Modeling Orbital Mechanics Through Various Methods in Python

Kyle Emmi, Braden Oh,
Madi Wyatt

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1 Introduction and Contextualization

Johannes Kepler first published his Laws of Planetary Motion in 1609 [1][2], creating the first accurate model of celestial orbits in our history. Over four centuries later, we still use these laws to calculate the trajectories that just sent Mars lander Insight to the surface of the Red Planet [3] or Japanese spacecraft Hayabusa 2 on the near Earth asteroid Ryugu [4]. These equations have helped us put men on the Moon, and also help us keep satellites in orbit around our planet, satellites that help us watch the weather, use GPS signals to get us from home to the store, and controls some of our television (satellite TV).We can even use these equations to help us calculate the mass of Earth, the other planets, and any other celestial bodies. Though we may not notice it, orbital mechanics has an impact on many parts of our everyday life.

In this project, we are creating two models of a two-body system, for example, a model of Earth and the Sun, or a satellite orbiting Earth. The first model is a Newtonian, Euler simulation that uses basic Newtonian physics and timesteps to create an orbit around the larger body. The second model uses the conservation of angular momentum to produce an equation derived orbital trajectory. If the two models create the same result, we have validated our models. We will be focusing only on a two-body system because it is easy to validate, as a three-body system, while more realistic (for example, an Earth, Moon, satellite trajectory) is not perfectly modeled mathematically.

2 Mathematical Background

The following mathematical proof comes from *Orbital Mechanics for Engineering Students* [5].

2.1 Law of Gravitation

We began with the vector form of Newton's basic law of gravitation from two points of view:

$$\vec{F_1} = \frac{Gm_1m_2}{r^3}\vec{r} \tag{1}$$

$$\vec{F_2} = \frac{-Gm_1m_2}{r^3}\vec{r} \tag{2}$$

where object 1 of mass m_1 and object 2 of mass m_2 are pulling on each other with forces F_1 and F_2 respectively. $\vec{r}$ is the vector starting at object 1 and extending to object 2. We assumed object 1 is a planet, such as the Earth, and object 2 is a satellite orbiting that planet.

2.2 Relative Motion

Once we defined our forces, we moved to acceleration. Recall that for a body with mass m, $\vec{F} = m\vec{a}$. When we rearranged this equation to solve for the acceleration of the body, we found that $\vec{a} = \frac{\vec{F}}{m}$. By substituting equations 1 and 2 for gravitational force into this equation for both a planet with mass m_1 and orbiting satellite with mass m_2 we found the individual accelerations of each.

$$\vec{a_1} = \frac{1}{m_1}\frac{Gm_1m_2}{r^3}\vec{r} = \frac{Gm_2}{r^3}\vec{r} \tag{3}$$

$$\vec{a_2} = \frac{1}{m_2}\frac{-Gm_1m_2}{r^3}\vec{r} = \frac{-Gm_1}{r^3}\vec{r} \tag{4}$$

Note that in these equations, $\vec{a_1}$ and $\vec{a_2}$ are measured relative to a global inertial frame's origin while r and $\vec{r}$ are the scalar and vector forms of the distance between the two objects. This is illustrated in Figure 1.

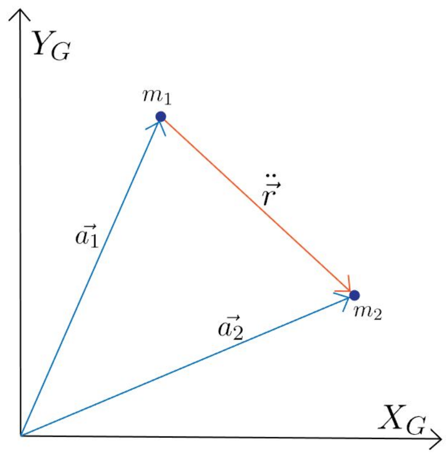

Figure 1: A visualization of the interactions between a_1, a_2, and $\ddot{\vec{r}}$

With the accelerations of two objects in a global frame, we could find the acceleration between the two by subtracting their global acceleration vectors.

$$\ddot{\vec{r}} = \vec{a_2} - \vec{a_1} \tag{5}$$

Substituting Equations 3 and 4 into equation 5 gave us

$$\ddot{\vec{r}} = -\frac{G(m_1 + m_2)}{r^3}\vec{r} \tag{6}$$

and an opportunity to create a new variable. We defined the gravitational parameter, μ, as

$$\mu = G(m_1 + m_2) \tag{7}$$

and used it to simplify Equation 6 into an equation used throughout orbital mechanics:

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r} \tag{8}$$

2.3 Momentum of Orbiting Bodies

In this section we used a form of angular momentum: specific angular momentum. This is simply momentum with the mass component divided away. Think of it as angular momentum per unit mass. It's equation is

$$\vec{h} = \vec{r} \times \dot{\vec{r}} \tag{9}$$

where $\vec{h}$ is the specific angular momentum of the body in orbit, $\vec{r}$ is vector from the planet to the orbiting body, and $\dot{\vec{r}}$ is the orbiting body's velocity vector. Figure 2 illustrates these vectors.

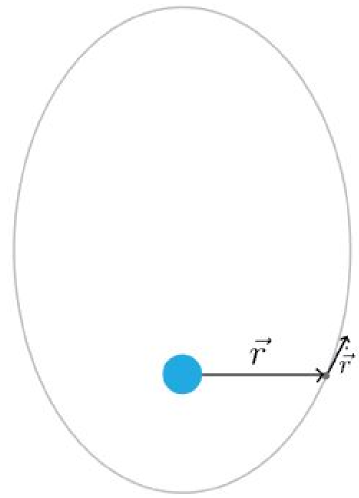

Figure 2: Definitions of the orbital vectors where $\vec{r}$ is position and $\dot{\vec{r}}$ is velocity

Once we had a definition for momentum, we utilized Equation 8 from the previous section and took the cross product of both sides with $\vec{h}$.

$$\ddot{\vec{r}} \times \vec{h} = -\frac{\mu}{r^3}\vec{r} \times \vec{h} \tag{10}$$

We could simplify the right side of Equation 10 by first splitting apart $\vec{h}$ using Equation 9

$$\frac{-\mu}{r^3}\vec{r} \times \vec{h} = \frac{-\mu}{r^3}[\vec{r} \times (\vec{r} \times \dot{\vec{r}})] \tag{11}$$

then using the bac-cab rule $[A \times (B \times C) = B(A \cdot C) - C(A \cdot B)]$.

$$\frac{-\mu}{r^3}\vec{r} \times \vec{h} = \frac{-\mu}{r^3}[\vec{r}(\vec{r} \cdot \dot{\vec{r}}) - \dot{\vec{r}}(\vec{r} \cdot \vec{r})] \tag{12}$$

We knew that $\vec{r} \cdot \dot{\vec{r}} = r\dot{r}$ and $\vec{r} \cdot \vec{r} = r^2$ so we could simplify to

$$\frac{-\mu}{r^3}\vec{r} \times \vec{h} = \frac{-\mu(\vec{r}\dot{r} - \dot{\vec{r}}r)}{r^2} \tag{13}$$

which we could further simplify with the quotient rule:

$$\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = -\frac{\vec{r}\dot{r} - r\dot{\vec{r}}}{r^2} \tag{14}$$

to give us a derivative.

$$\frac{-\mu}{r^3}\vec{r} \times \vec{h} = \frac{d}{dt}\left(\frac{\mu\vec{r}}{r}\right) \tag{15}$$

We could now use Equation 15 to substitute back into Equation 10, giving us

$$\ddot{\vec{r}} \times \vec{h} = \frac{d}{dt}\left(\frac{\mu\vec{r}}{r}\right) \tag{16}$$

We then simplified the left side by first taking the time derivative of $\dot{\vec{r}} \times \vec{h}$ with the product rule

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \dot{\vec{r}} \times \dot{\vec{h}} + \ddot{\vec{r}} \times \vec{h} \tag{17}$$

and using the fact that specific angular momentum is conserved ($\dot{\vec{h}} = 0$) to simplify to

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \ddot{\vec{r}} \times \vec{h} \tag{18}$$

and substitute back into Equation 16 giving us

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h}\right) = \frac{d}{dt}\left(\frac{\mu\vec{r}}{r}\right) \tag{19}$$

which can be factored to

$$\frac{d}{dt}\left(\dot{\vec{r}} \times \vec{h} - \frac{\mu\vec{r}}{r}\right) = 0 \tag{20}$$

This equation is stating that the quantity inside the derivative is unchanging since its time derivative is zero. We could therefore state that

$$\dot{\vec{r}} \times \vec{h} - \frac{\mu\vec{r}}{r} = \vec{C} \tag{21}$$

where $\vec{C}$ is some constant to be determined. Note here that $\vec{C}$ gained a vector hat. This is because both of the components of the left side of the equation have direction, therefore C must have direction as well.

We then defined a new variable, eccentricity or $\vec{e}$. This variable is a key characteristic of an orbit and allows simple classification of an orbit. An $\vec{e}$ value of less than one signifies an elliptical orbit, a value of zero signifies a circular orbit, and a value of greater than one signifies a hyperbolic trajectory. The equation is

$$\vec{e} = \frac{\vec{C}}{\mu} \tag{22}$$

A final note about $\vec{e}$: as it is a vector, it has direction. This direction happens to be a line in the plane of the orbit that passes from the planet being orbited through the periapse. This line is called the apse line (as it runs from periapse to apoapse on the other side of the orbit) and happens to be an excellent starting point for measuring an orbit (think of the starting point in the unit circle that everything is measured from). See Figure 3 for a visualization of the apse line.

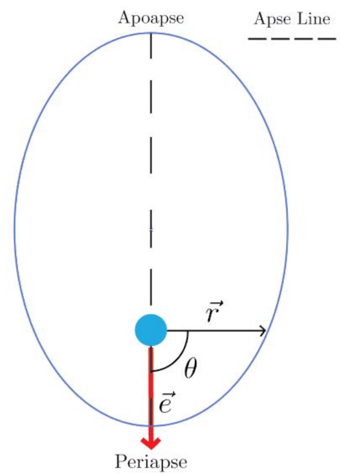

Figure 3: A map of how orbital position is defined using the Apse Line

Reordering Equation 22 into $\vec{C} = \mu\vec{e}$ and placing it into Equation 21 gave us

$$\dot{\vec{r}} \times \vec{h} - \mu\frac{\vec{r}}{r} = \mu\vec{e} \tag{23}$$

which could be simplified to

$$\frac{\vec{r}}{r} + \vec{e} = \frac{\dot{\vec{r}} \times \vec{h}}{\mu} \tag{24}$$

Next, we took the dot product of both sides of Equation 24 with $\vec{r}$. Which gave us

$$\frac{\vec{r}\cdot\vec{r}}{r} + \vec{r}\cdot\vec{e} = \frac{\vec{r}\cdot(\dot{\vec{r}} \times \vec{h})}{\mu} \tag{25}$$

The identity, $A \cdot (B \times C) = (A \times B) \cdot C$, and Equation 9 allowed us to simplify the right side of the equation with the following steps

$$\frac{\vec{r}\cdot(\dot{\vec{r}} \times \vec{h})}{\mu} = \frac{(\vec{r} \times \dot{\vec{r}})\cdot\vec{h}}{\mu} \tag{26}$$

$$= \frac{\vec{h}\cdot\vec{h}}{\mu} \tag{27}$$

$$= \frac{h^2}{\mu} \tag{28}$$

Substituting this back into Equation 25 and simplifying that dot product on the left gave us

$$r + \vec{r} \cdot \vec{e} = \frac{h^2}{\mu} \tag{29}$$

If we use the geometric version of the dot product (which replaces the vectors with magnitudes) and rearrange to solve for r, we achieve

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\cos\theta} \tag{30}$$

where θ is the angle between the apse line and the current orientation of the orbiting body. We were now very close to our final goal of position as a function of time.

We needed to derive one final equation before we could solve for time. The equation for angular momentum of a point mass is

$$L = Iw = r^2 m w \tag{31}$$

where L is angular momentum, I is the moment of inertia, m is mass, and w is angular velocity. We have been dealing with specific angular momentum so the mass component divides out to give us

$$h = r^2 \dot{\theta} \tag{32}$$

Notice how we replaced w with $\dot{\theta}$. This will come in handy later.

2.4 Incorporating Time

Once we had thoroughly tackled the problem of momentum, we were one integral away from having a time dependent definition of θ and consequently a time dependent definition of position.

When we took Equation 32 and rewrote it, we had

$$\frac{d\theta}{dt} = \frac{h}{r^2} \tag{33}$$

Substituting r from Equation 30 into this gave us

$$\frac{d\theta}{dt} = h\frac{\mu^2}{h^4}(1 + e\cos\theta)^2 \tag{34}$$

which was simplified to

$$\frac{\mu^2}{h^3} dt = \frac{d\theta}{(1 + e\cos\theta)^2} \tag{35}$$

This is a differential equation, and is set up to solve in Equation 36.

$$\int_0^t \frac{\mu^2}{h^3} dt = \int_0^\theta \frac{d\theta}{(1 + e\cos\theta)^2} \tag{36}$$

When we solved Equation 36, we found that

$$\frac{\mu^2}{h^3} t = \int_0^\theta \frac{d\theta}{(1 + e\cos\theta)^2} \tag{37}$$

Note that the limits of integration start at zero and go to the current time on the left and go from zero (at periapse) to the current angle on the right. For a circular orbit (where e is zero), the solution is trivial:

$$\frac{\mu^2}{h^3} t = \int_0^\theta \frac{d\theta}{(1 + 0)^2} \tag{38}$$

$$\theta = \frac{\mu^2}{h^3} t \tag{39}$$

However for an ellipse where e is less than one, we had to use the following integral proof

$$\int \frac{dx}{(a + b\cos x)^2} = d \tag{40}$$

$$d = \frac{\left(2a \tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2} - \frac{b\sqrt{a^2-b^2}\sin x}{a+b\cos x}\right)}{(a^2 - b^2)^{3/2}} \tag{41}$$

where b is always less than a. When we substituted our variables in, we found that

$$M_e = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1 + e\cos\theta} \tag{42}$$

where M_e, the mean anomaly, equals

$$\frac{\mu^2}{h^3}(1 - e^2)^{\frac{3}{2}} t \tag{43}$$

When we solved for theta we got

$$\theta = M_e + (2e - \frac{e^3}{4})\sin M_e + \frac{5}{4}e^2 \sin 2M_e + \frac{13}{12}e^3 \sin 3M_e \tag{44}$$

which will work for any circular or elliptical orbit.

With a combination of Equations 44 and 30 we could now define all orbits as a functions of time.

3 Quantitative Calculation

Once we had calculated the equations of motion we could now use them as functions to plot an orbit given an initial set of conditions: initial position and velocity. The code started with a program written by Carrie Nugent and evolved into our own creation. All of our parameters can be found in the table of Section 6. We assumed all orbits were occurring around the Earth. We utilized Python to create GIFs of orbital paths with varying initial parameters. A frame from one of these is shown below in Figure 4.

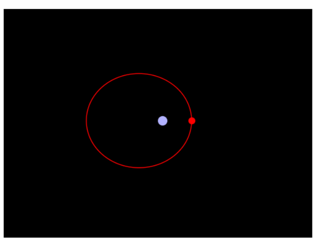

Figure 4: Orbit generated by equations of motion in Python showing an elliptical orbit

To ensure that this orbital path was correct, we created a second Python simulation. This simulation also took in initial position and velocity and created an orbital path with the Euler Method. For every time step, it evaluated Equation 4 with the Earth as the only other mass in the simulation. It then used the found acceleration to calculate a new velocity and position and continued on. Over enough time steps, this yielded a complete orbital path that we could compare with the equation derived path. We used the same parameters from Figure 4 to create an Euler simulation path shown in Figure 5

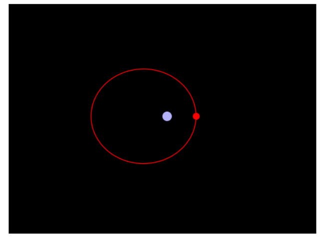

Figure 5: Orbit generated by Euler Method in Python showing a similar elliptical orbit to Figure 4

4 Results and Discussion

You can see that Figures 4 and 5 are indeed very similar. This is a good indicator that our equations of motion were derived correctly, however for additional proof we calculated the average error in position between the equation method and the Euler Method in ten different orbital configurations. This is shown in the table below.

Orbital Configuration	Average Error km
1	443 946
2	259 252
3	235 759
4	153 757
5	154 326
6	557 106
7	2 990 965
8	440 766
9	133 383
10	129 682

In addition to calculating error, we also overlaid the two simulations onto the same plot. One of these is shown in Figure 6. The rest are in the Additional Figures Section.

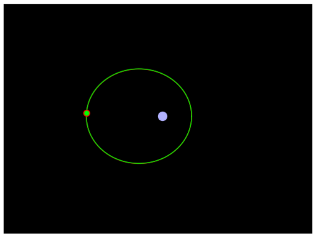

Figure 6: Overlapped orbits generated by Python

5 Conclusion

This project served as an introduction to the complex world of orbital mechanics. Various physical and mathematical models for orbit plotting exist and we have demonstrated that two of these models yield nearly identical results. The Newtonian force-gravity model of orbital mechanics has definite advantages in simulation, as it essentially Euler-steps through time, updating the state of each particle in a system at each time step with a finite set of rules (forces and gravitational interaction) that define how those particles interact. The orbital elements model of orbital mechanics has advantages when analyzing a particular orbit; discrete equations for a particular orbital scenario yield the position of a satellite as functions of orbital elements, angles, and times, allowing easy analysis of that satellite under a set of arbitrary conditions. We plotted various two-body problems with each mathematical model, comparing the outputs to each other.

Two clear next steps for these simulations are to implement a 3 dimensional orbit (the simulation actually already supports three dimensions, but for the sake of comparing orbits we only plotted in 2 dimensions) and to extend the Newtonian equation mode for n-bodies, as it currently supports only two.

Our simulations demonstrate that both models yield nearly identical plots of a 2-body problem (one satellite orbiting one celestial body). From interacting with both types of simulations we understand that various models of physics are strong in different ways, and that performing quantitative analysis with multiple models, depending on the application and to cross-check each other, is a powerful engineering tool.

6 Detailed Orbital Parameters

Configuration Number	Orbit Name	Initial Height (m)	Initial Velocity (m/s)
1	Geostationary	$4.215\,71 \times 10^7$	3.07×10^3
2	Low Earth	6.471×10^6	7.8×10^3
3	Medium Low Earth	2.6571×10^7	3.887×10^3
4	Heliosynchronous	7.071×10^6	7.713×10^3
5	ISS	6.791×10^6	7.66×10^3
6	Hubble	6.728×10^6	7.59×10^3
7	Ellipse 1	6.571×10^6	10.0×10^3
8	Ellipse 2	7.871×10^6	8.54×10^3
9	Ellipse 3	6.451×10^6	8.45×10^3
10	Ellipse 4	6.771×10^6	8.2×10^3

7 Additional Figures

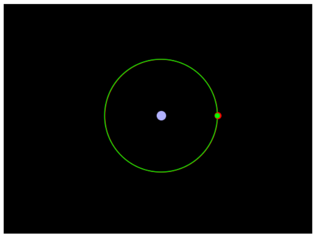

Figure 7: Orbital configuration 1

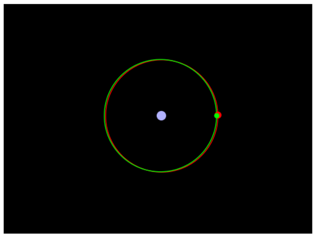

Figure 8: Orbital configuration 2

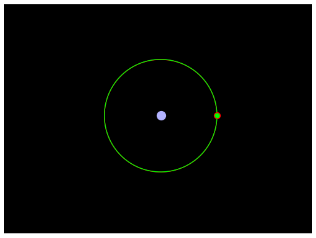

Figure 9: Orbital configuration 3

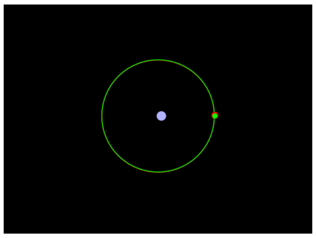

Figure 10: Orbital configuration 4

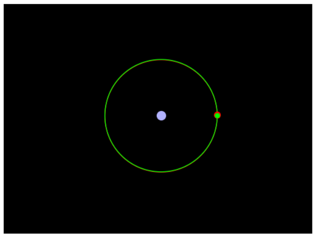

Figure 11: Orbital configuration 5

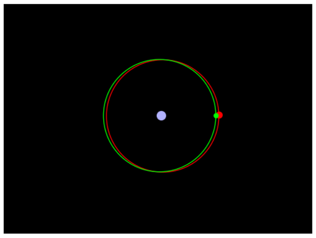

Figure 12: Orbital configuration 6

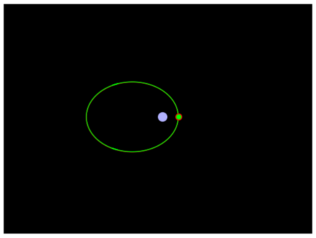

Figure 13: Orbital configuration 7

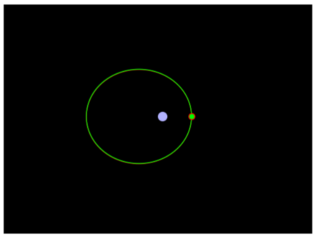

Figure 14: Orbital configuration 8

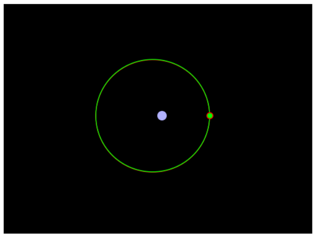

Figure 15: Orbital configuration 9

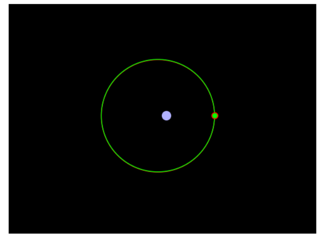

Figure 16: Orbital configuration 10

References

[1] Johannes Kepler. *Astronomia nova*. 1609

[2] Johannes Kepler. *Epitome Astronomiae Copernicanae*. 1617.

[3] "NASA's InSight Mars Lander," *NASA*, 06-Dec-2018. [Online]. Available: https://mars.nasa.gov/insight/.

[4] "Asteroid Explorer "Hayabusa2"" *Japan Aerospace Exploration Agency*, 2013 [Online]. Available: http://www.hayabusa2.jaxa.jp/en/

[5] Howard D. Curtis. *Orbital Mechanics for Engineering Students*, 2nd ed. Elsevier, Amsterdam, 2009.