Vadique Myself

PHYSICS *of* ELASTIC CONTINUA

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chapter 2

CLASSICAL MECHANICS

When relativistic mechanics (for the very fast) and quantum mechanics (for the very small) emerged at the beginning of the XX$^{\text{th}}$ century, the equations of mechanics existed prior to that, still perfectly suitable for describing objects of everyday sizes and speeds, needed a new name. So the "classical" in mechanics doesn't refer to antiquity. This was just chosen as the name for description of reality without any quantum and relativistic effects influencing it.

§1. Discrete collection of particles

Classical mechanics models physical objects by discretizing them into a collection of particles ("pointlike masses", "material points"*).

In a collection of N particles, each k-th particle has its nonzero mass $m_k = \mathsf{constant} > 0$ and the motion function $\boldsymbol{r}_k(t)$. The function $\boldsymbol{r}_k(t)$ is measured relative to the chosen reference system.

The "reference system" (or "reference frame") consists of (figure 1)

- ✓ some "null" reference point o,
- ✓ a set of coordinates, which give the units of spatial measurements,
- ✓ a clock.

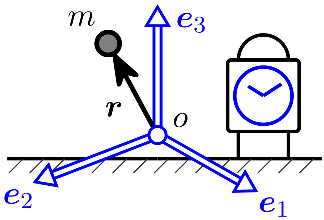

figure 1

Long time ago, the reference system was some "absolute space", empty at first, and then filled with the continuous elastic medium — the æther. Later, it

* The point mass (pointlike mass, material point) is the concept of an object, typically matter, that has the nonzero mass and is (or is being thought of as) infinitesimal in its volume (dimensions).

became clear that any frame of reference can be used for classical mechanics, but the preference is given to the so called "inertial" frames, where a particle in the absence of external interactions (or applied forces) moves "in free motion" — along a straight line with a constant velocity ($\dot{\boldsymbol{r}} = \textbf{constant}$), thence without acceleration ($\ddot{\boldsymbol{r}} = \boldsymbol{0}$)

$$\dot{\boldsymbol{r}} = \textbf{constant} = \dot{x}_i \boldsymbol{e}_i \;\;\Rightarrow\;\; \dot{x}_i = \text{constant} \;\;\Leftarrow\;\; \boldsymbol{e}_i = \textbf{constant}$$

The measure of interaction in mechanics is the vector of force $\boldsymbol{F}$. In the widely known* Newton's equation

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \tag{1.1}$$

the force $\boldsymbol{F}$ can depend only on position, velocity and explicitly on time, whereas acceleration $\ddot{\boldsymbol{r}}$ is directly proportional to force $\boldsymbol{F}$ with coefficient $^1\!/m$.

Here're theses of the dynamics of a collection of particles.

The force $\boldsymbol{F}_k$, acting on the k-th particle (figure 2)

$$m_k \ddot{\boldsymbol{r}}_k = \boldsymbol{F}_k ,$$
$$\boldsymbol{F}_k = \boldsymbol{F}_k^{(e)} + \sum_j \boldsymbol{F}_{kj}^{(i)}. \tag{1.2}$$

$\boldsymbol{F}_k^{(e)}$ is the external force — such forces emanate from objects outside the system being considered. The second addend is the sum of internal forces (force $\boldsymbol{F}_{kj}^{(i)}$ is the interaction induced by the j-th particle on the k-th particle). Internal interactions happen only between

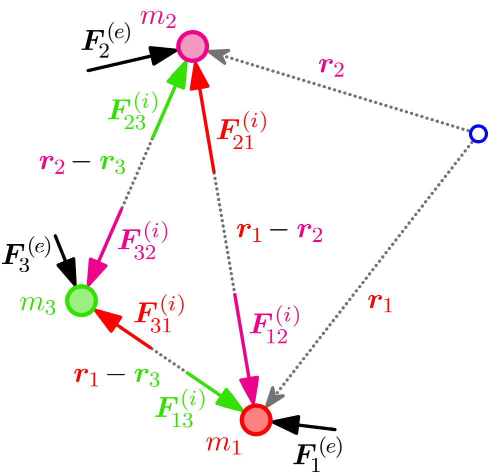

figure 2

* "Axiomata sive Leges Motus" ("Axioms or Laws of Motion") were written by Isaac Newton in his Philosophiæ Naturalis Principia Mathematica, first published in 1687. Reprint (en Latin), 1871. Translated into English by Andrew Motte, 1846.

elements of the system and don't affect (mechanically) anything other. Neither particle interacts with itself, $\boldsymbol{F}_{kk}^{(i)} = \mathbf{0}\ \forall k$.

AXIOMATA
SIVE
LEGES MOTUS

Lex. I.

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Projectilia perseverant in motibus suis nisi quatenus a resistentia aeris retardantur & vi gravitatis impelluntur deorsum. Trochus, cujus partes cohærendo perpetuo retrahunt sese a motibus rectilineis, non cessat rotari nisi quatenus ab aere retardatur. Majora autem Planetarum & Cometarum corpora motus suos & progressivos & circulares in spatiis minus resistentibus factos conservant diutius.

Lex. II.

Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur.

Si vis aliqua motum quemvis generet, dupla duplum, tripla triplum generabit, sive simul & semel, sive gradatim & successive impressa fuerit. Et hic motus quoniam in eandem semper plagam cum vi generatrice determinatur, si corpus antea movebatur, motui ejus vel conspiranti additur, vel contrario subducitur, vel obliquo oblique adjicitur, & cum eo secundum utriusq; determinationem componitur.

Lex. III.

Actioni contrariam semper & æqualem esse reactionem : sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.

Quicquid premit vel trahit alterum, tantundem ab eo premitur vel trahitur. Siquis lapidem digito premit, premitur & hujus digitus a lapide. Si equus lapidem funi allegatum trahit, retrahetur etiam & equus aequaliter in lapidem : nam funis utrinq; distentus eodem relaxandi se conatu urgebit Equum versus lapidem, ac lapidem versus equum, tantumq; impediet progressum unius quantum promovet progressum alterius. Si corpus aliquod in corpus aliut impingens, motum ejus vi sua quomodocunq: mutaverit, idem quoque vicissim in motu proprio eandem mutationem in partem contrariam vi alterius (ob æqualitatem pressionis mutuæ) subibit. His actionibus æquales fiunt mutationes non velocitatum sed motuum, (scilicet in corporibus non aliunde impeditis:) Mutationes enim velocitatum, in contrarias itidem partes factæ, quia motus æqualiter mutantur, sunt corporibus reciproce proportionales.

.... Rene Descartes' mechanics ....

Measuring motions in mechanics is, however, more controversial than measuring interactions. The discord and the extensive polemic on this topic dates back to the times of Newton and Leibniz. In those days, exploring how objects of various masses change the speed and velocity* of their motion when various forces are applied to it, both Newton and Leibnitz were looking for a useful invariant that would fit the observations.

- "*mv, the product of mass and velocity, is a useful quantity that remains constant*" thought Newton.
- "*mv^2, the product of mass and velocity squared, is a useful quantity that remains constant*" thought Leibniz.

And each of them believed that the quantity he proposed is more useful, more fundamental and more "fruitful".

Newton named mv as "quantitas motus" ("quantity of motion")

momentum is a measure of mechanical motion of an object

momentum depends on the weight (i.e. quantity) and velocity of an object.

Momentum is the product of mass and velocity, so

when either an object's mass or its velocity changes , then the momentum will change

....

* *Speed* is the time rate of motion, that is *how fast* a thing moves along some path, a scalar. *Velocity* is the movement's rate and direction, that's *how fast* ***and where*** a thing moves, a vector.

what is momentum? The measure of motion in mechanics is called "momentum"...

....hmmmmm https://hsm.stackexchange.com/questions/769/when-and-by-whom-was-the-term-momentum-introduced

why "mass by velocity" measures the amount of motion ? there are two momenta known, the linear (translational) one and the angular (rotational) one, why?

Why is momentum defined as mass times velocity?

https://physics.stackexchange.com/a/577486/377185

If you read the history, you'll find there was much discussion, rivalry, and even bad blood as each pushed the benefits of their particular view. Each thought that their quantity was more fundamental, or more important.

Now, we see that both are useful, just in different contexts.

I'm sure somebody briefly toyed with the expressions like mv^3 and maybe m^2v before quickly finding that they didn't stay constant under any reasonable set of constraints, so had no predictive power. That's why they're not named, or used for anything.

So why has the quantity mv been given a name? Because it's useful, it's conserved, and it allows us to make predictions about some parameters of a mechanical system as it undergoes interactions with other things.

from *Leibniz and the Vis Viva Controversy by Carolyn Iltis (1971)*

Roger Boscovich in 1745 and Jean d'Alembert in 1758 both pointed out that vis viva mv^2 and momentum mv were equally valid.

The momentum of a body is actually the Newtonian force F acting through a time, since $v = at$ and $mv = mat = Ft$.

The kinetic energy is the Newtonian force acting over a space, since $v^2 = 2as$ and $mv^2 = 2mas$ or $\frac{1}{2}mv^2 = Fs$.

$$v \equiv \dot{s}, \quad a \equiv \dot{v} \equiv \ddot{s}$$

$$\begin{aligned} ds &= \dot{s}dt & d\dot{s} &= \ddot{s}dt \\ &= vdt & dv &= adt \end{aligned}$$

$$\begin{aligned} vdv &= vadt = avdt = ads \\ \dot{s}d\dot{s} &= \dot{s}\ddot{s}dt = \ddot{s}\dot{s}dt = \ddot{s}ds \end{aligned}$$

$mdv = madt = Fdt$

$d(v^2) = 2vdv = 2ads$ and $md(v^2) = 2mads = 2Fds$

The amount of movement of some object is the product of the mass and velocity of that object.

When two objects collide, ...

the (linear, translational) momentum

$$\begin{aligned} &m_k \dot{\boldsymbol{r}}_k \;\; \text{— for the } k\text{-th particle}\,, \\ &\sum_k m_k \dot{\boldsymbol{r}}_k \;\; \text{— for the whole discrete system} \end{aligned} \tag{1.3}$$

and the angular (rotational) momentum

$$\begin{aligned} &\boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \;\; \text{— for the } k\text{-th particle}\,, \\ &\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \;\; \text{— for the whole discrete system}\,. \end{aligned} \tag{1.4}$$

From (1.2) together with the action–reaction principle

$$\boldsymbol{F}^{(i)}_{kj} = -\boldsymbol{F}^{(i)}_{jk} \;\; \forall k,j \;\; \Rightarrow \;\; \sum_{k,j} \boldsymbol{F}^{(i)}_{kj} = \boldsymbol{0} \tag{1.5}$$

ensues the balance of linear momentum

$$\Big(\sum_k m_k \dot{\boldsymbol{r}}_k \Big)^{\bullet} = \sum_k m_k \ddot{\boldsymbol{r}}_k = \sum_k \boldsymbol{F}^{(e)}_k . \tag{1.6}$$

And here's the balance of angular momentum*

$$\Big(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \Big)^{\bullet} = \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k \tag{1.7}$$

— is the sum $\sum \boldsymbol{M}_k$ of moments. The moment $\boldsymbol{M}_k$, acting on the k-th particle

$$\boldsymbol{M}_k = \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k = \boldsymbol{r}_k \times \boldsymbol{F}^{(e)}_k + \boldsymbol{r}_k \times \sum_j \boldsymbol{F}^{(i)}_{kj} . \tag{1.8}$$

When in addition to the action–reaction principle, all internal interactions between particles are assumed to be central, that is

$$\boldsymbol{F}^{(i)}_{kj} \parallel \big(\boldsymbol{r}_k - \boldsymbol{r}_j \big) \;\; \Leftrightarrow \;\; \big(\boldsymbol{r}_k - \boldsymbol{r}_j \big) \times \boldsymbol{F}^{(i)}_{kj} = \boldsymbol{0} , \tag{1.9}$$

* $\Big(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \Big)^{\bullet} = \sum_k \dot{\boldsymbol{r}}_k \times m_k \dot{\boldsymbol{r}}_k + \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k = \sum_k \boldsymbol{r}_k \times m_k \ddot{\boldsymbol{r}}_k$

$\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0} \;\; \forall \boldsymbol{a} \;\; \Rightarrow \;\; \dot{\boldsymbol{r}}_k \times \dot{\boldsymbol{r}}_k = \boldsymbol{0}$

the balance of rotational (angular) momentum becomes*

$$\Big(\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k\Big)^{\bullet} = \sum_k \boldsymbol{r}_k \times \boldsymbol{F}_k^{(e)}. \tag{1.10}$$

Thus, all changes in the linear and angular momenta are due only to external forces $\boldsymbol{F}_k^{(e)}$, not internal ones.

Unlike for momenta, the balance of kinetic energy $\mathrm{K} \equiv {}^1\!/\!_2 \sum m_k \dot{\boldsymbol{r}}_k \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k$ (mv^2 is Leibniz's "vis viva") includes the power of internal forces as well

$$\begin{aligned}\dot{\mathrm{K}} = \Big(\tfrac{1}{2}\sum_k m_k \dot{\boldsymbol{r}}_k \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k\Big)^{\bullet} &= \tfrac{1}{2}\sum_k \big(m_k \ddot{\boldsymbol{r}}_k \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k + m_k \dot{\boldsymbol{r}}_k \boldsymbol{\cdot} \ddot{\boldsymbol{r}}_k\big) \\ &= \sum_k m_k \ddot{\boldsymbol{r}}_k \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k = \sum_k \boldsymbol{F}_k \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k = \sum_k \Big(\boldsymbol{F}_k^{(e)} + \sum_j \boldsymbol{F}_{kj}^{(i)}\Big) \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k \\ &= \sum_k \boldsymbol{F}_k^{(e)} \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k + \sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k \end{aligned} \tag{1.11}$$

or, using the action–reaction principle (1.5),

$$\dot{\mathrm{K}} - \sum_k \boldsymbol{F}_k^{(e)} \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k = \tfrac{1}{2}\sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \boldsymbol{\cdot} \big(\dot{\boldsymbol{r}}_k + \dot{\boldsymbol{r}}_k\big) = \tfrac{1}{2}\sum_{k,j} \Big(\boldsymbol{F}_{kj}^{(i)} \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k - \boldsymbol{F}_{jk}^{(i)} \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k\Big),$$

and since $\sum_{k,j} \boldsymbol{F}_{jk}^{(i)} \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k = \sum_{j,k} \boldsymbol{F}_{kj}^{(i)} \boldsymbol{\cdot} \dot{\boldsymbol{r}}_j = \sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \boldsymbol{\cdot} \dot{\boldsymbol{r}}_j$

$$\dot{\mathrm{K}} = \sum_k \boldsymbol{F}_k^{(e)} \boldsymbol{\cdot} \dot{\boldsymbol{r}}_k + \tfrac{1}{2}\sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \boldsymbol{\cdot} \big(\dot{\boldsymbol{r}}_k - \dot{\boldsymbol{r}}_j\big). \tag{1.12}$$

....

all bodies that are limited in free motion possess potential energy

.....

* $\forall k, j \;\; \boldsymbol{F}_{kj}^{(i)} = -\boldsymbol{F}_{jk}^{(i)}$ and $\big(\boldsymbol{r}_k - \boldsymbol{r}_j\big) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0} \;\; \Rightarrow$

$$\sum_k \boldsymbol{r}_k \times \sum_j \boldsymbol{F}_{kj}^{(i)} = \tfrac{1}{2}\sum_{k,j} \big(\boldsymbol{r}_k + \boldsymbol{r}_k\big) \times \boldsymbol{F}_{kj}^{(i)} = \tfrac{1}{2}\sum_{k,j} \big(\boldsymbol{r}_k - \boldsymbol{r}_j\big) \times \boldsymbol{F}_{kj}^{(i)} = \boldsymbol{0}$$

$$\sum_{k,j} \boldsymbol{r}_k \times \boldsymbol{F}_{kj}^{(i)} = -\sum_{k,j} \boldsymbol{r}_k \times \boldsymbol{F}_{jk}^{(i)} = -\sum_{j,k} \boldsymbol{r}_j \times \boldsymbol{F}_{kj}^{(i)} = -\sum_{k,j} \boldsymbol{r}_j \times \boldsymbol{F}_{kj}^{(i)}$$

Work

$$W(\boldsymbol{F}, \boldsymbol{u}) = \boldsymbol{F} \bullet \boldsymbol{u}$$

as the exact (full) differential

$$dW = \frac{\partial W}{\partial \boldsymbol{F}} \bullet d\boldsymbol{F} + \frac{\partial W}{\partial \boldsymbol{u}} \bullet d\boldsymbol{u}$$

by "product rule"

$$dW = d\big(\boldsymbol{F} \bullet \boldsymbol{u}\big) = d\boldsymbol{F} \bullet \boldsymbol{u} + \boldsymbol{F} \bullet d\boldsymbol{u}$$

$\dfrac{\partial W}{\partial \boldsymbol{F}} = \boldsymbol{u}, \ \dfrac{\partial W}{\partial \boldsymbol{u}} = \boldsymbol{F}$

...

Center of mass

- ✓ This is the unique point at any time where the weighted relative position of the distributed mass sums to zero.
- ✓ This is the point to which a force may be applied to cause a linear acceleration without an angular acceleration.
- ✓ This is a hypothetical point where the entire mass of an object may be assumed to be concentrated to visualise its motion.
- ✓ This is the particle equivalent of a given object for application of Newton's laws of motion.

When formulated for the center of mass, formulas in mechanics are often simplified.

...

— Are there any scenarios for which the center of mass is not almost exactly equivalent to the center of gravity?

— Non-uniform gravity field. In a uniform gravitational field, the center of mass is equal to the center of gravity.

...

Imposed on the positions and velocities of particles, there are restrictions of a geometrical or kinematical nature, called constraints.

Holonomic constraints are relations between position variables (and possibly time) which can be expressed as equality like

$$f(q^1, q^2, q^3, \ldots, q^n, t) = 0,$$

where $q^1, q^2, q^3, \ldots, q^n$ are n parameters (coordinates) that fully describe the system.

A constraint that cannot be expressed as such is nonholonomic.

Holonomic constraint depends only on coordinates and time. It does not depend on velocities or any higher time derivatives.

Velocity-dependent constraints like

$$f(q^1, q^2, \ldots, q^n, \dot{q}^1, \dot{q}^2, \ldots, \dot{q}^n, t) = 0$$

are mostly not holonomic.

For example, the motion of a particle constrained to lie on a sphere's surface is subject to a holonomic constraint, but if the particle is able to fall off a sphere under the influence of gravity, the constraint becomes non-holonomic. For the first case the holonomic constraint may be given by the equation: $r^2 - a^2 = 0$, where r is the distance from the centre of a sphere of radius a. Whereas the second non-holonomic case may be given by: $r^2 - a^2 \geq 0$.

Three examples of nonholonomic constraints are: when the constraint equations are nonintegrable, when the constraints have inequalities, or with complicated non-conservative forces like friction.

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q^1, q^2, \ldots, q^n, t)$$

(assuming n independent parameters/coordinates)

.....

§ 2. The principle of virtual work

Mécanique analytique (1788–89) is a two volume French treatise on analytical mechanics, written by Joseph Louis Lagrange, and published 101 years following Isaac Newton's *Philosophiæ Naturalis Principia Mathematica*.

Joseph Louis Lagrange. Mécanique analytique. Nouvelle édition, revue et augmentée par l'auteur. Tome premier. Mme Ve Courcier, Paris, 1811. 490 pages.

Joseph Louis Lagrange. Mécanique analytique. Troisième édition, revue, corrigée et annotée par M. J. Bertrand. Tome second. Mallet-Bachelier, Paris, 1855. 416 pages.

The historical transition from geometrical methods, as presented in Newton's Principia, to methods of mathematical analysis.

Consider the exact differential of any set of location vectors $\boldsymbol{r}_i$, that are functions of other variable parameters (coordinates) $q^1, q^2, ..., q^n$ and time t.

The actual displacement is the differential

$$d\boldsymbol{r}_i = \frac{\partial \boldsymbol{r}_i}{\partial t}\,dt + \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q^j}\,dq^j$$

Now, imagine an arbitrary path through the configuration space/manifold. This means it has to satisfy the constraints of the system but not the actual applied forces

$$\delta\boldsymbol{r}_i = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q^j}\,\delta q^j$$

A virtual infinitesimal displacement of a system of particles refers to a change in the configuration of a system as the result of any arbitrary infinitesimal change of location vectors (or coordinates) $\delta\boldsymbol{r}_k$, consistent with the forces and constraints imposed on the system at the current/given instant t. This displacement is called "virtual" to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the forces and constraints may be changing.

Assume the system is in equilibrium, that is the full force on each particle vanishes, $\boldsymbol{F}_i = \boldsymbol{0}\ \forall i$. Then clearly the term $\boldsymbol{F}_i \bullet \delta\boldsymbol{r}_i$, which is the virtual work of force $\boldsymbol{F}_i$ in displacement $\delta\boldsymbol{r}_i$, also vanishes for each particle, $\boldsymbol{F}_i \bullet \delta\boldsymbol{r}_i = 0\ \forall i$. The sum of these vanishing products over all particles is likewise equal to zero:

$$\sum_i \boldsymbol{F}_i \bullet \delta\boldsymbol{r}_i = 0.$$

Decompose the full force $\boldsymbol{F}_i$ into the applied (active) force $\boldsymbol{F}_i^{(a)}$ and the force of constraint $\boldsymbol{C}_i$,

$$\boldsymbol{F}_i = \boldsymbol{F}_i^{(a)} + \boldsymbol{C}_i$$

We now restrict ourselves to systems for which the net virtual work of the force of every constraint is zero:

$$\sum_i \boldsymbol{C}_i \bullet \delta\boldsymbol{r}_i = 0.$$

We therefore have as the condition for equilibrium of a system that the virtual work of all applied forces vanishes:

$$\sum_i \boldsymbol{F}_i^{(a)} \bullet \delta\boldsymbol{r}_i = 0.$$

— the principle of virtual work.

Note that coefficients $\boldsymbol{F}_i^{(a)}$ can no longer be thought equal to zero: in common $\boldsymbol{F}_i^{(a)} \neq 0$, since $\delta\boldsymbol{r}_i$ are not independent but are bound by constraints.

A virtual displacement of a particle with vector radius $\boldsymbol{r}_k$ is variation $\delta\boldsymbol{r}_k$ — any infinitesimal change of vector $\boldsymbol{r}_k$, which is compatible with the constraints. If the system is free, that is there are no constraints, then virtual displacements $\delta\boldsymbol{r}_k$ are perfectly random.

A constraint can be holonomic, aka geometric, bounding only positions (displacements) — a function of only coordinates and probably time

$$c(\boldsymbol{r}, t) = 0. \tag{2.1}$$

A nonholonomic, or differential, constraint includes time derivatives of coordinates

$$c(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) = 0 \tag{2.2}$$

and is not integrable into a geometric constraint. A system with nonholonomic constraints depends on the path — the intermediate values of trajectory, not only upon the initial and final locations, and thus cannot be represented by a potential function $\Pi(\boldsymbol{r}, t)$.

When all constraints are holonomic, then the system is completely independent of the trajectory of transition, of any intermediate values along the path. Here, the virtual displacements of the k-th particle satisfy the equation

$$\sum_{j=1}^{m} \frac{\partial c_j}{\partial \boldsymbol{r}_k} \bullet \delta \boldsymbol{r}_k = 0. \tag{2.3}$$

In constrained (non-free) systems, all forces can be divided into two groups : the active forces $\boldsymbol{F}_k^{(a)}$ and the constraint (or reaction) forces.

Reaction $\boldsymbol{C}_k$ acts on the k-th particle from all material limiters and changes according to equation (2.1) for each constraint.

The constraints are assumed to be ideal, that is

$$\sum_{k} \boldsymbol{C}_k \bullet \delta \boldsymbol{r}_k = 0 \tag{2.4}$$

— the work of constraint (reaction) forces is equal to zero on any virtual displacement.

The principle of virtual work is

$$\sum_{k} \left(\boldsymbol{F}_k^{(a)} - m_k \ddot{\boldsymbol{r}}_k \right) \bullet \delta \boldsymbol{r}_k = 0, \tag{2.5}$$

where $\boldsymbol{F}_k^{(a)}$ are only active forces, without reactions of constraints.

Differential variational equation (2.5) may seem like a trivial consequence of the Newton's law (1.1) and the ideality of constraints (2.4). Nevertheless, the contents of (2.5) is incomparably broader. In this book the reader will see that principle (2.5) can be put in the foundation of mechanics [92]. The various models of media, being described in this book, are based on that principle.

...

Проявилась замечательная особенность (2.5) : это уравнение эквивалентно системе такого порядка, каково число степеней свободы системы, то есть сколько независимых вариаций $\boldsymbol{\delta r}_k$ мы имеем. Если система N точек имеет m связей, то число степеней свободы $n = 3N - m$.

...

§ 3. Balance of momentum, rotational momentum, and energy

Эти уравнения баланса могут быть связаны со свойствами пространства and времени [93]. Сохранение импульса (количества движения) в за́мкнутой (изолированной)* системе выводится из однородности пространства *(при любом параллельном переносе — трансляции — замкнутой системы как целого свойства этой системы не меняются)*. Сохранение момента импульса — следствие изотропии пространства *(свойства замкнутой системы не меняются при любом повороте этой системы как целого)*. Энергия же изолированной системы сохраняется, так как время однородно** (energy $\mathrm{E} \equiv \mathrm{K}(q, \dot{q}) + \Pi(q)$ такой системы не зависит явно от времени).

* *За́мкнутая (изолированная) система* это такая система частиц, которые взаимодействуют only друг с другом, but ни с какими другими телами.

** Характеристики “однородность” and “изотропность” пространства, “однородность” времени не фигурируют среди аксиом классической механики.

The balance equations can be derived from the principle of virtual work (2.5). Перепишем его, выделив внешние силы $\boldsymbol{F}_k^{(e)}$ и виртуальную работу внутренних сил $\delta W^{(i)} = \sum_{k,j} \boldsymbol{F}_{kj}^{(i)} \cdot \delta \boldsymbol{r}_k$

$$\sum_k \Big(\boldsymbol{F}_k^{(e)} - m_k \ddot{\boldsymbol{r}}_k \Big) \cdot \delta \boldsymbol{r}_k + \delta W^{(i)} = 0. \tag{3.1}$$

It's assumed that internal forces don't do any work on virtual displacements of a system as a rigid whole ($\boldsymbol{\delta\rho}$ and $\boldsymbol{\delta o}$ are some constant vectors describing translation and rotation)

$$\begin{matrix} \delta \boldsymbol{r}_k = \boldsymbol{\delta\rho} + \boldsymbol{\delta o} \times \boldsymbol{r}_k, \\ \boldsymbol{\delta\rho} = \textbf{constant}, \ \boldsymbol{\delta o} = \textbf{constant} \end{matrix} \ \Rightarrow \ \delta W^{(i)} = 0. \tag{3.2}$$

Premises and considerations for this assumption are as follows. The first — for the case of potential, such as elastic, forces. A variation of the work of potential internal forces $W^{(i)}$ is a variation of the potential Π with the opposite sign,

$$\delta W^{(i)} = -\delta\Pi. \tag{3.3}$$

And it's quite obvious that Π alters only by deforming. The second consideration — the internal forces are balanced in the sense that the net vector (the resultant force) and the net moment (the resultant couple) are $\mathbf{0}$,

(1.5) & (1.9)

$$\sum \dots$$

...

Принимая (3.2) и подставляя в (3.1) сначала $\delta \boldsymbol{r}_k = \boldsymbol{\delta\rho}$ (трансляция), а затем $\delta \boldsymbol{r}_k = \boldsymbol{\delta o} \times \boldsymbol{r}_k$ (поворот), получаем баланс импульса (...) and баланс момента импульса (...).

...

§ 4. Hamilton's principle and Lagrange's equations

The two branches of analytical mechanics are Lagrangian mechanics (operating with generalized coordinates and corresponding generalized velocities in configuration space) and Hamiltonian mechanics (operating with coordinates and corresponding momenta in phase space). Both formulations are equivalent by a Legendre transformation on the generalized coordinates, velocities and momenta, therefore both contain the same information for describing the dynamics of a system.

Variational equation (2.5) is satisfied at any moment of time. Проинтегрируем его (оттуда лишь равенство F = ma)* по какому-либо промежутку $[t_1, t_2]$

$$\int_{t_1}^{t_2} \Bigl(\delta\mathrm{K} + \sum_k \boldsymbol{F}_k \cdot \delta \boldsymbol{r}_k \Bigr) dt - \Bigl[\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k \Bigr]_{t_1}^{t_2} = 0. \tag{4.1}$$

Без ущерба для общности можно принять $\delta \boldsymbol{r}_k(t_1) = \delta \boldsymbol{r}_k(t_2) = \boldsymbol{0}$, then the non-integral term vanishes.

Generalized coordinates q^i ($i = 1, \ldots, n$ — the number of degrees o'freedom) are introduced. Location vectors become functions like $\boldsymbol{r}_k(q^i, t)$, тождественно удовлетворяющими уравнениям связей (2.1). Если связи стационарны, то есть уравнения (2.1) не содержат t, то остаётся $\boldsymbol{r}_k(q^i)$. Kinetic energy превращается в функцию $\mathrm{K}(q^i, \dot{q}^i, t)$, где явно входящее t характерно лишь для нестационарных связей.

Hello to the essential concept of generalized forces. They originate from the virtual work $\boldsymbol{F}_k \cdot \delta \boldsymbol{r}_k$. With variation $\delta \boldsymbol{r}_k$ of the k-th point's location vector, expanded for generalized coordinates q^i,

$$\delta \boldsymbol{r}_k = \sum_i \frac{\partial \boldsymbol{r}_k}{\partial q^i} \delta q^i , \tag{4.2}$$

* $\delta\mathrm{K} = \sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta \dot{\boldsymbol{r}}_k$, $\Bigl(\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k \Bigr)^{\bullet} = \sum_k m_k \ddot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k + \underbrace{\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta \dot{\boldsymbol{r}}_k}_{\delta\mathrm{K}}$

$$\Rightarrow \quad \Bigl[\sum_k m_k \dot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k \Bigr]_{t_1}^{t_2} = \int_{t_1}^{t_2} \sum_k m_k \ddot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k \, dt + \int_{t_1}^{t_2} \delta\mathrm{K} dt$$

the virtual work can be written as

$$\sum_k \boldsymbol{F}_k \cdot \delta \boldsymbol{r}_k = \sum_k \boldsymbol{F}_k \cdot \sum_i \frac{\partial \boldsymbol{r}_k}{\partial q^i}\,\delta q^i = \sum_{i,k} \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i}\,\delta q^i$$
$$= \sum_i \Bigl(\sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i} \Bigr) \delta q^i = \sum_i Q_i \,\delta q^i, \quad (4.3)$$

where

$$Q_i \equiv \sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i}\,. \qquad (4.4)$$

It's worth to accentuate once more the origin of generalized forces from work. Having chosen the generalized coordinates q^i for the problem, the applied forces $\boldsymbol{F}_k$ are then grouped into the sets of generalized forces Q_i.

The particular case of potential forces is very relevant for this book. A force is *potential* when the work done by it depends only on locations of points, but not on paths between them. Then the potential energy Π can be introduced as a scalar field, also dubbed a "potential field" or just "a potential", because it is a function of only coordinates $\Pi = \Pi(q^i)$ (and possibly time, $\Pi(q^i, t)$ — the explicit dependence on time t may appear due to non-stationary constraints or because the physical fields themselves depend on time). $\delta \Pi = \sum \frac{\partial \Pi}{\partial q^i}\,\delta q^i$ is a variation of energy Π. The function Π is usually defined so that a positive work is a reduction in the potential. Thus, if generalized forces are potential, then

$$\sum_i Q_i \,\delta q^i = -\,\delta \Pi, \quad Q_i = -\frac{\partial \Pi}{\partial q^i}\,. \qquad (4.5)$$

...

Lagrange's equations of the first kind

Since there are Lagrange's equations "of the second kind", the reader may guess that equations "of the first kind" exist too. Yes, they are known. And they are worth mentioning at least because the derivation method, founded on these equations, is used in this book many times.

When constraints (2.1) are imposed on a system, the equality $\boldsymbol{F}_k = m_k \ddot{\boldsymbol{r}}_k$ doesn't follow from the variational equation (2.5), since virtual displacements $\delta\boldsymbol{r}_k$ are then not independent. Having m constraints and therefore m conditions (2.3) for variations, each of these conditions is multiplied by some scalar λ_a $(a = 1, \ldots, m)$ and added to (2.5), turning into

$$\sum_{k=1}^{N}\left(\boldsymbol{F}_k^{(a)} + \sum_{a=1}^{m}\lambda_a \frac{\partial c_a}{\partial \boldsymbol{r}_k} - m_k \ddot{\boldsymbol{r}}_k\right) \cdot \delta\boldsymbol{r}_k = 0\,. \tag{4.6}$$

Among $3N$ components of variations $\delta\boldsymbol{r}_k$, m are dependent. Aha, and the number of Lagrange multipliers λ_a is m too. Если выбрать λ_α такие, что coefficients(??какúе?) for dependent variations обращаются в нуль, то тогда у остальных вариаций коэффициенты(??) также будут нулевые из-за независимости. Следовательно, все выражения в скобках $(\cdots)$ равны нулю — это и есть Lagrange's equations of the first kind.

Since for each particle

...

§ 5. Statics

Let there is a mechanical system with stationary (constant over time) constraints under the action of static (not changing with time) active forces $\boldsymbol{F}_k$. In equilibrium all $\boldsymbol{r}_k =$ **constant**, hence $\delta\boldsymbol{r}_k = \boldsymbol{0}$, $\dfrac{\partial \boldsymbol{r}_k}{\partial q^i} = \boldsymbol{0}$, and the principle of virtual work is formulated as

$$\sum_k \boldsymbol{F}_k \cdot \delta\boldsymbol{r}_k = 0 \quad \Leftrightarrow \quad \sum_k \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q^i} = Q_i = 0. \tag{5.1}$$

Both pieces are essential : and the variational equation, and zeros in the generalized forces.

Relations (5.1) are the most generic and universal equations of statics. In literature, the narrow conception of the equilibrium equations as the balance of forces and moments is widespread. But in that case too, as in any other, the set of the equilibrium equations exactly matches with the generalized coordinates. “The resultant force” (also referred to as “the net (full) force” or “the net vector”) and “the resultant couple” (“the net couple”, “the net moment”) figure in the equilibrium equations* just because the system has translational and rotational degrees o’freedom. The huge popularity of forces and moments (force couples) comes not as much from the prevalence of statics of a perfectly non-deformable (ideally rigid) solid body, but more from the fact that the virtual work of internal forces on all movements of the system as a rigid whole is always equal to zero for any medium.

Let two kinds o’forces act in the system : potential, with the coordinate-dependent energy $\Pi(q^i)$, and plus external ones $\overset{\circ}{Q}{}_j^{(e)} \equiv P_j$. From (5.1) follow the equilibrium equations

$$\frac{\partial \Pi}{\partial q^i} = P_i \tag{5.2}$$

and the exact differential of $\Pi(q^i)$ (time independent) is

$$d\Pi = \sum_i \frac{\partial \Pi}{\partial q^i} dq^i = \sum_i P_i \, dq^i. \tag{5.3}$$

Equations (5.2) formulate the problem of statics, non-linear in overall, about the relation of the equilibrium position $q^i_\circ$ with the external loads P_i.

A linear system with quadratic potential Π as a function of coordinates

$$\Pi = \frac{1}{2} \sum_{i,k} C_{ik} \, q^i q^k \tag{5.4}$$

$$\sum_k C_{ik} \, q^k = P_i . \tag{5.5}$$

Here figure elements C_{ik} of “the stiffness matrix”, coordinates q^k and external loads P_i.

* Since **Louis Poinsot** described the reduction of any set of forces, acting on the same absolutely rigid system, into the single force and the single couple in his book “Élémens de statique”, first published in 1803 — Éléments de statique, chez Calixte-Volland, 1803 (an XII). 267 p. Onzième (11ème) édition : Gauthier-Villars, 1873.

Structures (both human-made artificial and in the nature) most often have a positive-definite stiffness matrix C_{ik}. Then $\det C_{ik} > 0$, the solution of a linear algebraic system (5.5) is unique, and this solution can be substituted by minimization of the quadratic form

$$\mathscr{E}(q^j) \equiv \Pi - \sum_i P_i q^i = \frac{1}{2} \sum_{i,k} q^i C_{ik} q^k - \sum_i P_i q^i \,\rightarrow\, \min. \tag{5.6}$$

However, the design may be so unfortunate that the stiffness matrix becomes singular (noninvertible) with $\det C_{ik} = 0$ (or the determinant is very close to zero, $\det C_{ik} \approx 0$ — the nearly singular matrix). Then the solution of the linear problem of statics (5.5) exists only when external loads P_i are orthogonal to all linearly independent solutions of the homogeneous conjugate system

...

The famous theorems of statics for linear continua (§ **??**.**??**) can be easily proved for a finite number of degrees o'freedom. The Clapeyron's theorem looks here like

...

The reciprocal work theorem ("the work W_{12} of the first set's forces on displacements from the forces of the second is equal to the work W_{21} of the second set's forces on displacements from the forces of the first") instantly derives from (5.5) :

(....)

Here the symmetry of the stiffness matrix $C_{ij} = C_{ji}$ is essential — that the system is conservative.

.....

Turning back to the problem (5.2), sometimes called the Lagrange's theorem. Inverted by Legendre transform(ation), it translates into

$$d\Big(\sum_i P_i q^i\Big) = \sum_i d\big(P_i q^i\big) = \sum_i \big(q^i dP_i + P_i dq^i\big),$$

$$\sum_i d\big(P_i q^i\big) - \sum_i P_i dq^i = \sum_i q^i dP_i\,,$$

$$d\Big(\sum_i P_i q^i - \Pi\Big) = \sum_i q^i dP_i = \sum_i \frac{\partial Ш}{\partial P_i} dP_i\,,$$

where appears the exact differential of the so-called "complementary energy" $\amalg$

$$q^i = \frac{\partial \amalg}{\partial P_i}, \quad \amalg(P_i) = \sum_i P_i q^i - \Pi. \tag{5.7}$$

This is known as the Castigliano theorem*. For a linear system (5.5) $\Rightarrow$ $\amalg(P_i) = \Pi(q^i)$. Theorem (5.7) is sometimes very useful — when the complementary energy as the function of external loads $\amalg(P_i)$ is easy to find. Someone may come across the so-called "statically determinate" structures (systems), for which all internal forces can luckily be found just only from the balance (equilibrium) equations for forces and moments. For such structures, (5.7) is effective.

Unlike the linear problem (5.5), the nonlinear problem (5.2) may have no solutions at all or may have several of them.

....

The overview of statics in classical mechanics I am ending with the d'Alembert's principle** : the dynamic equations differ from the static ones only in additional "inertia forces" ("fictitious forces") $m_k \ddot{\boldsymbol{r}}_k$. The d'Alembert's principle is pretty obvious, but applying it everytime & everywhere is a mistake. As example, the equations of motion for a viscous fluid (Navier–Stokes equations) in statics and in dynamics differ not only in inertial adjunct. Nevertheless, for solid elastic bodies the d'Alembert's principle always apply.

* **Carlo Alberto Castigliano**. Intorno ai sistemi elastici, Dissertazione presentata da Castigliano Alberto alla Commissione Esaminatrice della R. Scuola d'Applicazione degli Ingegneri in Torino per ottenere la Laurea di Ingegnere Civile. Torino, Vincenzo Bona, 1873.

** **Jean Le Rond d'Alembert**. Traité de Dynamique, dans lequel les Loix de l'Equilibre & du mouvement des Corps sont réduites au plus petit nombre possible, & démontrées d'une maniére nouvelle, & où l'on donne un Principe général pour trouver le Mouvement de plusieurs Corps qui agissent les uns sur les autres, d'une maniére quelconque. Paris : David l'aîné, MDCCXLIII (1743).

$d\Pi$

$d\amalg$

§ 6. Small oscillations (vibrations)

If the statics of a linear system is described by equation (5.5), then in the dynamics we have

$$\sum_k \left(A_{ik} \ddot{q}_k + C_{ik} q^k \right) = P_i(t), \tag{6.1}$$

where A_{ik} is the symmetric and positive "matrix of kinetic energy".

Any description of oscillations almost always includes the term "mode". A mode of vibration can be defined as a way of vibrating or a pattern of vibration. A normal mode is a pattern of periodic motion, when all parts of an oscillating system move sinusoidally with the same frequency and with a fixed phase relation. The free motion described by the normal modes takes place at fixed frequencies — the natural resonant frequencies of an oscillating system.

The most generic motion of an oscillating system is some superposition of normal modes of this system.*

A research of an oscillating system most often begins with orthogonal (normal) "modes" — harmonics, free (without any driving or damping force) sinusoidal oscillations

$$q^k(t) = \overset{*}{q}_k \sin \omega_k t.$$

Multipliers $\overset{*}{q}_k = \mathsf{constant}$ are orthogonal (normal) "modes" of oscillation, ω_k are natural (resonant, eigen-) frequencies. This set, dependent on the structure of an oscillating object, the materials and the boundary conditions, is found from the eigenvalue problem

$$\begin{gathered} P_i = 0, \;\; \ddot{q}_k = -\omega_k^2 \, \overset{*}{q}_k \sin \omega_k t, \;\; (6.1) \;\; \Rightarrow \\ \Rightarrow \; \sum_k \left(C_{ik} - A_{ik} \, \omega_k^2 \right) \overset{*}{q}_k \sin \omega_k t = 0 \end{gathered} \tag{6.2}$$

* The modes are "normal" in the sense that they move independently, and an excitation of one mode will never cause a motion of another mode. In mathematical terms, normal modes are orthogonal to each other. In music, normal modes of vibrating instruments (strings, air pipes, percussion and others) are called "harmonics" or "overtones".

...

The Duhamel's integral is a way of calculating the response of linear systems to an arbitrary time-varying external perturbation.

...

§ 7. Perfectly rigid undeformable solid body

"Absolutely rigid", aka "absolutely solid" and "absolutely durable" — the pipe dream of any engineer.

One more concept, modeled in classical generic mechanics, is the (perfectly) rigid body. That is a solid* body, in which deformation is zero (or is negligibly small — so small that it can be neglected). The distance between any two points of a non-deformable rigid body remains constant regardless of external forces exerted on it.

A non-deformable rigid body is modeled using the "continual approach" as a continuous distribution of mass (a material continuum, a continuous medium), rather than using the "discrete approach" (that is modeling a body as a discrete collection of particles, § 1).

The mass of a material continuum is distributed continuously throughout its volume,

$$dm \equiv \rho d\mathcal{V} \tag{7.1}$$

(ρ is a volume(tric) mass density and $d\mathcal{V}$ is an infinitesimal volume).

A formula with summation over discrete points becomes a formula for a continuous body by replacing the masses of particles with the mass (7.1) of an infinitesimal volume element $d\mathcal{V}$ with integration over the entire volume of a body. In particular, here are the formulas for the (linear) momentum

$$\sum_k m_k \dot{\boldsymbol{r}}_k \;\text{ becomes }\; \int\limits_{\mathcal{V}} \dot{\boldsymbol{r}}\, dm \tag{7.2}$$

* "Rigid" is inelastic and not flexible, and "solid" is not fluid. A solid substance retains its size and shape without a container (as opposed to a fluid substance, a liquid or a gas).

and for the angular momentum

$$\sum_k \boldsymbol{r}_k \times m_k \dot{\boldsymbol{r}}_k \;\text{ becomes }\; \int_{\mathcal{V}} \boldsymbol{r} \times \dot{\boldsymbol{r}}\, dm. \tag{7.3}$$

To fully describe the location (position, place) of any non-deformable body with all its points, it's enough to choose some unique point as the "pole", to find or to set the location $\boldsymbol{p} = \boldsymbol{p}(t)$ of the chosen point, as well as the angular orientation of a body relative to the pole (figure 3). As a result, any motion of an undeformable rigid body is either a rotation around the chosen pole, or an equal displacement of the pole and all body's points — a translation (a linear motion)*, or a combination of them both.

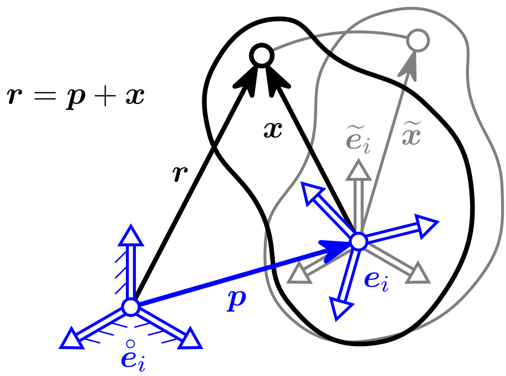

figure 3

$\mathring{\boldsymbol{e}}_i$ — the triplet of mutually perpendicular unit vectors, called the "basis vectors", immovable relatively to the absolute (or to any inertial) reference system

✓ $\mathring{\boldsymbol{e}}_i$ is the immovable (stationary) basis

✓ $\boldsymbol{e}_i$ is the basis which moves along with the body

By adding the basis $\boldsymbol{e}_i$ (it moves together with the body), the body's angular orientation can be determined by the rotation tensor $\boldsymbol{O} \equiv \boldsymbol{e}_i \widetilde{\boldsymbol{e}}_i$.

* A translation can also be thought of as a rotation with the revolution center at infinity.

Then any motion of a body is completely described by two functions, $\boldsymbol{p}(t)$ and $\boldsymbol{O}(t)$.

The location vector of some body's point

$$\boldsymbol{r} = \boldsymbol{p} + \boldsymbol{x} \tag{7.4}$$

$\widetilde{\boldsymbol{x}} = x_i \widetilde{\boldsymbol{e}}_i,\ \boldsymbol{x} = x_i \boldsymbol{e}_i$

(??), § ??.??

$\boldsymbol{x} = \boldsymbol{O} \cdot \widetilde{\boldsymbol{x}}$

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{p}} + \dot{\boldsymbol{x}},$$

For a non-deformable rigid body, components x_i don't depend on time: $x_i = \mathsf{constant}(t)$ and $\dot{\boldsymbol{x}} = x_i \dot{\boldsymbol{e}}_i$

$\dot{\boldsymbol{x}} = \dot{\boldsymbol{O}} \cdot \mathring{\boldsymbol{x}}$

$x_i \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{O}} \cdot x_i \mathring{\boldsymbol{e}}_i \;\Leftrightarrow\; \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{O}} \cdot \mathring{\boldsymbol{e}}_i$

...

The linear momentum and the rotational (angular) momentum of a non-deformable continuous body are described by the following integrals

...

...

$$\int\limits_{\mathcal{V}} \boldsymbol{p}\, dm = \boldsymbol{p} \int\limits_{\mathcal{V}} dm = \boldsymbol{p} m$$

$$\int\limits_{\mathcal{V}} \boldsymbol{x}\, dm = \boldsymbol{\Xi} m,\ \ \boldsymbol{\Xi} \equiv m^{-1} \int\limits_{\mathcal{V}} \boldsymbol{x}\, dm$$

Three inertial characteristics of the body:

- ✓ integral mass $m = \int_{\mathcal{V}} dm = \int_{\mathcal{V}} \rho d\mathcal{V}$ — the mass of the whole body,
- ✓ eccentricity vector $\boldsymbol{\Xi}$ — measures the offset of the chosen pole from the body's "center of mass",

✓ inertia tensor ${}^2\boldsymbol{\mathfrak{J}}$.

The eccentricity vector is equal to the null vector only when the chosen pole coincides with the "center of mass" — the unique point within a body with location vector $\boldsymbol{\mathcal{r}}$, in short

$$\boldsymbol{\Xi} = \mathbf{0} \;\Leftrightarrow\; \boldsymbol{p} = \boldsymbol{\mathcal{r}}.$$

$$\boldsymbol{x} = \boldsymbol{r} - \boldsymbol{p},\quad \boldsymbol{\Xi} m = \int\limits_{\mathcal{V}} (\boldsymbol{r} - \boldsymbol{\mathcal{r}})\, dm = \mathbf{0},$$

$$\int\limits_{\mathcal{V}} \boldsymbol{r}\, dm - \boldsymbol{\mathcal{r}} \int\limits_{\mathcal{V}} dm = \mathbf{0} \;\Rightarrow\; \boldsymbol{\mathcal{r}} = m^{-1} \int\limits_{\mathcal{V}} \boldsymbol{r}\, dm$$

...

Introducing the (pseudo)vector of angular velocity $\boldsymbol{\omega}$, ...

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i$$

...

inertia tensor ${}^2\boldsymbol{\mathfrak{J}}$

$${}^2\boldsymbol{\mathfrak{J}} \equiv -\int\limits_{\mathcal{V}} (\boldsymbol{x} \times \boldsymbol{E}) \cdot (\boldsymbol{x} \times \boldsymbol{E})\, dm = \int\limits_{\mathcal{V}} (\boldsymbol{x} \cdot \boldsymbol{x} \boldsymbol{E} - \boldsymbol{x}\boldsymbol{x})\, dm$$

It is assumed (can be proven?) that the inertia tensor changes only due to a rotation

$${}^2\boldsymbol{\mathfrak{J}} = \boldsymbol{O} \cdot {}^2\mathring{\boldsymbol{\mathfrak{J}}} \cdot \boldsymbol{O}^{\mathsf{T}}$$

and if some basis $\boldsymbol{e}_j$ is moving along with the body, the inertia components in that basis don't change over time

$${}^2\boldsymbol{\mathfrak{J}} = \mathfrak{J}_{ab} \boldsymbol{e}_a \boldsymbol{e}_b,\; \mathfrak{J}_{ab} = \mathsf{constant}(t),$$

thus the time derivative is

$$\begin{aligned} {}^2\dot{\boldsymbol{\mathfrak{J}}} &= \mathfrak{J}_{ab} (\dot{\boldsymbol{e}}_a \boldsymbol{e}_b + \boldsymbol{e}_a \dot{\boldsymbol{e}}_b) = \mathfrak{J}_{ab} (\boldsymbol{\omega} \times \boldsymbol{e}_a \boldsymbol{e}_b + \boldsymbol{e}_a \boldsymbol{\omega} \times \boldsymbol{e}_b) \\ &= \boldsymbol{\omega} \times \mathfrak{J}_{ab} \boldsymbol{e}_a \boldsymbol{e}_b - \mathfrak{J}_{ab} \boldsymbol{e}_a \boldsymbol{e}_b \times \boldsymbol{\omega} = \boldsymbol{\omega} \times {}^2\boldsymbol{\mathfrak{J}} - {}^2\boldsymbol{\mathfrak{J}} \times \boldsymbol{\omega} \end{aligned}$$

Substitution of (....) into (1.6) and (**??**) gives equations of balance — the balance of linear momentum and the balance of rotational momentum — for the model of a continuous non-deformable rigid body

...

here $\boldsymbol{f}$ is the external force per mass unit, $\boldsymbol{F}$ is the resultant of external forces (also called the “equally acting force” or the “main vector”), $\boldsymbol{M}$ is the resultant of external couples (the “main couple”, the “main moment”).

...

Рассмотрим моделирование совершенно жёсткого (недеформируемого) твёрдого тела по принципу виртуальной работы (2.5)

.... (7.4) $\Rightarrow \delta\boldsymbol{r} = \delta\boldsymbol{p} + \delta\boldsymbol{x}$

(begin copied from § **??**.**??**)

Варьируя тождество (**??**), получим $\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = -\boldsymbol{O}\cdot\delta\boldsymbol{O}^{\mathsf{T}}$. Этот тензор антисимметричен, и потому выражается через свой сопутствующий вектор $\boldsymbol{\delta\mathrm{o}}$ как $\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}} = \boldsymbol{\delta\mathrm{o}}\times\boldsymbol{E}$. Приходим к соотношениям

$$\delta\boldsymbol{O} = \boldsymbol{\delta\mathrm{o}}\times\boldsymbol{O},\quad \boldsymbol{\delta\mathrm{o}} = -\frac{1}{2}\Big(\delta\boldsymbol{O}\cdot\boldsymbol{O}^{\mathsf{T}}\Big)_{\times}, \tag{7.5}$$

(end of copied from § **??**.**??**)

....

§ 8. Mechanics of relative motion

До этого не ставился вопрос о системе отсчёта, всё рассматривалось в некой “абсолютной” системе или одной из инерциальных систем (§ 1). Теперь представим себе две системы: “абсолютную” и “подвижную”

...

$$\overset{\circ}{\boldsymbol{r}} = \boldsymbol{r} + \boldsymbol{x}$$
$$\boldsymbol{r} = \rho_i \overset{\circ}{\boldsymbol{e}}_i, \;\; \boldsymbol{x} = x_i \boldsymbol{e}_i$$
$$\dot{\overset{\circ}{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}}$$
$$\dot{\boldsymbol{r}} = \dot{\rho}_i \overset{\circ}{\boldsymbol{e}}_i, \;\; \dot{\boldsymbol{x}} = \big(x_i \boldsymbol{e}_i\big)^{\bullet} = \dot{x}_i \boldsymbol{e}_i + x_i \dot{\boldsymbol{e}}_i$$

$x_i \neq \mathsf{constant} \Rightarrow \dot{x}_i \neq 0$
By (??, § ??.??)

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \;\Rightarrow\; x_i \dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times x_i \boldsymbol{e}_i = \boldsymbol{\omega} \times \boldsymbol{x}$$

$\dot{\boldsymbol{x}} = \dot{x}_i \boldsymbol{e}_i + \boldsymbol{\omega} \times \boldsymbol{x}$

$$\boldsymbol{v} \equiv \dot{\overset{\circ}{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}} = \underbrace{\dot{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{x}}_{\boldsymbol{v}_e} \underbrace{- \boldsymbol{\omega} \times \boldsymbol{x} + \dot{\boldsymbol{x}}}_{\boldsymbol{v}_{rel}}$$

$\dot{\boldsymbol{x}} - \boldsymbol{\omega} \times \boldsymbol{x} = \dot{x}_i \boldsymbol{e}_i \equiv \boldsymbol{v}_{rel}$ — relative velocity, $\dot{\boldsymbol{r}} + \boldsymbol{\omega} \times \boldsymbol{x} \equiv \boldsymbol{v}_e$

$$\boldsymbol{v} = \boldsymbol{v}_e + \boldsymbol{v}_{rel} \tag{8.1}$$

...

$$\dot{\overset{\circ}{\boldsymbol{r}}} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{x}}$$
$$\ddot{\overset{\circ}{\boldsymbol{r}}} = \ddot{\boldsymbol{r}} + \ddot{\boldsymbol{x}}$$
$$\boldsymbol{w} \equiv \dot{\boldsymbol{v}} = \ddot{\overset{\circ}{\boldsymbol{r}}} = \ddot{\boldsymbol{r}} + \ddot{\boldsymbol{x}}$$
$$\ddot{\boldsymbol{r}} = \ddot{\rho}_i \overset{\circ}{\boldsymbol{e}}_i, \;\; \ddot{\boldsymbol{x}} = \big(x_i \boldsymbol{e}_i\big)^{\bullet\bullet} = \big(\dot{x}_i \boldsymbol{e}_i + x_i \dot{\boldsymbol{e}}_i\big)^{\bullet} = \ddot{x}_i \boldsymbol{e}_i + \dot{x}_i \dot{\boldsymbol{e}}_i + \dot{x}_i \dot{\boldsymbol{e}}_i + x_i \ddot{\boldsymbol{e}}_i$$

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \;\Rightarrow\; \ddot{\boldsymbol{e}}_i = \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)^{\bullet} = \dot{\boldsymbol{\omega}} \times \boldsymbol{e}_i + \boldsymbol{\omega} \times \dot{\boldsymbol{e}}_i = \dot{\boldsymbol{\omega}} \times \boldsymbol{e}_i + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)$$

$$x_i \ddot{\boldsymbol{e}}_i = x_i \big(\boldsymbol{\omega} \times \boldsymbol{e}_i\big)^{\bullet} = \dot{\boldsymbol{\omega}} \times x_i \boldsymbol{e}_i + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times x_i \boldsymbol{e}_i\big) = \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + \boldsymbol{\omega} \times \big(\boldsymbol{\omega} \times \boldsymbol{x}\big)$$

$$\dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \boldsymbol{e}_i \;\Rightarrow\; \dot{x}_i \dot{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \dot{x}_i \boldsymbol{e}_i = \boldsymbol{\omega} \times \boldsymbol{v}_{rel}$$

$\ddot{x}_i \boldsymbol{e}_i \equiv \boldsymbol{w}_{rel}$ — relative acceleration

$2\dot{x}_i\dot{\boldsymbol{e}}_i = 2\boldsymbol{\omega}\times\boldsymbol{v}_{rel} \equiv \boldsymbol{w}_{Cor}$ — Coriolis acceleration

$$\ddot{\boldsymbol{x}} = \boldsymbol{w}_{rel} + \boldsymbol{w}_{Cor} + x_i\ddot{\boldsymbol{e}}_i$$

$$\big(x_i\dot{\boldsymbol{e}}_i\big)^{\bullet} = \dot{x}_i\dot{\boldsymbol{e}}_i + x_i\ddot{\boldsymbol{e}}_i = \frac{1}{2}\boldsymbol{w}_{Cor} + x_i\ddot{\boldsymbol{e}}_i$$
$$\big(x_i\dot{\boldsymbol{e}}_i\big)^{\bullet} = \big(\boldsymbol{\omega}\times\boldsymbol{x}\big)^{\bullet} = \dot{\boldsymbol{\omega}}\times\boldsymbol{x} + \boldsymbol{\omega}\times\dot{\boldsymbol{x}}$$

$$\boldsymbol{\omega}\times\dot{\boldsymbol{x}} = \boldsymbol{\omega}\times\big(\dot{x}_i\boldsymbol{e}_i + \boldsymbol{\omega}\times\boldsymbol{x}\big) = \underbrace{\boldsymbol{\omega}\times\dot{x}_i\boldsymbol{e}_i}_{\dot{x}_i\dot{\boldsymbol{e}}_i = \frac{1}{2}\boldsymbol{w}_{Cor}} + \boldsymbol{\omega}\times\big(\boldsymbol{\omega}\times\boldsymbol{x}\big)$$

...

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In a long list of the books about the classical mechanics, the reader can find the works of both the specialists in mechanics [87, 88, 94, 95, 96] and the broadly oriented theoretical physicists [93, 89]. The book by Felix R. Gantmacher (Феликс Р. Гантмахер) [92] with the compact but complete narration of the fundamentals is pretty interesting.

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