V}}\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\boldsymbol{q}\Theta^{-1}\right)d\mathcal{V}$$

$$-\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\boldsymbol{q}\Theta^{-1}\right) = -\left(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q}\right)\Theta^{-1}-\boldsymbol{q}\boldsymbol{\cdot}\left(\boldsymbol{\nabla}\frac{1}{\Theta}\right)$$

$$-\boldsymbol{\nabla}\frac{1}{\Theta} = \frac{1}{\Theta^2}\boldsymbol{\nabla}\Theta$$

$$-\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\boldsymbol{q}\Theta^{-1}\right) = -\left(\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q}\right)\Theta^{-1}+\left(\boldsymbol{q}\boldsymbol{\cdot}\boldsymbol{\nabla}\Theta\right)\Theta^{-2}$$

$$\rho\dot{s} \geq \left(-\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q}+\rho b\right)\Theta^{-1}+\left(\boldsymbol{q}\boldsymbol{\cdot}\boldsymbol{\nabla}\Theta\right)\Theta^{-2}$$
$$(2.3)$$

The Clausius–Duhem inequality is also called the dissipation inequality. For a real matter, the dissipation is always greater than zero, it can never be negative and can't be zero whenever irreversible processes are present.

...

Helmholtz free energy per mass unit
$$a \equiv e - \Theta s, \qquad (2.4)$$
$$\dot{a} = \dot{e} - \Theta\dot{s} - \dot{\Theta}s$$

§ 3. Constitutive equations

К балансу импульса, балансу момента импульса и законам термодинамики нужно добавить определяющие уравнения, выражающие свойства среды. Эти уравнения

...

Термоупругим называется материал, в котором свободная энергия a

и энтропия s — функции деформации $\boldsymbol{C}$ и температуры Θ

$$a = a(\boldsymbol{C}, \Theta)$$

$$\dot{a} = \frac{\partial a}{\partial \boldsymbol{C}} \cdot\cdot \dot{\boldsymbol{C}} + \frac{\partial a}{\partial \Theta} \dot{\Theta}$$

...

§ 4. Heat equation

In mathematical physics, a parabolic differential equation, similar to

$$k \triangle \Theta + B = c \dot{\Theta}, \qquad (4.1)$$

is declared as a "heat equation". Here k is thermal conductivity, $B = \rho b$ is rate of heat transfer by radiation per volume unit, c is thermal capacity per volume unit. Boundary conditions most often are external temperature $\Theta_1^{(e)}$ on part O_1 of the surface and heat flux $q^{(e)}$ from the outside of part O_2 of the surface:

$$\Theta\big|_{O_1} = \Theta_1^{(e)}, \quad k\,\partial_n\Theta\big|_{O_2} = q^{(e)}.$$

Sometimes, flux $q^{(e)}$ is thought to be proportional to the difference between temperature $\Theta^{(e)}$ of the ambient and body temperature Θ

$$k\,\partial_n\Theta + \hbar\left(\Theta - \Theta^{(e)}\right) = 0.$$

If heat transfer coefficient $\hbar$ is infinitely large, it turns into the first condition $\Theta = \Theta^{(e)}$, and when $\hbar \to 0$ — into condition $\partial_n\Theta = 0$ of thermal insulation.

But how is equation (4.1) related to fundamental principles of balance? Since there's no special "thermal energy", but there is internal energy, changing

according to the first law of thermody-
namics ...

...

$$e = a + \Theta s \;\Rightarrow\; \dot{e} = \dot{a} + \dot{\Theta}s + \Theta\dot{s}$$

$$\rho\dot{e} = \rho\big(\dot{a} + \dot{\Theta}s + \Theta\dot{s}\big) = \rho\Big(\overbrace{\frac{\partial a}{\partial \boldsymbol{C}}\boldsymbol{\cdot}\dot{\boldsymbol{C}} + \frac{\partial a}{\partial \Theta}\dot{\Theta}}^{\dot{a}(\boldsymbol{C},\Theta)} + \underbrace{\dot{\Theta}s}_{=\,0\;\Leftarrow\;s\,=\,-\frac{\partial}{\partial}} + \Theta\dot{s}$$

...

§ 5. Linear thermoelasticity

Квадратичная аппроксимация свобод-
ной энергии наиболее естественна
в линейной теории

...

§ 6. Equations
for displacements

Полагая поле температуры известным

...

§ 7. Thermal stress

Это напряжение сто́ит рассмотреть
детально, хотя оно и определяется
очевидным образом полями смещений
и температуры. For an equilibrium
свободного тела без внешних нагрузок

...

§ 8. Variational formulations

Когда температура постоянна, уравнения термоупругости выглядят так же, как в механике.

...

Для переноса вариационного метода на термоупругость достаточно заменить in the (Lagrange) principle of minimum potential energy $\Pi(\boldsymbol{C})$ на Helmholtz free energy $A(\boldsymbol{C}, \Theta)$, а в принципе Reissner'a–Hellinger'a заменить $\widehat{\Pi}(\boldsymbol{\tau})$ на Gibbs free enthalpy (Gibbs function) $G(\boldsymbol{\tau}, \Theta)$.

The *Gibbs free energy* (the *Gibbs energy* or the *Gibbs function* or the *free enthalpy**) is a thermodynamic potential which measures the maximum of the reversible work produced with a constant temperature and pressure.

...

...

Более сложные вариационные постановки для нестационарных задач можно найти, например, в книге [89].

Bibliography

Шириной и глубиной описания термоупругости выделяются книги W. Nowacki [39, 40], книга E. Melan'a и H. Parkus'a [33] и монография H. Parkus'a [45]. C. Truesdell [60]

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* To distinguish it from the Helmholtz free energy.

внёс большой вклад в создание и распространение новых взглядов на термодинамику сплошной среды. Чёткое изложение основных законов есть у C. Teodosiu [56]. Методы расчёта температурных полей представлены у Н. М. Беляева и А. А. Рядно [89].

MAGNETOELASTICITY

Problems, which are solved using models of an elastic continua, usually include already known external loads. How to find these external loads is another problem. When electric currents flow in a body, the magnetic field creates a load. The magnetic field is changing because of body's deformation. If the field's sensitivity to deformations is high, then a problem of elasticity along with magnetism emerges.

§ 1. Electromagnetic field

Here is the summary of the theory of electromagnetism.

This theory describes the two vector fields, the electric one $\mathbb{E}(\boldsymbol{r}, t)$ and the magnetic one $\mathbb{B}(\boldsymbol{r}, t)$. What are vector $\mathbb{E}$ and pseudovector $\mathbb{B}$ can be figured out from the expression of the electromagnetic force (the Lorentz force) $\boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t, q)$ acting on a point-like charge — an infinitesimal in size particle that contains an electric charge q and moves with velocity $\dot{\boldsymbol{r}}$

$$\boldsymbol{F} = q\left(\mathbb{E} + \dot{\boldsymbol{r}} \times \mathbb{B}\right). \qquad (1.1)$$

In essence, the part of the electromagnetic force arising from an interaction with a moving charge — the magnetic force $q\dot{\boldsymbol{r}} \times \mathbb{B}$ — reveals the magnetic field $\mathbb{B}$, while the other part — the

electric force $q\mathbb{E}$ — reveals the electric field $\mathbb{E}$.

The acute question *"in which exactly frame of reference is velocity $\dot{\boldsymbol{r}}$ of a charged particle measured?"* leads to the special theory of relativity[*].

the classical concept about the existence of absolute space and time as the preferred frame of reference

a vacuum is space without matter, "free space"

...

continuous charge distribution

a charge density $\varrho(\boldsymbol{r}, t)$ is an electric charge per volume unit

$$dq = \varrho d\mathcal{V}, \quad q = \int_{\mathcal{V}} \varrho d\mathcal{V}$$

the density of electric current is the flux of electric charge $\boldsymbol{j}(\boldsymbol{r}, t) \equiv \varrho \dot{\boldsymbol{r}}$

в вакууме нет ϱ and $\boldsymbol{j}$

in a region with no charges ($\varrho = 0$) and no currents ($\boldsymbol{j} = \boldsymbol{0}$), such as a vacuum, ...

...

the ponderomotive force is the electromagnetic force (the Lorentz force) per volume unit

In a material continuum with charges and currents acts the volume "ponderomotive" force
$$\boldsymbol{f} = \varrho\mathbb{E} + \boldsymbol{j} \times \mathbb{B}. \qquad (1.2)$$

---

[*] **Albert Einstein**. Zur Elektrodynamik bewegter Körper. // Annalen der Physik, IV. Folge, Band 17, 1905. Seiten 891–921.

It is the differential (the local, the microscopic, the continual) version of (1.1).

...

Electromagnetic phenomena are usually described by the Maxwell's equations. Differential versions of these equations are

$$\boldsymbol{\nabla} \cdot \mathbb{E} = \frac{\varrho}{\varepsilon_0}$$

Gauss's theorem for e

$$(1.3^{\alpha})$$

$$\boldsymbol{\nabla} \times \mathbb{E} = -\dot{\mathbb{B}}$$

Maxwell–Faraday equ
(Faraday's law of ind
$$(1.3^{\beta})$$

$$\boldsymbol{\nabla} \cdot \mathbb{B} = 0$$

Gauss's theorem for r
$$(1.3^{\gamma})$$

$$c^2 \boldsymbol{\nabla} \times \mathbb{B} = \frac{\boldsymbol{j}}{\varepsilon_0} + \dot{\mathbb{E}}$$

Ampère's circuital la
Maxwell's term $\dot{\mathbb{E}}$ for
of electric charge
$$(1.3^{\delta})$$

....

speed of light in vacuum
$c = 299\ 792\ 458\ ^{\mathrm{m}}/_{\mathrm{s}}$

"electric constant", vacuum permittivity ε_0

$\varepsilon_0 \approx 8.8541878 \cdot 10^{-12}\ \mathrm{F} \cdot \mathrm{m}^{-1}$ (farads per metre)

"magnetic constant", vacuum permeability $\mu_0 = \dfrac{1}{\varepsilon_0 c^2}$

...

With μ_0 equation (1.3^{δ}) is sometimes written as
$\boldsymbol{\nabla} \times \mathbb{B} = \mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \dot{\mathbb{E}}$ or $\boldsymbol{\nabla} \times \mathbb{B} = \mu_0 \boldsymbol{j} + \dfrac{1}{c^2} \dot{\mathbb{E}}$.

...

The balance of electric charge — the continuity equation for electric

charge — mathematically follows from Maxwell's equations

$$\begin{aligned}
\boldsymbol{\nabla}\boldsymbol{\cdot}(1.3^{\delta}) &\Rightarrow c^2\,\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\boldsymbol{\nabla}\times\mathbb{B}\big) = \frac{\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{j}}{\varepsilon_0} + \boldsymbol{\nabla}\boldsymbol{\cdot}\dot{\mathbb{E}} \\
(1.3^{\alpha})^{\bullet} &\Rightarrow \boldsymbol{\nabla}\boldsymbol{\cdot}\dot{\mathbb{E}} = \frac{\dot{\varrho}}{\varepsilon_0} \\
& \qquad \boldsymbol{\nabla}\boldsymbol{\cdot}\big(\boldsymbol{\nabla}\times\boldsymbol{a}\big) = 0\ \forall \boldsymbol{a},\ \ \boldsymbol{j} \equiv \varrho\dot{\boldsymbol{r}}
\end{aligned} \Bigg\}$$

$$\Rightarrow\ \boldsymbol{\nabla}\boldsymbol{\cdot}\big(\varrho\dot{\boldsymbol{r}}\big) + \dot{\varrho} = 0. \tag{1.4}$$

An electromagnetic field acts upon a continuum with the vector of a ponderomotive force (1.2).

But there's also another expression of interaction, the bivalent "Maxwell" stress tensor

$$^2\boldsymbol{M} \equiv \varepsilon_0\Big(\mathbb{E}\mathbb{E} + c^2\,\mathbb{B}\mathbb{B} - \tfrac{1}{2}\big(\mathbb{E}\boldsymbol{\cdot}\mathbb{E} + c^2\,\mathbb{B}\boldsymbol{\cdot}\mathbb{B}\big)\boldsymbol{E}\Big). \tag{1.5}$$

It derives from (1.2) and Maxwell's equations

$$\begin{aligned}
(1.3^{\alpha}) &\Rightarrow \varrho = \varepsilon_0\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{E} \\
(1.3^{\delta}) &\Rightarrow \boldsymbol{j} = \varepsilon_0 c^2\,\boldsymbol{\nabla}\times\mathbb{B} - \varepsilon_0\dot{\mathbb{E}}
\end{aligned}$$

$$(1.2) \Rightarrow \boldsymbol{f} = \varepsilon_0\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{E}\mathbb{E} + \big(\varepsilon_0 c^2\,\boldsymbol{\nabla}\times\mathbb{B} - \varepsilon_0\dot{\mathbb{E}}\big)\times\mathbb{B} =$$
$$= \varepsilon_0\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{E}\mathbb{E} + \varepsilon_0 c^2\big(\boldsymbol{\nabla}\times\mathbb{B}\big)\times\mathbb{B} - \varepsilon_0\dot{\mathbb{E}}\times\mathbb{B}$$

$$\left.\begin{aligned}
\big(\mathbb{E}\times\mathbb{B}\big)^{\bullet} &= \dot{\mathbb{E}}\times\mathbb{B} + \mathbb{E}\times\dot{\mathbb{B}} \\
(1.3^{\beta}) &\Rightarrow \dot{\mathbb{B}} = -\boldsymbol{\nabla}\times\mathbb{E}
\end{aligned}\right\} \Rightarrow \dot{\mathbb{E}}\times\mathbb{B} = \big(\mathbb{E}\times\mathbb{B}\big.$$

Then

$$\boldsymbol{f} = \varepsilon_0\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{E}\mathbb{E} - \varepsilon_0 c^2\,\mathbb{B}\times\big(\boldsymbol{\nabla}\times\mathbb{B}\big) - \varepsilon_0\mathbb{E}\times\big(\boldsymbol{\nabla}\times\mathbb{E}\big) - \varepsilon_0\big(\mathbb{I}$$

For the symmetry with $\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{E}\mathbb{E}$, the null vector

$$(1.3^{\gamma}) \Rightarrow \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{B}\mathbb{B} = \boldsymbol{0},\ c^2\,\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{B}\mathbb{B} = \boldsymbol{0}$$

is added to $\boldsymbol{f}$.

...

$$(16.9, \S\,1.16) \;\Rightarrow\; \boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{a}\boldsymbol{a}) = (\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{a})\boldsymbol{a} + \boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{a}$$

$$(16.11, \S\,1.16) \;\Rightarrow\; \boldsymbol{\nabla}(\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{a}) = 2\,\boldsymbol{\nabla}\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{a}$$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{a}\,\boldsymbol{E}) = \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{E}\,(\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{a}) = \boldsymbol{\nabla}(\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{a}) = 2\,\boldsymbol{\nabla}\boldsymbol{a}\boldsymbol{\cdot}\boldsymbol{a}$$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}^2 M = \varepsilon_0\Big(\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbb{E}\mathbb{E}) + c^2\,\boldsymbol{\nabla}\boldsymbol{\cdot}(\mathbb{B}\mathbb{B}) - \tfrac{1}{2}\,\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\mathbb{E}\boldsymbol{\cdot}\mathbb{E}\,\boldsymbol{E} +$$

$$= \varepsilon_0\Big(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{E}\mathbb{E} + \mathbb{E}\boldsymbol{\cdot}\boldsymbol{\nabla}\mathbb{E} + c^2\,\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{B}\mathbb{B} + c^2\,\mathbb{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\mathbb{B} - $$
$$- \boldsymbol{\nabla}\mathbb{E}\boldsymbol{\cdot}\mathbb{E} - c^2\,\boldsymbol{\nabla}\mathbb{B}\boldsymbol{\cdot}\mathbb{B}\Big)$$

...

§2. Electromagnetic waves

To derive wave equations

$$\boldsymbol{\nabla}\times(1.3^{\beta}) \;\Rightarrow\; \boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbb{E}\big) = -\boldsymbol{\nabla}\times\dot{\mathbb{B}}$$

$$\boldsymbol{\nabla}\times(1.3^{\delta}) \;\Rightarrow\; \boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbb{B}\big) = \boldsymbol{\nabla}\times\left(\frac{\boldsymbol{j}}{\varepsilon_0 c^2} + \frac{\dot{\mathbb{E}}}{c^2}\right)$$

$$\boldsymbol{\nabla}\times\dot{\mathbb{E}} = \big(\boldsymbol{\nabla}\times\mathbb{E}\big)^{\bullet}, \; (1.3^{\beta}) \;\Rightarrow\; \boldsymbol{\nabla}\times\dot{\mathbb{E}} = -\ddot{\mathbb{B}}$$

$$\boldsymbol{\nabla}\times\dot{\mathbb{B}} = \big(\boldsymbol{\nabla}\times\mathbb{B}\big)^{\bullet}, \; (1.3^{\delta}) \;\Rightarrow\; \boldsymbol{\nabla}\times\dot{\mathbb{B}} = \frac{\boldsymbol{j}^{\bullet}}{\varepsilon_0 c^2} + $$

$$\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbb{E}\big) = \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{E} - \triangle\mathbb{E}$$
$$\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbb{B}\big) = \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{B} - \triangle\mathbb{B}$$

$$\triangle\mathbb{E} - \boldsymbol{\nabla}\underbrace{\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{E}}_{\varrho/\varepsilon_0\,(1.3^{\alpha})} = \frac{\ddot{\mathbb{E}}}{c^2} + \frac{\boldsymbol{j}^{\bullet}}{\varepsilon_0 c^2}$$

$$\triangle\mathbb{B} - \boldsymbol{\nabla}\underbrace{\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{B}}_{0\,(1.3^{\gamma})} = \frac{\ddot{\mathbb{B}}}{c^2} - \frac{\boldsymbol{\nabla}\times\boldsymbol{j}}{\varepsilon_0 c^2}$$

...

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\boldsymbol{\nabla}\times\boldsymbol{a}\big) = 0 \;\; \forall\boldsymbol{a}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla} = \boldsymbol{0}, \;\; \boldsymbol{\nabla}\times\boldsymbol{\nabla}\alpha = \boldsymbol{0} \;\; \forall\alpha$$

vector potential $\mathbb{A}$

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{B} = 0 \;\; (1.3^{\gamma}) \;\Leftrightarrow\; \mathbb{B} = \boldsymbol{\nabla}\times\mathbb{A}$$

potential $\mathbb{A}$ is not unique and has gauge freedom $\mathbb{A} + \boldsymbol{\nabla} a$

$$\mathbb{B} = \boldsymbol{\nabla}\times\big(\mathbb{A}+\boldsymbol{\nabla} a\big) \;\Leftrightarrow\; \boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{B} = 0 \quad (1.3^{\gamma})$$

scalar potential ϕ

$$\boldsymbol{\nabla}\times\mathbb{E} = -\dot{\mathbb{B}} \;\;(1.3^{\beta}) \;\Rightarrow\; \boldsymbol{\nabla}\times\mathbb{E} = -\boldsymbol{\nabla}\times\big(\dot{\mathbb{A}}+\boldsymbol{\nabla}\dot{a}$$
$$\Rightarrow\; \boldsymbol{\nabla}\times\big(\mathbb{E}+\dot{\mathbb{A}}+\boldsymbol{\nabla}\dot{a}\big) = \mathbf{0} \;\Rightarrow\; -\boldsymbol{\nabla}\phi = \mathbb{E}+\dot{\mathbb{A}}+\boldsymbol{\nabla}\dot{a}$$
$$\Rightarrow\; \mathbb{E} = -\boldsymbol{\nabla}\big(\phi + \dot{a}\big) - \dot{\mathbb{A}}. \quad (2.1)$$

And

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{E} = \frac{\varrho}{\varepsilon_0} \;\;(1.3^{\alpha}) \;\Rightarrow\; \begin{array}{l} -\triangle\big(\phi + \dot{a}\big) - \boldsymbol{\nabla}\boldsymbol{\cdot}\dot{\mathbb{A}} = \frac{\varrho}{\varepsilon} \\[2mm] -\triangle\phi - \boldsymbol{\nabla}\boldsymbol{\cdot}\big(\dot{\mathbb{A}} + \boldsymbol{\nabla}\dot{a}\big) = \frac{\varrho}{\varepsilon} \end{array}$$
$$(2.2)$$

$$\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbb{A}\big) = \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{A} - \triangle\mathbb{A}$$

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla} a - \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\nabla} a = \mathbf{0} \qquad \text{(partial}$$
derivatives of a smooth function commute)

$$c^2\,\boldsymbol{\nabla}\times\mathbb{B} = \frac{\boldsymbol{j}}{\varepsilon_0} + \dot{\mathbb{E}} \;\;(1.3^{\delta}) \;\Rightarrow$$
$$\Rightarrow\; c^2\,\boldsymbol{\nabla}\times\big(\boldsymbol{\nabla}\times\mathbb{A}\big) = \frac{\boldsymbol{j}}{\varepsilon_0}-\boldsymbol{\nabla}\big(\dot{\phi}+\ddot{a}\big)-\ddot{\mathbb{A}} \;\Rightarrow$$
$$\Rightarrow\; c^2\big(\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{A}-\triangle\mathbb{A}\big) = \frac{\boldsymbol{j}}{\varepsilon_0}-\boldsymbol{\nabla}\dot{\phi}-\boldsymbol{\nabla}\ddot{a}-\ddot{\mathbb{A}}.$$
$$(2.3)$$

With a gauge freedom it's possible to simplify the wave equations (2.3) and (2.2), assuming that

$$-\boldsymbol{\nabla}\ddot{a} = \boldsymbol{\nabla}\dot{\phi} + c^2\,\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{A} \;\Rightarrow\; \ddot{a} = -\dot{\phi} - c^2\,\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{A}$$
$$-\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\dot{\mathbb{A}} + \boldsymbol{\nabla}\dot{a}\big) = \frac{1}{c^2}\ddot{\phi} \;\Rightarrow\; \dot{\phi} = -c^2\,\boldsymbol{\nabla}\boldsymbol{\cdot}\big(\mathbb{A} +$$
$$\Rightarrow\; c^2\,\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{A} + c^2\triangle a - c^2\,\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbb{A} = \ddot{a},$$

finally presenting as the homogeneous wave equation for a

$$\ddot{a} = c^2\triangle a. \qquad (2.4)$$

The more popular condition is even more stiff

$$\triangle a = 0 \;\Rightarrow\; \ddot{a} = 0,\;\; \dot{\phi} + c^2 \boldsymbol{\nabla}\cdot\mathbb{A} = 0.$$

This Lorenz gauge condition gives the same effect, being just the particular — the harmonic — case of (2.4).

Following from (2.2) and (2.3) with condition (2.4), equations of electromagnetic waves in the potential formulation are

$$
\begin{aligned}
-\triangle\phi + \frac{1}{c^2}\ddot{\phi} &= \frac{\varrho}{\varepsilon_0}, \\
-c^2\triangle\mathbb{A} &= \frac{\boldsymbol{j}}{\varepsilon_0} - \ddot{\mathbb{A}}.
\end{aligned}
\tag{2.5}
$$

...

§ 3. Electrostatics

Рассмотрение этого вопроса полезно и для последующего описа́ния магнетизма. В статике

$$\dot{\boldsymbol{r}} = \boldsymbol{0}\;\Rightarrow\;\mathbb{B} = \boldsymbol{0}$$

...

The volume "ponderomotive" force, с которой электростатическое поле действует на среду ...

...

Maxwell stress tensor (1.5) in electrostatics is

$${}^2\boldsymbol{M} = \varepsilon_0\left(\mathbb{E}\mathbb{E} - \frac{1}{2}\,\mathbb{E}\cdot\mathbb{E}\boldsymbol{E}\right)$$

...

§ 4. Dielectrics

Начнём с рассмотрения электростатического поля

...

В диэлектриках нет свободных зарядов: charge density $\varrho = 0$. Здесь вводится плотность дипольного момента

...

§ 5. Magnetostatics

Если поле (а с ним ...)

...

§ 6. Magnetics

Выяснив законы магнитостатики в общем случае, обратимся к веществу — некий опыт у нас уже есть в электростатике диэлектриков.

Начнём с рассмотрения

...

...

Насколько соответствует поведение реальных материалов представленным здесь формальным построениям — сей вопрос is out of scope этой книги.

§ 7. Magnetic rigidity

В электротехнике распространены обмотки всевозможной формы, в кото-

рых провод намотан так, что образуется некое массивное тело. Такие обмотки есть в статоре генератора автомобиля (да и в роторе), в больших промышленных электромагнитах и в магнитных системах установок "токама́к" (**то**роидальная **ка**мера с **ма**гнитными **к**атушками) для управляемого термоядерного синтеза — примеров много. Сочетание токопровода и изоляции образует периодический композит, и одной из главных нагрузок для него является пондеромоторная магнитная сила. Рассчитывая деформации и механические напряжения в обмотке, начинают с определения магнитных сил. Поскольку распределение токов задано известной геометрией проводов, достаточно интегрирования по формуле Био-Савара (**??**). Термин "магнитоупругость" при этом неуместен, так как задачи магнитостатики и упругости решаются раздельно.

Однако при деформации обмотки меняются и поле $\boldsymbol{j}$, и вызываемое им поле $\mathbb{B}$. Объёмная сила становится равной

$$\boldsymbol{f} = (\boldsymbol{j} \times \mathbb{B})_0 + \ldots \qquad (7.1)$$

Подчёркнутое слагаемое соответствует недеформированному состоянию. Обусловленное деформацией изменение объёмной силы линейно связано с малым смещением $\boldsymbol{u}$, поэтому матричное (после дискретизации) уравнение в смещениях можно представить в виде

$$(C + C_m)u = \overset{\circ}{F}. \qquad (7.2)$$

К "обычному" оператору линейной упругости C добавилась магнитная жёсткость C_m, $\overset{\circ}{F}$ — силы в недоформированном состоянии.

Добавка C_m пропорциональна квадрату тока и может стать весьма существенной в магнитных системах с сильным полем. Учёт её необходим и при недостаточной величине C. В номинальном режиме конструкция может держать нагрузку, но дополнительная нагрузка неблагоприятного направления может оказаться "невыносимой".

The role магнитной жёсткости is very important в задачах устойчивости. Матрица C_m симметрична, так как магнитные силы are potential. The critical parameters can be found using статический метод Euler'a.

Как иллюстрацию рассмотрим простую задачу о балке в продольном магнитном поле. Балка располагается на декартовой оси z, концы $z=0$ и $z=l$ закреплены, магнитная индукция $\mathbb{B} = B\boldsymbol{k} = $ constant, по балке течёт постоянный (по величине) ток I. В классической модели балки при равных жёсткостях на изгиб для прогиба $\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2$ легко получить следующую постановку:

...

Вводя комплéксную комбинацию $u \equiv u_1 + iu_2$, будем иметь

...

с общим решением

...

Подстановка в граничные условия приводит к однородной системе для постоянных A_k. Приравняв нулю определитель, придём к характеристическому уравнению

...

Наименьший положительный корень $x = 3.666$, так что критическая комбинация параметров такова:

$$\left(IBl^3/a\right)_* = 394.2.$$

Поле $\mathbb{B}$ в этом решении считалось внешним и не варьировалось. Но если собственное поле тока в стержне сравнимо с $\mathbb{B}$, то решение измéнится and усложнúтся.

Bibliography

Основы электродинамики хорошо изложены во многих книгах [90, 85], но для приложений в механике выделяется книга И. Е. Тамма [96]. Растёт список литературы по связанным задачам электромагнетизма и упругости [47, 49]. Как введение в эту область может быть полезна монография В. Новацкого [41].

PERTURBATION METHODS (ASYMPTOTIC METHODS)

Of approximate approaches to the analysis of nonlinear systems, perturbation methods are applied the most often.

§1. Asymptotic decompositions

Until now, the arguments of a function were only coordinates and time. Dimensions of a body, elastic moduli, ranges of action changes in space and time — all these parameters were considered known. All asymptotic methods are based on studying how the solution depends on parameters.

A decomposition of perturbation usually consists of the first two terms. If a problem with unknown u contains parameter a, then, assuming

$$a = a_0 + \chi a_1,$$

a solution is sought as series

$$u = u_0 + \chi u_1 + \chi^2 u_2 + \ldots \qquad (1.1)$$

Additional argument χ is called a formal small parameter.

A decomposition of perturbation may diverge, but at the same time it can be more useful description of the solution than a representation that converges uniformly and absolutely.

Decomposition (1.1) looks like a common power series. However, pertur-

bation methods use another approach: series are considered as asymptotic with the convergence $\chi \to 0$, not $n \to \infty$ (n shows how many terms to hold).

$$\phi = \sum_{k=0}^{\infty} \phi_k(\chi), \quad \lim_{\chi \to 0} \frac{\phi - \sum_{k=0}^{n} \phi_k}{\phi_n} = 0.$$

(1.2)

In other words, the remainder of series is a higher order infinitesimal in comparison with the last withheld term. Obviously, a power series (1.1) also converges as an asymptotic series.

But the decomposable unknowns usually also depend on the "main" arguments — on coordinates and time. The convergence for $\chi \to 0$ must be uniform across these "main" arguments — this is a requirement for effective use of asymptotic decompositions. For example
$$\sin(1+\chi)t = \sin t + \chi t \cos \chi t - \tfrac{1}{2}\chi^2 t^2 \sin \chi t + \dots$$
doesn't satisfy the requirement of uniformity, because with $t \to \infty$ subsequent terms prevail over previous ones.

Why asymptotic methods are so attractive? As example take a solution of the equation
$$f(u, \chi) = 0.$$

Substituting   decomposition   (1.1) into (1.2) and equating coefficients at the same degrees of χ, we get

$$f(u_0, 0) = 0,$$
$$\left(\partial_u f\right)_0 + u_1\left(\partial_\chi f\right)_0 = 0,$$
$$\left(\partial_u f\right)_0 u_2 + \tfrac{1}{2}\left(\partial_u^2 f\right)_0 u_1^2 + \tfrac{1}{2}\left(\partial_\chi^2 f\right)_0 + \left(\partial_u \partial_\chi f\right)_0 u_1 =$$
$$\dots$$

If the problem for u_0 is uniquely solvable at the first step, then subsequent steps will give corrections $u_1, u_2, \dots$ Small corrections are barely important and u_0 is enough, but then the asymptotic analysis disappears, since formally small terms in (1.2) are just discarded. However, it happens that these corrections contain in self some important information, absent in u_0 — then they play the main role. It's worth mentioning that all corrections follow from the linear problem with the same operator $\left(\partial_u f\right)_0$.

But all non-trivial and effective solutions are obtained in other ways, non-uniqueness of the solution at the first step characterizes them. About this — in the next sections.

And yes, asymptotic methods change the initially complex formulation of the problem (1.2) to a simpler one. Essential is that this does not happen by "simple" discarding of terms, but quite correctly — equalities remain equalities. However, the convergence is not proven, so there is no complete mathematical accuracy.

In previous chapters, asymptotic problems have already arisen. The linear theory follows from the nonlinear theory thru an asymptotic decomposition by the value of load (§ 4.1). The momentless theory derives (accurate to edge effects) from the moment (micropolar) theory (chapter 5) when the "moment" stiffnesses are approaching infinity. In the thermoelasticity (chapter 6), the use of the heat equation (4.1, § 6.4) instead of the whole entire balance of energy needs to be proved by asymptotic methods.

To a certain degree the introduction of a small parameter $\chi \to 0$ is the weakness of all asymptotic approaches. A protestation like "infinitesimal parameters do not exist, all quantities are finite" is barely constructive here. The more relevant question is: what is a small parameter? Usually a problem is reformulated in "dimensionless" quantities, then that "dimensionless" parameter is taken as a small parameter χ, which turns out to be small. But another way is possible too: if it's known that some parameter ω influences the solution only a little, then, redesignating it as $\chi\omega$, do asymptotic analysis for $\chi \to 0$.

Surely, these are not "laws" of asymptotics, not recommendations, but only considerations. There's no common theory of asymptotic methods, their application is to some extent an art.

A deeper description of asymptotic methods is written in the books by Ali Hasan Nayfeh [106, 107].

§ 2. Splitting in a linear algebraic system

This simple case illustrates asymptotic methods well.

A linear system is considered

$$C_{ij} u_j = f_i, \ \ C_{ij} = C_{ij}^{(0)} + \chi C_{ij}^{(1)} \quad (2.1)$$

с матрицей C_{ij} и столбцами неизвестных u_j и нагрузок f_i. Процесс построения асимптотического решения определяется тем, вырождена матрица $C_{ij}^{(0)}$ или нет. Three cases are possible.

1° $\det C_{ij}^{(0)} \neq 0$. Однородная задача

$$C_{ij}^{(0)} u_j = 0 \qquad (2.2)$$

имеет лишь тривиальное (нулевое) решение. Матрица $C_{ij}^{(0)}$ обратима, неоднородная задача всегда однозначно реши́ма. Решение строится так:

$$u_j = \dots \qquad (2.3)$$

...

2° $\det C_{ij}^{(0)} = 0$

...

3° $\det C_{ij}^{(0)} = 0$

...

§ 3. Poincaré method

This method associated with the name Jules Henri Poincaré is widely known in the theory of nonlinear oscillations. It is intended, in particular, to determine periodic solutions of equation

$$\ddot{u} + u = \chi f(u, \dot{u}) \qquad (3.1)$$

....

§ 4. Van der Pol averaging method

Опять рассматривается уравнение (3.1), но теперь ищутся не только периодические решения. Вводится фазовая плоскость

...

Процедура осреднения применяется в разных разделах теории упругости (тонкие тела, композиты и др.). Осреднение вне асимптотических методов приводит обычно к незамкнутым системам уравнений; для замыкания приходится принимать некие гипотезы, что умаляет убедительность теории. Иная ситуация в асимптотике: условия разрешимости для поправочных членов с необходимостью приводят к соответствующим интегральным соотношениям.

§ 5. Coalescence
of asymptotic
decompositions

Основоположник метода сращивания внешних и внутренних асимптотических разложений — Ludwig Prandtl. Рассматривая течение вязкой жидкости, он заметил, что влияние малой вязкости локализовано у края — в тонком слое на краю. Вдали́ от края жидкость ведёт себя как идеальная. Одни и те же уравнения Navier–Stokes по-разному упрощаются вдали от края и около него [...]

Метод сращивания состоит из трёх процедур: построения внешнего разложения, построения внутренних разложений и сращивания внешнего разложения с внутренними. Метод предназначен для дифференциальных уравнений с малым параметром при старших производных. Вдали от края решение меняется плавно, формально малые члены можно отбросить, уравнение имеет пониженный порядок — всё это характерно для внешнего разложения. У края наоборот: решение меняется быстро, первостепенную роль играют старшие производные, хотя имеют малые коэффициенты. Но внешнее и внутреннее разложения — это разные формы одного решения, они должны быть состыкованы процедурой сращивания. Рассмотрим пример.

Задача о прогибе $u(x)$ натянутой струны с закреплёнными концами под действием равномерно распределённой нагрузки может быть поставлена так:

...

§ 6. Multiple-scale analysis (method of multiple scales)

Этот метод привлекателен, естественен и — как написано у Ali Hasan Nayfeh

...

§ 7. Equations with slowly varying parameters

Рассмотрим гармонический осциллятор, собственная частота которого медленно меняется во времени

...

§ 8. Thin bodies

Задачи теории упругости часто ставятся для тонких тел — стержней, пластин и оболочек. Таковы многие элементы конструкций, но и в природе вне человека тонкие тела встречаются довольно часто.

Решение задач упругости для тонких тел многие десятилетия основывалось на неких гипотезах о распределении решения по толщине и о по-

рядках одних неизвестных относительно других. Построенные так теории сыграли большу́ю роль в практике инженерных расчётов. Однако, им не хватало логической стройности и убедительности, их хотелось обосновать, уточнить — а в последнее время и уничтожить (в связи́ с появлением великолепных компьютеров). Но открытие асимптотического расщепления прояснило картину: в тонком теле трёхмерная задача расщепляется на задачи меньшей размерности. Классические теории тонких тел получили и подтверждение, и развитие.

Рассмотрим задачу о кручении из ...

...

Bibliography

Ali Hasan Nayfeh's book [107] is an excellent introduction to perturbation methods (asymptotic methods).
Всё разнообразие асимптотических методов представлено в монографиях ...

chapter 9
RODS

§1. Initial concepts

R od is a thin long body. It is thought of (and modeled) as a spatial curve — the axis of rod, coated with a material (a figure).

The axis of rod is described as a curve by parameterizing the location vector of points of a curve. This is a morphism (a function) of one variable coordinate s,

$$\boldsymbol{r} = \boldsymbol{r}(s). \qquad (1.1)$$

Material coating gives at each rod's point a plane figure, perpendicular to the axis — normal section $\Omega(s)$.

...

$\boldsymbol{c} = \boldsymbol{c}(t)$ is a parametric curve parameterized by parameter t. If $dt \neq d\ell$, then a parameterization is not natural. For the natural parameterization $dt = d\ell$, where

$$d\ell = \sqrt{\left(dq^1\right)^2 + \left(dq^2\right)^2 + \left(dq^3\right)^2}.$$

Many different functions draw the same curve. But among various parametrizations of a curve, the parametrization by the arc length is special, it is also called the *natural parametrization*.

The length of an infinitesimal piece of a curve is described by the Πυθαγόρας-formula

$$dℓ = \sqrt{dx^2 + dy^2 + dz^2}$$

where $dx \equiv dq^1$, $dy \equiv dq^2$, $dz \equiv dq^3$ are infinitesimal changes of coordinates. $dℓ$ is called the differential length, that is the length of an almost straight very small piece of a curve.

$\boldsymbol{c}(s)$ is a parametric curve parameterized by the arc length (the natural parametrization), its derivative by the arc length parameter is denoted as $\boldsymbol{c}' \equiv \dfrac{d\boldsymbol{c}}{ds}$.

If the arc length (natural) parametrization $\boldsymbol{r} = \boldsymbol{r}(s)$ is used, then the length of derivative $\boldsymbol{r}'(s) \equiv \dfrac{d\boldsymbol{r}(s)}{ds}$ (of the tangent vector) is always equal to the one unit of length:

$$\boldsymbol{r}(s) = q_i(s)\,\boldsymbol{e}_i(s) = q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 + q_3\boldsymbol{e}_3,$$

$$\boldsymbol{r}'(s) = \frac{d\boldsymbol{r}(s)}{ds} = \frac{dq_i(s)}{ds}\,\boldsymbol{e}_i(s) = q_i'(s)\,\boldsymbol{e}_i(s),$$

$$\|\boldsymbol{r}'(s)\|^2 \equiv \boldsymbol{r}'(s) \boldsymbol{\cdot} \boldsymbol{r}'(s) = q_i'(s)\,\boldsymbol{e}_i \boldsymbol{\cdot} q_j'(s)\,\boldsymbol{e}_j = q_i'(s)$$

$$ds = \sqrt{(dq_1)^2 + (dq_2)^2 + (dq_3)^2} \;\;\Rightarrow\;\; ds^2 = \sum_{i=1}^{3}(dq_i(s)$$

$$\|\boldsymbol{r}'(s)\|^2 = \frac{\sum_{i=1}^{3}(dq_i(s))^2}{(ds)^2} = 1 \;\;\Rightarrow\;\; \|\boldsymbol{r}'(s)\| = 1.$$

...

In each section we select two perpendicular axes x_α with co-directed unit vectors $\boldsymbol{e}_\alpha$ $(\alpha = 1, 2)$. The reason of selection for all rod sections is the same,

for example, the main axes of inertia of the section are chosen everywhere.

Th
axis and the init

When vector $\mathring{\boldsymbol{e}}$
tangent to the init
it is written as $\mathring{\boldsymbol{r}}' \equiv$

Vector $\boldsymbol{e}_3 \equiv \boldsymbol{k}$ is
gent (tangentially)

Together with the unit vector, tangent to the actual axis

$$\boldsymbol{r}' \equiv \frac{d\boldsymbol{r}}{ds} \equiv \boldsymbol{r}_{\partial s} \equiv \boldsymbol{e}_3 \equiv \boldsymbol{k},$$

we'll get for each s a triple of mutually perpendicular unit vectors.

The curvature and the torsion of the rod's axis can be described by vector $\boldsymbol{\psi} = \psi_j \boldsymbol{e}_j$:

$$\boldsymbol{e}_j' = \boldsymbol{\psi} \times \boldsymbol{e}_j, \quad \boldsymbol{\psi} = \frac{1}{2} \boldsymbol{e}_j \times \boldsymbol{e}_j'. \quad (1.2)$$

For a cylindrical (prismatic) rod $\boldsymbol{\psi} = \boldsymbol{0}$.

However, (1.2) is only an initial concept of vector $\boldsymbol{\psi}$ as of geometric features. Further in §2, after adopting the material structure of a rod, a concept of $\boldsymbol{\psi}$ will change.

Moreover, at each point of the rod's axis, thought of as a curve, there's also another triple of mutually perpendicular unit vectors, the one with the normal and the binormal vectors.

Tangent $\boldsymbol{T}$, normal $\boldsymbol{N}$ and binormal $\boldsymbol{B}$ vectors, together called the *Frenet–Serret frame*, are defined as:

✓ $\boldsymbol{T}$ is the unit vector tangent to a curve. The length of the tangent vector is always one unit, if the natural (arc length) parametrization of a curve is used. The tangent vector points to where a curve continues further.

✓ $\boldsymbol{N}$ is the normal unit vector, the derivativeof $\boldsymbol{T}$ by the curve's parameter (for instance, the arc length of a curve). The normal vector is always perpendicular to the tangent vector and points towards the center of curvature. It is divided by its length $\|\boldsymbol{N}\|$ to be the one unit long.

✓ $\boldsymbol{B}$ is the binormal unit vector, the "×"-product ("cross product") of $\boldsymbol{T}$ and $\boldsymbol{N}$, $\boldsymbol{B} \equiv \boldsymbol{T} \times \boldsymbol{N}$.

The Frenet–Serret formulas describe the derivatives of tangent, normal and binormal unit vectors through relations with each other.

$$\frac{d\boldsymbol{T}}{ds} = \kappa\boldsymbol{N},$$
$$\frac{d\boldsymbol{N}}{ds} = -\kappa\boldsymbol{T} + \tau\boldsymbol{B},$$
$$\frac{d\boldsymbol{B}}{ds} = -\tau\boldsymbol{N},$$

where d/ds denotes the derivative by the arc length, κ is the curvature and τ is the curve's torsion. The associated collection — $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, κ, τ — is called the Frenet–Serret apparatus.

Two scalars κ and τ effectively define the curvature and the torsion of a space curve. Intuitively, the curvature

measures the deviation of a curve from a straight line, while the torsion measures the deviation of a curve from being planar.

Two functions $\kappa(s)$ and $\tau(s)$ completely define the geometry of a curve, because they are the coefficients of a system of ordinary differential equations for $\boldsymbol{T}$, $\boldsymbol{N}$ and $\boldsymbol{B}$. Knowing $\boldsymbol{T}(s)$, we'll obtain $\boldsymbol{r}(s)$ via an integration of what? with an accuracy up to a constant rigid movement without deformations.

...

$$\boldsymbol{T} = \frac{d\boldsymbol{r}}{ds} \quad \text{or} \quad \boldsymbol{T} = \boldsymbol{r}'$$

The derivative of $\boldsymbol{T}$ consists of two multipliers — the curvature κ and the unit normal vector $\boldsymbol{N}$

$$\frac{d\boldsymbol{T}}{ds} = \kappa\boldsymbol{N} \quad \text{or} \quad \boldsymbol{T}' = \kappa\boldsymbol{N}$$

Curvature κ is equal to the magnitude (length) of vector $\boldsymbol{N}$ (the derivative of vector $\boldsymbol{T}$, the second derivative of location vector $\boldsymbol{r}$)

$$\kappa = \|\boldsymbol{N}\| = \|\boldsymbol{T}'\| = \left\|\frac{d\boldsymbol{T}}{ds}\right\| = \|\boldsymbol{r}''\| = \left\|\frac{d^2\boldsymbol{r}}{ds^2}\right\|$$

Vector $\boldsymbol{N}$ itself is divided by its length thus its length is equal to the one unit.

$$\boldsymbol{N} = \frac{\frac{d\boldsymbol{T}}{ds}}{\left\|\frac{d\boldsymbol{T}}{ds}\right\|} \quad \text{or} \quad \boldsymbol{N} = \frac{\boldsymbol{T}'}{\|\boldsymbol{T}'\|}$$

The radius of curvature is the reciprocal of curvature.

$$\frac{1}{\kappa}\frac{d\boldsymbol{T}}{ds} = \boldsymbol{N} \quad \text{or} \quad \frac{1}{\kappa}\boldsymbol{T}' = \boldsymbol{N}$$

...

$$\kappa \geq 0$$

The Frenet–Serret frame is defined only if the curvature is nonzero ($\kappa > 0$), it is not defined if $\kappa = 0$.

A line with the nonzero curvature $\kappa \neq 0$ is considered a curve.

The zero curvature implies that a line is straight, and it lies in a plane, making the torsion equal to zero too ($\tau = 0$).

...

T always has the unit magnitude (length). Since the length of T is constant, then N — the derivative of T and the second derivative of location vector r — is always perpendicular to T

$$T \cdot T = 1 \;\Rightarrow\; T' \cdot T = 0 \;\Rightarrow\; N \cdot T = 0.$$

Vectors of the Frenet–Serret frame make an orthonormal basis f_i:

$$f_1 = T,\;\; f_2 = N,\;\; f_3 = B.$$

The location vector in the Frenet–Serret basis

$$r(s) = q_j(s)\,f_j(s) = q_1(s)\underbrace{f_1(s)}_{T} + q_2(s)\underbrace{f_2(s)}_{N} + q_3(s)$$

The tensor version of the Frenet–Serret formulas

$$f_i' = {}^2d \cdot f_i. \qquad (1.3)$$

The Frenet–Serret formulas written using the matrix notation

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Tensor $^2\boldsymbol{d}$ is skew-symmetric, so it can be represented via the companion pseudovector (§ 1.7). This pseudovector is known as the Darboux vector.

$$\boldsymbol{D} = \tau\boldsymbol{T} + \kappa\boldsymbol{B}$$

$$\boldsymbol{D} = \tau\boldsymbol{T} + 0\boldsymbol{N} + \kappa\boldsymbol{B}$$

With the Darboux vector, the Frenet–Serret formulas turn into the following:

$$\begin{aligned} \boldsymbol{T}' &= \boldsymbol{D} \times \boldsymbol{T}, \\ \boldsymbol{N}' &= \boldsymbol{D} \times \boldsymbol{N}, \\ \boldsymbol{B}' &= \boldsymbol{D} \times \boldsymbol{B} \end{aligned}$$

or as the vector version of (1.3)

$$\boldsymbol{f}_i' = \boldsymbol{D} \times \boldsymbol{f}_i. \qquad (1.4)$$

The Darboux vector is the angular velocity vector of the Frenet–Serret frame.

...

Approximate applied theories of rods like the "strength of materials" use such concepts as internal force $\boldsymbol{Q}$ and internal moment $\boldsymbol{M}$. The following relations connect them with the stress tensor

$$\boldsymbol{Q}(s) = \int_\Omega \boldsymbol{t}_{(\boldsymbol{k})}\, d\Omega = \int_\Omega \boldsymbol{k} \cdot \boldsymbol{\tau}\, d\Omega, \quad (1.5)$$

$$\boldsymbol{M}(s) = \int_\Omega \boldsymbol{x} \times \boldsymbol{t}_{(\boldsymbol{k})}\, d\Omega = \int_\Omega \boldsymbol{x} \times \boldsymbol{k} \cdot \boldsymbol{\tau}\, d\Omega.$$

$$(1.6)$$

$$\boldsymbol{k} \equiv \boldsymbol{e}_3, \quad \boldsymbol{x} = x_\alpha \boldsymbol{e}_\alpha, \quad \alpha = 1, 2$$

...

These thoughts about geometry and about mechanics — in particular, about

internal force (1.5) and internal moment (1.6) — concern only some unique single configuration of a rod. It's meaningless to continue these thoughts, because in reality plane and normal sections do not remain plane and normal after deforming.

synonyms fo

warp, warping =
deformation, strain
ние (distort, disto
перекосить, перека
покоробить, коро
become twisted or

And the addition of some assumptions-hypotheses to the model, like "there are no warping (deplanations)" and initially plane sections remain plane* introduces essential contradictions with reality. Enough to recall just one fact that without deplanations it's impossible to acceptably describe the torsion of a rod (and not only torsion).

The very reasonable approach to modeling deformations of an elastic rod consists in asymptotic splitting of

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* The two very popular beam models exist which postulate the hypothesis about the absence of deplanations. In the Euler–Bernoulli beam theory, shear deformations are neglected, plane sections remain plane and perpendicular to the axis. In the Timoshenko beam theory there's a constant transverse shear along the section, so plane sections still remain plane, but they are no longer perpendicular to the axis.

the three-dimensional problem with a small thickness. But for a complex asymptotic procedure it would be much simpler to have whatever solution version aforehand. And the direct approach, when the one-dimensional model of a rod is a material line, gives such a version.

The primary question for building the one-dimensional model: what degrees of freedom — besides translation — do the particles of a material line possess?

It is known that rods are sensitive to the moment loads. And the presence of moments among generalized forces indicates the presence of rotational degrees of freedom. Therefore as the one-dimensional model of a rod it is reasonable to take the Cosserat line — it consists of infinitesimal absolutely rigid bodies. However, another new degrees of freedom may also appear — as for thin-walled rods, described in the dedicated chapter (chapter 10).

In the mechanics of a continuous elastic media, the place of rods is special. At first, the moments play here the main role, not the role of the small additions as in a three-dimensional Cosserat continuum. At second, the rods can be used to test models with the additional degrees of freedom, before the presence of these degrees will be researched for the three-dimensional models.

The following section presents and describes a simple one-dimensional Cosserat-like moment model.

§ 2. Kinematics of Cosserat lines

Model described further is a simplified version of chapter 5.

There's no more a triple of material coordinates q_i, but the only one — s. It may be the arc length parameter in the initial configuration. The motion of a particle over time is described by the location vector $\boldsymbol{r}(s,t)$ and the rotation tensor $\boldsymbol{O}(s,t)$. Linear and angular (11.8, § 1.11) velocities of a rod's particle are

$$\dot{\boldsymbol{r}} = \boldsymbol{v}, \qquad (2.1)$$

$$\dot{\boldsymbol{O}} = \boldsymbol{\omega} \times \boldsymbol{O} \;\Leftrightarrow\; \boldsymbol{\omega} = -\tfrac{1}{2}(\dot{\boldsymbol{O}} \cdot \boldsymbol{O}^\mathsf{T})_\times. \qquad (2.2)$$

Deformation of a rod as a Cosserat line is defined by two vectors

$$\boldsymbol{\Gamma} \equiv \boldsymbol{r}' - \boldsymbol{O} \cdot \overset{\circ}{\boldsymbol{r}}{}', \qquad (2.3)$$

$$\boldsymbol{\kappa} \equiv -\tfrac{1}{2}(\boldsymbol{O}' \cdot \boldsymbol{O}^\mathsf{T})_\times \;\Leftrightarrow\; \boldsymbol{O}' = \boldsymbol{\kappa} \times \boldsymbol{O}. \qquad (2.4)$$

$$\left(\overset{\circ}{\boldsymbol{r}}(s) \equiv \boldsymbol{r}(s,0) \right)$$

$$\|\boldsymbol{e}_3\| = \|\overset{\circ}{\boldsymbol{e}}_3\| = 1 = \mathsf{constant}$$

(2.3) and (2.4) are really the deformation vectors, it follows from their equality to zero on movements of a body as a rigid whole (.... add some equation(s) here

describing movements as a rigid whole
.....).

Further we will clarify the idea of the first deformation vector $\mathbf{\Gamma}$. Without a loss of universality, the parameter s is the initial arc length, the third initial basis vector $\mathring{\boldsymbol{e}}_3$ is directed along the tangent in the initial configuration: $\mathring{\boldsymbol{e}}_3 = \mathring{\boldsymbol{r}}'$. And then

$$\mathbf{\Gamma} \equiv \boldsymbol{r}' - \boldsymbol{O} \boldsymbol{\cdot} \mathring{\boldsymbol{r}}', \ \ \mathring{\boldsymbol{r}}' = \mathring{\boldsymbol{e}}_3, \ \ \boldsymbol{O} \boldsymbol{\cdot} \mathring{\boldsymbol{r}}' = \boldsymbol{O} \boldsymbol{\cdot} \mathring{\boldsymbol{e}}_3 = \boldsymbol{e}_3$$
$$\Rightarrow \ \mathbf{\Gamma} = \boldsymbol{r}' - \boldsymbol{e}_3, \quad (2.5)$$

$$\boldsymbol{r}' = \mathbf{\Gamma} + \boldsymbol{e}_3, \ \ \|\boldsymbol{r}'\|^2 = \boldsymbol{r}' \boldsymbol{\cdot} \boldsymbol{r}' = (\mathbf{\Gamma} + \boldsymbol{e}_3) \boldsymbol{\cdot} (\mathbf{\Gamma} + \boldsymbol{e}_3)$$
$$= \mathbf{\Gamma} \boldsymbol{\cdot} \mathbf{\Gamma} + 2\mathbf{\Gamma} \boldsymbol{\cdot} \boldsymbol{e}_3 + \boldsymbol{e}_3 \boldsymbol{\cdot} \boldsymbol{e}_3,$$

$$\|\boldsymbol{r}'\| = \sqrt{\|\mathbf{\Gamma}\|^2 + 2\Gamma_3 + 1},$$

$$\|\boldsymbol{r}'\| - 1 = \sqrt{\|\mathbf{\Gamma}\|^2 + 2\Gamma_3 + 1} - 1 = \Gamma_3 + \infty^{-1}(\|\mathbf{\Gamma}\|^2)$$
$$(2.6)$$

Equality (2.6) describes a relative elongation. Roughly speaking, component $\Gamma_3 \equiv \mathbf{\Gamma} \boldsymbol{\cdot} \boldsymbol{e}_3$ can be considered an elongation, and components $\Gamma_1 \equiv \mathbf{\Gamma} \boldsymbol{\cdot} \boldsymbol{e}_1$, $\Gamma_2 \equiv \mathbf{\Gamma} \boldsymbol{\cdot} \boldsymbol{e}_2$ present a transverse shear. It's more accurate to rely on formulas (2.5) and (2.6).

......

The Cosserat-like model of a rod doesn't have a section as a plane figure.

....

§3. Balance
of forces and moments

Possible loads acting on a rod as the Cosserat line are forces and moments: on infinitesimal element ds of a rod act external force $\boldsymbol{q}ds$ and external moment $\boldsymbol{m}ds$. The internal interactions will be force $\boldsymbol{Q}(s)$ and moment $\boldsymbol{M}(s)$ — this is the action of the particle with coordinate $s+0$ on the particle with $s-0$. The action–reaction principle gives that a reverse (a change of direction) of coordinate s changes signs of $\boldsymbol{Q}$ and $\boldsymbol{M}$.

...

§4. Principle of virtual
work and consequences

For a piece of rod $s_0 \leq s \leq s_1$ formulation of the principle is as follows

......

Conventionally $\boldsymbol{a}$ is the tensor of stiffness for bending and twisting, $\boldsymbol{b}$ is the tensor of stiffness for (ex)tension and shear, and $\boldsymbol{c}$ is the tensor of crosslinks.

Stiffness tensors rotate along with a particle:

...

§ 5. Classical
Kirchhoff's model

It is also called the *Kirchhoff's rod theory.*

Until now functions $\boldsymbol{r}(s,t)$ and $\boldsymbol{O}(s,t)$ were independent. The Kirchhoff's classical theory postulates the internal constraint

$$\boldsymbol{\Gamma} = \boldsymbol{0} \;\Leftrightarrow\; \boldsymbol{r}_{\partial s} = \boldsymbol{O}\boldsymbol{\cdot}\mathring{\boldsymbol{r}}_{\partial s} \text{ or } \boldsymbol{r}' = \boldsymbol{O}\boldsymbol{\cdot}\mathring{\boldsymbol{r}}'. \tag{5.1}$$

Having the idea of vector $\boldsymbol{\Gamma}$ (2.5), here we can tell that: (1) a rod is non-extensible, (2) there are no transverse shears.

If basis vector $\mathring{\boldsymbol{e}}_3$ was directed along the tangent to the axis in the initial configuration, then it will remain on the tangent also after deforming. The rod particles rotate only together with the tangent to the axis and around it.

Уравнения баланса сил и моментов (импульса и момента импульса) не меняются от введения связи (5.1). Но локальное вариационное соотношение (...) становится короче:

...

§ 6. Euler's problem
about stability of rods

Рассматривается прямой стержень, защемлённый на одном конце и нагруженный силой $\boldsymbol{P}$ на другом (рисунок

?? 123 ??). Сила "мёртвая" (не меня-
ется в процессе деформирования)

...

§ 7. Variational equations

In the nonlinear mechanics of elastic
media variational equations are useful,
which describe a small change in the
current configuration (§ 3.12).

Variating equations of the complete
system of the Cosserat model, we get
$$\delta\boldsymbol{Q}' + \delta\boldsymbol{q} = \rho\left(\boldsymbol{u} + \delta\boldsymbol{\tau}\times\boldsymbol{\varepsilon}\right)^{\bullet\bullet},$$
$$\delta\boldsymbol{M}' + \boldsymbol{u}'\times\boldsymbol{Q} + \boldsymbol{r}'\times\delta\boldsymbol{Q} + \delta\boldsymbol{m} = \ldots$$

...

§ 8. Non-shear model with (ex)tension

Kirchhoff's model with internal
constraint $\boldsymbol{\Gamma} = \boldsymbol{0}$ (5.1) doesn't
describe the simplest case of
extension/compression of a straight
rod. This nuisance disappears
with "softening" of the constraint,
for example adding the possibility of
(ex)tension and inhibiting only the
transverse shear
$$\boldsymbol{\Gamma} = \Gamma\boldsymbol{e}_3 \;\Leftrightarrow\; \Gamma_\alpha = 0 \qquad (8.1)$$

...

§9. Mechanics of flexible thread

A thread is a momentless rod.

A flexible thread (chain) is simpler than a rod, because its particles are "simple" material points with only translational degrees of freedom. Therefore среди нагрузок нет моментов, только "линейные" силы — внешние распределённые $\boldsymbol{q}$ и внутренние сосредоточенные $\boldsymbol{Q}$. Движение нити полностью определяется одним вектором-радиусом $\boldsymbol{r}(s,t)$, а инерционные свойства — линейной плотностью $\rho(s)$.

Вот принцип виртуальной работы для куска́ нити $s_0 \le s \le s_1$

$$\int\limits_{s_0}^{s_1}\Big((\boldsymbol{q}-\rho\ddot{\boldsymbol{r}})\boldsymbol{\cdot}\delta\boldsymbol{r}-\delta\Pi\Big)ds+\Big[\boldsymbol{Q}\boldsymbol{\cdot}\delta\boldsymbol{r}\Big]_{s_0}^{s_1}=0.$$

(9.1)

...

Механика нити детально описана в книге [34].

§10. Linear theory

В линейной теории внешние воздействия считаются малыми, а отсчётная конфигурация — ненапряжённым состоянием покоя. Уравнения в вариациях в этом случае дают

...

§11. Case
of small thickness

Когда относительная толщина стержня мала́, тогда модель типа Cosserat уступает место классической. Понятие "толщина" определяется соотношением жёсткостей: a, b and c — разных единиц измерения; полагая $a = \hbar^2 \widehat{a}$ and $c = \hbar \widehat{c}$, where $\hbar$ is some scope of length, получим тензоры $\widehat{a}$, b and $\widehat{c}$ с одной и той же единицей. Подбирая $\hbar$ так, чтобы сблизились характерные значения тензоров $\widehat{a}$, b and $\widehat{c}$, найдём equivalent thickness h стержня (для реальных трёхмерных стержней h где-то на уровне диаметра сечения).

Представив Q and M через векторы бесконечномалой линейной деформации

...

Переход модели типа Cosserat в классическую кажется более очевидным, если непосредственно интегрировать уравнения (...)

...

§12. Saint-Venant's
problem

Трудно переоценить ту роль, которую играет в механике стержней классическое решение Saint-Venant. О нём уже́ шла речь в §4.13.

Вместо условий ...

...

§ 13. Finding
stiffness by energy

Для определения тензоров жёсткости $\boldsymbol{a}$, $\boldsymbol{b}$ и $\boldsymbol{c}$ одномерной модели достаточно решений трёхмерных задач для стержня. Но тут возникают два вопроса: какие именно задачи рассматривать и что нужно взять из решений?

Saint-Venant's problem выделяется среди прочих, ведь оттуда берётся жёсткость на кручение.

Вдобавок есть много точных решений, получаемых таким путём: задаётся поле $\boldsymbol{u}(\boldsymbol{r})$, определяется $\breve{\boldsymbol{\tau}} = {}^{4}\boldsymbol{\mathcal{C}} \boldsymbol{\cdot\cdot} \boldsymbol{\nabla u}$, затем находятся объёмные $\boldsymbol{f} = -\boldsymbol{\nabla} \boldsymbol{\cdot} \breve{\boldsymbol{\tau}}$ и поверхностные $\boldsymbol{p} = \boldsymbol{n} \boldsymbol{\cdot} \breve{\boldsymbol{\tau}}$ нагрузки.

But что делать с решением? Ясно, что $\boldsymbol{Q}$ and $\boldsymbol{M}$ в стержне — это интегралы по сечению (...). And совсем не ясно, what принять за смещение и поворот in a one-dimensional model. Если взять, for example, such a version (the index of $\boldsymbol{u}$ shows the dimensions of the model)

$$\boldsymbol{u}_1(z) = \Omega^{-1} \int\limits_{\Omega} \boldsymbol{u}_3(\boldsymbol{x}, z)\, d\Omega, \ \boldsymbol{\theta}(z) = \tfrac{1}{2}\, \Omega^{-1} \int\limits_{\Omega} \boldsymbol{\nabla} \times \boldsymbol{u}$$

то чем другие возможные представления хуже?

Помимо $\boldsymbol{Q}$ and $\boldsymbol{M}$, есть ещё величина, не вызывающая сомнений —

упругая энергия. Естественно потребовать, чтобы в одномерной и в трёхмерной моделях энергии на единицу длины совпали. При этом, чтобы уйти от различий в трактовках $\boldsymbol{u}_1$ и $\boldsymbol{\theta}$, будем исходить из complementary energy $\widehat{\Pi}(\boldsymbol{M}, \boldsymbol{Q})$:

$$\widehat{\Pi}(\boldsymbol{M}, \boldsymbol{Q}) = \int\limits_{\Omega} \Pi_3 \, d\Omega$$

...

§14. Variational method of building one-dimensional model

Мы только что определили жёсткости стержня, полагая, что одномерная модель линии типа Cosserat адекватно описывает поведение трёхмерной модели. "Одномерные" представления ассоциируются со следующей картиной смещений в сечении:

$$\boldsymbol{u}(s, \boldsymbol{x}) = \boldsymbol{U}(s) + \boldsymbol{\theta}(s) \times \boldsymbol{x}. \quad (14.1)$$

Однако, такое поле $\boldsymbol{u}$ не удовлетворяет уравнениям трёхмерной теории(??добавить, каким именно). Невозможно пренебречь возникающими невязками в дифференциальных уравнениях и краевых условиях.

Формально "чистым" является вариационный метод сведения трёхмерной проблемы к одномерной, называемый иногда методом внутренних связей. Аппроксимация (14.1) подстав-

ляется в трёхмерную формулировку вариационного принципа минимума потенциальной энергии (9.1, § 4.9)

$$\mathscr{E}(\boldsymbol{u}) = \int\limits_{\mathcal{V}}\Big(\Pi(\boldsymbol{u}) - \boldsymbol{f}\boldsymbol{\cdot}\boldsymbol{u}\Big)d\mathcal{V} - \int\limits_{o_2}\boldsymbol{p}\boldsymbol{\cdot}\boldsymbol{u}\,do \to \min,$$

которая после интегрирования по сечению становится одномерной. Если $\boldsymbol{U}$ и $\boldsymbol{\theta}$ варьируются независимо, получаем модель типа Коссера. В случае $\boldsymbol{U}' = \boldsymbol{\theta} \times \boldsymbol{t}$ приходим к классической модели.

Метод внутренних связей привлекателен, его продолжают "переоткрывать". С его помощью возможно моделировать тела с неоднородностью и анизотропией, он легко обобщается на динамику, если $\boldsymbol{f}$ дополнить неварьируемой динамической добавкой до $\boldsymbol{f} - \rho\ddot{\boldsymbol{u}}$. Можно рассматривать и стержни переменного сечения, и даже нелинейно упругие, ведь вариационная постановка есть (гл. 3).

Аппроксимацию (14.1) можно дополнить слагаемыми с внутренними степенями свободы. Понимая необходимость учёта депланаций, некоторые авторы

...

...

For variational construction of one-dimensional models it's convenient to use the Reissner–Hellinger principle (§ 4.11) with independent approximation of stresses [17]. In this case,

some consistency between $\boldsymbol{u}$ and $\boldsymbol{\tau}$ is needed.

Many advantages of the variational method are opposed by the one, but significant disadvantage. Introducing approximations within cross-sections, we impose our unreal simplifications on reality. The variational method is more suitable for applied calculations.

§ 15. Asymptotic splitting of three-dimensional problem

Asymptotic splitting can be considered fundamental for describing the mechanics of rods. One-dimensional models paint only the part of a picture, two-dimensional problems in cross-sections paint the other part, and together they present the solution of a three-dimensional problem for a small thickness.

How to introduce a small parameter χ into a three-dimensional problem? The easiest way to do it is through representation of the location vector (§ 1):

$$\boldsymbol{R}(x_\alpha, s) = \chi^{-1}\boldsymbol{r}(s) + \boldsymbol{x}, \quad \boldsymbol{x} \equiv x_\alpha \boldsymbol{e}_\alpha(s), \quad \alpha = 1, 2.$$

For an orthonormal basis, upper and lower indices do not differ

$$q_i = q^i, \ \boldsymbol{R}^i = \boldsymbol{R}_i = \boldsymbol{R}_{\partial i} = \frac{\partial \boldsymbol{R}}{\partial q^i}.$$

Three coordinates are

$$q^1 = x_1, \ q^2 = x_2, \ q^3 = s.$$

The basis vectors are

$$\boldsymbol{e}_1, \ \boldsymbol{e}_2, \ \boldsymbol{t} \equiv \boldsymbol{e}_3.$$

Representation of the Hamilton operator $\boldsymbol{\nabla}$

$$\boldsymbol{\nabla} = \boldsymbol{R}^i \frac{\partial}{\partial q^i} = \boldsymbol{\nabla}_\perp + \mathrm{v}^{-1}\boldsymbol{t}\left(\partial_s - \psi_t \mathrm{D}\right), \ \ \boldsymbol{\nabla}_\perp \equiv \boldsymbol{e}_\alpha \frac{i}{\partial}$$

$$\mathrm{v} \equiv \boldsymbol{R}_{\partial 1} \times \boldsymbol{R}_{\partial 2} \cdot \boldsymbol{R}_{\partial 3} = \chi^{-1} + \boldsymbol{t}\cdot\boldsymbol{\psi}_\perp \times \boldsymbol{x}, \ \ \mathrm{D} \equiv \boldsymbol{t}$$

$$\boldsymbol{\psi}_\perp + \psi_t \boldsymbol{t} = \boldsymbol{\psi}$$

(the meaning of vector $\boldsymbol{\psi}$ is the same as in §1).

The Cauchy stress tensor

$$\boldsymbol{\tau} = \boldsymbol{\tau}_\perp + \sigma_t \boldsymbol{t}\boldsymbol{t} + 2\boldsymbol{t}\boldsymbol{\tau}_\|^\mathsf{S}, \ \ \boldsymbol{\tau}_\perp \equiv \tau_{\alpha\beta}\boldsymbol{e}_\alpha\boldsymbol{e}_\beta, \ \ \boldsymbol{\tau}_\| \equiv \tau_{3\alpha}$$

...

§16. Thermal deformation and stress

The direct approach, so efficient for making one-dimensional Cosserat and Kirchhoff models, isn't applicable for problems of thermoelasticity. The transition from the three-dimensional model to the one-dimensional can be realized either by variational or asymptotic way.

Описанный в §14 вариационный метод целиком переносится на термоупругость — включая задачи с неоднородностью и анизотропией, переменным сечением, динамические и даже нелинейные. Для этого нужно (§6.8) in the Lagrange principle of minimum potential energy заменить потенциал $\Pi(\boldsymbol{\varepsilon})$ свободной энергией

...

Bibliography

Unlike other topics of the theory of elasticity, rods are presented in books very modestly. Narration style of "strength of materials" prevails, more exact and perfect approaches seem impossible or unnecessary to many authors.

But many interesting articles have been published, their reviews are presented, for example, by S. Antman [1], В. В. Елисеев [17] and А. А. Илюхин [?].

THIN-WALLED RODS

§1. Variational approach

I n chapter 9, rods with massive cross-sections were described. But other rods are also widely used — thin-walled ones, with sections of narrow bands of various shapes: corner, Z-beam, channel (C-beam), I-beam, ... If rods are like lines, then in a thin-walled rod the section itself also looks like line. The three dimensions — the thickness and the length of a section, as well as the length of a rod — have different decimal orders.

The applied theories of thin-walled rods are known, they are described, for example, in books ...

There is also the exact theory, based on asymptotic splitting of the three-dimensional problem [17]. In sum, the complex asymptotic analysis confirmed most of the hypotheses of applied theories.

As introduction to the mechanics of thin-walled rods, here is the following variational procedure with warping (deplanation) of cross-sections.

Consider the simplest case of a cylindrical rod with a thin, simply connected section (... рисунок ...). Площадь сечения do, боковая поверхность свободна, нагрузка в объёме — $\boldsymbol{f}$, на торце $z = z_1$ известны поверхност-

ные силы $\boldsymbol{p}(\boldsymbol{x})$ $(\boldsymbol{x} = x_\alpha \boldsymbol{e}_\alpha)$, торец $z = z_0$ закреплён.

...

§ 2. Equations with small parameter

Рассмотрим призматический стержень с односвязным сечением в виде тонкой криволинейной полоски постоянной толщины h. Радиус-вектор в объёме предста́вим следующим образом:

...

§ 3. First step of the asymptotic procedure

Outer decomposition

Из системы

...

Inner decomposition near s_0

Выпишем уравнения для

...

Coalescence

Стыковка внутреннего и внешнего разложений

...

§ 4. Second step

Outer decomposition

Из системы

...

Inner decomposition near $s = s_0$

Из общей системы

...

Coalescence

Поскольку рассматриваются попра-
вочные члены асимптотических раз-
ложений

...

§ 5. Third step

Outer decomposition

Из системы

....

Inner decomposition around $s = s_0$

Как ужé отмечалось, внутренние раз-
ложения нужны для постановки кра-
евых условий на концах

...

Coalescence

В плоской задаче имеем следующее
двучленное внешнее разложение:

....

§ 6. Fourth step

Здесь понадобится лишь внешнее разложение. Более того: в этом приближении мы не будем искать решения уравнений — будет достаточно лишь условий разрешимости. Напомним, что философия наша такова: разыскиваются лишь главные члены асимптотических разложений, но для полного их определения могут понадобиться

...

§ 7. Displacements

Расписывая тензорное соотношение

...

§ 8. Results
of asymptotic analysis

Нахождение главных членов асимптотики of stresses and displacements для тонкостенных стержней оказалось намного сложнее, чем для массивных сечений. Дадим сводку полученных выше итоговых результатов.

Смещение:
$$\boldsymbol{u}_\perp = \lambda^{-4}\big(\boldsymbol{U}_\perp(z)+\theta(z)\boldsymbol{k}\times\boldsymbol{r}\big)+\infty^{-1}(\lambda^{-3}),\tag{8.1}$$

$$u_{[z]} = \lambda^{-3}\big(-\dot{\boldsymbol{U}}_\perp\boldsymbol{\cdot}\boldsymbol{r} + \dot{\theta}\omega(s) + U_z(z)\big) + \infty^{-1}(\lambda^{-2}).\tag{8.2}$$

Напряжения:

...

Помимо книг В. З. Власов'а [8],
и Я. Г. Пановко [15], отметим ста
и Е. А. Бейлина [43].
Материал этой главы содержится в ди
ра В. Елисеева [17], там также есть обширн
тему тонкостенных стержней.

SHELLS AND PLATES

§1. Surface geometry

A surface is described by a function (a morphism)

$$\boldsymbol{r} = \boldsymbol{r}(q^\alpha), \ \alpha = 1, 2 \qquad (1.1)$$

of two mutually independent variable parameters (coordinates) q^α, then $\boldsymbol{r}$ is location vector (radius vector) of surface's points.

Examples

✓ a linear mapping is a plane $\boldsymbol{r}(a,b) = a\boldsymbol{e}_1 + b$

✓ a helicoid $\boldsymbol{r}(u,v) = u\sin v\,\boldsymbol{e}_1 + u\cos v\,\boldsymbol{e}_2 + v\boldsymbol{e}$

✓ a cone $\boldsymbol{r}(u,v) = u\sin v\,\boldsymbol{e}_1 + u\cos v\,\boldsymbol{e}_2 + u\boldsymbol{e}_3$

✓ a cylinder of radius $r = \mathsf{constant}$
$$\boldsymbol{r}(u,v) = r\big(\cos u\,\boldsymbol{e}_1 + \sin u\,\boldsymbol{e}_2\big)$$

✓ a torus of revolution with radii r and R
$$\boldsymbol{r}(p,q) = \boldsymbol{e}_1\big(r\cos p + R\big)\cos q + \boldsymbol{e}_2\big(r\cos p$$

✓ a 2-sphere — a torus with $R = 0$
$$\boldsymbol{r}(p,q) = r\big(\cos p\cos q\,\boldsymbol{e}_1 + \mathsf{c}$$

✓ a paraboloid $\boldsymbol{r}(u,v) = u\boldsymbol{e}_1 + v\boldsymbol{e}_2 + \big(u^2 + v^2\big)$
rameterization $\boldsymbol{r}(\rho,\vartheta) = \rho\cos\vartheta\,\boldsymbol{e}_1 + \rho\sin\vartheta\,\boldsymbol{e}_2$
of revolution.

Continuous change of one coordinate q^1, while second coordinate $q^2 = u^* = \mathsf{constant}$ is "frozen", gives coordinate line $\overset{(1)}{\boldsymbol{r}}(q^1) = \boldsymbol{r}(q^1, u^*)$. A crossing of two coordinate lines $q^1 = v^*$

and $q^2 = w^*$ uniquely identifies a point $\boldsymbol{r}(v^*, w^*)$ of the surface.

Vectors
$$\boldsymbol{r}_{\partial\alpha} \equiv \partial_\alpha \boldsymbol{r}, \ \ \partial_\alpha \equiv \frac{\partial}{\partial q^\alpha} \qquad (1.2)$$
are tangent to coordinate lines. If they are linearly independent (that is $\boldsymbol{r}_{\partial 1} \times \boldsymbol{r}_{\partial 2} \neq \boldsymbol{0})^*$, they compose a local basis for representing any vector $\overset{(2)}{\boldsymbol{v}}$ in the tangent plane as linear combination
$$\overset{(2)}{\boldsymbol{v}} = v^\alpha \boldsymbol{r}_{\partial\alpha} = v_\alpha \boldsymbol{r}^\alpha,$$
$$v^\alpha = \overset{(2)}{\boldsymbol{v}} \boldsymbol{\cdot} \boldsymbol{r}^\alpha, \ \ v_\alpha = \overset{(2)}{\boldsymbol{v}} \boldsymbol{\cdot} \boldsymbol{r}_{\partial\alpha}, \ \ \boldsymbol{r}^\alpha \boldsymbol{\cdot} \boldsymbol{r}_{\partial\beta} = \delta^\alpha_\beta. \qquad (1.3)$$
Here $\boldsymbol{r}^\alpha$ is local reciprocal basis in the (co)tangent plane.

Field of unit normal vectors $\boldsymbol{n}(q^\alpha)$ adds at every point of the surface $(\forall \boldsymbol{r}(q^\alpha) \Leftrightarrow \forall q^\alpha)$ a unit** normal
$$\boldsymbol{n} = \frac{\boldsymbol{r}_{\partial 1} \times \boldsymbol{r}_{\partial 2}}{\|\boldsymbol{r}_{\partial 1} \times \boldsymbol{r}_{\partial 2}\|}. \qquad (1.4)$$
At non-singular points, three vectors $\boldsymbol{r}_{\partial 1}$, $\boldsymbol{r}_{\partial 2}$ and $\boldsymbol{n}$ can be taken as a basis for the entire three-dimensional space, giving a decomposition of any vector (and any tensor too), for example
$$\overset{(3)}{\boldsymbol{u}} = u^\alpha \boldsymbol{r}_{\partial\alpha} + u^n \boldsymbol{n}.$$
$$u^n = u_n$$

---

* Sometimes somewhere — at so-called singular points — it is not so. As example, for a 2-sphere of unit radius
$$\boldsymbol{r}(p, q) = \cos p \cos q \, \boldsymbol{e}_1 + \cos p \sin q \, \boldsymbol{e}_2 + \sin p \, \boldsymbol{e}_3,$$
$$\boldsymbol{r}_{\partial p} = \partial_p \boldsymbol{r} = -\sin p \cos q \, \boldsymbol{e}_1 - \sin p \sin q \, \boldsymbol{e}_2 + \cos p \, \boldsymbol{e}_3,$$
$$\boldsymbol{r}_{\partial q} = \partial_q \boldsymbol{r} = -\cos p \sin q \, \boldsymbol{e}_1 + \cos p \cos q \, \boldsymbol{e}_2 + 0 \, \boldsymbol{e}_3,$$
a pole $p = \pm\frac{\pi}{2}$ is a singular point: $\boldsymbol{r}_{\partial q}\big|_{p=\pm\pi/2} = \boldsymbol{0}$.

** $\|\boldsymbol{a}\| \equiv \sqrt{\boldsymbol{a} \boldsymbol{\cdot} \boldsymbol{a}}$ is the length of vector $\boldsymbol{a}$.

for a 2-sphere of unit radius

$$\boldsymbol{r}_{\partial p} \times \boldsymbol{r}_{\partial q} = -\det \begin{bmatrix} -\sin p \cos q & \boldsymbol{e}_1 & -\cos p \\ -\sin p \sin q & \boldsymbol{e}_2 & \cos p \, c \\ \cos p & \boldsymbol{e}_3 & 0 \end{bmatrix}$$

Единичные ("метрические") бивалентные тензоры $\boldsymbol{E}$ в пространстве and $\boldsymbol{I}$ в касательной плоскости

$$\boldsymbol{E} = \boldsymbol{I} + \boldsymbol{n}\boldsymbol{n}, \boldsymbol{I} \equiv \boldsymbol{E} - \boldsymbol{n}\boldsymbol{n} = \boldsymbol{r}_{\partial \alpha}\boldsymbol{r}^{\alpha} = \boldsymbol{r}^{\alpha}\boldsymbol{r}_{\partial \alpha}.$$

Представление вектора положения $\overset{(3)}{\boldsymbol{n}}$ любой точки пространства на расстоянии h от поверхности $(\partial_\alpha h = 0)$

$$\overset{(3)}{\boldsymbol{n}}(q^\alpha, h) = \boldsymbol{r}(q^\alpha) + h\boldsymbol{n}(q^\alpha) \qquad (1.5)$$

даёт локальный касательный базис

$$\overset{(3)}{\boldsymbol{n}}_{\partial n} = \boldsymbol{n} = \overset{(3)}{\boldsymbol{n}}{}^n,$$
$$\overset{(3)}{\boldsymbol{n}}_{\partial \alpha} \equiv \partial_\alpha \overset{(3)}{\boldsymbol{n}} = \partial_\alpha \boldsymbol{r} + h\partial_\alpha \boldsymbol{n} = \boldsymbol{r}_{\partial \alpha} + h\boldsymbol{r}_{\partial \alpha} \cdot \boldsymbol{r}^\beta \partial_\beta \boldsymbol{n}$$
...

дифференциальный оператор "набла"

в пространстве $\boldsymbol{\nabla} \equiv \overset{(3)}{\boldsymbol{n}}{}^i \partial_i$

в касательной плоскости $\overset{(2)}{\boldsymbol{\nabla}} \equiv \boldsymbol{r}^\alpha \partial_\alpha$

...

$$\overset{(3)}{\boldsymbol{n}}_{\partial \alpha} = \boldsymbol{r}_{\partial \alpha} \cdot \Big(\boldsymbol{r}^\beta \boldsymbol{r}_{\partial \beta} + h\boldsymbol{r}^\beta \partial_\beta \boldsymbol{n}\Big) = \boldsymbol{r}_{\partial \alpha} \cdot \Big(\boldsymbol{I} + h\overset{(2)}{\boldsymbol{\nabla}}\boldsymbol{n}\Big) =$$

Two-coordinate bivalent tensor

$$2\overset{(2)}{\boldsymbol{c}} \equiv -\overset{(2)}{\boldsymbol{\nabla}}\boldsymbol{n} = -\boldsymbol{r}^\alpha \partial_\alpha \boldsymbol{n} \qquad (1.6)$$

characterizes the surface's curvature.

...
кобазис $\overset{(3)}{\boldsymbol{n}}{}^\alpha \cdot \overset{(3)}{\boldsymbol{n}}_{\partial \beta} = \delta_\beta^\alpha$, $\overset{(3)}{\boldsymbol{n}}{}^i \cdot \overset{(3)}{\boldsymbol{n}}_{\partial j} = \delta_j^i$

$$\overset{(3)}{\boldsymbol{n}}{}^\alpha = \Big(\boldsymbol{I} + h\overset{(2)}{\boldsymbol{\nabla}}\boldsymbol{n}\Big)^{-1} \cdot \boldsymbol{r}^\alpha, \ \overset{(3)}{\boldsymbol{n}}{}^n = \boldsymbol{n}$$

связь между $\boldsymbol{\nabla}$ и $\overset{(2)}{\boldsymbol{\nabla}}$

$$\boldsymbol{\nabla} = \Big(\boldsymbol{I} + h\overset{(2)}{\boldsymbol{\nabla}}\boldsymbol{n}\Big)^{-1} \cdot \overset{(2)}{\boldsymbol{\nabla}} + \boldsymbol{n}\partial_n$$

...

§ 2. The model of a shell

С моделями трёхмерного моментного континуума и одномерных стержней не так уж тяжело to figure out the mechanics of shells.

As a geometrical object, the shell is defined срединной поверхностью и толщиной h, так что в формуле (1.5)
$$-h/_2 \leq h \leq h/_2.$$

...

§ 3. The balance of forces and moments for a shell

При $\delta\boldsymbol{u} = $ **constant** и $\delta\boldsymbol{\varphi} = \boldsymbol{0}$ (трансляция) ...

...

§ 4. Shells: The relations of elasticity

Локальное соотношение (**??**) после вывода уравнений баланса ...

...

§ 5. The classical theory of shells

Вышеизложенная теория (напоминающая балку Тимошенко и континуумы Cosserat) рассматривает rotations $\boldsymbol{\varphi}$

независимо от displacements $\boldsymbol{u}$. Но опыт подсказывает, что материальный элемент, нормальный к срединной поверхности до деформации, остаётся таким и после деформации (кинематическая гипотеза Kirchhoff'a). В классической теории Kirchhoff'a, Арона и Love'a field $\boldsymbol{\varphi}$ выражается через $\boldsymbol{u}$, что в конце концов даёт свести всё к одному векторному уравнению для $\boldsymbol{u}$.

Предположим, что в основе классической теории лежит внутренняя связь

...

§ 6. Shells: A plate

Plate is the simplest kind of shell. Единичный перпендикуляр $\boldsymbol{n} = \boldsymbol{k}$ направлен по декартовой оси z, в качестве координат ...

...

§ 7. Shells: Approach with Lagrange multipliers

Уязвимым местом этого изложения теории оболочек являются формулы

...

§ 8. Cylindrical shell

Существуют разные уравнения цилиндрической оболочки. Приводятся громоздкие выкладки с отбрасыванием некоторых малых членов, и не всегда ясно, какие именно члены действительно можно отбросить.

Предлагаемая читателю теория оболочек иного свойства: лишних членов нет, все уравнения записаны в компактной тензорной форме — остаётся лишь грамотно действовать с компонентами тензоров. В качестве иллюстрации рассмотрим цилиндрическую оболочку.

В декартовой системе

...

§ 9. Shells:
Common theorems

Пусть край закреплён

...

§ 10. Shells:
Boundary conditions

В рамках рассматриваемого прямого подхода к оболочкам как материальным поверхностям наиболее надёжным способом вывода граничных условий представляется вариационный. Исходим из вариационного уравнения:

...

§ 11. Shells of revolution

Surface of revolution (reference surface of shell of revolution) is created by rotating a plane curve (the meridian, the generatrix) about a straight line in the plane of curve (an axis of rotation).

Разберёмся в геометрии поверхности вращения (рисунок). Меридиан можно задать зависимостью декартовых координат

...

§ 12. Momentless theory of shells

В отличие от пластины, оболочка способна выдерживать нормальную распределённую нагрузку без появления внутренних моментов. В безмоментном состоянии напряжения равномерно распределены по толщине оболочки, безмоментные оболочечные конструкции можно считать оптимально спроектированными.

Уравнения безмоментной теории

...

§ 13. Shells: Nonlinear
momentless theory

Вышеизложенную безмоментную теорию оболочек возможно просто и корректно обобщить на нелинейную постановку. Материальная поверхность состоит из частиц

...

§ 14. Shells: Other
variant of classical theory

Выше при изложении моментной теории оболочек частицы материальной поверхности считались твёрдыми телами с шестью степенями свободы

...

§ 15. Plates:
Overall concepts

Пластиной называется тонкое трёхмерное тело, ограниченное двумя параллельными плоскостями и боковой цилиндрической поверхностью (?? рисунок ??). В декартовых координатах x_1, x_2, z поперечная координата ...

...

В теории пластин рассматриваются двумерные задачи. Переход от трёхмерной задачи наиболее достоверен на пути асимптотики. Но логическая стройность и эффективность присуща и вариационному подходу, основанному на аппроксимации по толщине решения трёхмерной вариационной

задачи. Самое же простое корректное изложение теории пластин характерно для прямого подхода к ним как материальным плоскостям.

...

§ 16. Timoshenko-like model of a plate (direct approach)

Пластина рассматривается как материальная плоскость, частицы которой

...

§ 17. Kirchhoff's classical theory of plates

Принимается внутренняя связь

...

§ 18. Plates: Asymptotic matching of two-dimensional models

При малой толщине из теории типа Тимошенко следует классическая теория. Толщина h определяется отношением жёсткостей. Перепишем

...

§19. Plates: Variational transition from the three-dimensional model

Using the variational principles by Lagrange or by Hellinger and Reissner с аппроксимацией решения по толщине, можно получить двумерные вариационные формулировки проблем. Из этих вариационных принципов вытекают и соотношения внутри области, and the naturalboundary conditions.

For example, here is the model of the Timoshenko type with the approximation of displacements

...

...

The variational transitions can be easily generalized for the cases of the temperature deformations, the inhomogeneity (heterogeneity) and anisotropy материала, the dynamics. The advantage of the Hellinger–Reissner principle состоит в явном представлении напряжений. Зато принцип Лагранжа применим and к нелинейным задачам (in the chapter 3 описана трёхмерная постановка).

§ 20. Plates: Splitting of three-dimensional bending problem

Двумерная классическая теория изгиба пластин легко выводится из трёхмерной постановки с малым параметром. Представив радиус-вектор в объёме

...

§ 21. Circular plates

В качестве иллюстрации рассмотрим широко представленный в литературе вопрос об уравнениях теории Kirchhoff'а в полярных координатах.

...

§ 22. Plates: Plane stress

Это вторая из двух задач, о которых говорилось в § 15. Силы

...

Теория оболочек изложена в монографиях А. Л
В. В. Новожилова [42], А. И. Лурье [31], В. С.
других. Достоинства этих книг перекрывают не
ного аппарата. Переход от трёхмерной модели с
рассмотрен у ...
...
Техническая теория изгиба пластин изложена
...

OSCILLATIONS AND WAVES

§1. Vibrations of three-dimensional bodies

$\mathbf{D}$ ynamic problem of the classical linear elasticity is

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\sigma} + \boldsymbol{f} = \rho\ddot{\boldsymbol{u}}, \;\; \boldsymbol{\sigma} = {}^{4}\!\boldsymbol{\mathcal{A}}\boldsymbol{\cdot\cdot}\boldsymbol{\nabla}\boldsymbol{u},$$
$$\boldsymbol{u}\big|_{o_1} = \boldsymbol{0}, \;\; \boldsymbol{n}\boldsymbol{\cdot}\boldsymbol{\sigma}\big|_{o_2} = \boldsymbol{p},$$
$$\boldsymbol{u}\big|_{t=0} = \boldsymbol{u}°, \;\; \dot{\boldsymbol{u}}\big|_{t=0} = \dot{\boldsymbol{u}}°.$$
$$(1.1)$$

According to the common theory (§ 2.8), we begin with the analysis of harmonics (orthogonal oscillations):

$$\boldsymbol{f} = \boldsymbol{0}, \;\; \boldsymbol{p} = \boldsymbol{0}, \;\; \boldsymbol{u}(\boldsymbol{r},t) = \boldsymbol{U}(\boldsymbol{r})\sin\omega t,$$
$$\boldsymbol{\nabla}\boldsymbol{\cdot}\big({}^{4}\!\boldsymbol{\mathcal{A}}\boldsymbol{\cdot\cdot}\boldsymbol{\nabla}\boldsymbol{U}\big) + \rho\omega^2\boldsymbol{U} = \boldsymbol{0}. \quad (1.2)$$

If a homogeneous problem has a nontrivial solution, then values of ω are natural resonant frequencies, and $\boldsymbol{U}(\boldsymbol{r})$ are orthogonal (normal) "modes".

Time independent equation (1.2) looks like equation of linear elastostatics (**??**, § 4.**??**) when the volume load is equal to $\omega^2\rho\boldsymbol{U}$. The surface load on o_2 is equal to zero. Clapeyron's identity (4.1, § 4.4) gives

$$\omega^2\int_{\mathcal{V}}\rho\boldsymbol{U}\boldsymbol{\cdot}\boldsymbol{U}\,d\mathcal{V} = 2\int_{\mathcal{V}}\Pi\big(\boldsymbol{\nabla}\boldsymbol{U}^{\mathsf{s}}\big)d\mathcal{V}. \quad (1.3)$$

It also means that $\omega^2 \geq 0$. The zero here means that a continuum moves U as a rigid whole. When even a small part of a surface is fixed, then all $\omega_i > 0$.

And here we assume that ω^2 and U are real numbers. This can be proven "by contradiction". If $\Im m\, \omega^2 \neq 0$, then conjugate frequency $\overline{\omega}^2$ is part of the oscillation spectrum too, and "mode" $\overline{U}$ for this frequency has conjugate components. Using then the reciprocal work theorem (**??**, § 4.4) for U and $\overline{U}$, we have

$$\omega^2 \int_{\mathcal{V}} \rho\, U{\cdot}\overline{U}\, d\mathcal{V} = \overline{\omega}^2 \int_{\mathcal{V}} \rho\, \overline{U}{\cdot}U\, d\mathcal{V} \;\Rightarrow\; \omega^2 = \overline{\omega}^2 \;\Rightarrow\; \Im$$
...

However, a bright picture with decomposition by modes is of little use for practical calculations of oscillations (vibrations) of a three-dimensional elastic body. The reason is density of the spectrum, driven oscillations excite many modes. When the natural frequency density is high, even a small friction qualitatively changes the resonance curve. Damping (decrease in amplitudes) in real bodies is also important. In addition, wave nature of non-stationary processes hinders to just transfer the theory of oscillations of discrete systems to the continuum: in case of sudden local excitation it's more correct to consider waves instead of superposing modes.

The way from a continuous dynamic model to a discrete one goes through the variational approach.

$$\int_{\mathcal{V}}\left((\boldsymbol{f}-\rho\ddot{\boldsymbol{u}})\cdot\delta\boldsymbol{u}-\boldsymbol{\sigma}\cdot\delta\boldsymbol{\varepsilon}\right)d\mathcal{V}+\int_{o_2}\boldsymbol{p}\cdot\delta\boldsymbol{u}\,do = 0.$$

(1.4)

This is the principle of virtual work with forces of inertia. Looking for an approximate solution in series

$$\boldsymbol{u}(\boldsymbol{r},t) = \sum_{k=1}^{N}\alpha_k(t)\varphi_k(\boldsymbol{r}),$$

where φ_k are given $\left(\varphi_k\big|_{o_1} = \boldsymbol{0}\right)$, and $\alpha_k(t)$ are varying. The solution???? is a/?????the system of ordinary equations

...

In place of the principle of virtual work (1.4) one can use the Hellinger–Reissner mixed formulation with an independent approximation of stresses.

In the dynamic theory of elasticity, the Laplace integral transform is often applied. For simple shaped bodies it is sometimes possible to find an analytical solution in transforms. The original can be found by the numerical inversion, but sometimes it's possible to take the Riemann–Mellin integral[*] using the "saddle-point method" (or "method of

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[*] The Riemann–Mellin integral determines the inverse Laplace transform $F(s) \mapsto f(t)$.

steepest descents") with a deformation of contour in the complex plane [50, 75].

§ 2. Vibrations of rods

In the linear dynamics of rods we have the following system for forces $\boldsymbol{Q}$, force couples (moments) $\boldsymbol{M}$, displacements $\boldsymbol{u}$ and rotations $\boldsymbol{\theta}$ (§ 9.10):

$$\boldsymbol{Q}' + \boldsymbol{q} = \rho\big(\ddot{\boldsymbol{u}} + \ddot{\boldsymbol{\theta}} \times \boldsymbol{\varepsilon}\big), \ \ \boldsymbol{M}' + \boldsymbol{r}' \times \boldsymbol{Q} + \boldsymbol{m} = \boldsymbol{J} \cdot$$

$$(2.1)$$

$$\ldots \qquad\qquad (2.2)$$

...

§ 3. Small perturbations of parameters

Рассмотрим задачу об определении собственных частот и форм with small perturbations масс и жёсткостей:

$$\big(C_{ij} - \omega^2 A_{ij}\big)U_j = 0,$$
$$C_{ij} = C_{ij}^{(0)} + \chi C_{ij}^{(1)}, \ \ A_{ij} = A_{ij}^{(0)} + \chi A_{ij}^{(1)}, \ \ \chi \to 0.$$
$$(3.1)$$

Находя́ решение в виде
$$\omega = \omega^{(0)} + \chi\omega^{(1)} + \ldots, \ \ U_j = U_j^{(0)} + \chi U_j^{(1)} + \ldots,$$
получаем последовательность задач

...

§ 4. Vibrations of shells

Динамика оболочек рассматривалась многими

...

§ 5. Waves
in an elastic continuum

Рассмотрим линейные уравнения динамики однородной изотропной среды без объёмных сил

...

§ 6. Waves in a rod

Рассмотрим прямой стержень. Продольная деформация описывается уравнениями

...

§ 7. Nonlinear oscillations

Рассмотрим простой пример: продольные колебания прямого стержня с ма́лой нелинейной добавкой в соотношениях упругости

...

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Методы решения динамических задач упругости представлены в книгах Л. И. Слепяна [75] и В. Б. Поручикова [50]. О ма́лых линейных колебаниях (вибрациях) написано у С. П. Тимошенко, D. H. Young'а и W. Weaver'а [67], И. М. Бабакова [68], В. Л. Бидермана [69], В. Т. Гринченко и В. В. Мелешко [71]. Асимптотические проблемы колебаний оболочек освещены у ...

chapter 13
STABILITY

§1. The
various approaches
to the problem of stability

There's the well developed theory of stability by Ляпунов (Lya-punov). It says that if the initial deviations*are "close (small) enough", and afterwards they don't increase, remaining "close enough" forever, then the process is stable. This applies to the condition of equilibrium too. This is the **dynamic approach** to the problem of stability, and this approach is the most reasonable.

However, for problems about the stability of the equilibrium, another approach has gained popularity. It is called the **static** approach, the Leonhard Euler's name is associated with it. When the equations of statics give a nontrivial solution for the small deviations-displacements, then the values of parameters are assumed to be "critical". In other words, if the equilibrium isn't isolated, then it's considered the "critical". At the same time, many adjacent forms of the equilibrium appear**
Using this approach, it's enough to solve the problem about the eigenvalues.

---

* From the equilibrium configuration.

** This is known as the "indifferent equilibrium".

There are also more approaches. For example, the **method of imperfections**: if small random changes of the initial shape, the stiffnesses, the loads and other variables cause only a small change of a deformed configuration, then this equilibrium is stable. Or the **energy approach**: when a loss of stability becomes energetically beneficial, that is when it leads to a decrease of energy, then this loss of stability happens.

The mentioned approaches draw a motley picture. Yet it is pretty easy to visualize for a model with a finite number of degrees of freedom.

Большо́й общностью обладают

...

§ 2. Classical problems with rods

Состояние перед варьированием описывается уравнениями нелинейной теории стержней Kirchhoff'a

...

§ 3. "Tracking" loads

В проблемах устойчивости весьма весо́мо поведение нагрузки в процессе деформирования. Ведь в уравнения входит вариация (of what?) $\delta\boldsymbol{q}$, она равна нулю лишь для "dead" loads. Распространены the "tracking" loads,

which change in a certain way along with the deviations (displacements) of particles.

The Euler's static approach to the problem of stability

...

§4. The role of additional pliabilities

Для прямого консольного стержня, сжатого постоянной силой $\boldsymbol{F}$ на свободном конце, критическая нагрузка определяется формулой Euler'a

...

§5. Variational formulations

Во всех разделах линейной теории упругости большу́ю роль играют вариационные постановки. Среди прочего, они составляют основу метода конечных элементов как варианта метода Ritz'a.

Менее развиты вариационные постановки для проблем устойчивости. Здесь получил популярность метод

...

§6. Nonconservative problems

В уравнении динамики (...) матрица позиционных сил

...

§7. Case of multiple roots

Вернёмся к проблеме устойчивости (...) в случае циркуляционных сил. Как уже отмечалось(где??), критическая ситуация характеризуется

...

Bibliography

Увлекательные вопросы устойчивости упругих систем освещены в книгах ...

chapter 14
DEFECTS

§1. Volterra dislocations

Consider the classical linear three-dimensional elastic medium (chapter 4). As shown in §4.7, the equation of compatibility of deformations

...

§2. Straight-line dislocations

A dislocation line may be any curve, closed inside a body or with ends on the surface. For a dislocation of a random shape in a limitless medium, it's not so difficult to obtain an appropriate solution [18]. Мы же ограничимся простейшим случаем прямолинейной дислокации. Ищется решение

...

§3. Action of stress field on dislocation

Рассмотрим среду, содержащую внутри себя дислокацию с замкнутой линией C. Тело нагружено объёмными $\boldsymbol{v}$

и поверхностными p силами. Обозначим

...

§4. About movement of dislocations

Рассмотрим это явление, следуя [25]. Ограничимся случаем прямолинейной винтовой дислокации, движущейся с постоянной скоростью

...

§5. Point defects

Речь пойдёт о континуальной модели таких явлений как вакансии, примесные частицы или междоузельные атомы в кристаллической решётке. В случае дислокации рассматривались

...

§6. Force acting on a point defect

Дефект находится в континууме, нагруженном объёмными v и поверхностными p силами. Суперпозиция

...

§7. Continuously distributed dislocations

Начнём со сложения векторов Бюргерса. При обходе сразу двух дислокаций (рис. ?? 40 ??) по контуру

...

§8. Stress during winding of coil

Не только дислокации и точечные дефекты, но и макроскопические факторы могут быть источниками собственных напряжений. При намотке катушки (рис. ?? 42 ??) в ней возникают напряжения от натяжения ленты. Расчёт этих напряжений очень сложен, если рассматривать детально процесс укладки ленты.

Но существует чёткий алгоритм Southwell'a [51] расчёта напряжений в катушке: укладка каждого нового витка вызывает внутри катушки приращения напряжений, определяемые соотношениями линейной упругости. There are two stages, and the first one consists in solving the Lamé problem about deformation of a hollow cylinder under external pressure (рис. ?? 43 ??)

...

Bibliography

Dislocations and point defects in linear elastic bodies have been considered by many authors: John Eshelby [18], Roland deWit [14], Cristian Teodosiu [56], Alan Cottrell [25]. The theory of self-stresses (eigenspannungen) is explained by Ekkehart Kröner in [26]. Calculation of stresses when winding a coil is described by Richard Southwell in his book [51].

§1. Traditional
criteria of durability

K ак судить о прочности тела после определения напряжения в нём? При одноосном растяжении напряжением σ есть, очевидно, некий "предел прочности" σ_*, выше которого материал разрушается. Прочность считают достаточной при $\sigma \leq \sigma_*/k$, где k есть так называемый коэффициент запаса. Но такой подход не вполне удовлетворяет, поскольку получаемые из опытов значения σ_* have a big scatter, а выбор коэффициента запаса sometimes becomes a bureaucratic act.

Тем не менее, подобные взгляды на анализ прочности широко́ распространены. Воздерживаясь от критики, сто́ит упомянуть самые популярные из них.

Критерий максимального нормального напряжения: разрушение происходит при $\sigma_1 = \sigma_*$ (наибольшее из главных напряжений достигает предельного значения). Но это положение несправедливо для одноосного сжатия, при котором $\sigma_1 = 0$.

Критерий максимального касательного напряжения (Tresca criterion): разрушение наступает при $\sigma_1 - \sigma_3 = \sigma_*$ (σ_3 — наименьшее из главных напряжений). Это более соответствует началу пластического течения.

Критерий максимального удлинения: наибольшее из собственных значений тензора деформации $\varepsilon_1 = \varepsilon_*$. Это приемлемо и при сжатии с $\varepsilon_1 > 0$.

Критерий энергии деформации: $\Pi = \Pi_*$. Здесь учитывается, что разрушение требует энергии, а источником её может быть лишь само́ деформированное тело. Однако достаточный запас энергии — необходимое, но не единственное условие разрушения; должен включиться некий механизм преобразования упругой энергии в работу разрушения.

Критерий энергии формоизменения (von Mises yield criterion): $\boldsymbol{s} \boldsymbol{\cdot\cdot} \boldsymbol{s} = 2\tau_*^2$, $\boldsymbol{s} \equiv \boldsymbol{\tau} - \frac{1}{3}\boldsymbol{E}\,\mathrm{trace}\,\boldsymbol{\tau}$ ("девиатор напряжений"). Здесь не играет роли энергия объёмной деформации. Richard von Mises предложил[*] этот критерий как гладкую аппроксимацию условия Henri Tresca.

Критерий Mohr'a. Представим себе множество предельных состояний ...

...

§ 2. Antiplane deformation of continuum with a crack

Любая регулярная функция компле́ксного переменного $z = x + iy$ содержит в себе решение какой-либо антиплоской задачи статики без ...

...

§ 3. Crack in plane deformation

Рассмотрим плоскую область произвольного очертания с трещиной внутри; нагрузка приложена и "в объёме", и

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[*] **R. von Mises**. Mechanik der festen Körper im plastisch-deformablen Zustand. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-physikalische Klasse.* 1913, Seiten 582–592.

на внешнем крае. Как и при антиплоской деформации, решение строится в два этапа

...

§ 4. Crack-driving force

Это едва ли не основное понятие механики трещин. Рассмотрим его, следуя

...

§ 5. Criterion
of crack growth

Связанная с энергией $\mathscr{E}$ трещинодвижущая сила F — не единственное воздействие на передний край трещины. Должна быть ещё некая сила сопротивления F_*; рост трещины начинается при условии

...

§ 6. J-integral

Одно из самых известных понятий в механике трещин выражается интегралом

$$J = \ldots \qquad (6.1)$$

...

§ 7. Stress intensity factors

Расчёт прочности тела с трещиной сводится к определению коэффици-

ентов интенсивности напряжений. Методы расчёта таких коэффициентов — как аналитические, так и численные — хорошо освещены в литературе.

Рассмотрим ещё один подход к задачам механики трещин, разработанный

...

§ 8. Barenblatt's model

An unlimited increase of stress at the edge of a crack seems indeed dubious. Singular solutions desire support by some additional reasonings or by using of another model. And such a support далá работа

...

§ 9. Deformational criterion

D. S. Dugdale[*], а также М. Я. Леонов и В. В. Панасюк[**] предложили модель, напоминающую построения Баренблатта. Также есть силы сцепления q и равен нулю итоговый коэффициент интенсивности напряжений. Но, во-первых, q имеет иной вид:

[*] **Dugdale, D. S.** Yielding of steel sheets containing slits. *Journal of the Mechanics and Physics of Solids.* 1960, Volume 8, Issue 2, pages 100–104.

[**] **Леонов М. Я.**, **Панасюк В. В.** Развитие мельчайших трещин в твёрдом теле. *Прикладная механика.* 1959, Т. 5, № 4, с. 391–401.

...

Второе отличие рассматриваемой модели — в формулировке критерия прочности: трещина начинает расти, когда расхождение берегов в конце свободного участка достигает критического значения δ_* (этот параметр — константа материала), то есть при

...

§ 10. Growth of cracks

Пусть нагрузка на тело с трещиной выросла настолько, что выполняется условие

...

§ 11. Elastic field ahead of a moving crack

Рассмотрим этот вопрос

...

§ 12. Balance of energy for a moving crack

Уравнение баланса энергии в линейной теории ($\Pi = \frac{1}{2}\,\boldsymbol{\varepsilon} \cdot\cdot {}^{4}\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\varepsilon}$, $\mathrm{K} = \frac{1}{2}\,\rho\,\dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}$):

$$\int\limits_{\mathcal{V}} \left(\mathrm{K} + \Pi\right)^{\bullet} d\mathcal{V} = \int\limits_{\mathcal{V}} \boldsymbol{f}\cdot\dot{\boldsymbol{u}}\,d\mathcal{V} + \int\limits_{\mathcal{O}} \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\dot{\boldsymbol{u}}\,d\mathcal{O}.$$

$$(12.1)$$

...

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Список книг по механике трещин уже́ вели́к. В нём сто́ит отметить работы Л. М. Качанова [22], Н. Ф. Морозова [37], В. З. Партона и Е. М. Морозова [48], Г. П. Черепанова [64]. Обзор статей есть у ... Экспериментальные данные представлены, например, в [23].

chapter 16
COMPOSITES

§1. Introductory thoughts

W hen the clay is used as a building material, a shredded straw is added to it. Working with an epoxy gum, it's useful to blend with a filler before solidifying, which can be a powder, fibers, pieces o' fabric. These were examples of composites (composite materials, composite mixtures). The new types of composites are used more and more widely, replacing the steel, the aluminum alloys and another popular homogeneous materials.

The composites can be defined as the micro-inhomogeneous materials, where happens some averaging together with the new properties. The "usual" mechanics of continua is applicable, surely, to the composite materials too. But it's barely possible and is absurd to model the every aspect of a composite material. The new approach is needed that will deal with the complexity of material structure. As for a gas, when instead of describing the dynamics of individual molecules we introduce the parameters, such as "the pressure", "the temperature" and the others.

The mechanics of a homogeneous non-composite continuum uses the only single, macroscopic length ℓ. Thus the characteristic dimension is the vol-

ume divided by the surface area), assuming that any small volumes have the same properties as the finite volumes.

For a composite it's different, there we have the three scopes of length: $\ell \gg \ell' \gg \ell_0$. The largest ℓ presents the macroscopic dimensions of a body. The smallest ℓ_0 is near to the size of elements of the material structure, for example particles of a filler powder). The intermediate scope ℓ' shows the size of a so-called "representative" volume, of "the unit cell": that small enough volume, where we can see the specific properties of this composite.

In composites, the complex problem with an inhomogeneous body is split into the two: for a body as a whole (the macrolevel) and for a "representative" volume (the microlevel). At the macrolevel (scope ℓ), a composite is modeled as a homogeneous medium with "the effective properties", and the "representative" volumes play there the role of particles. At the microlevel (scope ℓ_0) fields are very inhomogeneous, but some averaging by the "representative" volume leads to the macrolevel. The complex problem for a composite as an inhomogeneous body splits into the two: for a representative volume (the microlevel) and for a body as a whole (the macrolevel).

These thoughts, however, are not perfectly convincing. To convince more, we can use "the random field theory"

...

The mechanics of composites appeared not very long ago, but yet it is developing pretty fast. Due to the high fracture toughness, the fracture mechanics of composites is very popular.

§2. Effective fields

Any field in a composite is usually represented by the sum $u = u* + u'$, where $u*$ is some smoothed "effective" field, and u' is fast oscillating fluctuation. It's often assumed

$$u*(A) = <u> \equiv \mathcal{V}^{-1} \int_{\mathcal{V}} u d\mathcal{V}, \quad (2.1)$$

where $<u>$ is an average within a representative volume centered at point A. Averaging (2.1)

...

§3. Boundary value problems for representative volume

How the elastic moduli are determined for a homogeneous medium? Without the real possibility to get a relation between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ for a point of infinitesimal size, experiments are carried out with bodies of finite volumes under a so-called "homogeneous stress" —

when stress is the same at all points of a body. In composites, the role of a point plays a "representative" volume.

...

§ 4. Hill's fork

Using Voigt and Reuss theories, Hill derived upper and lower bounds on the effective properties of a composite material [**Hill, R. W.** The elastic behaviour of a crystalline aggregate. *Proceedings of the Physical Society*, Section A, Volume 65, Issue 5 (May 1952). Pages 349–354.]

The scale separation is motivated by the material properties, at both scales continuum mechanics models the underlying system. Such an approach uses energy equivalence at both scales as proposed in Hill (1972).

Hill R (1972) On constitutive macro-variables for heterogeneous solids at finite strain (pages 131–147)

For a composite material, at least two different materials with known material models and parameters, generate a homogenized material modeled with a predetermined constitutive equation. Determination of material parameters of the homogenized material is a challenging task.

Отметив, что

...

§5. Eshelby formulas

Итак, эффективные модули определяются энергией представительного объёма в первой или второй задачах:

...

§6. Effective moduli for material with spherical inclusions

В однородной матрице случайным образом, но достаточно равномерно, распределены сферические включения радиусом a. Получившийся композит на макроуровне будет изотропным, его упругие свойства полностью определяются ...

...

§7. Self-consistent method

Выше мы опирались на две задачи for a representative volume и определяли effective modules from the equality of energies. В основе метода самосогласования лежит новая идея: представительный объём помещается в безграничную среду с эффективными свойствами, на бесконечности состояние считается однородным, эффективные модули находятся из некоторых дополнительных условий самосогласования.

Обратимся снова к вопросу об объёмном модуле среды со сферически-

ми включениями. Задача сферически симметрична; для включения по-прежнему

...

§8. Hashin–Shtrikman principle

Hashin and Shtrikman derived upper and lower bounds for the effective elastic properties of quasi-isotropic and quasi-homogeneous multiphase materials using a variational approach [**Hashin, Z.**; **Shtrikman, S.** A variational approach to the theory of the elastic behaviour of multiphase materials. // Journal of the Mechanics and Physics of Solids. Volume 11, Issue 2 (March–April 1963). Pages 127–140.]

Hashin Z., Shtrikman S. (1962) On some variational principles in anisotropic and nonhomogeneous elasticity. Journal of the Mechanics and Physics of Solids 10(4): pages 335–342

Вилка Hill'а основана на обычных экстремальных принципах теории упругости. Специально для механики композитов Hashin и Shtrikman построили очень своеобразный функционал, который на некотором точном решении может иметь как максимум, так и минимум, давая возможность с двух сторон оценивать эффективные модули [66].

Рассмотрим первую из двух задач для представительного объёма

...

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Книги R. Christensen'а [77] and Б. Е. Победри [79] содержат и основы механики композитов, и постановку не теряющих актуальности проблем. Для са́мого требовательного читателя представляет интерес монография Т. Д. Шермергора [66]. Немало книг посвящено механике разрушения композитов, здесь сто́ит отметить труд Г. П. Черепанова [80].

chapter 17

PERIODIC COMPOSITES

A periodic composite is made of a repetition,
finite or infinite, of a unit cell.

§ 1. One-dimensional problem

В одномерной задаче статики имеем уравнение

...

§ 2. Three-dimensional continuum

Исходим из уравнений в смещениях

...

§ 3. Fibrous structure

Тензор ${}^4\!\boldsymbol{A}$ в этом случае постоянен вдоль оси

...

§ 4. Statics of a periodic rod

В уравнениях линейной статики стержня

...

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Лежащий в основе этой главы асимптотический метод представлен с разной степенью математической скрупулёзности в книгах [81, 82, 78, 79].

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