TT-Geometry Notes
Fall 2024
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WARNING FOR THE READER: These notes are in progress and they are evolving as my understanding of the subject evolves. In fact, so

This is aimed at those who have some experience with homological algebra, (some kind of) homotopy theory, and hopefully a little representation theory. I will try not to assume too much familiarity with triangulated categories.

TO-DO:

- Add section on Idempotents and The Main Classification theorem

QUESTIONS:

- Ask Julia about the Brown Representability theorem and conditions that guarantee an intenal Hom.

Contents

1 Introduction

[ADD: Write a brief introduction.]

1.1 Definitions

Definition 1.1 (Monoidal Category). a **monoidal category** is a category $\mathcal{C}$ equipped with

1. A functor
$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$
called the **tensor product**.

2. an object $1 \in \mathcal{C}$ called the **unit object** or **tensor unit**,

3. a natural isomorphism
$$\alpha : ((-) \otimes (-)) \otimes (-) \xrightarrow{\sim} (-) \otimes ((-) \otimes (-))$$
of the form
$$\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z)$$
called the **associator**,

4. natural isomorphisms
$$\lambda : (1 \otimes (-)) \xrightarrow{\sim} (-) \quad \text{and} \quad \rho : (-) \otimes 1 \xrightarrow{\sim} (-)$$
of the form
$$\lambda_x : 1 \otimes x \to x \quad \text{and} \quad \rho_x : x \otimes 1 \to x$$
such that the following diagrams commute

$$
\begin{array}{ccc}
(x \otimes 1) \otimes y & \xrightarrow{\alpha_{x,1,y}} & x \otimes (1 \otimes y) \\
& \searrow{\scriptstyle \rho_x \otimes \mathrm{id}_y} \quad \swarrow{\scriptstyle \mathrm{id}_x \otimes \lambda_y} & \\
& x \otimes y &
\end{array}
$$

$$(w \otimes x) \otimes (y \otimes z)$$

$$((w \otimes x) \otimes y) \otimes z \xrightarrow{\alpha_{w \otimes x, y, z}} \qquad \xrightarrow{\alpha_{w, x, y \otimes z}} w \otimes (x \otimes (y \otimes z))$$

$$\Big\downarrow a_{w,x,y} \otimes \mathrm{id}_z \qquad\qquad \mathrm{id}_w \otimes \alpha_{x,y,z} \Big\uparrow$$

$$(w \otimes (x \otimes z)) \otimes z \xrightarrow{\qquad \alpha_{w, x \otimes y, z} \qquad} w((x \otimes y) \otimes z)$$

We say that $\mathcal{C}$ is a **symmetric monoidal category** or a **tensor category** if it is a braided monoidal category for which the braiding $B_{x,y} : x \otimes y \to y \otimes x$ has the property $B_{y,x} \circ B_{x,y} = \mathrm{id}_{x \otimes y}$ for all objects x, y in $\mathcal{C}$.[1]

[ADD EXAMPLES]

Definition 1.2. A **triangulated category** is a triple $(\mathcal{T}, \Sigma, \Delta)$ where $\mathcal{T}$ is an additive category equipped with an auto-equivalence

$$\Sigma : \mathcal{T} \to \mathcal{T}$$

called the *shift* or *suspension* and a class Δ of **distinguished** or **exact** triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

sometimes rendered as

$$X \xrightarrow{\quad f \quad} Y$$
$$h \nwarrow \qquad \swarrow g$$
$$Z$$

sometimes written more concisely as $(X, Y, Z; f, g, h)$, satisfying the following axioms:

TC1. (Bookkeeping)

    a. $X \xrightarrow{\mathrm{id}} X \to 0 \to \Sigma X$ is distinguished for each object $X \in \mathcal{T}$.

    b. Distinguished triangles are closed under isomorphisms.

    c. $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is distinguished if and only if $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$ is distinguished. This is called *rotation*.

TC2. (Cones) For each morphism $X \xrightarrow{f} Y$ there is an object Z (unique up to non-unique isomorphism) sometimes written $\mathrm{cone}(f)$ which fits into a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

---

[1]Not covered here are the finer details of what a braided monoidal category are.

TC3. (Extension) Given two distinguished triangles $(X, Y, Z; f, g, h)$ and $(X', Y', Z'; f', g', h')$ and arrows $X \xrightarrow{\alpha} X'$ and $Y' \xrightarrow{\beta} Y'$ such that $\beta \circ f = f' \circ \alpha$, then there exists $Z \xrightarrow{\gamma} Z'$ making the diagram below commute
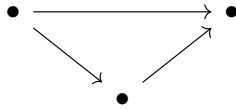
$$
\begin{array}{ccccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ g\ } & Z & \xrightarrow{\ h\ } & \Sigma X \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \exists \gamma} & & \downarrow{\scriptstyle \Sigma \alpha} \\
X' & \xrightarrow{\ f'\ } & Y' & \xrightarrow{\ g'\ } & Z' & \xrightarrow{\ h'\ } & \Sigma X'
\end{array}
$$

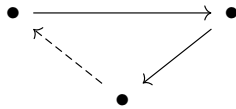Such a commutative diagram is called a **morphism of triangles**.

TC4. (Octohedral) Given a composition $X \xrightarrow{f} Y \xrightarrow{h} Z$ there exists dotted arrows which form an octohedral diagram,
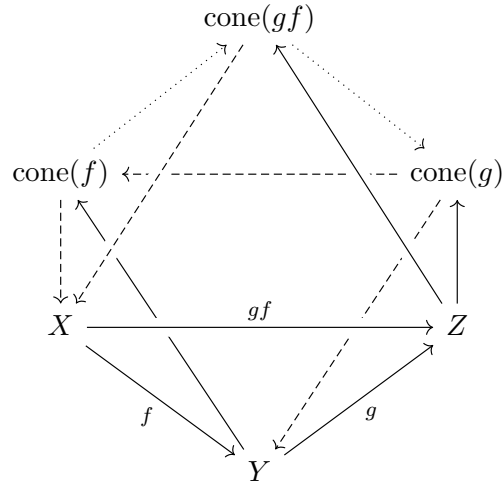
where each face is either a simplex commuting up to suspension,

or a distinguished triangle

For other more precise presentations of the octohedral axiom, refer to Appendix A.

[ADD EXAMPLES]

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Definition 1.3 (Tensor Triangulated Category). A **tensor triangulated category** (or tt-category) is a category $\mathcal{T}$ equipped with a tensor structure $(\mathcal{T}, \otimes, 1)$ and a triangulated structure $(\mathcal{T}, \Sigma, \Delta)$ and a natural isomorphism

$$e : \Sigma(-) \otimes Y \xrightarrow{\sim} \Sigma(- \otimes -)$$

with components

$$e_{X,Y} : \Sigma X \otimes Y \xrightarrow{\sim} \Sigma(X \otimes Y)$$

such that

1. $\otimes$ is additive and exact in both arguments (takes distinguished triangles to distinguished triangles),

2. e satisfies the following commutative coherence pentagon for all objects X, Y, Z in $\mathcal{T}$.

3. the topologist's sign rule is satisfied:

[ADD EXAMPLES]

Definition 1.4. Let $\mathcal{T}$ and $\mathcal{T}'$ be tt-categories. A functor $F : \mathcal{T} \to \mathcal{T}'$ is called an **exact functor** if it is equipped with functorial isomorphisms $\xi : F(\Sigma X) \to \Sigma F(X)$ such that an exact triangle $(X, Y, Z; f, g, h)$ in $\mathcal{T}$ is taken to an exact triangle $(F(X), F(Y), F(Z); F(f), F(g), \xi_X F(h))$ in $\mathcal{T}'$.

Furthermore, F is called $\otimes$-**exact** if $F(1_{\mathcal{T}}) = 1_{\mathcal{T}'}$ and if $F(X \otimes Y) \cong F(X) \otimes F(Y)$.

Definition 1.5. Given a triangulated category $(\mathcal{T}, \Sigma, \Delta)$, a **triangulated subcategory** is a pair $(\mathcal{C}, \Delta')$ such that

1. $\mathcal{C}$ is an full additive subcategory of $\mathcal{T}$ which is preserved under Σ such that $\Sigma : \mathcal{C} \to \mathcal{C}$ is an autoequivalence,

2. $\Delta' = \Delta \cap \mathcal{C}$, and

3. $(\mathcal{C}, \Sigma, \Delta')$ is a triangulated category.

Remark 1.6. Note that the above definition immediately implies that if $X \to Y \to Z \to \Sigma X$ is a distinguished triangle in $\mathcal{T}$ and any two of X, Y, Z are in $\mathcal{C}$ then the third is in $\mathcal{C}$.

We are now ready for our first proposition.

Proposition 1.7. *For objects $a, b, c \in \mathcal{T}$ the distributive property holds:*

$$(a \oplus b) \otimes c \cong (a \otimes c) \oplus (b \otimes c)$$

Proof. By corollary 3.9 $a \xrightarrow{i_a} a \oplus b \xrightarrow{p_b} b \xrightarrow{0} \Sigma(a)$ is a distinguished triangle. By applying exactness of the $- \otimes c$ functor,

$$a \otimes c \xrightarrow{i_a \otimes id_c} (a \oplus b) \otimes c \xrightarrow{p_b \otimes \mathrm{id}_c} b \otimes c \xrightarrow{0} \Sigma(a) \otimes c \tag{1}$$

is distinguished, as is

$$a \otimes c \xrightarrow{i_{a \otimes c}} (a \otimes c) \oplus (b \otimes c) \xrightarrow{p_{b \otimes c}} b \otimes c \xrightarrow{0} \Sigma(a \otimes c) \tag{2}$$

by another application of corollary 3.9. By applying TC1(rotation), TC2, and TC3 we can get a morphism of distinguished triangles between the triangles on lines (1) and (2) above that is the identity on the first and third components and an isomorphism on the fourth as $\Sigma(a \otimes c) \cong \Sigma(a) \otimes c$. By proposition 3.5 $(a \oplus b) \otimes c \cong (a \otimes c) \oplus (b \otimes c)$. $\square$

Remark 1.8. Tensor Triangular geometry is largely inspired by the rich geometry found in the realm of ring spectra, though I would argue that it is not immediately obvious from the definitions above that one should be able to develop such a theory on tt-categories. It's also worth noting that we could have proved this proposition if our category had an internal-hom structure (more on this later), but this proof makes it clear that distributivity of the tensor product over direct sums directly comes from the triangular structure communicating well with the tensor structure, which is a hint that our tt-category might secretly behave somewhat like a ring. With this in mind, we shall seek out other parallels between tt-categories and commutative rings.

Definition 1.9. If $\mathcal{T}$ is a triangulated category then we call $\mathcal{C} \subseteq T$ a **thick subcategory** if it is a triangulated subcategory of $\mathcal{T}$ and $\mathcal{C}$ is closed under sums, meaning that $a, b \in \mathcal{C}$ if and only if $a \oplus b \in \mathcal{C}$.

If $\mathcal{C}$ is as thick subcategory of $\mathcal{T}$ that is closed under $\otimes$, i.e. $a \in \mathcal{C}$ and $b \in \mathcal{T}$ implies that $a \times b \in \mathcal{C}$, then we say that it is a **thick $\otimes$-ideal**. Given an object $a \in \mathcal{T}$, we write $\langle a \rangle^{\otimes}$ to denote the $\otimes$-ideal generated by a. Equivalently, this is the smallest $\otimes$-ideal containing a.

It is important to keep in mind that to describe a full subcategory it is sufficient to merely describe all the objects within the subcategory, which we will do regularly.

Remark 1.10. Though not immediately obvious from the definition, it's worth noting that if $\mathcal{I}$ is a triangulated subcategory of $\mathcal{T}$ then $a, b \in \mathcal{C}$ implies that $a \otimes b \in \mathcal{C}$, so $\mathcal{C}$ being thick only adds the requirement that direct sums of objects in $\mathcal{T}$ decompose within $\mathcal{C}$.

Additionally, note that we are not distinguishing between left, right, or two-sided ideals. This is because our monoidal structure is symmetric, though it should be said that there is no reason that we couldn't consider a non-commutative analogue to the definition above in the scenario that we relax the condition that $\otimes$ be symmetric.

2 The Spectrum of $\mathcal{T}$

In the following section we will assume that $\mathcal{T}$ is an essentially small tt-category.

2.1 Definition of Spc$\mathcal{T}$

Definition 2.1. A thick $\otimes$-ideal $\mathcal{P} \subseteq \mathcal{T}$ is called **prime** if for any $a, b \in \mathrm{Obj}(\mathcal{T})$ such that $a \otimes b \in \mathcal{P}$ then either $a \in \mathcal{P}$ or $b \in \mathcal{P}$. We define $\mathrm{Spc}\,\mathcal{T}$ to be the collection of all prime $\otimes$-ideals in $\mathcal{T}$. A **maximal** $\otimes$-ideal is a proper $\otimes$-ideal of $\mathcal{T}$ that is maximal with respect to inclusion.

Later on we will define $\mathrm{Spec}\,\mathcal{T}$ to be the set $\mathrm{Spc}\,\mathcal{T}$ equipped with a particular ringed space structure, so we will hold the notation $\mathrm{Spec}\,\mathcal{T}$ in reserve until then. To put a ringed space structure on $\mathrm{Spc}\,\mathcal{T}$, we first need to put a topology on $\mathcal{T}$.

Definition 2.2. For any $\mathcal{S} \subseteq \mathrm{Obj}(\mathcal{T})$, we write $\mathcal{Z}(S) := \{\mathcal{P} \in \mathrm{Spc}\,\mathcal{T} \mid \mathcal{S} \cap \mathcal{P} = \emptyset\}$.

Proposition 2.3. *For any tt-category $\mathcal{T}$ the following hold:*

1. *$\bigcap_{i \in I} \mathcal{Z}(\mathcal{S}_i) = \mathcal{Z}(\bigcup_{i \in I} \mathcal{S}_i)$ for a family $\{\mathcal{S}_i\}_{i \in I}$ where $\mathcal{S}_i \subseteq \mathrm{Obj}(\mathcal{T})$ for each $i \in I$.*

2. *$\mathcal{Z}(\mathcal{S}_1 \oplus \mathcal{S}_2) = \mathcal{Z}(\mathcal{S}_1) \cup \mathcal{Z}(\mathcal{S}_2)$*

3. *$\mathcal{Z}(\mathcal{T}) = \emptyset$*

4. *$\mathcal{Z}(\emptyset) = \mathrm{Spc}\,\mathcal{T}$*

where $\mathcal{S}_1 \oplus \mathcal{S}_2 = \{a \oplus b \mid a \in \mathcal{S}_1, b \in \mathcal{S}_2\}$.

Proof.

1. $\mathcal{P} \in \bigcap_{i \in I} \mathcal{Z}(\mathcal{S}_i)$ if and only if $\mathcal{P} \cap \mathcal{S}_i = \emptyset$ for all $i \in I$ if and only if $\mathcal{P} \cap \left(\bigcup_{i \in I} \mathcal{S}_i\right) = \emptyset$ if and only if $\mathcal{P} \in \mathcal{Z}(\bigcup_{i \in I} \mathcal{S}_i)$.

2. If $\mathcal{P} \notin \mathcal{Z}(\mathcal{S}_1 \oplus \mathcal{S}_2)$ then $\exists a \oplus b \in \mathcal{P}$ where $a \in \mathcal{S}_1, b \in \mathcal{S}_2$. But as $\mathcal{P}$ is a thick $\otimes$-ideal $a, b \in \mathcal{P}$, so $\mathcal{P}$ intersects both $\mathcal{S}_1$ and $\mathcal{S}_2$ nontrivially and therefore $\mathcal{P} \notin \mathcal{Z}(\mathcal{S}_1) \cup \mathcal{Z}(\mathcal{S}_2)$. On the other hand, if $\mathcal{P}$ is not in either $\mathcal{Z}(\mathcal{S}_1)$ or $\mathcal{Z}(\mathcal{S}_2)$ then $\mathcal{P}$ must intersect both $\mathcal{S}_1$ and $\mathcal{S}_2$, so take $a \in \mathcal{P} \cap \mathcal{S}_1$ and $b \in \mathcal{P}\,\mathcal{S}_2$. Then $a \oplus b \in \mathcal{P}$ and therefore $\mathcal{P} \cap \mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$, so $\mathcal{P} \notin \mathcal{Z}(\mathcal{S}_1 \oplus \mathcal{S}_2)$.

3. and 4. are obvious from the definition. $\qquad\square$

It immediately follows that we can define a Zariski topology on $\mathrm{Spc}\,\mathcal{T}$ with closed sets $\mathcal{Z}(S)$.

Definition 2.4. For any $a \in \mathrm{Ob}(\mathcal{T})$, we define $\mathrm{Supp}(a) = \{\mathcal{P} \mid a \notin \mathcal{P}\}$.

We could also define the topology using open sets of the form below:

Definition 2.5. Given $\mathcal{Z}(\mathcal{S})$ a closed set in $\mathrm{Spc}\,\mathcal{T}$ define

$$U(\mathcal{S}) := \mathrm{Spc}\,\mathcal{T} \setminus \mathcal{Z}(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}\,\mathcal{T} : \mathcal{S} \cap \mathcal{P} \neq \emptyset\}$$

Definition 2.6. A subset $\mathcal{S} \subseteq \mathcal{T}$ is called a $\otimes$-**multiplicative** subset if for any $a.b \in \mathcal{S}$, $a \otimes b \in \mathcal{S}$.

Proposition 2.7. *Let $\mathcal{T}$ be a non-zero tt-category. Then,*

1. *If $S \subseteq \mathcal{T}$ is a $\otimes$-multiplicative subset which does not contain 0, then there exists $\mathcal{P} \in \mathrm{Spc}\,\mathcal{T}$ such that $\mathcal{S} \cap \mathcal{P} = \emptyset$.*

2. If $\mathcal{C} \subseteq \mathcal{T}$ is a proper thick $\otimes$-ideal then there exists a proper maximal $\otimes$-ideal $\mathcal{M} \subseteq \mathcal{T}$ such that $\mathcal{C} \subseteq \mathcal{M} \subset \mathcal{T}$.

3. $\operatorname{Spc} \mathcal{T} \neq \emptyset$

The proof of this proposition follows easily from an application of the lemma below to the ideals generated by 0, $\mathcal{C}$, and 1 respectively.

Lemma 2.8. *Let $\mathcal{I}$ be a $\otimes$-ideal of $\mathcal{T}$ and $\mathcal{S} \subseteq \mathcal{T}$ a $\otimes$-multiplicative subset. If $\mathcal{I} \cap \mathcal{S} = \emptyset$ then there exists $\mathcal{P} \in \operatorname{Spc} \mathcal{T}$ such that $\mathcal{I} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S} = \emptyset$.*

Proof. Define $\mathcal{J} := \{a \in \mathcal{T} : \exists s \in S, a \otimes s \in \mathcal{I}\}$. I claim that this is a thick $\otimes$-ideal containing $\mathcal{I}$. Clearly $\mathcal{I} \subseteq \mathcal{J}$ since $a \otimes x \in \mathcal{I}$ for any $a \in \mathcal{I}$ and any $x \in \mathcal{T}$. Now suppose that $a \xrightarrow{f} b$ is a morphism in $\mathcal{J}$. Since $a, b \in \mathcal{J}$ there are $s, s' \in S$ such that $a \otimes s, b \otimes s \in \mathcal{I}$. In fact, we can just take $s = s'$ by replacing s with $s \otimes s'$ since $a \otimes s \in \mathcal{I}$ implies that $a \otimes s \otimes s' \in \mathcal{I}$ and $s \otimes s' \in S$. Now examine the diagram below:

$$
\begin{array}{ccccccc}
a \otimes s & \xrightarrow{f \otimes \mathrm{id}_s} & b \otimes s & \longrightarrow & \operatorname{cone}(f \otimes \mathrm{id}_s) & \longrightarrow & \Sigma(a \otimes s) \\
\| & & \| & & \updownarrow & & \| \\
a \otimes s & \xrightarrow{f \otimes \mathrm{id}_s} & b \otimes s & \longrightarrow & \operatorname{cone}(f) \otimes s & \longrightarrow & \Sigma(a) \otimes s
\end{array}
$$

the dotted arrow exists by TC3 of Definition 1.2. By proposition 3.5 $\operatorname{cone}(f) \otimes s \cong \operatorname{cone}(f \otimes \mathrm{id}_s)$. Since $\mathcal{I}$ is a thick subcategory and $a \otimes s, b \otimes s \in \mathcal{I}$ it follows that $\operatorname{cone}(f) \otimes s \cong \operatorname{cone}(f \otimes \mathrm{id}_s) \in \mathcal{I}$. Therefore, $\operatorname{cone}(f) \in \mathcal{J}$, and by the same token we can see that $\mathcal{J}$ is closed under Σ, so $\mathcal{J}$ is a triangulated subcategory of $\mathcal{T}$. Now suppose that $a, b \in \mathcal{J}$. It follows from corollary 3.10 that $a \oplus b \in \mathcal{J}$, so $\exists s \in S$ such that $(a \oplus b) \otimes s \in \mathcal{I}$. Then, since $(a \oplus b) \otimes s \cong (a \otimes s) \oplus (b \otimes s)$ it follows that $a \otimes s, b \otimes s \in \mathcal{I}$, so $a, b \in \mathcal{J}$, so $\mathcal{J}$ is a thick triangulated subcategory. To see that $\mathcal{J}$ is a $\otimes$-ideal, let $a \in \mathcal{J}$ with $s \in \mathcal{S}$ such that $a \otimes s \in \mathcal{I}$, and since $\mathcal{I}$ is a $\otimes$-ideal we see that for any $b \in \mathcal{T}$, $(a \otimes s) \otimes b \in \mathcal{I}$, but then by associativity of $\otimes$ we have $a \otimes b \in \mathcal{J}$. Additionally, $\mathcal{J} \cap \mathcal{S} = \emptyset$ since if $a \in \mathcal{S} \cap \mathcal{J}$ then $\exists s \in \mathcal{S}$ such that $a \otimes s \in \mathcal{I}$, but then $a \otimes s \in \mathcal{I} \cap \mathcal{S}$ as $a, s \in \mathcal{S}$ implies that $a \otimes s$ as $\mathcal{S}$ is $\otimes$-multiplicatively closed, which is a contradiction.

Now let C be the collection of thick $\otimes$-ideals $\mathcal{J}$ where

(1) $\mathcal{J} \cap \mathcal{S} = \emptyset$,

(2) If $a \in \mathcal{T}$ such that there is some $s \in \mathcal{S}$ where $a \otimes s \in \mathcal{J}$, then $a \in \mathcal{J}$

(3) $\mathcal{I} \subseteq \mathcal{J}$

The paragraph above shows that C is nonempty. Now let $\{J_i\}$ be an ascending chain (via containment) within C. Then clearly the full subcategory of $\mathcal{T}$ generated by the union of the objects within $\{J_i\}$ is also a thick $\otimes$-ideal satisfying the conditions above, so by Zorn's lemma there exists a maximal element $\mathcal{P}$ of C. To see that $\mathcal{P}$ is prime, suppose that $a \otimes b \in \mathcal{P}$ and $b \notin P$. Then define $(\mathcal{P} : a)$ to be the full subcategory of $\mathcal{T}$ generated by $\{x \in \mathcal{T} : x \otimes a \in \mathcal{P}\}$. This subcategory is easily shown to be a thick $\otimes$-ideal of $\mathcal{T}$ by essentially the same methods used to show that $\mathcal{J}$ above was a thick $\otimes$-ideal. If $x \in \mathcal{P}$ then $x \otimes a \in \mathcal{P}$ by definition of a $\otimes$-ideal, so $\mathcal{P} \subset (\mathcal{P} : a)$, but $b \in (\mathcal{P} : a) \setminus \mathcal{P}$, so the containment is proper. Hence $(\mathcal{P} : a) \notin C$, so one of the three properties above fails for $(\mathcal{P} : a)$. Clearly it can't be condition (3), so suppose that $x \in \mathcal{T}$ such that there

is some $s \in \mathcal{S}$ where $x \otimes s \in (\mathcal{P} : a)$. Then $x \otimes s \otimes a \in \mathcal{P}$, but by condition (2) this implies that $x \otimes a \in \mathcal{P}$, and therefore $x \in (\mathcal{P} : a)$. Therefore the only condition that $(\mathcal{P} : a)$ could fail is (2), which means that $\exists s \in (\mathcal{P} : a) \cap S$, so $a \otimes s \in \mathcal{P}$. But then $a \in \mathcal{P}$ by condition (2). Hence $\mathcal{P}$ is prime. $\qquad\square$

Corollary 2.9. *Maximal thick $\otimes$-ideals are prime.*

By definition a is contained in every $\mathcal{P}$ in $\operatorname{Spc}\mathcal{T}$ if and only if $U(a) = \operatorname{Spc}\mathcal{T}$. Similarly, a is contained in every $\mathcal{P} \in \operatorname{Spc}\mathcal{T}$ if and only if $\operatorname{Supp} a = \emptyset$.

Corollary 2.10.

$$\bigcap_{\mathcal{P}\in\operatorname{Spc}\mathcal{T}} \mathcal{P} = \{a \in \mathcal{T} : U(a) = \operatorname{Spc}\mathcal{T}\}$$
$$= \{a \in \mathcal{T} : \operatorname{Supp}(a) = \emptyset\}$$
$$= \{a \in \mathcal{T} : a^{\otimes n} = 0 \text{ for some } n \in \mathbb{N}\}$$

Proof. One direction is obvious, as $0 \in \mathcal{P}$ for any $\mathcal{P} \in \operatorname{Spc}\mathcal{T}$. Now suppose that $a^{\otimes n} \neq 0$ for any $n \in \mathbb{N}$ and let $\mathcal{S} = \{a^{\otimes n}\}_{n\in\mathbb{N}}$. $\mathcal{S}$ does not contain 0 so by proposition 2.7, there exists $\mathcal{P} \in \operatorname{Spc}\mathcal{T}$ such that $\mathcal{S} \cap \mathcal{P} = \emptyset$, but then $\operatorname{Supp}(a) \neq \emptyset$. By the contrapositive we both containments. $\qquad\square$

Corollary 2.11. *An object $a \notin \mathcal{P}$ for all $\mathcal{P} \in \operatorname{Spc}\mathcal{T}$ if and only if $\langle a \rangle^{\otimes} = \mathcal{T}$*

Proposition 2.12. *Some properties of open sets of the form $U(a) = \{\mathcal{P} \in \operatorname{Spc}\mathcal{T} : a \in \mathcal{P}\}$ where $a \in \operatorname{Obj}(\mathcal{T})$.*

  1. $U(0) = \operatorname{Spc}\mathcal{T}$

  2. $U(1) = \emptyset$

  3. $U(a \otimes b) = U(a) \cap U(b)$

  4. $U(\Sigma a) = U(a)$

  5. $U(a) \supset U(b) \cap U(c)$ *For any distinguished triangle $(a, b, c; f, g, h)$.*

  6. $U(a \otimes b) = U(a) \cup U(b)$

Remark 2.13. The interaction between the tt-structure and the Zariski topology can be interpreted as the support remembering more than just the additive structure.

Corollary 2.14. *The collection $\{U(a) : a \in \mathcal{T}\}$ provides an (open) basis for the topology. The collection $\{\operatorname{Supp}(a) = \operatorname{Spc}\mathcal{T} \setminus U(a)\}$ provides a (closed) basis for the topology.*

2.2 $\operatorname{Spc}\mathcal{T}$ is a Spectral Space

So far, we have defined prime ideals ideals and equipped $\mathcal{T}$ with a topology that is analogous to the Zariski topology found in the world of commutative rings. It is natural for us to then ask if this topology is merely analogous to spectra on rings. Does the connection go deeper than analogy? To answer this question we have to first know what $\operatorname{Spec} T$ looks like for a commutative ring R. It turns out that for any R a commutative ring, $\operatorname{Spec} R$ is what is called a *spectral space*.

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Definition 2.15. A topological space X is called **irreducible** if it cannot be written as the union of two proper closed subsets of X.

A topological space X is called **sober** if every nonempty irreducible closed subset of X is the closure of exactly one point of X. Given such an irreducible closed subset Y of X the unique point $y \in X$ such that $\bar{y} = Y$, the point y is called the **generic point** of Y.

Definition 2.16. Let X be a topological space and $\mathcal{K}^\circ(X)$ its set of quasi-compact open subsets of X. X is **spectral** if

1. X is quasi-compact and T_0.

2. $\mathcal{K}^\circ(X)$ is a basis for X.

3. $\mathcal{K}^\circ(X)$ is closed under finite intersections.

4. X is sober.

Given spectral spaces X and Y, a **spectral map** $X \xrightarrow{f} Y$ is a continuous map such that for any quasi-compact open $U \subseteq Y$ the preimage $f^{-1}(U)$ is quasi-compact.

Remarkably, Hochster showed in [Hoc69] that any spectral space may be realized as the spectrum of a ring. We will now proceed to show that $\operatorname{Spc} \mathcal{T}$ is also spectral.

Proposition 2.17. *Let* $\mathcal{P} \in \operatorname{Spc} \mathcal{T}$. *Then* $\overline{P} = \{\mathcal{Q} \in \operatorname{Spc} \mathcal{T} : \mathcal{Q} \subseteq \mathcal{P}\}$.

Proof. Let $\mathcal{S} = \mathcal{T} \setminus \mathcal{P}$. Then,

$$
\begin{aligned}
\mathcal{Z}(\mathcal{S}) &= \mathcal{Z}(\mathcal{T} \setminus \mathcal{P}) \\
&= \{\mathcal{Q} : \mathcal{Q} \cap (\mathcal{T} \setminus \mathcal{P}) = \emptyset\} \\
&= \{\mathcal{Q} : \mathcal{Q} \subseteq \mathcal{P}\}
\end{aligned}
$$

Clearly $\mathcal{P} \in \mathcal{Z}(\mathcal{S})$ and by definition $\mathcal{Z}(\mathcal{P})$ is closed, so $\overline{P} \subseteq \mathcal{Z}(\mathcal{S})$. Now suppose that $\mathcal{P} \subseteq \mathcal{Z}(\mathcal{S}')$ where $\mathcal{S}'$ is some other collection of objects in $\mathcal{T}$. Then $\mathcal{S}' \cap \mathcal{P} = \emptyset$, so $\mathcal{S}' \subseteq \mathcal{T} \setminus \mathcal{P} = \mathcal{S}$, which implies that $\mathcal{Z}(\mathcal{S}) \subseteq \mathcal{Z}(\mathcal{S}')$, and therefore $\mathcal{Z}(\mathcal{S})$ is minimal amongst closed sets containing $\mathcal{P}$. $\square$

Remark 2.18. This may be confusing to people used to thinking about $\operatorname{Spec} R$ for a commutative ring R, so one must be carful when switching between these two contexts. However, it is a hint towards the relationship between prime ideals in R and the prime $\otimes$-ideals of $\operatorname{Spc} D^{\mathrm{perf}}(R)$.

Corollary 2.19. *If* $\mathcal{P}_1, \mathcal{P}_2 \in \operatorname{Spc} \mathcal{T}$ *and* $\overline{\mathcal{P}_1} = \overline{\mathcal{P}_2}$, *then* $\mathcal{P}_1 = \mathcal{P}_2$. *Consequently,* $\operatorname{Spc} \mathcal{T}$ *is* T_0.

Proof. Immediate from proposition 2.17. $\square$

Proposition 2.20. *There exists a minimal prime in* $\mathcal{T}$, *that is, there exists a prime* $\otimes$*-ideal* $\mathcal{P} \in \operatorname{Spc} \mathcal{T}$ *such that if* $\mathcal{Q} \in \operatorname{Spc} \mathcal{T}$ *where* $\langle 0 \rangle^\otimes \subseteq \mathcal{Q} \subseteq \mathcal{P}$, *then either* $\mathcal{Q} = \langle 0 \rangle^\otimes$ *or* $\mathcal{Q} = \mathcal{P}$.

Proof. The proof is more or less identical to the proof using Zorn's lemma used in standard ring theory. $\square$

Remark 2.21. A point is called **closed** if $x = \bar{x}$. By proposition 2.17 the only closed points of $\operatorname{Spc} \mathcal{T}$ are exactly minimal primes. As a result, any closed set of $\operatorname{Spc} \mathcal{T}$ contains a closed point.

Lemma 2.22. *Let $a \in \mathcal{T}$ and $\mathcal{S} \subseteq \mathcal{T}$ be a collection of objects. Then the following are equivalent.*

(a) $U(a) \subseteq U(\mathcal{S})$ (or equivalently $\mathcal{Z}(\mathcal{S}) \subseteq \mathrm{Supp}(a)$).

(b) There exist finitely many objects $b_1, ..., b_n \in \mathcal{S}$ such that $\langle a \rangle^{\otimes}$.

Proof. $(a) \implies (b)$. Let $\mathcal{I} = \langle a \rangle^{\otimes}$ and $\mathcal{S}'$ be all finite tensor products of elements in $\mathcal{S}$ and suppose that $\mathcal{I} \cap \mathcal{S}' = \emptyset$. Then by lemma 2.8 there exists $\mathcal{P} \in \mathrm{Spc}\,\mathcal{T}$ such that $\mathcal{I} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{S}' = \emptyset$. Therefore $\mathcal{P} \in U(a)$, but $\mathcal{P} \notin U(\mathcal{S})$ as $U(\mathcal{S}) \subseteq U(\mathcal{S}')$. By the contrapositive, $U(a) \subseteq U(\mathcal{S})$ implies that $I \cap \mathcal{S}' \neq \emptyset$.

$(b) \implies (a)$. Assume (b). Let $\mathcal{P} \in U(a)$. Then $a \in \mathcal{P}$ so $\langle a \rangle^{\otimes} \subseteq \mathcal{P}$. In particular, $b_1, ..., b_n \in \mathcal{P}$. Since $\mathcal{P}$ is a prime ideal, there exists some $b_j \in \mathcal{P}$, so $\mathcal{P} \cap \mathcal{S} \neq \emptyset$. Hence $\mathcal{P} \subseteq U(\mathcal{S})$. $\square$

Proposition 2.23.

(a) $\mathrm{Spc}\,\mathcal{T}$ is quasi-compact.

(b) $U(a)$ is quasi-compact for all $a \in \mathcal{T}$.

(c) Any quasi-compact open set has the form $U(a)$ for some $a \in \mathcal{T}$.

Proof. Since $U(0) = \mathrm{Spc}\,\mathcal{T}$ we get (a) from (b).

We will first prove (b). Let $a \in \mathcal{T}$ and $U(a) \subseteq \bigcup_{i \in I} U(\mathcal{S}_i)$ an open cover where I is an indexing set. Then set $\mathcal{S} := \bigcup_{i \in I} \mathcal{S}_i$. Then $U(\mathcal{S}) = \bigcup_{i \in I} U(\mathcal{S})$. By lemma 2.22 there exists a finite subset $\mathcal{S}_1, ..., \mathcal{S}_n$ and objects $b_1, ..., b_n$ where $b_i \in \mathcal{S}$ for $1 \leq i \leq n$ such that $b_1 \otimes ... \otimes b_n \in \langle a \rangle^{\otimes}$. Now let $\mathcal{P} \in U(a)$. Then $\langle a \rangle^{\otimes} \subseteq \mathcal{P}$ and therefore $b_1 \otimes ... \otimes b_n \in \mathcal{P}$ and since $\mathcal{P}$ is prime there is some $b_j \in \mathcal{P}$. Thus $\mathcal{P} \cap \mathcal{S}_j \neq \emptyset$ and so $\mathcal{P} \subseteq U(S_j)$. Hence, $U(a)$ is covered by the finite subcover $U(\mathcal{S})_1, ..., U(\mathscr{S})_n$.

Now let $U(\mathcal{S})$ be a quasi-compact set. Then

$$U(\mathcal{S}) = \bigcup_{a \in \mathcal{S}} U(a)$$
$$= U\left(\bigcup_{a \in \mathcal{S}} a\right)$$
$$= U(a_1) \cup ... \cup U(a_n) \text{by } (b)$$
$$= U(a_1 \otimes ... \otimes a_n)$$

Remark 2.24. Given proposition 2.23 might be tempting to think that $\mathrm{Spc}\,\mathcal{T}$ is Noetherian (as a topological space) if and only if any $\mathcal{Z} \subseteq \mathrm{Spc}\,\mathcal{T}$ can be realized as $\mathrm{Supp}(a)$ for some $a \in \mathcal{T}$, but this is not the case as has been shown in [ADD REFERENCE].

$\square$

Corollary 2.25. *The set of quasi-compact open subsets of $\mathrm{Spc}\,\mathcal{T}$ form a basis for the space, and this basis is closed under intersections.*

It remains to show that $\operatorname{Spc}\mathcal{T}$ is a sober space.

Proposition 2.26. *Any closed irreducible subset $\mathcal{Z} \subseteq \operatorname{Spc}\mathcal{T}$ has a unique generic point, making $\operatorname{Spc}\mathcal{T}$ a sober space.*

Proof. Our goal is to be able to find for any $\mathcal{Z}$ a $\mathcal{P} \in \operatorname{Spc}\mathcal{T}$ such that $\mathcal{Z} = \overline{\mathcal{P}}$. Uniqueness of the generic point $\mathcal{P}$ will follow from corollary 2.19. Suggestively denote $\mathcal{P} := \{a \in \mathcal{T} : U(a) \cap \mathcal{Z} = \emptyset\}$. We will show that $\mathcal{P}$ is a prime $\otimes$-ideal and that $\mathcal{Z} = \mathcal{P}$.

Currently $\mathcal{P}$ is merely a full subcategory of $\mathcal{T}$ and it must be upgraded to the status of prime $\otimes$-ideal. For closure of $\oplus$, let $a, b \in \mathcal{P}$. Suppose that $a \oplus b \notin \mathcal{P}$. Then

$$\emptyset = U(a \oplus b) \cap \mathcal{Z} = (U(a) \cap U(b)) \cap \mathcal{Z} = (U(a) \cap \mathcal{Z}) \cap (U(b) \cap \mathcal{Z})$$

This is a nonempty decomposition of $\mathcal{Z}$ into two disjoint and open subsets, which cannot happen as $\mathcal{Z}$ is irreducible, so $a \oplus b \in \mathcal{P}$.

Now let $a \xrightarrow{f} b$ be a morphism in $\mathcal{P}$. By TC1 f fits into a distinguished triangle $a \xrightarrow{f} b \xrightarrow{g} \operatorname{cone}(f) \xrightarrow{h} \Sigma a$ distinguished, and we want to see that $c \in \mathcal{P}$. Since $\langle a \oplus b \rangle^{\otimes}$ is the smallest $\otimes$-ideal containing a and b it follows that $c \in \langle a \oplus b \rangle^{\otimes}$. Hence $U(a \oplus b) \subseteq U(c)$, and since $a \oplus b \in \mathcal{P}$ by the paragraph above it follows that

$$\emptyset \neq U(a \oplus b) \cap \mathcal{Z} \subseteq U(c) \cap \mathcal{Z}$$

and therefore $c \in \mathcal{P}$.

Closure under the translation functor Σ is easy, as $U(\Sigma a) = U(a)$ by proposition 2.12, and so $a \in \mathcal{P}$ iff $\Sigma a \in \mathcal{P}$. We now have that $\mathcal{P}$ is a triangulated thick subcategory. Now let $a \in \mathcal{P}$ and $b \in \mathcal{T}$. Then

$$U(a \otimes b) \cap \mathcal{Z} = (U(a) \cup U(b)) \cap \mathcal{Z} = (U(a) \cap \mathcal{Z}) \cup (U(b) \cap \mathcal{Z})$$

Note that $U(a) \cap \mathcal{Z} \neq \emptyset$ as $\mathcal{P}$, so the intersection above is nonempty and therefore $a \otimes b \in \mathcal{P}$. The expression above also shows us that $\mathcal{P}$ is prime, as $U(a \otimes b) \cap \mathcal{Z} \neq \emptyset$ implies that either $U(a) \cap \mathcal{Z}$ or $U(b) \cap \mathcal{Z}$ is nonempty and therefore either a or b are found in $\mathcal{P}$.

It remains to show that $\mathcal{Z} = \overline{\mathcal{P}}$. By proposition 2.17 $\overline{\mathcal{P}} = \{\mathcal{Q} : \mathcal{Q} \subseteq \mathcal{P}\}$. Let $\mathcal{Q} \in \mathcal{Z}$ and $a \in \mathcal{Q}$. Since $\mathcal{Q} \in U(a)$, $U(a) \cap \mathcal{Z} \neq \emptyset$. Then $a \in \mathcal{P}$ and so $\mathcal{Q} \subseteq \mathcal{P}$ and therefore $\mathcal{Q} \in \overline{P}$ and therefore $\mathcal{Z} \subseteq \overline{\mathcal{P}}$. Since $\mathcal{Z}$ is itself closed it suffices to show that $\mathcal{P} \in \mathcal{Z}$ to finish the proof. By corollary 2.14 we can write

$$\mathcal{Z} = \bigcap_{\mathcal{Z} \subset \operatorname{Supp}(a)} \operatorname{Supp}(a)$$

If a is an object such that $\mathcal{Z} \subseteq \operatorname{Supp}(a)$ then $a \notin \mathcal{P}$, and thus $\mathcal{P} \in \operatorname{Supp} a$. As a was picked generally, $\mathcal{P} \in \mathcal{Z}$, and hence $\mathcal{Z} = \overline{\mathcal{P}}$. $\qquad\square$

Corollary 2.27. $\operatorname{Spc}\mathcal{T}$ *is a spectral space.*

2.3 Support Data

We have proved some abstract results about $\operatorname{Spc} \mathcal{T}$, but how do we compute $\operatorname{Spc} \mathcal{T}$?

Definition 2.28. A **support data** on a tt-category $\mathcal{T}$ is a pair (X, σ) where X is a topological space and σ is an assignment which associates to any object $a \in \mathcal{T}$ a closed subset $\sigma(a) \subseteq X$ such that:

(SD 1) $\sigma(0) = \emptyset$ and $\sigma(1) = X$,

(SD 2) and $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$.

(SD 3) $\sigma(\Sigma a) = \sigma(a)$,

(SD 4) $\sigma(b) \subseteq \alpha(a) \cup \sigma(c)$ whenever $a \to b \to c \to \Sigma a$ is distinguished

(SD 5) $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$,

A morphism $f : (X, \sigma) \to (Y, \tau)$ of support data on $\mathcal{T}$ is a continuous map $f : X \to Y$ such that $\sigma(a) = f^{-1}(\tau(a))$ for all objects $a \in \mathcal{T}$. A morphisms of support data is an isomorphism if and only if f is a homeomorphism.

Lemma 2.29. *Given a support data (X, σ) for $\mathcal{T}$, the full subcategory of $\mathcal{T}$ consisting of objects $\{a \in \mathcal{T} : \sigma(a) \subseteq Y\}$ is a thick $\otimes$-ideal in $\mathcal{T}$ for any subset $Y \subseteq X$.*

Proof. Denote $\mathcal{C}$ as the full subcategory generated by the objects $\{a \in \mathcal{T} : \sigma(a) \subseteq Y\}$ for some fixed $Y \subseteq X$. If $a \to b \to c \to \Sigma a$ is a distinguished triangle in $\mathcal{T}$ with any two of a, b, or c in $\mathcal{C}$ then the third must be in $\mathcal{C}$ via application of (SD 3),(SD 4), and rotation of the triangle, so $\mathcal{C}$ is triangulated. If $x \subseteq a \oplus b$ and $x \in \mathcal{C}$ then $a, b \in \mathcal{C}$ via (SD 2). Finally, if $a \in \mathcal{C}$ and $b \in \mathcal{T}$ then $a \otimes b \in \mathcal{C}$ by (SD 5). $\qquad\square$

Lemma 2.30. *Let X be a set and let $f_1, f_2 : X \to \operatorname{Spc}(\mathcal{T})$ be two maps such that $f_1^{-1}(\operatorname{Supp}(a)) = f_2^{-1}(\operatorname{Supp}(a))$ for all $a \in \mathcal{T}$. Then $f_1 = f_2$.*

Proof. Let $x \in X$. For any $a \in \mathcal{T}$ our assumption on f_1 and f_2 means $f_1(x) \in \operatorname{Supp}(a) \iff f_2(x) \in \operatorname{Supp}(a)$. It follows that

$$\bigcap_{f_1(x) \in \operatorname{Supp}(a)} \operatorname{Supp}(a) = \bigcap_{f_2(x) \in \operatorname{Supp}(a)} \operatorname{Supp}(a)$$

which is a closed set. But then the set above is equal to $\overline{\{f_1(x)\}}$ by [corollary 2.14](#). But then $\overline{f_1(x)} = \overline{f_2(x)}$, and since $\operatorname{Spc}(\mathcal{T})$ is a sober space, it must be that $f_1(x) = f_2(x)$. $\qquad\square$

Theorem 2.31. *Let $\mathcal{T}$ be a tt-category. The support data $(\operatorname{Spc} \mathcal{T}, \operatorname{Supp})$ is the final support data on $\mathcal{T}$, meaning that $(\operatorname{Spc} \mathcal{T}, \operatorname{Supp})$ is a support data on $\mathcal{T}$ and for any support data (X, σ) on $\mathcal{T}$ there exists a unique support map $f : (X, \sigma) \to (\operatorname{Spc} \mathcal{T}, \operatorname{Supp})$ where for $x \in X$*

$$f(x) = \{a \in \mathcal{T} : x \notin \sigma(a)\}$$

Proof. From [proposition 2.12](#) it follows that $(\operatorname{Spc}(\mathcal{T}), \operatorname{Supp})$ is a support data for $\mathcal{T}$ as $\operatorname{Supp}(a) = \operatorname{Spc}(\mathcal{T}) \setminus U(a)$. Let (X, σ) be a support data on $\mathcal{T}$ and let f be the map in the theorem statement. We must check that f is a support data morphism. Since $f(x) = \{a \in \mathcal{T} : \sigma(a) \subseteq X \setminus \{x\}\}$ it follows from [lemma 2.29](#) that $f(x)$ is indeed a thick $\otimes$-ideal. To see that it is prime, let $a \otimes b \in f(x)$.

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Then $x \notin \sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ and then $x \notin \sigma(a)$ or $x \notin \sigma(b)$, so $a \in f(x)$ or $b \in f(x)$ so $f(x)$ is prime. Then we check

$$f^{-1}(\mathrm{Supp}(a)) = \{x \in X : a \notin f(x)\} = \{x \in X : x \in \sigma(a)\} = \sigma(a)$$

This also yields continuity by the definition of the topology on $\mathrm{Spc}(\mathcal{T})$. $\qquad \square$

Example 2.32. If R is a commutative unital ring and M an R-module then the support of M is defined as $\mathrm{Supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) : M_{\mathfrak{p}} \neq 0\} \subseteq \mathrm{Spec}(R)$. If M is finitely generated then $\mathrm{Supp}(M) = V(\mathrm{Ann}_R(M))$ making $\mathrm{Supp}(M)$ closed in $\mathrm{Spec}(R)$. Additionally, if L, M, N are R-modules and $0 \to L \to M \to N \to 0$ is exact then $\mathrm{Supp}(M) = \mathrm{Supp}(L) \cup \mathrm{Supp}(N)$. This suggests a support data structure $(\mathrm{Spec}(R), \mathrm{Supp})$. However, the category of finitely generated R-modules is not triangulated. To fix this, we may generalize this by passing to the big derived category $D(R)$ and defining $\mathrm{Supp}(C^{\bullet}) := \{\mathfrak{p} \in \mathrm{Spec}(R) : C^{\bullet} \otimes_R^L k(p) \neq 0\}$. By regarding M as a complex concentrated in degree 0 we have that the two definitions agree. If we restrict our attention to $D^{\mathrm{perf}}(R)$ we then it is straightforward to check that $(\mathrm{Spec}(R), \mathrm{Supp})$ is a support data for $D^{\mathrm{perf}}(R)$. The same holds for $D^{\mathrm{perf}}(X)$ for X a "sufficiently nice" scheme. NOTE TO SELF: what does sufficiently nice mean?

Now we want to show that the universal support data is functorial. Going forward, if we wish to distinguish between different tt-categories we will differentiate supports by the notations $\mathrm{Supp}_{\mathcal{T}}(a)$ for the tt-category $\mathcal{T}$.

Proposition 2.33. *The spectrum is functorial, meaning that given a $\otimes$-exact $F : \mathcal{T} \to \mathcal{K}$, the map*

$$\mathrm{Spc}\, F : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{T})$$

given by $\mathcal{Q} \mapsto F^{-1}(\mathcal{Q})$ is well-defined and continuous. Furthermore, for all objects $a \in \mathcal{T}$,

$$(\mathrm{Spc}\, F)^{-1}(\mathrm{Supp}_{\mathcal{T}}(a)) = \mathrm{Supp}_{\mathcal{K}}(F(a))$$

This assignment defines a contravariant functor $\mathrm{Spc}(-)$ from essentially small tensor triangulated categories to the category of topological spaces.

Proof. Immediate from the construction. $\qquad \square$

Corollary 2.34. *If two $\otimes$-exact functors $F_1, F_2 : \mathcal{T} \to \mathcal{K}$ have the property that $\langle F_1(a) \rangle = \langle F_2(a) \rangle$ in $\mathcal{L}$ for any object $a \in \mathcal{T}$, then $\mathrm{Spc}\, F_1 = \mathrm{Spc}\, F_2$. In particular, agreement of F_1 and F_2 on objects implies that they have the same map on spectra.*

Proof. If $\mathcal{Q} \in \mathrm{Spc}(\mathcal{K})$ and $a \in \mathrm{Spc}(\mathcal{T})$ then

$$a \in F_i^{-1}(\mathcal{Q}) \iff \langle F_i(a) \rangle \subseteq \mathcal{Q}$$

Therefore, $\langle F_1(a) \rangle = \langle F_2(a) \rangle$ implies that $F_1^{-1}(\mathcal{Q}) = F_2^{-1}(\mathcal{Q})$ making $\mathrm{Spc}(F_1) = \mathrm{Spc}(F_2)$. $\qquad \square$

Corollary 2.35. *Suppose that a $\otimes$-exact functor $F : \mathcal{T} \to \mathcal{K}$ is essentially surjective. Then $\mathrm{Spc}\, F$ is a homeomorphism onto its image.*

Proof. Any thick $\otimes$-ideal is replete, i.e. closed under isomorphism, so $\langle F(F^{-1}(\mathcal{Q})) \rangle = \mathcal{Q}$ as F is essentially surjective. Therefore if $F^{-1}(\mathcal{Q}_1) = F^{-1}(Q_2)$ then $\mathcal{Q}_1 = \mathcal{Q}_2$, so $\mathrm{Spc}\, F$ is injective. The rest of the proof procedes similarly to the proof of the analogous statement for commutative rings. $\square$

NEXT: IDEMPOTENTS AND THE MAIN CLASSIFICATION THEOREM

3 Appendix A - Triangulated Categories

3.1 The Octohedral Axiom

Here is an undiagrammatic presentation of TC4: Given a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ and distinguished triangles $(X, Y, Z'; f_1, p_1, d_1)$, $(X, Z, Y'; g \circ f, p_2, d_2)$, and $(Y, Z, X'; g, p_3, d_3)$, there exist morphisms $Z' \xrightarrow{a} Y'$ and $Y' \xrightarrow{b} X'$ such that

(a) $(Z', Y', X'; a, b, \Sigma p_1 \circ d_3)$ is exact,

(b) the triple (id_X, g, a) is a morphism of triangles $(X, Y, Z'; f, p_1, d_1) \to (X, Z, Y'; g \circ f, p_2, d_2)$, and

(c) the triple (f, id_Z, b) is a morphism of triangles $(X, Z, Y'; g \circ f, p_2, d_2) \to (Y, Z, X'; g, p_3, d_3)$.

Here one should think of $Z' = \mathrm{cone}(f)$, $Y' = \mathrm{cone}(gf)$ and $X' = \mathrm{cone}(g)$.

Here is yet another presentation, utilizing a different diagram.
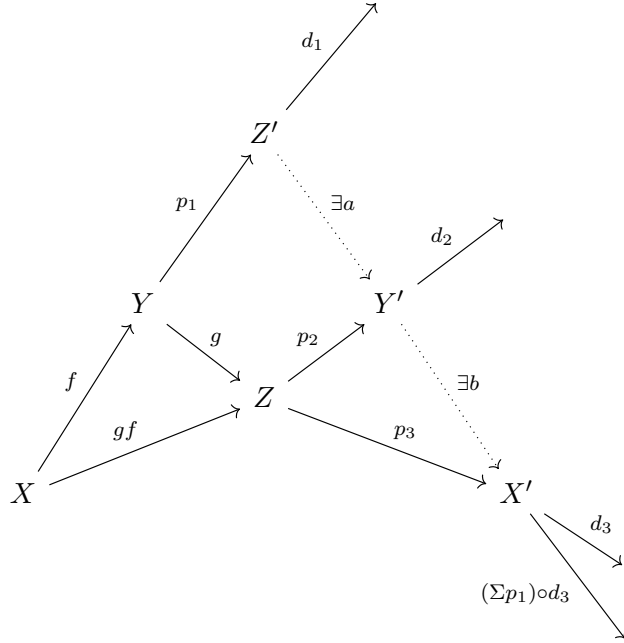
This view emphasizes some of the intuition of the octohedral axiom a little more clearly. Since exact triangles are supposed to be thought of as playing the role of exact sequences, we might want to think of our cones as quotients, i.e. $Z' = Y/X$, $Y' = Z/X$. For X' we have $X' = Z/Y$ from the triangle $Y \to Z \to X' \to \dots$ and $X' = Y'/Z'$ from the triangle $Z' \to Y' \to X' \to \dots$. Then,

$$(Z/X)/(Y/X) \cong (Y')/(Z') \cong X' \cong Z/Y$$

This looks a lot like the third isomorphism theorem, and so one way to view the octohedral axiom is as a coherence condition enforcing the kind of quotient isomorphisms that we expect to see in algebraic settings.

3.2 Definitions and Elementary Results

Much of the exposition here is adapted from [Sta18, Section 05QN].

Definition 3.1. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H : \mathcal{T} \to \mathcal{A}$ is called **homological** if for every distinguished triangle $(X, Y, Z; f, g, h)$ the sequence given by the image

$$H(X) \to H(Y) \to H(Z)$$

is exact in $\mathcal{A}$. An additive functor $H : \mathcal{T}^{\mathrm{opp}} \to \mathcal{A}$ is called **cohomological** if the corresponding opposite functor $\mathcal{T} \to \mathcal{A}^{\mathrm{opp}}$ is homological.

If $H : \mathcal{T} \to \mathcal{A}$ is homological then we'll write $H_n(X) := H(\Sigma^n X)$ and $H_0(X) := H(X)$. Then, this means that for every distinguished triangle $(X, Y, Z; f, g, h)$ we get a long exact sequence

$$\ldots \to H_{-1}(Z) \to H_0(X) \to H_0(Y) \to H_0(Z) \to H_0(\Sigma X) = H_1(X) \to \ldots$$

The long exact sequence associated to $(X, Y, Z; f, g, h)$ by H is called the **long exact sequence** associated to the triangle by H.

Definition 3.2. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ an abelian category. A δ-**functor** between $\mathcal{A}$ and $\mathcal{T}$ is functor $G : \mathcal{A} \to \mathcal{T}$ and functorial assignment from short exact sequences $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ to distinguished triangles in $\mathcal{T}$. Explicitly, for any $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ there is a morphism $\delta_{f,g} : G(C) \to \Sigma G(A)$ such that

   i. $(G(A), G(B), G(C); G(f), G(g), \delta_{f,g})$ is a distinguished triangle, and

   ii. For any morphism of short exact sequences $\phi : (A \xrightarrow{f} B \xrightarrow{g} C) \to (A' \xrightarrow{f'} B' \xrightarrow{g'} C')$ the diagram

$$
\begin{array}{ccc}
G(C) & \xrightarrow{\delta_{fg}} & \Sigma G(A) \\
{\scriptstyle G(\phi_C)}\downarrow & & \downarrow{\scriptstyle \Sigma G(\phi_A)} \\
G(C') & \xrightarrow[\delta_{f'g'}]{} & \Sigma G(A')
\end{array}
$$

The two definitions above are of critical importance as they axiomatize the relationship between short exact sequences and their derived long exact counterparts. As one should expect, for any $A \in \mathcal{T}$ the functor $\mathrm{Hom}_{\mathcal{T}}(A, -)$ is homological and $\mathrm{Hom}_{\mathcal{T}}(-, A)$ is cohomological.

Lemma 3.3. *If $\mathcal{T}$ is a triangulated category and $(X, Y, Z; f, g, h)$ is a distinguished triangle then $g \circ f, h \circ g, \Sigma f \circ h$ are all the zero map.*

Proof. From TC1 we know that $(X, X, 0; \mathrm{id}_X, 0, 0)$ is distinguished. Then by TC3 we know that the dashed map below exists making the diagram commute

$$
\begin{array}{ccccccc}
X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 & \longrightarrow & \Sigma X \\
\| & & \downarrow{\scriptstyle f} & & \vdots & & \| \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X
\end{array}
$$

but the dashed map must be the zero map, and so by commutivity $g \circ f = 0$. The other compositions follow from rotation of the triangle. □

Proposition 3.4. *For any object $A \in \mathcal{T}$ the functor $\mathrm{Hom}_{\mathcal{T}}(A, -)$ is homological and $\mathrm{Hom}_{\mathcal{T}}(-, A)$ is cohomological.*

Proof. Since $\mathcal{T}$ is an additive category $\mathrm{Hom}_{\mathcal{T}}(-, A)$ is an additive functor. By the lemma previous $\mathrm{Hom}_{\mathcal{T}}(-, A)$ takes distinguished triangles to chain complexes of abelian groups, so it remains to show exactness. Let $(X, Y, Z; f, g, h)$ be a distinguished triangle. Then using T3 and rotation, given $\phi \in \mathrm{Hom}_{\mathcal{T}}(A, Z)$ we can find ψ such that the diagram below commutes:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \exists\psi} & & \downarrow{\scriptstyle \phi} & & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X
\end{array}
$$

Therefore, $g \circ \psi = \phi$. We can do the same for each position in the triangle, and therefore we have exactness of the long exact sequence induced by $\mathrm{Hom}_{\mathcal{T}}(A, -)$. The proof for $\mathrm{Hom}_{\mathcal{T}}(-, A)$ is analogous. □

Proposition 3.5. *If $(\alpha, \beta, \gamma) : (X, Y, Z; f, g, h) \to (X', Y', Z'; f', g', h')$ is a morphisms of distinguished triangles such that any two of α, β, γ are isomorphisms, then the third is also an isomorphism.*

Proof. Without loss of generality assume that α and γ are isomorphisms. Then let $A \in \mathrm{Obj}(\mathcal{T})$. Abbreviate $\mathrm{Hom}_{\mathcal{T}}(A, -)$ as H_A. Then all the maps in the diagram below are isomorphisms, save for the middle one:

$$
\begin{array}{ccccccccc}
H_A(\Sigma Z) & \longrightarrow & H_A(X) & \longrightarrow & H_A(Y) & \longrightarrow & H_A(Z) & \longrightarrow & H_A(\Sigma X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_A(Z') & \longrightarrow & H_A(X') & \longrightarrow & H_A(Y') & \longrightarrow & H_A(Z') & \longrightarrow & H_A(\Sigma X')
\end{array}
$$

Then, by the 5-lemma, the middle map is an isomorphism, so $\mathrm{Hom}_{\mathcal{T}}(A, Y) \cong \mathrm{Hom}_{\mathcal{T}}(A, Y')$ via $\mathrm{Hom}_{\mathcal{T}}(\beta, A)$ for any arbitrary A. By Yoneda's lemma it follows that $Y \xrightarrow{\beta} Y'$ is an isomorphism. □

Definition 3.6. Let

Remark 3.7. This proof actually give us a little more than advertised. It says that if we have a morphism of (not necessarily distinguished) triangles $(\alpha, \beta, \gamma) : (X, Y, Z; f, g, h) \to (X', Y', Z'; f', g', h')$ such that the long exact sequences coming from $\mathrm{Hom}_{\mathcal{T}}(W, -)$ on the triangles are exact for all $W \in \mathrm{Obj}(\mathcal{T})$, then any two of α, β, γ being isomorphisms implies that the third is an isomorphism. We will call such triangles **special triangles** It's worth pointing this out since this conclusion is slightly stronger as there are triangles for which this condition holds that are not distinguished.

Corollary 3.8. *Given triangles $(X, Y, Z; f, g, h)$ and $(X', Y', Z'; f', g', h')$, the triangle*

$$(X \oplus X', Y \oplus Y', Z \oplus Z'; f + f', g + g', h + h')$$

is distinguished if and only if both $(X, Y, Z; f, g, h)$ and $(X', Y', Z'; f', g', h')$ are distinguished.

Proof. Assume that the two individual triangles are distinguished. By T2 there exists Q such that $(X \oplus X', Y \oplus Y', Q; f + f', g'', h'')$ is distinguished. By TC3 there are morphisms from $(X, Y, Z; f, g, h)$ and $(X', Y', Z'; f', g', h')$ into $(X \oplus X', Y \oplus Y', Q; f + f', g'', h'')$ induced by the inclusion maps from the components of the triangles into their direct sums. This in turn induces a map

$$(X \oplus X', Y \oplus Y', Z \oplus Z'; f + f', g + g', h + h') \xrightarrow{(\mathrm{id}, \mathrm{id}, \alpha)} (X \oplus X', Y \oplus Y', Q; f + f', g'', h'')$$

By proposition 3.5, α is an isomorphism and therefore the direct sum of triangles is distinguished.

Now suppose that the direct sum is distinguished. We will show that $(X, Y, Z; f, g, h)$ is distinguished, and by symmetry the result will follow. Using TC2 and TC3 there is a distinguished triangle $(X, Y, Q; f, g'', h'')$ and a morphism of distinguished triangles $(\pi_X, \Pi_Y, p) : (X \oplus X', Y \oplus Y', Z \oplus Z'; f + f', g + g', h + h') \to (X, Y, Q; f, g'', h'')$. We can then get a morphism of triangles $(X, Y, Z; f, g, h) \to (X, Y, Q; f, g'', h'')$ by precomposing with the inclusion maps coming from X, Y, Z. The long exact sequence on $(X \oplus X', Y \oplus Y', Z \oplus Z'; f + f', g + g', h + h')$ coming from $\mathrm{Hom}_{\mathcal{T}}(W, -)$ will split into the direct sum of exact complexes coming from $(X, Y, Z; f, g, h)$ and $(X', Y', Z'; f', g', h')$ and therefore $(X, Y, Z; f, g, h)$ satisfies the condition in remark 3.7. Therefore, the morphism $(X, Y, Z; f, g, h) \to (X, Y, Q; f, g'', h'')$ of triangles is a morphism on the third component since the other two components are the identity map. Hence, $(X, Y, Z; f, g, h)$ is isomorphism to a distinguished triangle and is therefore itself distinguished. $\square$

Corollary 3.9. *For any two objects A and B of $\mathcal{T}$, the triangle $(A, A \oplus B, B; (\mathrm{id}_A, 0), (0, \mathrm{id}_B), 0)$ is distinguished.*

Proof. Apply corollary 3.8 to the distinguished triangles $(A, A, 0; \mathrm{id}_A, 0, 0)$ and $(0, B, B; 0, \mathrm{id}_B, 0)$. $\square$

Corollary 3.10. *If $\mathcal{T}'$ is a triangulated subcategory of $\mathcal{T}$ and a, b are objects in $\mathcal{T}'$, then $a \oplus b \in \mathcal{T}'$.*

Proof. As a, b in $\mathcal{T}'$ it follows from corollary 3.9 and remark 1.6 that $a \oplus b \in \mathcal{T}'$. $\square$

References

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