Shri Shankaracharya Institute of Professional Management & Technology, Raipur

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Roll No.:

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Enrollment No.:

B	J	4	5	9	9

Course: B.Tech **Semester:** 2nd

Branch: COMPUTER SCIENCE AND ENGINEERING

Subject Name: Mathematics-2

Subject Code:

A	0	0	0	2	1	2	(	0	1	4	)

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Q1>

Sol^n → we have, $u \to$ Real part

$\Rightarrow u = \log\sqrt{x^2+y^2}$

from C-R Eq^n,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \times \frac{2x}{2\sqrt{x^2+y^2}}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}}$$

$$\phi_1(z,0) = \frac{z}{z^2} = \frac{1}{z}$$

$$\Rightarrow \cancel{\frac{\partial v}{\partial y}} \quad \frac{\partial u}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} \times \frac{2y}{2\sqrt{x^2+y^2}}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}}$$

$$\phi_2(z,0) = 0$$

$$f(z) = \int_C (\phi_1(z,0) + i\phi_2(z,0))\,dz$$

$$= \int_C \left(\frac{1}{z} + i(0)\right) dz$$

$$= \int_C \left(\frac{1}{z}\right) dz$$

$$\boxed{f(z) = \log z + c}$$

$$\boxed{f(z) = \log(u+iv) + c} \quad \text{Ans}$$

Q2>

Sol^n → Let $z = i^{i^{i\cdots\infty}} = A + iB$

Taking log on both sides,

$\therefore \log z = i^{i^{\cdots\infty}} \log i = z \log i = \log (A+iB)$

$$\therefore z = \frac{\log z}{\log i}$$

$$\therefore (A+iB) = \frac{\log(A+iB)}{\log i}$$

$$\therefore A+iB = \frac{\frac{1}{2}\log(A^2+B^2) + i\tan^{-1}\frac{B}{A}}{i\frac{\pi}{2}}$$

$$= \frac{\frac{i}{2}\log(A^2+B^2) + i^2\tan^{-1}\frac{B}{A}}{i^2\frac{\pi}{2}}$$

$$= \frac{\tan^{-1}\frac{B}{A}}{\frac{\pi}{2}} - i\frac{1}{\pi}\log(A^2+B^2)$$

$\because \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

comparing real and imaginary parts,

$$\therefore A = \frac{\tan^{-1}\frac{B}{A}}{\pi/2}$$

$$\therefore \frac{\pi A}{2} = \tan^{-1}\frac{B}{A}$$

$$\therefore \tan\left(\frac{\pi A}{2}\right) = \frac{B}{2}$$

Hence, proved.

Also, $B = -i\frac{1}{\pi}\log(A^2+B^2)$

$\therefore \log(A^2+B^2) = -\pi B$

$A^2+B^2 = e^{-\pi B}$

Hence, proved.

Q3>

Sol^n → To prove : $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Proof: Let $f(z) = u+iv$

$\overline{f(z)} = u-iv$

$\therefore f(z)\,\overline{f(z)} = (u+iv)(u-iv) = u^2+v^2$

$\therefore |f(z)|^2 = u^2+v^2$

Now,

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2)+\frac{\partial^2}{\partial x^2}(v^2)+\frac{\partial^2}{\partial y^2}(u^2)+\frac{\partial^2}{\partial y^2}(v^2) \quad \text{— (1)}$$

Now consider,

$\frac{\partial}{\partial x}(u^2) = 2uu_x$

$\therefore \frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}(2uu_x) = 2uu_{xx}+2u_x^2$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2uu_{yy}+2u_y^2$

$$\therefore \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = 2u(u_{xx}+u_{yy}) + 2(u_x^2 + u_y^2)$$

$$= 2\left[u(0) + u_x^2 + u_y^2\right] \quad \left[\begin{array}{l} \because f(z) \text{ is analytic} \\ \& \ u \text{ is harmonic} \end{array}\right]$$

$$= 2\left[u_x^2 + (-v_x)^2\right] \quad \left[\begin{array}{l} \because f(z) \text{ is analytic} \\ \Rightarrow \text{C-R eq}^n \text{ satisfied} \end{array}\right]$$

$$= 2\left[u_x^2 + u_y^2\right]$$

$$= 2\,|f'(z)|^2 \quad \left(\begin{array}{l} \because f'(z) = u_x + i v_x \\ \Rightarrow |f'(z)| = \sqrt{u_x^2 + v_x^2} \end{array}\right)$$

Similarly, $\dfrac{\partial^2 v^2}{\partial x^2} + \dfrac{\partial^2 v^2}{\partial y^2} = 2|f'(z)|^2$

from (1)

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\,|f'(z)|^2 + 2|f'(z)|^2$$

$$= 4\,|f'(z)|^2$$

Hence, proved.

Q1>

Sol → Here, $f(x) = \frac{\sin^2 z}{(z-\pi/6)^3}$ is a analytic inside the circle c.

$|z| = 1$ and the point $a = \frac{\pi}{6}$

($\approx$ 0.5 approx) lies ~~between~~ within the circle 'c'.

∴ By cauchy's Integral formula,

$$= \int_c \frac{f(x)}{z-a} dz$$

$$= 2\pi i \times f(a)$$

$$= \int_c \frac{\sin^2 z}{\left(z - \frac{\pi}{6}\right)^3} dz$$

$$= 2\pi i \times \frac{1}{2!} \left[\frac{d^2}{dz^2} (\sin^2 z) \right]_{z = \pi/6}$$

$$= \pi i \left[2 \sin z \cos z \frac{d}{dz} \right]$$

$$= \pi i \left[(2 \cos 2z)_{z=\pi/6} \right]$$

$$= 2\pi i \cos \frac{\pi}{3}$$

$$= 2\pi i \cos \frac{\pi}{3}$$

$$= 2\pi i \times \frac{1}{2} = \pi i$$

Q3>

Sol^n→ $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$

$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$ is analytic

within the circle $|z| = 3$ excepting the pole $z=1$ and $z=2$,

Since, $z=1$ is a pole of order 2.

$\therefore$ Res $f(1) = \frac{1}{1!} \left[\frac{d}{dz} (z-1)^2 + (z)_{z=1} \right]$

$= \left[\frac{d}{dz} \left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-2} \right) \right]_{z=1}$

$= \left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2} \right]_{z=1}$

$= (-1)(-2\pi) - (-1)$

$= 2\pi + 1$

Also Res $(z) = \lim_{z \to 2} [(z-2) f(z)]$

$= \lim_{z \to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2}$

$= 1$

Hence by residue theorem,

$\int_C f(z) dz = 2\pi i [\text{Res } f(1) + \text{Res } f(2)]$

$= 2\pi i [2\pi + 1 + 1]$

$= 4\pi i(\pi + 1)$

Q4>

Sol^n →

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$$

Putting $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$ $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$\cos 3\theta = \frac{1}{2}\left(e^{3i\theta} + e^{-3i\theta}\right)$$

$$= \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right)$$

∴ The given integral

$$I = \int_C \frac{\frac{1}{2}\left(z^3+\frac{1}{z^3}\right)}{5-2\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$$

$$I = -\frac{1}{2i}\int_C \frac{z^6+1}{z^3(2z^2-5z+2)}\, dz$$

$$= -\frac{1}{2i}\int_C \frac{z^6+1}{z^3(2z-1)(z-2)}\, dz$$

$$= -\frac{1}{2i}\int_C f(z)\, dz$$

where c is unit circle $|z|=1$

Now $f(z)$ has a pole of order 3 at $z=0$ and simple poles at $z=\frac{1}{2}$ and $z=2$ of there only $z=0$ and $z=\frac{1}{2}$ lie within the unit ~~scale~~ circle.

$$\therefore \operatorname{Res} f(1/2) = \lim_{z \to \frac{1}{2}} \frac{\left(z-\frac{1}{2}\right)(z^6+1)}{z^3(2z-1)(z-2)}$$

$$= \lim_{z \to \frac{1}{2}} \left[\frac{z^6+1}{2z^3(z-2)}\right]$$

$$= -\frac{65}{24}$$

$$\operatorname{Res} f(0) = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\left\{(z-0)^n f(z)\right\}\right]_{z=0}$$

$\therefore$ when, $n=3$

$$= \frac{1}{2}\left[\frac{d^2}{dz^2}\left(\frac{z^6+1}{2z^2-5z+2}\right)\right]_{z=0}$$

$$= \frac{21}{8}$$

Hence,

$$I = \frac{-1}{2i}\left\{2\pi i\left[\operatorname{Res} f\left(\tfrac{1}{2}\right) + \operatorname{Res} f(0)\right]\right\}$$

$$= -\pi\left[-\frac{65}{24} + \frac{21}{8}\right]$$

$$= \frac{\pi}{12}$$