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PHYSICS *of* ELASTIC CONTINUA

CONTENTS

chapter 3

NONLINEAR ELASTIC MOMENTLESS CONTINUUM

§ 1. Continuum and the two approaches to describe it

According to the atomic theory, a substance is composed of discrete particles — atoms. Therefore a model of a system of particles with masses m_k and location vectors $\boldsymbol{r}_k(t)$ may seem suitable yet despite an unimaginable number of degrees of freedom, because amounts of memory and the speed of modern computers are characterized also by astronomical numbers.

But anyway, maybe it's worth choosing a fundamentally and qualitatively different model — a model of the material continuum, or the continuous medium, where the mass is distributed continuously within a volume, and the finite volume $\mathcal{V}$ contains the mass

$$m = \int_{\mathcal{V}} \rho d\mathcal{V}, \quad dm = \rho d\mathcal{V}, \tag{1.1}$$

here ρ is the volume(tric) mass density and $d\mathcal{V}$ is the infinitesimal volume element.

A real matter is modeled as a continuum, which can be thought of as an infinite set of vanishingly small particles, joined together.

A space of material points is only the first and simple idea of a continuous distribution of mass. More complex models are possible too, where particles have more degrees of freedom: not only of translation, but also of rotation, of internal deformation, and others. Knowing that such models are attracting more and more interest, in this chapter we will consider the classical concept of a continuous medium as "made of simple points".

At every moment of time t, a deformable continuum occupies a certain volume $\mathcal{V}$ of the space. This volume moves and deforms, but the set of particles inside this volume is constant. It is the balance of mass (“matter is neither created nor annihilated”)

$$dm = \rho d\mathcal{V} = \rho' d\mathcal{V}' = \mathring{\rho} d\mathring{\mathcal{V}}, \quad m = \int\limits_{\mathcal{V}} \rho d\mathcal{V} = \int\limits_{\mathcal{V}'} \rho' d\mathcal{V}' = \int\limits_{\mathring{\mathcal{V}}} \mathring{\rho} d\mathring{\mathcal{V}}. \tag{1.2}$$

Introducing some variable parameters q^i — the curvilinear coordinates, we have a relation for locations of particles

$$\boldsymbol{r} = \boldsymbol{r}(q^i, t). \tag{1.3}$$

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Material description

at the initial moment, in the so-called initial (original, ~~reference~~, “material”) configuration

at some initial moment $t=0$

“запоминается” начальная (“материальная”) конфигурация — locations in space of particles at some arbitrarily chosen “initial” moment $t=0$

$$\mathring{\boldsymbol{r}}(q^i) \equiv \boldsymbol{r}(q^i, 0)$$

Morphism (function) $\mathring{\boldsymbol{r}} = \mathring{\boldsymbol{r}}(q^i)$

isomorphism (bijective mapping) (invertible one-to-one relation) (взаимно однозначное)

Subsequent locations in space of particles are then dependent variables — functions of time and of the initial (material, “Lagrangian”) coordinates/location $\mathring{\boldsymbol{r}}$

$$\boldsymbol{r} = \boldsymbol{r}(\mathring{\boldsymbol{r}}, t).$$

Для пространственного дифференцирования (постоянных во времени) отношений like $\varphi = \varphi(\mathring{\boldsymbol{r}})$

вводится локальный касательный базис $\mathring{\boldsymbol{r}}_{\partial i}$ и взаимный базис $\mathring{\boldsymbol{r}}^i$

$$\mathring{\boldsymbol{r}}_{\partial i} \equiv \partial_i \mathring{\boldsymbol{r}} \ \left(\partial_i \equiv \frac{\partial}{\partial q^i}\right), \quad \mathring{\boldsymbol{r}}^i \cdot \mathring{\boldsymbol{r}}_{\partial j} = \delta^i_j$$

"материальный" оператор Hamilton'а $\mathring{\boldsymbol{\nabla}}$

$$\boldsymbol{E} = \mathring{\boldsymbol{r}}^i \mathring{\boldsymbol{r}}_{\partial i} = \mathring{\boldsymbol{r}}^i \partial_i \mathring{\boldsymbol{r}} = \mathring{\boldsymbol{\nabla}} \mathring{\boldsymbol{r}}, \quad \mathring{\boldsymbol{\nabla}} \equiv \mathring{\boldsymbol{r}}^i \partial_i, \tag{1.4}$$

тогда $d\varphi = d\mathring{\boldsymbol{r}} \cdot \mathring{\boldsymbol{\nabla}} \varphi$.

...

But yet another approach may be effective — *the spatial (or "Eulerian") description*, when instead of focusing on how particles of a continuum move from the initial configuration through space and time, processes are considered at fixed points in space as time progresses. With relations like $\rho = \rho(\boldsymbol{r}, t)$, we track what's happening exactly in this place. Various particles, continuously leaving and coming here, do not confuse us.

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the balance of mass in spatial description (the continuity equation for mass)

....

Jaumann derivative ("corotational time derivative") was first introduced by Gustav Jaumann*

Es sei $\frac{\partial}{\partial t}$ der Operator der lokalen Fluxion, d. i. der partiellen Fluxion in einem gegen das Koordinatensystem ruhenden Punkte des Raumes. Ferner sei $\frac{d}{dt}$ der Operator der totalen Fluxion, welcher definiert wird durch

$$\frac{da}{dt} = \frac{\partial a}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} a,$$

$$\frac{d\boldsymbol{a}}{dt} \overset{3}{=} \frac{\partial \boldsymbol{a}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}; \boldsymbol{a} - \tfrac{1}{2} \big(\operatorname{rot} \boldsymbol{v}\big) \times \boldsymbol{a},$$

$$\frac{d\boldsymbol{\alpha}}{dt} \overset{9}{=} \frac{\partial \boldsymbol{\alpha}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}; \boldsymbol{\alpha} - \tfrac{1}{2} \big(\operatorname{rot} \boldsymbol{v} \times \boldsymbol{\alpha} - \boldsymbol{\alpha} \times \operatorname{rot} \boldsymbol{v}\big).$$

* **Gustav Jaumann**. Geschlossenes System physikalischer und chemischer Differentialgesetze (I. Mitteilung) // Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Klasse, Abteilung IIa, Band CXX, 1911. Seiten 385–530.

Endlich verwenden wir die körperliche Fluxion eines Skalars:
$$\frac{\delta}{\delta t} a = \frac{\partial}{\partial t} a + \operatorname{div} a\boldsymbol{v} = \frac{d}{dt} a + a \operatorname{div} \boldsymbol{v}.$$

körperliche — bodily / телесная, material / вещественная(материальная), physical / физическая

$\boldsymbol{\nabla} \cdot (a\boldsymbol{v}) = a \boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} a$

...

Пусть $\upsilon(\mathring{\boldsymbol{r}}, t)$ — какое-либо поле (?? только в материальном описании от $\mathring{\boldsymbol{r}}$??). Найдём скорость изменения интеграла по объёму

$$\Upsilon \equiv \int_{\mathcal{V}} \rho \upsilon \, d\mathcal{V}$$

("υ is Υ per mass unit"). Seemingly difficult calculation of $\dot{\Upsilon}$ (since $\mathcal{V}$ is deforming) is actually quite simple with the balance of mass (1.2) :

$$\Upsilon = \int_{\mathring{\mathcal{V}}} \mathring{\rho} \upsilon \, d\mathring{\mathcal{V}} \;\Rightarrow\; \dot{\Upsilon} = \int_{\mathring{\mathcal{V}}} \mathring{\rho} \dot{\upsilon} \, d\mathring{\mathcal{V}} = \int_{\mathcal{V}} \rho \dot{\upsilon} \, d\mathcal{V}. \tag{1.5}$$

$$\Psi = \int_{\mathcal{V}} \rho \psi \, d\mathcal{V} = \int_{\mathcal{V}'} \rho' \psi \, d\mathcal{V}' \Rightarrow \dot{\Psi} = \int_{\mathcal{V}} \rho \dot{\psi} \, d\mathcal{V} = \int_{\mathcal{V}'} \rho' \dot{\psi} \, d\mathcal{V}'$$

...

It is not worth it to contrapose the material and the spatial descriptions. In this book both are used, depending on the situation.

§ 2. Motion gradient

Having the motion function $\boldsymbol{r} = \boldsymbol{r}(q^i, t)$, $\mathring{\boldsymbol{r}}(q^i) \equiv \boldsymbol{r}(q^i, 0)$, the "nabla" operators $\boldsymbol{\nabla} \equiv \boldsymbol{r}^i \partial_i$, $\mathring{\boldsymbol{\nabla}} \equiv \mathring{\boldsymbol{r}}^i \partial_i$ and looking at differential relations for a certain infinitesimal vector in two configurations, the current with $d\boldsymbol{r}$ and the initial with $d\mathring{\boldsymbol{r}}$

$$\begin{aligned} d\boldsymbol{r} &= d\mathring{\boldsymbol{r}} \cdot \overbrace{\mathring{\boldsymbol{\nabla}} \boldsymbol{r}}^{\substack{\boldsymbol{F}^{\mathsf{T}} \\ \mathring{\boldsymbol{r}}^i \boldsymbol{r}_{\partial i}}} = \overbrace{\mathring{\boldsymbol{\nabla}} \boldsymbol{r}^{\mathsf{T}}}^{\substack{\boldsymbol{F} \\ \boldsymbol{r}_{\partial i} \mathring{\boldsymbol{r}}^i}} \cdot \, d\mathring{\boldsymbol{r}} \\ d\mathring{\boldsymbol{r}} &= d\boldsymbol{r} \cdot \underbrace{\boldsymbol{\nabla} \mathring{\boldsymbol{r}}}_{\substack{\boldsymbol{r}^i \mathring{\boldsymbol{r}}_{\partial i} \\ \boldsymbol{F}^{-\mathsf{T}}}} = \underbrace{\boldsymbol{\nabla} \mathring{\boldsymbol{r}}^{\mathsf{T}}}_{\substack{\mathring{\boldsymbol{r}}_{\partial i} \boldsymbol{r}^i \\ \boldsymbol{F}^{-1}}} \cdot \, d\boldsymbol{r} \end{aligned} \tag{2.1}$$

here comes to mind to introduce the "motion gradient"*, picking one of these tensor multipliers for it : $\boldsymbol{F} \equiv \mathring{\boldsymbol{\nabla}} \boldsymbol{r}^{\mathsf{T}} = \boldsymbol{r}_{\partial i} \mathring{\boldsymbol{r}}^i$.

Why this one? The reason to choose $\mathring{\boldsymbol{\nabla}} \boldsymbol{r}^{\mathsf{T}}$ is another expression for the differential

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial \mathring{\boldsymbol{r}}} \cdot d\mathring{\boldsymbol{r}} \qquad \boldsymbol{F} = \frac{\partial \boldsymbol{r}}{\partial \mathring{\boldsymbol{r}}}$$

$$d\mathring{\boldsymbol{r}} = \frac{\partial \mathring{\boldsymbol{r}}}{\partial \boldsymbol{r}} \cdot d\boldsymbol{r} \qquad \boldsymbol{F}^{-1} = \frac{\partial \mathring{\boldsymbol{r}}}{\partial \boldsymbol{r}}$$

$$\frac{\partial \zeta}{\partial \mathring{\boldsymbol{r}}} = \partial_i \boldsymbol{\zeta} \mathring{\boldsymbol{r}}^i \qquad \frac{\partial \zeta}{\partial \boldsymbol{r}} = \partial_i \boldsymbol{\zeta} \boldsymbol{r}^i$$

....

$$\boldsymbol{E} = \underbrace{\mathring{\boldsymbol{\nabla}} \mathring{\boldsymbol{r}}}_{\frac{\partial \mathring{\boldsymbol{r}}}{\partial \mathring{\boldsymbol{r}}}} = \underbrace{\boldsymbol{\nabla} \boldsymbol{r}}_{\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}}}$$

...

For cartesian coordinates with orthonormal basis $\boldsymbol{e}_i =$ **constant**

$$\boldsymbol{r} = x_i(t)\, \boldsymbol{e}_i, \;\; \mathring{\boldsymbol{r}} = x_i(0)\, \boldsymbol{e}_i = \mathring{x}_i \boldsymbol{e}_i, \;\; \mathring{x}_i \equiv x_i(0),$$

* Tensor $\boldsymbol{F}$ doesn't well suit the more popular name "deformation gradient", because this tensor describes not only the deformation itself, but also the rotation of a body as a whole without deformation.

$$\overset{\circ}{\boldsymbol{\nabla}} = \boldsymbol{e}_i \frac{\partial}{\partial \overset{\circ}{x}_i} = \boldsymbol{e}_i \overset{\circ}{\partial}_i , \;\; \boldsymbol{\nabla} = \boldsymbol{e}_i \frac{\partial}{\partial x_i} = \boldsymbol{e}_i \partial_i ,$$

$$\overset{\circ}{\boldsymbol{\nabla}} \boldsymbol{r} = \boldsymbol{e}_i \frac{\partial \boldsymbol{r}}{\partial \overset{\circ}{x}_i} = \boldsymbol{e}_i \frac{\partial (x_j \boldsymbol{e}_j)}{\partial \overset{\circ}{x}_i} = \frac{\partial x_j}{\partial \overset{\circ}{x}_i} \boldsymbol{e}_i \boldsymbol{e}_j = \overset{\circ}{\partial}_i x_j \boldsymbol{e}_i \boldsymbol{e}_j ,$$
$$\boldsymbol{\nabla} \overset{\circ}{\boldsymbol{r}} = \boldsymbol{e}_i \frac{\partial \overset{\circ}{\boldsymbol{r}}}{\partial x_i} = \frac{\partial \overset{\circ}{x}_j}{\partial x_i} \boldsymbol{e}_i \boldsymbol{e}_j = \partial_i \overset{\circ}{x}_j \boldsymbol{e}_i \boldsymbol{e}_j$$

...

By the polar decomposition theorem (§ **??.??**), the motion gradient decomposes into the rotation tensor $\boldsymbol{O}$ and the symmetric positive stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$:

$$\boldsymbol{F} = \boldsymbol{O} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{O}$$

...

When there's no rotation ($\boldsymbol{O} = \boldsymbol{E}$), then $\boldsymbol{F} = \boldsymbol{U} = \boldsymbol{V}$.

....

§ 3. Measures (tensors) of deformation

And this is where the extra complexity arose. Although, multivariance is often seen as a big gift.

The motion gradient $\boldsymbol{F}$ characterizes both the deformation of a body and the rotation of a body as a whole. The deformation-only tensors are the stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ from the polar decomposition $\boldsymbol{F} = \boldsymbol{O} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{O}$, as well as another tensors, originating from $\boldsymbol{U}$ or (and) $\boldsymbol{V}$.

The widely used ones are the "squares" of $\boldsymbol{U}$ and $\boldsymbol{V}$

$$\begin{aligned} \left(\boldsymbol{U}^2 =\right) \, \boldsymbol{U} \cdot \boldsymbol{U} = \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \equiv \boldsymbol{G}, \\ \left(\boldsymbol{V}^2 =\right) \, \boldsymbol{V} \cdot \boldsymbol{V} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathsf{T}} \equiv \boldsymbol{\Phi}. \end{aligned} \tag{3.1}$$

These are the Green's deformation tensor (or the right Cauchy–Green tensor) $\boldsymbol{G}$ and the Finger's deformation tensor (or the left Cauchy–Green tensor) $\boldsymbol{\Phi}$. They have the convenient link with the motion gradient $\boldsymbol{F}$, without calculating square roots (as it's needed for $\boldsymbol{U}$

and $\boldsymbol{V}$). That's the big reason why tensors $\boldsymbol{G}$ and $\boldsymbol{\Phi}$ are so widely used.

Tensor $\boldsymbol{G}$ was first used by George Green*.

An inversion of $\boldsymbol{\Phi}$ and $\boldsymbol{G}$ gives the two more deformation tensors

$$\begin{aligned} \boldsymbol{V}^{-2} = \boldsymbol{\Phi}^{-1} = \left(\boldsymbol{F} \cdot \boldsymbol{F}^{\mathsf{T}}\right)^{-1} = \boldsymbol{F}^{-\mathsf{T}} \cdot \boldsymbol{F}^{-1} \equiv {}^{2}\boldsymbol{c}, \\ \boldsymbol{U}^{-2} = \boldsymbol{G}^{-1} = \left(\boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F}\right)^{-1} = \boldsymbol{F}^{-1} \cdot \boldsymbol{F}^{-\mathsf{T}} \equiv {}^{2}\boldsymbol{f}, \end{aligned} \tag{3.2}$$

each of which is sometimes called the Piola tensor or the Finger tensor. The inverse of the left Cauchy–Green tensor $\boldsymbol{\Phi}$ is known as the Cauchy deformation tensor ${}^{2}\boldsymbol{c}$.

The components of these tensors are

$$\begin{aligned} \boldsymbol{G} &= \mathring{\boldsymbol{r}}^{i} \boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j} \mathring{\boldsymbol{r}}^{j} = G_{ij} \mathring{\boldsymbol{r}}^{i} \mathring{\boldsymbol{r}}^{j}, \;\; G_{ij} \equiv \boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j}, \\ {}^{2}\boldsymbol{f} &= \mathring{\boldsymbol{r}}_{\partial i} \boldsymbol{r}^{i} \cdot \boldsymbol{r}^{j} \mathring{\boldsymbol{r}}_{\partial j} = G^{ij} \mathring{\boldsymbol{r}}_{\partial i} \mathring{\boldsymbol{r}}_{\partial j}, \;\; G^{ij} \equiv \boldsymbol{r}^{i} \cdot \boldsymbol{r}^{j}, \\ {}^{2}\boldsymbol{c} &= \boldsymbol{r}^{i} \mathring{\boldsymbol{r}}_{\partial i} \cdot \mathring{\boldsymbol{r}}_{\partial j} \boldsymbol{r}^{j} = g_{ij} \boldsymbol{r}^{i} \boldsymbol{r}^{j}, \;\; g_{ij} \equiv \mathring{\boldsymbol{r}}_{\partial i} \cdot \mathring{\boldsymbol{r}}_{\partial j}, \\ \boldsymbol{\Phi} &= \boldsymbol{r}_{\partial i} \mathring{\boldsymbol{r}}^{i} \cdot \mathring{\boldsymbol{r}}^{j} \boldsymbol{r}_{\partial j} = g^{ij} \boldsymbol{r}_{\partial i} \boldsymbol{r}_{\partial j}, \;\; g^{ij} \equiv \mathring{\boldsymbol{r}}^{i} \cdot \mathring{\boldsymbol{r}}^{j}, \end{aligned}$$

and they coincide with the components of the unit (metric) tensor

$$\begin{aligned} \boldsymbol{E} = \boldsymbol{r}_{\partial i} \boldsymbol{r}^{i} = G_{ij} \boldsymbol{r}^{i} \boldsymbol{r}^{j} = \boldsymbol{r}^{i} \boldsymbol{r}_{\partial i} = G^{ij} \boldsymbol{r}_{\partial i} \boldsymbol{r}_{\partial j} \\ = \mathring{\boldsymbol{r}}^{i} \mathring{\boldsymbol{r}}_{\partial i} = g^{ij} \mathring{\boldsymbol{r}}_{\partial i} \mathring{\boldsymbol{r}}_{\partial j} = \mathring{\boldsymbol{r}}_{\partial i} \mathring{\boldsymbol{r}}^{i} = g_{ij} \mathring{\boldsymbol{r}}^{i} \mathring{\boldsymbol{r}}^{j}, \end{aligned}$$

but the components' bases are different. Using only the index notation, it's easy to get confused due to the similarity between the unit tensor $\boldsymbol{E}$ and the strain tensors $\boldsymbol{G}$, $\boldsymbol{\Phi}$, ${}^{2}\boldsymbol{f}$, ${}^{2}\boldsymbol{c}$. The direct indexless notation has the obvious advantage here.

As was mentioned in § **??**.**??**, the invariants of the stretch tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ are the same. If w_i are the three eigenvalues of $\boldsymbol{U}$ and $\boldsymbol{V}$,

* **Green, George**. (1839) On the propagation of light in crystallized media. *Transactions of the Cambridge Philosophical Society.* 1842, vol. 7, part II, pages 121–140.

that is the roots of the characteristic equations for these tensors, then here are their invariants:

$$
\begin{gathered}
\mathrm{I}(\boldsymbol{U}) = \mathrm{I}(\boldsymbol{V}) = \operatorname{trace}\boldsymbol{U} = \operatorname{trace}\boldsymbol{V} = \sum U_{jj} = \sum V_{jj} = \sum w_i,\\
\mathrm{II}(\boldsymbol{U}) = \mathrm{II}(\boldsymbol{V}) = w_1 w_2 + w_1 w_3 + w_2 w_3,\\
\mathrm{III}(\boldsymbol{U}) = \mathrm{III}(\boldsymbol{V}) = w_1 w_2 w_3.
\end{gathered}
$$

The invariants of $\boldsymbol{G}$ and $\boldsymbol{\Phi}$ coincide too:

$$\mathrm{I}(\boldsymbol{G}) = \mathrm{I}(\boldsymbol{\Phi}), \ldots$$

Without deformation

$$\boldsymbol{F} = \boldsymbol{U} = \boldsymbol{V} = \boldsymbol{G} = \boldsymbol{\Phi} = {}^2\boldsymbol{f} = {}^2\boldsymbol{c} = \boldsymbol{E},$$

thus as characteristics of deformation it's worth taking the differences like $\boldsymbol{U} - \boldsymbol{E}$, $\boldsymbol{U} \cdot \boldsymbol{U} - \boldsymbol{E}$, ...

...

The right Cauchy–Green deformation tensor

George Green discovered a deformation tensor known as the right Cauchy–Green deformation tensor or Green's deformation tensor

$$\boldsymbol{G} = \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} = \boldsymbol{U}^2 \quad \text{or} \quad G_{ij} = F_{k'i} F_{k'j} = \frac{\partial x_{k'}}{\partial \mathring{x}_i} \frac{\partial x_{k'}}{\partial \mathring{x}_j}.$$

This tensor gives the "square" of local change in distances due to deformation: $d\boldsymbol{r} \cdot d\boldsymbol{r} = d\mathring{\boldsymbol{r}} \cdot \boldsymbol{G} \cdot d\mathring{\boldsymbol{r}}$

The most popular invariants of $\boldsymbol{G}$ are

$$
\begin{aligned}
\mathrm{I}(\boldsymbol{G}) &\equiv \operatorname{trace}\boldsymbol{G} = G_{ii} = \gamma_1^2 + \gamma_2^2 + \gamma_3^2\\
\mathrm{II}(\boldsymbol{G}) &\equiv \tfrac{1}{2}\left(G_{jj}^2 - G_{ik}G_{ki}\right) = \gamma_1^2\gamma_2^2 + \gamma_2^2\gamma_3^2 + \gamma_3^2\gamma_1^2\\
\mathrm{III}(\boldsymbol{G}) &\equiv \det\boldsymbol{G} = \gamma_1^2\gamma_2^2\gamma_3^2
\end{aligned}
$$

where γ_i are stretch ratios for unit fibers that are initially oriented along directions of eigenvectors of the right stretch tensor $\boldsymbol{U}$.

The inverse of Green's deformation tensor

Sometimes called the Finger tensor or the Piola tensor, the inverse of the right Cauchy–Green deformation tensor

$$^2\boldsymbol{f} = \boldsymbol{G}^{-1} = \boldsymbol{F}^{-1} \bullet \boldsymbol{F}^{-\mathsf{T}} \quad \text{or} \quad f_{ij} = \frac{\partial \mathring{x}_i}{\partial x_{k'}} \frac{\partial \mathring{x}_j}{\partial x_{k'}}$$

The left Cauchy–Green or Finger deformation tensor

Swapping multipliers in the formula for the right Green–Cauchy deformation tensor leads to the left Cauchy–Green deformation tensor, defined as

$$\boldsymbol{\Phi} = \boldsymbol{F} \bullet \boldsymbol{F}^{\mathsf{T}} = \boldsymbol{V}^2 \quad \text{or} \quad \Phi_{ij} = \frac{\partial x_i}{\partial \mathring{x}_k} \frac{\partial x_j}{\partial \mathring{x}_k}$$

The left Cauchy–Green deformation tensor is often called the Finger's deformation tensor, named after Josef Finger (1894).

Invariants of $\boldsymbol{\Phi}$ are also used in expressions for strain energy density functions. The conventional invariants are defined as

$$\begin{aligned}
I_1 &\equiv \Phi_{ii} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\
I_2 &\equiv \tfrac{1}{2}\left(\Phi_{ii}^2 - \Phi_{jk}\Phi_{kj}\right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \\
I_3 &\equiv \det \boldsymbol{\Phi} = \mathcal{J}^2 = \lambda_1^2\lambda_2^2\lambda_3^2
\end{aligned}$$

($\mathcal{J} \equiv \det \boldsymbol{F}$ is the Jacobian, determinant of the motion gradient)

The Cauchy deformation tensor

The Cauchy deformation tensor is defined as the inverse of the left Cauchy–Green deformation tensor

$$^2\boldsymbol{c} = \boldsymbol{\Phi}^{-1} = \boldsymbol{F}^{-\mathsf{T}} \bullet \boldsymbol{F}^{-1} \quad \text{or} \quad c_{ij} = \frac{\partial \mathring{x}_k}{\partial x_i} \frac{\partial \mathring{x}_k}{\partial x_j}$$

$$d\mathring{\boldsymbol{r}} \bullet d\mathring{\boldsymbol{r}} = d\boldsymbol{r} \bullet {}^2\boldsymbol{c} \bullet d\boldsymbol{r}$$

This tensor is also called the Piola tensor or the Finger tensor in rheology and fluid dynamics literature.

Finite strain tensors

The concept of *strain* is used to evaluate how much a given displacement differs locally from a body displacement as a whole (a "rigid body displacement"). One of such strains for large (finite) deformations is the *Green strain tensor* (*Green–Lagrangian strain tensor*, *Green–Saint-Venant strain tensor*). It measures how much $\boldsymbol{G}$ differs from $\boldsymbol{E}$

$$\boldsymbol{C} = \tfrac{1}{2}\bigl(\boldsymbol{G} - \boldsymbol{E}\bigr) = \tfrac{1}{2}\bigl(\boldsymbol{F}^{\mathsf{T}}\!\cdot\boldsymbol{F} - \boldsymbol{E}\bigr) \tag{3.3}$$

or as the function of the displacement gradient tensor

$$\boldsymbol{C} = \tfrac{1}{2}\Bigl(\mathring{\boldsymbol{\nabla}}\boldsymbol{u} + \mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}} + \mathring{\boldsymbol{\nabla}}\boldsymbol{u}\cdot\mathring{\boldsymbol{\nabla}}\boldsymbol{u}^{\mathsf{T}}\Bigr),$$

in cartesian coordinates

$$C_{ij} = \tfrac{1}{2}\Bigl(\frac{\partial x_{k'}}{\partial \mathring{x}_i}\frac{\partial x_{k'}}{\partial \mathring{x}_j} - \delta_{ij}\Bigr) = \tfrac{1}{2}\Bigl(\frac{\partial u_j}{\partial \mathring{x}_i} + \frac{\partial u_i}{\partial \mathring{x}_j} + \frac{\partial u_k}{\partial \mathring{x}_i}\frac{\partial u_k}{\partial \mathring{x}_j}\Bigr).$$

The *Almansi–Hamel strain tensor*, referenced to the deformed configuration ("Eulerian description"), is defined as

$$^{2}\boldsymbol{a} = \tfrac{1}{2}\bigl(\boldsymbol{E} - {}^{2}\boldsymbol{c}\bigr) = \tfrac{1}{2}\bigl(\boldsymbol{E} - \boldsymbol{\Phi}^{-1}\bigr) \quad \text{or} \quad a_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial \mathring{x}_k}{\partial x_i}\frac{\partial \mathring{x}_k}{\partial x_j}\right)$$

or as function of the displacement gradient

$$^{2}\boldsymbol{a} = \tfrac{1}{2}\bigl(\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{T}} + \boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{T}}\bigr)$$

$$a_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

Seth–Hill family of abstract strain tensors

B. R. Seth was the first to show that the Green and Almansi strain tensors are special cases of a more abstract measure of deformation. The idea was further expanded upon by Rodney Hill in 1968 (publication??). The Seth–Hill family of strain measures (also called Doyle–Ericksen tensors) is expressed as

$$\boldsymbol{D}_{(m)} = \frac{1}{2m}\left(\boldsymbol{U}^{2m} - \boldsymbol{E}\right) = \frac{1}{2m}\left(\boldsymbol{G}^{m} - \boldsymbol{E}\right)$$

For various m it gives

$$\begin{aligned}
\boldsymbol{D}_{(1)} &= \tfrac{1}{2}\left(\boldsymbol{U}^{2} - \boldsymbol{E}\right) = \tfrac{1}{2}(\boldsymbol{G} - \boldsymbol{E}) && \text{Green strain tensor}\\
\boldsymbol{D}_{(1/2)} &= \boldsymbol{U} - \boldsymbol{E} = \boldsymbol{G}^{1/2} - \boldsymbol{E} && \text{Biot strain tensor}\\
\boldsymbol{D}_{(0)} &= \ln \boldsymbol{U} = \tfrac{1}{2}\ln \boldsymbol{G} && \text{logarithmic strain, Hencky strain}\\
\boldsymbol{D}_{(-1)} &= \tfrac{1}{2}\left(\boldsymbol{E} - \boldsymbol{U}^{-2}\right) && \text{Almansi strain}
\end{aligned}$$

The second-order approximation of these tensors is

$$\boldsymbol{D}_{(m)} = \boldsymbol{\varepsilon} + \tfrac{1}{2}\boldsymbol{\nabla}\boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{T}} - (1-m)\,\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}$$

where $\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}\boldsymbol{u}^{\mathsf{S}}$ is the infinitesimal deformation tensor.

Many other different definitions of measures $\boldsymbol{D}$ are possible, provided that they satisfy these conditions:

- ✓ $\boldsymbol{D}$ vanishes for any movement of a body as a rigid whole
- ✓ dependence of $\boldsymbol{D}$ on displacement gradient tensor $\nabla\boldsymbol{u}$ is continuous, continuously differentiable and monotonic
- ✓ it's desired that $\boldsymbol{D}$ reduces to the infinitesimal linear deformation tensor $\boldsymbol{\varepsilon}$ when $\boldsymbol{\nabla}\boldsymbol{u} \to 0$

For example, tensors from the set

$$\boldsymbol{D}^{(n)} = \left(\boldsymbol{U}^{n} - \boldsymbol{U}^{-n}\right)/\,2n$$

aren't from the Seth–Hill family, but for any n they have the same 2nd-order approximation as Seth–Hill measures with $m = 0$.

Wikipedia, the free encyclopedia — Finite strain theory

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Heinrich Hencky. Über die Form des Elastizitätsgesetzes bei ideal elastischen Stoffen. Zeitschrift für technische Physik, Vol. 9 (1928), Seiten 215–220.

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§ 4. Velocity field

This topic is discussed in nearly any book about continuum mechanics, however for solid elastic continua it's not very vital. Among various models of a material continuum, an elastic solid body is distinguished by interesting possibility of deriving the complete set (system) of equations for it via the single logically flawless procedure. But now we follow the way, usual for fluid continuum mechanics.

So, there's velocity field in spatial description $\boldsymbol{v} \equiv \dot{\boldsymbol{r}} = \boldsymbol{v}(\boldsymbol{r}, t)$. Decomposition of tensor $\boldsymbol{\nabla}\boldsymbol{v} = \boldsymbol{\nabla}\dot{\boldsymbol{r}} = \boldsymbol{r}^i \partial_i \dot{\boldsymbol{r}} = \boldsymbol{r}^i \dot{\boldsymbol{r}}_{\partial i}$* into symmetric and skewsymmetric parts (§ ??.??)

$$\boldsymbol{\nabla}\dot{\boldsymbol{r}} = \boldsymbol{\nabla}\dot{\boldsymbol{r}}^{\mathsf{S}} - \tfrac{1}{2}\big(\boldsymbol{\nabla} \times \dot{\boldsymbol{r}}\big) \times \boldsymbol{E}$$

or, introducing the rate of deformation tensor (rate of stretching tensor, strain rate tensor) $\boldsymbol{\mathcal{D}}$ and the vorticity tensor (rate of rotation tensor, spin tensor) $\boldsymbol{\mathcal{W}}$

$$\begin{gathered}
\boldsymbol{\nabla}\boldsymbol{v} = \boldsymbol{\mathcal{D}} - \boldsymbol{\mathcal{W}}, \\
\boldsymbol{\mathcal{D}} \equiv \boldsymbol{\nabla}\boldsymbol{v}^{\mathsf{S}} = \boldsymbol{\nabla}\dot{\boldsymbol{r}}^{\mathsf{S}} = \tfrac{1}{2}\big(\boldsymbol{r}^i \dot{\boldsymbol{r}}_{\partial i} + \dot{\boldsymbol{r}}_{\partial i} \boldsymbol{r}^i\big), \\
-\boldsymbol{\mathcal{W}} \equiv \boldsymbol{\nabla}\boldsymbol{v}^{\mathsf{A}} = -\,\boldsymbol{w} \times \boldsymbol{E}, \quad \boldsymbol{w} \equiv \frac{1}{2} \boldsymbol{\nabla} \times \boldsymbol{v} = \frac{1}{2} \boldsymbol{r}^i \times \dot{\boldsymbol{r}}_{\partial i},
\end{gathered} \tag{4.1}$$

where also figures the vorticity (pseudo)vector $\boldsymbol{w}$, the companion of $\boldsymbol{\mathcal{W}}$.

* For sufficiently smooth functions, partial derivatives always commute, space and time ones too. Thus

$$\frac{\partial}{\partial q^i}\frac{\partial \boldsymbol{r}}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \boldsymbol{r}}{\partial q^i} \quad \text{or} \quad \partial_i \dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_{\partial i}$$

Components of the rate of deformation tensor in the current configuration's basis

$$\boldsymbol{\mathcal{D}} = \mathcal{D}_{ij} \boldsymbol{r}^i \boldsymbol{r}^j, \;\; \mathcal{D}_{ij} = \boldsymbol{r}_{\partial i} \cdot \boldsymbol{\mathcal{D}} \cdot \boldsymbol{r}_{\partial j} = \frac{1}{2}\, \boldsymbol{r}_{\partial i} \cdot \left(\boldsymbol{r}^k \dot{\boldsymbol{r}}_k + \dot{\boldsymbol{r}}_k \boldsymbol{r}^k \right) \cdot \boldsymbol{r}_{\partial j} = \\ = \frac{1}{2} \left(\dot{\boldsymbol{r}}_{\partial i} \cdot \boldsymbol{r}_{\partial j} + \boldsymbol{r}_{\partial i} \cdot \dot{\boldsymbol{r}}_{\partial j} \right) = \frac{1}{2} \Big(\boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j} \Big)^{\bullet}$$

...

$$\dot{G}_{ij}$$

$$G_{ij} \equiv \boldsymbol{r}_{\partial i} \cdot \boldsymbol{r}_{\partial j}$$

...

For elastic solid media, there's no need to discuss about rotations : the true representation appears along the way of logically harmonious conclusions and without additional hypotheses.

§ 5. Area vector. Surface change

Take an infinitesimal surface. The area vector by length is equal to the surface's area and is directed along the normal to this surface.

In the initial (original, undeformed, "material", ~~reference~~) configuration, the area vector can be represented as $\mathring{\boldsymbol{n}} do$. Surface's area do is infinitely small, and $\mathring{\boldsymbol{n}}$ is unit normal vector.

In the present (current, actual, deformed, "spatial") configuration, the same surface has area vector $\boldsymbol{n} d\mathcal{O}$.

With differential precision, these infinitesimal surfaces are parallelograms, thus

$$\begin{aligned} \mathring{\boldsymbol{n}} do &= d\mathring{\boldsymbol{r}}' \times d\mathring{\boldsymbol{r}}'' = \frac{\partial \mathring{\boldsymbol{r}}}{\partial q^i} dq^i \times \frac{\partial \mathring{\boldsymbol{r}}}{\partial q^j} dq^j = \mathring{\boldsymbol{r}}_{\partial i} \times \mathring{\boldsymbol{r}}_{\partial j} \, dq^i dq^j, \\ \boldsymbol{n} d\mathcal{O} &= d\boldsymbol{r}' \times d\boldsymbol{r}'' = \frac{\partial \boldsymbol{r}}{\partial q^i} dq^i \times \frac{\partial \boldsymbol{r}}{\partial q^j} dq^j = \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j} \, dq^i dq^j. \end{aligned} \tag{5.1}$$

Applying the transformation of volume (**??**), we have

$$d\mathcal{V} = \mathcal{J} d\mathring{\mathcal{V}} \;\Rightarrow\; \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j} \cdot \boldsymbol{r}_{\partial k} = \mathcal{J} \mathring{\boldsymbol{r}}_{\partial i} \times \mathring{\boldsymbol{r}}_{\partial j} \cdot \mathring{\boldsymbol{r}}_{\partial k} \;\Rightarrow$$
$$\Rightarrow\; \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j} \cdot \boldsymbol{r}_{\partial k} \boldsymbol{r}^{k} = \mathcal{J} \mathring{\boldsymbol{r}}_{\partial i} \times \mathring{\boldsymbol{r}}_{\partial j} \cdot \mathring{\boldsymbol{r}}_{\partial k} \boldsymbol{r}^{k} \;\Rightarrow$$
$$\Rightarrow\; \boldsymbol{r}_{\partial i} \times \boldsymbol{r}_{\partial j} = \mathcal{J} \mathring{\boldsymbol{r}}_{\partial i} \times \mathring{\boldsymbol{r}}_{\partial j} \cdot \boldsymbol{F}^{-1}.$$

Hence with (5.1) we come to the relation

$$\boldsymbol{n} d\mathcal{O} = \mathcal{J} \mathring{\boldsymbol{n}} do \cdot \boldsymbol{F}^{-1}, \tag{5.2}$$

called the Nanson's formula.

§ 6. Forces in continuum. Existence of the Cauchy stress tensor

Augustin-Louis Cauchy founded the *continuum mechanics* with the idea that two adjoining parts of a body interact with each other by means of contact forces on a dividing surface.

Assuming that these contact forces depend only on the perpendicular to the dividing surface and that surface contact forces are balanced by some volumetric force density, including inertia, Cauchy played with tetrahedrons and proved the existænce of the stress tensor.

"De la pression ou tension dans un corps solide." dans (i) *Exercices de mathématiques*, par M. **Augustin-Louis Cauchy**. Seconde année: 1827. Paris, Chez de Bure frères. Pages 42 à 59. (ii) *Œuvres complètes* d'**Augustin Cauchy**. Série 2, tome 7. Pages 60 à 78.

The particles of a momentless model of a continuum are points that have only translational degrees of freedom*. Thus there're no moments among generalized forces, and there can't be any external force couples.

Force $\rho \boldsymbol{f} d\mathcal{V}$ acts on infinitesimal volume $d\mathcal{V}$. If $\boldsymbol{f}$ is a mass force (acting per unit of mass), then $\rho \boldsymbol{f}$ is a volume one. Such forces

* The translational degrees of freedom come from the particle's ability to move freely in space.

originate from force fields, for example: the gravitational forces ("weight"), the forces of inertia in a non-inertial reference system, the electromagnetic forces in a medium with charges and currents.

Surface force $\boldsymbol{p}d\mathcal{O}$ acts on infinitesimal surface $d\mathcal{O}$. It may be a contact pressure or/and a friction, an electrostatic force with charges concentrated on the surface.

In a material continuum, like in any mechanical system, the external and the internal forces are distinguished. The internal forces balance the action of the external forces, and they are transmitted continuously from point to point. Since the times of Euler and Cauchy, the internal forces are assumed to be the surface short-range contact forces: on an infinitesimal surface $\boldsymbol{n}d\mathcal{O}$ acts the force $\boldsymbol{t}_{(\boldsymbol{n})}d\mathcal{O}$. It acts from?? that side of the two where the unit normal $\boldsymbol{n}$ is directed.

By the action–reaction principle, a traction vector $\boldsymbol{t}_{(\boldsymbol{n})}$ is reversed (alters direction) together with a unit normal vector $\boldsymbol{n}$: $\boldsymbol{t}_{(-\boldsymbol{n})} = -\boldsymbol{t}_{(\boldsymbol{n})}$. Sometimes this thesis is called "the Cauchy pillbox argument" and is proved thru the balance of momentum for an infinitely short cylinder with bases $\boldsymbol{n}d\mathcal{O}$ and $-\boldsymbol{n}d\mathcal{O}$.

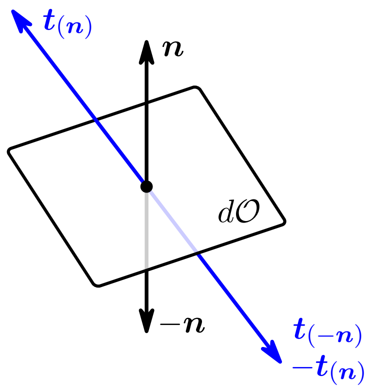

figure 1

Traction vector $\boldsymbol{t}_{(\boldsymbol{n})}$ on the surface with the unit normal $\boldsymbol{n}$ is called the surface traction vector or force-stress vector. However, $\boldsymbol{t}_{(\boldsymbol{n})}$ is not a vector field: traction $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{n}, \boldsymbol{r}, t)$ depends not only on location $\boldsymbol{r}$ of the point, but also on the local direction (defined by $\boldsymbol{n}$) of the surface element. An infinite number of surfaces of any direction contain the same point, and there are infinitely many traction vectors $\boldsymbol{t}_{(\boldsymbol{n})}$ at each point.

The stress at a point of continuum is *not a vector field*. Such a field is more complex, it is an infinite collection of all traction vectors for all infinitesimal surfaces of any direction, containing that point.

And in fact, an infinite collection of all traction vectorsat a point is completely defined by the only one single second complexity tensor — the Cauchy stress tensor $\boldsymbol{\tau}$.

The derivation of this thesis is described in many books. It is known as the theorem about existænce of the Cauchy stress tensor — the one with the impressive tetrahedron argument.

The Cauchy tetrahedron argument and the proof of the existænce of the Cauchy stress tensor.

On the surface of an infinitesimal material tetrahedron ...

......

The traction vector $\boldsymbol{t}$ and its projections, $\boldsymbol{t}_{\perp}$ and $\boldsymbol{t}_{\|}$

✓ the projection of the traction vector on the unit normal vector

$$\boldsymbol{t}_{\perp} = \boldsymbol{t}_{\boldsymbol{n}} = \boldsymbol{t} \cdot \boldsymbol{n} \tag{6.1}$$

(is perpendicular to the cross-section area),

✓ the projection of the traction vector on the surface

$$\boldsymbol{t}_{\|} = \boldsymbol{t} - \boldsymbol{t}_{\perp} \tag{6.2}$$

....

§ 7. Balance of momentum and angular momentum

Consider some random finite volume $\mathcal{V}$ of an elastic medium, contained within surface $\mathcal{O}(\partial\mathcal{V})$. It is loaded with external forces, surface contact ones $\boldsymbol{p}d\mathcal{O}$ and body (mass or volume) ones $\boldsymbol{f}dm = \rho\boldsymbol{f}d\mathcal{V}$.

The integral formulation of the balance of momentum is as follows

$$\Bigg(\int\limits_{\mathcal{V}} \rho\boldsymbol{v}d\mathcal{V}\Bigg)^{\bullet} = \int\limits_{\mathcal{V}} \rho\boldsymbol{f}d\mathcal{V} + \oint\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{p}d\mathcal{O}. \tag{7.1}$$

... $\boldsymbol{p} = \boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{\tau}$ ...

The derivative of the momentum on the left can be found as in (1.5), and the integral over the surface turns into the volume integral by the divergence theorem. This gives

$$\int\limits_{\mathcal{V}} \Bigl(\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \bigl(\boldsymbol{f} - \dot{\boldsymbol{v}} \bigr) \Bigr) d\mathcal{V} = \boldsymbol{0}.$$

But volume $\mathcal{V}$ is random, and therefore the integrand itself is also equal to the null vector — the equation of balance of momentum (forces) in local (differential) form

$$\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \bigl(\boldsymbol{f} - \dot{\boldsymbol{v}} \bigr) = \boldsymbol{0}. \tag{7.2}$$

....

Now about the balance of the angular (rotational) momentum. Here is the integral formulation :

$$\Biggl(\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \boldsymbol{v} d\mathcal{V} \Biggr)^{\bullet} = \int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \boldsymbol{f} d\mathcal{V} + \oint\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{r} \times \boldsymbol{p} d\mathcal{O}. \tag{7.3}$$

Дифференцируя левую часть ($\boldsymbol{v} \equiv \dot{\boldsymbol{r}}$)

$$\Biggl(\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \dot{\boldsymbol{r}} \, d\mathcal{V} \Biggr)^{\bullet} = \int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \ddot{\boldsymbol{r}} \, d\mathcal{V} + \int\limits_{\mathcal{V}} \underbrace{\dot{\boldsymbol{r}} \times \rho \dot{\boldsymbol{r}}}_{\boldsymbol{0}} \, d\mathcal{V},$$

применяя теорему о дивергенции к поверхностному интегралу ....

(... $\boldsymbol{p} = \boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{\tau}$ ...)

$$\boldsymbol{r} \times (\boldsymbol{n} \cdot \boldsymbol{\tau}) = -(\boldsymbol{n} \cdot \boldsymbol{\tau}) \times \boldsymbol{r} = -\boldsymbol{n} \cdot (\boldsymbol{\tau} \times \boldsymbol{r}) \Rightarrow$$
$$\Rightarrow \oint\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{r} \times (\boldsymbol{n} \cdot \boldsymbol{\tau}) \, d\mathcal{O} = -\int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \times \boldsymbol{r}) \, d\mathcal{V},$$

...

$$\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \ddot{\boldsymbol{r}} \, d\mathcal{V} = \int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \boldsymbol{f} \, d\mathcal{V} - \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \times \boldsymbol{r}) \, d\mathcal{V},$$

$$\int\limits_{\mathcal{V}} \boldsymbol{r} \times \rho \big(\boldsymbol{f} - \ddot{\boldsymbol{r}} \big) \, d\mathcal{V} - \int\limits_{\mathcal{V}} \boldsymbol{\nabla} \boldsymbol{\cdot} (\boldsymbol{\tau} \times \boldsymbol{r}) \, d\mathcal{V} = \boldsymbol{0},$$

...

$$\underbrace{\boldsymbol{\nabla} \boldsymbol{\cdot} (\boldsymbol{\tau} \times \boldsymbol{r})}_{\boldsymbol{r}^i \boldsymbol{\cdot} \partial_i (\boldsymbol{\tau} \times \boldsymbol{r})} = \underbrace{(\boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\tau}) \times \boldsymbol{r}}_{\boldsymbol{r}^i \boldsymbol{\cdot} (\partial_i \boldsymbol{\tau}) \times \boldsymbol{r}} + \boldsymbol{r}^i \boldsymbol{\cdot} (\boldsymbol{\tau} \times \partial_i \boldsymbol{r})$$

$\boldsymbol{\tau} = \boldsymbol{e}_i \boldsymbol{t}_{(i)}$, $\boldsymbol{e}_i =$ **constant**

$$\begin{aligned} \boldsymbol{r}^i \boldsymbol{\cdot} (\boldsymbol{\tau} \times \partial_i \boldsymbol{r}) = \boldsymbol{r}^i \boldsymbol{\cdot} (\boldsymbol{e}_j \boldsymbol{t}_{(j)} \times \boldsymbol{r}_i) = \boldsymbol{r}^i \boldsymbol{\cdot} \boldsymbol{e}_j \boldsymbol{t}_{(j)} \times \boldsymbol{r}_i = \\ = - \boldsymbol{e}_j \boldsymbol{\cdot} \boldsymbol{r}^i \boldsymbol{r}_i \times \boldsymbol{t}_{(j)} = - \boldsymbol{e}_j \boldsymbol{\cdot} \boldsymbol{E} \times \boldsymbol{t}_{(j)} = - \boldsymbol{e}_j \times \boldsymbol{t}_{(j)} = - \boldsymbol{\tau}_{\times} \end{aligned}$$

...

§ 8. Eigenvalues of the Cauchy stress tensor. Mohr's circles

Like any symmetric bivalent tensor, the Cauchy stress tensor $\boldsymbol{\tau}$ has три вещественных собственных числа́ σ_i, а также тройку взаимно перпендикулярных собственных векторов единичной длины (§ **??**.**??**). Собственные числа тензора $\boldsymbol{\tau}$ называются главными напряжениями (principal stresses).

In the representation $\boldsymbol{\tau} = \sum \sigma_i \boldsymbol{e}_i \boldsymbol{e}_i$, the values of σ_i are most often sorted descending, $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and the triple $\boldsymbol{e}_i$ is oriented as "right".

Известна теорема о кругах Мора (Mohr's circles)*

...

Чтобы замкнуть набор (систему) уравнений модели сплошной среды, нужно добавить определяющие отношения (constitutive relations) — уравнения, связывающие напряжение с деформацией (и другие необходимые связи). However, for a solid elastic continuum такой длинный путь построения модели излишен, что читатель и увидит ниже.

* Mohr's circles, named after Christian Otto Mohr, is a two-dimensional graphical representation of transformation for the Cauchy stress tensor.

§ 9. Principle of virtual work (without Lagrange multipliers)

According to the principle of virtual work for some finite volume of a continuous medium

$$\int\limits_{\mathcal{V}} \Bigl(\rho \boldsymbol{f} \cdot \delta \boldsymbol{r} + \delta W^{(i)} \Bigr) d\mathcal{V} + \oint\limits_{\mathcal{O}(\partial \mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\tau} \cdot \delta \boldsymbol{r} \, d\mathcal{O} = 0. \tag{9.1}$$

Here $\delta W^{(i)}$ is the work of internal forces per volume unit in the current configuration, $\boldsymbol{f}$ is the mass force (including dynamics, $\boldsymbol{f} \equiv \boldsymbol{f}_{*} - \ddot{\boldsymbol{r}}$), $\boldsymbol{p} = \boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{\tau}$ is the surface force.

Applying the divergence theorem to the surface integral, using*

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\tau} \cdot \delta \boldsymbol{r}) = \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \cdot \delta \boldsymbol{r} + \boldsymbol{\tau} \cdot\cdot \boldsymbol{\nabla} \delta \boldsymbol{r}^{\mathsf{T}}$$

and the randomness of $\mathcal{V}$, here comes the local differential version of (9.1)

$$\Bigl(\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \boldsymbol{f} \Bigr) \cdot \delta \boldsymbol{r} + \boldsymbol{\tau} \cdot\cdot \boldsymbol{\nabla} \delta \boldsymbol{r}^{\mathsf{T}} + \delta W^{(i)} = 0. \tag{9.2}$$

When a body virtually moves as a rigid whole, the work of internal forces nullifies

$$\delta \boldsymbol{r} = \delta \boldsymbol{\rho} + \delta \mathbf{o} \times \boldsymbol{r} \;\Rightarrow\; \delta W^{(i)} = 0,$$

$$\bigl(\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \boldsymbol{f} \bigr) \cdot \bigl(\delta \boldsymbol{\rho} + \delta \mathbf{o} \times \boldsymbol{r} \bigr) + \boldsymbol{\tau}^{\mathsf{T}} \cdot\cdot \boldsymbol{\nabla} \bigl(\delta \boldsymbol{\rho} + \delta \mathbf{o} \times \boldsymbol{r} \bigr) = 0,$$

$$\delta \boldsymbol{\rho} = \textsf{constant} \;\Rightarrow\; \boldsymbol{\nabla} \delta \boldsymbol{\rho} = {}^{2}\mathbf{0}, \quad \delta \mathbf{o} = \textsf{constant} \;\Rightarrow\; \boldsymbol{\nabla} \delta \mathbf{o} = {}^{2}\mathbf{0},$$

$$\begin{aligned} \boldsymbol{\nabla} \bigl(\delta \boldsymbol{\rho} + \delta \mathbf{o} \times \boldsymbol{r} \bigr) = \boldsymbol{\nabla} \bigl(\delta \mathbf{o} \times \boldsymbol{r} \bigr) = \boldsymbol{\nabla} \delta \mathbf{o} \times \boldsymbol{r} - \boldsymbol{\nabla} \boldsymbol{r} \times \delta \mathbf{o} = \\ = - \boldsymbol{\nabla} \boldsymbol{r} \times \delta \mathbf{o} = - \boldsymbol{E} \times \delta \mathbf{o} \end{aligned}$$

$$\ldots$$

Assuming $\delta \mathbf{o} = \mathbf{0}$ (just a translation) $\Rightarrow \boldsymbol{\nabla} \delta \boldsymbol{r} = \boldsymbol{\nabla} \delta \boldsymbol{\rho} = {}^{2}\mathbf{0}$, it turns into the balance of forces (of momentum)

$$\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \boldsymbol{f} = \mathbf{0}. \tag{9.3}$$

* $\boldsymbol{r}^{i} \cdot \partial_{i} (\boldsymbol{\tau} \cdot \delta \boldsymbol{r}) = \boldsymbol{r}^{i} \cdot (\partial_{i} \boldsymbol{\tau}) \cdot \delta \boldsymbol{r} + \boldsymbol{r}^{i} \cdot \boldsymbol{\tau} \cdot \partial_{i} (\delta \boldsymbol{r}),$
$\boldsymbol{r}^{i} \cdot \boldsymbol{\tau} \cdot \partial_{i} (\delta \boldsymbol{r}) = \boldsymbol{\tau} \cdot\cdot \partial_{i} (\delta \boldsymbol{r}) \boldsymbol{r}^{i} = \boldsymbol{\tau} \cdot\cdot (\boldsymbol{r}^{i} \partial_{i} \delta \boldsymbol{r})^{\mathsf{T}}$

If $\delta \boldsymbol{r} = \delta\mathbf{o} \times \boldsymbol{r}$ (just a rotation) with $\delta\mathbf{o} = \textbf{constant}$, then

$$(??, \S\, ??.??) \;\Rightarrow\; \boldsymbol{\nabla}\delta\boldsymbol{r} = \boldsymbol{\nabla}\delta\mathbf{o} \times \boldsymbol{r} - \boldsymbol{\nabla}\boldsymbol{r} \times \delta\mathbf{o} = -\boldsymbol{E} \times \delta\mathbf{o},$$
$$\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}} = \boldsymbol{E} \times \delta\mathbf{o}$$

With

$$(??, \S\, ??.??) \;\Rightarrow\; \boldsymbol{\tau}_{\times} = -\boldsymbol{\tau} \cdot\cdot\, {}^{3}\boldsymbol{\epsilon},$$
$$\boldsymbol{\tau} \cdot\cdot \left(\boldsymbol{E} \times \delta\mathbf{o}\right) = \boldsymbol{\tau} \cdot\cdot \left(-{}^{3}\boldsymbol{\epsilon} \cdot \delta\mathbf{o}\right) = \left(-\boldsymbol{\tau} \cdot\cdot\, {}^{3}\boldsymbol{\epsilon}\right) \cdot \delta\mathbf{o} = \boldsymbol{\tau}_{\times} \cdot \delta\mathbf{o}$$

....

In an elastic continuum, the internal forces are potential

$$\delta W^{(i)} = -\rho\, \delta\widetilde{\Pi} \tag{9.4}$$

....

Variational equation (9.2) with the balance (9.3) of linear momentum (of forces) and the balance of angular momentum $\boldsymbol{\tau}_{\times} = 0 \;\Leftrightarrow\; \boldsymbol{\tau}^{\mathsf{T}} = \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{S}}$ for an elastic (9.4) continuum

$$\boldsymbol{\tau} \cdot\cdot \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = -\delta W^{(i)} = \rho\, \delta\widetilde{\Pi}. \tag{9.5}$$

What the potential energy $\widetilde{\Pi}$ per mass unit looks like is yet unknown, but it's obvious that $\widetilde{\Pi}$ is determined by deformation. The potential energy per unit volume $\mathring{\Pi}$ in the undeformed configuration can be presented as

$$\mathring{\Pi} \equiv \mathring{\rho}\, \widetilde{\Pi} \;\Rightarrow\; \delta\mathring{\Pi} = \mathring{\rho}\, \delta\widetilde{\Pi}. \tag{9.6}$$

With the balance of mass $\rho \mathcal{J} = \mathring{\rho} \Leftrightarrow \rho = \mathcal{J}^{-1} \mathring{\rho}$ ($\mathcal{J} \equiv \det \boldsymbol{F}$ is the Jacobian, determinant of the motion gradient)

$$\rho\, \delta\widetilde{\Pi} = \mathcal{J}^{-1} \delta\mathring{\Pi}.$$

> "The elastic potential energy density per volume unit", becomes when shorting "The elastic potential ...."
>
> Плотность упругой потенциальной энергии, запасённой|накопленной в единице объёма тела (среды́).
>
> Дословный перевод с english на русский фразы "the elastic potential" даёт "упругий потенциал".

Полным аналогом (...) является равенство

...

§ 10. Constitutive relations of elasticity

The fundamental relation of elasticity (**??**)

...

$$\Pi(\boldsymbol{C}) = \int\limits_{0}^{\boldsymbol{C}} \boldsymbol{\tau} \cdot\cdot\, d\boldsymbol{C}$$

If the strain energy density is path independent, then it acts as a potential for stress, that is

$$\boldsymbol{\tau} = \frac{\partial \Pi(\boldsymbol{C})}{\partial \boldsymbol{C}}$$

For adiabatic processes, Π is equal to the change in internal energy per unit of volume.

For isothermal processes, Π is equal to the Helmholtz free energy per unit of volume.

The natural configuration of a body is defined as the configuration in which the body is in stable thermal equilibrium with no external loads and zero stress and strain.

When we apply energy methods in elasticity, we implicitly assume that a body returns to its natural configuration after loads are removed. This implies that the Gibbs' condition is satisfied:

$$\Pi(\boldsymbol{C}) \geq 0 \;\text{ with }\; \Pi(\boldsymbol{C}) = 0 \;\text{ iff }\; \boldsymbol{C} = 0$$

...

Начальная конфигурация считается естественной (natural configuration) — недеформированной ненапряжённой : $\boldsymbol{C} = {}^2\boldsymbol{0} \Leftrightarrow \boldsymbol{\tau} = {}^2\boldsymbol{0}$, поэтому в Π нет линейных членов.

Тензор жёсткости ${}^4\!\boldsymbol{\mathcal{A}}$

...

A rubber-like material (an elastomer)

Для материала типа резины (эластомера) характерны больши́е деформации. Функция $\Pi(\mathrm{I}, \mathrm{II}, \mathrm{III})$ для такого материала бывает весьма сложной*.

Преимущества использования $\boldsymbol{u}$ и $\boldsymbol{C}$ исчезают, если деформации больши́е (коне́чные) — проще остаться с вектором-радиусом $\boldsymbol{r}$ ...

...

* **Harold Alexander**. A constitutive relation for rubber-like materials // International Journal of Engineering Science, volume 6 (September 1968), pages 549–563.

§ 11. Piola–Kirchhoff stress tensors and other measures of stress

Соотношение Nanson'а $\boldsymbol{n} d\mathcal{O} = \mathcal{J} \mathring{\boldsymbol{n}} do \cdot \boldsymbol{F}^{-1}$ между векторами бесконечно малой площáдки в начальной ($\mathring{\boldsymbol{n}} do$) и в текущей ($\boldsymbol{n} d\mathcal{O}$) конфигурациях*

$$(5.2) \;\Rightarrow\; \boldsymbol{n} d\mathcal{O} \cdot \boldsymbol{\tau} = \mathcal{J} \mathring{\boldsymbol{n}} do \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \;\Rightarrow\; \boldsymbol{n} \cdot \boldsymbol{\tau} d\mathcal{O} = \mathring{\boldsymbol{n}} \cdot \mathcal{J} \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} do$$

gives the dual expression of a surface force

$$\boldsymbol{n} \cdot \boldsymbol{\tau} d\mathcal{O} = \mathring{\boldsymbol{n}} \cdot \boldsymbol{T} do, \;\; \boldsymbol{T} \equiv \mathcal{J} \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau}. \tag{11.1}$$

Тензор $\boldsymbol{T}$ называется первым (несимметричным) тензором напряжения Piola–Kirchhoff, иногда — “номинальным напряжением” (“nominal stress”) или “инженерным напряжением” (“engineering stress”). Бывает и когда какое-либо из этих (на)именований даётся транспонированному тензору

$$\boldsymbol{T}^{\mathsf{T}} = \mathcal{J} \boldsymbol{\tau}^{\mathsf{T}} \cdot \boldsymbol{F}^{-\mathsf{T}} = \mathcal{J} \boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathsf{T}}.$$

Обращение (11.1)

$$\mathcal{J}^{-1} \boldsymbol{F} \cdot \boldsymbol{T} = \mathcal{J}^{-1} \boldsymbol{F} \cdot \mathcal{J} \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \;\Rightarrow\; \boldsymbol{\tau} = \mathcal{J}^{-1} \boldsymbol{F} \cdot \boldsymbol{T}$$

...

$$\delta\Pi = \boldsymbol{T} \cdot\cdot\, \delta \mathring{\boldsymbol{\nabla}} \boldsymbol{r}^{\mathsf{T}} \;\Rightarrow\; \Pi = \Pi(\mathring{\boldsymbol{\nabla}} \boldsymbol{r}) \tag{11.2}$$

— этот немного неожиданный результат получился благодаря коммутативности δ и $\mathring{\boldsymbol{\nabla}}$: $\mathring{\boldsymbol{\nabla}} \delta \boldsymbol{r}^{\mathsf{T}} = \delta \mathring{\boldsymbol{\nabla}} \boldsymbol{r}^{\mathsf{T}}$ ($\boldsymbol{\nabla}$ and δ don't commute).

Тензор $\boldsymbol{T}$ оказался энергетически сопряжённым с $\boldsymbol{F} \equiv \mathring{\boldsymbol{\nabla}} \boldsymbol{r}^{\mathsf{T}}$

$$\boldsymbol{T} = \frac{\partial \Pi}{\partial \mathring{\boldsymbol{\nabla}} \boldsymbol{r}^{\mathsf{T}}} = \frac{\partial \Pi}{\partial \boldsymbol{F}}. \tag{11.3}$$

* Like before, $\boldsymbol{F} = \frac{\partial \boldsymbol{r}}{\partial \mathring{\boldsymbol{r}}} = \boldsymbol{r}_{\partial i} \mathring{\boldsymbol{r}}^i = \mathring{\boldsymbol{\nabla}} \boldsymbol{r}^{\mathsf{T}}$ is the motion gradient, $\mathcal{J} \equiv \det \boldsymbol{F}$ is the Jacobian (the Jacobian determinant).

Второй (симметричный) тензор напряжения Piola–Kirchhoff $\boldsymbol{S}$ энергетически сопряжён с $\boldsymbol{G} \equiv \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F}$ и $\boldsymbol{C} \equiv \frac{1}{2}(\boldsymbol{G} - \boldsymbol{E})$

$$\begin{gathered} \delta\Pi(\boldsymbol{C}) = \boldsymbol{S} \cdot\cdot\, \delta\boldsymbol{C} \;\Rightarrow\; \boldsymbol{S} = \frac{\partial\Pi}{\partial\boldsymbol{C}}, \\ d\boldsymbol{G} = 2 d\boldsymbol{C} \;\Rightarrow\; \delta\Pi(\boldsymbol{G}) = \tfrac{1}{2}\boldsymbol{S} \cdot\cdot\, \delta\boldsymbol{G},\;\; \boldsymbol{S} = 2\frac{\partial\Pi}{\partial\boldsymbol{G}}. \end{gathered} \tag{11.4}$$

Связь между первым и вторым тензорами

$$\boldsymbol{S} = \boldsymbol{T} \cdot \boldsymbol{F}^{-\mathsf{T}} = \boldsymbol{F}^{-1} \cdot \boldsymbol{T}^{\mathsf{T}} \;\Leftrightarrow\; \boldsymbol{T} = \boldsymbol{S} \cdot \boldsymbol{F}^{\mathsf{T}},\;\; \boldsymbol{T}^{\mathsf{T}} = \boldsymbol{F} \cdot \boldsymbol{S}$$

и между тензором $\boldsymbol{S}$ и тензором напряжения Cauchy $\boldsymbol{\tau}$

$$\boldsymbol{S} = \mathcal{J}\boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathsf{T}} \;\Leftrightarrow\; \mathcal{J}^{-1}\boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^{\mathsf{T}} = \boldsymbol{\tau}.$$

...

$$\boldsymbol{T} = \frac{\partial\Pi}{\partial\boldsymbol{C}} \cdot \boldsymbol{F}^{\mathsf{T}} = 2\frac{\partial\Pi}{\partial\boldsymbol{G}} \cdot \boldsymbol{F}^{\mathsf{T}}$$

$$\delta\boldsymbol{S} = \frac{\partial\boldsymbol{S}}{\partial\boldsymbol{C}} \cdot\cdot\, \delta\boldsymbol{C} = \frac{\partial^2\Pi}{\partial\boldsymbol{C}\partial\boldsymbol{C}} \cdot\cdot\, \delta\boldsymbol{C}$$

$$\delta\boldsymbol{T} = \delta\boldsymbol{S} \cdot \boldsymbol{F}^{\mathsf{T}} + \boldsymbol{S} \cdot \delta\boldsymbol{F}^{\mathsf{T}}$$

...

The quantity $\boldsymbol{\kappa} = \mathcal{J}\boldsymbol{\tau}$ is called the *Kirchhoff stress tensor* and is used widely in numerical algorithms in metal plasticity (where there's no change in volume during plastic deformation). Another name for it is *weighted Cauchy stress tensor*.

...

Here's balance of forces (of momentum) with tensor $\boldsymbol{T}$ for any undeformed volume $\mathring{\mathcal{V}}$

$$\int\limits_{\mathcal{V}} \rho\boldsymbol{f}\, d\mathcal{V} + \int\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{n} \cdot \boldsymbol{\tau}\, d\mathcal{O} = \int\limits_{\mathring{\mathcal{V}}} \mathring{\rho}\boldsymbol{f}\, d\mathring{\mathcal{V}} + \int\limits_{o(\partial\mathring{\mathcal{V}})} \mathring{\boldsymbol{n}} \cdot \boldsymbol{T}\, do = \int\limits_{\mathring{\mathcal{V}}} \Big(\mathring{\rho}\boldsymbol{f} + \mathring{\boldsymbol{\nabla}} \cdot \boldsymbol{T}\Big) d\mathring{\mathcal{V}} = \boldsymbol{0}$$

or in the local (differential) version

$$\mathring{\boldsymbol{\nabla}} \cdot \boldsymbol{T} + \mathring{\rho} \boldsymbol{f} = \boldsymbol{0}. \tag{11.5}$$

Advantages of this equation in comparison with (7.2) are : here figures the known mass density $\mathring{\rho}$ of an undeformed volume $\mathring{\mathcal{V}}$, and the operator $\mathring{\boldsymbol{\nabla}} \equiv \mathring{\boldsymbol{r}}^i \partial_i$ is defined through the known vectors $\mathring{\boldsymbol{r}}^i$. The appearance of $\boldsymbol{T}$ presents the specific property of an elastic solid body — "to retain" its initial configuration. Tensor $\boldsymbol{T}$ is unlikely useful in fluid mechanics.

The principle of virtual work for an arbitrary volume $\mathring{\mathcal{V}}$ of elastic ($\delta W^{(i)} = -\delta \Pi$) continuum :

$$\int\limits_{\mathring{\mathcal{V}}} \Big(\mathring{\rho} \boldsymbol{f} \cdot \delta \boldsymbol{r} - \delta \Pi \Big) d\mathring{\mathcal{V}} + \int\limits_{o(\partial \mathring{\mathcal{V}})} \mathring{\boldsymbol{n}} \cdot \boldsymbol{T} \cdot \delta \boldsymbol{r} \, do = 0,$$

$$\mathring{\boldsymbol{\nabla}} \cdot (\boldsymbol{T} \cdot \delta \boldsymbol{r}) = \mathring{\boldsymbol{\nabla}} \cdot \boldsymbol{T} \cdot \delta \boldsymbol{r} + \boldsymbol{T}^{\mathsf{T}} \cdot\cdot\, \mathring{\boldsymbol{\nabla}} \delta \boldsymbol{r}, \;\; \boldsymbol{T}^{\mathsf{T}} \cdot\cdot\, \mathring{\boldsymbol{\nabla}} \delta \boldsymbol{r} = \boldsymbol{T} \cdot\cdot\, \mathring{\boldsymbol{\nabla}} \delta \boldsymbol{r}^{\mathsf{T}}$$

$$\delta \Pi = \Big(\mathring{\rho} \boldsymbol{f} + \mathring{\boldsymbol{\nabla}} \cdot \boldsymbol{T} \Big) \cdot \delta \boldsymbol{r} + \boldsymbol{T} \cdot\cdot\, \mathring{\boldsymbol{\nabla}} \delta \boldsymbol{r}^{\mathsf{T}}$$

....

The first one is non-symmetric, it links forces in the deformed stressed configuration to the underfomed geometry and mass (volumes, areas, densities as they were initially), and it is energetically conjugate to the motion gradient (often mistakenly called the "deformation gradient", forgetting about rigid rotations). The first (or sometimes its transpose) is also known as "nominal stress" and "engineering stress".

The second one is symmetric, it links loads in the initial undeformed configuration to the initial mass and geometry, and it is conjugate to the right Cauchy–Green deformation tensor (and thus to the Cauchy–Green–Venant measure of deformation).

The first is simplier when you use just the motion gradient and is more universal, but the second is simplier when you prefer right Cauchy–Green deformation and its offsprings.

There's also popular Cauchy stress, which relates forces in the deformed configuration to the deformed geometry and mass.

"energetically conjugate" means that their product is kind of energy, here: elastic potential energy per unit of volume

......

In the case of finite deformations, the Piola–Kirchhoff tensors $\boldsymbol{T}$ and $\boldsymbol{S}$ describe the stress relative to the initial configuration. In contrast with them, the Cauchy stress tensor $\boldsymbol{\tau}$ describes the stress relative to the current configuration. For infinitesimal deformations, the Cauchy and Piola–Kirchhoff stress tensors are identical.

1st Piola–Kirchhoff stress tensor

The 1st Piola–Kirchhoff stress tensor $\boldsymbol{T}$ relates forces in the current (present, "spatial") configuration with areas in the initial ("material") configuration

$$\boldsymbol{T} = \mathcal{J}\, \boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathsf{T}}$$

where $\boldsymbol{F}$ is the motion gradient and $\mathcal{J} \equiv \det \boldsymbol{F}$ is the Jacobi determinant, Jacobian.

Because it relates different coordinate systems, the 1st Piola–Kirchhoff stress is a two-point tensor. Commonly, it's not symmetric.

The 1st Piola–Kirchhoff stress is the 3D generalization of the 1D concept of engineering stress.

If the material rotates without a change in stress (rigid rotation), the components of the 1st Piola–Kirchhoff stress tensor will vary with material orientation.

The 1st Piola–Kirchhoff stress is energy conjugate to the motion gradient.

2nd Piola–Kirchhoff stress tensor

The 2nd Piola–Kirchhoff stress tensor $\boldsymbol{S}$ relates forces in the initial configuration to areas in the initial configuration. The force in the initial configuration is obtained via mapping that preserves the relative relationship between the force direction and the area normal in the initial configuration.

$$\boldsymbol{S} = \mathcal{J}\, \boldsymbol{F}^{-1} \boldsymbol{\cdot} \boldsymbol{\tau} \boldsymbol{\cdot} \boldsymbol{F}^{-\mathsf{T}}$$

This tensor is a one-point tensor and it is symmetric.

If the material rotates without a change in stress (rigid rotation), the 2nd Piola–Kirchhoff stress tensor remain constant, irrespective of material orientation.

The 2nd Piola–Kirchhoff stress tensor is energy conjugate to the Green–Lagrange finite strain tensor.

....

§ 12. Variation of the present configuration

Usually the two configurations of a nonlinear elastic medium are considered : the initial one with location vectors $\mathring{\boldsymbol{r}}$ and the present (current) one with $\boldsymbol{r}$.

The following equations describe a small change of the current configuration with infinitesimal changes to the location vector $\delta\boldsymbol{r}$, to the vector of mass forces $\delta\boldsymbol{f}$, to the first Piola–Kirchhoff stress tensor $\delta\boldsymbol{T}$ and to the Green strain tensor $\delta\boldsymbol{C}$.

Varying (11.5), (....) and (3.3)* gives

$$\begin{gathered}
\mathring{\boldsymbol{\nabla}} \cdot \delta\boldsymbol{T} + \mathring{\rho}\,\delta\boldsymbol{f} = \boldsymbol{0}, \\
\delta\boldsymbol{T} = \left(\frac{\partial^2 \Pi}{\partial\boldsymbol{C}\partial\boldsymbol{C}} \cdot\cdot\, \delta\boldsymbol{C} \right) \cdot \boldsymbol{F}^{\mathsf{T}} + \frac{\partial\Pi}{\partial\boldsymbol{C}} \cdot \delta\boldsymbol{F}^{\mathsf{T}}, \\
\delta\boldsymbol{F}^{\mathsf{T}} = \delta\mathring{\boldsymbol{\nabla}}\boldsymbol{r} = \mathring{\boldsymbol{\nabla}}\delta\boldsymbol{r} = \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{\nabla}\delta\boldsymbol{r}, \;\; \delta\boldsymbol{F} = \delta\mathring{\boldsymbol{\nabla}}\boldsymbol{r}^{\mathsf{T}} = \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}} \cdot \boldsymbol{F}, \\
\delta\boldsymbol{C} = \tfrac{1}{2}\,\delta\big(\boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F}\big) = \tfrac{1}{2}\Big(\big(\delta\boldsymbol{F}^{\mathsf{T}}\big) \cdot \boldsymbol{F} + \boldsymbol{F}^{\mathsf{T}} \cdot \big(\delta\boldsymbol{F}\big)\Big), \\
\delta\boldsymbol{C} = \boldsymbol{F}^{\mathsf{T}} \cdot \delta\boldsymbol{\varepsilon} \cdot \boldsymbol{F}, \;\; \delta\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}}.
\end{gathered} \tag{12.1}$$

.....

$$\begin{gathered}
(5.2) \;\; \Rightarrow \;\; \mathring{\boldsymbol{n}} do = \mathcal{J}^{-1} \boldsymbol{n} d\mathcal{O} \cdot \boldsymbol{F} \;\; \Rightarrow \;\; \mathring{\boldsymbol{n}} \cdot \delta\boldsymbol{T} do = \mathcal{J}^{-1} \boldsymbol{n} \cdot \boldsymbol{F} \cdot \delta\boldsymbol{T} d\mathcal{O} \\
\text{or} \;\; \mathring{\boldsymbol{n}} \cdot \delta\boldsymbol{T} do = \boldsymbol{n} \cdot \delta\boldsymbol{\tau} d\mathcal{O}, \;\; \delta\boldsymbol{\tau} \equiv \mathcal{J}^{-1} \boldsymbol{F} \cdot \delta\boldsymbol{T}
\end{gathered}$$

— tensor $\delta\boldsymbol{\tau}$ introduced here is related to variation $\delta\boldsymbol{T}$ just alike $\boldsymbol{\tau}$ is related to $\boldsymbol{T}$ ($\boldsymbol{\tau} = \mathcal{J}^{-1} \boldsymbol{F} \cdot \boldsymbol{T}$). From (12.1) and ...

.....

... and adjusting the coefficients of the linear function $\delta\boldsymbol{\tau}(\delta\boldsymbol{\varepsilon})$ (...)

§ 13. Internal constraints

До сих пор деформация считалась свободной, мера деформации $\boldsymbol{C}$ могла быть любой. Однако, существуют материалы со значи-

* $\boldsymbol{\nabla} = \boldsymbol{\nabla} \cdot \mathring{\boldsymbol{\nabla}}\mathring{\boldsymbol{r}} = \boldsymbol{r}^i \partial_i \cdot \mathring{\boldsymbol{r}}^j \partial_j \mathring{\boldsymbol{r}} \overset{?}{=} \boldsymbol{r}^i \partial_i \mathring{\boldsymbol{r}} \cdot \mathring{\boldsymbol{r}}^j \partial_j = \boldsymbol{\nabla}\mathring{\boldsymbol{r}} \cdot \mathring{\boldsymbol{\nabla}} = \boldsymbol{F}^{-\mathsf{T}} \cdot \mathring{\boldsymbol{\nabla}}$

$\mathring{\boldsymbol{\nabla}} = \mathring{\boldsymbol{\nabla}} \cdot \boldsymbol{\nabla}\boldsymbol{r} = \mathring{\boldsymbol{r}}^i \partial_i \cdot \boldsymbol{r}^j \partial_j \boldsymbol{r} \overset{?}{=} \mathring{\boldsymbol{r}}^i \partial_i \boldsymbol{r} \cdot \boldsymbol{r}^j \partial_j = \mathring{\boldsymbol{\nabla}}\boldsymbol{r} \cdot \boldsymbol{\nabla} = \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{\nabla}$

тельным сопротивлением некоторым видам деформации. Резина, например, изменению формы сопротивляется намного меньше, чем изменению объёма — некоторые виды резины можно считать несжимаемым материалом.

Понятие геометрической связи, развитое в общей механике ...

...

for incompressible materials $\Pi = \Pi(\mathrm{I}, \mathrm{II})$

Mooney–Rivlin model of incompressible material

$$\Pi = c_1 \bigl(\mathrm{I} - 3\bigr) + c_2 \bigl(\mathrm{II} - 3\bigr)$$

incompressible Treloar (neo-Hookean) material

$$c_2 = 0 \;\;\Rightarrow\;\; \Pi = c_1 \bigl(\mathrm{I} - 3\bigr)$$

...

§ 14. Hollow sphere under pressure

Решение этой относительно простой задачи описано во многих книгах. В начальной (ненапряжённой) конфигурации имеем сферу с внутренним радиусом r_0 и наружным r_1. Давление равно p_0 внутри и p_1 снаружи.

Введём удобную для этой задачи сферическую систему координат в отсчётной конфигурации $q^1 = \theta$, $q^2 = \phi$, $q^3 = r$ (рисунок **??**). Эти же координаты будут и материальными. Имеем

...

§ 15. Stresses as Lagrange multipliers

The application of the principle of virtual work, described in § 9, was preceded by the introduction of the Cauchy stress tensor through the balance of forces for an infinitesimal tetrahedron (§ 6). But now the reader will see that this principle may be as well applied without any tetrahedrons.

Considering a continuum/body — not only elastic, with any virtual work of internal forces $\delta W^{(i)}$ (per unit mass) — loaded with external forces, mass ones $\boldsymbol{f}\,dm = \boldsymbol{f}\rho\,d\mathcal{V}$ (for brevity just $\boldsymbol{f}$, meaning $\boldsymbol{f} \equiv \boldsymbol{f}_{*} - \ddot{\boldsymbol{r}}$ in dynamics) and surface ones $\boldsymbol{p}\,d\mathcal{O}$. Then the variational equation of the principle of virtual work is

$$\int\limits_{\mathcal{V}} \rho\Big(\boldsymbol{f} \cdot \delta\boldsymbol{r} + \delta W^{(i)}\Big)\,d\mathcal{V} + \int\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{p} \cdot \delta\boldsymbol{r}\,d\mathcal{O} = 0. \tag{15.1}$$

Further, it's assumed that internal forces ("stresses") do not produce work when a continuum/body virtually moves (with $\delta\boldsymbol{r}$) as a whole without deformations (when $\delta\boldsymbol{\varepsilon} \equiv \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = {}^{2}\boldsymbol{0}$), that is

$$\boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = {}^{2}\boldsymbol{0} \;\;\Rightarrow\;\; \delta W^{(i)} = 0. \tag{15.2}$$

(15.1) with condition (15.2) and without $\delta W^{(i)}$ becomes a variational equation with constraint.

The method of Lagrange multipliers makes $\delta\boldsymbol{r}$ random (independent) variations. Since at each point the constraint appears as a symmetric bivalent tensor, the Lagrange multiplier ${}^{2}\boldsymbol{\lambda}$ will likewise be such a tensor, bivalent and symmetric. The equation with this multiplier looks like

$$\int\limits_{\mathcal{V}} \Big(\rho\boldsymbol{f} \cdot \delta\boldsymbol{r} - {}^{2}\boldsymbol{\lambda} \cdot\cdot\, \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}}\Big)\,d\mathcal{V} + \int\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{p} \cdot \delta\boldsymbol{r}\,d\mathcal{O} = 0. \tag{15.3}$$

The symmetry of ${}^{2}\boldsymbol{\lambda}$ gives*

$${}^{2}\boldsymbol{\lambda} = {}^{2}\boldsymbol{\lambda}^{\mathsf{T}} \;\;\Rightarrow\;\; {}^{2}\boldsymbol{\lambda} \cdot\cdot\, \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = {}^{2}\boldsymbol{\lambda} \cdot\cdot\, \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{T}},$$

* $\boldsymbol{\Lambda}^{\mathsf{S}} \cdot\cdot\, \boldsymbol{X} = \boldsymbol{\Lambda}^{\mathsf{S}} \cdot\cdot\, \boldsymbol{X}^{\mathsf{T}} = \boldsymbol{\Lambda}^{\mathsf{S}} \cdot\cdot\, \boldsymbol{X}^{\mathsf{S}}$, $\quad \boldsymbol{\nabla} \cdot (\boldsymbol{B} \cdot \boldsymbol{a}) = (\boldsymbol{\nabla} \cdot \boldsymbol{B}) \cdot \boldsymbol{a} + \boldsymbol{B}^{\mathsf{T}} \cdot\cdot\, \boldsymbol{\nabla}\boldsymbol{a}$

$$ {}^2\boldsymbol{\lambda} \cdot\cdot \boldsymbol{\nabla}\delta\boldsymbol{r}^{\mathsf{S}} = \boldsymbol{\nabla}\cdot\big({}^2\boldsymbol{\lambda}\cdot\delta\boldsymbol{r}\big) - \boldsymbol{\nabla}\cdot{}^2\boldsymbol{\lambda}\cdot\delta\boldsymbol{r}. $$

Substituting this into (15.3) and applying the divergence theorem*, and the variational equation with multiplier ${}^2\boldsymbol{\lambda}$ becomes

$$ \int\limits_{\mathcal{V}} \Big(\rho\boldsymbol{f} + \boldsymbol{\nabla}\cdot{}^2\boldsymbol{\lambda}\Big)\cdot\delta\boldsymbol{r}\,d\mathcal{V} + \int\limits_{\mathcal{O}(\partial\mathcal{V})} \Big(\boldsymbol{p} - \boldsymbol{n}\cdot{}^2\boldsymbol{\lambda}\Big)\cdot\delta\boldsymbol{r}\,d\mathcal{O} = 0. \tag{15.4} $$

But $\delta\boldsymbol{r}$ is random both on a surface and in a volume, thus

$$ \boldsymbol{p} = \boldsymbol{n}\cdot{}^2\boldsymbol{\lambda}, \quad \boldsymbol{\nabla}\cdot{}^2\boldsymbol{\lambda} + \rho\boldsymbol{f} = \boldsymbol{0} \tag{15.5} $$

— the symmetric multiplier ${}^2\boldsymbol{\lambda}$, introduced formally, is in fact precisely the Cauchy stress tensor !

A similar introduction of stresses was presented in the book [47]. Here are no new results, but the very possibility of simultaneously deriving those equations of continuum mechanics, that were previously considered independent, is quite interesting. In subsequent chapters this technique is used for building new continuum models.

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* $\displaystyle\int\limits_{\mathcal{V}} \boldsymbol{\nabla}\cdot\big({}^2\boldsymbol{\lambda}\cdot\delta\boldsymbol{r}\big)d\mathcal{V} = \int\limits_{\mathcal{O}(\partial\mathcal{V})} \boldsymbol{n}\cdot\big({}^2\boldsymbol{\lambda}\cdot\delta\boldsymbol{r}\big)d\mathcal{O}, \quad \boldsymbol{n}\cdot\big({}^2\boldsymbol{\lambda}\cdot\delta\boldsymbol{r}\big) = \big(\boldsymbol{n}\cdot{}^2\boldsymbol{\lambda}\big)\cdot\delta\boldsymbol{r} = \boldsymbol{n}\cdot{}^2\boldsymbol{\lambda}\cdot\delta\boldsymbol{r}$

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