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PHYSICS *of* ELASTIC CONTINUA

CONTENTS

chapter 4

THE CLASSICAL LINEAR ELASTICITY

This chapter is about the geometrically linear model with infinitesimal displacements, where

- ✓ $\mathcal{V} = \mathring{\mathcal{V}}$, $\rho = \mathring{\rho}$ — "the equations can be written in the initial configuration" (sometimes it's called "the principle of initial dimensions"),
- ✓ operators $\mathring{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ are indistinguishable,
- ✓ operators δ and $\boldsymbol{\nabla}$ commute, thus for example $\delta \boldsymbol{\nabla} \boldsymbol{u} = \boldsymbol{\nabla} \delta \boldsymbol{u}$.

§ 1. The complete set of equations

Equations of the nonlinear elasticity, even in their simplest cases, lead to the mathematically complex problems. Therefore the linear theory of infinitesimal displacements is applied everywhere. This theory's equations were derived in the first half of the XIX$^{\text{th}}$ century by Cauchy, Navier, Lamé, Clapeyron, Poisson, Saint-Venant, George Green and the other scientists.

The complete closed set of equations of the classical linear theory in the direct invariant tensor notation, consisting of

- ✓ the balance of forces (of momentum),
- ✓ the stress–strain relations for a material,
- ✓ displacement $\boldsymbol{u} \mapsto \boldsymbol{\varepsilon}$ relative deformation,

is

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} = \boldsymbol{0}, \quad \boldsymbol{\sigma} = \frac{\partial \Pi}{\partial \boldsymbol{\varepsilon}} = {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \boldsymbol{\nabla} \boldsymbol{u}^{\mathsf{S}}. \tag{1.1}$$

Here $\boldsymbol{\sigma}$ is the linear stress tensor, $\boldsymbol{g}$ is the resultant vector of volume loads*, $\boldsymbol{u}$ is the vector of displacement (the "absolute" displacement

* In linear theory, the "per volume unit" loads $\boldsymbol{g} d\mathcal{V}$ are used much more often than the "per mass unit" $\boldsymbol{f} dm$ ones ($\boldsymbol{g} d\mathcal{V} = \boldsymbol{f} dm$, $\boldsymbol{g} = \rho \boldsymbol{f}$), because $d\mathcal{V} = d\mathring{\mathcal{V}}$, $\rho d\mathcal{V} = \mathring{\rho} d\mathring{\mathcal{V}}$ and $\boldsymbol{f} dm = \boldsymbol{f} \mathring{\rho} d\mathring{\mathcal{V}} = \boldsymbol{f} \rho d\mathcal{V} = \boldsymbol{g} d\mathcal{V}$.

or absolute deformation), $\boldsymbol{\varepsilon}$ is the tensor of infinitesimal relative deformation (relative displacement, or strain*), $\Pi(\boldsymbol{\varepsilon})$ is the potential energy of deformation per volume unit and ${}^4\!\boldsymbol{\mathcal{A}}$ is the stiffness tensor. The latter is tetravalent with the following symmetry

$$ {}^4\!\boldsymbol{\mathcal{A}}_{12\rightleftarrows 34} = {}^4\!\boldsymbol{\mathcal{A}}, \quad {}^4\!\boldsymbol{\mathcal{A}}_{1\rightleftarrows 2} = {}^4\!\boldsymbol{\mathcal{A}}, \quad {}^4\!\boldsymbol{\mathcal{A}}_{3\rightleftarrows 4} = {}^4\!\boldsymbol{\mathcal{A}}. $$

But where does all this come from?

The equations (1.1) are exact, they can be derived by varying the equations of the nonlinear theory. Varying from an arbitrary configuration is described in § ??.??. The linear theory is the result of varying from the initial unstressed configuration, when

$$ \begin{gathered} \boldsymbol{F} = \boldsymbol{E}, \;\; \boldsymbol{C} = {}^2\boldsymbol{0}, \;\; \delta\boldsymbol{C} = \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}}, \\ \boldsymbol{\tau} = {}^2\boldsymbol{0}, \;\; \delta\boldsymbol{\tau} = \delta\boldsymbol{T} = \frac{\partial^2\Pi}{\partial\boldsymbol{C}\partial\boldsymbol{C}} \cdot\cdot\, \delta\boldsymbol{C}, \;\; \boldsymbol{\nabla}\cdot\delta\boldsymbol{\tau} + \rho\,\delta\boldsymbol{f} = \boldsymbol{0}. \end{gathered} \tag{1.2} $$

It remains to change

- ✓ $\delta\boldsymbol{r}$ to $\boldsymbol{u}$,
- ✓ $\delta\boldsymbol{C}$ to $\boldsymbol{\varepsilon}$,
- ✓ $\delta\boldsymbol{\tau}$ to $\boldsymbol{\sigma}$,
- ✓ $\partial^2\Pi/\partial\boldsymbol{C}\partial\boldsymbol{C}$ to ${}^4\!\boldsymbol{\mathcal{A}}$,
- ✓ $\rho\,\delta\boldsymbol{f}$ to $\boldsymbol{g}$.

If the derivation (1.2) via varying seems abstruse to the reader, it's possible to proceed from the following equations

$$ \begin{gathered} \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\boldsymbol{f} = \boldsymbol{0}, \;\; \boldsymbol{\nabla} = \boldsymbol{F}^{-\mathsf{T}}\cdot\mathring{\boldsymbol{\nabla}}, \;\; \boldsymbol{F} = \boldsymbol{E} + \mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}, \\ \boldsymbol{\tau} = J^{-1}\boldsymbol{F}\cdot\frac{\partial\Pi}{\partial\boldsymbol{C}}\cdot\boldsymbol{F}^{\mathsf{T}}, \;\; \boldsymbol{C} = \mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{S}} + \tfrac{1}{2}\mathring{\boldsymbol{\nabla}}\boldsymbol{u}\cdot\mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}. \end{gathered} \tag{1.3} $$

Assuming the displacement $\boldsymbol{u}$ is small (infinitesimal), we'll move from (1.3) to (1.1).

* As written in the first sentence of "Historical introduction" in the Augustus Edward Hough Love's book [26].

Or so. Instead of $\boldsymbol{u}$ to take some small enough parameter $\chi\boldsymbol{u}$, $\chi \to 0$. And to represent thereafter the unknowns by the series in the integer exponents of parameter χ

$$\boldsymbol{\tau} = \boldsymbol{\tau}^{(0)} + \chi\boldsymbol{\tau}^{(1)} + \ldots, \;\; \boldsymbol{C} = \boldsymbol{C}^{(0)} + \chi\boldsymbol{C}^{(1)} + \ldots,$$
$$\boldsymbol{\nabla} = \mathring{\boldsymbol{\nabla}} + \chi\boldsymbol{\nabla}^{(1)} + \ldots, \;\; \boldsymbol{F} = \boldsymbol{E} + \chi\mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}, \;\; J = 1 + \chi J^{(1)} + \ldots$$

The complete set of equations (1.1) comes from the first (zeroth) terms of these series. In the book [54] this is called "formal approximation".

It is impossible to tell unambiguously how small the parameter χ should be — the answer depends on the situation and is determined by whether the linear model describes the effect we are interested in or not. When, as example, I'm interested in the relation between the frequency of a freely vibrating motion after the initial displacement, then a nonlinear model is needed.

A linear problem is posed in the initial volume $\mathcal{V} = \mathring{\mathcal{V}}$, bounded by the surface o with the area vector $\boldsymbol{n}do$ ("the principle of initial dimensions").

The boundary conditions (that is, conditions on the surface) most often are : on the part o_1 of the surface displacements are known, and on another part o_2 the forces are known.

$$\boldsymbol{u}\Big|_{o_1} = \boldsymbol{u}_0\,, \quad \boldsymbol{n} \cdot \boldsymbol{\sigma}\Big|_{o_2} = \boldsymbol{p}. \tag{1.4}$$

The more complex combinations happen too, if we know the certain components of the both $\boldsymbol{u}$ and $\boldsymbol{t}_{(n)} = \boldsymbol{n} \cdot \boldsymbol{\sigma}$ simultaneously. For example, on a flat face $x = \mathsf{constant}$ when pressing a stamp with a smooth surface $u_x = \nu(y,z)$, $\tau_{xy} = \tau_{xz} = 0$ (the function ν is determined by the stamp's shape).

In dynamics, vector $\boldsymbol{g}$ also includes the inertial addend $-\rho\ddot{\boldsymbol{u}}$. The conditions on the surface (boundary conditions) here may depend on time. And the initial conditions for dynamic problems commonly are : positions $\boldsymbol{u}$ and velocities $\dot{\boldsymbol{u}}$, known at the specific point in time $t = 0$.

The linearity gives the principle of superposition (or independence) of the action of loads. When there are several loads, the problem can be solved for each load separately, and the complete solution is then obtained by summation. For statics this means, for example, that if external loads $\boldsymbol{g}$ and $\boldsymbol{p}$ increase by m times (the body is fixed on o_1), then $\boldsymbol{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ will increase by m times too. The potential energy Π will increase by m^2 times. In reality, this is observed only when the loads are small.

The potential energy

With linearity, the potential energy density Π as a function of infinitesimal deformation $\boldsymbol{\varepsilon}$ is a quadratic form

$$\Pi(\boldsymbol{\varepsilon}) = \tfrac{1}{2}\, \boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}\,, \tag{1.5}$$

the variation of which is

$$\delta\Pi = \tfrac{1}{2}\,\delta\big(\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}\big) = \tfrac{1}{2}\big(\delta\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \delta\boldsymbol{\varepsilon}\big) = \underbrace{\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}}}_{\frac{\partial\Pi}{\partial\boldsymbol{\varepsilon}} = \boldsymbol{\sigma}} \cdot\cdot\, \delta\boldsymbol{\varepsilon}$$

and the second variation

$$\delta^2\Pi = \delta\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \delta\boldsymbol{\varepsilon}\,.$$

The relation between the stiffness tensor and the potential energy of elastic deformation is now clear :

$${}^4\!\boldsymbol{\mathcal{A}} = \frac{\partial^2\Pi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}}\,,$$

because $\delta\Pi(\boldsymbol{\varepsilon}) = \dfrac{\partial\Pi}{\partial\boldsymbol{\varepsilon}} \cdot\cdot\, \delta\boldsymbol{\varepsilon}$ and $\delta^2\Pi(\boldsymbol{\varepsilon}) = \delta\boldsymbol{\varepsilon} \cdot\cdot\, \dfrac{\partial^2\Pi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}} \cdot\cdot\, \delta\boldsymbol{\varepsilon}$.

Adding that here, as well as for 1-variable calculus*, $\delta^2\Pi = 2\Pi(\delta\boldsymbol{\varepsilon})$, the first and second variations of Π may be written as

$$\begin{aligned} \delta\Pi(\boldsymbol{\varepsilon}) &= \frac{\partial\Pi}{\partial\boldsymbol{\varepsilon}} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot\cdot\, \delta\boldsymbol{\varepsilon}\,, \\ \delta^2\Pi(\boldsymbol{\varepsilon}) &= \delta\boldsymbol{\varepsilon} \cdot\cdot\, \frac{\partial^2\Pi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = \delta\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = 2\Pi(\delta\boldsymbol{\varepsilon})\,. \end{aligned} \tag{1.6}$$

* for $y(x) = \frac{1}{2}\alpha x^2 = \frac{1}{2} x\alpha x$ $\quad dy = \frac{1}{2}\big(dx\alpha x + x\alpha dx\big) = x\alpha dx$

$d^2y = dx\alpha dx = \alpha(dx)^2$

that is $d^2y = 2y(dx)$

The d'Alembert–Lagrange principle of virtual work (**??**, § **??**.**??**), which can be used as the foundation of mechanics, applies to the linear theory as well. Since internal forces in any elastic medium are potential ($\delta W^{(i)} = -\delta\Pi$), the principle is formulated as

$$\int\limits_{\mathcal{V}} \Big((\boldsymbol{g} - \rho \ddot{\boldsymbol{u}}) \cdot \delta \boldsymbol{u} - \delta\Pi \Big) d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p} \cdot \delta \boldsymbol{u} \, do = 0, \quad \boldsymbol{u}\big|_{o_1} = \boldsymbol{0}. \tag{1.7}$$

In addition, if the medium is elastic *linearly*, then

$$\delta\boldsymbol{\varepsilon} = \delta\big(\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}\big) = \boldsymbol{\nabla}\big(\delta\boldsymbol{u}^{\mathsf{S}}\big) = \boldsymbol{\nabla}\delta\boldsymbol{u}^{\mathsf{S}},$$

$$\delta\Pi = \boldsymbol{\sigma} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla}\delta\boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \delta\boldsymbol{u}) - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \cdot \delta\boldsymbol{u},$$

$$\int\limits_{\mathcal{V}} \delta\Pi \, d\mathcal{V} = \oint\limits_{o(\partial\mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \delta\boldsymbol{u} \, do - \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \cdot \delta\boldsymbol{u} \, d\mathcal{V},$$

and (1.7) becomes

$$\int\limits_{\mathcal{V}} \Big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} - \rho\ddot{\boldsymbol{u}} \Big) \cdot \delta\boldsymbol{u} \, d\mathcal{V} + \int\limits_{o_2} \Big(\boldsymbol{p} - \boldsymbol{n} \cdot \boldsymbol{\sigma} \Big) \cdot \delta\boldsymbol{u} \, do = 0.$$

Here, virtual displacements $\delta\boldsymbol{u}$ are compatible with the boundary condition for displacements in (1.7), $\delta\boldsymbol{u}\big|_{o_1} = \boldsymbol{0}$.

§ 2. The uniqueness of the solution in dynamics

Usually the uniqueness theorem is proven "by contradiction". Assume that there are two solutions : $\boldsymbol{u}_1(\boldsymbol{r}, t)$ and $\boldsymbol{u}_2(\boldsymbol{r}, t)$. If the difference $\boldsymbol{u}^* \equiv \boldsymbol{u}_1 - \boldsymbol{u}_2$ will be equal to $\boldsymbol{0}$, then these solutions coincide, that is the solution is unique.

But at first we'll make sure of the existænce of the energy integral — by deriving the balance of mechanical energy equation for the linear model of the small displacements theory

$$\int_{\mathcal{V}} \bigl(\mathrm{K} + \Pi\bigr)^{\bullet} d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{g} \cdot \dot{\boldsymbol{u}}\, d\mathcal{V} + \int_{o_2} \boldsymbol{p} \cdot \dot{\boldsymbol{u}}\, do, \tag{2.1}$$

$$\boldsymbol{u}\bigr|_{o_1} = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \boldsymbol{\sigma}\bigr|_{o_2} = \boldsymbol{p},$$

$$\boldsymbol{u}\bigr|_{t=0} = \boldsymbol{u}^{\circ}, \quad \dot{\boldsymbol{u}}\bigr|_{t=0} = \dot{\boldsymbol{u}}^{\circ}.$$

For the left-hand side we have

$$\dot{\mathrm{K}} = \tfrac{1}{2}\bigl(\rho \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}\bigr)^{\bullet} = \tfrac{1}{2}\rho\bigl(\dot{\boldsymbol{u}} \cdot \ddot{\boldsymbol{u}} + \ddot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}\bigr) = \rho \ddot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}},$$

$$\dot{\Pi} = \tfrac{1}{2}\underbrace{\bigl(\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\varepsilon}\bigr)^{\bullet}}_{2\boldsymbol{\varepsilon} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}}} = \boldsymbol{\sigma} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla} \dot{\boldsymbol{u}}^{\mathsf{S}} = \boldsymbol{\nabla} \cdot \bigl(\boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\bigr) - \underbrace{\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}}_{-(\boldsymbol{g} - \rho \ddot{\boldsymbol{u}})} \cdot\, \dot{\boldsymbol{u}} = $$
$$= \boldsymbol{\nabla} \cdot \bigl(\boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\bigr) + \bigl(\boldsymbol{g} - \rho \ddot{\boldsymbol{u}}\bigr) \cdot \dot{\boldsymbol{u}}$$

(the balance of momentum $\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} - \rho \ddot{\boldsymbol{u}} = \boldsymbol{0}$ is used),

$$\dot{\mathrm{K}} + \dot{\Pi} = \boldsymbol{\nabla} \cdot \bigl(\boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\bigr) + \boldsymbol{g} \cdot \dot{\boldsymbol{u}}.$$

Applying the divergence theorem

$$\int_{\mathcal{V}} \boldsymbol{\nabla} \cdot \bigl(\boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\bigr)\, d\mathcal{V} = \oint_{o(\partial\mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{u}}\, do$$

and the boundary condition $\boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{p}$ on o_2, we get (2.1).

From (2.1) it follows that without loads (when there're no external forces, neither volume nor surface), and the full mechanical energy doesn't change :

$$\boldsymbol{g} = \boldsymbol{0} \text{ and } \boldsymbol{p} = \boldsymbol{0} \;\Rightarrow\; \int_{\mathcal{V}} \bigl(\mathrm{K} + \Pi\bigr)\, d\mathcal{V} = \mathsf{constant}(t). \tag{2.2}$$

If at the moment $t = 0$ there was unstressed ($\Pi = 0$) rest ($\mathrm{K} = 0$), then

$$\int_{\mathcal{V}} \bigl(\mathrm{K} + \Pi\bigr)\, d\mathcal{V} = 0. \tag{2.2'}$$

The kinetic energy is positive : $\mathrm{K} > 0$ if $\dot{\boldsymbol{u}} \neq \mathbf{0}$ and vanishes (nullifies) only when $\dot{\boldsymbol{u}} = \mathbf{0}$ — this ensues from its definition $\mathrm{K} \equiv \frac{1}{2} \rho \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}}$. The potential energy, being a quadratic form $\Pi(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon} \cdot\cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\varepsilon}$ (1.5), is positive too : $\Pi > 0$ if $\boldsymbol{\varepsilon} \neq {}^2\mathbf{0}$. Such is a priori requirement of the positive definiteness for stiffness tensor ${}^4\!\boldsymbol{\mathcal{A}}$. This is one of the "additional inequalities in the theory of elasticity" [27, 54].

Since K and Π are positive definite, (2.2′) gives

$$\mathrm{K} = 0,\ \Pi = 0 \ \Rightarrow\ \dot{\boldsymbol{u}} = \mathbf{0},\ \boldsymbol{\varepsilon} = \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} = {}^2\mathbf{0} \ \Rightarrow\ \boldsymbol{u} = \boldsymbol{u}^{\circ} + \boldsymbol{\omega}^{\circ} \times \boldsymbol{r}$$

($\boldsymbol{u}^{\circ}$ and $\boldsymbol{\omega}^{\circ}$ are some constants of translation and rotation). With an immobile part of the surface

$$\boldsymbol{u}\big|_{o_1} = \mathbf{0} \ \Rightarrow\ \boldsymbol{u}^{\circ} = \mathbf{0} \text{ and } \boldsymbol{\omega}^{\circ} = \mathbf{0} \ \Rightarrow\ \boldsymbol{u} = \mathbf{0} \text{ everywhere.}$$

Now back to the two solutions, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Their difference $\boldsymbol{u}^* \equiv \boldsymbol{u}_1 - \boldsymbol{u}_2$ is a solution of the entirely "homogeneous" (with no constant terms at all) linear problem : in a volume $\boldsymbol{g} = \mathbf{0}$, in boundary and in initial conditions — zeroes. Therefore $\boldsymbol{u}^* = \mathbf{0}$, and the uniqueness is proven.

As for the existænce of a solution — it cannot be proven for the generic case by simple conclusions. I could only tell that a dynamic problem is evolutional, it describes the progress of a process in time.

The balance (the conservation) of momentum gives the acceleration $\ddot{\boldsymbol{u}}$. Then, moving to the "next time layer" $t + dt$:

$$\begin{gathered}
\dot{\boldsymbol{u}}(\boldsymbol{r}, t+dt) = \dot{\boldsymbol{u}}(\boldsymbol{r}, t) + \ddot{\boldsymbol{u}} dt, \\
\boldsymbol{u}(\boldsymbol{r}, t+dt) = \boldsymbol{u}(\boldsymbol{r}, t) + \dot{\boldsymbol{u}} dt, \\
\boldsymbol{\varepsilon}(\boldsymbol{r}, t+dt) = \big(\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{r}, t+dt)\big)^{\mathsf{S}} \ \Rightarrow\ \boldsymbol{\sigma}, \\
\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} = \rho \ddot{\boldsymbol{u}}(\boldsymbol{r}, t+dt)
\end{gathered}$$

and so forth. Surely, these considerations lack the "mathematical scrupulosity". If the reader is looking for such, there is, for example, the monograph by Philippe Ciarlet [22].

That relation between the stress and the deformation (strain), which in the XVII$^{\text{th}}$ century Robert Hooke could only phrase pretty vaguely*, is written as part of the complete set of equations (1.1) and is implemented via the stiffness tensor

$$
{}^4\!\boldsymbol{\mathcal{A}} = \frac{\partial^2 \Pi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}} = A^{ijkl} \boldsymbol{r}_{\partial i} \boldsymbol{r}_{\partial j} \boldsymbol{r}_{\partial k} \boldsymbol{r}_{\partial l} \,, \quad A^{ijkl} = \frac{\partial^2 \Pi}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \,. \tag{3.1}
$$

The stiffness tensor is the partial derivative of the scalar elastic potential energy density Π twice by the same bivalent infinitesimal deformation tensor $\boldsymbol{\varepsilon}$. It is symmetric in the pairs of indices : ${}^4\!\boldsymbol{\mathcal{A}}_{12\rightleftarrows 34} = {}^4\!\boldsymbol{\mathcal{A}} \Leftrightarrow A^{ijkl} = A^{klij}$. Therefrom 36 constants out of $3^4 = 81$ "have a twin" and only 45 are independent. Furthermore, due to the symmetry of the infinitesimal deformation tensor $\boldsymbol{\varepsilon}$, the stiffness tensor ${}^4\!\boldsymbol{\mathcal{A}}$ is symmetric inside each pair of indices : $A^{ijkl} = A^{jikl} = A^{ijlk}$ $\left(= A^{jilk}\right)$. This reduces the number of the independent constants (the "elastic moduli") to 21 :

$$
A^{abcd} = A^{cdab} = A^{bacd} = A^{abdc}
$$

$$
\begin{aligned}
&A^{1111} \\
&A^{1112} = A^{1121} = A^{1211} = A^{2111} \\
&A^{1113} = A^{1131} = A^{1311} = A^{3111} \\
&A^{1122} = A^{2211} \\
&A^{1123} = A^{1132} = A^{2311} = A^{3211} \\
&A^{1133} = A^{3311} \\
&A^{1212} = A^{1221} = A^{2112} = A^{2121} \\
&A^{1213} = A^{1231} = A^{1312} = A^{1321} = A^{2113} = A^{2131} = A^{3112} = A^{3121} \\
&A^{1222} = A^{2122} = A^{2212} = A^{2221} \\
&A^{1223} = A^{1232} = A^{2123} = A^{2132} = A^{2312} = A^{2321} = A^{3212} = A^{3221} \\
&A^{1233} = A^{2133} = A^{3312} = A^{3321} \\
&A^{1313} = A^{1331} = A^{3113} = A^{3131}
\end{aligned}
$$

* "*ceiiinosssttuu, id est, Ut tensio sic vis*" — **Robert Hooke**. Lectures de Potentia Restitutiva, Or of Spring Explaining the Power of Springing Bodies. London, 1678. 56 pages.

$$
\begin{aligned}
&A^{1322} = A^{2213} = A^{2231} = A^{3122}\\
&A^{1323} = A^{1332} = A^{2313} = A^{2331} = A^{3123} = A^{3132} = A^{3213} = A^{3231}\\
&A^{1333} = A^{3133} = A^{3313} = A^{3331}\\
&A^{2222}\\
&A^{2223} = A^{2232} = A^{2322} = A^{3222}\\
&A^{2233} = A^{3322}\\
&A^{2323} = A^{2332} = A^{3223} = A^{3232}\\
&A^{2333} = A^{3233} = A^{3323} = A^{3332}\\
&A^{3333}
\end{aligned}
$$

The moduli of the tetravalent stiffness tensor are often written as the symmetric 6×6 matrix

$$
\underset{6\times 6}{[\mathcal{A}]} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix} \equiv \begin{bmatrix} A^{1111} & A^{1122} & A^{1133} & A^{1112} & A^{1113} & A^{1123} \\ A^{2211} & A^{2222} & A^{2233} & A^{1222} & A^{1322} & A^{2223} \\ A^{3311} & A^{3322} & A^{3333} & A^{1233} & A^{1333} & A^{2333} \\ A^{1211} & A^{2212} & A^{3312} & A^{1212} & A^{1213} & A^{1223} \\ A^{1311} & A^{2213} & A^{3313} & A^{1312} & A^{1313} & A^{1323} \\ A^{2311} & A^{2322} & A^{3323} & A^{2312} & A^{2313} & A^{2323} \end{bmatrix}
$$

Even in Cartesian coordinates x, y, z, the quadratic form (1.5) looks pretty huge :

$$
\begin{aligned}
2\Pi = a_{11}\varepsilon_x^2 + a_{22}\varepsilon_y^2 + a_{33}\varepsilon_z^2 + a_{44}\varepsilon_{xy}^2 + a_{55}\varepsilon_{xz}^2 + a_{66}\varepsilon_{yz}^2 \\
+ 2\big[\varepsilon_x\big(a_{12}\varepsilon_y + a_{13}\varepsilon_z + a_{14}\varepsilon_{xy} + a_{15}\varepsilon_{xz} + a_{16}\varepsilon_{yz}\big) \\
+ \varepsilon_y\big(a_{23}\varepsilon_z + a_{24}\varepsilon_{xy} + a_{25}\varepsilon_{xz} + a_{26}\varepsilon_{yz}\big) \\
+ \varepsilon_z\big(a_{34}\varepsilon_{xy} + a_{35}\varepsilon_{xz} + a_{36}\varepsilon_{yz}\big) \\
+ \varepsilon_{xy}\big(a_{45}\varepsilon_{xz} + a_{46}\varepsilon_{yz}\big) + a_{56}\varepsilon_{xz}\varepsilon_{yz}\big].
\end{aligned}
\tag{3.2}
$$

When a material symmetry is added, then the number of the independent moduli of tensor ${}^4\!\boldsymbol{\mathcal{A}}$ decreases.

One plane of material symmetry, a monoclinic material

For a material with a symmetry plane of the elastic properties, for example $z = \mathsf{constant}$.

The change of signs of x and y coordinates does not change the potential energy density Π. And this is possible only when

$$\Pi\Big|_{\substack{\varepsilon_{xz}=-\varepsilon_{xz}\\ \varepsilon_{yz}=-\varepsilon_{yz}}} = \Pi \qquad \Leftrightarrow \qquad \begin{aligned} 0 = a_{15} = a_{16} = a_{25} = a_{26} \\ = a_{35} = a_{36} = a_{45} = a_{46} \end{aligned} \tag{3.3}$$

— the number of independent coefficients lowers to 13.

An orthotropic material

Let there be then the two planes of symmetry : $z = \mathsf{constant}$ and $y = \mathsf{constant}$. Because energy Π in such a case is not sensitive to the signs of ε_{yx} and ε_{yz}, in addition to (3.3) we have

$$a_{14} = a_{24} = a_{34} = a_{56} = 0 \tag{3.4}$$

— 9 constants remained.

A material with the three mutually orthogonal planes of symmetry — let these be the x and y, z planes — is called the orthotropic (orthogonally anisotropic). It's easy to see that (3.3) and (3.4) is the whole set of zero constants, in this case as well. So, an orthotropic material is characterized by the nine elastic moduli, and for orthotropy the two mutually perpendicular planes of symmetry are enough. The expression for the elastic energy density here can be simplified to

$$\Pi = \tfrac{1}{2}a_{11}\varepsilon_x^2 + \tfrac{1}{2}a_{22}\varepsilon_y^2 + \tfrac{1}{2}a_{33}\varepsilon_z^2 + \tfrac{1}{2}a_{44}\varepsilon_{xy}^2 + \tfrac{1}{2}a_{55}\varepsilon_{xz}^2 + \tfrac{1}{2}a_{66}\varepsilon_{yz}^2 \\ + a_{12}\varepsilon_x\varepsilon_y + a_{13}\varepsilon_x\varepsilon_z + a_{23}\varepsilon_y\varepsilon_z\,.$$

For an orthotropic material, the shear (angular) deformations ε_{xy}, ε_{xz}, ε_{yz} are not linked to the normal stresses $\sigma_x = {}^{\partial\Pi}\!/_{\partial\varepsilon_x}$, $\sigma_y = {}^{\partial\Pi}\!/_{\partial\varepsilon_y}$, $\sigma_z = {}^{\partial\Pi}\!/_{\partial\varepsilon_z}$ (and vice versa).

The popular orthotropic material is wood. Elastic properties there differ along three mutually perpendicular lines : by the radius, along the circumference and along the trunk height.

A transversely isotropic material

One more case of anisotropy is a transversely isotropic material. It is characterized by an axis of anisotropy — let it be z. Then any plane which is parallel* to z is a plane of material symmetry. It is clear that this material is orthotropic. But more than that, any rotation of the deformation tensor $\boldsymbol{\varepsilon}$ around the z axis doesn't change the elastic potential energy density Π. Thus

$$\frac{\partial \Pi}{\partial \boldsymbol{\varepsilon}} \cdot\!\cdot \left(\boldsymbol{k} \times \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon} \times \boldsymbol{k} \right) = 0, \tag{3.5}$$

because for any small rotation with vector $\boldsymbol{\delta o}$, the variation of the infinitesimal deformation tensor $\boldsymbol{\varepsilon}$ is $\boldsymbol{\delta o} \times \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon} \times \boldsymbol{\delta o}$, and $\boldsymbol{\delta o}$ goes along z with the unit vector $\boldsymbol{k} \equiv \boldsymbol{e}_z$. The equation (3.5) is true for any infinitesimal deformation $\boldsymbol{\varepsilon}$. In components

$$\begin{aligned} \left(a_{11}\varepsilon_x + a_{12}\varepsilon_y + a_{13}\varepsilon_z\right)\left(-2\varepsilon_{xy}\right) + \left(a_{12}\varepsilon_x + a_{22}\varepsilon_y + a_{23}\varepsilon_z\right)2\varepsilon_{xy} \\ + 2a_{44}\varepsilon_{xy}\left(\varepsilon_x - \varepsilon_y\right) + 2a_{55}\varepsilon_{xz}\left(-\varepsilon_{yz}\right) + 2a_{66}\varepsilon_{yz}\varepsilon_{xz} = 0 \end{aligned}$$

$$\Rightarrow \quad a_{11} = a_{12} + a_{44} = a_{22}\,, \quad a_{13} = a_{23}\,, \quad a_{55} = a_{66}\,.$$

Writing the stress tensor like

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\perp} + \boldsymbol{s}\boldsymbol{k} + \boldsymbol{k}\boldsymbol{s} + \sigma_{zz}\boldsymbol{k}\boldsymbol{k}, \tag{3.6}$$

where

$$\begin{gathered} \boldsymbol{\sigma}_{\perp} \equiv \sigma_{\alpha\beta}\boldsymbol{e}_{\alpha}\boldsymbol{e}_{\beta} = \sigma_{xx}\boldsymbol{i}\boldsymbol{i} + \sigma_{xy}\left(\boldsymbol{i}\boldsymbol{j} + \boldsymbol{j}\boldsymbol{i}\right) + \sigma_{yy}\boldsymbol{j}\boldsymbol{j}, \\ \boldsymbol{s} \equiv \sigma_{\alpha z}\boldsymbol{e}_{\alpha} = \sigma_{xz}\boldsymbol{i} + \sigma_{yz}\boldsymbol{j} \\ \left(\, \alpha, \beta \text{ are } x \text{ or } y, \quad \boldsymbol{e}_x = \boldsymbol{i}, \quad \boldsymbol{e}_y = \boldsymbol{j} \,\right), \end{gathered}$$

the Hooke's law for a transversely isotropic material may be presented as

$$\begin{gathered} \boldsymbol{\sigma}_{\perp} = a_{44}\boldsymbol{\varepsilon}_{\perp} + \left(a_{12}\varepsilon_{\alpha\alpha} + a_{13}\varepsilon_z\right)\boldsymbol{E}_{\perp}, \quad \boldsymbol{s} = a_{55}\boldsymbol{\epsilon}, \quad \sigma_{zz} = a_{33}\varepsilon_z + a_{13}\varepsilon_{\alpha\alpha} \\ \left(\text{here } \boldsymbol{\varepsilon}_{\perp} \equiv \varepsilon_{\alpha\beta}\boldsymbol{e}_{\alpha}\boldsymbol{e}_{\beta}, \quad \varepsilon_{\alpha\alpha} = \operatorname{trace}\boldsymbol{\varepsilon}_{\perp} = \varepsilon_x + \varepsilon_y, \right. \\ \left. \boldsymbol{\epsilon} \equiv \varepsilon_{\alpha z}\boldsymbol{e}_{\alpha}, \quad \boldsymbol{E}_{\perp} \equiv \boldsymbol{e}_{\alpha}\boldsymbol{e}_{\alpha} = \boldsymbol{i}\boldsymbol{i} + \boldsymbol{j}\boldsymbol{j} \,\right) \end{gathered} \tag{3.7}$$

* If a plane is parallel to a line, this plane's normal vector is perpendicular to that line.

It comes that a transversely isotropic material is characterized by five non-null mutually independent components, the elastic moduli $a_{12} = A^{1122}$, $a_{13} = A^{1133}$, $a_{33} = A^{3333}$, $a_{44} = A^{1212}$, $a_{55} = A^{1313}$.

A crystal symmetry

There are only seven kinds of various primitive parallelepiped lattices (Bravais lattices) — the seven syngonies*, namely triclinic, monoclinic, orthorhombic (or just rhombic), rhombohedral (or trigonal), tetragonal, hexagonal and cubic.

Each case of crystal symmetry is characterized by the set of orthogonal** tensors $\boldsymbol{Q}$, for which the following equation

$$ {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \left(\boldsymbol{Q} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{Q}^{\mathsf{T}} \right) = \boldsymbol{Q} \cdot \left({}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\varepsilon} \right) \cdot \boldsymbol{Q}^{\mathsf{T}} \quad \forall \boldsymbol{\varepsilon} \tag{3.8} $$

is true (for any infinitesimal deformation $\boldsymbol{\varepsilon}$).

Inverse relations

....

$$ \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \frac{\partial Ш}{\partial \boldsymbol{\sigma}} = {}^4\!\boldsymbol{\mathcal{B}} \cdot\cdot \boldsymbol{\sigma}, \quad Ш(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\varepsilon} - \Pi(\boldsymbol{\varepsilon}) \tag{3.9} $$

For the linear model

$$ 2\Pi = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\varepsilon}, \quad Ш = \Pi = \tfrac{1}{2} \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\varepsilon} \tag{3.10} $$

— the complementary energy density is numerically equal to the elastic potential energy density.

....

The theory, describing the linear properties of anisotropic materials — crystals, composites, wood and others.

* syngony = lattice system = crystallographic system = crystal symmetry

** Orthogonal tensors are those that satisfy the equality $\boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathsf{T}} = \boldsymbol{E}$ (**??**, § **??**.**??**), describing rotations and mirror flippings.

Many physical phenomena are described by tensors, including thermal, mechanical, electrical and magnetic properties.

The "material tensors" define the physical properties of bodies and media, kind of

✓ the elasticity,
✓ the thermal expansion,
✓ the thermal conductivity,
✓ the electrical conductivity,
✓ the piezoelectric effect.

Piezoelectricity (the piezoelectric effect) is the coupling (for example, linear) between the mechanical strain and the electric charge in a material, it is the transduction of electrical and mechanical energy.

Certain materials generate an electrical charge when mechanical stress is applied to them. Piezoelectric materials directly transduce electrical and mechanical energy. The most famous piezoelectric material is quartz crystal. Certain ceramics are piezoelectric as well, piezoelectricity is often associated with ceramic materials. Piezoelectric behaviour is also observed in many polymers. And biomatter, such as bone and various proteins, too.

§ 4. Hooke's law for isotropic medium

When a material is isotropic, (3.8) is satisfied for any orthogonal tensor $\boldsymbol{Q}$. So here, for a linear elastic isotropic medium/body, it's easier to assume that the potential energy density $\Pi(\boldsymbol{\varepsilon})$ becomes an isotropic function*

$$\Pi = \alpha \mathrm{I}^2 + \beta \mathrm{II}. \tag{4.1}$$

(II is not ${}_{\text{cha}}\mathrm{II}$)

$\Pi(\boldsymbol{\varepsilon})$ is a quadratic function (or a "quadratic form") with terms of the second degree.

"quadratic form" = "homogeneous polynomial of the second degree" = "function with quadratic terms only"

* As a function depending only on invariants, such as the coefficients from the solution of the characteristic equation (**??**, § **??**.**??**). Any function whose arguments are only the invariants is isotropic.

The isotropic function

$$\mathrm{I}(\boldsymbol{\varepsilon}), \mathrm{II}(\boldsymbol{\varepsilon}) \mapsto \Pi(\boldsymbol{\varepsilon}), \quad \Pi = \Pi(\mathrm{I}, \mathrm{II})$$

has terms of only the 2^{nd} degree (I^2 and II), not of the 1^{st} and not higher. And there's no third invariant $\mathrm{III}(\boldsymbol{\varepsilon})$ among the arguments of Π.

> **But why not add $\mathrm{III}^{\frac{2}{3}}$?**
>
> By excluding dependence on III, we simplify constitutive models while retaining accuracy for most practical applications.
>
> For most isotropic elastic materials, it suffices to model their behavior using only I and II, which capture volumetric and distortional effects without explicitly including volume change through III.
>
> In many cases of isotropic elastic materials, particularly those that are nearly incompressible (e.g., rubber-like materials), volume changes are negligible during deformation. There $\mathrm{III} = 1$.
>
> For small deformations or incompressibility constraints, changes in $\Pi(\mathrm{I}, \mathrm{II})$ already account for all physically relevant behaviors. Including $\Pi(\mathrm{III})$ would unnecessarily complicate calculations without adding meaningful information about material response.
>
> The third invariant III, which relates directly to volume changes, becomes redundant unless large compressive or expansive deformations occur. In linear elasticity (small strains), dependence on only I and II aligns with linear stress-strain relationships (the Hooke's law).

$$\mathrm{I}(\boldsymbol{\varepsilon}) = \operatorname{trace} \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\bullet} = \boldsymbol{\nabla} \cdot \boldsymbol{u}$$
$$\mathrm{II}(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} \cdot\cdot\, \boldsymbol{\varepsilon}$$

.....

$$2\Pi(\boldsymbol{\varepsilon}) = A\, \boldsymbol{\varepsilon}_{\bullet} \boldsymbol{\varepsilon}_{\bullet} + B\, \boldsymbol{\varepsilon} \cdot\cdot\, \boldsymbol{\varepsilon} \tag{4.2}$$

......

Since $\boldsymbol{\varepsilon}_{\bullet} = \boldsymbol{\varepsilon} \cdot\cdot\, \boldsymbol{E} = \boldsymbol{E} \cdot\cdot\, \boldsymbol{\varepsilon}$ (??, § ??.??), the derivative of trace by the tensor itself is the unit dyad

$$\frac{\partial(\boldsymbol{\varepsilon} \cdot\cdot\, \boldsymbol{E})}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \boldsymbol{\varepsilon}} \cdot\cdot\, \boldsymbol{E} + \boldsymbol{\varepsilon} \cdot\cdot\, \frac{\partial \boldsymbol{E}}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{E} \cdot\cdot\, \boldsymbol{E} + \boldsymbol{\varepsilon} \cdot\cdot\, {}^{2}\mathbf{0} = \boldsymbol{E} + {}^{2}\mathbf{0}$$

$$\Rightarrow \frac{\partial \boldsymbol{\varepsilon}_{\bullet}}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{E} \tag{4.3}$$

An isotropic medium is characterized by the two non-zero elastic constants (“elastic moduli”)

$$\Pi(\boldsymbol{\varepsilon}) = \alpha\,\mathrm{I}^2(\boldsymbol{\varepsilon}) + \beta\,\mathrm{II}(\boldsymbol{\varepsilon}) \tag{4.4}$$

$$\frac{\partial \Pi}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma}$$

$$\frac{\partial(\boldsymbol{\varepsilon}_{\bullet}\boldsymbol{\varepsilon}_{\bullet})}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \boldsymbol{\varepsilon}_{\bullet}}{\partial \boldsymbol{\varepsilon}}\boldsymbol{\varepsilon}_{\bullet} + \boldsymbol{\varepsilon}_{\bullet}\frac{\partial \boldsymbol{\varepsilon}_{\bullet}}{\partial \boldsymbol{\varepsilon}} = 2\boldsymbol{\varepsilon}_{\bullet}\boldsymbol{E}$$

or

$$\frac{\partial(\mathrm{I}^2)}{\partial \boldsymbol{\varepsilon}} = 2\,\mathrm{I}\,\frac{\partial \mathrm{I}}{\partial \boldsymbol{\varepsilon}} = 2\,\mathrm{I}\,\boldsymbol{E}\ \ \big(\text{where } \mathrm{I} = \boldsymbol{\varepsilon}_{\bullet}\big)$$

....

$$\boldsymbol{\sigma} = \lambda\,\boldsymbol{\varepsilon}_{\bullet}\boldsymbol{E} + 2\mu\,\boldsymbol{\varepsilon}$$

.....

In components for an isotropic medium

$$A_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu\big(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}\big) \tag{4.5}$$

— these are components of an isotropic tensor of the fourth complexity, which don’t change when the basis rotates.

....

Pairs of elastic moduli

There are versions of the Hooke’s law for the various pairs of elastic constants (elastic moduli). λ and μ are the Lamé parameters, μ (sometimes G) is the shear modulus, E — the Young’s modulus (the modulus of tension or compression), ν is the Poisson’s ratio, K — the bulk modulus.

......

The a priori conditions for the values of elastic moduli are

$$
\begin{aligned}
E > 0 &\ \text{— if something is stretched, it elongates,}\\
\mu > 0 &\ \text{— the shear goes in the same way as the tangen-}\\
&\ \quad\text{tial (shear) stress component,}\\
K > 0 &\ \text{— due to external pressure the volume decreases.}
\end{aligned}
\tag{4.6}
$$

Inequalities for the elastic moduli (4.6) are sufficient for the positivity of Π.

When $\nu \to \frac{1}{2}$, the material becomes incompressible with an infinitely large bulk modulus $K \to \infty$. Negative values of ν are possible* too.

....

Inverse relations, the complementary energy

$$2\Pi = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \frac{\partial Ш}{\partial \boldsymbol{\sigma}} = {}^4\!\boldsymbol{\mathcal{B}} \cdot\cdot\, \boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{B}}. \tag{4.7}$$

the Legendre transform

The complementary energy Ш

$$Ш(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon} - \Pi(\boldsymbol{\varepsilon}). \tag{4.8}$$

In the linear theory, the “complementary energy” is numerically equal to the elastic potential energy

$$\underbrace{2\Pi}_{\boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon}} - \underbrace{\Pi(\boldsymbol{\varepsilon})}_{\frac{1}{2}\boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon}} = Ш(\boldsymbol{\sigma}).$$

$$Ш(\boldsymbol{\sigma}) = \Pi(\boldsymbol{\varepsilon})$$

....

* Such a material, called an auxetic, becomes thicker when it stretches.

§ 5. Theorems of statics

Clapeyron's theorem

In equilibrium with the external forces, the volume ones $\boldsymbol{g}$ and the surface ones $\boldsymbol{p}$, the work of these "statically frozen" (that is constant along time) forces on the actual displacements is equal to the double of* the energy of deformation

$$2\int_{\mathcal{V}} \Pi \, d\mathcal{V} = \int_{\mathcal{V}} \boldsymbol{g} \cdot \boldsymbol{u} \, d\mathcal{V} + \int_{o_2} \boldsymbol{p} \cdot \boldsymbol{u} \, do. \tag{5.1}$$

○ $2\Pi = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon} = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{u}) - \underbrace{\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}}_{-\boldsymbol{g}} \cdot\, \boldsymbol{u} \;\Rightarrow$

$$\Rightarrow\; 2\int_{\mathcal{V}} \Pi \, d\mathcal{V} = \int_{o_2} \underbrace{\boldsymbol{n} \cdot \boldsymbol{\sigma}}_{\boldsymbol{p}} \cdot\, \boldsymbol{u} \, do + \int_{\mathcal{V}} \boldsymbol{g} \cdot \boldsymbol{u} \, d\mathcal{V} \quad ●$$

From (5.1) also follows, that without loading $\int_{\mathcal{V}} \Pi \, d\mathcal{V} = 0$. Because Π is positive, then the stress $\boldsymbol{\sigma}$, and deformation $\boldsymbol{\varepsilon}$ without a load are equal to zero.

$$2\Pi = \boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon}$$
$$\dot{\Pi} = \boldsymbol{\sigma} \cdot\cdot\, \dot{\boldsymbol{\varepsilon}}$$
$$\delta\Pi = \boldsymbol{\sigma} \cdot\cdot\, \delta\boldsymbol{\varepsilon}$$

Π is equal to only the half of the work of the external forces.

The accumulated potential energy of deformation Π is equal to only the half of the work done by the external forces, acting from

* "Ce produit représentait d'ailleurs le double de la force vive que le ressort pouvait absorber par l'effet de sa flexion et qui était la mesure naturelle de sa puissance." —
Benoît Paul Émile Clapeyron. Mémoire sur le travail des forces élastiques dans un corps solide élastique déformé par l'action de forces extérieures. *Comptes rendus*, Tome XLVI, Janvier–Juin 1858. Pagine 208–212.

the unstressed configuration to the equilibrium with the external forces.

Clapeyron's theorem implies that the accumulated elastic energy accounts for only the half of the energy spent on the deformation. The remaining half of the work, done by the external forces, is lost somewhere before reaching the equilibrium.

Roger Fosdick and **Lev Truskinovsky**. About Clapeyron's Theorem in Linear Elasticity. *Journal of Elasticity*, Volume 72, July 2003. Pages 145–172.

In theory, the concept of the "static loading" is common. It's when the external load is applied infinitely slow (sounds like forever, yeah).

The work of the external forces on the actual displacements is equal to the double of the potential energy density 2Π.

Если снять внешние воздействия мгновенно (бесконечно быстро), то тело будет колебаться. Но из-за сопротивления среды (внутреннего трения) спустя некоторое время тело придёт в состояние равновесия.

Yes, only the half of the linear elastic energy is stored. The second half is the " additional energy ", which is lost before reaching of the equilibrium on the dynamics — on the internal energy of the particles (of the dissipation), on the vibrations and waves.

But any real loading would be neither a sudden loading nor an infinitely slow loading. These are the two extremes. The real dynamics of applying the loads will always be different from the theory.

Within infinitesimal variations, applied to an elastic medium real external forces work on virtual displacements $\delta\boldsymbol{\varepsilon}$ and produce work $\delta W^{(e)}$ that is exactly equal (for linear elastic only or for non-linear too??) to the variation of the elastic potential energy density

$$\delta W^{(e)} = \boldsymbol{\sigma} \cdot\cdot\, \delta\boldsymbol{\varepsilon} = \delta\Pi .$$

A linear elastic medium is medium, where the variation of work of the internal forces (that is stresses) is the variation of the potential energy density with the opposite sign $-\delta W^{(i)} = \delta\Pi = \delta W^{(e)}$, when only the displacements vary (while the stress loads do not).

It is necessary that the virtual work of the real external forces on variations of displacements be equal to the variation of internal energy with the opposite sign (for an elastic media — the variation of internal energy).

The uniqueness of the solution theorem

As in dynamics (§ 2), we suppose the existænce of the two solutions and are looking for their difference

$$\ldots\ldots\ldots\ldots \tag{5.2}$$

The uniqueness of the solution, discovered by Gustav Kirchhoff for bodies with the simply connected contour*, is contrary to, as it seems, the everyday experience. Imagine a straight rod, clamped at the one end (the “ cantilever ”) and compressed at the second end with a longitudinal force (figure 1). When the load is large enough, the problem of statics has the two solutions, “straight” and “bent”. Such a contradiction with the uniqueness theorem comes from the nonlinearity of this problem. If a load is small (infinitesimal), then the solution is described by the linear equations and is unique.

...

...

Reciprocal work theorem

Proposed by Enrico Betti**.

For a body with a fixed part o_1 of the surface, the two cases are considered : the first one with loads $\boldsymbol{g}_1$, $\boldsymbol{p}_1$ and the second one with loads $\boldsymbol{g}_2$, $\boldsymbol{p}_2$. The verbal formulation of the theorem is the same as in § **??**.**??**. The mathematical notation

$$\overbrace{\int\limits_{\mathcal{V}} \boldsymbol{g}_1 \cdot \boldsymbol{u}_2 \, d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p}_1 \cdot \boldsymbol{u}_2 \, do}^{W_{12}} = \overbrace{\int\limits_{\mathcal{V}} \boldsymbol{g}_2 \cdot \boldsymbol{u}_1 \, d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p}_2 \cdot \boldsymbol{u}_1 \, do}^{W_{21}}. \qquad (5.3)$$

...

The reciprocal work theorem, also known as the Betti’s theorem, claims that for a linear elastic structure subject to the two sets of forces, P and Q, the work done by set P through displacements produced by set Q is equal to the work done by set Q through displacements produced by set P. This

* **Gustav Robert Kirchhoff**. Über das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes. *Journal für die reine und angewandte Mathematik (Crelle’s journal)*, 56. Band (1859). Seiten 285–313. (Seite 291)

** **Enrico Betti**. Teoria della elasticità. *Il Nuovo Cimento* (1869–1876), VII e VIII (1872). Pagina 69.

theorem has applications in structural engineering where it is used to define influence lines and derive the boundary element method.

...

The reciprocal work theorem finds unexpected and effective applications. For example, consider a rod-beam clamped at one end (“cantilever”) and bent by the two forces with the integral values P_1 and P_2 (figure 2). While the linear theory is applied, displacements-deflections can be represented as

$$u_1 = \alpha_{11} P_1 + \alpha_{12} P_2\,,$$
$$u_2 = \alpha_{21} P_1 + \alpha_{22} P_2\,.$$

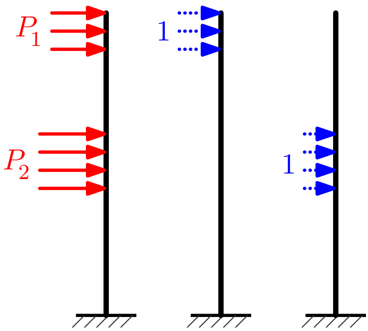

figure 2

...

§ 6. Equations in displacements

The complete set of equations (1.1) contains unknowns $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{u}$. Excluding $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, we get the formulation in displacements (symmetrization of $\boldsymbol{\nabla u}$ is redundant due to the symmetry ${}^4\!\boldsymbol{\mathcal{A}}_{3\rightleftarrows 4} = {}^4\!\boldsymbol{\mathcal{A}}$).

$$\begin{gathered} \boldsymbol{\nabla} \cdot \left({}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u} \right) + \boldsymbol{g} = \boldsymbol{0}\,, \\ \boldsymbol{u}\Big|_{o_1} = \boldsymbol{u}_0\,,\;\; \boldsymbol{n} \cdot {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u}\Big|_{o_2} = \boldsymbol{p}\,. \end{gathered} \tag{6.1}$$

For an isotropic medium (6.1) becomes

....

The solution for the homogeneous part* of equation (...) was found by

Пётр Ф. Папкович (Pjotr F. Papkowitsch) Papkovich

* A homogeneous differential equation contains a differentiation and a homogeneous function with a set of variables.

Пётр Ф. Папкович. Выражение общего интеграла основных уравнений теории упругости через гармонические функции // Известия Академии наук СССР. Отделение математических и естественных наук. 1932, выпуск 10, страницы 1425–1435.

and Heinz Neuber

Heinz Neuber. Ein neuer Ansatz zur Lösung räumlicher Probleme der Elastizitätstheorie. Der Hohlkegel unter Einzellast als Beispiel // Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), 1934, Band 14, Nr. 4, Seiten 203–212.

Пётр Ф. Папкович (Pjotr F. Papkowitsch) in 1932 and Heinz Neuber in 1934 proposed to represent displacements as harmonic functions and, therefore, to make use of a wide catalogue of particular solutions to the Laplace's equation. And sometimes the problem of elasticity can be reduced, at least partially, to one of the classical problems of the theory of harmonic functions (potential theory).

Приведя полную совокупность уравнений (1.1) к дифференциальному уравнению второй степени (в компонентах — к трём таким уравнениям) for a twice-differentiable function (трёх функций в компонентах), описывающей перемещение точек тела, Пётр Папкович и Hanz Neuber смогли записать общее решение, однако, с существенным ограничением класса объёмных сил — рассматривались только потенциальные. Классические силы механического характера (силы тяжести, силы инерции при равномерном вращении тел, силы гравитационного взаимодействия) потенциальны.

Для консервативных объёмных сил возможно аналитическое решение. Подход Папковича–Neuber'а не применим, если воздействия механической и иной физической природы не потенциальны.

.....

Three-dimensional elasticity based on quaternion-valued potentials

....

§ 7. Concentrated (point) force

A concentrated force acting on a point is a handy abstraction to simplify reality. However, it does not exist in the real world, where all forces either act in a volume — volume forces, or act over an area — surface (contact) forces.

Here is a rhetorical question : *why an elastic body withstands an applied load, "bears" it?* The book [12] by James Gordon gives the following answer : the body deforms, and thus the internal forces appear, called "stresses", which can compensate (balance, equilibrate) an external load.

But a linear elastic body cannot take the load of a point force.

.... in the balance of forces (of momentum)

$$\int_{\mathcal{V}} (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g}) d\mathcal{V} = \boldsymbol{0}$$

$$\mathcal{V} \sim r^3,\ \boldsymbol{g} \sim \tfrac{1}{r^3},\ \boldsymbol{\sigma} \sim \tfrac{1}{r^2}$$

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\big(\text{trace}\ \boldsymbol{\varepsilon}\big)\boldsymbol{E} \Rightarrow \boldsymbol{\varepsilon} \sim \tfrac{1}{r^2},\ \boldsymbol{u} \sim \tfrac{1}{r},\ \text{thus}\ \boldsymbol{u} \to \boldsymbol{\infty}\ \text{when}\ r \to 0$$

...

the solution by Kelvin–Somigliana (William Thompson aka Lord Kelvin*, Carlo Somigliana)

... in an infinite medium

......

the Saint-Venant's principle

Real things can have non-linearities at the places where the external loads are applied. But away from such places only the resulants are important.

As example for rods, the lengths of the non-linear loading regions are comparable with the sizes of the cross sections.

....

§ 8. Finding displacements by deformations

Like any bivalent tensor, the displacement gradient can be decomposed into the sum of the symmetric and antisymmetric parts

$$\boldsymbol{\nabla}\boldsymbol{u} = \overbrace{\boldsymbol{\varepsilon}}^{\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}} - \overbrace{\boldsymbol{\omega} \times \boldsymbol{E}}^{-\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{A}}},\quad \boldsymbol{\omega} \equiv \tfrac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{u}, \tag{8.1}$$

The symmetric part $\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}$ is the linear deformation tensor $\boldsymbol{\varepsilon}$.

* William "Lord Kelvin" Thompson. Note on the Integration of the Equations of Equilibrium of an Elastic Solid. The Cambridge and Dublin Mathematical Journal, 1848 volume iii (vii), pages 87–89

The antisymmetric part $\boldsymbol{\nabla u}^{\mathsf{A}}$ can be denoted as $\boldsymbol{\Omega}$ and called the tensor of small rotations. Since any skew-symmetric bivalent tensor is uniquely representable by the vector (§ **??**.**??**), one more field — the vector field of rotations $\boldsymbol{\omega}(\boldsymbol{r})$ — is needed to find displacements $\boldsymbol{u}$ by deformations $\boldsymbol{\varepsilon}$.

....

The deformation compatibility condition in the linear elasticity

The compatibility condition represent the integrability conditions for a symmetric bivalent tensor field. When such a tensor field is compatible, then it describes some deformation (strain).

In the displacement ↦ deformation relation $\boldsymbol{\varepsilon} = \boldsymbol{\nabla u}^{\mathsf{S}}$, the six components ε_{ij} of deformation $\boldsymbol{\varepsilon}$ originate from the only three components u_k of the displacement vector $\boldsymbol{u}$.

...

$$\operatorname{inc} \boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \boldsymbol{\varepsilon} \right)^{\mathsf{T}} \quad \text{or} \quad \operatorname{inc} \boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla} \times \boldsymbol{\varepsilon} \times \boldsymbol{\nabla}$$

....

A contour here is arbitrary, therefore

$$\operatorname{inc} \boldsymbol{\varepsilon} = {}^{2}\mathbf{0}. \tag{8.2}$$

This relation is called the deformation compatibility condition or the deformation continuity equation(s).

...

Expression (8.2) constraints the possible types of the deformation (strain) field. The compatibility (continuity) condition ensures that no gaps and/or overlaps appear as the result of deformation.

(... add a picture here, the figure where the whole is cut into squares ...)

...

Tensor $\operatorname{inc} \boldsymbol{\varepsilon}$ is symmetric together with $\boldsymbol{\varepsilon}$

....

The Saint-Venant's compatibility equations

It was previously proven that

$$\operatorname{inc}\boldsymbol{\varepsilon} = {}^2\mathbf{0} \quad \Leftrightarrow \quad \triangle\boldsymbol{\varepsilon} + \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{\varepsilon}_{\bullet} = 2\big(\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\varepsilon}\big)^{\mathsf{S}}$$

in index notation for rectangular coordinates

$$\partial_m\partial_m\varepsilon_{ij} + \partial_i\partial_j\varepsilon_{mm} = \big(\partial_i\partial_m\varepsilon_{mj} + \partial_j\partial_m\varepsilon_{mi}\big)$$

with summations expanded

$$\left(\frac{\partial^2\varepsilon_{ij}}{\partial x_1^2} + \frac{\partial^2\varepsilon_{ij}}{\partial x_2^2} + \frac{\partial^2\varepsilon_{ij}}{\partial x_3^2}\right) + \left(\frac{\partial^2\varepsilon_{11}}{\partial x_i\partial x_j} + \frac{\partial^2\varepsilon_{22}}{\partial x_i\partial x_j} + \frac{\partial^2\varepsilon_{33}}{\partial x_i\partial x_j}\right)$$
$$= \left(\frac{\partial^2\varepsilon_{1j}}{\partial x_i\partial x_1} + \frac{\partial^2\varepsilon_{2j}}{\partial x_i\partial x_2} + \frac{\partial^2\varepsilon_{3j}}{\partial x_i\partial x_3}\right) + \left(\frac{\partial^2\varepsilon_{1i}}{\partial x_j\partial x_1} + \frac{\partial^2\varepsilon_{2i}}{\partial x_j\partial x_2} + \frac{\partial^2\varepsilon_{3i}}{\partial x_j\partial x_3}\right)$$

for $i = j\ (= a)$

$$\left(\frac{\partial^2\varepsilon_{aa}}{\partial x_1^2} + \frac{\partial^2\varepsilon_{aa}}{\partial x_2^2} + \frac{\partial^2\varepsilon_{aa}}{\partial x_3^2}\right) + \left(\frac{\partial^2\varepsilon_{11}}{\partial x_a^2} + \frac{\partial^2\varepsilon_{22}}{\partial x_a^2} + \frac{\partial^2\varepsilon_{33}}{\partial x_a^2}\right)$$
$$= 2\left(\frac{\partial^2\varepsilon_{1a}}{\partial x_a\partial x_1} + \frac{\partial^2\varepsilon_{2a}}{\partial x_a\partial x_2} + \frac{\partial^2\varepsilon_{3a}}{\partial x_a\partial x_3}\right) \quad \cancel{\sum_a},\ \ a = 1, 2, 3$$

.....

$$\begin{cases}
\dfrac{\partial^2\varepsilon_{22}}{\partial x_1^2} + \dfrac{\partial^2\varepsilon_{11}}{\partial x_2^2} = 2\,\dfrac{\partial^2\varepsilon_{21}}{\partial x_1\partial x_2} \\[2ex]
\dfrac{\partial^2\varepsilon_{33}}{\partial x_2^2} + \dfrac{\partial^2\varepsilon_{22}}{\partial x_3^2} = 2\,\dfrac{\partial^2\varepsilon_{32}}{\partial x_2\partial x_3} \\[2ex]
\dfrac{\partial^2\varepsilon_{11}}{\partial x_3^2} + \dfrac{\partial^2\varepsilon_{33}}{\partial x_1^2} = 2\,\dfrac{\partial^2\varepsilon_{13}}{\partial x_3\partial x_1} \\[2ex]
\dfrac{\partial^2\varepsilon_{23}}{\partial x_1^2} + \dfrac{\partial^2\varepsilon_{11}}{\partial x_2\partial x_3} = \dfrac{\partial^2\varepsilon_{13}}{\partial x_2\partial x_1} + \dfrac{\partial^2\varepsilon_{12}}{\partial x_3\partial x_1} \\[2ex]
\dfrac{\partial^2\varepsilon_{13}}{\partial x_2^2} + \dfrac{\partial^2\varepsilon_{22}}{\partial x_1\partial x_3} = \dfrac{\partial^2\varepsilon_{23}}{\partial x_1\partial x_2} + \dfrac{\partial^2\varepsilon_{21}}{\partial x_3\partial x_2} \\[2ex]
\dfrac{\partial^2\varepsilon_{12}}{\partial x_3^2} + \dfrac{\partial^2\varepsilon_{33}}{\partial x_1\partial x_2} = \dfrac{\partial^2\varepsilon_{32}}{\partial x_1\partial x_3} + \dfrac{\partial^2\varepsilon_{31}}{\partial x_2\partial x_3}
\end{cases}$$

— the deformation continuity (compatibility) equations in "classical" notation for rectangular coordinates (the six Saint-Venant's equations).

The last three can also be written as

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left(\frac{\partial \varepsilon_{13}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} - \frac{\partial \varepsilon_{23}}{\partial x_1}\right)$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{21}}{\partial x_3} - \frac{\partial \varepsilon_{13}}{\partial x_2}\right)$$

$$\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_{32}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right)$$

Isaac Todhunter. The Elastical Researches of Barré de Saint-Venant. Cambridge University Press, 1889.

[110.] *L'institut*, Vol. 26, 1858, pp. 178–9. Further results on Torsion communicated to the *Société Philomathique* (April 24 and May 15, 1858) and afterwards incorporated in the *Leçons de Navier* (pp. 305–6, 273–4). They relate to cross-sections in the form of doubly symmetrical quartic curves and to torsion about an external axis : see our Arts. 49 (c), 182 (b), 181 (d), and 182 (a).

[111.] Vol. 27, 1860, of same Journal, pp. 21–2. Saint-Venant presents to the *Société Philomathique* the model *de la surface décrite par une corde vibrante transportée d'un mouvement rapide perpendiculaire à son plan de vibration.* Copies of this as well as some other of Saint-Venant's models may still be obtained of M. Delagrave in Paris and are of considerable value for class-lectures on the vibration of elastic bodies.

[112.] Vol. 28, 1861, of same Journal, pp. 294–5. This gives an account of a paper of Saint-Venant's read before the *Société Philomathique* (July 28, 1860). In this he deduces the *conditions of compatibility*, or the six differential relations of the types :

$$2\,\frac{d^2 s_x}{dy\,dz} = \frac{d}{dx}\left(\frac{d\sigma_{xz}}{dy} + \frac{d\sigma_{xy}}{dz} - \frac{d\sigma_{yz}}{dx}\right)$$

$$\frac{d^2 \sigma_{yz}}{dy\,dz} = \frac{d^2 s_y}{dz^2} + \frac{d^2 s_z}{dy^2}$$

which must be satisfied by the strain-components. These conditions enable us in many cases to dispense with the consideration of the shifts. A proof of these conditions by Boussinesq will be found in the *Journal de Liouville*, Vol. 16, 1871, pp. 132–4. At the same meeting Saint-Venant extended his results on torsion to : (1) prisms on any base with at each point only one plane of symmetry perpendicular to the sides, (2) prisms on an elliptic base with or without any plane of symmetry whatever ; see our Art. 190 (d).

What about nonlinear theory?

All equations of the linear theory have an analogue — the primary source — in the nonlinear theory. To find it for (8.2), remember the Cauchy–Green deformation tensor (§ **??**.**??**) and the curvature tensors (§ **??**.**??**)

....

§ 9. Equations in stresses

The balance of forces (or of momentum)

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} = \boldsymbol{0} \tag{9.1}$$

does not quite yet determine the stresses. It's necessary as well that deformations (strains) $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ corresponding to stresses (3.9)

$$\boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \frac{\partial Ш}{\partial \boldsymbol{\sigma}} = {}^4\!\boldsymbol{\mathcal{B}} \cdot\cdot\, \boldsymbol{\sigma} \tag{9.2}$$

were compatible (§ 8)

$$\operatorname{inc} \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) \equiv \boldsymbol{\nabla} \times \Big(\boldsymbol{\nabla} \times \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) \Big)^{\mathsf{T}} = {}^2\boldsymbol{0}. \tag{9.3}$$

Gathered together, (9.1), (9.2) and (9.3) present the complete closed set (system) of equations in stresses.

...

§ 10. The principle of the minimum potential energy

When the existænce of the deformation energy function is assured, and the external forces are assumed to be constant during varying of displacements, then the principle of virtual work leads to the principle of the minimum potential energy.

The formulation of the principle:

$$\mathscr{E}(\boldsymbol{u}) \equiv \int\limits_{\mathcal{V}} \Bigl(\Pi(\boldsymbol{u}) - \boldsymbol{g} \cdot \boldsymbol{u}\Bigr) d\mathcal{V} - \int\limits_{o_2} \boldsymbol{p} \cdot \boldsymbol{u}\, do \to \min, \;\; \boldsymbol{u}\big|_{o_1} = \boldsymbol{u}_0. \quad (10.1)$$

The functional $\mathscr{E}(\boldsymbol{u})$, called the (full) potential energy enof a linear-elastic body, is minimal when displacements $\boldsymbol{u}$ are true — that is for the solution of a problem (6.1). The input functions $\boldsymbol{u}$ must satisfy the geometrical condition on o_1 (so they don't break the existing constraints and can be continuous or else $\Pi(\boldsymbol{u})$ will not be integrable)
.

For the true field of displacements $\boldsymbol{u}$, the quadratic function

$$\Pi(\boldsymbol{u}) = \tfrac{1}{2} \boldsymbol{\nabla u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla u}$$

becomes equal to the true potential energy of deformation. Then

$$\mathscr{E} = \mathscr{E}_{\min},$$

which according to the Clapeyron's theorem (5.1) is

$$\mathscr{E}_{\min} = \int\limits_{\mathcal{V}} \Pi(\boldsymbol{u})\, d\mathcal{V} - \Biggl(\int\limits_{\mathcal{V}} \boldsymbol{g} \cdot \boldsymbol{u}\, d\mathcal{V} + \int\limits_{o_2} \boldsymbol{p} \cdot \boldsymbol{u}\, do \Biggr) = - \int\limits_{\mathcal{V}} \Pi(\boldsymbol{u})\, d\mathcal{V}.$$

Taking a some other satisfactory field of displacements $\boldsymbol{u}'$, look at the finite difference

$$\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) = \int\limits_{\mathcal{V}} \Bigl(\Pi(\boldsymbol{u}') - \Pi(\boldsymbol{u}) - \boldsymbol{g} \cdot (\boldsymbol{u}' - \boldsymbol{u}) \Bigr) d\mathcal{V} - \int\limits_{o_2} \boldsymbol{p} \cdot (\boldsymbol{u}' - \boldsymbol{u})\, do,$$

seeking $\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) \geq 0$ or (ditto) $\mathscr{E}(\boldsymbol{u}') \geq \mathscr{E}(\boldsymbol{u})$.

$\boldsymbol{g} =$ **constant** and $\boldsymbol{p} =$ **constant**

$\Pi(\boldsymbol{a}) = \frac{1}{2} \boldsymbol{\nabla a} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla a}$ (but *not* the linear $\frac{1}{2} \boldsymbol{\nabla u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot \boldsymbol{\nabla a}$ — this means $\Pi(\boldsymbol{a}) \neq \frac{1}{2} \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\nabla a}$)

Constraints don't change: $(\boldsymbol{u}'-\boldsymbol{u})\big|_{o_1}=\boldsymbol{u}_0-\boldsymbol{u}_0=\boldsymbol{0}$. External surface force $\boldsymbol{p}\big|_{o_2}=\boldsymbol{t}_{(\boldsymbol{n})}=\boldsymbol{n}\cdot\boldsymbol{\sigma}$ on o_2 and $=\boldsymbol{0}$ elsewhere on $o(\partial\mathcal{V})$. $\boldsymbol{\sigma}=\boldsymbol{\nabla}\boldsymbol{u}\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}={}^2$**constant** along with constant $\boldsymbol{p}$ and $\boldsymbol{g}$. Therefore

$$\int\limits_{o_2}\boldsymbol{p}\cdot(\boldsymbol{u}'-\boldsymbol{u})\,do=\oint\limits_{o(\partial\mathcal{V})}\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot(\boldsymbol{u}'-\boldsymbol{u})\,do=\int\limits_{\mathcal{V}}\boldsymbol{\nabla}\cdot\Big(\boldsymbol{\sigma}\cdot(\boldsymbol{u}'-\boldsymbol{u})\Big)d\mathcal{V}=$$
$$=\int\limits_{\mathcal{V}}(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\cdot(\boldsymbol{u}'-\boldsymbol{u})\,d\mathcal{V}+\int\limits_{\mathcal{V}}\boldsymbol{\sigma}^{\mathsf{T}}\cdot\cdot\,\boldsymbol{\nabla}(\boldsymbol{u}'-\boldsymbol{u})\,d\mathcal{V}.$$

Due to symmetry $\boldsymbol{\sigma}^{\mathsf{T}}=\boldsymbol{\sigma}\;\Rightarrow\;\boldsymbol{\sigma}^{\mathsf{T}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{a}=\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{a}=\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{a}^{\mathsf{S}}\;\forall\boldsymbol{a}$. Разность преобразуется до

$$\mathscr{E}(\boldsymbol{u}')-\mathscr{E}(\boldsymbol{u})=$$
$$=\int\limits_{\mathcal{V}}\Big(\Pi(\boldsymbol{u}')-\Pi(\boldsymbol{u})-\big(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\boldsymbol{g}\big)\cdot(\boldsymbol{u}'-\boldsymbol{u})-\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\nabla}(\boldsymbol{u}'-\boldsymbol{u})\Big)d\mathcal{V}.$$

And with the balance of momentum $\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}+\boldsymbol{g}=\boldsymbol{0}$

$$\mathscr{E}(\boldsymbol{u}')-\mathscr{E}(\boldsymbol{u})=\int\limits_{\mathcal{V}}\Big(\Pi(\boldsymbol{u}')-\Pi(\boldsymbol{u})-\boldsymbol{\sigma}\cdot\cdot\,\boldsymbol{\nabla}(\boldsymbol{u}'-\boldsymbol{u})\Big)d\mathcal{V}.$$

Here

$$\Pi(\boldsymbol{u}')=\tfrac{1}{2}\boldsymbol{\nabla}\boldsymbol{u}'\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{u}',\;\;\Pi(\boldsymbol{u})=\tfrac{1}{2}\boldsymbol{\nabla}\boldsymbol{u}\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{u},$$

$$\Pi(\boldsymbol{u}')-\Pi(\boldsymbol{u})=\tfrac{1}{2}\Big(\boldsymbol{\nabla}\boldsymbol{u}'\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{u}'-\boldsymbol{\nabla}\boldsymbol{u}\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{u}\Big)$$

$${}^4\!\boldsymbol{\mathcal{A}}_{12\rightleftarrows34}={}^4\!\boldsymbol{\mathcal{A}}\;\Rightarrow\;\boldsymbol{\nabla}\boldsymbol{u}\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{u}'=\boldsymbol{\nabla}\boldsymbol{u}'\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{u}$$

$$\tfrac{1}{2}\Big(\boldsymbol{\nabla}\boldsymbol{u}'\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{u}'-\boldsymbol{\nabla}\boldsymbol{u}\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}\boldsymbol{u}\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{u}'-\boldsymbol{\nabla}\boldsymbol{u}'\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\boldsymbol{u}\Big)$$

$$\big(\boldsymbol{\nabla}\boldsymbol{u}'-\boldsymbol{\nabla}\boldsymbol{u}\big)=\boldsymbol{\nabla}(\boldsymbol{u}'-\boldsymbol{u})$$

for a finite difference of potentials

$$\tfrac{1}{2}\boldsymbol{\nabla}\big(\boldsymbol{u}'+\boldsymbol{u}\big)\cdot\cdot\,{}^4\!\boldsymbol{\mathcal{A}}\cdot\cdot\,\boldsymbol{\nabla}\big(\boldsymbol{u}'-\boldsymbol{u}\big)=\Pi(\boldsymbol{u}')-\Pi(\boldsymbol{u}),$$

adding to which

$$-\boldsymbol{\nabla}\boldsymbol{u} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u}) = -\boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u})$$

we get

$$\tfrac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u}) \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}(\boldsymbol{u}' - \boldsymbol{u}) = \Pi(\boldsymbol{u}' - \boldsymbol{u})$$

and finally*

$$\mathscr{E}(\boldsymbol{u}') - \mathscr{E}(\boldsymbol{u}) = \int\limits_{\mathcal{V}} \Pi(\boldsymbol{u}' - \boldsymbol{u})\, d\mathcal{V}.$$

Since ${}^4\!\boldsymbol{\mathcal{A}}$ is positive definite (§ 2) $\Pi(\boldsymbol{w}) = \tfrac{1}{2}\boldsymbol{\nabla}\boldsymbol{w} \cdot\cdot\, {}^4\!\boldsymbol{\mathcal{A}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{w} \geq 0 \;\forall \boldsymbol{w}$ (and $= 0$ only if $\boldsymbol{\nabla}\boldsymbol{w} = \boldsymbol{0} \Leftrightarrow \boldsymbol{w} = \textbf{constant}$: for a case of translation as a whole without deformation.

...

$\delta\boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{\nabla}\delta\boldsymbol{u}$

...

the Ritz method

The minimum functional problem $\mathscr{E}(\boldsymbol{u})$ is approximately solved as

....

the finite element method

....

§ 11. The principle of the minimum complementary energy

When the stress–strain relations (the Hooke's law) assure the existænce of a complementary energy function and the geometrical boundary conditions are assumed constant during variation of stresses, then the principle of minimum complementary energy emerges.

* $b^2 - a^2 - 2a(b-a) = (b+a)(b-a) - 2a(b-a) = (b-a)^2$

"Complementary" work (energy) is named so as not to be confused with the "full" work by Clapeyron (5.1) $W = F(\int du) = 2\Pi$, where $F = \mathsf{constant}$.

The complementary energy of a linearly elastic body is the following functional over the field of stresses :

$$\mathscr{D}(\boldsymbol{\sigma}) \equiv \int\limits_{\mathcal{V}} \amalg(\boldsymbol{\sigma})\, d\mathcal{V} - \int\limits_{o_1} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{u}_0\, do, \;\; \boldsymbol{u}_0 \equiv \boldsymbol{u}\big|_{o_1}, \tag{11.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g} = \boldsymbol{0}, \;\; \boldsymbol{n} \cdot \boldsymbol{\sigma}\big|_{o_2} = \boldsymbol{p}.$$

...

The variation of the balance of force equation

$$\delta\big(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{g}\big) = \boldsymbol{\nabla} \cdot \delta\boldsymbol{\sigma} = \boldsymbol{0}$$

...

The principle of the minimum complementary energy is very useful for estimating inexact (approximate) solutions. But for computations it isn't so essential as the (Lagrange) principle of minimum potential energy (10.1).

To derive the variational principles it is natural to use the principle of the virtual work (§ **??**.**??**) as a foundation.

§ 12. Mixed principles of stationarity

Prange–Hellinger–Reissner Variational Principle,
named after *Ernst Hellinger*, *Georg Prange* and *Eric Reissner*.
Working independently of Hellinger and Prange, Eric Reissner published his famous six-page paper "On a variational theorem in elasticity" in 1950. In this paper he develops — without, however, considering Hamilton–Jacobi theory — a variational principle same to that of Prange and Hellinger.

Hu–Washizu Variational Principle,
named as *Hu Haichang* and *Kyuichiro Washizu*.

The following functional over the displacements and stresses

$$\mathcal{R}(\boldsymbol{u},\boldsymbol{\sigma}) = \int\limits_{\mathcal{V}} \Big[\boldsymbol{\sigma} \cdot\cdot \boldsymbol{\nabla} \boldsymbol{u}^{\mathsf{S}} - \text{Ш}(\boldsymbol{\sigma}) - \boldsymbol{g} \cdot \boldsymbol{u} \Big] d\mathcal{V} - \\ - \int\limits_{o_1} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \big(\boldsymbol{u} - \boldsymbol{u}_0 \big) do - \int\limits_{o_2} \boldsymbol{p} \cdot \boldsymbol{u} \, do \quad (12.1)$$

carries names of Reissner, Prange and Hellinger.

...

The advantage of the Reissner–Hellinger principle — freedom of variation. But it also has a drawback: on the true solution the functional has no extremum, but only stationarity.

Принцип можно использовать для построения приближённых решений методом Ritz (Ritz method). Задавая аппроксимации

...

Принцип Hu–Washizu [102] формулируется так:

$$\delta \mathcal{W}(\boldsymbol{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma}) = 0,$$

$$\mathcal{W} \equiv \int\limits_{\mathcal{V}} \Big[\boldsymbol{\sigma} \cdot\cdot \Big(\boldsymbol{\nabla} \boldsymbol{u}^{\mathsf{S}} - \boldsymbol{\varepsilon} \Big) + \Pi(\boldsymbol{\varepsilon}) - \boldsymbol{g} \cdot \boldsymbol{u} \Big] d\mathcal{V} - \\ - \int\limits_{o_1} \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \big(\boldsymbol{u} - \boldsymbol{u}_0 \big) do - \int\limits_{o_2} \boldsymbol{p} \cdot \boldsymbol{u} \, do. \quad (12.2)$$

Как и в принципе Reissner'a–Hellinger'a, здесь нет ограничений ни в объёме, ни на поверхности, но добавляется третий независимый аргумент $\boldsymbol{\varepsilon}$. Поскольку $\text{Ш} = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\varepsilon} - \Pi$, то (12.1) and (12.2) кажутся почти одним и тем же.

From the Hu–Washizu principle ensues the whole complete set of equations with boundary conditions, так как

.....

§ 13. Antiplane shear

This is such a problem of the linear theory of elasticity, where the non-trivial results are obtained by the simple outputs* .

This problem is about an isotropic elastic continuum in the cartesian coordinates

$$x_\alpha, \qquad \alpha = 1, 2, \qquad x_1 \text{ and } x_2.$$

The plane x_1, x_2 is a cross-section of a rod, the third coordinate x_3 is perpendicular to the section. The basis vectorsare

$$\boldsymbol{e}_i = \partial_i \boldsymbol{r}, \quad \boldsymbol{r} = x_i \boldsymbol{e}_i, \quad \boldsymbol{e}_i \boldsymbol{e}_i = \boldsymbol{E} \Leftrightarrow \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}.$$

In a case of an antiplane strain (an antiplane shear), the field of displacements $\boldsymbol{u}(\boldsymbol{r})$ is parallel to the third coordinate x_3:

$$\boldsymbol{u} = \upsilon \boldsymbol{e}_3,$$

and υ doesn't depend on x_3:

$$\upsilon = \upsilon(x_1, x_2), \quad \partial_3 \upsilon = 0.$$

The deformation

$$\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla} \boldsymbol{u}^{\mathsf{S}} = \boldsymbol{\nabla} \big(\upsilon \boldsymbol{e}_3\big)^{\mathsf{S}} = \boldsymbol{e}_3 \boldsymbol{\nabla} \upsilon^{\mathsf{S}} + \upsilon \underbrace{\boldsymbol{\nabla} \boldsymbol{e}_3}_{{}^2\mathbf{0}}{}^{\mathsf{S}} = \tfrac{1}{2} \big(\boldsymbol{\nabla} \upsilon \, \boldsymbol{e}_3 + \boldsymbol{e}_3 \boldsymbol{\nabla} \upsilon\big) \quad (13.1)$$

In the plane x_1, x_2 of the section

$$\mu = \mu(x_1, x_2), \partial_3 \mu = 0$$

is a possible inhomogeneity of the medium.

* Non-trivial in the theory of elasticity is, for example, when the division of a force by an area gives an infinitely large error in the calculation of the stresses.

§ 14. The torsion of rods

M. de Saint-Venant. Memoire sur la torsion des prismes (1853)

Adhémar-Jean-Claude Barré de Saint-Venant. Mémoire sur la torsion des prismes, avec des considérations sur leur flexion ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exerçant simultanément. 1856. 327 pages.

1. Memoire sur la torsion des prismes, avec des considerations sur leur flexion, etc. Memoires presentes par divers savants a l'Academie des scienees, t. 14, 1856.

2. Memoire sur la flexion des prismes, etc. Journal de mathematiques pures et appliquees, publie par J. Liouville, 2me serie, t. 1, 1856.

Перевод на русский язык: **Сен-Венан Б.** Мемуар о кручении призм. Мемуар об изгибе призм. М.: Физматгиз, 1961. 518 страниц.

стр. 379–494

This problem, which was studied in detail by Adhémar-Jean-Claude Barré de Saint-Venant, is contained in almost every book on the linear elasticity. It considers a cylinder of some section, loaded only by the surface forces at the ends (... add a figure ...)

$$\begin{aligned} z = \ell :&\ \boldsymbol{k} \cdot \boldsymbol{\sigma} = \boldsymbol{p}(x_\alpha), \\ z = 0 :&\ -\boldsymbol{k} \cdot \boldsymbol{\sigma} = \boldsymbol{p}_0(x_\alpha), \end{aligned}$$

where $\boldsymbol{k} \equiv \boldsymbol{e}_3$, $\alpha = 1, 2$, $\boldsymbol{x} \equiv x_\alpha \boldsymbol{e}_\alpha$. Coordinates are x_1, x_2, z.

The resultant (the sum) of the external forces is equal to $\mathbf{0}$, and the resultant couple is directed along the z axis :

$$\int\limits_o \boldsymbol{p} do = \mathbf{0}, \quad \int\limits_o \boldsymbol{x} \times \boldsymbol{p} do = M \boldsymbol{k}.$$

On torsion, the tangential stress components $\tau_{z1} \equiv \boldsymbol{k} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{e}_1$ and $\tau_{z2} \equiv \boldsymbol{k} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{e}_2$ arise. Assuming that only these components of $\boldsymbol{\sigma}$ are non-zero

$$\boldsymbol{\sigma} = \boldsymbol{s}\boldsymbol{k} + \boldsymbol{k}\boldsymbol{s}, \quad \boldsymbol{s} \equiv \tau_{z\alpha} \boldsymbol{e}_\alpha.$$

The solution of this problem simplifies if the equations in stresses are used.

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mathbf{0} \;\Rightarrow\; \boldsymbol{\nabla}_{\perp} \cdot \boldsymbol{s} = \mathbf{0} \;\; (\boldsymbol{\nabla}_{\perp} \equiv \boldsymbol{e}_{\alpha} \partial_{\alpha}), \;\; \partial_z \boldsymbol{s} = \mathbf{0}, \tag{14.1}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \boldsymbol{\sigma} + \frac{1}{1+\nu} \boldsymbol{\nabla} \boldsymbol{\nabla} \sigma = {}^{2}\mathbf{0} \;\Rightarrow\; \triangle_{\perp} \boldsymbol{s} = \mathbf{0} \;\; (\triangle_{\perp} \equiv \partial_{\alpha} \partial_{\alpha}). \tag{14.2}$$

The independence of $\boldsymbol{s}$ from z makes it possible to replace the three-dimensional operators with the two-dimensional ones.

...

§ 15. Plane deformation

Here the displacement vector $\boldsymbol{u}$ is parallel to the plane x_1, x_2 and does not depend on the third coordinate z.

For example рассмотрим полуплоскость с сосредоточенной нормальной силой Q на краю (?? рисунок ??)

...

Bibliography

There are several dozen books on the classical linear theory of elasticity that haven't lost their relevance over time. First of all, the fundamental monograph by Анатолий Лурье (Anatoliy Lurie) [28] and his earlier book [29] about solving spatial problems. Quite rich in content is the Witold Nowacki's book [38]. There the author spent many pages describing the problems of both statics and "elastokinetics" (that is dynamics), and the last chapter of this book describes the linear Cosserat continuum — that's what the next chapter is about. Being mathematically capacious and saturated, the theory of elasticity attracts mathematicians, as it happened with the monograph [22] by Philippe Ciarlet. The Augustus Love's book [26] cannot go unmentioned as well. Климентий Черны́х (Klimentiy Chernih) described in [57] how to model linear elastic media with anisotropy.

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